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Randomly stopped sums with consistently varying distributions

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Abstract Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables, and η be a counting random variable independent of this sequence. We consider conditions for $\{\xi_1, \xi_2, \ldots\}$ and η under which the distribution function of the random sum $S_\eta = \xi_1 + \xi_2 + \cdots + \xi_\eta$ belongs to the class of consistently varying distributions. In our consideration, the random variables $\{\xi_1, \xi_2, \ldots\}$ are not necessarily identically distributed.

Keywords Heavy tail, consistently varying tail, randomly stopped sum, inhomogeneous distributions, convolution closure, random convolution closure

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1 Introduction

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting r.v., that is, an integer-valued, nonnegative, and nondegenerate at zero r.v. In addition, suppose that the r.v. η and r.v.s $\{\xi_1, \xi_2, \ldots\}$ are independent. Let $S_0 = 0$, $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \in \mathbb{N}$,

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and let

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the randomly stopped sum of r.v.s $\{\xi_1, \xi_2, \dots\}$.

We are interested in conditions under which the d.f. of S_η,

$$F_{S_\eta}(x) = \mathbb{P}(S_\eta \leqslant x) = \sum_{n=0}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n \leqslant x), \tag{1}$$

belongs to the class of consistently varying distributions.

Throughout this paper, $f(x) = o(g(x))$ means that $\lim_{x\to\infty} f(x)/g(x) = 0$, and $f(x) \sim g(x)$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$ for two vanishing (at infinity) functions f and g. Also, we denote the support of a counting r.v. η by

$$\mathrm{supp}(\eta) := \big\{n \in \mathbb{N}_0 : \mathbb{P}(\eta = n) > 0\big\}.$$

Before discussing the properties of F_{S_η}, we recall the definitions of some classes of heavy-tailed d.f.s, where $\overline{F}(x) = 1 - F(x)$ for all real x and a d.f. F.

- *A d.f. F is heavy-tailed ($F \in \mathcal{H}$) if for every fixed $\delta > 0$,*

$$\lim_{x\to\infty} \overline{F}(x)\mathrm{e}^{\delta x} = \infty.$$

- *A d.f. F is long-tailed ($F \in \mathcal{L}$) if for every y (equivalently, for some $y > 0$),*

$$\overline{F}(x + y) \sim \overline{F}(x).$$

- *A d.f. F has a dominatedly varying tail ($F \in \mathcal{D}$) if for every fixed $y \in (0, 1)$ (equivalently, for some $y \in (0, 1)$),*

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

- *A d.f. F has a consistently varying tail ($F \in \mathcal{C}$) if*

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

- *A d.f. F has a regularly varying tail ($F \in \mathcal{R}$) if*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}$$

  for some $\alpha \geqslant 0$ and any fixed $y > 0$.

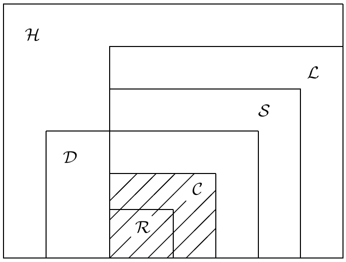

Fig. 1. Classes of heavy-tailed distributions.

- *A d.f. F supported on the interval $[0, \infty)$ is subexponential ($F \in \mathcal{S}$) if*

$$\lim_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2. \tag{2}$$

If a d.f. G is supported on $\mathbb{R}$, then we suppose that G is subexponential ($G \in \mathcal{S}$) if the d.f. $F(x) = G(x)\mathbb{1}_{[0,\infty)}(x)$ satisfies relation (2).

It is known (see, e.g., [4, 11, 13], and Chapters 1.4 and A3 in [8]) that these classes satisfy the following inclusions:

$$\mathcal{R} \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}, \qquad \mathcal{D} \subset \mathcal{H}.$$

These inclusions are depicted in Fig. 1 with the class $\mathcal{C}$ highlighted.

There exist many results on sufficient or necessary and sufficient conditions in order that the d.f. of the randomly stopped sum (1) belongs to some heavy-tailed distribution class. Here we present a few known results concerning the belonging of the d.f. F_{S_η} to some class. The first result on subexponential distributions was proved by Embrechts and Goldie (see Theorem 4.2 in [9]) and Cline (see Theorem 2.13 in [5]).

Theorem 1. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent copies of a nonnegative r.v. ξ with subexponential d.f. F_ξ. Let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If $\mathbb{E}(1+\delta)^\eta < \infty$ for some $\delta > 0$, then the d.f. $F_{S_\eta} \in \mathcal{S}$.*

Similar results for the class $\mathcal{D}$ can be found in Leipus and Šiaulys [14]. We present the statement of Theorem 5 from this work.

Theorem 2. *Let $\{\xi_1, \xi_2, \ldots\}$ be i.i.d. nonnegative r.v.s with common d.f. $F_\xi \in \mathcal{D}$ and finite mean $\mathbb{E}\xi$. Let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ with d.f. F_η and finite mean $\mathbb{E}\eta$. Then d.f. $F_{S_\eta} \in \mathcal{D}$ iff $\min\{F_\xi, F_\eta\} \in \mathcal{D}$.*

We recall only that the d.f. F belongs to the class $\mathcal{D}$ if and only if the upper Matuszewska index $J_F^+ < \infty$, where, by definition,

$$J_F^+ = -\lim_{y\to\infty} \frac{1}{\log y} \log\left(\liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}\right).$$

The random convolution closure for the class $\mathcal{L}$ was considered, for instance, in [1, 14, 16, 17]. We now present a particular statement of Theorem 1.1 from [17].

Theorem 3. *Let $\{\xi_1, \xi_2, \dots\}$ be independent r.v.s, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$ with d.f. F_η. Then the d.f. $F_{S_\eta} \in \mathcal{L}$ if the following five conditions are satisfied:*

(i) $\mathbb{P}(\eta \geqslant \kappa) > 0$ *for some* $\kappa \in \mathbb{N}$*;*

(ii) *for all* $k \geqslant \kappa$*, the d.f.* F_{S_k} *of the sum* S_k *is long tailed;*

(iii) $\sup\limits_{k\geqslant 1} \sup\limits_{x\in\mathbb{R}} (F_{S_k}(x) - F_{S_k}(x-1))\sqrt{k} < \infty$*;*

(iv) $\limsup\limits_{z\to\infty} \sup\limits_{k\geqslant\kappa} \sup\limits_{x\geqslant k(z-1)+z} \dfrac{\overline{F}_{S_k}(x-1)}{\overline{F}_{S_k}(x)} = 1$*;*

(v) $\overline{F}_\eta(ax) = o(\sqrt{x}\,\overline{F}_{S_\kappa}(x))$ *for each* $a > 0$.

We observe that the case of identically distributed r.v.s is considered in Theorems 1 and 2. In Theorem 3, r.v.s $\{\xi_1, \xi_2, \dots\}$ are independent but not necessarily identically distributed. A similar result for r.v.s having d.f.s with dominatedly varying tails can be found in [6].

Theorem 4 ([6], Theorem 2.1). *Let r.v.s $\{\xi_1, \xi_2, \dots\}$ be nonnegative independent, not necessarily identically distributed, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Then the d.f F_{S_η} belongs to the class $\mathcal{D}$ if the following three conditions are satisfied:*

(i) $F_{\xi_\kappa} \in \mathcal{D}$ *for some* $\kappa \in \operatorname{supp}(\eta)$*,*

(ii) $\limsup\limits_{x\to\infty} \sup\limits_{n\geqslant\kappa} \dfrac{1}{n\overline{F}_{\xi_\kappa}(x)} \sum\limits_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty$*,*

(iii) $\mathbb{E}\eta^{p+1} < \infty$ *for some* $p > J^+_{F_{\xi_\kappa}}$.

In this work, we consider randomly stopped sums of independent and not necessarily identically distributed r.v.s. As noted before, we restrict ourselves on the class $\mathcal{C}$. If r.v.s $\{\xi_1, \xi_2, \dots\}$ are not identically distributed, then different collections of conditions on $\{\xi_1, \xi_2, \dots\}$ and η imply that $F_{S_\eta} \in \mathcal{C}$. We suppose that some r.v.s from $\{\xi_1, \xi_2, \dots\}$ have distributions belonging to the class $\mathcal{C}$, and we find minimal conditions on $\{\xi_1, \xi_2, \dots\}$ and η for the distribution of the randomly stopped sum S_η to remain in the same class. It should be noted that we use the methods developed in [6] and [7].

The rest of the paper is organized as follows. In Section 2, we present our main results together with two examples of randomly stopped sums S_η with d.f.s having consistently varying tails. Section 3 is a collection of auxiliary lemmas, and the proofs of the main results are presented in Section 4.

2 Main results

In this section, we present three statements in which we describe the belonging of a randomly stopped sum to the class $\mathcal{C}$. In the conditions of Theorem 5, the counting r.v. η has a finite support. Theorem 6 describes the situation where no moment conditions on the r.v.s $\{\xi_1, \xi_2, \dots\}$ are required, but there is strict requirement for η. Theorem 7 deals with the opposite case: the r.v.s $\{\xi_1, \xi_2, \dots\}$ should have finite means, whereas the requirement for η is weaker. It should be noted that the case of real-valued r.v.s $\{\xi_1, \xi_2, \dots\}$ is considered in Theorems 5 and 6, whereas Theorem 7 deals with nonnegative r.v.s.

Theorem 5. *Let $\{\xi_1, \xi_2, \dots, \xi_D\}$, $D \in \mathbb{N}$, be independent real-valued r.v.s, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots, \xi_D\}$. Then the d.f. F_{S_η} belongs to the class $\mathcal{C}$ if the following conditions are satisfied:*

(a) $\mathbb{P}(\eta \leqslant D) = 1$,

(b) $F_{\xi_1} \in \mathcal{C}$,

(c) *for each $k = 2, \dots, D$, either $F_{\xi_k} \in \mathcal{C}$ or $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$.*

Theorem 6. *Let $\{\xi_1, \xi_2, \dots\}$ be independent real-valued r.v.s, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Then the d.f. F_{S_η} belongs to the class $\mathcal{C}$ if the following conditions are satisfied:*

(a) $F_{\xi_1} \in \mathcal{C}$,

(b) *for each $k \geqslant 2$, either $F_{\xi_k} \in \mathcal{C}$ or $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$,*

(c) $\limsup\limits_{x\to\infty} \sup\limits_{n\geqslant 1} \dfrac{1}{n\overline{F}_{\xi_1}(x)} \sum\limits_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty$,

(d) $\mathbb{E}\eta^{p+1} < \infty$ *for some* $p > J^+_{F_{\xi_1}}$.

When $\{\xi_1, \xi_2, \dots\}$ are identically distributed with common d.f. $F_\xi \in \mathcal{C}$, conditions (a), (b), and (c) of Theorem 6 are satisfied obviously. Hence, we have the following corollary.

Corollary 1 (See also Theorem 3.4 in [3])**.** *Let $\{\xi_1, \xi_2, \dots\}$ be i.i.d. real-valued r.v.s with d.f. $F_\xi \in \mathcal{C}$, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Then the d.f. F_{S_η} belongs to the class $\mathcal{C}$ if $\mathbb{E}\eta^{p+1} < \infty$ for some $p > J^+_{F_\xi}$.*

Theorem 7. *Let $\{\xi_1, \xi_2, \dots\}$ be independent nonnegative r.v.s, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Then the d.f. F_{S_η} belongs to the class $\mathcal{C}$ if the following conditions are satisfied:*

(a) $F_{\xi_1} \in \mathcal{C}$,

(b) *for each $k \geqslant 2$, either $F_{\xi_k} \in \mathcal{C}$ or $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$,*

(c) $\mathbb{E}\xi_1 < \infty$,

(d) $\overline{F}_{\eta}(x) = o(\overline{F}_{\xi_1}(x))$,

(e) $\limsup\limits_{x\to\infty} \sup\limits_{n\geqslant 1} \dfrac{1}{n\overline{F}_{\xi_1}(x)} \sum\limits_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty$,

(f) $\lim\limits_{u\to\infty} \sup\limits_{n\geqslant 1} \dfrac{1}{n} \sum\limits_{\substack{k=1 \\ \mathbb{E}\xi_k \geqslant u}}^{n} \mathbb{E}\xi_k = 0$.

Similarly to Corollary 1, we can formulate the following statement. We note that, in the i.i.d. case, conditions (a), (b), (e), and (f) of Theorem 7 are satisfied.

Corollary 2. *Let $\{\xi_1, \xi_2, \ldots\}$ be i.i.d. nonnegative r.v.s with common d.f. $F_\xi \in \mathcal{C}$, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then the d.f. F_{S_η} belongs to the class $\mathcal{C}$ under the following two conditions: $\mathbb{E}\xi < \infty$ and $\overline{F}_\eta(x) = o(\overline{F}_\xi(x))$.*

Further in this section, we present two examples of r.v.s $\{\xi_1, \xi_2, \ldots\}$ and η for which the random sum F_{S_η} has a consistently varying tail.

Example 1. Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.s such that ξ_k are exponentially distributed for all even k, that is,

$$\overline{F}_{\xi_k}(x) = \mathrm{e}^{-x}, \quad x \geqslant 0,\ k \in \{2, 4, 6, \ldots\},$$

whereas, for each odd k, ξ_k is a copy of the r.v.

$$(1 + \mathcal{U})\, 2^{\mathcal{G}},$$

where $\mathcal{U}$ and $\mathcal{G}$ are independent r.v.s, $\mathcal{U}$ is uniformly distributed on the interval [0, 1], and $\mathcal{G}$ is geometrically distributed with parameter $q \in (0, 1)$, that is,

$$\mathbb{P}(\mathcal{G} = l) = (1 - q)\, q^l, \quad l = 0, 1, \ldots.$$

In addition, let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ and distributed according to the Poisson law.

Theorem 6 implies that the d.f. of the randomly stopped sum S_η belongs to the class $\mathcal{C}$ because:

(a) $F_{\xi_1} \in \mathcal{C}$ due to considerations in pp. 122–123 of [2],

(b) $F_{\xi_k} \in \mathcal{C}$ for $k \in \{3, 5, \ldots\}$, and $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$ for $k \in \{2, 4, 6, \ldots\}$,

(c) $\limsup\limits_{x\to\infty} \sup\limits_{n\geqslant 1} \dfrac{1}{n\overline{F}_{\xi_1}(x)} \sum\limits_{i=1}^{n} \overline{F}_{\xi_i}(x) \leqslant 1$,

(d) all moments of the r.v. η are finite.

Note that ξ_1 does not satisfy condition (c) of Theorem 7 in the case $q \geqslant 1/2$. Hence, Example 1 describes the situation where Theorem 6 should be used instead of Theorem 7.

Example 2. Let $\{\xi_1, \xi_2, \dots\}$ be independent r.v.s such that ξ_k are distributed according to the Pareto law (with tail index $\alpha = 2$) for all odd k, and ξ_k are exponentially distributed (with parameter equal to 1) for all even k, that is,

$$\overline{F}_{\xi_k}(x) = \frac{1}{x^2}, \quad x \geqslant 1,\ k \in \{1, 3, 5, \dots\},$$
$$\overline{F}_{\xi_k}(x) = \mathrm{e}^{-x}, \quad x \geqslant 0,\ k \in \{2, 4, 6, \dots\}.$$

In addition, let η be a counting r.v independent of $\{\xi_1, \xi_2, \dots\}$ that has the Zeta distribution with parameter 4, that is,

$$\mathbb{P}(\eta = m) = \frac{1}{\zeta(4)} \frac{1}{(m+1)^4}, \quad m \in \mathbb{N}_0,$$

where ζ denotes the Riemann zeta function.

Theorem 7 implies that the d.f. of the randomly stopped sum S_η belongs to the class $\mathcal{C}$ because:

(a) $F_{\xi_1} \in \mathcal{C}$,

(b) $F_{\xi_k} \in \mathcal{C}$ for $k \in \{3, 5, \dots\}$, and $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$ for $k \in \{2, 4, 6, \dots\}$,

(c) $\mathbb{E}\xi_1 = 2$,

(d) $\overline{F}_\eta(x) = o(\overline{F}_{\xi_1}(x))$,

(e) $\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n\overline{F}_{\xi_1}(x)} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) \leqslant 1$,

(f) $\max_{k\in\mathbb{N}} \mathbb{E}\xi_k = 2$.

Regarding condition (d), it should be noted that the Zeta distribution with parameter 4 is a discrete version of Pareto distribution with tail index 3.

Note that η does not satisfy the condition (d) of Theorem 6 because $J^+_{F_{\xi_1}} = 2$ and $\mathbb{E}\eta^3 = \infty$. Hence, Example 2 describes the situation where Theorem 7 should be used instead of Theorem 6.

3 Auxiliary lemmas

This section deals with several auxiliary lemmas. The first lemma is Theorem 3.1 in [3] (see also Theorem 2.1 in [15]).

Lemma 1. *Let $\{X_1, X_2, \dots X_n\}$ be independent real-valued r.v.s. If $F_{X_k} \in \mathcal{C}$ for each $k \in \{1, 2, \dots, n\}$, then*

$$\mathbb{P}\biggl(\sum_{i=1}^{n} X_i > x\biggr) \sim \sum_{i=1}^{n} \overline{F}_{X_i}(x).$$

The following statement about nonnegative subexponential distributions was proved in Proposition 1 of [10] and later generalized to a wider distribution class in Corollary 3.19 of [12].

Lemma 2. *Let* $\{X_1, X_2, \dots X_n\}$ *be independent real-valued r.v.s. Assume that* $\overline{F}_{X_i}/\overline{F}(x) \underset{x\to\infty}{\to} b_i$ *for some subexponential d.f.* F *and some constants* $b_i \geqslant 0$, $i \in \{1, 2, \dots n\}$. *Then*

$$\frac{\overline{F_{X_1} * F_{X_2} * \cdots * F_{X_n}}(x)}{\overline{F}(x)} \underset{x\to\infty}{\to} \sum_{i=1}^{n} b_i .$$

In the next lemma, we show in which cases the convolution $F_{X_1} * F_{X_2} * \cdots * F_{X_n}$ belongs to the class $\mathcal{C}$.

Lemma 3. *Let* $\{X_1, X_2, \dots, X_n\}$, $n \in \mathbb{N}$, *be independent real-valued r.v.s. Then the d.f.* F_{Σ_n} *of the sum* $\Sigma_n = X_1 + X_2 + \cdots + X_n$ *belongs to the class* $\mathcal{C}$ *if the following conditions are satisfied:*

(a) $F_{X_1} \in \mathcal{C}$,

(b) *for each* $k = 2, \dots, n$, *either* $F_{X_k} \in \mathcal{C}$ *or* $\overline{F}_{X_k}(x) = o(\overline{F}_{X_1}(x))$.

Proof. Evidently, we can suppose that $n \geqslant 2$. We split our proof into two parts.

First part. Suppose that $F_{X_k} \in \mathcal{C}$ for all $k \in \{1, 2, \dots, n\}$. In such a case, the lemma follows from Lemma 1 and the inequality

$$\frac{a_1 + a_2 + \cdots + a_m}{b_1 + b_2 + \cdots + b_m} \leqslant \max\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_m}{b_m}\right\} \tag{3}$$

for $a_i \geqslant 0$ and $b_i > 0$, $i = 1, 2, \dots, m$.

Namely, using the relation of Lemma 1 and estimate (3), we get that

$$\begin{aligned}\limsup_{x\to\infty} \frac{\overline{F}_{\Sigma_n}(xy)}{\overline{F}_{\Sigma_n}(x)} &= \limsup_{x\to\infty} \frac{\sum_{k=1}^{n} \overline{F}_{X_k}(xy)}{\sum_{k=1}^{n} \overline{F}_{X_k}(x)} \\ &\leqslant \max_{1\leqslant k\leqslant n} \limsup_{x\to\infty} \frac{\overline{F}_{X_k}(xy)}{\overline{F}_{X_k}(x)}\end{aligned}$$

for arbitrary $y \in (0, 1)$.

Since $F_{X_k} \in \mathcal{C}$ for each k, the last estimate implies that the d.f. F_{Σ_n} has a consistently varying tail, as desired.

Second part. Now suppose that $F_{X_k} \notin \mathcal{C}$ for some of indexes $k \in \{2, 3, \dots, n\}$. By the conditions of the lemma we have that $\overline{F}_{X_k}(x) = o(\overline{F}_{X_1}(x))$ for such k. Let $\mathcal{K} \subset \{2, 3, \dots, n\}$ be the subset of indexes k such that

$$F_{X_k} \notin \mathcal{C} \quad \text{and} \quad \overline{F}_{X_k}(x) = o\big(\overline{F}_{X_1}(x)\big).$$

By Lemma 2,

$$\overline{F}_{\widehat{\Sigma}_n}(x) \sim \overline{F}_{X_1}(x),$$

where

$$\widehat{\Sigma}_n = X_1 + \sum_{k\in\mathcal{K}} X_k .$$

Hence,

$$\limsup_{x\to\infty} \frac{\overline{F}_{\widehat{\Sigma}_n}(xy)}{\overline{F}_{\widehat{\Sigma}_n}(x)} = \limsup_{x\to\infty} \frac{\overline{F}_{X_1}(xy)}{\overline{F}_{X_1}(x)} \tag{4}$$

for every $y \in (0, 1)$.

Equality (4) implies immediately that the d.f. $F_{\widehat{\Sigma}_n}$ belongs to the class $\mathcal{C}$. Therefore, the d.f. F_{Σ_n} also belongs to the class $\mathcal{C}$ according to the first part of the proof because

$$\Sigma_n = \widehat{\Sigma}_n + \sum_{k\notin\mathcal{K}} X_k$$

and $F_{X_k} \in \mathcal{C}$ for each $k \notin \mathcal{K}$. The lemma is proved. □

The following two statements about dominatedly varying distributions are Lemma 3.2 and Lemma 3.3 in [6]. Since any consistently varying distribution is also dominatingly varying, these statements will be useful in the proofs of our main results concerning the class $\mathcal{C}$.

Lemma 4. *Let* $\{X_1, X_2, \dots\}$ *be independent real-valued r.v.s, and* $F_{X_\nu} \in \mathcal{D}$ *for some* $\nu \geqslant 1$. *Suppose, in addition, that*

$$\limsup_{x\to\infty} \sup_{n\geqslant\nu} \frac{1}{n\overline{F}_{X_\nu}(x)} \sum_{i=1}^{n} \overline{F}_{X_i}(x) < \infty.$$

Then, for each $p > J^+_{F_{X_\nu}}$, *there exists a positive constant* c_1 *such that*

$$\overline{F}_{S_n}(x) \leqslant c_1 n^{p+1} \overline{F}_{X_\nu}(x) \tag{5}$$

for all $n \geqslant \nu$ *and* $x \geqslant 0$.

In fact, Lemma 4 is proved in [6] for nonnegative r.v.s. However, the lemma remains valid for real-valued r.v.s. To see this, it suffices to observe that $\mathbb{P}(X_1 + X_2 + \cdots + X_n > x) \leqslant \mathbb{P}(X_1^+ + X_2^+ \cdots + X_n^+ > x)$ and $\mathbb{P}(X_k > x) = \mathbb{P}(X_k^+ > x)$, where $n \in \mathbb{N}$, $k \in \{1, 2, \dots, n\}$, $x \geqslant 0$, and a^+ denotes the positive part of a.

Lemma 5. *Let* $\{X_1, X_2, \dots\}$ *be independent real-valued r.v.s, and* $F_{X_\nu} \in \mathcal{D}$ *for some* $\nu \geqslant 1$. *Let, in addition,*

$$\lim_{u\to\infty} \sup_{n\geqslant\nu} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\big(|X_k| \mathbb{1}_{\{X_k\leqslant -u\}}\big) = 0,$$

$$\limsup_{x\to\infty} \sup_{n\geqslant\nu} \frac{1}{n\overline{F}_{X_\nu}(x)} \sum_{i=1}^{n} \overline{F}_{X_i}(x) < \infty,$$

and $\mathbb{E}X_k = \mathbb{E}X_k^+ - \mathbb{E}X_k^- = 0$ *for* $k \in \mathbb{N}$. *Then, for each* $\gamma > 0$, *there exists a positive constant* $c_2 = c_2(\gamma)$ *such that*

$$\mathbb{P}(S_n > x) \leqslant c_2 n \overline{F}_{X_\nu}(x)$$

for all $x \geqslant \gamma n$ *and all* $n \geqslant \nu$.

4 Proofs of the main results

Proof of Theorem 5. It suffices to prove that

$$\limsup_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}_{S_\eta}(xy)}{\overline{F}_{S_\eta}(x)}\leqslant 1. \tag{6}$$

According to estimate (3), for $x>0$ and $y\in(0,1)$, we have

$$\frac{\overline{F}_{S_\eta}(xy)}{\overline{F}_{S_\eta}(x)}=\frac{\sum_{\substack{n=1\\ n\in\mathrm{supp}(\eta)}}^{D}\mathbb{P}(S_n>xy)\mathbb{P}(\eta=n)}{\sum_{\substack{n=1\\ n\in\mathrm{supp}(\eta)}}^{D}\mathbb{P}(S_n>x)\mathbb{P}(\eta=n)}\leqslant\max_{\substack{1\leqslant n\leqslant D\\ n\in\mathrm{supp}(\eta)}}\frac{\mathbb{P}(S_n>xy)}{\mathbb{P}(S_n>x)}.$$

Hence, by Lemma 3,

$$\begin{aligned}\limsup_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}_{S_\eta}(xy)}{\overline{F}_{S_\eta}(x)}&\leqslant\limsup_{y\uparrow 1}\limsup_{x\to\infty}\max_{\substack{1\leqslant n\leqslant D\\ n\in\mathrm{supp}(\eta)}}\frac{\overline{F}_{S_n}(xy)}{\overline{F}_{S_n}(x)}\\&\leqslant\max_{\substack{1\leqslant n\leqslant D\\ n\in\mathrm{supp}(\eta)}}\limsup_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}_{S_n}(xy)}{\overline{F}_{S_n}(x)}=1,\end{aligned}$$

which implies the desired estimate (6). The theorem is proved. □

Proof of Theorem 6. As in Theorem 5, it suffices to prove inequality (6). For all $K\in\mathbb{N}$ and $x>0$, we have

$$\mathbb{P}(S_\eta>x)=\left(\sum_{n=1}^{K}+\sum_{n=K+1}^{\infty}\right)\mathbb{P}(S_n>x)\mathbb{P}(\eta=n).$$

Therefore, for $x>0$ and $y\in(0,1)$, we have

$$\begin{aligned}\frac{\mathbb{P}(S_\eta>xy)}{\mathbb{P}(S_\eta>x)}&=\frac{\sum_{n=1}^{K}\mathbb{P}(S_n>xy)\mathbb{P}(\eta=n)}{\mathbb{P}(S_\eta>x)}\\&\quad+\frac{\sum_{n=K+1}^{\infty}\mathbb{P}(S_n>xy)\mathbb{P}(\eta=n)}{\mathbb{P}(S_\eta>x)}\\&=:\mathcal{J}_1+\mathcal{J}_2.\end{aligned}\tag{7}$$

The random variable η is not degenerate at zero, so there exists $a\in\mathbb{N}$ such that $\mathbb{P}(\eta=a)>0$. If $K\geqslant a$, then using inequality (3), we get

$$\mathcal{J}_1\leqslant\frac{\sum_{\substack{n=1\\ n\in\mathrm{supp}(\eta)}}^{K}\mathbb{P}(S_n>xy)\mathbb{P}(\eta=n)}{\sum_{\substack{n=1\\ n\in\mathrm{supp}(\eta)}}^{K}\mathbb{P}(S_n>x)\mathbb{P}(\eta=n)}\leqslant\max_{\substack{1\leqslant n\leqslant K\\ n\in\mathrm{supp}(\eta)}}\frac{\mathbb{P}(S_n>xy)}{\mathbb{P}(S_n>x)}.$$

Similarly as in the proof of Theorem 5, it follows that

$$\limsup_{y\uparrow 1}\limsup_{x\to\infty}\mathcal{J}_1\leqslant\max_{\substack{1\leqslant n\leqslant K\\ n\in\mathrm{supp}(\eta)}}\limsup_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}_{S_n}(xy)}{\overline{F}_{S_n}(x)}=1. \tag{8}$$

Since $\mathcal{C} \subset \mathcal{D}$, we can use Lemma 4 for the numerator of $\mathcal{J}_2$ to obtain

$$\sum_{n=K+1}^{\infty} \mathbb{P}(S_n > xy)\mathbb{P}(\eta = n) \leqslant c_3 \overline{F}_{\xi_1}(xy) \sum_{n=K+1}^{\infty} n^{p+1}\mathbb{P}(\eta = n)$$

with some positive constant c_3. For the denominator of $\mathcal{J}_2$, we have that

$$\begin{aligned}\mathbb{P}(S_\eta > x) &= \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n) \\ &\geqslant \mathbb{P}(S_a > x)\mathbb{P}(\eta = a).\end{aligned}$$

The conditions of the theorem imply that

$$S_a = \xi_1 + \sum_{k \in \mathcal{K}_a} \xi_k + \sum_{k \notin \mathcal{K}_a} \xi_k,$$

where $\mathcal{K}_a = \{k \in \{2, \dots, a\} : F_{\xi_k} \notin \mathcal{C}, \overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))\}$.

By Lemma 2

$$\overline{F}_{\widehat{S}_a}(x)/\overline{F}_{\xi_1}(x) \underset{x\to\infty}{\to} 1,$$

where $F_{\widehat{S}_a}$ is the d.f. of the sum

$$\widehat{S}_a = \xi_1 + \sum_{k \in \mathcal{K}_a} \xi_k$$

In addition, by Lemma 3 we have that the d.f. $F_{\widehat{S}_a}$ belongs to the class $\mathcal{C}$.

If $k \notin \mathcal{K}_a$, then $F_{\xi_k} \in \mathcal{C}$ by the conditions of the theorem. This fact and Lemma 1 imply that

$$\liminf_{x\to\infty} \frac{\mathbb{P}(S_a > x)}{\overline{F}_{\xi_1}(x)} \geqslant 1 + \sum_{k \notin \mathcal{K}_a} \liminf_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)}.$$

Hence,

$$\mathbb{P}(S_\eta > x) \geqslant \frac{1}{2}\overline{F}_{\xi_1}(x)\mathbb{P}(\eta = a) \tag{9}$$

for x sufficiently large. Therefore,

$$\begin{aligned}&\limsup_{y\uparrow 1} \limsup_{x\to\infty} \mathcal{J}_2 \\ &\quad\leqslant \frac{2c_3}{\mathbb{P}(\eta = a)} \left(\limsup_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_1}(xy)}{\overline{F}_{\xi_1}(x)} \right) \sum_{n=K+1}^{\infty} n^{p+1}\mathbb{P}(\eta = n).\end{aligned} \tag{10}$$

Estimates (7), (8), and (10) imply that

$$\limsup_{y\uparrow 1} \limsup_{x\to\infty} \frac{\mathbb{P}(S_\eta > xy)}{\mathbb{P}(S_\eta > x)} \leqslant 1 + \frac{2c_3}{\mathbb{P}(\eta = a)} \mathbb{E}\eta^{p+1}\mathbb{1}_{\{\eta > K\}}$$

for arbitrary $K \geqslant a$.

Letting K tend to infinity, we get the desired estimate (6) due to condition (d). The theorem is proved. □

Proof of Theorem 7. Once again, it suffices to prove inequality (6).

By condition (e) we have that there exist two positive constants c_4 and c_5 such that

$$\sum_{i=1}^{n} \overline{F}_{\xi_i}(x) \leqslant c_5 n \overline{F}_{\xi_1}(x), \quad x \geqslant c_4,\ n \in \mathbb{N}.$$

Therefore,

$$\mathbb{E}S_n = \sum_{j=1}^{n} \mathbb{E}\xi_j = \sum_{j=1}^{n} \left(\int_0^{c_4} + \int_{c_4}^{\infty} \right) \overline{F}_{\xi_j}(u)du \leqslant c_4 n + c_5 n \mathbb{E}\xi_1 =: c_6 n \tag{11}$$

for a positive constant c_6 and all $n \in \mathbb{N}$.

If $K \in \mathbb{N}$ and $x > 4Kc_6$, then we have

$$\begin{aligned}\mathbb{P}(S_\eta > x) &= \mathbb{P}(S_\eta > x, \eta \leqslant K) \\ &\quad + \mathbb{P}\left(S_\eta > x, K < \eta \leqslant \frac{x}{4c_6} \right) \\ &\quad + \mathbb{P}\left(S_\eta > x, \eta > \frac{x}{4c_6} \right).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\mathbb{P}(S_\eta > xy)}{\mathbb{P}(S_\eta > x)} &= \frac{\mathbb{P}(S_\eta > xy, \eta \leqslant K)}{\mathbb{P}(S_\eta > x)} \\ &\quad + \frac{\mathbb{P}\big(S_\eta > xy, K < \eta \leqslant \frac{xy}{4c_6}\big)}{\mathbb{P}(S_\eta > x)} \\ &\quad + \frac{\mathbb{P}\big(S_\eta > xy, \eta > \frac{xy}{4c_6}\big)}{\mathbb{P}(S_\eta > x)} \\ &=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3\end{aligned} \tag{12}$$

if $xy > 4Kc_6,\ x > 0$, and $y \in (0, 1)$.

The random variable η is not degenerate at zero, so $\mathbb{P}(\eta = a) > 0$ for some $a \in \mathbb{N}$. If $K \geqslant a$, then

$$\limsup_{y \uparrow 1} \limsup_{x \to \infty} \mathcal{I}_1 \leqslant 1 \tag{13}$$

similarly to estimate (8) in Theorem 6.

For the numerator of $\mathcal{I}_2$, we have

$$\begin{aligned}\mathcal{I}_{2,1} &:= \mathbb{P}\left(S_\eta > xy, K < \eta \leqslant \frac{xy}{4c_6} \right) \\ &= \sum_{K < n \leqslant \frac{xy}{4c_6}} \mathbb{P}\left(\sum_{i=1}^{n} (\xi_i - \mathbb{E}\xi_i) > xy - \sum_{j=1}^{n} \mathbb{E}\xi_j \right) \mathbb{P}(\eta = n) \\ &\leqslant \sum_{K < n \leqslant \frac{xy}{4c_6}} \mathbb{P}\left(\sum_{i=1}^{n} (\xi_i - \mathbb{E}\xi_i) > \frac{3}{4} xy \right) \mathbb{P}(\eta = n)\end{aligned} \tag{14}$$

by inequality (11).

The random variables $\xi_1 - \mathbb{E}\xi_1, \xi_2 - \mathbb{E}\xi_2, \ldots$ satisfy the conditions of Lemma 5. Namely, $\mathbb{E}(\xi_k - \mathbb{E}\xi_k) = 0$ for $k \in \mathbb{N}$ and $F_{\xi_1 - \mathbb{E}\xi_1} \in \mathcal{C} \subset \mathcal{D}$ obviously. In addition,

$$\limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n\,\mathbb{P}(\xi_1 - \mathbb{E}\xi_1 > x)} \sum_{k=1}^{n} \mathbb{P}(\xi_i - \mathbb{E}\xi_i > x) < \infty$$

by conditions (a), (c) and (e). Finally,

$$\begin{aligned}
&\limsup_{u\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}(|\xi_k - \mathbb{E}\xi_k| \mathbb{1}_{\{\xi_k - \mathbb{E}\xi_k \leqslant -u\}}) \\
&\quad = \limsup_{u\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\big((\mathbb{E}\xi_k - \xi_k) \mathbb{1}_{\{\xi_k - \mathbb{E}\xi_k \leqslant -u\}}\big) \\
&\quad \leqslant \limsup_{u\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{\substack{1\leqslant k\leqslant n \\ \mathbb{E}\xi_k \geqslant u}} \mathbb{E}\xi_k = 0
\end{aligned}$$

because of condition (f). So, applying the estimate of Lemma 5 to (14), we get

$$\begin{aligned}
\mathcal{I}_{2,1} &\leqslant c_7 \sum_{K < n \leqslant \frac{xy}{4c_6}} n \overline{F}_{\xi_1}\left(\frac{3}{4}xy + \mathbb{E}\xi_1\right) \mathbb{P}(\eta = n) \\
&\leqslant\ c_7 \overline{F}_{\xi_1}\Big(\frac{3}{4}xy\Big) \mathbb{E}\eta \mathbb{1}_{\{\eta > K\}}
\end{aligned}$$

with a positive constant c_7. For the denominator of $\mathcal{I}_2$, we can use the inequality

$$\begin{aligned}
\mathbb{P}(S_\eta > x) &= \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n) \\
&\geqslant \sum_{n=1}^{\infty} \mathbb{P}(\xi_1 > x)\mathbb{P}(\eta = n) \\
&\geqslant \overline{F}_{\xi_1}(x)\mathbb{P}(\eta = a)
\end{aligned} \tag{15}$$

since the r.v.s $\{\xi_1, \xi_2, \ldots\}$ are nonnegative by assumption. Hence,

$$\mathcal{I}_2 \leqslant \frac{c_7}{\mathbb{P}(\eta = a)} \mathbb{E}\eta \mathbb{1}_{\{\eta > K\}} \frac{\overline{F}_{\xi_1}\big(\frac{3}{4}xy\big)}{\overline{F}_{\xi_1}(x)}.$$

If $y \in (1/2, 1)$, then the last estimate implies that

$$\limsup_{x\to\infty} \mathcal{I}_2 \leqslant \frac{c_7}{\mathbb{P}(\eta = a)} \mathbb{E}\eta \mathbb{1}_{\{\eta > K\}} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_1}\big(\frac{3}{8}x\big)}{\overline{F}_{\xi_1}(x)} \leqslant c_8 \mathbb{E}\eta \mathbb{1}_{\{\eta > K\}} \tag{16}$$

with some positive constant c_8 because $F_{\xi_1} \in \mathcal{C} \subset \mathcal{D}$.

Using inequality (15) again, we obtain

$$\mathcal{I}_3 \leqslant \frac{\mathbb{P}(\eta > \frac{xy}{4c_6})}{\mathbb{P}(S_\eta > x)} \leqslant \frac{1}{\mathbb{P}(\eta = a)} \frac{\overline{F}_\eta(\frac{xy}{4c_6})}{\overline{F}_{\xi_1}(\frac{xy}{4c_6})} \frac{\overline{F}_{\xi_1}(\frac{xy}{4c_6})}{\overline{F}_{\xi_1}(x)}.$$

Therefore, for $y \in (1/2, 1)$, we get

$$\limsup_{x\to\infty} \mathcal{I}_3 \leqslant \frac{1}{\mathbb{P}(\eta = a)} \limsup_{x\to\infty} \frac{\overline{F}_\eta(\frac{xy}{4c_6})}{\overline{F}_{\xi_1}(\frac{xy}{4c_6})} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_1}(\frac{xy}{4c_6})}{\overline{F}_{\xi_1}(x)} = 0 \tag{17}$$

by condition (d).

Estimates (12), (13), (16), and (17) imply that

$$\limsup_{y\uparrow 1} \limsup_{x\to\infty} \frac{\mathbb{P}(S_\eta > xy)}{\mathbb{P}(S_\eta > x)} \leqslant 1 + c_8 \mathbb{E}\eta \mathbb{1}_{\{\eta > K\}}$$

for $K \geqslant a$.

Letting K tend to infinity, we get the desired estimate (6) because $\mathbb{E}\eta < \infty$ by conditions (c) and (d). The theorem is proved. □

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