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Properties of Tail Moments of Heavy-Tailed Distributions

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Saulius Paukštys

Sunkiauodegių skirstinių momentų uodegų savybės

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Contents

Notation

$\mathbb{N}$	Set of natural numbers $\{1,2,3,...\}$.
$\mathbb{N}_0$	Set $\{0\} \cup \mathbb{N}$.
$\mathbb{R}$	Set of real numbers (the real line).
$\mathbb{R}^+$	Set of nonnegative real numbers (the positive half-line $[0, \infty)$).
$\mathbb{1}_{\{A\}}$	Indicator function of A: $\mathbb{1}_{\{A\}} = 1$, if A holds, $\mathbb{1}_{\{A\}} = 0$ otherwise.
$:=$ $(=:)$	Quantity on the left (on the right) is defined to be equal to the quantity on the right (on the left).
$\lfloor z \rfloor$	Integer part of the real number z.
$\{z\}$	Fractional part of the real number z $(z - \lfloor z \rfloor)$.
z^+	Positive part of z $(z^+ = \max\{z, 0\})$.
z^-	Negative part of z $(z^- = \max\{-z, 0\})$.
$\mathbb{P}(B)$	Probability of the event B.
$F_\xi(x)$	Distribution function of the random variable ξ.
$\overline{F}_\xi(x)$	Tail of the distribution function F_ξ.
$\mathbb{E}(\xi)$	Mean of the random variable ξ.
$F_\xi^{\leftarrow}(q)$	Quantile function of random variable ξ: $\inf\{x \in \mathbb{R} : F_\xi(x) \geqslant q\}$.
$F_\xi * F_\eta$	Convolution of distribution functions F_ξ and F_η. If ξ and η are independent, then $F_\xi * F_\eta(x) = \mathbb{P}(\xi + \eta \leqslant x)$, and $\overline{F_\xi * F_\eta}(x) = \mathbb{P}(\xi + \eta > x)$.
F_ξ^{*n}	n-fold convolution (convolution power) of a distribution function F_ξ. If $\xi_1, \ldots, \xi_n$ are independent copies of ξ, then $F_\xi^{*n}(x) = \mathbb{P}(\xi_1 + \cdots + \xi_n \leqslant x)$, and $\overline{F_\xi^{*n}}(x) = \mathbb{P}(\xi_1 + \cdots + \xi_n > x)$.
$\mathcal{H}$	Class of heavy-tailed distribution functions.
$\mathcal{L}$	Class of long-tailed distribution functions.
$\mathcal{L}_\gamma$	Class of exponential-type distribution functions.
$\mathcal{OL}$	Class of $\mathcal{O}$-exponential distribution functions.
$\mathcal{D}$	Class of dominatedly varying-tailed distribution functions.
$\mathcal{S}$	Class of subexponential distribution functions.
$\mathcal{S}^*$	Class of strong subexponential distribution functions.

$\mathcal{C}$	Class of consistently varying-tailed distribution functions.
$\mathcal{R}$	Class of regularly varying-tailed distribution functions.
S_n^{ξ}	Sum of random variables $\xi_1, \dots, \xi_n$: $\xi_1 + \cdots + \xi_n$.
$S_n^{\theta\xi}$	Weighted sum of random variables $\xi_1, \dots, \xi_n$: $\theta_1\xi_1 + \cdots + \theta_n\xi_n$, where $\theta_1, \dots, \theta_n$ are nonnegative, nondegenerate at zero random variables.

Abbreviations:

r.v. (r.v.s)	Abbreviation for "random variable" ("random variables").
d.f. (d.f.s)	Abbreviation for "distribution function" ("distribution functions").
i.i.d.	Abbreviation for "independent and identically distributed".

For two positive functions f and g, we write:

$$
\begin{aligned}
f(x) &= o(g(x)), && \text{if } \lim_{x\to\infty} \tfrac{f(x)}{g(x)} = 0,\\
f(x) &= O(g(x)), && \text{if } \limsup_{x\to\infty} \tfrac{f(x)}{g(x)} < \infty,\\
f(x) &\sim g(x), && \text{if } \lim_{x\to\infty} \tfrac{f(x)}{g(x)} = 1,\\
f(x) &\asymp g(x), && \text{if } 0 < \liminf_{x\to\infty} \tfrac{f(x)}{g(x)} \leqslant \limsup_{x\to\infty} \tfrac{f(x)}{g(x)} < \infty,\\
f(x) &\lesssim g(x), && \text{if } \limsup_{x\to\infty} \tfrac{f(x)}{g(x)} \leqslant 1.
\end{aligned}
$$

1. Introduction and preliminaries

1.1. Introduction

Interest in heavy-tailed distributions was increasingly growing during the last few decades. We could find many reasons for this but amongst most popular explanations are quick spread of information and communication technologies, increased need for financial models that would better correspond to real issues and constantly growing statistical evidence for their appropriateness in natural sciences. Many researchers in insurance and finance are particularly interested in such distributions – see Embrechts, Klüppelberg, Mikosch [21] for a review. Some classic, but still very important examples of heavy-tailed distributions are Pareto, lognormal and Weibull distributions. Among less known examples we find generalised Peter and Paul, discrete Weibull, Cauchy distributions. We will later attribute these and some other examples to specific classes of heavy-tailed distribution functions. Also, for illustrational purposes, we will construct some mixed distributions that would be in or outside specific class. In our study we are mainly interested in class $\mathcal{D}$ of dominatedly varying-tailed distributions but we will also analyse several properties of related classes, like $\mathcal{C}$ (consistently varying-tailed distributions), class $\mathcal{R}$ (regularly varying-tailed distributions), class $\mathcal{L}$ (long-tailed distributions) and class $\mathcal{S}$ (subexponential distributions).

Since many models include cumulative effects, analysis of sums of random variables, their distributions, moments and other probabilistic properties is of high importance. Let $n \in \mathbb{N} := \{1, 2, \dots\}$ and let $\xi_1, \dots, \xi_n$ be possibly dependent, heavy-tailed, real-valued random variables (r.v.s.), called primary r.v.s. Also, let $\theta_1, \dots, \theta_n$ be nonnegative, nondegenerate at zero r.v.s, called random weights. Our main object of study is the tail moment,

$$\mathbb{E}((S_n^\xi)^\alpha \mathbb{1}_{\{S_n^\xi > x\}}), \tag{1.1}$$

where $\alpha \in [0, \infty)$ and

$$S_n^\xi := \xi_1 + \xi_2 + \dots + \xi_n.$$

In addition, some results will be formulated for weighted counterpart of tail moment (1.1):

$$\mathbb{E}((S_n^{\theta\xi})^\alpha \mathbb{1}_{\{S_n^{\theta\xi}>x\}}), \tag{1.2}$$

where $\alpha \in [0, \infty)$ and

$$S_n^{\theta\xi} := \theta_1\xi_1 + \theta_2\xi_2 + \cdots + \theta_n\xi_n. \tag{1.3}$$

Sometimes we will limit exponent α to the case of nonnegative integers ($\mathbb{N}_0 := \{0\} \cup \mathbb{N}$) and for more visual discernment, in that case, we will use letter m instead of α. Also, let us note, that in the literature the term **truncated moment** is often used instead of **tail moment**.

Inspired by recent results by Leipus et al. [40], we seek to bound asymptotically the tail moments (1.1) and (1.2) by the sums of individual tail moments $\mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\{\xi_k>x\}}\big)$ and $\mathbb{E}\big((\theta_k\xi_k)^\alpha \mathbb{1}_{\{\theta_k\xi_k>x\}}\big)$, respectively, with some specific correcting constants. Compared to their theorems, we obtain more precise asymptotic bounds, showing that each summand in the approximating sums can be accompanied by a separate L-index (defined later in the introduction) of the corresponding distribution function. A novelty of our result, also, comes from employing a more abstract dependence structure and considering nonnegative real exponent in the tail moment instead of a natural one. In case when primary random variables are nonnegative, we prove that correcting constants can be omitted.

There are many papers addressing two special cases $\alpha = 0$ and $\alpha = 1$. If we suppose that $\alpha = 0$, then tail moments in (1.1) and (1.2) are equal to respective tail probabilities

$$\mathbb{P}\big(S_n^{\xi} > x\big) \quad \text{and} \quad \mathbb{P}\big(S_n^{\theta\xi} > x\big).$$

Studies show that under various conditions (see Section 3.1 for a brief overview), these tail probabilities are asymptotically equal to the sums of tail probabilities of individual summands,

$$\sum_{k=1}^{n} \mathbb{P}\big(\xi_k > x\big) \quad \text{and} \quad \sum_{k=1}^{n} \mathbb{P}\big(\theta_k\xi_k > x\big)$$

respectively. If we suppose that $\alpha = 1$, then tail moments in (1.1) and (1.2) are equal to respective tail expectations

$$\mathbb{E}\big(S_n^{\xi} \mathbb{1}_{\{S_n^{\xi}>x\}}\big) \quad \text{and} \quad \mathbb{E}\big(S_n^{\theta\xi} \mathbb{1}_{\{S_n^{\theta\xi}>x\}}\big).$$

The corresponding question to those mentioned above is investigated in this case too.

We will provide a brief review of the literature on related studies further in the text, after giving all the necessary definitions.

During our research we noticed that there are some regularities in the behaviour of the tail moment in relation with the tail of a distribution, which generated that moment. This evolved into a discussion about closure properties of the classes of heavy-tailed and related distributions, with respect to calculation of moments. As our research shows, for the tail moment to have a property, which defines a specific class, it is sufficient for corresponding distribution to belong to the same class, but it is not always necessary.

Before finishing this subsection, we would like to mention one interesting example, which illustrates how knowledge about randomly weighted sums (1.3) can be applied in financial or actuarial context. Let primary random variable ξ_k correspond to the net losses (total claim amount minus total premium income) of an insurance company during period $(k-1, k]$, calculated at the moment k, and random weight θ_k correspond to the stochastic discount factor, from the moment k to the present moment 0, for all $k = 1, \ldots, n$. In such a scenario, sum $S_n^{\theta\xi}$ could be treated as the total discounted net loss of a company in the time interval $(0, n]$.

When quantifying random losses of any portfolio or company, various risk measures are employed – Value at Risk (VaR), Conditional Value at Risk (CVaR), Haezendonck–Goovaerts (HG) risk measure among many others. To read more about the mentioned risk measures, risk measurement theory in general and its development, the reader is referred to Sections 4.1 and 4.2. To provide examples of application, we found asymptotic formulas specifically for the Haezendonck–Goovaerts risk measure by combining our main results with the essential theorem from the paper by Tang and Yang [63, Theorem 2.1].

The main results in Sections 2 and 3.2 are original and can be considered as new. These results are achieved by the author of the dissertation together with the co-authors. In essence, Section 3.2 is based on the paper by Leipus, Paukštys, Šiaulys [38] and the paper by Dirma, Paukštys, Šiaulys [20]. The new closure properties in Section 2 are published in the paper by Paukštys, Šiaulys, Leipus [49].

1.2. Publication of results

The main results of this thesis are published in the following scientific papers:

- R. Leipus, S. Paukštys, J. Šiaulys, *Tails of higher-order moments of sums with heavy-tailed increments and application to the Haezendonck–Goovaerts risk measure*, Statistics & Probability Letters 170 (2021), 1–12;
- M. Dirma, S. Paukštys, J. Šiaulys, *Tails of the Moments for Sums with Dominatedly Varying Random Summands*, Mathematics 9 (2021), 1–26;
- S. Paukštys, J. Šiaulys, R. Leipus, *Truncated Moments for Heavy-Tailed and Related Distribution Classes*, Mathematics 11 (2023), 1–15.

The results obtained during the preparation of the thesis were presented in the following conferences and seminars:

- S. Paukštys, *Atsitiktinių dydžių sumų momentų uodegų asimptotinis elgesys ir taikymai Haezendonck–Goovaerts rizikos matui*, Finansų ir draudimo matematikos seminaras, 23 February 2021, Vilnius;
- S. Paukštys, *Atsitiktinių dydžių sumų nupjautinių momentų ribinis elgesys*, Finansų ir draudimo matematikos seminaras, 9 November 2021, Vilnius;
- S. Paukštys, *Atsitiktinių dydžių sumų nupjautinių momentų ribinio elgesio tyrimai ir taikymai rizikos teorijoje*, 10-asis Lietuvos jaunųjų matematikų susitikimas, 28 December 2021, Vilnius;
- S. Paukštys, *Tails of moments of sums with heavy-tailed summands and applications to the Haezendonck–Goovaerts risk measure*, European Actuarial Journal Conference 2022 Tartu, 22 August 2022., Tartu;
- S. Paukštys, *Sunkiauodegių skirstinių momentų uodegų savybės (disertacijos pristatymas)*, Finansų ir draudimo matematikos seminaras, 13 March 2023, Vilnius.

1.3. Structure of the thesis

Section 1 is for introduction, preliminaries and providing the context of our work. In Section 1.1 we already introduced the problem and main objects of study. In Sections 1.4 and 1.5 we give definitions of distribution classes of interest and used dependence structures. In addition, we give several examples for illustrational purposes. We also describe known interrelationships among the classes. The introduction ends with Section 1.6, where we summarise known closure properties of the classes of heavy-tailed and related distributions.

Section 2 is dedicated for novel closure properties of the popular distribution classes $\mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{L}_\gamma$, $\gamma \geqslant 0$ and $\mathcal{OL}$. Main results concerning those properties and their proofs are given in Section 2.2.

In Section 3.1 we quote some interesting results that are related to the main results of this thesis.

In Section 3.2 we present the main results of our thesis, that is, asymptotic results for tail moment of the sum of possibly dependent random variables. In Section 3.3 we present the proofs of those results.

In Section 4 we briefly discuss risk measure theory and show how our main results can be applied to achieve asymptotic formulas for the Haezendonck–Goovaerts risk measure. Illustration in case when primary r.v.s are distributed according to the Pareto law is given in Section 4.3, and illustration in case when primary r.v.s. are distributed according to the generalised Peter and Paul distribution is given in Section 4.4.

1.4. Definitions

1.4.1. Heavy-tailed and related distributions

In this subsection we provide definitions of concepts that are used in our research. Most of them will be used directly in the presentation of main results, but some of them are given for illustrative purposes and for better understanding of the context. We will try to make a clear distinction where necessary. Before we start, we recall that distribution function (d.f.) F is called "on $\mathbb{R}^+$" if $F(-0) = 0$. A d.f. F is called "on $\mathbb{R}$" if the condition $F(-0) = 0$ can fail.

Let us begin by introducing some important classes of distributions. First, we define heavy-tailed distributions.

Definition 1.1. *A distribution function F on $\mathbb{R}$ is called heavy-tailed, written as $F \in \mathcal{H}$, if for every $h > 0$, it holds that*

$$\int_{-\infty}^{\infty} \mathrm{e}^{hx} \mathrm{d}F(x) = \infty.$$

The following proposition gives us an alternative way to look at heavy-tailed distributions. It corresponds to usual intuition that we have about them.

Proposition 1.1. *A distribution function F on $\mathbb{R}$ is heavy-tailed if and only if for every $h > 0$, it holds that*

$$\limsup_{x \to \infty} \overline{F}(x) \mathrm{e}^{hx} = \infty. \tag{1.4}$$

Intuition becomes more clear by noticing that condition (1.4) is equivalent to

$$\liminf_{x \to \infty} \frac{\mathrm{e}^{-hx}}{\overline{F}(x)} = 0$$

for every $h > 0$. We see that the tail of d.f. F vanishes slower (is "heavier") than the tail of any exponential distribution.

We say that distribution is light-tailed, if it is not heavy-tailed.

We define class $\mathcal{L}$ (well-known subclass of $\mathcal{H}$) in a same manner as in Foss, Korshunov, Zachary [27].

Definition 1.2. *A positive function f is called long-tailed if*

$$\lim_{x \to \infty} \frac{f(x+y)}{f(x)} = 1$$

for all $y > 0$.

Definition 1.3. *A distribution function F on $\mathbb{R}$ is called long-tailed, written as $F \in \mathcal{L}$, if for any fixed $y > 0$,*

$$\overline{F}(x+y) \sim \overline{F}(x) \tag{1.5}$$

as $x \to \infty$.

From the definitions above, one can see that distribution F is long-tailed if and only if its tail function $\overline{F}$ is a long-tailed function.

To determine whether the distribution function is long-tailed, it suffices to verify that the relation (1.5) is satisfied for one chosen value of y which is not zero.

One may use the concept of slowly varying functions to prove that every long-tailed distribution function is heavy-tailed.

Definition 1.4. *A positive function g is called slowly varying at infinity if*

$$\lim_{x\to\infty} \frac{g(ax)}{g(x)} = 1 \tag{1.6}$$

for any fixed $a > 0$.

The above-mentioned fact that every long-tailed distribution function is heavy-tailed is well-known in the literature. Since we will often use this result as an auxiliary statement (many times implicitly), for the reader's convenience, we state it here together with a detailed proof.

Lemma 1.1. *Every long-tailed d.f. is heavy-tailed, that is, $\mathcal{L} \subset \mathcal{H}$.*

Proof. Since $F \in \mathcal{L}$ and $\lim\limits_{x\to\infty} \log x = \infty$,

$$\lim_{x\to\infty} \frac{\overline{F}(\log x + y)}{\overline{F}(\log x)} = 1$$

for all $y \in \mathbb{R}$. Taking $y = \log z$, $z > 0$, we have

$$\lim_{x\to\infty} \frac{\overline{F}(\log x + \log z)}{\overline{F}(\log x)} = 1.$$

Then

$$\lim_{x\to\infty} \frac{\overline{F}(\log(zx))}{\overline{F}(\log x)} = 1$$

for any fixed $z > 0$. Thus, by Definition 1.4, $\overline{F}(\log x)$ is slowly varying function at infinity.

Using Proposition 1.3.6 (v) by Bingham, Goldie, Teugels [11], we get

$$\lim_{x\to\infty} \overline{F}(\log x)x^h = \infty, \tag{1.7}$$

where $h > 0$. Replacing x with e^x in (1.7), we have

$$\lim_{x\to\infty} \overline{F}(x)\mathrm{e}^{hx} = \infty$$

for every $h > 0$. This implies $F \in \mathcal{H}$. □

Proposition 1.3.6 (v) in [11] additionally gives us intuition that slowly varying tails are extremely heavy.

Further, we provide definitions of exponential-type distributions (see, for instance, Ragulina, Šiaulys [54]) and so called $\mathcal{O}$-exponential distributions (see, for instance, Xu, Foss, Wang [67]).

Definition 1.5. *A distribution function F on $\mathbb{R}$ is said to be exponentially-tailed, written as $F \in \mathcal{L}_\gamma$, $\gamma > 0$, if for any $y > 0$ (equivalently for all $y \in \mathbb{R}$),*

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-\gamma y}. \tag{1.8}$$

Definition 1.6. *A distribution function F on $\mathbb{R}$ is said to be $\mathcal{O}$-exponential, written as $F \in \mathcal{OL}$, if for any $y > 0$,*

$$\limsup_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} < \infty.$$

The classes of exponentially-tailed distributions $\mathcal{L}_\gamma$ were introduced by Embrechts and Goldie [22], while class $\mathcal{OL}$ was first defined by Shimura and Watanabe [60]. Notice that class $\mathcal{L}$ can be obtained by supposing $\gamma = 0$ in (1.8). Hence, we can write $\mathcal{L}_0 = \mathcal{L}$. As it is noted in [67], class $\mathcal{OL}$ includes all classes $\mathcal{L}_\gamma, \gamma \geqslant 0$.

Next we define an important subclass of heavy-tailed distributions, known as the class $\mathcal{D}$. It was introduced by Feller [25]. Our main results will be formulated for this class of distributions.

Definition 1.7. *A distribution function F on $\mathbb{R}$ is said to be dominatedly varying-tailed, written as $F \in \mathcal{D}$, if for any fixed $0 < y < 1$, its tail $\overline{F} = 1 - F$ satisfies the condition*

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

Further we define a few other classes related to those mentioned above but before that we would like to introduce concept of max-sum-equivalence, which helps to acquire deeper understanding about heavy-tailed distributions. For that we need to remember the concept of convolution. Recall that the convolution $F * G$ of any two distributions F and G is given by

$$F*G(x) = (F*G)(x) = \int_{-\infty}^{\infty} F(x-y)\mathrm{d}G(y) = \int_{-\infty}^{\infty} G(x-y)\mathrm{d}F(y),\ x \in \mathbb{R}.$$

If ξ and η are independent random variables with respective distribution functions F and G, then

$$F * G(x) = \mathbb{P}(\xi + \eta \leqslant x), \quad \forall x \in \mathbb{R}.$$

The tail function of the convolution (also known as the convolution tail) of distributions F and G is then given by

$$\overline{F*G}(x) = \int_{-\infty}^{\infty} \overline{F}(x-y)\mathrm{d}G(y) = \int_{-\infty}^{\infty} \overline{G}(x-y)\mathrm{d}F(y).$$

Clearly, if ξ and η are independent r.v.s with respective d.f.s F and G, then

$$\overline{F*G}(x) = \mathbb{P}(\xi+\eta>x), \quad \forall x \in \mathbb{R}.$$

We define max-sum-equivalent distributions as follows.

Definition 1.8. *Distributions (or distribution functions) F_1 and F_2 are said to be max-sum-equivalent, written as $F_1 \sim_M F_2$, if*

$$\overline{F_1 * F_2}(x) \sim \overline{F}_1(x) + \overline{F}_2(x) \tag{1.9}$$

as $x \to \infty$.

From Foss et al. [27, Theorem 2.11], we get that for any distributions F_1 and F_2 on $\mathbb{R}^+$ with unbounded supports ($\overline{F}(x) > 0\ \forall x \in \mathbb{R}$), it is true that

$$\liminf_{x\to\infty} \frac{\overline{F_1 * F_2}(x)}{\overline{F_1}(x) + \overline{F_2}(x)} \geqslant 1. \tag{1.10}$$

Therefore, to prove that requirement (1.9) holds, it is enough to show that the limit superior of the same ratio as in (1.10) is less or equal to 1. It is noted in the same monograph that for distributions on $\mathbb{R}$ property (1.10) in general is not true.

As it is noted in Cai and Tang [12], if two independent random variables ξ_1 and ξ_2 have distributions F_1 and F_2, then property $F_1 \sim_M F_2$ is equivalent to

$$\mathbb{P}(\xi_1+\xi_2>x) \sim \mathbb{P}(\max\{\xi_1,\xi_2\}>x), \tag{1.11}$$

as $x \to \infty$, which means that the distribution of the sum of two independent random variables is asymptotically determined by that of the maximum of the two random variables. The equivalency of (1.9) and

(1.11) follows from these equalities:

$$\begin{aligned}\lim_{x\to\infty}\frac{\mathbb{P}(\xi_1+\xi_2>x)}{\mathbb{P}(\max\{\xi_1,\xi_2\}>x)} &= \lim_{x\to\infty}\frac{\overline{F_1*F_2}(x)}{\overline{F_1}(x)+\overline{F_2}(x)-\overline{F_1}(x)\overline{F_2}(x)}\\ &= \lim_{x\to\infty}\frac{\overline{F_1*F_2}(x)}{\overline{F_1}(x)+\overline{F_2}(x)}\\ &\quad\times\lim_{x\to\infty}\frac{\overline{F_1}(x)+\overline{F_2}(x)}{\overline{F_1}(x)+\overline{F_2}(x)-\overline{F_1}(x)\overline{F_2}(x)}\\ &= \lim_{x\to\infty}\frac{\overline{F_1*F_2}(x)}{\overline{F_1}(x)+\overline{F_2}(x)}.\end{aligned}$$

The term "max-sum-equivalence" comes from relation (1.11). In [12] it is mentioned that this is an interesting property used in modelling extremal events and in describing heavy-tailed distributions.

Usually, class $\mathcal{S}$, one of the most important heavy-tailed classes, is defined in the following manner.

Definition 1.9. *A distribution function F on $\mathbb{R}^+$ is said to be subexponential, written as $F\in\mathcal{S}$, if*

$$\overline{F*F}(x)\sim 2\overline{F}(x) \tag{1.12}$$

as $x\to\infty$.

If F is on $\mathbb{R}^+$ and F is max-sum-equivalent to itself, $F\sim_M F$, then we see that F is a subexponential distribution.

The following proposition gives us an alternative way to look at subexponential distributions.

Proposition 1.2. *A distribution function F on $\mathbb{R}^+$ is subexponential if and only if*

$$\overline{F^{*n}}(x)\sim n\overline{F}(x)$$

as $x\to\infty$, for any $n\geqslant 2$, where F^{*n} denotes the n-fold convolution of F.

It is obvious that relation in Proposition 1.2 implies that one in Definition 1.9 but the proof of reverse implication is quite complicated.

Now we would like to extend Definition 1.9 to distributions on the whole line of real numbers. To require a distribution on $\mathbb{R}$ satisfy condition (1.12) is not enough for an adequate definition. Example 3.3 in [27] gives us a distribution for which

$$\overline{F*F}(x)\underset{x\to\infty}{\sim}2\overline{F}(x),$$

but the respective d.f. F is not long-tailed or even heavy-tailed. So defining subexponentiality for d.f.s of any random variable it is required that both condition (1.12) and condition of long-tailedness are met. The definition, which is used more often, is as follows.

Definition 1.10. *A d.f. F_ξ on $\mathbb{R}$ of a real-valued r.v. ξ is said to be subexponential, written as $F_\xi \in \mathcal{S}$, if the distribution function F_{ξ^+} of $\xi^+ = \max\{\xi, 0\}$ satisfies relation* (1.12)*.*

Interestingly for any heavy-tailed distribution F supported on $\mathbb{R}$,

$$\limsup_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} \geqslant 2.$$

This result can be obtained from Theorem 1.1 in Yu, Wang, Cui [66] by taking constant r.v. τ such that $\mathbb{P}(\tau = 2) = 1$.

It follows from (1.10) that for any distribution F on $\mathbb{R}^+$ with unbounded support ($\overline{F}(x) > 0$ for all x),

$$\liminf_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} \geqslant 2. \tag{1.13}$$

In the monograph by Foss et al. [27], we find the following related result for the smaller class of heavy-tailed distributions (see also Foss and Korshunov [26]).

Theorem 1.1 (Foss et al. [27, Theorem 2.12])**.** *Let F be a heavy-tailed distribution on $\mathbb{R}^+$. Then*

$$\liminf_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2.$$

The "if and only if" statement does not hold. Foss et al. [27] gave a light-tailed example for which limit inferior is also equal to two.

Furthermore, note that this theorem could not be extended to the case of distributions on $\mathbb{R}$. Yu et al. [66] gave an example of two-sided heavy-tailed distribution such that

$$\liminf_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} < 2.$$

We would like to add that, if a distribution on $\mathbb{R}$ is long-tailed, property (1.13) remains valid (see [27, Corollary 2.30]).

Theorem 1.1 implies that subexponentiality of a distribution F on $\mathbb{R}^+$ is equivalent to heavy-tailedness and the reasonable regularity condition that

$$\lim_{x\to\infty}\frac{\overline{F*F}(x)}{\overline{F}(x)}$$

exists.

Next we define a class $\mathcal{S}^*$ with a stronger regularity condition than imposed on the distributions in the class $\mathcal{S}$. It was introduced by Klüppelberg [37, p. 133].

Definition 1.11. *A distribution function F on $\mathbb{R}$ with finite mean $\mathbb{E}(\xi^+)$ is said to be strong subexponential, written as $F \in \mathcal{S}^*$, if*

$$\int_0^x \overline{F}(x-y)\overline{F}(y)\mathrm{d}y \sim 2\mathbb{E}(\xi^+)\overline{F}(x),$$

as $x \to \infty$.

It has been shown that $\mathcal{D} \cap \mathcal{L} \subseteq \mathcal{S}$. In a monograph by Foss, Korshunov, Zachary [27], it is given together with a couple of other interesting conclusions. We present their result as follows.

Theorem 1.2 (Foss et al. [27, Theorem 3.29])**.** *Suppose that F is a d.f. of a real-valued r.v. Let $F \in \mathcal{L} \cap \mathcal{D}$. Then*

(i) F is subexponential.

(ii) $F \in \mathcal{S}^*$, provided F has a finite mean $\mathbb{E}(\xi^+)$.

(iii) $F_\mu \in \mathcal{S}$, for all nonnegative measures μ on $\mathbb{R}^+$ such that

$$\int_0^\infty \overline{F}(t)\mu(\mathrm{d}t) \tag{1.14}$$

is finite. Here distribution function F_μ is defined by its tail:

$$\overline{F}_\mu(x) := \min\Big(1, \int_0^\infty \overline{F}(x+t)\mu(\mathrm{d}t)\Big),\ x \geqslant 0. \tag{1.15}$$

Next we define a useful subclass of $\mathcal{L} \cap \mathcal{D}$, called a class of distributions with consistently varying tails or, in short, class $\mathcal{C}$.

Definition 1.12. *A distribution function F on $\mathbb{R}$ is said to be consistently varying-tailed, written as $F \in \mathcal{C}$, if*

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

Equivalently, $F \in \mathcal{C}$, if

$$\lim_{y \downarrow 1} \liminf_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

An important subclass of class $\mathcal{C}$ is the famous class of regularly varying-tailed distributions.

Definition 1.13. *A distribution function F on $\mathbb{R}$ is said to be regularly varying-tailed with index $\alpha \geqslant 0$, written as $F \in \mathcal{R}_\alpha$, if for any $y > 0$, it holds that*

$$\lim_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}.$$

We write

$$\mathcal{R} := \bigcup_{\alpha \geqslant 0} \mathcal{R}_\alpha.$$

We remark that class $\mathcal{C}$ is strictly larger than class $\mathcal{R}$ (see, for example, Cline and Samorodnitsky [17]). For a full discussion about relations between classes, see the end of this section.

In many cases Karamata theory helps to solve problems concerning distributions of class $\mathcal{R}$, but more abstract classes require different techniques. Later we will see that this is the case in this thesis, too.

Now we introduce an alternative form of Karamata theorem as found in the book by Bingham, Goldie and Teugels [11]. This theorem will be applied later on in the text. Before that we remind the reader about some concepts of bounded variation as given in the same work by Bingham et al. [11].

Definition 1.14. *Let f be real-valued function defined on $E \subseteq \mathbb{R}$. The total variation of f on E is*

$$V(f; E) := \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

where the supremum is taken over all finite sequences $x_0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ in E.

Definition 1.15. *Let I be an interval in $\mathbb{R}$. Class $BV_{loc}(I)$ includes all functions $f : I \to \mathbb{R}$ that are right continuous and are locally of bounded variation on I, i.e. $V(f; J) < \infty$ for each compact $J \subseteq I$.*

Theorem 1.3 (Bingham et al. [11, Theorem 1.6.5]). *Let f be positive, and $f \in BV_{loc}[0,\infty)$. If f is regularly varying with index ρ, and $\sigma+\rho<0$, then*

$$\lim_{x\to\infty} \frac{\int_x^\infty t^\sigma \mathrm{d}f(t)}{x^\sigma f(x)} = -\frac{\rho}{\sigma+\rho}. \tag{1.16}$$

Conversely if $\sigma \neq 0$, and (1.16) *holds for some (finite) ρ with $\sigma+\rho<0$, then f varies regularly with index ρ.*

Here a measurable function $f > 0$ is called regularly varying with index ρ, if

$$\lim_{x\to\infty} \frac{f(yx)}{f(x)} = y^\rho$$

for all $y > 0$.

In relation with Definition 1.13, we say that distribution function F on $\mathbb{R}$ is regularly varying-tailed with index $\alpha \geqslant 0$, if its tail $\overline{F}$ is a regularly varying function with index $-\alpha$.

As the class $\mathcal{L}$ is linked to slow variation (see the proof of Lemma 1.1), classes $\mathcal{L}_\gamma, \gamma \geqslant 0$, are linked to regular variation, in general. The following result is well-known, but we haven't found an article with the proof to refer to, so we state short argumentation here.

Proposition 1.3. *Distribution function $F \in \mathcal{L}_\gamma$, $\gamma \geqslant 0$, if and only if $\overline{F}(\log x)$ is a regularly varying function with index $-\gamma$.*

Proof. For any index $\gamma \geqslant 0$, we have

$$\begin{aligned}
F \in \mathcal{L}_\gamma &\Leftrightarrow \lim_{x\to\infty} \frac{\overline{F}(\log x + y)}{\overline{F}(\log x)} = \mathrm{e}^{-\gamma y} \text{ for all } y \in \mathbb{R} \\
&\Leftrightarrow \lim_{x\to\infty} \frac{\overline{F}(\log x + \log z)}{\overline{F}(\log x)} = \mathrm{e}^{-\gamma \log z} \text{ for all } z > 0 \\
&\Leftrightarrow \lim_{x\to\infty} \frac{\overline{F}(\log(zx))}{\overline{F}(\log x)} = z^{-\gamma} \text{ for all } z > 0 \\
&\Leftrightarrow \overline{F}(\log x) \text{ is a regularly varying function with index } -\gamma.
\end{aligned}$$

□

The classes $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{D}$ can be characterised by specific indices. The first index is the so-called *upper Matuszewska index*, defined as

$$J_F^+ = \inf_{y>1} \left\{ -\frac{1}{\log y} \log \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \right\}.$$

Another index, so-called *L-index*, is defined as

$$L_F = \lim_{y \downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.$$

The aforementioned indices give important characterisations for dominatedly varying and consistently varying d.f.s:

$$L_F > 0 \;\Leftrightarrow\; F \in \mathcal{D} \;\Leftrightarrow\; J_F^+ < \infty; \qquad L_F = 1 \;\Leftrightarrow\; F \in \mathcal{C}.$$

For a d.f. $F \in \mathcal{R}_\alpha$, we have that $J_F^+ = \alpha$ and $L_F = 1$. These two equalities can be verified in just a few steps. If $F \in \mathcal{R}_\alpha$, then

$$L_F \;=\; \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \;=\; \lim_{y\downarrow 1} y^{-\alpha} = 1$$

and

$$J_F^+ \;=\; \inf_{y>1} \left\{ -\frac{1}{\log y} \log \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \right\} \;=\; \inf_{y>1} \left\{ -\frac{1}{\log y} \log y^{-\alpha} \right\} = \alpha.$$

It is also known (see Tang and Tsitsiashvili [61, Lemma 3.5]) that for a d.f. $F \in \mathcal{D}$ it is true that

$$x^{-\alpha} = o(\overline{F}(x))$$

for any $\alpha > J_F^+$.

We finish this section by summarising the interrelationships of the heavy-tailed distribution classes:

$$\mathcal{R} \subsetneq \mathcal{C} \subsetneq \mathcal{L} \cap \mathcal{D} \subsetneq \mathcal{S} \subsetneq \mathcal{L} \subsetneq \mathcal{H}; \quad \mathcal{D} \subsetneq \mathcal{H}; \quad \mathcal{D} \not\subset \mathcal{S} \text{ and } \mathcal{S} \not\subset \mathcal{D}. \tag{1.17}$$

These relationships are illustrated in Figure 1.1.

- Inclusion $\mathcal{R} \subset \mathcal{C}$ is evident from the definitions of respective classes.
- An example from $\mathcal{C} \backslash \mathcal{R}$ was first constructed by Cline and Samorodnitsky [17, Section 3]. In 2004 Cai and Tang proposed a simpler example showing that $\mathcal{C}$ is strictly larger than $\mathcal{R}$ (see [12, Section 2]. We call it the Cai and Tang distribution and discuss it in Section 1.5.6.
- The inclusion $\mathcal{C} \subset \mathcal{L} \cap \mathcal{D}$ is mentioned by Foss et al. [27, Section 3.5].
- The result $\mathcal{L} \cap \mathcal{D} \subset \mathcal{S}$ is established by Goldie [30, Theorem 1] for d.f.s on $\mathbb{R}^+$. The general case is addressed in Foss et al. [27, Theorem 3.29]. In the same paper [30], one can find examples justifying that $\mathcal{D} \not\subset$

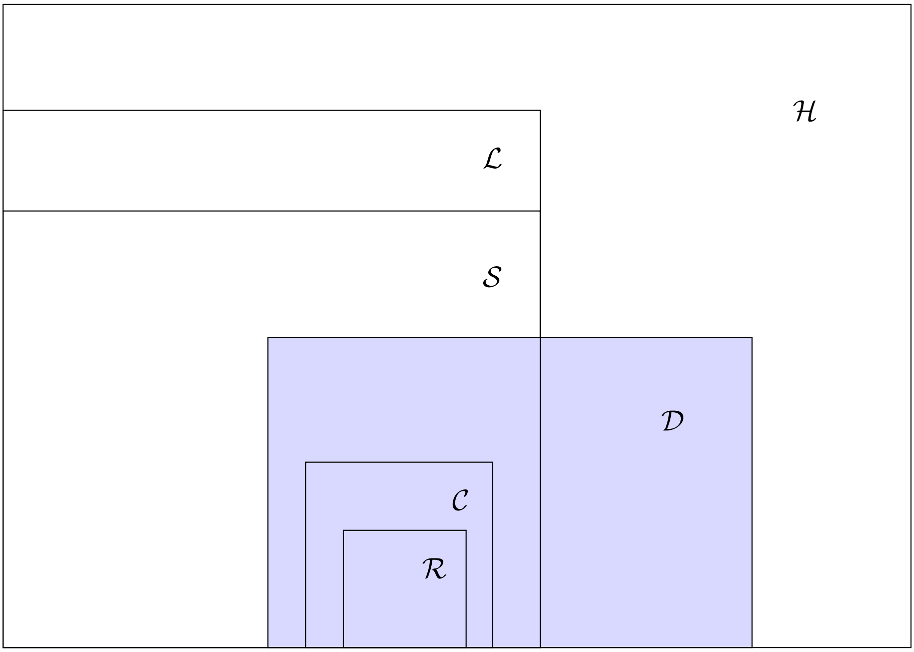

Figure 1.1: Visualisation of interrelationships among the subclasses of heavy-tailed d.f.s.

$\mathcal{S}$ and $\mathcal{S} \not\subset \mathcal{D}$. An example from $\mathcal{D} \backslash \mathcal{S}$ is the Peter and Paul distribution, which we discuss in detail in Section 1.5.2. An example from $\mathcal{S} \backslash \mathcal{D}$ is given by the d.f. $F(x) = 1 - \exp\{-\sqrt{x}\}$, $x > 0$. This is known as Weibull distribution with scale parameter $\lambda = 1$ and shape parameter $\tau = 1/2$. Actually, any Weibull distribution with shape parameter $0 < \tau < 1$ is subexponential but doesn't have dominatedly varying tail (see Section 1.5.4). A related family of examples is given by Embrechts and Omey [24]. Another well-known example from $\mathcal{S} \backslash \mathcal{D}$ is the lognormal distribution (see Section 1.5.7 below).

• The result $\mathcal{S} \subset \mathcal{L}$ can be found in Foss et al. [27, Lemma 3.4]. An example from $\mathcal{L} \backslash \mathcal{S}$ was proposed by Embrechts and Goldie [22, Section 3]. See Section 1.5.5 of our thesis for definition and comments. In the same year another example was proposed by Pitman [53, Section 3], although from a different methodological perspective, which, in some sense, takes the negative logarithms of tail functions as a basis for research.

• In the previously mentioned article by Cline and Samorodnitsky [17, Section 3], we find examples from the sets $(\mathcal{L} \cap \mathcal{D}) \backslash \mathcal{C}$, $\mathcal{S} \backslash (\mathcal{L} \cap \mathcal{D})$

(also showing that $\mathcal{S} \not\subset \mathcal{D}$) and $\mathcal{D} \backslash \mathcal{L}$ (also showing that $\mathcal{D} \not\subset \mathcal{S}$).

• For an argument that $\mathcal{L} \subset \mathcal{H}$, see the proof of Lemma 1.1. The same elements (above) from $\mathcal{D} \backslash \mathcal{L}$ serve the purpose of showing that class $\mathcal{L}$ is a proper subset of $\mathcal{H}$.

• The same elements (above) from $\mathcal{L} \backslash \mathcal{D}$ serve the purpose of showing that class $\mathcal{D}$ is a proper subset of $\mathcal{H}$.

Similarly, it is well known that for class $\mathcal{OL}$,

$$\bigcup_{\gamma \geqslant 0} \mathcal{L}_\gamma \cup \mathcal{D} \subsetneq \mathcal{OL}.$$

This relationship is illustrated in Figure 1.2.

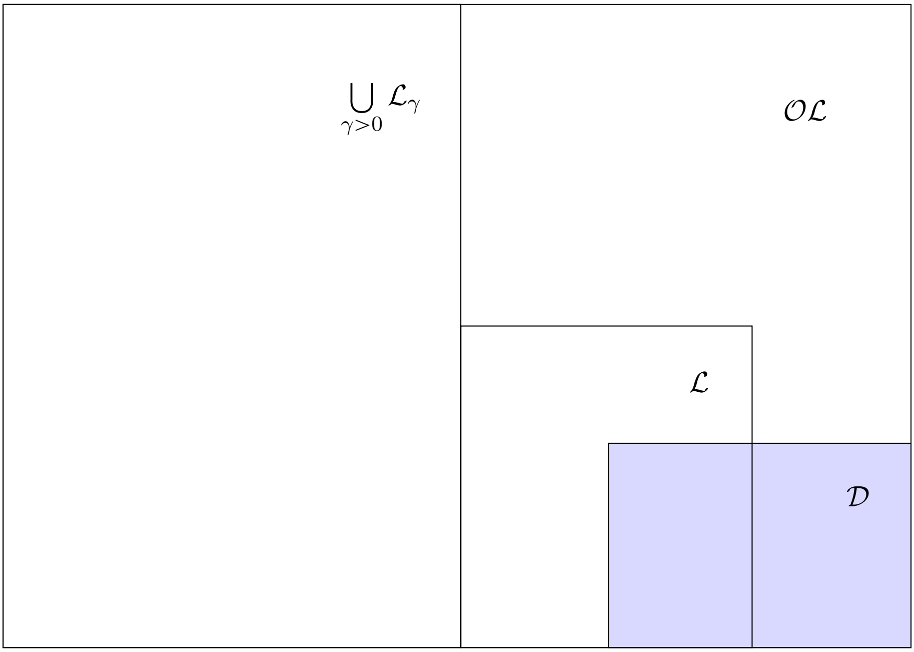

Figure 1.2: Visualisation of interrelationships among the subclasses of $\mathcal{O}$-exponential d.f.s.

1.4.2. Dependence structures

There is a vast amount of literature about asymptotic tail behaviour of distributions of sums of independent random variables. In various settings there is already established that the tail probability of a sum of r.v.s is asymptotically equal to the sum of tail probabilities of individual random summands. We will discuss more about this further in the text. For now we would like to note that, since there is this almost universal agreement that independence assumption is often too unrealistic for applied problems, our goal is to get asymptotic results under a more general dependence structure.

The main dependence assumption about r.v.s $\xi_1, \dots, \xi_n$ used in formulation of the main results is called **pairwise quasi-asymptotic independence**. This dependence assumption was first introduced by Chen and Yuen [13].

Definition 1.16. *Real-valued r.v.s $\xi_1, \dots, \xi_n$ are called pairwise quasi-asymptotically independent (pQAI) if for any $k, l \in \{1, 2, \dots, n\}$, $k \neq l$,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^+ > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} = \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^- > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} = 0. \tag{1.18}$$

We note that the definitions of all classes under study in this thesis guarantee the condition $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$. Therefore, the sums in the denominators of the ratios in (1.18) are not equal to zero. A similar remark applies to the definitions of other dependence structures later in the text.

When proposing QAI dependence structure, Chen and Yuen, as they note themselves, borrowed the term "asymptotic independence" from Resnick (see [55],[56]) and redefined respective definition for non-identically distributed random variables. Here we define pairwise asymptotic independence as follows.

Definition 1.17. *Random variables $\xi_1, \dots, \xi_n$ are called pairwise asymptotically independent (pAI) if for any $k, l = 1, \dots, n$, $k \neq l$,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, \xi_l > x)}{\mathbb{P}(\xi_k > x)} = \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, \xi_l > x)}{\mathbb{P}(\xi_l > x)} = 0.$$

If r.v.s $\xi_1, \dots, \xi_n$ are nonnegative, it is not difficult to see that pAI implies pQAI. To prove that the first limit in (1.18) is equal to zero, we write

$$\begin{aligned} 0 \leqslant \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^+ > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} &= \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, \xi_l > x)}{\mathbb{P}(\xi_k > x) + \mathbb{P}(\xi_l > x)} \\ &\leqslant \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, \xi_l > x)}{\mathbb{P}(\xi_l > x)} = 0 \end{aligned}$$

for any $k, l \in \{1, 2, \dots, n\}$, $k \neq l$. From here, obviously,

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^+ > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} = 0.$$

Proving that the second limit in (1.18) is equal to zero, we observe that for $x \geqslant 0$, the set $\{\xi_l^- > x\} = \varnothing$ and, therefore,

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^- > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} = \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, \varnothing)}{\mathbb{P}(\xi_k > x) + \mathbb{P}(\xi_l > x)} = 0.$$

The following condition is sufficient for pAI ⇒ pQAI to hold in the general case of real-valued r.v.s.

ASSUMPTION $\mathcal{A}$. Random variables $\xi_1, \dots, \xi_n$ satisfy a tail condition

$$\lim_{x\to\infty} \frac{\overline{F}_{\xi_l^-}(x)}{\overline{F}_{\xi_l}(x)} = 0 \quad \text{for all} \quad l \in \{1, \dots, n\}.$$

Since $\{\xi^+ > x\} = \{\xi > x\}$ for any $x \geqslant 0$, we prove that the first limit in (1.18) is equal to zero in the same manner as in the nonnegative case. To prove that the second limit is equal to zero, we employ Assumption $\mathcal{A}$:

$$\begin{aligned}\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^- > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} &= \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, \xi_l^- > x)}{\mathbb{P}(\xi_k > x) + \mathbb{P}(\xi_l > x)} \\ &\leqslant \lim_{x\to\infty} \frac{\overline{F}_{\xi_l^-}(x)}{\overline{F}_{\xi_l}(x)} = 0\end{aligned}$$

for any $k, l \in \{1, 2, \dots, n\}$, $k \neq l$.

In the paper by Leipus et al. [40], a new dependence structure is proposed. It is given as follows.

ASSUMPTION $\mathcal{B}$. Random variables $\xi_1, \dots, \xi_n$ for all $k, l = 1, \dots, n$, $k \neq l$, satisfy

$$\begin{aligned}\lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^+ > x \mid \xi_l^+ > u) &= \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^- > x \mid \xi_l^+ > u) \\ &= \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^+ > x \mid \xi_l^- > u) = 0.\end{aligned}$$

Assumption $\mathcal{B}$ and previously defined pairwise quasi-asymptotic independence are related. We can show that Assumption $\mathcal{B}$ implies the pQAI condition. Indeed, for any ξ_k, ξ_l, $1 \leqslant k \neq l \leqslant n$,

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^+ > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} \leqslant \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^+ > x \mid \xi_l^+ > u) = 0,$$

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^- > x, \xi_l^+ > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} \leqslant \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^- > x \mid \xi_l^+ > u) = 0.$$

In the end of this subsection, we would like to give two examples of r.v.s possessing pQAI dependence structure, which is generated by copulas. The first example is constructed using well-known Farlie–Gumbel–Morgenstein copula.

Example 1.1. *Let $\{\xi_1, \ldots, \xi_n\}$ be r.v.s with infinite right supports and corresponding marginal d.f.s $\{F_1, \ldots, F_n\}$. Consider the Farlie–Gumbel–Morgenstein (FGM) copula:*

$$C_\theta(u, v) = uv + \theta uv(1-u)(1-v), \; u, v \in [0,1], \quad \theta \in [-1,1] .$$

Let r.v.s ξ_i, ξ_j have a joint d.f.

$$\mathbb{P}(\xi_i \leqslant x_1, \xi_j \leqslant x_2) = C_{\theta_i}(F_i(x_1), F_j(x_2))$$

with some $\theta_i \in [-1,1]$, if

$$\max\{i,j\} - \min\{i,j\} = 1, \quad \min\{i,j\} = 2k-1$$

for some $k \in \mathbb{N}$ and be independent otherwise. Then r.v.s $\{\xi_1, \ldots, \xi_n\}$ are pQAI.

It follows from Sklar's theorem (see, for instance, Nelsen [46, Theorem 2.3.3]) that for any given marginal d.f.s F_1, F_2 and an arbitrary copula $C(u_1, u_2)$, function $F(x_1, x_2) := C(F_1(x_1), F_2(x_2))$ is a bivariate d.f. with marginal d.f.s F_1, F_2. If ξ_i, ξ_j, $i, j = 1, \ldots, n$ are independent, then obviously they are pQAI. If $\max\{i,j\} - \min\{i,j\} = 1$, $\min\{i,j\} = 2k-1$ for some $k \in \mathbb{N}$, then

$$\begin{aligned} \frac{\mathbb{P}(\xi_i > x, \xi_j > x)}{\mathbb{P}(\xi_i > x) + \mathbb{P}(\xi_j > x)} &= \frac{1 - F_i(x) - F_j(x) + C_{\theta_i}(F_i(x), F_j(x))}{\overline{F}_i(x) + \overline{F}_j(x)} \\ &= \frac{\overline{F}_i(x)\overline{F}_j(x)\left(1 + \theta_i F_i(x) F_j(x)\right)}{\overline{F}_i(x) + \overline{F}_j(x)} \\ &\leqslant 2\overline{F}_i(x). \end{aligned} \tag{1.19}$$

Similarly, by observing that

$$\begin{aligned} \mathbb{P}\left(\xi_i > x, \xi_j^- > x\right) &\leqslant \mathbb{P}\left(\xi_i > x, \xi_j^- \geqslant x\right) \\ &= \mathbb{P}(\xi_j \leqslant -x) - \mathbb{P}(\xi_i \leqslant x, \xi_j \leqslant -x) \end{aligned}$$

for positive x, we get

$$\frac{\mathbb{P}(\xi_i > x, \xi_j^- > x)}{\mathbb{P}(\xi_i > x) + \mathbb{P}(\xi_j > x)} \leqslant \frac{F_j(-x) - C_{\theta_i}(F_i(x), F_j(-x))}{\overline{F}_i(x) + \overline{F}_j(x)}$$
$$= \frac{\overline{F}_i(x) F_j(-x) \left(1 - \theta_i F_i(x) \overline{F}_j(-x)\right)}{\overline{F}_i(x) + \overline{F}_j(x)}$$
$$\leqslant 2F_j(-x). \tag{1.20}$$

Estimates (1.19) and (1.20) imply (1.18). It follows that random variables $\{\xi_1, \ldots, \xi_n\}$ in Example 1.1 are pQAI.

Another example shows that popular Ali–Mikhail–Haq copula (see [3]) also generates pQAI dependence structure.

Example 1.2. *Let ξ_1, ξ_2 be r.v.s with corresponding d.f.s F_1, F_2 and let random vector (ξ_1, ξ_2) have a bivariate d.f.*

$$F(x_1, x_2) := C_\theta(F_1(x_1), F_2(x_2)),$$

where C_θ is the Ali–Mikhail–Haq copula:

$$C_\theta(u, v) = \frac{uv}{1 - \theta(1-u)(1-v)}, \quad u, v \in [0, 1], \quad \theta \in [-1, 1].$$

Similarly as in Example 1.1, it can be shown that r.v.s ξ_1, ξ_2 are QAI.

Indeed, for positive x, we have

$$\frac{\mathbb{P}(\xi_1 > x, \xi_2 > x)}{\mathbb{P}(\xi_1 > x) + \mathbb{P}(\xi_2 > x)} = \frac{1 - F_1(x) - F_2(x) + C_\theta(F_1(x), F_2(x))}{\overline{F}_1(x) + \overline{F}_2(x)}$$
$$= \frac{\overline{F}_1(x)\overline{F}_2(x)\left(1 + \theta\left(F_2(x) - \overline{F}_1(x)\right)\right)}{(\overline{F}_1(x) + \overline{F}_2(x))(1 - \theta\overline{F}_1(x)\overline{F}_2(x))}$$
$$\leqslant \frac{2\overline{F}_2(x)}{1 - \theta\overline{F}_1(x)\overline{F}_2(x)}.$$

In the same fashion for positive x, we obtain

$$\frac{\mathbb{P}(\xi_1 > x, \xi_2^- > x)}{\mathbb{P}(\xi_1 > x) + \mathbb{P}(\xi_2 > x)} \leqslant \frac{F_2(-x) - C_\theta(F_1(x), F_2(-x))}{\overline{F}_1(x) + \overline{F}_2(x)}$$
$$= \frac{F_2(-x)\overline{F}_1(x)\left(1 - \theta\overline{F}_2(-x)\right)}{(\overline{F}_1(x) + \overline{F}_2(x))(1 - \theta\overline{F}_1(x)\overline{F}_2(-x))}$$
$$\leqslant \frac{2F_2(-x)}{1 - \theta\overline{F}_1(x)\overline{F}_2(-x)}.$$

The derived estimates imply that r.v.s ξ_1 and ξ_2 are QAI.

1.5. Examples of heavy-tailed and related distributions

In this section we define some wel-known and some less-known (but still important) distributions and assign them to specific distribution classes. The results are summarised in Table 1.

Ditribution \ Class	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{S}$	$\mathcal{L}$	$\mathcal{D}$	$\mathcal{H}$	$\mathcal{OL}$
1. Pareto distribution	+	+	+	+	+	+	+
2. Peter and Paul distribution	-	-	-	-	+	+	+
3. Example of a mixed distribution (see Section 1.5.3)	-	-	-	-	+	+	+
4. Weibull distribution (shape parameter $0 < \tau < 1$)	-	-	+	+	-	+	+
5. Weibull distribution ($\tau = 1$; also known as Exponential distribution)	-	-	-	-	-	-	+
6. Weibull distribution ($\tau > 1$)	-	-	-	-	-	-	-
7. Embrechts and Goldie distribution	-	-	-	+	-	+	+
8. Cai and Tang distribution	-	+	+	+	+	+	+
9. Lognormal distribution	-	-	+	+	-	+	+

Table 1: Examples of heavy-tailed and related distributions.

1.5.1. Pareto distribution

Suppose that r.v. ξ is distributed according to the Pareto law:

$$F_\xi(x) = \left(1 - \left(\frac{\lambda}{x}\right)^\alpha\right) \mathbb{1}_{[\lambda,\infty)}(x),$$

where scale parameter $\lambda > 0$, shape parameter $\alpha > 1$.

For sufficiently large x, ratio of the tails is

$$\frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} = \frac{\left(\frac{\lambda}{xy}\right)^\alpha}{\left(\frac{\lambda}{x}\right)^\alpha} = y^{-\alpha}.$$

We see that the condition

$$\lim_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} = y^{-\alpha},$$

for any $y > 0$, is satisfied. And, therefore, $F_\xi \in \mathcal{R}_\alpha \subset \mathcal{R}$.

From relations (1.17) it follows that F_ξ belongs, also, to other classes: $\mathcal{C}$, $\mathcal{S}$, $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{H}$.

1.5.2. Peter and Paul distribution

Suppose that r.v. ξ is distributed according to the generalised Peter and Paul law:

$$\mathbb{P}(\xi = a^{-j\beta}) = a^{j-1}(1-a),\ j = 1, 2, \ldots,$$

where $a \in (0,1)$ and $\beta > 0$. For the properties of this distribution, see Berkes et al. [9].

For $x > 0$,

$$\mathbb{P}(\xi > x) = \sum_{j: a^{-j\beta} > x} a^{j-1}(1-a) = a^{\left\lfloor \frac{\log x}{\beta \log(1/a)} \right\rfloor}.$$

Here and further $\lfloor z \rfloor$ denotes the integer part of the real number z.

First we will prove that Peter and Paul distribution doesn't belong to class $\mathcal{L}$. It is enough to show that relation (1.5) doesn't hold for any single $y > 0$. Let us choose $y = 1$. Then for sufficiently large x,

$$\begin{aligned}\frac{\overline{F}_\xi(x+1)}{\overline{F}_\xi(x)} &= a^{\left\lfloor \frac{\log(x+1)}{\beta\log(1/a)} \right\rfloor - \left\lfloor \frac{\log x}{\beta \log(1/a)} \right\rfloor} \\ &= a^{\frac{\log(x+1)}{\beta\log(1/a)} - \frac{\log x}{\beta\log(1/a)}} a^{-\left\{ \frac{\log(x+1)}{\beta\log(1/a)} \right\} + \left\{ \frac{\log x}{\beta\log(1/a)} \right\}} \\ &= a^{\frac{\log(1+1/x)}{\beta\log(1/a)}} a^{-\left(\left\{ \frac{\log(x+1)}{\beta\log(1/a)} \right\} - \left\{ \frac{\log x}{\beta\log(1/a)} \right\} \right)}. \end{aligned} \tag{1.21}$$

Here and further $\{z\}$ denotes the fractional part of the real number z, that is, $\{z\} = z - \lfloor z \rfloor$. From the expression (1.21), we can see that

$$\limsup_{x\to\infty} \frac{\overline{F}_\xi(x+1)}{\overline{F}_\xi(x)} = 1$$

and

$$\liminf_{x\to\infty} \frac{\overline{F}_\xi(x+1)}{\overline{F}_\xi(x)} = a \in (0,1).$$

Necessary limit doesn't exist and, therefore, $F_\xi \notin \mathcal{L}$. As an immediate consequence, we get that $F_\xi \notin \mathcal{S}$.

Now we will prove that Peter and Paul distribution function belongs to class $\mathcal{D}$. For the floor function, it is true that, for every real a and b, the following inequalities hold:

$$\lfloor a \rfloor + \lfloor b \rfloor \leqslant \lfloor a + b \rfloor \leqslant \lfloor a \rfloor + \lfloor b \rfloor + 1. \tag{1.22}$$

Similarly as before, for any $0 < y < 1$ and sufficiently large x, by (1.22), we get

$$\begin{aligned}\frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} &= a^{\left\lfloor \frac{\log(xy)}{\beta\log(1/a)}\right\rfloor - \left\lfloor \frac{\log x}{\beta\log(1/a)}\right\rfloor} \\ &= a^{\left\lfloor \frac{\log x}{\beta\log(1/a)} + \frac{\log y}{\beta\log(1/a)}\right\rfloor - \left\lfloor \frac{\log x}{\beta\log(1/a)}\right\rfloor} \\ &\leqslant a^{\left\lfloor \frac{\log x}{\beta\log(1/a)}\right\rfloor + \left\lfloor \frac{\log y}{\beta\log(1/a)}\right\rfloor - \left\lfloor \frac{\log x}{\beta\log(1/a)}\right\rfloor} \\ &= a^{\left\lfloor \frac{\log y}{\beta\log(1/a)}\right\rfloor},\end{aligned}$$

since $a \in (0,1)$. Thus, for any fixed $0 < y < 1$,

$$\limsup_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} \leqslant a^{\left\lfloor \frac{\log y}{\beta\log(1/a)}\right\rfloor} < \infty,$$

implying that $F_\xi \in \mathcal{D}$.

Now let us turn to class $\mathcal{C}$. Since every distribution function with consistently varying tail is also long-tailed and for Peter and Paul distribution it is true that $F_\xi \notin \mathcal{L}$, we conclude that $F_\xi \notin \mathcal{C}$. It is also possible to demonstrate this proposition in a straightforward way. For sufficiently large x,

$$\begin{aligned}\frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} &= a^{\left\lfloor \frac{\log(xy)}{\beta\log(1/a)}\right\rfloor - \left\lfloor \frac{\log x}{\beta\log(1/a)}\right\rfloor} \\ &= a^{\frac{\log x}{\beta\log(1/a)} + \frac{\log y}{\beta\log(1/a)} - \left\{ \frac{\log x}{\beta\log(1/a)} + \frac{\log y}{\beta\log(1/a)}\right\} - \frac{\log x}{\beta\log(1/a)} + \left\{ \frac{\log x}{\beta\log(1/a)}\right\}} \\ &= a^{\frac{\log y}{\beta\log(1/a)}} a^{-\left(\left\{ \frac{\log x}{\beta\log(1/a)} + \frac{\log y}{\beta\log(1/a)}\right\} - \left\{ \frac{\log x}{\beta\log(1/a)}\right\}\right)},\end{aligned}$$

for any $y \in (0,1)$. Therefore,

$$\begin{aligned}&\lim_{y\uparrow 1}\limsup_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} \\ &= \lim_{y\uparrow 1}\limsup_{x\to\infty} a^{\frac{\log y}{\beta\log(1/a)}} a^{-\left(\left\{ \frac{\log x}{\beta\log(1/a)} + \frac{\log y}{\beta\log(1/a)}\right\} - \left\{ \frac{\log x}{\beta\log(1/a)}\right\}\right)} \\ &= \lim_{y\uparrow 1} a^{\frac{\log y}{\beta\log(1/a)}} a^{-\left(\left\{ \frac{\log y}{\beta\log(1/a)}\right\} + 1\right)} \\ &= a^{-1} \neq 1.\end{aligned}$$

Thus, $F_\xi \notin \mathcal{C}$. And, therefore, $F_\xi \notin \mathcal{R}$.

1.5.3. Example of a mixed distribution

Previously we have given examples of purely continuous and discrete distributions. Here we give an example of a mixed distribution which belongs to class $\mathcal{D}\backslash\mathcal{L}$. It is constructed by "breaking" Pareto distribution function at the points from a diverging sequence of real numbers.

Suppose that r.v. ξ has the following distribution function:

$$F_\xi(x) = \begin{cases} 0, & x < 1, \\ 1 - \frac{1}{2^{n-1}x^\alpha}, & 2^{n-1} \leqslant x < 2^n, \ n \in \mathbb{N}, \ \alpha > 1. \end{cases}$$

Then the tail function

$$\overline{F}_\xi(x) = \begin{cases} 1, & x < 1, \\ \frac{1}{2^{n-1}x^\alpha}, & 2^{n-1} \leqslant x < 2^n, \ n \in \mathbb{N}, \ \alpha > 1. \end{cases} \tag{1.23}$$

First we will prove that F_ξ is not long-tailed. By definition, for $x \in [2^n - 1, 2^n), \ n \in \mathbb{N}$,

$$\frac{\overline{F}_\xi(x+1)}{\overline{F}_\xi(x)} = \frac{\frac{1}{2^n(x+1)^\alpha}}{\frac{1}{2^{n-1}x^\alpha}} = \frac{1}{2\left(1+\frac{1}{x}\right)^\alpha} \leqslant \frac{1}{2}.$$

There exists a diverging sequence $x_n = 2^n - 1, \ n \in \mathbb{N}$, such that

$$\frac{\overline{F}_\xi(x_n+1)}{\overline{F}_\xi(x_n)} \leqslant \frac{1}{2},$$

and, therefore, $\frac{\overline{F}_\xi(x+1)}{\overline{F}_\xi(x)}$ doesn't converge to 1 as $x \to \infty$. Thus, conditions of Definition 1.3 are not satisfied - the distribution function F_ξ is not long-tailed.

The values of the ratio $\frac{\overline{F}_\xi(x+1)}{\overline{F}_\xi(x)}$, in the case $\alpha = 2$, are depicted in Figure 1.3. The graph illustrates the fact that $\liminf\limits_{x\to\infty} \frac{\overline{F}_\xi(x+1)}{\overline{F}_\xi(x)} = 1/2$ and $\limsup\limits_{x\to\infty} \frac{\overline{F}_\xi(x+1)}{\overline{F}_\xi(x)} = 1$.

Next we prove that F_ξ is dominatedly varying-tailed. Condition $\limsup\limits_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$, for all $y \in (0,1)$, is equivalent to $\limsup\limits_{x\to\infty} \frac{\overline{F}(\frac{1}{2}x)}{\overline{F}(x)} < \infty$. Thus, we will consider the ratio $\frac{\overline{F}_\xi(\frac{1}{2}x)}{\overline{F}_\xi(x)}, \ x \in \mathbb{R}$. We get

$$\frac{\overline{F}_\xi(\frac{1}{2}x)}{\overline{F}_\xi(x)} = \begin{cases} 1, & x < 1, \\ x^\alpha, & 1 \leqslant x < 2, \\ 2^{\alpha+1}, & x \geqslant 2. \end{cases}$$

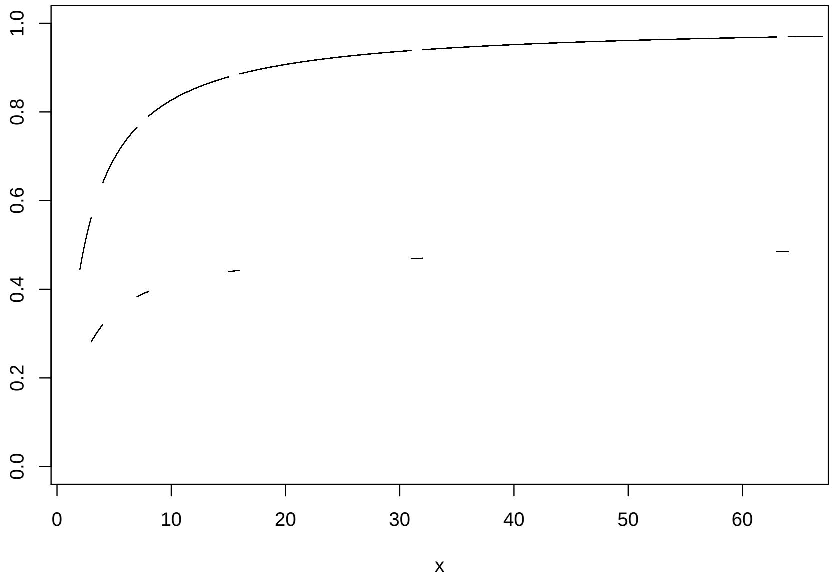

Figure 1.3: Values of the ratio of tails in the x range $(2, 65)$.

Indeed, when $x < 1$, then, also, $\frac{1}{2}x < 1$. Thus, by (1.23), $\frac{\overline{F}_\xi(\frac{1}{2}x)}{\overline{F}_\xi(x)} = 1$. When $1 \leqslant x < 2$, then $\frac{1}{2} \leqslant \frac{1}{2}x < 1$, and by (1.23),

$$\frac{\overline{F}_\xi(\frac{1}{2}x)}{\overline{F}_\xi(x)} = \frac{1}{\frac{1}{x^\alpha}} = x^\alpha.$$

Finally, when $2^{n-1} \leqslant x < 2^n$, $n \in \mathbb{N}\backslash\{1\}$, then $2^{n-2} \leqslant \frac{1}{2}x < 2^{n-1}$. Thus,

$$\frac{\overline{F}_\xi(\frac{1}{2}x)}{\overline{F}_\xi(x)} = \frac{\frac{1}{2^{n-2}(x/2)^\alpha}}{\frac{1}{2^{n-1}x^\alpha}} = 2^{\alpha+1}.$$

It is clear that $\limsup_{x\to\infty} \frac{\overline{F}_\xi(\frac{1}{2}x)}{\overline{F}_\xi(x)} = 2^{\alpha+1} < \infty$. Therefore, $F_\xi \in \mathcal{D}$, and we conclude that $F_\xi \in \mathcal{D}\backslash\mathcal{L}$.

1.5.4. Weibull distribution

Further we explore the classical Weibull distribution. Whether distribution is heavy-tailed or not depends on the value of its shape parameter τ.

The distribution function of Weibull random variable ξ with scale parameter $\lambda > 0$ and shape parameter $\tau > 0$ is given by

$$F_\xi(x) = (1 - \mathrm{e}^{-\lambda x^\tau})\mathbb{1}_{\{x \geqslant 0\}}.$$

Notice that in case $\tau = 1$ it is a distribution function of the famous exponential distribution.

First, we will show that in case $\tau \geqslant 1$ d.f. F_ξ is not heavy-tailed. Remembering Definition 1.1, we have to show that there exists a number $h > 0$ such that

$$\limsup_{x\to\infty} \overline{F}(x)\mathrm{e}^{hx} < \infty.$$

Let us take $h = \lambda$. Then for $x \geqslant 0$,

$$\overline{F}(x)\mathrm{e}^{hx} = \mathrm{e}^{-\lambda x^\tau}\mathrm{e}^{\lambda x} = \mathrm{e}^{\lambda(-x^\tau + x)}.$$

Thus,

$$\limsup_{x\to\infty} \overline{F}(x)\mathrm{e}^{hx} = \begin{cases} 1, & \tau = 1, \\ 0, & \tau > 1. \end{cases}$$

Therefore, in case $\tau \geqslant 1$ d.f. $F_\xi \notin \mathcal{H}$ and, clearly, F_ξ doesn't belong to any subclass of $\mathcal{H}$. Belonging to the class $\mathcal{OL}$, though, depends on the choice of $\tau \geqslant 1$. Indeed, in case $\tau = 1$, with $x \geqslant 0$ and $y > 0$, we have

$$\frac{\overline{F}_\xi(x+y)}{\overline{F}_\xi(x)} = \frac{\mathrm{e}^{-\lambda(x+y)}}{\mathrm{e}^{-\lambda x}} = \mathrm{e}^{-\lambda y}$$

and, therefore,

$$\lim_{x\to\infty} \frac{\overline{F}_\xi(x+y)}{\overline{F}_\xi(x)} = \mathrm{e}^{-\lambda y}$$

for any $y > 0$. Thus, exponential d.f. F_ξ with parameter $\lambda > 0$ belongs to class $\mathcal{L}_\gamma$ with $\gamma = \lambda$ and, also, belongs to class $\mathcal{OL}$.

Further we analyse the case $\tau > 1$. The ratio of interest is

$$\frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)} = \frac{\mathrm{e}^{-\lambda(x-1)^\tau}}{\mathrm{e}^{-\lambda x^\tau}} = \mathrm{e}^{\lambda(x^\tau - (x-1)^\tau)}.$$

Since with $\tau > 1$ the function $x^\tau - (x-1)^\tau \to \infty$, when $x \to \infty$,

$$\limsup_{x\to\infty} \frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)} = \infty,$$

and, clearly, $F_\xi \notin \mathcal{OL}$.

Now let us turn to the case $0 < \tau < 1$. First, we will show that $F_\xi \notin \mathcal{D}$. When $x \geqslant 0$,

$$\frac{\overline{F}_\xi(\frac{1}{2}x)}{\overline{F}_\xi(x)} = \frac{\mathrm{e}^{-\lambda(\frac{1}{2}x)^\tau}}{\mathrm{e}^{-\lambda x^\tau}} = \mathrm{e}^{\lambda x^\tau(1-(\frac{1}{2})^\tau)}.$$

Since $\lambda > 0$, $0 < \tau < 1$, $0 < 1 - (\frac{1}{2})^\tau < 1$, we have

$$\limsup_{x\to\infty} \frac{\overline{F}_\xi(\frac{1}{2}x)}{\overline{F}_\xi(x)} = \infty.$$

Therefore, F_ξ does not belong to class $\mathcal{D}$, nor does it belong to classes $\mathcal{R}$ and $\mathcal{C}$.

It is well known that Weibull distribution with shape parameter $0 < \tau < 1$ is subexponential. See, for example, Klüppelberg [37] or Shimura, Watanabe [60]. Consequently, it is also long-tailed and belongs to the general class of heavy-tailed distributions $\mathcal{H}$.

1.5.5. Embrechts and Goldie distribution

In 1980 Embrechts and Goldie [22, Section 3] gave an example showing that $\mathcal{S}$ is a proper subset of $\mathcal{L}$. Here, for the sake of discernibility, we call this distribution by the names of mentioned authors.

Let $\{a_n\}$ be a sequence of positive numbers satisfying $a_n \to \infty$, $a_n < \frac{1}{2}(n+1)!$, and let's define d.f. F_ξ by its tail function:

$$\overline{F}_\xi(x) = \begin{cases} 1, & -\infty < x \leqslant 2, \\ \frac{1}{(n+1)!}, & (n+1)! + na_n \leqslant x \leqslant (n+2)!, \ \ n = 1, 2, \ldots, \end{cases}$$

$$\overline{F}_\xi((n+1)! + u) = \frac{1 + n - u/a_n}{(n+1)!}, \ \ 0 \leqslant u \leqslant na_n, \ \ n = 1, 2, \ldots$$

The idea behind the construction can be observed in the graph of the tail $\overline{F}_\xi$, when taking some specific sequence $\{a_n\}$, satisfying the conditions above (see Figure 1.4). For the proof that $F_\xi \in \mathcal{L}$ but $F_\xi \notin \mathcal{S}$, look into the mentioned paper of Embrechts and Goldie [22]. The proof that F_ξ is long-tailed emerges from the idea that line segment slopes flatten, when x increases, and flat segments in between become longer. The proof that F_ξ is not subexponential is much more complicated and requires specifying a sequence a_n, so that $a_n \to \infty$ "very slowly".

Since $F_\xi \in \mathcal{L}$, it is clear that F_ξ is heavy-tailed. Since $F_\xi \notin \mathcal{S}$ and interrelationships (1.17) hold, we conclude, also, that F_ξ doesn't belong to any of the classes $\mathcal{D}$, $\mathcal{C}$ or $\mathcal{R}$.

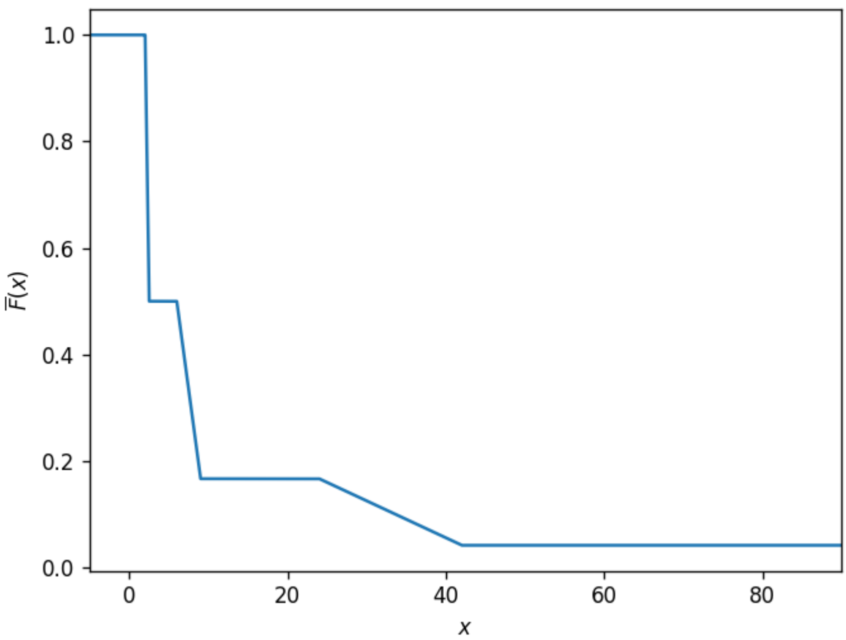

Figure 1.4: Values of the tail of Embrechts and Goldie distribution in the x range $(-5, 90)$.

1.5.6. Cai and Tang distribution

The proposition that class $\mathcal{C}$ is strictly larger than class $\mathcal{R}$ was first shown by Cline and Samorodnitsky [17, Section 3]. Cai and Tang [12, Section 2] provided a simpler example. Since there is no name proposed for the distribution of interest, we call it by the names of authors of the mentioned paper. Consider a random variable, written, simply, as

$$\xi = (1 + \eta)2^N,$$

where η and N are independent random variables, η is uniformly distributed on $(0, 1)$ and N is a geometric random variable with $\mathbb{P}(N = k) = (1 - p)p^k$ for $0 < p < 1$ and $k = 0, 1, \ldots$ Let F_ξ be the distribution of ξ.

Showing that $F_\xi \notin \mathcal{R}$, Cai and Tang chose two different sequences $\{x_n = 2^n,\ n = 1, 2, \ldots\}$ and $\{y_n = 2^{n+1}/3,\ n = 1, 2, \ldots\}$. It appears that for all $0 < p < 1$,

$$\lim_{n\to\infty} \frac{\overline{F}_\xi(1.5x_n)}{\overline{F}_\xi(x_n)} \neq \lim_{n\to\infty} \frac{\overline{F}_\xi(1.5y_n)}{\overline{F}_\xi(y_n)}.$$

The limit

$$\lim_{x\to\infty} \frac{\overline{F}_\xi(1.5x)}{\overline{F}_\xi(x)}$$

does not exist and, therefore, $F_\xi \notin \mathcal{R}$.

The proof that $F_\xi \in \mathcal{C}$ involves noticing that for any $x > 1$, there is a unique integer $n = n(x)$ such that $2^n < x \leqslant 2^{n+1}$, and, using this, evaluating ratio $\frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)}$ by employing well-known properties from probability theory. Since $F_\xi \in \mathcal{C}$, belonging to classes $\mathcal{S}$, $\mathcal{L}$, $\mathcal{D}$, $\mathcal{H}$ and $\mathcal{OL}$ is implied.

Furthermore, Cai and Tang [12] remark that, for any positive m, it is possible to choose $0 < p = p(m) < 1$ such that ξ has the finite moment of order m. So, in some sense, the tail of F_ξ is moderately heavy.

1.5.7. Lognormal distribution

Finally, we mention the lognormal distribution. We are interested in the d.f. of random variable $\mathrm{e}^{\sigma X}$, where X is a standard normal r.v. and $\sigma > 0$.

The tail of lognormal distribution is

$$\overline{F}(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{\log x}^{\infty} \exp\Big\{-\frac{u^2}{2\sigma^2}\Big\}\mathrm{d}u,\ x > 0.$$

Short, but interesting proof that $F \in \mathcal{S}$ is given by Embrechts et al. [23, Section 6]. They use a standard estimate, $\overline{F}(x) \sim \overline{F}_0(x)$, where

$$\overline{F}_0(x) = \frac{\sigma}{\sqrt{2\pi}\log x} \exp\Big\{-\frac{(\log x)^2}{2\sigma^2}\Big\},\ x > e.$$

Then $F \in \mathcal{S}$ if and only if $F_0 \in \mathcal{S}$ (see Theorem 1.10 (a) and the references in the following subsection). Using a characterisation in [64, Theorem 2], they found that $F_0 \in \mathcal{S}$ and, therefore, $F \in \mathcal{S}$.

The fact that lognormal d.f. $F \in \mathcal{S}$ is well-known. Besides [23] we found it mentioned in [17],[19],[29],[37],[60],[69]. On the other hand, it is mentioned in [29], [60], [69] that $F \notin \mathcal{D}$.

1.6. Summary of known closure properties of the classes of heavy-tailed and related distributions

Here we provide a summary of known closure properties of heavy-tailed and related distributions' classes, that is, the classes that we already defined in Section 1.4. Some of these results will be used in the proofs of our propositions further in the text.

• We say that a class of distribution functions $\mathcal{A}$ is closed under convolutions, if for any $F_1 \in \mathcal{A}$ and $F_2 \in \mathcal{A}$, it holds that $F_1 * F_2 \in \mathcal{A}$.

• We say that the max-sum-equivalence holds in class $\mathcal{A}$, if for any $F_1 \in \mathcal{A}$ and $F_2 \in \mathcal{A}$, it holds that $F_1 \sim_M F_2$.

• We say that class $\mathcal{A}$ is closed under convolution power, if for any $F \in \mathcal{A}$, it holds that $F^{*n} \in \mathcal{A}$ for all integer $n \geqslant 2$.

• We say that class $\mathcal{A}$ is closed under convolution root, if $F^{*n} \in \mathcal{A}$ for some $n \geqslant 2$ implies $F \in \mathcal{A}$.

The following closure properties are already established in the literature.

The class of regularly varying-tailed distributions $\mathcal{R}$ is closed under convolutions and the max-sum-equivalence holds in this class.

Theorem 1.4 (Cai, Tang [12, Proposition 1.2]). *Suppose that F_1,F_2 are d.f.s corresponding to distributions supported on $\mathbb{R}^+$. If $F_1 \in \mathcal{R}$ and $F_2 \in \mathcal{R}$, then $F_1 * F_2 \in \mathcal{R}$ and $F_1 \sim_M F_2$.*

For the bigger class $\mathcal{C}$, which includes all elements of $\mathcal{R}$, we have an analogous result.

Theorem 1.5 (Cai, Tang [12, Theorem 2.2]). *Suppose that F_1,F_2 are d.f.s corresponding to distributions supported on $\mathbb{R}^+$. If $F_1 \in \mathcal{C}$ and $F_2 \in \mathcal{C}$, then $F_1 * F_2 \in \mathcal{C}$ and $F_1 \sim_M F_2$.*

When assuming independence, the following result holds for real-valued random variables.

Theorem 1.6 (Kizinevič et al. [36, Lemma 3]; Chen, Yuen [13, Theorem 3.1]; Wang, Tang [68, Theorem 2.1] (i.i.d. case)). *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent real-valued r.v.s. If $F_{\xi_k} \in \mathcal{C}$ for each $k \in \{1, 2, \ldots, n\}$, then the d.f. $F_{S_n^\xi} \in \mathcal{C}$ and*

$$\overline{F}_{S_n^\xi}(x) \sim \sum_{k=1}^{n} \overline{F}_{\xi_k}(x).$$

If ξ_1 and ξ_2 are independent, this is clearly an equivalent of Theorem 1.5 for real-valued random variables. For the sake of visual comparison, we formulate the statement as the following corollary.

Corollary 1.1. *Let ξ_1 and ξ_2 be independent real-valued r.v.s. with d.f.s. F_1 and F_2. If $F_1 \in \mathcal{C}$ and $F_2 \in \mathcal{C}$, then $F_1 * F_2 \in \mathcal{C}$ and $F_1 \sim_M F_2$.*

The set closure result for the class $\mathcal{D} \cap \mathcal{S} = \mathcal{D} \cap \mathcal{L}$ is established by Embrechts and Goldie [22].

Theorem 1.7 (Embrechts, Goldie [22, Proposition 2])**.** *Let F_1,F_2 be such d.f.s. that $F_1(0) = 0$ and $F_2(0) = 0$. If $F_1 \in \mathcal{D} \cap \mathcal{S}$ and $F_2 \in \mathcal{D} \cap \mathcal{S}$, then $F_1 * F_2 \in \mathcal{D} \cap \mathcal{S}$.*

The result that max-sum-equivalence holds in $\mathcal{D} \cap \mathcal{L}$ is due to Cai, Tang [12, Theorem 2.1] and Omey [47]. The closure of class $\mathcal{L} \cap \mathcal{D}$ under convolution power follows also from Leipus, Šiaulys [39, Corollary 3].

In 1980, when Embrechts and Goldie wrote their paper, it was not known if the class $\mathcal{S}$ is closed under convolutions, but they achieved useful equivalent forms of convolution closure.

Theorem 1.8 (Embrechts, Goldie [22, Theorem 2])**.** *Let F_1,F_2 be such d.f.s. that $F_1(0) = 0$ and $F_2(0) = 0$. If $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{S}$, then the following are equivalent:*

(i) $F_1 * F_2 \in \mathcal{S}$,

(ii) $F_1 \sim_M F_2$,

(iii) $pF_1 + (1-p)F_2 \in \mathcal{S}$ for some (all) p satisfying $0 < p < 1$.

In the same article we find the theorem about convolution closure of classes $\mathcal{L}_\gamma, \gamma \geqslant 0$. We remind that $\mathcal{L}_0 = \mathcal{L}$, that is, in case of $\gamma = 0$ we have in mind long-tailed distributions.

Theorem 1.9 (Embrechts, Goldie [22, Theorem 3])**.** *Let F_1 and F_2 be d.f.s of real-valued random variables.*

(a) If $F_1 \in \mathcal{L}_\gamma$ and $\overline{F_2} = o(\overline{F_1})$, in particular if $F_2 \in \mathcal{L}_{\gamma'}$ for $\gamma' > \gamma$, then $F_1 * F_2 \in \mathcal{L}_\gamma$.

(b) If $F_1 \in \mathcal{L}_\gamma$ and $F_2 \in \mathcal{L}_\gamma$, then $F_1 * F_2 \in \mathcal{L}_\gamma$.

Thus, clearly, class $\mathcal{L}$ is closed under convolutions (see, also, [27, Corollary 2.42]). But the subclass $\mathcal{S} \subset \mathcal{L}$ is neither closed under convolutions nor max-sum-equivalence holds in it. Therefore, max-sum-equivalence is invalid in class $\mathcal{L}$, too. The result of non-closure of class $\mathcal{S}$ is due to Leslie [41].

Even though class $\mathcal{S}$ in general is not closed under convolutions, in case of dominated relations between tails of F_1 and F_2, the convolution $F_1 * F_2$ stays in class $\mathcal{S}$. Also nonnegative distribution with a tail asymptotically equal (or related) to the tail of subexponential distribution stays in class $\mathcal{S}$; this fact is not immediately clear because the definition of subexponential distributions involves convolution.

Theorem 1.10 (Pakes [48, Lemma 2]; Embrechts et al. [23, Proposition 1]; Pitman [53, Corollary 2]; Embrechts and Goldie [22, Lemma 2]; Cline [16, Corollary 1])**.** *Let F_1 and F_2 be two d.f.s on the half-line* $[0, \infty)$.

(a) *If $F_1 \in \mathcal{S}$, $\overline{F}_2(x) \underset{x\to\infty}{\sim} c\overline{F}_1(x),\ c > 0$, then $F_2 \in \mathcal{S}$.*

(b) *If $F_1 \in \mathcal{S}$ and $\overline{F}_2(x) = o(\overline{F}_1(x))$, then $F_1 * F_2 \in \mathcal{S}$ and $F_1 \sim_M F_2$.*

(c) *If $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{L}$, and $\overline{F}_2(x) = O(\overline{F}_1(x))$, then $F_1 * F_2 \in \mathcal{S}$ and $F_1 \sim_M F_2$.*

Result (a) was given in Pakes [48, Lemma 2]. Case $c = 1$ (closure under asymptotic equality) is also given in Teugels [64, Theorem 3]. For the proof of (b), see Embrechts et al. [23, Proposition 1] and Pitman [53, Corollary 2]; for the proof of (c), see Embrechts and Goldie [22, Lemma 2] and Cline [16, Corollary 1].

• Generalisation of Theorem 1.10 (a) for distributions on $\mathbb{R}$ is given in Foss et al. [27, Corollary 3.13]. Note that in the literature two distributions F_1 and F_2 satisfying condition $\overline{F}_2(x) \underset{x\to\infty}{\sim} c\overline{F}_1(x)$, for some constant $c > 0$, are sometimes called proportionally tail-equivalent. Also, in relation with Theorem 1.10 part (a), we write another result with a weaker condition of tail-equivalence.

Theorem 1.11 (Foss et al. [27, Theorem 3.11])**.** *Let F_1 and F_2 be two distributions on* $\mathbb{R}$.

If $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{L}$ and $\overline{F_1}(x) \asymp \overline{F_2}(x)$, then $F_2 \in \mathcal{S}$.

For the proof of this result, see [27, Theorem 3.11].

• Generalisation of Theorem 1.10 (b) for distributions on $\mathbb{R}$ is Corollary 3.18 in [27].

• Versions of Theorem 1.10 (c) for distributions on $\mathbb{R}$ are Lemma 3.2 in [61] and Corollaries 3.16 and 3.17 in [27].

It is well known that for any positive integer n,

$$F \in \mathcal{S} \Leftrightarrow F^{*n} \in \mathcal{S}. \tag{1.24}$$

Proofs can be found in Embrechts et al. [23, Theorem 2] and Chistyakov [15]. Left-to-right implication can also be checked employing Theorem 1.10 (c) above. For the implication $F \in \mathcal{S} \Rightarrow F^{*n} \in \mathcal{S}$, when corresponding distribution has support in $\mathbb{R}$, see [27, Corollary 3.20].

Similarly, for any positive integer n, $F \in \mathcal{L}_\gamma \Rightarrow F^{*n} \in \mathcal{L}_\gamma$, $\gamma \geqslant 0$. This follows straightforwardly from Theorem 1.9 (b). For d.f.s supported on $\mathbb{R}^+$, the result also follows from Theorem 3.1 in the paper by Albin [2].

Using convolution root closure property of class $\mathcal{S}$, that is, right-to-left implication in (1.24), we get the following proposition, which will be useful later in the text.

Proposition 1.4. *Classes $\mathcal{R}_\alpha$, $\alpha \geqslant 0$, are closed under convolution root.*

Proof. It follows from relations (1.17) that $\mathcal{R}_\alpha \subset \mathcal{S}$ for any $\alpha \geqslant 0$. From the convolution root closure of $\mathcal{S}$, we get that $F^{*n} \in \mathcal{R}_\alpha$ for some $n \geqslant 2$ implies $F \in \mathcal{S}$. Therefore,

$$\overline{F^{*n}}(x) \sim n\overline{F}(x), \tag{1.25}$$

as $x \to \infty$. Then, by (1.25) and the definition of the class $\mathcal{R}_\alpha$, we have $F \in \mathcal{R}_\alpha$. □

Convolution closure result for the class $\mathcal{D}$, in the case of distributions on $\mathbb{R}^+$, is mentioned in the literature, but we haven't found the full proof. For a more complete review of closure properties, we provide it here. First, we need the following lemma.

Lemma 1.2. *Assume that ξ_1 and ξ_2 are nonnegative r.v.s with d.f.s F_{ξ_1} and F_{ξ_2}. Then*

$$\max\{\overline{F_{\xi_1}}(x), \overline{F_{\xi_2}}(x)\} \leqslant \overline{F_{\xi_1} * F_{\xi_2}}(x) \leqslant \overline{F_{\xi_1}}(x/2) + \overline{F_{\xi_2}}(x/2),$$

for $x > 0$.

Proof. Let us prove, firstly, the upper inequality. Starting with convolution definition, we obtain

$$\begin{aligned}\overline{F_{\xi_1} * F_{\xi_2}}(x) &= \int_{-\infty}^{\infty} \overline{F_{\xi_1}}(x-y)\mathrm{d}F_{\xi_2}(y) = \int_{0}^{\infty} \overline{F_{\xi_1}}(x-y)\mathrm{d}F_{\xi_2}(y) \\ &= \int_{0}^{x/2} \overline{F_{\xi_1}}(x-y)\mathrm{d}F_{\xi_2}(y) + \int_{x/2}^{\infty} \overline{F_{\xi_1}}(x-y)\mathrm{d}F_{\xi_2}(y) \\ &\leqslant \overline{F_{\xi_1}}(x/2)\int_{0}^{x/2} \mathrm{d}F_{\xi_2}(y) + \int_{x/2}^{\infty} \mathrm{d}F_{\xi_2}(y) \\ &\leqslant \overline{F_{\xi_1}}(x/2) + \overline{F_{\xi_2}}(x/2), \quad x > 0.\end{aligned}$$

Similarly, for the lower inequality, we have

$$\begin{aligned}\overline{F_{\xi_1} * F_{\xi_2}}(x) &= \int_{0}^{\infty} \overline{F_{\xi_1}}(x-y)\mathrm{d}F_{\xi_2}(y) \\ &\geqslant \int_{0}^{\infty} \overline{F_{\xi_1}}(x)\mathrm{d}F_{\xi_2}(y) = \overline{F_{\xi_1}}(x)\end{aligned}$$

and

$$\begin{aligned}\overline{F_{\xi_1} * F_{\xi_2}}(x) &= \int_{0}^{\infty} \overline{F_{\xi_2}}(x-y)\mathrm{d}F_{\xi_1}(y) \\ &\geqslant \int_{0}^{\infty} \overline{F_{\xi_2}}(x)\mathrm{d}F_{\xi_1}(y) = \overline{F_{\xi_2}}(x).\end{aligned}$$

Therefore, $\overline{F_{\xi_1} * F_{\xi_2}}(x) \geqslant \max\{\overline{F_{\xi_1}}(x), \overline{F_{\xi_2}}(x)\}$. □

Now we prove convolution closure of class $\mathcal{D}$.

Theorem 1.12. *Assume that ξ_1, ξ_2 are nonnegative r.v.s with d.f.s. F_{ξ_1}, F_{ξ_2}. Then $F_{\xi_1} \in \mathcal{D}$ and $F_{\xi_2} \in \mathcal{D}$ implies $F_{\xi_1} * F_{\xi_2} \in \mathcal{D}$.*

Proof. Since $F_{\xi_1} \in \mathcal{D}$ and $F_{\xi_2} \in \mathcal{D}$, then $\limsup\limits_{x\to\infty} \frac{\overline{F_{\xi_1}}(\frac{1}{4}x)}{\overline{F_{\xi_1}}(x)} < \infty$ and $\limsup\limits_{x\to\infty} \frac{\overline{F_{\xi_2}}(\frac{1}{4}x)}{\overline{F_{\xi_2}}(x)} < \infty$. Employing Lemma 1.2, we get the following chain of inequalities:

$$\frac{\overline{F_{\xi_1} * F_{\xi_2}}(\frac{1}{2}x)}{\overline{F_{\xi_1} * F_{\xi_2}}(x)} \leqslant \frac{\overline{F_{\xi_1}}(\frac{1}{4}x) + \overline{F_{\xi_2}}(\frac{1}{4}x)}{\overline{F_{\xi_1} * F_{\xi_2}}(x)} \leqslant \frac{\overline{F_{\xi_1}}(\frac{1}{4}x)}{\overline{F_{\xi_1}}(x)} + \frac{\overline{F_{\xi_2}}(\frac{1}{4}x)}{\overline{F_{\xi_2}}(x)}, \ x > 0.$$

Therefore,

$$\limsup_{x\to\infty} \frac{\overline{F_{\xi_1} * F_{\xi_2}}(\frac{1}{2}x)}{\overline{F_{\xi_1} * F_{\xi_2}}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F_{\xi_1}}(\frac{1}{4}x)}{\overline{F_{\xi_1}}(x)} + \limsup_{x\to\infty} \frac{\overline{F_{\xi_2}}(\frac{1}{4}x)}{\overline{F_{\xi_2}}(x)} < \infty.$$

Thus, clearly, $F_{\xi_1} * F_{\xi_2} \in \mathcal{D}$. □

For the real-valued r.v.s, we have the following result by Tang and Yan [62, Lemma 1].

Theorem 1.13 (Tang and Yan [62, Lemma 1])**.** *Assume that* ξ_1, ξ_2 *are real-valued r.v.s with d.f.s.* F_{ξ_1}, F_{ξ_2}*. Let* $F_{\xi_1} \in \mathcal{D}$*. If* F_{ξ_2} *is a d.f. such that* $\overline{F_{\xi_2}}(x) = O(\overline{F_{\xi_1}}(x))$*, then* $F_{\xi_1} * F_{\xi_2} \in \mathcal{D}$ *and* $\overline{F_{\xi_1} * F_{\xi_2}}(x) \asymp \overline{F_{\xi_1}}(x)$*.*

Using induction, the following corollary can be obtain from the theorem above. See, also, Tang and Yan [62, Lemma 2]. Class $\mathcal{D}$ is closed under convolution power.

Corollary 1.2. *Let* ξ *be a real-valued r.v. with d.f.* $F_\xi \in \mathcal{D}$*. Then* $F_\xi^{*n} \in \mathcal{D}$ *for all integer* $n \geqslant 1$ *and* $\overline{F_\xi^{*n}}(x) \asymp \overline{F_\xi}(x)$*.*

We summarise the main results of this section (in the case, when r.v.s are nonnegative) in the following table.

Class of distributions	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{S}$	$\mathcal{L}$	$\mathcal{D}$
1. Is the class closed under convolutions?	Yes	Yes	No	Yes	Yes
2. Is the class closed under convolution power?	Yes	Yes	Yes	Yes	Yes
3. Does the max-sum-equivalence hold in the class?	Yes	Yes	No	No	No

2. New closure properties of the classes of distributions

In the previous section we concentrated on the results related to convolution closure of heavy-tailed and related classes. Analysing specific distributions for our examples, we noticed that tails of moments of random variables retain properties that ensure they belong to the same class. Indeed, later in the text we will prove that most of the classes that we mentioned in the introduction are closed with respect to calculation of moments (see Definition 2.1 below). Though limit properties that define distribution classes can clearly be used to define a wider set of functions, we stick to definitions that are usual in the papers of applied probability theory. Thus, for the sake of formality, we “cut” tail moment function, so that usual formulations would hold. Since we are interested in the right tail properties, the following definition is suitable for the presentation of our results.

Definition 2.1. *Let, as previously, ξ be a real-valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution function F_ξ. Suppose that moment*

$$\mathbb{E}(\xi^+)^m = \int_{[0,\infty)} x^m \mathrm{d}F_\xi(x)$$

is finite for $m \geqslant 0$. In such a case, the function

$$F_{\xi,m}(x) = \max\left\{0, 1 - \mathbb{E}(\xi^m \mathbb{1}_{\{\xi > x\}})\right\},\ x \geqslant 0,$$

is a new distribution function with the tail function

$$\overline{F}_{\xi,m}(x) = \min\left\{1, \mathbb{E}(\xi^m \mathbb{1}_{\{\xi > x\}})\right\},\ x \geqslant 0.$$

We will prove that if F_ξ belongs to some specific class, then $F_{\xi,m}$ belongs to the same class of regularity with all valid moment orders m. Also, for some classes of distributions, we were able to find that, if $F_{\xi,m}$ belongs to some specific class, then the distribution, which generated the moment, F_ξ, is necessarily from the same class. Although, working in this direction is usually far more complicated, and some questions are still left unanswered.

In some classes of heavy-tailed distributions, we extended the results to the sums of random variables, in the sense of showing that, if d.f.s $\{F_{\xi_1}, F_{\xi_2}, \ldots, F_{\xi_n}\}$ belong to some specific class, then $F_{S_n^\xi,m}$ belongs to the same class. To achieve this, we applied some known results for independent or, more generally, pairwise quasi-asymptotically independent random variables. If, in addition, random variables $\xi_1, \ldots, \xi_n$ are identically distributed copies of ξ, the following question related to convolution root property also makes sense: if $F_{S_n^\xi,m}$ belongs to some specific class, is F_ξ in the same class?

These results with some generalisations and additional facts will be presented in the Section 2.2.

2.1. Some illustrative examples

Before presenting abstract results we would like to go through several illustrative examples. We will take some distributions from the Section

1.5, find formulas for their tail moments and attribute respective functions $F_{\xi,m}$ to relevant regularity classes.

2.1.1. Tail moment of Pareto distribution

Let Pareto distribution be defined as in Section 1.5.1. It is not difficult to calculate that, when $m < \alpha$, formula for tail moment is

$$\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>x\}}) = \begin{cases} \frac{\alpha\lambda^m}{\alpha-m}, & x < \lambda, \\ \frac{\alpha\lambda^\alpha}{\alpha-m}\frac{1}{x^{\alpha-m}}, & x \geqslant \lambda. \end{cases}$$

For sufficiently large x,

$$\frac{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>xy\}})}{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>x\}})} = \frac{\frac{\alpha\lambda^\alpha}{\alpha-m}\frac{1}{x^{\alpha-m}y^{\alpha-m}}}{\frac{\alpha\lambda^\alpha}{\alpha-m}\frac{1}{x^{\alpha-m}}} = y^{-(\alpha-m)}.$$

Thus,

$$\lim_{x\to\infty}\frac{\overline{F}_{\xi,m}(xy)}{\overline{F}_{\xi,m}(x)} = y^{-(\alpha-m)}$$

for any $y > 0$.

This means that $F_{\xi,m} \in \mathcal{R}_{\alpha-m} \subset \mathcal{R}$. This is not a coincidence. Later we will prove that $F_\xi \in \mathcal{R}$ if and only if $F_{\xi,m} \in \mathcal{R}$.

As in case $m = 0$, in general it is also true that $F_{\xi,m}$ belongs to other classes ($\mathcal{C}$, $\mathcal{S}$, $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{H}$), too.

2.1.2. Tail moment of mixed distribution

Let distribution of interest be defined as in Section 1.5.3. That is, suppose that r.v. ξ has the following distribution function:

$$F_\xi(x) = \begin{cases} 0, & x < 1, \\ 1 - \frac{1}{2^{n-1}x^\alpha}, & 2^{n-1} \leqslant x < 2^n,\ n \in \mathbb{N},\ \alpha > 1. \end{cases}$$

After some straightforward calculations, we get that tail moment of distribution F_ξ is

$$\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>x\}}) = \begin{cases} \frac{1}{\alpha-m}\left(\alpha - \frac{m}{2^{\alpha+1-m}-1}\right), & x < 1, \\ \frac{1}{\alpha-m}\left(\frac{\alpha}{2^{n-1}x^{\alpha-m}} - \frac{m}{(2^{\alpha+1-m}-1)(2^{\alpha+1-m})^{n-1}}\right), & 2^{n-1} \leqslant x < 2^n, \end{cases} \tag{2.1}$$

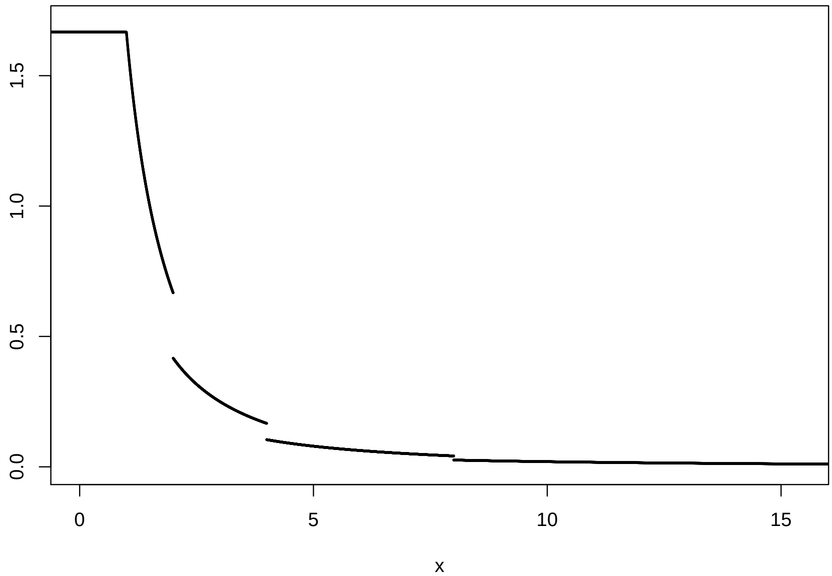

Figure 2.1: Values of tail expectation.

where $n \in \mathbb{N}$, $m < \alpha$. The graph of this function, in the case $\alpha = 2$, $m = 1$, is shown in Figure 2.1.

We will prove that, if $m < \alpha$, then $F_{\xi,m} \notin \mathcal{L}$. When $x \in [2^n - 1, 2^n)$, by (2.1), we have

$$\frac{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi > x+1\}})}{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi > x\}})} = \frac{\frac{\alpha}{2^n (x+1)^{\alpha-m}} - \frac{m}{(2^{\alpha+1-m}-1)(2^{\alpha+1-m})^n}}{\frac{\alpha}{2^{n-1} x^{\alpha-m}} - \frac{m}{(2^{\alpha+1-m}-1)(2^{\alpha+1-m})^{n-1}}}.$$

Let's take the same diverging sequence $x_n = 2^n - 1,\ n \in \mathbb{N}$, as before. Then

$$\begin{aligned}
\frac{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi > x_n+1\}})}{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi > x_n\}})} &= \frac{\frac{\alpha}{2^n 2^{n(\alpha-m)}} - \frac{m}{(2^{\alpha+1-m}-1)(2^{\alpha+1-m})^n}}{\frac{\alpha}{2^{n-1}(2^n-1)^{\alpha-m}} - \frac{m}{(2^{\alpha+1-m}-1)(2^{\alpha+1-m})^{n-1}}} \\
&= \frac{\alpha - \frac{m}{2^{\alpha+1-m}-1}}{\frac{\alpha 2^{1-n} 2^{n(\alpha+1-m)}}{(2^n-1)^{\alpha-m}} - \frac{m 2^{\alpha+1-m}}{2^{\alpha+1-m}-1}} \\
&= \frac{\alpha - \frac{m}{2^{\alpha+1-m}-1}}{\frac{2\alpha}{(1-\frac{1}{2^n})^{\alpha-m}} - \frac{m 2^{\alpha+1-m}}{2^{\alpha+1-m}-1}}.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\frac{\overline{F}_{\xi,m}(x_n+1)}{\overline{F}_{\xi,m}(x_n)}=\frac{\alpha-\frac{m}{2^{\alpha+1-m}-1}}{2\alpha-\frac{m2^{\alpha+1-m}}{2^{\alpha+1-m}-1}}=\frac{\alpha(2^{\alpha+1-m}-1)-m}{(2\alpha-m)2^{\alpha+1-m}-2\alpha}.$$

The ratio

$$\begin{aligned}
&\frac{\alpha(2^{\alpha+1-m}-1)-m}{(2\alpha-m)2^{\alpha+1-m}-2\alpha}=1\\
&\Leftrightarrow \alpha(2^{\alpha+1-m}-1)-m=(2\alpha-m)2^{\alpha+1-m}-2\alpha\\
&\Leftrightarrow (\alpha-m)(1-2^{\alpha+1-m})=0\\
&\Rightarrow m=\alpha \text{ or } m=\alpha+1.
\end{aligned}$$

Both solutions do not satisfy condition $m<\alpha$. Therefore, $\frac{\overline{F}_{\xi,m}(x+1)}{\overline{F}_{\xi,m}(x)}$ doesn't converge to 1 as $x\to\infty$. The d.f. $F_{\xi,m}$ is not long-tailed.

The values of the ratio $\frac{\mathbb{E}(\xi\mathbb{1}_{\{\xi>x+1\}})}{\mathbb{E}(\xi\mathbb{1}_{\{\xi>x\}})}$, in the case $\alpha=2$, are depicted in Figure 2.2. The graph illustrates the fact that $\liminf\limits_{x\to\infty}\frac{\overline{F}_{\xi,1}(x+1)}{\overline{F}_{\xi,1}(x)}=0.625$ and $\limsup\limits_{x\to\infty}\frac{\overline{F}_{\xi,1}(x+1)}{\overline{F}_{\xi,1}(x)}=1$.

2.1.3. Tail moment of exponential distribution

The distribution function of exponential random variable ξ with parameter $\lambda>0$ is given by $F_\xi(x)=(1-\mathrm{e}^{-\lambda x})\mathbb{1}_{\{x\geqslant 0\}}$. The tail moment of this distribution is

$$\mathbb{E}(\xi^m\mathbb{1}_{\{\xi>x\}})=\begin{cases}\frac{m!}{\lambda^m}, & x<0,\\ m!\mathrm{e}^{-\lambda x}\sum_{j=0}^m\frac{x^j}{j!}\lambda^{j-m}, & x\geqslant 0,\ m\in\mathbb{N}.\end{cases}$$

We will prove that $F_{\xi,m}\notin\mathcal{L}$ for any $m\in\mathbb{N}$. Indeed, when $x\geqslant 0$,

$$\begin{aligned}
\frac{\mathbb{E}(\xi^m\mathbb{1}_{\{\xi>x+1\}})}{\mathbb{E}(\xi^m\mathbb{1}_{\{\xi>x\}})}&=\frac{m!\mathrm{e}^{-\lambda(x+1)}\sum_{j=0}^m\frac{(x+1)^j}{j!}\lambda^{j-m}}{m!\mathrm{e}^{-\lambda x}\sum_{j=0}^m\frac{x^j}{j!}\lambda^{j-m}}\\
&=\mathrm{e}^{-\lambda}\frac{\sum_{j=0}^m\left(1+\frac{1}{x}\right)^j\frac{x^{j-m}}{j!}\lambda^{j-m}}{\sum_{j=0}^m\frac{x^{j-m}}{j!}\lambda^{j-m}}.
\end{aligned}$$

Therefore, the limit

$$\lim_{x\to\infty}\frac{\overline{F}_{\xi,m}(x+1)}{\overline{F}_{\xi,m}(x)}=\mathrm{e}^{-\lambda}\neq 1.$$

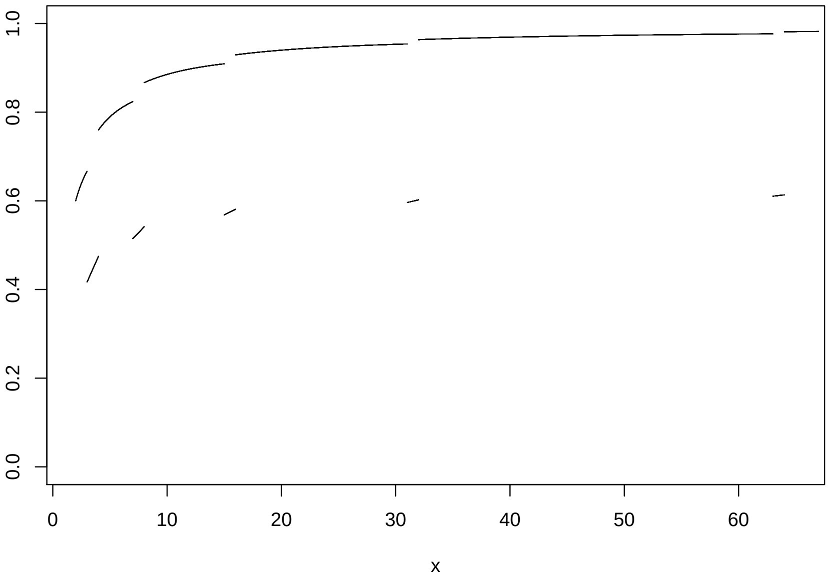

Figure 2.2: Values of the ratio of expectation tails in the x range $(2, 65)$.

Consequently, the distribution function $F_{\xi,m}$ is not long-tailed.

Similarly, when $x > 0$,

$$\frac{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi > \frac{1}{2}x\}})}{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi > x\}})} = \frac{m! \mathrm{e}^{-\frac{1}{2}\lambda x} \sum_{j=0}^{m} \frac{(\frac{1}{2}x)^j}{j!} \lambda^{j-m}}{m! \mathrm{e}^{-\lambda x} \sum_{j=0}^{m} \frac{x^j}{j!} \lambda^{j-m}} = \mathrm{e}^{\frac{1}{2}\lambda x} \frac{\sum_{j=0}^{m} (\frac{1}{2})^j \frac{\lambda^{j-m}}{j!} x^{j-m}}{\sum_{j=0}^{m} \frac{\lambda^{j-m}}{j!} x^{j-m}}$$

$$= \mathrm{e}^{\frac{1}{2}\lambda x} \frac{\frac{1}{m!}(\frac{1}{2})^m + \sum_{j=0}^{m-1} (\frac{1}{2})^j \frac{\lambda^{j-m}}{j!} \frac{1}{x^{m-j}}}{\frac{1}{m!} + \sum_{j=0}^{m-1} \frac{\lambda^{j-m}}{j!} \frac{1}{x^{m-j}}}.$$

Thus, $\limsup\limits_{x \to \infty} \frac{\overline{F}_{\xi,m}(\frac{1}{2}x)}{\overline{F}_{\xi,m}(x)} = \infty$, and $F_{\xi,m} \notin \mathcal{D}$, $m \in \mathbb{N}$.

2.2. Closure properties and their proofs

In this section we formulate and prove results concerning closure properties of introduced classes with respect to calculation of moments. First, we state theorem regarding sufficient conditions, so that $F_{\xi,m}$ would belong to respective class.

Theorem 2.1. *Let ξ be a real-valued r.v. with d.f. F_ξ and finite moment $\mathbb{E}(\xi^+)^m$ for some $m > 0$. Then the following relations hold:*

(i) $F_\xi \in \mathcal{R}_\alpha\ m < \alpha \ \Rightarrow\ F_{\xi,m} \in \mathcal{R}_{\alpha-m}\ (F_\xi \in \mathcal{R} \ \Rightarrow\ F_{\xi,m} \in \mathcal{R})$,

(ii) $F_\xi \in \mathcal{C} \ \Rightarrow\ F_{\xi,m} \in \mathcal{C}$,

(iii) $F_\xi \in \mathcal{D} \ \Rightarrow\ F_{\xi,m} \in \mathcal{D}$,

(iv) $F_\xi \in \mathcal{L} \ \Rightarrow\ F_{\xi,m} \in \mathcal{L}$,

(v) $F_\xi \in \mathcal{L}_\gamma,\ \gamma > 0 \ \Rightarrow\ F_{\xi,m} \in \mathcal{L}_\gamma$,

(vi) $F_\xi \in \mathcal{OL} \ \Rightarrow\ F_{\xi,m} \in \mathcal{OL}$.

In the second theorem we present the relationships in the other direction compared to Theorem 2.1. We proved that for $F_{\xi,m}$ to be in class $\mathcal{K}$ it is sufficient that distribution function F_ξ is in class $\mathcal{K}$. But, as we will see in the following theorem, it is not always necessary. For example, d.f. F_ξ outside class $\mathcal{D}$ may generate $F_{\xi,m} \in \mathcal{D}$. In some classes having $F_{\xi,m} \in \mathcal{K}$ implies, in general, $F_\xi \in \mathcal{K}$ or $F_\xi \notin \mathcal{K}$.

Theorem 2.2. *Let ξ be a real-valued r.v. with d.f. F_ξ and finite moment $\mathbb{E}(\xi^+)^m$ for some $m > 0$. Then, in general, the following relations hold:*

(i) $F_{\xi,m} \in \mathcal{R}_{\alpha-m},\ m < \alpha \ \Rightarrow\ F_\xi \in \mathcal{R}_\alpha\ (F_{\xi,m} \in \mathcal{R} \ \Rightarrow F_\xi \in \mathcal{R})$,

(ii) $F_{\xi,m} \in \mathcal{C} \ \nRightarrow\ F_\xi \in \mathcal{C}$,

(iii) $F_{\xi,m} \in \mathcal{D} \ \nRightarrow\ F_\xi \in \mathcal{D}$,

(iv) $F_{\xi,m} \in \mathcal{L} \ \nRightarrow\ F_\xi \in \mathcal{L}$,

(v) $F_{\xi,m} \in \mathcal{OL} \ \nRightarrow\ F_\xi \in \mathcal{OL}$.

2.2.1. Proof of Theorem 2.1.

To prove this theorem and several other results in the thesis we will employ the following lemma. Proof of it can be found in Leipus et al. [40].

Lemma 2.1 (Leipus et al. [40, Lemma 3])**.** *Let ξ be a real-valued r.v such that $\mathbb{E}(\xi^+)^p < \infty$ for some $p > 0$. Then, for all $x \geqslant 0$,*

$$\mathbb{E}\big(\xi^p \mathbb{1}_{\{\xi > x\}}\big) = x^p \mathbb{P}(\xi > x) + p\int_x^\infty u^{p-1}\mathbb{P}(\xi > u)\,\mathrm{d}u, \tag{2.2}$$

$$\mathbb{E}\big((\xi - x)^+\big)^p = p\int_x^\infty (u-x)^{p-1}\mathbb{P}(\xi > u)\,\mathrm{d}u. \tag{2.3}$$

Note that equation (2.3) follows directly from (2.2). Indeed,

$$\begin{aligned}\mathbb{E}((\xi - x)^+)^p &= \mathbb{E}((\xi - x)^p \mathbb{1}_{\{\xi - x > 0\}}) \\ &= 0^p \mathbb{P}(\xi - x > 0) + p \int_0^\infty u^{p-1} \mathbb{P}(\xi - x > u) \mathrm{d}u \\ &= p \int_x^\infty (u - x)^{p-1} \mathbb{P}(\xi > u) \mathrm{d}u.\end{aligned}$$

Also, we will use the classic min-max inequality:

$$\min\left\{\frac{a_1}{b_1}, \ldots, \frac{a_r}{b_r}\right\} \leqslant \frac{a_1 + \cdots + a_r}{b_1 + \cdots + b_r} \leqslant \max\left\{\frac{a_1}{b_1}, \ldots, \frac{a_r}{b_r}\right\}, \tag{2.4}$$

where $a_i \geqslant 0$, $b_i > 0$ for all $i = 1, \ldots, r$.

Proof of Theorem 2.1 (i).

First, notice that for all x,

$$\overline{F}_\xi(x) = -\int_x^\infty u^{-m} \mathrm{d}\Big(\int_u^\infty s^m \mathrm{d}F_\xi(s)\Big),$$

and for sufficiently large x,

$$\overline{F}_\xi(x) = -\int_x^\infty u^{-m} \mathrm{d}\overline{F}_{\xi,m}(u). \tag{2.5}$$

Assume that $F_\xi \in \mathcal{R}_\alpha$. Applying Theorem 1.3 for $f = \overline{F}_\xi$, we get

$$\lim_{x\to\infty} \frac{\int_x^\infty u^m \mathrm{d}\overline{F}_\xi(u)}{x^m \overline{F}_\xi(x)} = \frac{\alpha}{m - \alpha}.$$

Thus, plugging in (2.5), we have

$$\lim_{x\to\infty} \frac{-\overline{F}_{\xi,m}(x)}{x^m(-\int_x^\infty u^{-m} \mathrm{d}\overline{F}_{\xi,m}(u))} = \frac{\alpha}{m - \alpha},$$

or

$$\lim_{x\to\infty} \frac{\int_x^\infty u^{-m} \mathrm{d}\overline{F}_{\xi,m}(u)}{x^{-m}\overline{F}_{\xi,m}(x)} = -\frac{m - \alpha}{-m + m - \alpha}.$$

Since $m \neq 0$ and $-\alpha < 0$, by the converse part of Theorem 1.3, we arrive at the conclusion that $F_{\xi,m} \in \mathcal{R}_{\alpha - m}$.

Proof of Theorem 2.1 (ii).

Fix $\epsilon > 0$. Then there exists such a positive $y_0 = y_0(\epsilon)$ that

$$1 - \epsilon \leqslant \limsup_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} \leqslant 1 + \epsilon,$$

when $y_0 < y < 1$. For this y, there exists $x_0 = x_0(y)$ such that

$$\overline{F}_\xi(xy) \leqslant (1+\epsilon)\overline{F}_\xi(x),$$

when $x > x_0$.

From Lemma 2.1 and formulations above, we have that with $y > y_0$ and sufficiently large x,

$$\begin{aligned}
\frac{\overline{F}_{\xi,m}(xy)}{\overline{F}_{\xi,m}(x)} &= \frac{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>xy\}})}{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>x\}})} \\
&= \frac{x^m y^m \mathbb{P}(\xi > xy) + my^m \int_x^\infty u^{m-1}\mathbb{P}(\xi > uy)\mathrm{d}u}{x^m \mathbb{P}(\xi > x) + m\int_x^\infty u^{m-1}\mathbb{P}(\xi > u)\mathrm{d}u} \\
&\leqslant \frac{x^m y^m (1+\epsilon)\mathbb{P}(\xi > x) + my^m(1+\epsilon) \int_x^\infty u^{m-1}\mathbb{P}(\xi > u)\mathrm{d}u}{x^m \mathbb{P}(\xi > x) + m\int_x^\infty u^{m-1}\mathbb{P}(\xi > u)\mathrm{d}u} \\
&= (1+\epsilon)y^m. \qquad (2.6)
\end{aligned}$$

From (2.6), we get

$$\limsup_{y\uparrow 1}\limsup_{x\to\infty} \frac{\overline{F}_{\xi,m}(xy)}{\overline{F}_{\xi,m}(x)} \leqslant 1+\epsilon,$$

and letting $\epsilon \downarrow 0$, we achieve

$$\limsup_{y\uparrow 1}\limsup_{x\to\infty} \frac{\overline{F}_{\xi,m}(xy)}{\overline{F}_{\xi,m}(x)} \leqslant 1. \qquad (2.7)$$

Since function $\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>x\}})$ is not increasing in $\mathbb{R}^+$, we have that $\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>xy\}}) \geqslant \mathbb{E}(\xi^m \mathbb{1}_{\{\xi>x\}})$ with $y \in (0,1)$ and $x \geqslant 0$. Thus, for sufficiently large x,

$$\frac{\overline{F}_{\xi,m}(xy)}{\overline{F}_{\xi,m}(x)} = \frac{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>xy\}})}{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>x\}})} \geqslant 1.$$

Therefore,

$$\liminf_{y\uparrow 1}\limsup_{x\to\infty} \frac{\overline{F}_{\xi,m}(xy)}{\overline{F}_{\xi,m}(x)} \geqslant 1. \qquad (2.8)$$

Finally, from (2.7) and (2.8), we get that

$$\lim_{y\uparrow 1}\limsup_{x\to\infty} \frac{\overline{F}_{\xi,m}(xy)}{\overline{F}_{\xi,m}(x)} = 1.$$

This implies that $F_{\xi,m} \in \mathcal{C}$.

Proof of Theorem 2.1 (iii).

Let us fix $0 < y < 1$. By condition $F_\xi \in \mathcal{D}$, there exists a positive $c = c(y)$ such that for sufficiently large x,

$$\overline{F}_\xi(xy) \leqslant c\overline{F}_\xi(x).$$

From this and Lemma 2.1, we have that for sufficiently large x,

$$\begin{aligned}
\frac{\overline{F}_{\xi,m}(xy)}{\overline{F}_{\xi,m}(x)} &= \frac{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>xy\}})}{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>x\}})} \\
&= \frac{(xy)^m\overline{F}_\xi(xy) + m\int_{xy}^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u}{x^m\overline{F}_\xi(x) + m\int_x^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u} \\
&= \frac{(xy)^m\overline{F}_\xi(xy) + my^m\int_{x}^\infty u^{m-1}\overline{F}_\xi(uy)\mathrm{d}u}{x^m\overline{F}_\xi(x) + m\int_x^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u} \\
&\leqslant \frac{(xy)^m c\overline{F}_\xi(x) + my^m\int_{x}^\infty u^{m-1}c\overline{F}_\xi(u)\mathrm{d}u}{x^m\overline{F}_\xi(x) + m\int_x^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u} \\
&= cy^m < \infty.
\end{aligned}$$

Hence, for any fixed $0 < y < 1$,

$$\limsup_{x\to\infty} \frac{\overline{F}_{\xi,m}(xy)}{\overline{F}_{\xi,m}(x)} < \infty.$$

This implies that $F_{\xi,m} \in \mathcal{D}$ and concludes the proof of Theorem 2.1 part (iii).

Proof of Theorem 2.1 (iv).

Working on closure properties of class $\mathcal{L} = \mathcal{L}_0$, we noticed that it is possible to start the discussion from the more general statement about limits of tail functions.

Theorem 2.3. *Let ξ be a random variable with distribution function such that $\overline{F}_\xi(x) > 0$ for all x and, for any fixed $y > 0$, $\lim\limits_{x\to\infty} \frac{\overline{F}_\xi(x+y)}{\overline{F}_\xi(x)} = a$ $(0 < a \leqslant 1)$. Also, $\mathbb{E}(\xi^+)^m < \infty$ for some $m > 0$. Then for the same a,*

$$\lim_{x\to\infty} \frac{\overline{F}_{\xi,m}(x+y)}{\overline{F}_{\xi,m}(x)} = a \text{ for all } y > 0.$$

Proof. Let us fix $\epsilon > 0$ and $y > 0$. By conditions of the theorem, we have that for sufficiently large x,

$$(a-\epsilon)\overline{F}_\xi(x) \leqslant \overline{F}_\xi(x+y) \leqslant (a+\epsilon)\overline{F}_\xi(x). \tag{2.9}$$

Thus, applying Lemma 2.1 and the min-max inequality (2.4) in case $r = 2$, we get that for sufficiently large x,

$$\begin{aligned}
&\frac{\overline{F}_{\xi,m}(x+y)}{\overline{F}_{\xi,m}(x)} = \frac{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi > x+y\}})}{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi > x\}})} \\
&= \frac{(x+y)^m \overline{F}_\xi(x+y) + m\int_{x+y}^{\infty} u^{m-1}\overline{F}_\xi(u)\mathrm{d}u}{x^m \overline{F}_\xi(x) + m\int_x^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u} \\
&= \frac{(x+y)^m \overline{F}_\xi(x+y) + m\int_{x}^{\infty} (u+y)^{m-1}\overline{F}_\xi(u+y)\mathrm{d}u}{x^m \overline{F}_\xi(x) + m\int_x^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u} \\
&\leqslant \frac{(x+y)^m (a+\epsilon)\overline{F}_\xi(x) + m\int_{x}^{\infty} (u+y)^{m-1}(a+\epsilon)\overline{F}_\xi(u)\mathrm{d}u}{x^m \overline{F}_\xi(x) + m\int_x^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u} \\
&\leqslant \frac{(a+\epsilon)(1+\frac{y}{x})^m x^m \overline{F}_\xi(x) + (a+\epsilon)\sup\limits_{u\geqslant x}(1+\frac{y}{u})^{m-1} m\int_{x}^{\infty} u^{m-1}\overline{F}_\xi(u)\mathrm{d}u}{x^m \overline{F}_\xi(x) + m\int_x^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u} \\
&\leqslant \max\left\{(a+\epsilon)\Big(1+\frac{y}{x}\Big)^m, (a+\epsilon)\sup_{u\geqslant x}\Big(1+\frac{y}{u}\Big)^{m-1}\right\} \\
&= (a+\epsilon)\Big(1+\frac{y}{x}\Big)^m.
\end{aligned}$$

From this, we have that

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\overline{F}_{\xi,m}(x+y)}{\overline{F}_{\xi,m}(x)} &\leqslant \limsup_{x\to\infty}\left((a+\epsilon)\Big(1+\frac{y}{x}\Big)^m\right) \\
&= a+\epsilon.
\end{aligned}$$

Letting $\epsilon \downarrow 0$, we have

$$\limsup_{x\to\infty} \frac{\overline{F}_{\xi,m}(x+y)}{\overline{F}_{\xi,m}(x)} \leqslant a. \tag{2.10}$$

Using the same techniques, we get

$$\liminf_{x\to\infty} \frac{\overline{F}_{\xi,m}(x+y)}{\overline{F}_{\xi,m}(x)} \geqslant a. \tag{2.11}$$

Finally, from (2.10) and (2.11) for any fixed $y > 0$, we achieve the equality

$$\lim_{x\to\infty} \frac{\overline{F}_{\xi,m}(x+y)}{\overline{F}_{\xi,m}(x)} = a.$$

The theorem is proved. □

Taking $a = 1$ in Theorem 2.3, we get the result of **Theorem 2.1 part (iv)** as the following corollary.

Corollary 2.1. *Let ξ be a random variable with distribution function $F_\xi \in \mathcal{L}$ and $\mathbb{E}(\xi^+)^m < \infty$ for some $m > 0$. Then with this m, $F_{\xi,m} \in \mathcal{L}$.*

Also, using Theorem 2.3 we conclude that if $F_\xi \notin \mathcal{L}$ due to the reason that $\lim\limits_{x\to\infty} \frac{\overline{F}_\xi(x+y)}{\overline{F}_\xi(x)}$ exists but is not equal to 1, then $\lim\limits_{x\to\infty} \frac{\overline{F}_{\xi,m}(x+y)}{\overline{F}_{\xi,m}(x)}$ exists but is not equal to one as well. This means that, in this case, $F_{\xi,m} \notin \mathcal{L}$.

Proof of Theorem 2.1 (v).

Let us fix $\gamma \geqslant 0$, $y > 0$ and $\epsilon > 0$. By conditions of the theorem, we have that for sufficiently large x,

$$(\mathrm{e}^{-\gamma y} - \epsilon)\overline{F}_\xi(x) \leqslant \overline{F}_\xi(x+y) \leqslant (\mathrm{e}^{-\gamma y} + \epsilon)\overline{F}_\xi(x).$$

Similarly, as in the proof of Theorem 2.3, we get

$$\limsup_{x\to\infty} \frac{\overline{F}_{\xi,m}(x+y)}{\overline{F}_{\xi,m}(x)} \leqslant \mathrm{e}^{-\gamma y}. \tag{2.12}$$

Also,

$$\liminf_{x\to\infty} \frac{\overline{F}_{\xi,m}(x+y)}{\overline{F}_{\xi,m}(x)} \geqslant \mathrm{e}^{-\gamma y}. \tag{2.13}$$

From (2.12) and (2.13) for any fixed $y > 0$, we achieve the equality

$$\lim_{x\to\infty} \frac{\overline{F}_{\xi,m}(x+y)}{\overline{F}_{\xi,m}(x)} = \mathrm{e}^{-\gamma y}.$$

Part (v) of Theorem 2.1 is proved.

Proof of Theorem 2.1 (vi).

Let us fix $y > 0$. By condition $F_\xi \in \mathcal{OL}$, there exists a constant $c = c(y) > 0$ such that for sufficiently large x,

$$\overline{F}_\xi(x-y) \leqslant c\overline{F}_\xi(x).$$

From this inequality, Lemma 2.1 and the min-max inequality (2.4) in case r=2, we have that for sufficiently large $x \geqslant y$,

$$
\begin{aligned}
\frac{\overline{F}_{\xi,m}(x-y)}{\overline{F}_{\xi,m}(x)} &= \frac{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>x-y\}})}{\mathbb{E}(\xi^m \mathbb{1}_{\{\xi>x\}})} \\
&= \frac{(x-y)^m \overline{F}_\xi(x-y) + m\int_{x-y}^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u}{x^m\overline{F}_\xi(x) + m\int_x^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u} \\
&= \frac{(x-y)^m \overline{F}_\xi(x-y) + m\int_{x}^\infty (u-y)^{m-1}\overline{F}_\xi(u-y)\mathrm{d}u}{x^m\overline{F}_\xi(x) + m\int_x^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u} \\
&\leqslant \frac{(x-y)^m c\overline{F}_\xi(x) + m\int_{x}^\infty (u-y)^{m-1}c\overline{F}_\xi(u)\mathrm{d}u}{x^m\overline{F}_\xi(x) + m\int_x^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u} \\
&\leqslant \frac{c(1-\frac{y}{x})^m x^m \overline{F}_\xi(x) + c\sup\limits_{u\geqslant x}(1-\frac{y}{u})^{m-1} m\int_{x}^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u}{x^m\overline{F}_\xi(x) + m\int_x^\infty u^{m-1}\overline{F}_\xi(u)\mathrm{d}u} \\
&\leqslant c\max\left\{\left(1-\frac{y}{x}\right)^m, \sup_{u\geqslant x}\left(1-\frac{y}{u}\right)^{m-1}\right\} \\
&= \begin{cases} c\Big(1-\frac{y}{x}\Big)^{m-1}, & 0<m<1, \\ c, & m\geqslant 1. \end{cases}
\end{aligned}
$$

Hence, for any fixed $y>0$,

$$
\limsup_{x\to\infty} \frac{\overline{F}_{\xi,m}(x-y)}{\overline{F}_{\xi,m}(x)} < \infty.
$$

This implies that $F_{\xi,m} \in \mathcal{OL}$ and concludes the proof of Theorem 2.1 part (vi).

2.2.2. Proof of Theorem 2.2.

Proof of Theorem 2.2 (i).

As in the proof of Theorem 2.1 (i), for all x,

$$
\overline{F}_\xi(x) = -\int_x^\infty u^{-m}\mathrm{d}\Big(\int_u^\infty s^m \mathrm{d}F_\xi(s)\Big),
$$

and for sufficiently large x,

$$
\overline{F}_\xi(x) = -\int_x^\infty u^{-m}\mathrm{d}\overline{F}_{\xi,m}(u). \tag{2.14}
$$

Assume that $F_{\xi,m} \in \mathcal{R}_{\alpha-m}$. We apply Theorem 1.3 for function $f = \overline{F}_{\xi,m}$.

We get

$$\lim_{x\to\infty} \frac{\int_x^\infty u^{-m} \mathrm{d}\overline{F}_{\xi,m}(u)}{x^{-m}\overline{F}_{\xi,m}(x)} = -\frac{m-\alpha}{-m+m-\alpha}.$$

Remembering the definition of function $F_{\xi,m}$, we achieve

$$\lim_{x\to\infty} \frac{\int_x^\infty u^{m} \mathrm{d}\overline{F}_{\xi}(u)}{x^{m}\overline{F}_{\xi}(x)} = \frac{\alpha}{m-\alpha}.$$

Hence, $F_\xi \in \mathcal{R}_\alpha$.

Proof of Theorem 2.2 (ii).

The converse statement is generally not true in class $\mathcal{C}$. To demonstrate this, we construct a counterexample. Let us define the tail function of r.v. ξ as

$$\overline{F}_\xi(x) = \mathbb{1}_{(-\infty,2)}(x) + \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} \mathbb{1}_{[2^n,2^{n+1})}(x).$$

This tail function describes the distribution of r.v. ξ, for which

$$\mathbb{P}(\xi = 2) = \frac{1}{2}$$

and

$$\mathbb{P}(\xi = 2^{n+1}) = \frac{1}{n^2 2^n} - \frac{1}{(n+1)^2 2^{n+1}}, \quad n \in \{1,2,\ldots\}.$$

Then, computing the expectation of ξ^+, we get

$$\mathbb{E}(\xi^+) = \mathbb{E}(\xi) = 1 + 2\sum_{n=1}^{\infty} \Big(\frac{1}{n^2} - \frac{1}{2(n+1)^2}\Big) = 2 + \frac{\pi^2}{6} < \infty.$$

For $x \geqslant 2$,

$$\mathbb{E}(\xi \mathbb{1}_{\{\xi > x\}}) = 2 \sum_{n \geqslant \lfloor \log_2 x \rfloor} \Big(\frac{1}{n^2} - \frac{1}{2(n+1)^2}\Big).$$

Thus,

$$\overline{F}_{\xi,1}(x) \underset{x\to\infty}{\sim} \frac{1}{\log_2(x)}, \tag{2.15}$$

implying that $F_{\xi,1} \in \mathcal{R}_0 \subset \mathcal{C}$.

However, for the sequence $x_n = 2^n + \frac{1}{2}, n \in \mathbb{N}$,

$$\limsup_{x\to\infty} \frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)} = \lim_{n\to\infty} \frac{\overline{F}_\xi(x_n - 1)}{\overline{F}_\xi(x_n)} = 2. \tag{2.16}$$

This gives that $F_\xi \notin \mathcal{L}$. Since $\mathcal{C} \subset \mathcal{L}$, it follows that $F_\xi \notin \mathcal{C}$, which finishes the proof of part (ii) of the theorem.

Proof of Theorem 2.2 (iii).

The converse statement is generally not true in class $\mathcal{D}$. To prove this part of the theorem, we construct a counterexample.

Let us take two recursive sequences:

1) $\{a_n, n \geqslant 1\}: \ a_1 = 2, \ a_n = a_{n-1}2^n, \ n = 2, 3, 4, \dots;$

2) $\{b_n, n \geqslant 1\}: \ b_1 = 1, \ b_2 = 1/2, \ b_n = \frac{b_{n-1}}{2^{n+2}}, \ n = 3, 4, \dots$

We define distribution of interest by its tail:

$$\overline{F}_\xi(x) = \begin{cases} b_1, & x < a_1, \\ b_n, & a_{n-1} \leqslant x < a_n, \ n = 2, 3, 4, \dots \end{cases}$$

Then the ratio of tails is

$$\frac{\overline{F}_\xi(\frac{1}{2}x)}{\overline{F}_\xi(x)} = \begin{cases} 1, & x < a_1 \text{ or } 2a_{n-1} \leqslant x < a_n, \ n = 2, 3, 4, \dots, \\ 2, & a_1 \leqslant x < 2a_1, \\ 2^{n+3}, & a_n \leqslant x < 2a_n, \ n = 2, 3, 4, \dots \end{cases}$$

Clearly, $\limsup\limits_{x \to \infty} \frac{\overline{F}_\xi(\frac{1}{2}x)}{\overline{F}_\xi(x)} = \infty$ and $F_\xi \notin \mathcal{D}$.

After calculations, we get that first tail moment (tail expectation)

$$\mathbb{E}(\xi \mathbb{1}_{\{\xi > x\}}) = \begin{cases} 130/21, & x < a_1, \\ \frac{1}{3 \times 2^{2(n-4)}} - \frac{1}{7 \times 2^{3(n-2)}}, & a_{n-1} \leqslant x < a_n, \ n = 2, 3, 4, \dots \end{cases}$$

Thus,

$$\frac{\mathbb{E}(\xi \mathbb{1}_{\{\xi > \frac{1}{2}x\}})}{\mathbb{E}(\xi \mathbb{1}_{\{\xi > x\}})} = \begin{cases} 1, & x < a_1 \text{ or } 2a_{n-1} \leqslant x < a_n, \\ & \quad n = 2, 3, 4, \dots, \\ 130/109, & a_1 \leqslant x < 2a_1, \\ \frac{\frac{1}{3 \times 2^{2(n-4)}} - \frac{1}{7 \times 2^{3(n-2)}}}{\frac{1}{3 \times 2^{2(n-3)}} - \frac{1}{7 \times 2^{3(n-1)}}}, & a_n \leqslant x < 2a_n, \ n = 2, 3, 4, \dots \end{cases}$$

Since

$$\lim_{n \to \infty} \frac{\frac{1}{3 \times 2^{2(n-4)}} - \frac{1}{7 \times 2^{3(n-2)}}}{\frac{1}{3 \times 2^{2(n-3)}} - \frac{1}{7 \times 2^{3(n-1)}}} = \lim_{n \to \infty} \frac{\frac{1}{3 \times 2^{-8}} - \frac{1}{7 \times 2^{-6} \times 2^n}}{\frac{1}{3 \times 2^{-6}} - \frac{1}{7 \times 2^{-3} \times 2^n}} = 4,$$

it is clear that
$$\limsup_{x\to\infty}\frac{\mathbb{E}(\xi\mathbb{1}_{\{\xi>\frac{1}{2}x\}})}{\mathbb{E}(\xi\mathbb{1}_{\{\xi>x\}})}=4,$$
and, therefore,
$$\limsup_{x\to\infty}\frac{\overline{F}_{\xi,1}(\frac{1}{2}x)}{\overline{F}_{\xi,1}(x)}=4<\infty.$$
We conclude that $F_{\xi,1}\in\mathcal{D}$, but $F_\xi\notin\mathcal{D}$.

Proof of Theorem 2.2 (iv).

To prove this part of the theorem, we can use the same counterexample as in the proof of part (ii). According to relation (2.15), d.f. $F_{\xi,1}$ belongs to the class $\mathcal{L}$. On the other hand, the relation (2.16) gives that d.f. F_ξ does not belong to $\mathcal{L}$. Part (iv) is proved.

Proof of Theorem 2.2 (v).

The converse statement is generally not true in class $\mathcal{OL}$. To demonstrate this, we construct a counterexample.

Let us take two recursive sequences:

1) $\{a_n, n\geqslant 1\}:\ a_1=2,\ a_n=a_{n-1}2^n,\ n=2,3,4,\ldots;$

2) $\{b_n, n\geqslant 1\}:\ b_1=1,\ b_2=1/2,\ b_n=\frac{b_{n-1}}{2^{n+2}},\ n=3,4,\ldots$

We define distribution of interest by its tail:
$$\overline{F}_\xi(x)=b_1\mathbb{1}_{(-\infty,a_1)}(x)+\sum_{n=2}^{\infty}b_n\mathbb{1}_{[a_{n-1},a_n)}(x).$$
Then the ratio of tails is
$$\frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)}=\mathbb{1}_{(-\infty,a_1)}(x)+2\mathbb{1}_{[a_1,a_1+1)}(x)+\sum_{n=2}^{\infty}\mathbb{1}_{[a_{n-1}+1,a_n)}(x)$$
$$+\sum_{n=2}^{\infty}2^{n+3}\mathbb{1}_{[a_n,a_n+1)}(x).$$
Clearly, $\limsup\limits_{x\to\infty}\frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)}=\infty$ and $F_\xi\notin\mathcal{OL}$.

In the proof of part (ii), we have already shown that $F_{\xi,1}\in\mathcal{D}$. It is well known that $\mathcal{D}\subset\mathcal{OL}$. Thus, we conclude that $F_{\xi,1}\in\mathcal{OL}$, but $F_\xi\notin\mathcal{OL}$.

2.2.3. Corollaries and additional results

Though $F_\xi \in \mathcal{D} \Rightarrow F_{\xi,m} \in \mathcal{D}$ is true for every distribution with required moment condition, as we just learned, $F_{\xi,m} \in \mathcal{D} \nRightarrow F_\xi \in \mathcal{D}$ generally. But in the smaller space of distributions, the following statement about the closure of class $\mathcal{D}$ is valid.

Proposition 2.1. *Let ξ be a random variable with distribution function F_ξ such that ratio $\overline{F}_\xi(\frac{1}{2}u)/\overline{F}_\xi(u)$ is monotonically increasing and let $\mathbb{E}(\xi^+)^m < \infty$ for some $m > 0$. Then $F_\xi \in \mathcal{D}$ if and only if $F_{\xi,m} \in \mathcal{D}$.*

Proof. Necessity follows directly from Theorem 2.1. We will prove sufficiency by showing that in the space of distribution functions with mentioned monotonic increasingness condition the tail moment of any distribution outside class $\mathcal{D}$ remains outside class $\mathcal{D}$.

Inside class $\mathcal{D}$, d.f. F_ξ satisfies condition $\limsup\limits_{x\to\infty} \frac{\overline{F}_\xi(\frac{1}{2}x)}{\overline{F}_\xi(x)} < \infty$. Then any d.f. G outside of class $\mathcal{D}$ will satisfy condition

$$\limsup_{x\to\infty} \frac{\overline{G}(\frac{1}{2}x)}{\overline{G}(x)} = \infty. \tag{2.17}$$

Let's say that d.f. G_η satisfies the condition (2.17). This means that

$$\lim_{x\to\infty} \sup_{u\geqslant x} \frac{\overline{G}_\eta(\frac{1}{2}u)}{\overline{G}_\eta(u)} = \infty.$$

Thus, $\nexists\ M$ such that $\sup\limits_{u\geqslant x} \frac{\overline{G}_\eta(\frac{1}{2}u)}{\overline{G}_\eta(u)} < M$ for all $x \in \mathbb{R}$. To put it another way, it is true that for any M there exists $x_0 \in \mathbb{R}$ such that $\sup\limits_{u\geqslant x_0} \frac{\overline{G}_\eta(\frac{1}{2}u)}{\overline{G}_\eta(u)} > M$, and, therefore, for any M there exists $\tilde{x}_0$ such that $\overline{G}_\eta(\frac{1}{2}\tilde{x}_0) > M\overline{G}_\eta(\tilde{x}_0)$. By given assumptions, function $\overline{G}_\eta(\frac{1}{2}u)/\overline{G}_\eta(u)$ is monotonically increasing. Thus, it is also true that

$$\frac{\overline{G}_\eta(\frac{1}{2}u)}{\overline{G}_\eta(u)} > M,\ u \geqslant \tilde{x}_0. \tag{2.18}$$

By inequality (2.18), when $\tilde{x}_0$ is chosen big enough, we have

$$\begin{aligned}
\frac{\overline{G}_{\eta,m}(\frac{1}{2}\tilde{x}_0)}{\overline{G}_{\eta,m}(\tilde{x}_0)} &= \frac{\mathbb{E}(\eta^m \mathbb{1}_{\{\eta > \frac{1}{2}\tilde{x}_0\}})}{\mathbb{E}(\eta^m \mathbb{1}_{\{\eta > \tilde{x}_0\}})} \\
&= \frac{(\frac{1}{2}\tilde{x}_0)^m \overline{G}_\eta(\frac{1}{2}\tilde{x}_0) + \frac{m}{2^m}\int_{\tilde{x}_0}^{\infty} u^{m-1}\overline{G}_\eta(\frac{1}{2}u)\mathrm{d}u}{\tilde{x}_0^m \overline{G}_\eta(\tilde{x}_0) + m\int_{\tilde{x}_0}^{\infty} u^{m-1}\overline{G}_\eta(u)\mathrm{d}u} \\
&= \frac{(\frac{1}{2}\tilde{x}_0)^m \overline{G}_\eta(\frac{1}{2}\tilde{x}_0) + \frac{m}{2^m}\int_{\tilde{x}_0}^{\infty} u^{m-1}\overline{G}_\eta(u)\frac{\overline{G}_\eta(\frac{1}{2}u)}{\overline{G}_\eta(u)}\mathrm{d}u}{\tilde{x}_0^m \overline{G}_\eta(\tilde{x}_0) + m\int_{\tilde{x}_0}^{\infty} u^{m-1}\overline{G}_\eta(u)\mathrm{d}u} \\
&> \frac{\frac{M}{2^m}\tilde{x}_0^m \overline{G}_\eta(\tilde{x}_0) + \frac{M}{2^m} m\int_{\tilde{x}_0}^{\infty} u^{m-1}\overline{G}_\eta(u)\mathrm{d}u}{\tilde{x}_0^m \overline{G}_\eta(\tilde{x}_0) + m\int_{\tilde{x}_0}^{\infty} u^{m-1}\overline{G}_\eta(u)\mathrm{d}u} \\
&= \frac{M}{2^m}.
\end{aligned}$$

We found that for any M, there exists $\tilde{x}_0$ such that $\frac{\overline{G}_{\eta,m}(\frac{1}{2}\tilde{x}_0)}{\overline{G}_{\eta,m}(\tilde{x}_0)} > \frac{M}{2^m}$. From this it follows that $\limsup\limits_{x\to\infty} \frac{\overline{G}_{\eta,m}(\frac{1}{2}x)}{\overline{G}_{\eta,m}(x)} = \infty$. Therefore, $G_{\eta,m} \notin \mathcal{D}$. $\square$

Let us note that the set of d.f.s F for which the ratio $\overline{F}(\frac{1}{2}u)/\overline{F}(u)$ is monotonically increasing is not empty. In what follows, we give an example of such a d.f. Thus, such a condition in Proposition 2.1 is meaningful.

Let us take the recursive sequence $\{a_n, n \geqslant 1\}$ given by

$$a_1 = 1,\ a_2 = \frac{1}{2^2},\ a_n = \frac{a_{n-1}}{2^n},\ n = 3, 4, \ldots$$

Let us define the tail function of r.v. ξ as

$$\overline{F}_\xi(u) = a_1 \mathbb{1}_{(-\infty,2)}(u) + \sum_{n=2}^{\infty} a_n \mathbb{1}_{[2^{n-1},2^n)}(u).$$

Then the ratio of tails is

$$\frac{\overline{F}_\xi(\frac{1}{2}u)}{\overline{F}_\xi(u)} = \mathbb{1}_{(-\infty,2)}(u) + \sum_{n=2}^{\infty} 2^n \mathbb{1}_{[2^{n-1},2^n)}(u).$$

Clearly, the ratio $\frac{\overline{F}_\xi(\frac{1}{2}u)}{\overline{F}_\xi(u)}$ is monotonically increasing.

Next result follows directly from Theorems 1.2 and 2.1.

Corollary 2.2. *Let ξ be a real-valued random variable with distribution function $F_\xi \in \mathcal{L} \cap \mathcal{D}$. Also, assume that $\mathbb{E}(\xi^+)^m < \infty$ for some $m > 0$. Then with this m, $F_{\xi,m} \in \mathcal{S}$.*

Employing known closure properties from Section 1.6, we get the following corollary of Theorem 2.1.

Corollary 2.3. *Let* $\xi_1, \xi_2, \ldots, \xi_n$ *be independent nonnegative r.v.s with d.f.s* F_{ξ_k}, $k = 1, \ldots, n$. *Assume that for some* $m > 0$ *it holds that* $\mathbb{E}\xi_k^m < \infty$ *for all* $k = 1, \ldots, n$. *Then the following relations hold:*

(i) $F_{\xi_k} \in \mathcal{R}_\alpha,\ k = 1, \ldots, n,\ m < \alpha\ \Rightarrow\ F_{S_n^\xi, m} \in \mathcal{R}_{\alpha - m}$,

(ii) $F_{\xi_k} \in \mathcal{C},\ k = 1, \ldots, n\ \Rightarrow\ F_{S_n^\xi, m} \in \mathcal{C}$,

(iii) $F_{\xi_k} \in \mathcal{D},\ k = 1, \ldots, n\ \Rightarrow\ F_{S_n^\xi, m} \in \mathcal{D}$,

(iv) $F_{\xi_k} \in \mathcal{L},\ k = 1, \ldots, n\ \Rightarrow\ F_{S_n^\xi, m} \in \mathcal{L}$,

(v) $F_{\xi_k} \in \mathcal{L}_\gamma,\ \gamma > 0,\ k = 1, \ldots, n\ \Rightarrow\ F_{S_n^\xi, m} \in \mathcal{L}_\gamma$.

Proof. Let us prove part (iii). The other parts can be proved by analogy.

Since $\xi_1, \xi_2, \ldots, \xi_n$ are nonnegative r.v.s. with d.f.s $F_{\xi_k} \in \mathcal{D}$, $k = 1, \ldots, n$, applying Theorem 1.12, we get that convolution $F_{\xi_1} * F_{\xi_2} * \cdots * F_{\xi_n} \in \mathcal{D}$. Under the condition that $\xi_1, \xi_2, \ldots, \xi_n$ are independent, we have

$$F_{S_n^\xi} = F_{\xi_1} * F_{\xi_2} * \cdots * F_{\xi_n}.$$

It follows that $F_{S_n^\xi} \in \mathcal{D}$. Furthermore, the existence of the moment $\mathbb{E}\xi_k^m$ for some $m > 0$ and all $k = 1, \ldots, n$ implies that the moment $\mathbb{E}((S_n^\xi)^+)^m = \mathbb{E}(S_n^\xi)^m$ exists for the same $m > 0$. Thus, using Theorem 2.1, we get that $F_{S_n^\xi, m} \in \mathcal{D}$.

□

It is known that, if $F_{\xi_k} \in \mathcal{R}_\alpha$, $\alpha > 0$, for all $k = 1, \ldots, n$, then $F_{S_n^\xi} \in \mathcal{R}_\alpha$ with the same index α. No assumption about independence is needed. Thus, we can formulate the following broader statement for class $\mathcal{R}$.

Corollary 2.4. *Let* $\xi_1, \xi_2, \ldots, \xi_n$ *be real-valued r.v.s with d.f.s* F_{ξ_k}, $k = 1, \ldots, n$. *Assume that for some* $m > 0$ *it holds that* $\mathbb{E}(\xi_k^+)^m < \infty$ *for all* $k = 1, \ldots, n$. *Then* $F_{\xi_k} \in \mathcal{R}_\alpha$, $k = 1, \ldots, n$, $m < \alpha$, *implies* $F_{S_n^\xi, m} \in \mathcal{R}_{\alpha - m}$.

The converse half follows from the convolution root property of regularly varying-tailed distributions (see Proposition 1.4 in Section 1.6). Assume that $F_{S_n^\xi}$ is regularly varying-tailed with index $\alpha \geqslant 0$ and r.v.s $\xi_1, \xi_2, \ldots, \xi_n$ are i.i.d., and F_ξ denotes their distribution function. Then this F_ξ is also regularly varying-tailed with the same index $\alpha \geqslant 0$. Thus, we can formulate the following corollary of Theorem 2.2.

Corollary 2.5. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent copies of a random variable ξ with distribution function F_ξ. Assume that for some $m > 0$ it holds that $\mathbb{E}(\xi_k^+)^m < \infty$ for all $k = 1, \ldots, n$. If $F_{S_{n,m}^\xi} \in \mathcal{R}_{\alpha-m}$, $m < \alpha$, then $F_\xi \in \mathcal{R}_\alpha$.*

Employing Theorem 3.5 (see Section 3.2, where the main results are given), we get the following corollary of Theorem 2.1.

Corollary 2.6. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be pQAI real-valued random variables such that $F_{\xi_k} \in \mathcal{C}$ for each $k \in \{1, 2, \ldots, n\}$. Assume that for some $m > 0$ it holds that $\mathbb{E}(\xi_k^+)^m < \infty$ for all $k = 1, \ldots, n$. Then $F_{S_{n,m}^\xi} \in \mathcal{C}$.*

3. Main results

3.1. Related results

In this section we briefly review some of the existing related results to the theorems, which we propose in Section 3.2. Unless stated otherwise, we assume that the collections of r.v.s. $\{\xi_1, \ldots, \xi_n\}$ and $\{\theta_1, \ldots, \theta_n\}$ are independent.

As we mentioned earlier, while introducing needed dependence structures in Section 1.4.2, there is a vast amount of literature about asymptotic tail behaviour of distributions of sums of independent r.v.s or identically distributed r.v.s. See, for example, Tang and Tsitsiashvili [61], Wang and Tang [68] and the references therein. In this discussion we concentrate on the results, in which such restrictive assumptions have been weakened. First, we mention several interesting results concerning the equivalence

$$\mathbb{P}\big(S_n^\xi > x\big) \sim \sum_{k=1}^{n} \mathbb{P}\big(\xi_k > x\big). \tag{3.1}$$

Geluk and Tang [29] achieved (3.1) for distributions $F_{\xi_k} \in \mathcal{L} \cap \mathcal{D}$ (for all $k = 1, \ldots, n$), assuming a strong pairwise quasi-asymptotic independence between r.v.s $\xi_1, \ldots, \xi_n$ (see Assumption A in [29]).

In the same year Chen and Yuen [13] achieved (3.1) under a weaker dependence structure, called pairwise quasi-asymptotic independence (see Section 1.4.2 for the definition and comments). Distributions, though, were assumed to be from a smaller class $\mathcal{C}$ compared to the setting of Geluk and Tang [29].

Moreover, in the same article [13] Chen and Yuen extended the results to the case of randomly weighted sums, resulting in relation

$$\mathbb{P}\big(S_n^{\theta\xi} > x\big) \sim \sum_{k=1}^{n} \mathbb{P}\big(\theta_k \xi_k > x\big)$$

under an additional moment condition on nonnegative random weights:

$$\max\{\mathbb{E}\theta_1^p, \ldots, \mathbb{E}\theta_n^p\} < \infty \quad \text{for some } p > \max\{J_{\xi_1}^+, \ldots, J_{\xi_n}^+\}. \tag{3.2}$$

No requirements on the dependence structure of the weights were imposed.

Yi, Chen and Su in [65] considered the tail probability asymptotics of the randomly weighted sum $S_n^{\theta\xi}$, when r.v.s $\xi_1, \ldots, \xi_n$ belong to the class $\mathcal{D}$ and follow the same pQAI dependence structure as in Chen and Yuen [13]. It was shown that under (3.2) and additional assumption

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^- > x)}{\mathbb{P}(\xi_k > x)} = 0 \quad \text{for all } k = 1, \ldots n, \tag{3.3}$$

the following asymptotic bounds hold:

$$L_n^{\xi} \sum_{k=1}^{n} \mathbb{P}\big(\theta_k \xi_k > x\big) \underset{x\to\infty}{\lesssim} \mathbb{P}\big(S_n^{\theta\xi} > x\big) \underset{x\to\infty}{\lesssim} \frac{1}{L_n^{\xi}} \sum_{k=1}^{n} \mathbb{P}\big(\theta_k \xi_k > x\big), \tag{3.4}$$

where $L_n^{\xi} := \min\big\{L_{F_{\xi_1}}, \ldots, L_{F_{\xi_n}}\big\}$. Cheng in [14] tightened the bounds in (3.4) by putting the L-indices inside the sums and obtaining the following asymptotic bounds:

$$\sum_{k=1}^{n} L_{\xi_k} \mathbb{P}\big(\theta_k \xi_k > x\big) \underset{x\to\infty}{\lesssim} \mathbb{P}\big(S_n^{\theta\xi} > x\big) \underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{\xi_k}} \mathbb{P}\big(\theta_k \xi_k > x\big), \tag{3.5}$$

where $F_{\xi_k} \in \mathcal{D}$ for all $k = 1, \ldots, n$. The moment condition (3.2) was substituted by a weaker condition: for any $u > 0$,

$$\lim_{x\to\infty} \frac{\mathbb{P}(\theta_k > ux)}{\mathbb{P}(\theta_k \xi_k > x)} = 0, \quad k = 1, \ldots, n.$$

Yet the cost of the improvements is that pAI was considered instead of pQAI. The negligibility of the left tail condition, namely (3.3), is still needed. However, Cheng in the same paper proved that it is not needed when r.v.s $\xi_k,\ k = 1, \ldots, n$, are pairwise tail quasi-asymptotically independent (pTQAI) (see [14, Definition 1.3]).

Jaunė et al. in [35, Theorem 1] improved the mentioned result of Yi et al. [65], achieving (3.4) under the same conditions, except for the requirement of left tail negligibility (3.3).

3.2. Main results

Before presenting our main results, we formulate a couple of recent results from Leipus et al. [40]. We recall that Assumption $\mathcal{B}$, required in the theorems below, is given in Section 1.4.2. For the reader's convenience, we restate it here.

ASSUMPTION $\mathcal{B}$. Random variables $\xi_1, \dots, \xi_n$ for all $k, l = 1, \dots, n$, $k \neq l$, satisfy

$$\lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^+ > x \mid \xi_l^+ > u) = \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^- > x \mid \xi_l^+ > u) = \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^+ > x \mid \xi_l^- > u) = 0.$$

The following theorem follows from Theorems 3 and 4 in [40].

Theorem 3.1 (Leipus et al. [40, Theorems 3 and 4]). *Suppose that $\xi_1, \dots, \xi_n$ are real-valued r.v.s. If Assumption $\mathcal{B}$ holds, $F_{\xi_1} \in \mathcal{D}$, $\overline{F}_{\xi_k}(x) \asymp \overline{F}_{\xi_1}(x)$, $\overline{F}_{\xi_k^-}(x) = O(\overline{F}_{\xi_1}(x))$ for $k = 1, \dots, n$, and $\mathbb{E}\,|\xi_1|^m < \infty$ for some $m \in \mathbb{N}$, then*

$$L_{F_{\xi_k}} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{S_n^\xi > x\}}\big) \underset{x\to\infty}{\lesssim} \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big), \quad (3.6)$$

with $k = 1, \dots, n$, and

$$L_n^\xi \sum_{k=1}^n \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big((S_n^\xi)^m \mathbb{1}_{\{S_n^\xi > x\}}\big) \underset{x\to\infty}{\lesssim} \frac{1}{L_n^\xi} \sum_{k=1}^n \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big),$$

where (as before) $L_n^\xi := \min\{L_{F_{\xi_1}}, \dots, L_{F_{\xi_n}}\}$.

The second result is Theorem 5 in [40] with some small changes in notation.

Theorem 3.2 (Leipus et al. [40, Theorem 5]). *Let $\xi_1, \dots, \xi_n$ be r.v.s satisfying Assumption $\mathcal{B}$ such that $F_{\xi_1} \in \mathcal{D}$, $\mathbb{E}\,|\xi_1|^m < \infty$ for some $m \in \mathbb{N}$. Let $\theta_1, \dots, \theta_n$ be nonnegative, nondegenerate at zero, bounded r.v.s, independent of $\xi_1, \dots, \xi_n$. If $\overline{F}_{\theta_k\xi_k}(x) \asymp \overline{F}_{\theta_1\xi_1}(x)$, $\overline{F}_{\theta_k\xi_k^-}(x) = O(\overline{F}_{\theta_1\xi_1}(x))$ for all $k = 2, \dots, n$, then*

$$L_n^\xi \sum_{k=1}^n \mathbb{E}\big((\theta_k\xi_k)^m \mathbb{1}_{\{\theta_k\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big((S_n^{\theta\xi})^m \mathbb{1}_{\{S_n^{\theta\xi} > x\}}\big) \underset{x\to\infty}{\lesssim} \frac{1}{L_n^\xi} \sum_{k=1}^n \mathbb{E}\big((\theta_k\xi_k)^m \mathbb{1}_{\{\theta_k\xi_k > x\}}\big).$$

Firstly, we obtain a more precise result, compared to that in Theorem 3.1, showing that each summand in the approximating sums can be accompanied by a separate L-index of the corresponding distribution function. Also, we additionally include the case $m = 0$.

Theorem 3.3. *Let $\xi_1, \ldots, \xi_n$ be real-valued r.v.s satisfying the requirements of Theorem 3.1, where $\mathbb{E}\,|\xi_1|^m < \infty$ for some $m \in \mathbb{N}_0$. Then*

$$\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big(\big(S_n^\xi\big)^m \mathbb{1}_{\{S_n^\xi > x\}}\big)$$
$$\underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big).$$

In a later study, we found that it is possible to generalise results in the last theorem by weakening conditions for a dependence structure and exponent. Instead of nonnegative integer exponent, we will consider nonnegative real exponent and allow for a wider class of dependence, pQAI, instead of Assumption $\mathcal{B}$.

Theorem 3.4. *Let $\xi_1, \ldots, \xi_n$ be pQAI real-valued r.v.s. If $\mathbb{E}\,|\xi_k|^\alpha < \infty$, $F_{\xi_k} \in \mathcal{D}$ for all $k \in \{1, \ldots, n\}$ and some $\alpha \in [0, \infty)$, then*

$$\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\{\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big(\big(S_n^\xi\big)^\alpha \mathbb{1}_{\{S_n^\xi > x\}}\big)$$
$$\underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\{\xi_k > x\}}\big). \tag{3.7}$$

When assuming pairwise quasi-asymptotic independence, the following result follows from Theorem 3.4 for the real-valued random variables.

Theorem 3.5. *Let $\xi_1, \ldots, \xi_n$ be pQAI real-valued r.v.s. If $F_{\xi_k} \in \mathcal{C}$ for each $k \in \{1, \ldots, n\}$, then the d.f. $F_{S_n^\xi} \in \mathcal{C}$ and*

$$\overline{F}_{S_n^\xi}(x) \sim \sum_{k=1}^{n} \overline{F}_{\xi_k}(x). \tag{3.8}$$

Consider now the case where the ξ_k are nonnegative r.v.s. In such a case, the result in Theorem 3.4 can be improved omitting the corresponding L-indices in the lower asymptotic bound of (3.7).

Theorem 3.6. *Let $\xi_1, \ldots, \xi_n$ be nonnegative pQAI r.v.s and let $F_{\xi_k} \in \mathcal{D}$, $\mathbb{E}\xi_k^{\alpha} < \infty$ for all $k = 1, \ldots, n$ and some $\alpha \in [0, \infty)$. Then*

$$\sum_{k=1}^{n} \mathbb{E}\big(\xi_k^{\alpha} \mathbb{1}_{\{\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big(\big(S_n^{\xi}\big)^{\alpha} \mathbb{1}_{\{S_n^{\xi} > x\}}\big).$$

We finish this section with a generalisation of Theorem 3.4 to the case of randomly weighted sums $S_n^{\theta\xi}$. We can also look at it as a generalisation of Theorem 3.2 – in the following theorem we weaken the conditions for a dependence structure and exponent and remove quite a restrictive assumption about the boundedness of random weights $\theta_1, \ldots, \theta_n$.

Theorem 3.7. *Let $\xi_1, \ldots, \xi_n$ be pQAI real-valued r.v.s such that $F_{\xi_k} \in \mathcal{D}$ for all $k \in \{1, \ldots, n\}$, and let $\theta_1, \ldots, \theta_n$ be arbitrary dependent, nonnegative, nondegenerate at zero r.v.s with*

$$\max\{\mathbb{E}\theta_1^p, \ldots, \mathbb{E}\theta_n^p\} < \infty \text{ for some } p > \max\{J_{\xi_1}^+, \ldots J_{\xi_n}^+\}.$$

If collections $\{\xi_1, \ldots, \xi_n\}$ and $\{\theta_1, \ldots, \theta_n\}$ are independent, and

$$\mathbb{E}(\theta_k|\xi_k|)^{\alpha} < \infty \text{ for all } k \in \{1, \ldots, n\}$$

and some $\alpha \in [0, \infty)$, then

$$\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}\big((\theta_k\xi_k)^{\alpha} \mathbb{1}_{\{\theta_k\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big(\big(S_n^{\theta\xi}\big)^{\alpha} \mathbb{1}_{\{S_n^{\theta\xi} > x\}}\big)$$
$$\underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\big((\theta_k\xi_k)^{\alpha} \mathbb{1}_{\{\theta_k\xi_k > x\}}\big).$$

3.3. Proofs of main results

3.3.1. Auxiliary lemmas

In this subsection we state auxiliary lemmas, which are crucial in the proofs of the main results. The proof of the first can be found in Leipus et al. [40]. It is a slight modification of Lemma 4 therein.

Lemma 3.1 (Leipus et al. [40, Lemma 4])**.** *Let $\xi_1, \ldots, \xi_n$ be real-valued r.v.s satisfying Assumption $\mathcal{B}$. If $F_{\xi_1} \in \mathcal{D}$, $\overline{F}_{\xi_k}(x) \asymp \overline{F}_{\xi_1}(x)$, $\overline{F}_{\xi_k^-}(x) = O(\overline{F}_{\xi_1}(x))$, $\mathbb{E}\,|\xi_1|^m < \infty$ for all $k = 1, \ldots, n$ and some $m \geqslant 2$, then*

$$\big|\mathbb{E}\big(\xi_{k_1}^{m_1} \cdots \xi_{k_l}^{m_l} \mathbb{1}_{\{S_n^{\xi} > x\}}\big)\big| = o\bigg(\sum_{k=1}^{n} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big)\bigg)$$

for $1 \leqslant k_1, \ldots, k_l \leqslant n$, $l \geqslant 2$ *and* $m_1, \ldots, m_l \geqslant 1$, *such that* $m_1 + \cdots + m_l = m$.

Lemma 3.2 (Cline and Samorodnitsky [17, Lemma 3.1]; Jaunė et al. [35, Lemma 3])**.** *If* ξ *and* θ *are two independent r.v.s such that* $F_\xi \in \mathcal{D}$ *and* θ *is nonnegative, nondegenerate at zero r.v., then d.f.* $F_{\theta\xi}$ *of product* $\theta\xi$ *belongs to the class* $\mathcal{D}$. *If, in addition,* $\mathbb{E}\theta^p < \infty$ *for some* $p > J_\xi^+$, *then the inequality* $L_{F_{\theta\xi}} \geqslant L_{F_\xi}$ *holds for L-indices.*

Lemma 3.3 (Jaunė et al. [35, Lemma 4])**.** *Let two pairs of r.v.s* $\{\xi_1, \xi_2\}$ *and* $\{\theta_1, \theta_2\}$ *be independent. Let* ξ_1, ξ_2 *be QAI r.v.s such that* $F_{\xi_k} \in \mathcal{D}$, $k \in \{1, 2\}$, *and let* θ_1, θ_2 *be two arbitrarily dependent, nonnegative, nondegenerate at zero r.v.s with* $\max\{\mathbb{E}\theta_1^p, \mathbb{E}\theta_2^p\} < \infty$ *for some* $p > \max\{J_{\xi_1}^+, J_{\xi_2}^+\}$. *Then r.v.s* $\theta_1\xi_1$ *and* $\theta_2\xi_2$ *are QAI as well.*

Previously used Lemma 2.1 will be important in the proofs of the main results too, but avoiding confusion we do not repeat it here and refer the reader to Section 2.2.1.

3.3.2. Proofs

Proof of Theorem 3.3. Let us start from the case $m = 1$. By (3.6),

$$\mathbb{E}(\xi_k \mathbb{1}_{\{S_n^\xi > x\}}) \geqslant (L_{F_{\xi_k}}/2)\mathbb{E}(\xi_k \mathbb{1}_{\{\xi_k > x\}}) > 0$$

for all $k = 1, \ldots, n$ and large x. Thus, we can apply the min-max inequality (2.4) and get

$$\begin{aligned} \frac{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{E}\big(\xi_k \mathbb{1}_{\{\xi_k > x\}}\big)}{\mathbb{E}\big(S_n^\xi \mathbb{1}_{\{S_n^\xi > x\}}\big)} &= \frac{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{E}\big(\xi_k \mathbb{1}_{\{\xi_k > x\}}\big)}{\sum_{k=1}^n \mathbb{E}\big(\xi_k \mathbb{1}_{\{S_n^\xi > x\}}\big)} \\ &\leqslant \max_{1 \leqslant k \leqslant n} \frac{L_{F_{\xi_k}} \mathbb{E}\big(\xi_k \mathbb{1}_{\{\xi_k > x\}}\big)}{\mathbb{E}\big(\xi_k \mathbb{1}_{\{S_n^\xi > x\}}\big)} \end{aligned}$$

for large x. Hence, by (3.6),

$$\begin{aligned} \limsup_{x \to \infty} \frac{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{E}\big(\xi_k \mathbb{1}_{\{\xi_k > x\}}\big)}{\mathbb{E}\big(S_n^\xi \mathbb{1}_{\{S_n^\xi > x\}}\big)} &\leqslant \limsup_{x \to \infty} \max_{1 \leqslant k \leqslant n} \frac{L_{F_{\xi_k}} \mathbb{E}\big(\xi_k \mathbb{1}_{\{\xi_k > x\}}\big)}{\mathbb{E}\big(\xi_k \mathbb{1}_{\{S_n^\xi > x\}}\big)} \\ &\leqslant \max_{1 \leqslant k \leqslant n} \limsup_{x \to \infty} \frac{L_{F_{\xi_k}} \mathbb{E}\big(\xi_k \mathbb{1}_{\{\xi_k > x\}}\big)}{\mathbb{E}\big(\xi_k \mathbb{1}_{\{S_n^\xi > x\}}\big)} \\ &\leqslant 1. \end{aligned} \tag{3.9}$$

Similarly, the min-max inequality (2.4) and relation (3.6) imply that

$$\begin{aligned}\limsup_{x\to\infty}\frac{\mathbb{E}\big(S_n^\xi\mathbb{1}_{\{S_n^\xi>x\}}\big)}{\sum_{k=1}^n L_{F_{\xi_k}}^{-1}\mathbb{E}\big(\xi_k\mathbb{1}_{\{\xi_k>x\}}\big)} &= \limsup_{x\to\infty}\frac{\sum_{k=1}^n\mathbb{E}\big(\xi_k\mathbb{1}_{\{S_n^\xi>x\}}\big)}{\sum_{k=1}^n L_{F_{\xi_k}}^{-1}\mathbb{E}\big(\xi_k\mathbb{1}_{\{\xi_k>x\}}\big)}\\ &\leqslant \max_{1\leqslant k\leqslant n}\limsup_{x\to\infty}\frac{\mathbb{E}\big(\xi_k\mathbb{1}_{\{S_n^\xi>x\}}\big)}{L_{F_{\xi_k}}^{-1}\mathbb{E}\big(\xi_k\mathbb{1}_{\{\xi_k>x\}}\big)}\\ &\leqslant 1. \end{aligned}\tag{3.10}$$

Estimates (3.9) and (3.10) yield the statement of Theorem 3.3 in the case $m=1$.

Consider now the case $m\geqslant 2$. Note that

$$\begin{aligned}\mathbb{E}\left((S_n^\xi)^m\mathbb{1}_{\{S_n^\xi>x\}}\right) &= \sum_{k=1}^n\mathbb{E}\left(\xi_k^m\mathbb{1}_{\{S_n^\xi>x\}}\right)\\ &\quad+\sum\frac{m!}{m_1!\cdots m_n!}\mathbb{E}\left(\xi_1^{m_1}\cdots\xi_n^{m_n}\mathbb{1}_{\{S_n^\xi>x\}}\right),\end{aligned}\tag{3.11}$$

where only off-diagonal products are included in the second sum. As in the case $m=1$, by (3.6), with $m\geqslant 2$,

$$\mathbb{E}(\xi_k^m\mathbb{1}_{\{S_n^\xi>x\}})\geqslant(L_{F_{\xi_k}}/2)\mathbb{E}(\xi_k^m\mathbb{1}_{\{\xi_k>x\}})>0$$

for all $k=1,\ldots,n$ and large x. Thus, by the min-max inequality (2.4), we get that

$$\begin{aligned}&\frac{\mathbb{E}\left((S_n^\xi)^m\mathbb{1}_{\{S_n^\xi>x\}}\right)}{\sum_{k=1}^n L_{F_{\xi_k}}^{-1}\mathbb{E}(\xi_k^m\mathbb{1}_{\{\xi_k>x\}})}\\ &=\frac{\sum_{k=1}^n\mathbb{E}\left(\xi_k^m\mathbb{1}_{\{S_n^\xi>x\}}\right)+\sum\frac{m!}{m_1!\cdots m_n!}\mathbb{E}\left(\xi_1^{m_1}\cdots\xi_n^{m_n}\mathbb{1}_{\{S_n^\xi>x\}}\right)}{\sum_{k=1}^n L_{F_{\xi_k}}^{-1}\mathbb{E}(\xi_k^m\mathbb{1}_{\{\xi_k>x\}})}\\ &\leqslant\max_{1\leqslant k\leqslant n}\frac{\mathbb{E}\left(\xi_k^m\mathbb{1}_{\{S_n^\xi>x\}}\right)}{L_{F_{\xi_k}}^{-1}\mathbb{E}\left(\xi_k^m\mathbb{1}_{\{\xi_k>x\}}\right)}+\frac{\sum\frac{m!}{m_1!\cdots m_n!}\left|\mathbb{E}\left(\xi_1^{m_1}\cdots\xi_n^{m_n}\mathbb{1}_{\{S_n^\xi>x\}}\right)\right|}{\sum_{k=1}^n\mathbb{E}\left(\xi_k^m\mathbb{1}_{\{\xi_k>x\}}\right)}\end{aligned}$$

for large x.

Thus, by (3.6) and Lemma 3.1, we get

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\mathbb{E}\left((S_n^\xi)^m \mathbb{1}_{\{S_n^\xi > x\}}\right)}{\sum_{k=1}^n L_{F_{\xi_k}}^{-1}\,\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)} \\
&\leqslant \limsup_{x\to\infty} \max_{1\leqslant k\leqslant n} \frac{\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{S_n^\xi > x\}}\right)}{L_{F_{\xi_k}}^{-1}\,\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)} \\
&\quad + \limsup_{x\to\infty} \frac{\sum \frac{m!}{m_1!\cdots m_n!}\left|\mathbb{E}\left(\xi_1^{m_1}\cdots\xi_n^{m_n}\mathbb{1}_{\{S_n^\xi > x\}}\right)\right|}{\sum_{k=1}^n \mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)} \\
&\leqslant \max_{1\leqslant k\leqslant n} \limsup_{x\to\infty} \frac{\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{S_n^\xi > x\}}\right)}{L_{F_{\xi_k}}^{-1}\,\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)} \\
&\quad + \sum \frac{m!}{m_1!\cdots m_n!} \limsup_{x\to\infty} \frac{\left|\mathbb{E}\left(\xi_1^{m_1}\cdots\xi_n^{m_n}\mathbb{1}_{\{S_n^\xi > x\}}\right)\right|}{\sum_{k=1}^n \mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)} \\
&\leqslant 1.
\end{aligned} \tag{3.12}$$

Similarly, by (2.4), (3.11), (3.6), we have

$$\begin{aligned}
&\frac{\mathbb{E}\left((S_n^\xi)^m \mathbb{1}_{\{S_n^\xi > x\}}\right)}{\sum_{k=1}^n L_{F_{\xi_k}}\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)} \\
&= \frac{\sum_{k=1}^n \mathbb{E}\left(\xi_k^m \mathbb{1}_{\{S_n^\xi > x\}}\right) + \sum \frac{m!}{m_1!\cdots m_n!}\mathbb{E}\left(\xi_1^{m_1}\cdots\xi_n^{m_n}\mathbb{1}_{\{S_n^\xi > x\}}\right)}{\sum_{k=1}^n L_{F_{\xi_k}}\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)} \\
&\geqslant \min_{1\leqslant k\leqslant n} \frac{\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{S_n^\xi > x\}}\right)}{L_{F_{\xi_k}}\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)} - \left|\frac{\sum \frac{m!}{m_1!\cdots m_n!}\mathbb{E}\left(\xi_1^{m_1}\cdots\xi_n^{m_n}\mathbb{1}_{\{S_n^\xi > x\}}\right)}{\sum_{k=1}^n L_{F_{\xi_k}}\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)}\right| \\
&\geqslant \min_{1\leqslant k\leqslant n} \frac{\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{S_n^\xi > x\}}\right)}{L_{F_{\xi_k}}\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)} - \frac{\sum \frac{m!}{m_1!\cdots m_n!}\left|\mathbb{E}\left(\xi_1^{m_1}\cdots\xi_n^{m_n}\mathbb{1}_{\{S_n^\xi > x\}}\right)\right|}{\min\limits_{1\leqslant k\leqslant n} L_{F_{\xi_k}} \sum_{k=1}^n \mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)}
\end{aligned}$$

for large x. Remembering (3.11) and min-max inequality (2.4), one can see that

$$\begin{aligned}
&\liminf_{x\to\infty} \frac{\mathbb{E}\left((S_n^\xi)^m \mathbb{1}_{\{S_n^\xi > x\}}\right)}{\sum_{k=1}^n L_{F_{\xi_k}}\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)} \\
&\geqslant \liminf_{x\to\infty} \min_{1\leqslant k\leqslant n} \frac{\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{S_n^\xi > x\}}\right)}{L_{F_{\xi_k}}\mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)} \\
&\quad + \liminf_{x\to\infty}\left(- \frac{\sum \frac{m!}{m_1!\cdots m_n!}\left|\mathbb{E}\left(\xi_1^{m_1}\cdots\xi_n^{m_n}\mathbb{1}_{\{S_n^\xi > x\}}\right)\right|}{\min\limits_{1\leqslant k\leqslant n} L_{F_{\xi_k}} \sum_{k=1}^n \mathbb{E}\left(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\right)}\right).
\end{aligned}$$

Thus, by (3.6), Lemma 3.1 and properties of limit inferior and limit superior, we have

$$\begin{aligned}
&\liminf_{x\to\infty}\frac{\mathbb{E}\left((S_n^\xi)^m\mathbb{1}_{\{S_n^\xi>x\}}\right)}{\sum_{k=1}^n L_{F_{\xi_k}}\mathbb{E}\left(\xi_k^m\mathbb{1}_{\{\xi_k>x\}}\right)}\\
&\geqslant \min_{1\leqslant k\leqslant n}\liminf_{x\to\infty}\frac{\mathbb{E}\left(\xi_k^m\mathbb{1}_{\{S_n^\xi>x\}}\right)}{L_{F_{\xi_k}}\mathbb{E}\left(\xi_k^m\mathbb{1}_{\{\xi_k>x\}}\right)}\\
&\quad-\frac{1}{\min\limits_{1\leqslant k\leqslant n}L_{F_{\xi_k}}}\limsup_{x\to\infty}\frac{\sum\frac{m!}{m_1!\cdots m_n!}\left|\mathbb{E}\left(\xi_1^{m_1}\cdots\xi_n^{m_n}\mathbb{1}_{\{S_n^\xi>x\}}\right)\right|}{\sum_{k=1}^n\mathbb{E}\left(\xi_k^m\mathbb{1}_{\{\xi_k>x\}}\right)}\\
&\geqslant \min_{1\leqslant k\leqslant n}\liminf_{x\to\infty}\frac{\mathbb{E}\left(\xi_k^m\mathbb{1}_{\{S_n^\xi>x\}}\right)}{L_{F_{\xi_k}}\mathbb{E}\left(\xi_k^m\mathbb{1}_{\{\xi_k>x\}}\right)}\\
&\quad-\frac{1}{\min\limits_{1\leqslant k\leqslant n}L_{F_{\xi_k}}}\sum\frac{m!}{m_1!\cdots m_n!}\limsup_{x\to\infty}\frac{\left|\mathbb{E}\left(\xi_1^{m_1}\cdots\xi_n^{m_n}\mathbb{1}_{\{S_n^\xi>x\}}\right)\right|}{\sum_{k=1}^n\mathbb{E}\left(\xi_k^m\mathbb{1}_{\{\xi_k>x\}}\right)}\\
&\geqslant 1. \qquad (3.13)
\end{aligned}$$

The estimates (3.12) and (3.13) yield the proof for $m\geqslant 2$.

Finally, consider the case $m=0$. In case $n=1$, the relation of the theorem is obvious. Suppose that $n\geqslant 2$. Considering $x>0$ and $\delta_1\in(0,1)$, we have (for similar arguments, see the proof of Theorem 3.1 in Chen and Yuen [13])

$$\begin{aligned}
&\mathbb{P}\big(S_n^\xi>x\big)\\
&=\mathbb{P}\big(S_n^\xi>x,\Omega\big)\\
&=\mathbb{P}\left(S_n^\xi>x,\left\{\bigcup_{k=1}^n\{\xi_k>(1-\delta_1)x\}\right\}\cup\left\{\bigcap_{k=1}^n\{\xi_k\leqslant(1-\delta_1)x\}\right\}\right)\\
&=\mathbb{P}\left(S_n^\xi>x,\bigcup_{k=1}^n\{\xi_k>(1-\delta_1)x\}\right)+\mathbb{P}\left(S_n^\xi>x,\bigcap_{k=1}^n\{\xi_k\leqslant(1-\delta_1)x\}\right)\\
&\leqslant\mathbb{P}\left(\bigcup_{k=1}^n\{\xi_k>(1-\delta_1)x\}\right)+\mathbb{P}\left(S_n^\xi>x,\bigcap_{k=1}^n\{\xi_k\leqslant(1-\delta_1)x\}\right)\\
&=:\mathcal{A}_1(x)+\mathcal{A}_2(x).
\end{aligned}$$

It is easy to see that

$$\mathcal{A}_1(x)\leqslant\sum_{k=1}^n\mathbb{P}\big(\xi_k>(1-\delta_1)x\big).$$

Evaluation of $\mathcal{A}_2(x)$ is more complicated. Since

$$\{S_n^\xi > x\} \subset \bigcup_{k=1}^n \{\xi_k > x/n\}$$

and, similarly,

$$\{S_n^\xi - \xi_k > \delta_1 x\} \subset \bigcup_{j=1, j\neq k}^n \{\xi_j > \delta_1 x/(n-1)\},$$

we have

$$\begin{aligned}
&\mathcal{A}_2(x) \\
&= \mathbb{P}\Big(S_n^\xi > x, \bigcup_{k=1}^n \{\xi_k > x/n\}, \bigcap_{k=1}^n \{\xi_k \leqslant (1-\delta_1)x\}\Big) \\
&= \mathbb{P}\Big(\bigcup_{k=1}^n \Big\{\xi_k > x/n, S_n^\xi > x, \bigcap_{k=1}^n \{\xi_k \leqslant (1-\delta_1)x\}\Big\}\Big) \\
&\leqslant \sum_{k=1}^n \mathbb{P}\Big(\xi_k > x/n, S_n^\xi > x, \bigcap_{k=1}^n \{\xi_k \leqslant (1-\delta_1)x\}\Big) \\
&\leqslant \sum_{k=1}^n \mathbb{P}\Big(\xi_k > x/n, S_n^\xi > x, \xi_k \leqslant (1-\delta_1)x\Big) \\
&\leqslant \sum_{k=1}^n \mathbb{P}\big(\xi_k > x/n, S_n^\xi - \xi_k > \delta_1 x\big) \\
&\leqslant \sum_{k=1}^n \mathbb{P}\Big(\xi_k > x/n, \bigcup_{j=1, j\neq k}^n \{\xi_j > \delta_1 x/(n-1)\}\Big) \\
&\leqslant \sum_{k=1}^n \sum_{j=1, j\neq k}^n \mathbb{P}\big(\xi_k > x/n, \xi_j > \delta_1 x/(n-1)\big) \\
&\leqslant \sum_{k=1}^n \sum_{j=1, j\neq k}^n \mathbb{P}\big(\xi_k > \min\{x/n, \delta_1 x/(n-1)\}, \xi_j > \min\{x/n, \delta_1 x/(n-1)\}\big) \\
&\leqslant \sum_{k=1}^n \sum_{j=1, j\neq k}^n \mathbb{P}\big(\xi_k > \delta_2 x, \xi_j > \delta_2 x\big),
\end{aligned}$$

where $\delta_2 \in (0, \min\{1/n, \delta_1/(n-1)\})$.

Thus,

$$\frac{\mathbb{P}(S_n^{\xi} > x)}{\sum_{k=1}^{n} L_{F_{\xi_k}}^{-1} \mathbb{P}(\xi_k > x)} \leqslant \frac{\sum_{k=1}^{n} \mathbb{P}(\xi_k > (1-\delta_1)x)}{\sum_{k=1}^{n} L_{F_{\xi_k}}^{-1} \mathbb{P}(\xi_k > x)} + \frac{\sum_{k=1}^{n} \sum_{j=1, j\neq k}^{n} \mathbb{P}\big(\xi_k > \delta_2 x, \xi_j > \delta_2 x\big)}{\sum_{k=1}^{n} L_{F_{\xi_k}}^{-1} \mathbb{P}(\xi_k > x)}$$

$$\leqslant \max_{1\leqslant k\leqslant n} \frac{L_{F_{\xi_k}} \overline{F}_{\xi_k}((1-\delta_1)x)}{\overline{F}_{\xi_k}(x)} + \max_{1\leqslant k\leqslant n} \sum_{j=1, j\neq k}^{n} \frac{L_{F_{\xi_k}} \mathbb{P}\big(\xi_j > \delta_2 x, \xi_k > \delta_2 x\big)}{\mathbb{P}(\xi_k > x)}.$$

By conditions of the theorem and properties of limit superior,

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_n^{\xi} > x)}{\sum_{k=1}^{n} L_{F_{\xi_k}}^{-1} \mathbb{P}(\xi_k > x)}$$

$$\leqslant \max_{1\leqslant k\leqslant n} \limsup_{x\to\infty} \frac{L_{F_{\xi_k}} \overline{F}_{\xi_k}((1-\delta_1)x)}{\overline{F}_{\xi_k}(x)} + \max_{1\leqslant k\leqslant n} \sum_{j=1, j\neq k}^{n} \limsup_{x\to\infty} \frac{L_{F_{\xi_k}} \mathbb{P}\big(\xi_j^+ > \delta_2 x, \xi_k^+ > \delta_2 x\big)}{\mathbb{P}(\xi_k^+ > x)}$$

$$= \max_{1\leqslant k\leqslant n} \left\{ L_{F_{\xi_k}} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}((1-\delta_1)x)}{\overline{F}_{\xi_k}(x)} \right\} + \max_{1\leqslant k\leqslant n} \left\{ L_{F_{\xi_k}} \sum_{j=1, j\neq k}^{n} \limsup_{x\to\infty} \frac{\mathbb{P}\big(\xi_j^+ > \delta_2 x, \xi_k^+ > \delta_2 x\big)}{\mathbb{P}(\xi_k^+ > x)} \right\}$$

$$= \max_{1\leqslant k\leqslant n} \left\{ L_{F_{\xi_k}} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}((1-\delta_1)x)}{\overline{F}_{\xi_k}(x)} \right\}.$$

Therefore, by letting $\delta_1 \downarrow 0$, we get

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_n^{\xi} > x)}{\sum_{k=1}^{n} L_{F_{\xi_k}}^{-1} \mathbb{P}(\xi_k > x)} \leqslant 1,$$

since $\lim_{\delta_1\downarrow 0} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}((1-\delta_1)x)}{\overline{F}_{\xi_k}(x)} = 1/L_{F_{\xi_k}}$, $k = 1, \ldots, n$. Thus, the upper asymptotic bound is proved.

Now let's turn to the proof of lower asymptotic bound in case $m = 0$.

Considering $x > 0$, $\delta_1 > 0$, we have

$$\begin{aligned}\mathbb{P}\big(S_n^\xi > x\big) &\geqslant \mathbb{P}\Big(S_n^\xi > x, \bigcup_{k=1}^n \{\xi_k > (1+\delta_1)x\}\Big) \\ &= \mathbb{P}\Big(\bigcup_{k=1}^n \{S_n^\xi > x, \xi_k > (1+\delta_1)x\}\Big).\end{aligned}$$

Making use of Bonferroni inequality, we get that

$$\begin{aligned}&\mathbb{P}\Big(\bigcup_{k=1}^n \{S_n^\xi > x, \xi_k > (1+\delta_1)x\}\Big) \\ &\geqslant \sum_{k=1}^n \mathbb{P}\big(S_n^\xi > x, \xi_k > (1+\delta_1)x\big) \\ &\quad - \sum_{1\leqslant k<j\leqslant n} \mathbb{P}\big(S_n^\xi > x, \xi_k > (1+\delta_1)x, \xi_j > (1+\delta_1)x\big) \\ &\geqslant \sum_{k=1}^n \mathbb{P}\big(S_n^\xi > x, \xi_k > (1+\delta_1)x\big) \\ &\quad - \sum_{1\leqslant k<j\leqslant n} \mathbb{P}\big(\xi_k > (1+\delta_1)x, \xi_j > (1+\delta_1)x\big).\end{aligned}$$

Thus, for $x > 0$, $\delta_1 > 0$,

$$\begin{aligned}\mathbb{P}\big(S_n^\xi > x\big) \geqslant &\sum_{k=1}^n \mathbb{P}\big(S_n^\xi > x, \xi_k > (1+\delta_1)x\big) \\ &- \sum_{1\leqslant k<j\leqslant n} \mathbb{P}\big(\xi_k > (1+\delta_1)x, \xi_j > (1+\delta_1)x\big).\end{aligned} \tag{3.14}$$

Let us further analyse

$$\mathcal{A}_3(x) := \sum_{k=1}^n \mathbb{P}\big(S_n^\xi > x, \xi_k > (1+\delta_1)x\big). \tag{3.15}$$

To proceed, we need to remember that for any two events A and B, the following formula is true:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}(A \cap B^c), \tag{3.16}$$

where B^c marks the complement of an event B.

Using (3.16) and the fact that

$$\{S_n^\xi - \xi_k > -\delta_1 x, \xi_k > (1+\delta_1)x\} \subset \{S_n^\xi > x, \xi_k > (1+\delta_1)x\},$$

we get

$$
\begin{aligned}
&\mathcal{A}_3(x)\\
&\geqslant \sum_{k=1}^{n} \mathbb{P}\big(S_n^{\xi} - \xi_k > -\delta_1 x, \xi_k > (1+\delta_1)x\big)\\
&= \sum_{k=1}^{n} \mathbb{P}\big(\xi_k > (1+\delta_1)x\big) - \sum_{k=1}^{n} \mathbb{P}\big(\xi_k > (1+\delta_1)x, S_n^{\xi} - \xi_k \leqslant -\delta_1 x\big)\\
&\geqslant \sum_{k=1}^{n} \mathbb{P}\big(\xi_k > (1+\delta_1)x\big)\\
&\quad - \sum_{k=1}^{n} \mathbb{P}\Bigg(\xi_k > (1+\delta_1)x, \bigcup_{j=1, j\neq k}^{n} \big\{\xi_j \leqslant -\delta_1 x/(n-1)\big\}\Bigg)\\
&= \sum_{k=1}^{n} \mathbb{P}\big(\xi_k > (1+\delta_1)x\big)\\
&\quad - \sum_{k=1}^{n} \mathbb{P}\Bigg(\bigcup_{j=1, j\neq k}^{n} \big\{\xi_k > (1+\delta_1)x, \xi_j \leqslant -\delta_1 x/(n-1)\big\}\Bigg)\\
&\geqslant \sum_{k=1}^{n} \mathbb{P}\big(\xi_k > (1+\delta_1)x\big)\\
&\quad - \sum_{1\leqslant k\neq j\leqslant n} \mathbb{P}\big(\xi_k > (1+\delta_1)x, \xi_j \leqslant -\delta_1 x/(n-1)\big)\\
&\geqslant \sum_{k=1}^{n} \mathbb{P}\big(\xi_k > (1+\delta_1)x\big)\\
&\quad - \sum_{1\leqslant k\neq j\leqslant n} \mathbb{P}\big(\xi_j^{-} > \delta_3 x, \xi_k^{+} > \delta_3 x\big),
\end{aligned}
\tag{3.17}
$$

where $\delta_3 \in (0, \delta_1/(n-1))$. Therefore, from (3.14), (3.15) and (3.17), we have

$$
\begin{aligned}
&\frac{\mathbb{P}\big(S_n^{\xi} > x\big)}{\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{P}\big(\xi_k > x\big)}\\
&\geqslant \frac{\sum_{k=1}^{n} \mathbb{P}\big(\xi_k > (1+\delta_1)x\big)}{\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{P}\big(\xi_k > x\big)} - \frac{\sum_{1\leqslant k\neq j\leqslant n} \mathbb{P}\big(\xi_j^{-} > \delta_3 x, \xi_k^{+} > \delta_3 x\big)}{\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{P}\big(\xi_k^{+} > x\big)}\\
&\quad - \frac{\sum_{1\leqslant k< j\leqslant n} \mathbb{P}\big(\xi_k > (1+\delta_1)x, \xi_j > (1+\delta_1)x\big)}{\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{P}\big(\xi_k > x\big)}.
\end{aligned}
$$

Hence, by min-max inequality (2.4),

$$\begin{aligned}
&\frac{\mathbb{P}\big(S_n^\xi > x\big)}{\sum_{k=1}^n L_{F_{\xi_k}}\mathbb{P}\big(\xi_k > x\big)}\\
&\geqslant \min_{1\leqslant k\leqslant n}\frac{\overline{F}_{\xi_k}((1+\delta_1)x)}{L_{F_{\xi_k}}\overline{F}_{\xi_k}(x)} - \max_{1\leqslant k\leqslant n}\sum_{j=1,j\neq k}^{n}\frac{\mathbb{P}\big(\xi_j^- > \delta_3 x, \xi_k^+ > \delta_3 x\big)}{L_{F_{\xi_k}}\mathbb{P}\big(\xi_k^+ > x\big)}\\
&\quad - \max_{1\leqslant k\leqslant n}\sum_{j=k+1}^{n}\frac{\mathbb{P}\big(\xi_j > (1+\delta_1)x, \xi_k > (1+\delta_1)x\big)}{L_{F_{\xi_k}}\mathbb{P}\big(\xi_k > x\big)}.
\end{aligned}$$

By conditions of the theorem and properties of limit inferior,

$$\begin{aligned}
&\liminf_{x\to\infty}\frac{\mathbb{P}\big(S_n^\xi > x\big)}{\sum_{k=1}^n L_{F_{\xi_k}}\mathbb{P}\big(\xi_k > x\big)}\\
&\geqslant \liminf_{x\to\infty}\min_{1\leqslant k\leqslant n}\frac{\overline{F}_{\xi_k}((1+\delta_1)x)}{L_{F_{\xi_k}}\overline{F}_{\xi_k}(x)}\\
&\quad + \liminf_{x\to\infty}\Bigg(-\max_{1\leqslant k\leqslant n}\sum_{j=1,j\neq k}^{n}\frac{\mathbb{P}\big(\xi_j^- > \delta_3 x, \xi_k^+ > \delta_3 x\big)}{L_{F_{\xi_k}}\mathbb{P}\big(\xi_k^+ > x\big)}\Bigg)\\
&\quad + \liminf_{x\to\infty}\Bigg(-\max_{1\leqslant k\leqslant n}\sum_{j=k+1}^{n}\frac{\mathbb{P}\big(\xi_j^+ > (1+\delta_1)x, \xi_k^+ > (1+\delta_1)x\big)}{L_{F_{\xi_k}}\mathbb{P}\big(\xi_k^+ > x\big)}\Bigg)\\
&\geqslant \min_{1\leqslant k\leqslant n}\Bigg\{L_{F_{\xi_k}}^{-1}\liminf_{x\to\infty}\frac{\overline{F}_{\xi_k}((1+\delta_1)x)}{\overline{F}_{\xi_k}(x)}\Bigg\}\\
&\quad - \max_{1\leqslant k\leqslant n}\Bigg\{L_{F_{\xi_k}}^{-1}\sum_{j=1,j\neq k}^{n}\limsup_{x\to\infty}\frac{\mathbb{P}\big(\xi_j^- > \delta_3 x, \xi_k^+ > \delta_3 x\big)}{\mathbb{P}\big(\xi_k^+ > x\big)}\Bigg\}\\
&\quad - \max_{1\leqslant k\leqslant n}\Bigg\{L_{F_{\xi_k}}^{-1}\sum_{j=k+1}^{n}\limsup_{x\to\infty}\frac{\mathbb{P}\big(\xi_j^+ > (1+\delta_1)x, \xi_k^+ > (1+\delta_1)x\big)}{\mathbb{P}\big(\xi_k^+ > x\big)}\Bigg\}\\
&\geqslant 1,
\end{aligned}$$

since $\lim_{\delta_1\downarrow 0}\liminf_{x\to\infty}\frac{\overline{F}_{\xi_k}((1+\delta_1)x)}{\overline{F}_{\xi_k}(x)} = L_{F_{\xi_k}}$, $k = 1,\ldots,n$. Thus, the proof of the lower asymptotic bound is complete. □

Proof of Theorem 3.4. Let us begin with the case $\alpha = 0$. In this case we have to prove that

$$\sum_{k=1}^n L_{F_{\xi_k}}\overline{F}_{\xi_k}(x) \underset{x\to\infty}{\lesssim} \mathbb{P}(S_n^\xi > x) \underset{x\to\infty}{\lesssim} \sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}}\overline{F}_{\xi_k}(x). \tag{3.18}$$

The case $n = 1$ in (3.18) follows trivially from the definition of coefficient $L_{F_{\xi_k}}$. Now let $n \geqslant 2$. First, let us consider the upper asymptotic bound in (3.18). Identically to the proof of Theorem 3.3, for an arbitrary $\delta_1 \in (0,1)$ and $x > 0$,

$$\mathbb{P}\big(S_n^\xi > x\big) \leqslant \sum_{k=1}^{n} \mathbb{P}\big(\xi_k > (1-\delta_1)x\big) + \sum_{k=1}^{n} \sum_{j=1, j\neq k}^{n} \mathbb{P}\big(\xi_k > \delta_2 x, \xi_j > \delta_2 x\big),$$

where $\delta_2 \in (0, \min\{1/n, \delta_1/(n-1)\})$.

Consequently,

$$\begin{aligned}\frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^{n} L_{F_{\xi_k}}^{-1} \mathbb{P}(\xi_k > x)} &\leqslant \frac{\sum_{k=1}^{n} \mathbb{P}(\xi_k > (1-\delta_1)x)}{\sum_{k=1}^{n} L_{F_{\xi_k}}^{-1} \mathbb{P}(\xi_k > x)} \\ &\quad + \frac{\sum_{k=1}^{n} \sum_{j=1, j\neq k}^{n} \mathbb{P}(\xi_k > \delta_2 x, \xi_j > \delta_2 x)}{\sum_{k=1}^{n} L_{F_{\xi_k}}^{-1} \mathbb{P}(\xi_k > x)} \\ &=: \mathcal{I}_1(x,\delta_1) + \mathcal{I}_2(x,\delta_1).\end{aligned}$$

Using the min-max inequality (2.4), we get

$$\mathcal{I}_1(x,\delta_1) \leqslant \max_{1\leqslant k\leqslant n} \left\{ L_{F_{\xi_k}} \frac{\mathbb{P}(\xi_k > (1-\delta_1)x)}{\mathbb{P}(\xi_k > x)} \right\}.$$

By the same inequality (2.4) and observation

$$\sum_{k=1}^{n} \sum_{j=1, j\neq k}^{n} \big(\mathbb{P}(\xi_k > \delta_2 x) + \mathbb{P}(\xi_j > \delta_2 x)\big) = 2(n-1) \sum_{k=1}^{n} \mathbb{P}(\xi_k > \delta_2 x), \tag{3.19}$$

we obtain

$$\begin{aligned}\mathcal{I}_2(x,\delta_1) &= \frac{\sum_{k=1}^{n} \sum_{j=1, j\neq k}^{n} \mathbb{P}(\xi_k > \delta_2 x, \xi_j > \delta_2 x)}{\sum_{k=1}^{n} \sum_{j=1, j\neq k}^{n} \big(\mathbb{P}(\xi_k > \delta_2 x) + \mathbb{P}(\xi_j > \delta_2 x)\big)} \\ &\quad \times \frac{\sum_{k=1}^{n} \sum_{j=1, j\neq k}^{n} \big(\mathbb{P}(\xi_k > \delta_2 x) + \mathbb{P}(\xi_j > \delta_2 x)\big)}{\sum_{k=1}^{n} L_{F_{\xi_k}}^{-1} \mathbb{P}(\xi_k > x)} \\ &\leqslant \max_{1\leqslant k\neq j\leqslant n} \left\{ \frac{\mathbb{P}(\xi_k > \delta_2 x, \xi_j > \delta_2 x)}{\mathbb{P}(\xi_k > \delta_2 x) + \mathbb{P}(\xi_j > \delta_2 x)} \right\} \\ &\quad \times 2(n-1) \max_{1\leqslant k\leqslant n} \left\{ L_{F_{\xi_k}} \frac{\mathbb{P}(\xi_k > \delta_2 x)}{\mathbb{P}(\xi_k > x)} \right\}.\end{aligned}$$

The fact that $F_{\xi_k} \in \mathcal{D}$ for all $k \in \{1,\ldots,n\}$ and condition of pQAI for r.v.s $\{\xi_1, \ldots, \xi_n\}$ implies

$$\begin{aligned}\limsup_{x\to\infty} \mathcal{I}_1(x,\delta_1) &\leqslant \limsup_{x\to\infty} \max_{1\leqslant k\leqslant n} \left\{ L_{F_{\xi_k}} \frac{\mathbb{P}(\xi_k > (1-\delta_1)x)}{\mathbb{P}(\xi_k > x)} \right\} \\ &\leqslant \max_{1\leqslant k\leqslant n} \left\{ L_{F_{\xi_k}} \limsup_{x\to\infty} \frac{\mathbb{P}(\xi_k > (1-\delta_1)x)}{\mathbb{P}(\xi_k > x)} \right\}, \\ \limsup_{x\to\infty} \mathcal{I}_2(x,\delta_1) &\leqslant 2(n-1) \\ &\quad \times \max_{1\leqslant k\neq j\leqslant n} \left\{ \limsup_{x\to\infty} \frac{\mathbb{P}(\xi_k > \delta_2 x, \xi_j > \delta_2 x)}{\mathbb{P}(\xi_k > \delta_2 x) + \mathbb{P}(\xi_j > \delta_2 x)} \right\} \\ &\quad \times \max_{1\leqslant k\leqslant n} \left\{ L_{F_{\xi_k}} \limsup_{x\to\infty} \frac{\mathbb{P}(\xi_k > \delta_2 x)}{\mathbb{P}(\xi_k > x)} \right\} \\ &= 0.\end{aligned}$$

Thus,

$$\begin{aligned}\limsup_{x\to\infty} \frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^n L_{F_{\xi_k}}^{-1} \mathbb{P}(\xi_k > x)} &\leqslant \limsup_{x\to\infty} \mathcal{I}_1(x,\delta_1) + \limsup_{x\to\infty} \mathcal{I}_2(x,\delta_1) \\ &\leqslant \max_{1\leqslant k\leqslant n} \left\{ L_{F_{\xi_k}} \limsup_{x\to\infty} \frac{\mathbb{P}(\xi_k > (1-\delta_1)x)}{\mathbb{P}(\xi_k > x)} \right\}.\end{aligned}$$

Therefore, by letting $\delta_1 \downarrow 0$, and from the definition of indices $L_{F_{\xi_k}}$, we get

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^n L_{F_{\xi_k}}^{-1} \mathbb{P}(\xi_k > x)} \leqslant 1.$$

The upper bound in (3.18) is proved. Now let us turn to the lower asymptotic bound in (3.18). Again, similarly to the proof of Theorem 3.3, for an arbitrary $\delta_1 \in (0,1)$ and $x > 0$,

$$\begin{aligned}\mathbb{P}\big(S_n^\xi > x\big) \geqslant \sum_{k=1}^n \mathbb{P}\big(\xi_k > (1+\delta_1)x\big) &- \sum_{1\leqslant k\neq j\leqslant n} \mathbb{P}\big(\xi_k > \delta_3 x, \xi_j^- > \delta_3 x\big) \\ &- \sum_{1\leqslant k\neq j\leqslant n} \mathbb{P}\big(\xi_k > (1+\delta_1)x, \xi_j > (1+\delta_1)x\big),\end{aligned}$$

where $\delta_3 \in (0, \delta_1/(n-1))$.

Consequently,

$$\begin{aligned}
\frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}(\xi_k > x)} \geqslant\ & \frac{\sum_{k=1}^n \mathbb{P}(\xi_k > (1+\delta_1)x)}{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}(\xi_k > x)} \\
&- \frac{\sum_{1 \leqslant k \neq j \leqslant n} \mathbb{P}\big(\xi_k > \delta_3 x, \xi_j^- > \delta_3 x\big)}{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}\big(\xi_k > x\big)} \\
&- \frac{\sum_{1 \leqslant k \neq j \leqslant n} \mathbb{P}\big(\xi_k > (1+\delta_1)x, \xi_j > (1+\delta_1)x\big)}{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}\big(\xi_k > x\big)} \\
=:\ & \mathcal{I}_1(x,\delta_1) - \mathcal{I}_2(x,\delta_1) - \mathcal{I}_3(x,\delta_1).
\end{aligned}$$

Using the min-max inequality (2.4), we get

$$\mathcal{I}_1(x,\delta_1) \geqslant \min_{1 \leqslant k \leqslant n} \left\{ \frac{\mathbb{P}(\xi_k > (1+\delta_1)x)}{L_{F_{\xi_k}} \mathbb{P}(\xi_k > x)} \right\}.$$

By the same inequality (2.4) and observation analogous to (3.19), we obtain

$$\begin{aligned}
\mathcal{I}_2(x,\delta_1) =\ & \frac{\sum_{1 \leqslant k \neq j \leqslant n} \mathbb{P}(\xi_k > \delta_3 x, \xi_j^- > \delta_3 x)}{\sum_{1 \leqslant k \neq j \leqslant n} \big(\mathbb{P}(\xi_k > \delta_3 x) + \mathbb{P}(\xi_j > \delta_3 x)\big)} \\
& \times \frac{\sum_{1 \leqslant k \neq j \leqslant n} \big(\mathbb{P}(\xi_k > \delta_3 x) + \mathbb{P}(\xi_j > \delta_3 x)\big)}{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}(\xi_k > x)} \\
\leqslant\ & 2(n-1) \max_{1 \leqslant k \neq j \leqslant n} \left\{ \frac{\mathbb{P}(\xi_k > \delta_3 x, \xi_j^- > \delta_3 x)}{\mathbb{P}(\xi_k > \delta_3 x) + \mathbb{P}(\xi_j > \delta_3 x)} \right\} \\
& \times \max_{1 \leqslant k \leqslant n} \left\{ \frac{\mathbb{P}(\xi_k > \delta_3 x)}{L_{F_{\xi_k}} \mathbb{P}(\xi_k > x)} \right\}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathcal{I}_3(x,\delta_1) \leqslant\ & 2(n-1) \max_{1 \leqslant k \neq j \leqslant n} \left\{ \frac{\mathbb{P}(\xi_k > (1+\delta_1)x, \xi_j > (1+\delta_1)x)}{\big(\mathbb{P}(\xi_k > (1+\delta_1)x) + \mathbb{P}(\xi_j > (1+\delta_1)x)\big)} \right\} \\
& \times \max_{1 \leqslant k \leqslant n} \left\{ \frac{\mathbb{P}(\xi_k > (1+\delta_1)x)}{L_{F_{\xi_k}} \mathbb{P}(\xi_k > x)} \right\} \\
\leqslant\ & \max_{1 \leqslant k \leqslant n} \left\{ \frac{2(n-1)}{L_{F_{\xi_k}}} \right\} \\
& \times \max_{1 \leqslant k \neq j \leqslant n} \left\{ \frac{\mathbb{P}(\xi_k > (1+\delta_1)x, \xi_j > (1+\delta_1)x)}{\mathbb{P}(\xi_k > (1+\delta_1)x) + \mathbb{P}(\xi_j > (1+\delta_1)x)} \right\}.
\end{aligned}$$

Using the fact that $F_{\xi_k} \in \mathcal{D}$ for all $k \in \{1, \ldots, n\}$ and condition of pQAI for r.v.s $\{\xi_1, \ldots, \xi_n\}$, we get the following estimates:

$$\liminf_{x\to\infty} \mathcal{I}_1(x,\delta_1) \geqslant \min_{1\leqslant k\leqslant n} \left\{ \frac{1}{L_{F_{\xi_k}}} \liminf_{x\to\infty} \frac{\mathbb{P}(\xi_k > (1+\delta_1)x)}{\mathbb{P}(\xi_k > x)} \right\},$$

$$\begin{aligned}
&\limsup_{x\to\infty} \mathcal{I}_2(x,\delta_1)\\
&\leqslant 2(n-1) \max_{1\leqslant k\neq j\leqslant n} \left\{ \limsup_{x\to\infty} \frac{\mathbb{P}(\xi_k > \delta_3 x, \xi_j^- > \delta_3 x)}{\mathbb{P}(\xi_k > \delta_3 x) + \mathbb{P}(\xi_j > \delta_3 x)} \right\}\\
&\quad \times \max_{1\leqslant k\leqslant n} \left\{ \frac{1}{L_{F_{\xi_k}}} \limsup_{x\to\infty} \frac{\mathbb{P}(\xi_k > \delta_3 x)}{\mathbb{P}(\xi_k > x)} \right\}\\
&= 0,\\
&\limsup_{x\to\infty} \mathcal{I}_3(x,\delta_1)\\
&\leqslant \max_{1\leqslant k\leqslant n} \left\{ \frac{2(n-1)}{L_{F_{\xi_k}}} \right\}\\
&\quad \times \max_{1\leqslant k\neq j\leqslant n} \left\{ \limsup_{x\to\infty} \frac{\mathbb{P}(\xi_k > (1+\delta_1)x, \xi_j > (1+\delta_1)x)}{\mathbb{P}(\xi_k > (1+\delta_1)x) + \mathbb{P}(\xi_j > (1+\delta_1)x)} \right\}\\
&= 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\liminf_{x\to\infty} \frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}(\xi_k > x)} &\geqslant \liminf_{x\to\infty} \mathcal{I}_1(x,\delta_1) + \liminf_{x\to\infty} (-\mathcal{I}_2(x,\delta_1))\\
&\quad + \liminf_{x\to\infty} (-\mathcal{I}_3(x,\delta_1))\\
&= \liminf_{x\to\infty} \mathcal{I}_1(x,\delta_1) - \limsup_{x\to\infty} \mathcal{I}_2(x,\delta_1)\\
&\quad - \limsup_{x\to\infty} \mathcal{I}_3(x,\delta_1)\\
&\geqslant \min_{1\leqslant k\leqslant n} \left\{ \frac{1}{L_{F_{\xi_k}}} \liminf_{x\to\infty} \frac{\mathbb{P}(\xi_k > (1+\delta_1)x)}{\mathbb{P}(\xi_k > x)} \right\}.
\end{aligned}$$

Therefore, letting $\delta_1 \downarrow 0$, and from the definition of indices $L_{F_{\xi_k}}$, we obtain the lower asymptotic bound in (3.18):

$$\liminf_{x\to\infty} \frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}(\xi_k > x)} \geqslant 1.$$

The proof of the case $\alpha = 0$ is finished. Consider $\alpha > 0$. The case $n = 1$ follows trivially from the definition of index $L_{F_{\xi_1}}$. Let $n \geqslant 2$. By

Lemma 2.1 and the min-max inequality (2.4), we have

$$\begin{aligned}
&\frac{\mathbb{E}\big((S_n^\xi)^\alpha \mathbb{1}_{\{S_n^\xi > x\}}\big)}{\sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}}\mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\xi_k > x}\big)} \\
&= \frac{x^\alpha \mathbb{P}(S_n^\xi > x) + \alpha\int_x^\infty u^{\alpha-1}\mathbb{P}(S_n^\xi > u)\mathrm{d}u}{\sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}}(x^\alpha \mathbb{P}(\xi_k > x) + \alpha\int_x^\infty u^{\alpha-1}\mathbb{P}(\xi_k > u)\mathrm{d}u)} \\
&\leqslant \max\left\{\frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}}\mathbb{P}(\xi_k > x)}, \frac{\int_x^\infty u^{\alpha-1}\mathbb{P}(S_n^\xi > u)\mathrm{d}u}{\sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}}\int_x^\infty u^{\alpha-1}\mathbb{P}(\xi_k > u)\mathrm{d}u}\right\} \\
&= \max\left\{\frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}}\mathbb{P}(\xi_k > x)}, \frac{\int_x^\infty u^{\alpha-1}\mathbb{P}(S_n^\xi > u)\mathrm{d}u}{\int_x^\infty u^{\alpha-1}\sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}}\mathbb{P}(\xi_k > u)\mathrm{d}u}\right\} \\
&=: \max\{\mathcal{B}_1(x), \mathcal{B}_2(x)\}. \qquad (3.20)
\end{aligned}$$

We already already proved that $\limsup_{x\to\infty} \mathcal{B}_1(x) \leqslant 1$. For the term $\mathcal{B}_2(x)$, we have that

$$\begin{aligned}
\limsup_{x\to\infty} \mathcal{B}_2(x) &= \limsup_{x\to\infty} \frac{\int_x^\infty u^{\alpha-1}\mathbb{P}(S_n^\xi > u)\mathrm{d}u}{\int_x^\infty u^{\alpha-1}\mathbb{P}(S_n^\xi > u)\frac{\sum_{k=1}^n L_{F_{\xi_k}}^{-1}\mathbb{P}(\xi_k > u)}{\mathbb{P}(S_n^\xi > u)}\mathrm{d}u} \\
&\leqslant \limsup_{x\to\infty} \frac{\int_x^\infty u^{\alpha-1}\mathbb{P}(S_n^\xi > u)\mathrm{d}u}{\int_x^\infty u^{\alpha-1}\mathbb{P}(S_n^\xi > u)\inf_{u\geqslant x}\frac{\sum_{k=1}^n L_{F_{\xi_k}}^{-1}\mathbb{P}(\xi_k > u)}{\mathbb{P}(S_n^\xi > u)}\mathrm{d}u} \\
&= \limsup_{x\to\infty} \frac{1}{\inf_{u\geqslant x}\frac{\sum_{k=1}^n L_{F_{\xi_k}}^{-1}\mathbb{P}(\xi_k > u)}{\mathbb{P}(S_n^\xi > u)}} \\
&= \limsup_{x\to\infty}\sup_{u\geqslant x} \frac{\mathbb{P}(S_n^\xi > u)}{\sum_{k=1}^n L_{F_{\xi_k}}^{-1}\mathbb{P}(\xi_k > u)} \\
&= \limsup_{x\to\infty} \frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^n L_{F_{\xi_k}}^{-1}\mathbb{P}(\xi_k > x)} \\
&\leqslant 1.
\end{aligned}$$

The desired upper estimate now follows from (3.20).

Now let us turn to the lower estimate. In the same fashion, we obtain

$$
\begin{aligned}
&\frac{\mathbb{E}\big((S_n^\xi)^\alpha \mathbb{1}_{\{S_n^\xi > x\}}\big)}{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\xi_k > x}\big)} \\
&= \frac{x^\alpha \mathbb{P}(S_n^\xi > x) + \alpha \int_x^\infty u^{\alpha-1} \mathbb{P}(S_n^\xi > u)\mathrm{d}u}{\sum_{k=1}^n L_{F_{\xi_k}} (x^\alpha \mathbb{P}(\xi_k > x) + \alpha \int_x^\infty u^{\alpha-1} \mathbb{P}(\xi_k > u)\mathrm{d}u)} \\
&\geqslant \min\left\{ \frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}(\xi_k > x)}, \frac{\int_x^\infty u^{\alpha-1} \mathbb{P}(S_n^\xi > u)\mathrm{d}u}{\int_x^\infty u^{\alpha-1} \sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}(\xi_k > u)\mathrm{d}u} \right\} \\
&=: \min\{\mathcal{B}_3(x), \mathcal{B}_4(x)\}. \qquad (3.21)
\end{aligned}
$$

We have already proved that $\liminf_{x\to\infty} \mathcal{B}_3(x) \geqslant 1$. For the term $\mathcal{B}_4(x)$, we have

$$
\begin{aligned}
\liminf_{x\to\infty} \mathcal{B}_4(x) &= \liminf_{x\to\infty} \frac{\int_x^\infty u^{\alpha-1} \mathbb{P}(S_n^\xi > u)\mathrm{d}u}{\int_x^\infty u^{\alpha-1} \mathbb{P}(S_n^\xi > u) \frac{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}(\xi_k > u)}{\mathbb{P}(S_n^\xi > u)} \mathrm{d}u} \\
&\geqslant \liminf_{x\to\infty} \frac{\int_x^\infty u^{\alpha-1} \mathbb{P}(S_n^\xi > u)\mathrm{d}u}{\int_x^\infty u^{\alpha-1} \mathbb{P}(S_n^\xi > u) \sup\limits_{u \geqslant x} \frac{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}(\xi_k > u)}{\mathbb{P}(S_n^\xi > u)} \mathrm{d}u} \\
&= \liminf_{x\to\infty} \inf_{u \geqslant x} \frac{\mathbb{P}(S_n^\xi > u)}{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}(\xi_k > u)} \\
&= \liminf_{x\to\infty} \frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{P}(\xi_k > x)} \\
&\geqslant 1.
\end{aligned}
$$

The desired lower estimate now follows from (3.21). Theorem 3.4 is proved. □

Proof of Theorem 3.5. Result (3.8) is Theorem 3.1 in Chen, Yuen [13]. Since $\mathcal{C} \subset \mathcal{D}$ (see Section 1.4.1), relation (3.8) also follows from Theorem 3.4 in this dissertation. Consequently, for all $y \in (0,1)$,

$$
\overline{F}_{S_n^\xi}(xy) \sim \sum_{k=1}^n \overline{F}_{\xi_k}(xy). \qquad (3.22)
$$

We can write

$$
\frac{\overline{F}_{S_n^\xi}(xy)}{\overline{F}_{S_n^\xi}(x)} = \frac{\overline{F}_{S_n^\xi}(xy)}{\sum_{k=1}^n \overline{F}_{\xi_k}(xy)} \frac{\sum_{k=1}^n \overline{F}_{\xi_k}(xy)}{\sum_{k=1}^n \overline{F}_{\xi_k}(x)} \frac{\sum_{k=1}^n \overline{F}_{\xi_k}(x)}{\overline{F}_{S_n^\xi}(x)}.
$$

Using this equation together with (3.8) and (3.22), we get that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_n^\xi}(xy)}{\overline{F}_{S_n^\xi}(x)} = \limsup_{x\to\infty} \frac{\sum_{k=1}^n \overline{F}_{\xi_k}(xy)}{\sum_{k=1}^n \overline{F}_{\xi_k}(x)}. \tag{3.23}$$

Properties of limit inferior and limit superior, equation (3.23) and classic min-max inequality imply that the following chain of inequalities hold:

$$\begin{aligned}
1 \leqslant \liminf_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}_{S_n^\xi}(xy)}{\overline{F}_{S_n^\xi}(x)} &\leqslant \limsup_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}_{S_n^\xi}(xy)}{\overline{F}_{S_n^\xi}(x)} \\
&= \limsup_{y\uparrow 1} \limsup_{x\to\infty} \frac{\sum_{k=1}^n \overline{F}_{\xi_k}(xy)}{\sum_{k=1}^n \overline{F}_{\xi_k}(x)} \\
&\leqslant \max_{1\leqslant k\leqslant n} \left\{ \limsup_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}(xy)}{\overline{F}_{\xi_k}(x)} \right\} \\
&= 1.
\end{aligned}$$

Thus, $F_{S_n^\xi} \in \mathcal{C}$. □

Proof of Theorem 3.6. The case $\alpha = 0,\ n = 1$ is obvious. Let $\alpha = 0$ and $n \geqslant 2$. Since r.v.s are nonnegative, we have $\bigcup_{k=1}^{n} \{\xi_k > x\} \subset \{S_n^\xi > x\}$. Therefore, for any $x > 0$,

$$\begin{aligned}
\mathbb{P}(S_n^\xi > x) &\geqslant \mathbb{P}\left(\bigcup_{k=1}^n \{\xi_k > x\} \right) \\
&\geqslant \sum_{k=1}^n \mathbb{P}(\xi_k > x) - \sum_{1\leqslant k\neq j\leqslant n} \mathbb{P}(\xi_k > x, \xi_j > x).
\end{aligned} \tag{3.24}$$

Let us further analyse

$$\mathcal{A}_4(x) := \sum_{1\leqslant k\neq j\leqslant n} \mathbb{P}(\xi_k > x, \xi_j > x). \tag{3.25}$$

We have

$$\begin{aligned}
\mathcal{A}_4(x) &= \sum_{1\leqslant k\neq j\leqslant n} \frac{\mathbb{P}(\xi_k > x, \xi_j > x)}{\mathbb{P}(\xi_k > x) + \mathbb{P}(\xi_j > x)} (\mathbb{P}(\xi_k > x) + \mathbb{P}(\xi_j > x)) \\
&\leqslant \max_{1\leqslant i\neq j\leqslant n} \frac{\mathbb{P}(\xi_i > x, \xi_j > x)}{\mathbb{P}(\xi_i > x) + \mathbb{P}(\xi_j > x)} \sum_{1\leqslant k\neq l\leqslant n} (\mathbb{P}(\xi_k > x) + \mathbb{P}(\xi_l > x)) \\
&= \max_{1\leqslant i\neq j\leqslant n} \frac{\mathbb{P}(\xi_i > x, \xi_j > x)}{\mathbb{P}(\xi_i > x) + \mathbb{P}(\xi_j > x)}\, 2 \sum_{1\leqslant k< l\leqslant n} (\mathbb{P}(\xi_k > x) + \mathbb{P}(\xi_l > x)) \\
&\leqslant \max_{1\leqslant i\neq j\leqslant n} \frac{\mathbb{P}(\xi_i > x, \xi_j > x)}{\mathbb{P}(\xi_i > x) + \mathbb{P}(\xi_j > x)}\, 2n \sum_{k=1}^n \mathbb{P}(\xi_k > x).
\end{aligned} \tag{3.26}$$

Hence, from (3.24), (3.25) and (3.26), we get

$$\frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^n \mathbb{P}(\xi_k > x)} \geqslant 1 - 2n \max_{1\leqslant i\neq j\leqslant n} \frac{\mathbb{P}(\xi_i > x, \xi_j > x)}{\mathbb{P}(\xi_i > x) + \mathbb{P}(\xi_j > x)},$$

and, therefore,

$$\begin{aligned}
&\liminf_{x\to\infty} \frac{\mathbb{P}(S_n^\xi > x)}{\sum_{k=1}^n \mathbb{P}(\xi_k > x)} \\
&\geqslant 1 - 2n \max_{1\leqslant i\neq j\leqslant n} \left\{ \limsup_{x\to\infty} \frac{\mathbb{P}(\xi_i^+ > x, \xi_j^+ > x)}{\mathbb{P}(\xi_i^+ > x) + \mathbb{P}(\xi_j^+ > x)} \right\} \\
&= 1,
\end{aligned}$$

since r.v.s are pQAI.

Let now $\alpha > 0$. Conditions of the theorem imply $\mathbb{E}(S_n^\xi)^\alpha < \infty$. Thus, by Lemma 2.1 and estimate (2.4), we obtain that for $x > 0$,

$$\begin{aligned}
&\frac{\sum_{k=1}^n \mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\{\xi_k > x\}}\big)}{\mathbb{E}\big((S_n^\xi)^\alpha \mathbb{1}_{\{S_n^\xi > x\}}\big)} \\
&= \frac{\sum_{k=1}^n \big(x^\alpha \mathbb{P}(\xi_k > x) + \alpha \int_x^\infty u^{\alpha-1} \mathbb{P}(\xi_k > u)\mathrm{d}u\big)}{x^\alpha \mathbb{P}(S_n^\xi > x) + \alpha \int_x^\infty u^{\alpha-1}\mathbb{P}(S_n^\xi > u)\mathrm{d}u} \\
&= \frac{x^\alpha \sum_{k=1}^n \mathbb{P}(\xi_k > x) + \alpha \int_x^\infty u^{\alpha-1} \sum_{k=1}^n \mathbb{P}(\xi_k > u)\mathrm{d}u}{x^\alpha \mathbb{P}(S_n^\xi > x) + \alpha \int_x^\infty u^{\alpha-1}\mathbb{P}(S_n^\xi > u)\mathrm{d}u} \\
&\leqslant \max\left\{ \frac{\sum_{k=1}^n \mathbb{P}(\xi_k > x)}{\mathbb{P}(S_n^\xi > x)}, \frac{\int_x^\infty u^{\alpha-1} \sum_{k=1}^n \mathbb{P}(\xi_k > u)\mathrm{d}u}{\int_x^\infty u^{\alpha-1}\mathbb{P}(S_n^\xi > u)\mathrm{d}u} \right\} \\
&= \max\left\{ \frac{\sum_{k=1}^n \mathbb{P}(\xi_k > x)}{\mathbb{P}(S_n^\xi > x)}, \frac{\int_x^\infty u^{\alpha-1} \frac{\sum_{k=1}^n \mathbb{P}(\xi_k > u)}{\mathbb{P}(S_n^\xi > u)} \mathbb{P}(S_n^\xi > u)\mathrm{d}u}{\int_x^\infty u^{\alpha-1}\mathbb{P}(S_n^\xi > u)\mathrm{d}u} \right\} \\
&\leqslant \max\left\{ \frac{\sum_{k=1}^n \mathbb{P}(\xi_k > x)}{\mathbb{P}(S_n^\xi > x)}, \sup_{u\geqslant x} \frac{\sum_{k=1}^n \mathbb{P}(\xi_k > u)}{\mathbb{P}(S_n^\xi > u)} \right\} \\
&= \sup_{u\geqslant x} \frac{\sum_{k=1}^n \mathbb{P}(\xi_k > u)}{\mathbb{P}(S_n^\xi > u)}.
\end{aligned}$$

Therefore, using the fact that estimate in case $\alpha = 0$ is already proved, we get that for any $\alpha > 0$,

$$\limsup_{x\to\infty} \frac{\sum_{k=1}^n \mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\{\xi_k > x\}}\big)}{\mathbb{E}\big((S_n^\xi)^\alpha \mathbb{1}_{\{S_n^\xi > x\}}\big)} \leqslant \limsup_{x\to\infty} \frac{\sum_{k=1}^n \mathbb{P}(\xi_k > x)}{\mathbb{P}(S_n^\xi > x)} \leqslant 1.$$

This finishes the proof of Theorem 3.6. □

Proof of Theorem 3.7. It is easy to see that all conditions of Lemma 3.2 are satisfied, so that it implies $F_{\theta_k\xi_k} \in \mathcal{D}$ for all $k \in \{1, \ldots, n\}$. In addition, since we have that $\max\{\mathbb{E}\theta_1^p, \ldots, \mathbb{E}\theta_n^p\} < \infty$ for some $p > \max\{J_{\xi_1}^+, \ldots, J_{\xi_n}^+\}$, the same lemma implies that $L_{F_{\theta_k\xi_k}} \geqslant L_{F_{\xi_k}}$ for all $k \in \{1, \ldots, n\}$. By Lemma 3.3, we have that for any $k, l \in \{1, \ldots, n\}$, $k \neq l$, r.v.s $\theta_k\xi_k$ and $\theta_l\xi_l$ are QAI. In other words, r.v.s $\theta_1\xi_1, \ldots, \theta_n\xi_n$ are pQAI. Using these observations and condition $\mathbb{E}(\theta_k|\xi_k|)^\alpha < \infty$ for all $k \in \{1, \ldots, n\}$ and some $\alpha \in [0, \infty)$, we apply Theorem 3.4 for r.v.s $\theta_1\xi_1, \ldots, \theta_n\xi_n$ to obtain the desired result. □

4. Applications in risk measure theory

4.1. Brief introduction to risk measure theory

Before presenting results concerning Haezendonck–Goovaerts risk measure, we give a brief introduction to the risk measure theory in general. We shall follow the same definitions and notation as in Artzner et al. [5]. Denote by X the random variable defined on a sample space Ω. And let $\mathcal{G}$ be the set of all risks, that is, the set of all r.v.s defined on Ω.

Definition 4.1. *A measure of risk is a mapping from $\mathcal{G}$ into $\mathbb{R}$.*

Attempting to summarise random variable in one number is clearly not a recent idea and goes back to the time of introduction of expectation, variance (respectively standard deviation), etc. Thus, the history of risk measurement can be traced back to 16th century and even earlier. Many scholars agree, that speaking about risk measurement with modern economic connotation, the first groundbreaking work in this direction is Daniel Bernoulli's "Specimen theoriae novae de mensura sortis" ("Exposition of a new theory on the measurement of risk") [10]. Bernoulli noticed that risk taking depends not only on the risk but also on the risk takers resources. Before Bernoulli there was a wide acceptance of the expected value as a sufficient tool to evaluate the risk. Significant doubts and debates emerged when Nicolaus Bernoulli (cousin of Daniel Bernoulli) proposed famous St Petersburg paradox in 1713. The paradox, historical context and possible resolutions are well presented in O. Peters work [50]. Here we give just a few elements for the basic understanding.

The proposed lottery of N. Bernoulli is as follows. Assume that the player starts the game with unit capital 1. The coin is tossed. Every time the coin shows tails, the capital doubles. The game ends, when heads show up for the first time. Shortly, this means that in this lottery a player wins 2^{n-1} units of money, where n is the number of that toss, when the first heads appeared. For example, if the heads appear on the first try, the player leaves with 1 unit of money. If this happens on the second try, 2 units of money, and so on.

Deciding to play the game or not and what amount of money to pay for the lottery ticket is a risk taking decision. Since the coin is fair, the expected value of this game is $\sum_{n=1}^{\infty} 2^{n-1}(1/2)^n$. It is obvious that this series diverges. N. Bernoulli argued that a rational person should be willing to pay any price for the ticket. But in reality he noticed that people agree to pay only small sums of money. This contradiction constitutes the St. Petersburg paradox.

There were various resolutions proposed for this paradox (see [50]) but the most widely accepted one was proposed by Daniel Bernoulli [10]. Claiming that decision depends not only on the possible gain but also on the person's wealth, he proposed to calculate expectation of gain in "utility" instead of expected win itself. For that, he introduced the so-called utility function $u(w)$. It is required to have the property of concavity and $\mathrm{d}u(w)/\mathrm{d}w$ should be monotonically decreasing. The same win is worth less for a wealthy person than for a poor person. Bernoulli suggested the logarithmic function $u(w) = \ln(w)$ noticing that it would satisfy the equation $\mathrm{d}u/\mathrm{d}w = 1/w$. As it is stated in Peters [50], this means it would cover our intuition that the increase in wealth should correspond to an increase in utility that is inversely proportional to the wealth a person already has. This Bernoulli's response to the paradox was revolutionary at the time and gained quite a big prominence in economics.

With an emergence of a huge variety of financial instruments and portfolios becoming more and more complicated, there was a need for fresh ideas how to measure and understand risk. In the first half of the 20th century, some economists were suggesting to use variance as a measure of portfolio risk. In 1952 Harry Markowitz proposed very compelling theory for portfolio allocation under uncertainty (see [42]). Markowitz's Portfolio Theory can shortly be described as follows. Let's define the weighted expected return of a portfolio (μ_p) as

$$\mu_p = \sum_{i=1}^{N} \omega_i \mu_i,$$

then the portfolio's variance (σ_p^2) is

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} \omega_i \omega_j,$$

where N is the number of assets in a portfolio, μ_i is the expected return of asset i, ω_i is the asset weight ($0 \leqslant \omega_i \leqslant 1$ and $\omega_1 + \omega_2 + \cdots + \omega_N = 1$), σ_{ij} is the covariance between returns of assets i and j. And interesting ratio for measurement of portfolio quality was proposed by William F. Sharpe [59]. This ratio is called Sharpe ratio (S) and defined as

$$S = \frac{\mu_p - R_f}{\sigma_p},$$

where R_f is the risk-free rate of return, μ_p and σ_p are as defined in Markowitz's Portfolio Theory description.

Further in this thesis we will not refer to Markowitz's Portfolio Theory or Sharpe ratio anymore. We mentioned them here because of their significance to development of modern financial mathematics and risk measure theory.

Next important milestone in risk measurement was achieved by the introduction of Value at Risk (VaR) risk measure. To define VaR, we have to understand the concept of quantiles. We find the following definition given in Petrov [51, p. 9].

Definition 4.2. *Let X be a random variable, $q \in (0,1)$. A quantile of order q of the random variable X is called any number κ_q satisfying inequalities*

$$\mathbb{P}(X \leqslant \kappa_q) \geqslant q, \quad \mathbb{P}(X \geqslant \kappa_q) \geqslant 1 - q.$$

It is noted that either the random variable has just one quantile of order q, or the set of all quantiles of order q of this random variable coincides with some closed interval on the real line. If the distribution function of this random variable is strictly increasing on the real line, the r.v. X has just one quantile of arbitrary order q.

We defined the risk measure as a function which assigns only one real number to r.v. X. If the random variable has just one quantile of order q, the situation is clear. If the set of such quantiles of order q is a closed interval, we take the left endpoint of this interval as a value of VaR. Formally, we define VaR risk measure as follows.

Definition 4.3. *Given $q \in (0,1)$, the Value at Risk risk measure at level q (VaR_q) of random variable X with distribution function F_X is defined as the lower quantile of order q of distribution F_X:*

$$\mathrm{VaR}_q[X] = \inf\{x \in \mathbb{R} : F_X(x) \geqslant q\}.$$

In practical applications, typically q is chosen to be 0.9, 0.95 and 0.99. For discussions about VaR implementation, we refer the reader to Mausser and Rosen [43], [44] and the references therein.

In 1997 and 1999 a team of authors, P. Artzner, F. Delbaen, J.M. Eber and D. Heath, published two very important papers (see [4] and [5]), where they proposed and justified a unified framework for the analysis, construction and implementation of measures of risk. The approach to risk measurement became far less fragmented and to this day specialists in the field use elements of this theory to decide which risk measures are effective in managing and regulating risks, and which better should be changed or discarded. Most importantly, authors introduced the first axioms of risk measures. These axioms are called coherency axioms and risk measures satisfying these axioms are called coherent risk measures. Axioms on risk measures are quite intuitive to understand, however, before that acceptance sets were introduced, which Artzner et al. [5] claim to be the "fundamental object" in risk measurement. Let's start with a definition.

Definition 4.4. *An acceptance set $\mathcal{A} \subset \mathcal{G}$ is any set of risks that are acceptable to a regulator or supervisor. It satisfies four axioms:*

Axiom A1. $L_+ \subset \mathcal{A}$, *where* $L_+ = \{X \in \mathcal{G} : X \text{ is nonnegative}\}$.

Axiom A2. $\mathcal{A} \cap L_{--} = \varnothing$,
where $L_{--} = \{X \in \mathcal{G} : X \text{ is strictly negative random variable}\}$.

Axiom A3. *$\mathcal{A}$ is convex, that is, for all r.v.s X and Y in $\mathcal{A}$, r.v. $(1-\lambda)X + \lambda Y \in \mathcal{A}$ for all $\lambda \in [0,1]$.*

Axiom A4. *$\mathcal{A}$ is positively homogenious cone, that is, for each r.v. $X \in \mathcal{A}$ and $\lambda > 0$, the product $\lambda X \in \mathcal{A}$.*

Sometimes Axiom A2 is changed by a stronger axiom:

Axiom A2'. $\mathcal{A} \cap L_- = \{o\}$, where $L_- = \{X \in \mathcal{G} : X \text{ is not positive}\}$ and o is a r.v. degenerate at 0.

Axioms A1, A2 and A2' are easy to interpret. Positions with final net worth that is always nonnegative do not require extra capital and are acceptable. Positions with a net worth that is always negative are not acceptable. Axiom A2' is stricter in a sense that it does not allow any positions that have at least one outcome with negative net worth. As stated by authors, Axiom A3 reflects risk aversion of the regulator or supervisor, but they didn't give explicit interpretation for Axiom A4. We just find hints that, if currencies are changed, acceptable positions should remain acceptable. Without the cone property that wouldn't necessarily be guaranteed.

Let us remember that a risk measure ρ is any function from $\mathcal{G}$ into $\mathbb{R}$. Take $X \in \mathcal{G}$. If $\rho(X) > 0$, the number $\rho(X)$ can be interpreted as the minimum extra cash that has to be invested into "prudent" instrument and added to the risky position X, so that it would become acceptable. If $\rho(X) < 0$, then cash amount $-\rho(X)$ can be withdrawn from the position X. For the sake of clarity, we would like to note that in this thesis we will distance ourselves from such interpretations and simply analyse mathematical properties of risk measures.

Following the same paper by Artzner et al. [5], we define a correspondence between acceptance sets and measures of risk.

Definition 4.5. *Given the total rate of return r on a reference instrument, the risk measure associated with the acceptance set $\mathcal{A}$ is the mapping from $\mathcal{G}$ to $\mathbb{R}$ denoted by $\rho_{\mathcal{A},r}$ and defined by*

$$\rho_{\mathcal{A},r}(X) = \inf\{m : X + mr \in \mathcal{A}\}.$$

Reference instrument is an asset having the initial price 1 and a strictly positive price r in any scenario at date T.

Definition 4.6. *The acceptance set associated with a risk measure ρ is the set denoted by $\mathcal{A}_\rho$ and defined by*

$$\mathcal{A}_\rho = \{X \in \mathcal{G} : \rho(X) \leqslant 0\}.$$

Now let's turn to axioms stated above risk measure $\rho : \mathcal{G} \to \mathbb{R}$. As mentioned earlier, these axioms are called coherency axioms and are crucial in determining whether a risk measure is appropriate for practical use or should be taken with extreme caution. The first requirement is translation invariance.

Axiom T. Translation invariance. For all $X \in \mathcal{G}$ and all $c \in \mathbb{R}$,

$$\rho(X + c) = \rho(X) - c.$$

The second requirement is subadditivity. Speaking in economic terms, it ensures that risk measure is sensitive to diversification.

Axiom S. Subadditivity. For all $X_1, X_2 \in \mathcal{G}$,

$$\rho(X_1 + X_2) \leqslant \rho(X_1) + \rho(X_2).$$

Furthermore, Artzner et al. argues that this property of merger not creating extra risk is a natural requirement by giving substantial examples. Also, it can be used as a way for a firm to allocate its capital among desks or managers. Let's suppose that two desks in a firm compute in a decentralized way the measures $\rho(X_1)$ and $\rho(X_2)$ of the risks they have taken. If the function ρ is subadditive, the supervisor of the two desks can count on the fact that $\rho(X_1) + \rho(X_2)$ is a feasible guarantee relative to the global risk $X_1 + X_2$. If there is an amount m of cash available for their joint business, the supervisor can decentralize his cash constraint into two cash constraints (one per desk) m_1 and m_2: $m = m_1 + m_2$.

The remaining two axioms are very natural to insist on.

Axiom PH. Positive homogeneity. For all $\lambda \geqslant 0$ and all $X \in \mathcal{G}$,

$$\rho(\lambda X) = \lambda \rho(X).$$

Axiom M. Monotonicity. For all $X_1, X_2 \in \mathcal{G}$ with $X_1 \leqslant X_2$,

$$\rho(X_1) \geqslant \rho(X_2).$$

Now we can formally define a coherent risk measure.

Definition 4.7. *A risk measure satisfying the four axioms of translation invariance, subadditivity, positive homogeneity, and monotonicity is called coherent.*

Sometimes the notion of convexity is considered in the literature, too.

Axiom C. Convexity. For all $X_1, X_2 \in \mathcal{G}$,

$$\rho(\lambda X_1 + (1-\lambda)X_2) \leqslant \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$$

for $0 \leqslant \lambda \leqslant 1$.

It is said that a risk measure is weakly coherent if it is convex (satisfies Axiom C), translationally invariant and homogeneous. Coherency axioms ensure that a risk measure is convex. So it is obvious that every coherent measure is, also, weakly coherent.

It is obvious that axioms proposed by Artzener et al. are not restrictive enough to specify a unique risk measure. Instead, they characterise a large class of risk measures. Financial and economic context should be studied before selecting a specific risk measure from the class. We do not study this topic deeper not wanting to overcomplicate this introduction and go too far from theoretical boundaries of the thesis.

After introduction of the axioms, VaR, which we mentioned above, was no longer considered an adequate risk measure because it does not satisfy subadditivity axiom in general. In economic terms, managing risk using VaR may not stimulate diversification. So there were proposed VaR related risk measures: Conditional Value at Risk (CVaR) (see discussion below), Conditional Tail Expectation (CTE), Worst Conditional Expectation (WCE) (see, for example, Artzner et al. [5]) among others.

One risk measure that has significant advantages over VaR is Conditional Value at Risk (CVaR). Here we define it as it is given in Pflug [52]. Originally definition of this style was proposed by Rockafellar and Uryasev [57] in a continuous framework, and by Rockafellar and Uryasev [58] in a general framework.

Definition 4.8. *Given* $q \in (0,1)$*, Conditional Value at Risk risk measure at level* q *(*CVaR_q*) of random variable* X *is defined as follows:*

$$\mathrm{CVaR}_q[X] = \inf_{x\in\mathbb{R}} \left\{ x + \frac{1}{1-q}\mathbb{E}(X-x)^+ \right\}.$$

Rockafellar and Uryasev have shown that in the case when F_X is continuous,

$$\mathrm{CVaR}_q[X] = \mathbb{E}[X | X > \mathrm{VaR}_q(X)], \tag{4.1}$$

which was the usual definition of CVaR at the time. Note that (4.1) is sometimes called “expected shortfall” (see, for example, Mausser and Rosen [44]).

A very important advantage of CVaR (compared to VaR) is that it is coherent risk measure (see Pflug [52], Acerbi et al. [1], Rockafellar and Uryasev [58]). Another advantage can be seen directly from the definitions. VaR is a lower bound for values in the tail of the distribution, and, as Rockafellar and Uryasev [58] put it, "has a bias toward optimism instead of conservatism that ought to prevail in risk management". Indeed, obviously, for a chosen $q \in (0,1)$, $\mathrm{CVaR}_q \geqslant \mathrm{VaR}_q$. Also, it is mentioned in the same paper that VaR is unstable and difficult to work with numerically when losses are not normally distributed, which is often the case, since loss distributions are oftentimes heavy-tailed. Similar remark in the context of credit risks is made by Mausser and Rosen [45] – distributions of credit losses are claimed to be heavy-tailed in general.

We conclude this introduction by quoting two important propositions, which establish correspondence between the axioms on acceptance sets and the axioms on risk measures.

Proposition 4.1 (Artzner et al. [5, Proposition 2.1])**.** *If the set $\mathcal{B}$ satisfies Axioms A1, A2, A3, and A4, the risk measure $\rho_{\mathcal{B},r}$ is coherent. Moreover, $\mathcal{A}_{\rho_{\mathcal{B},r}} = \overline{\mathcal{B}}$, where $\overline{\mathcal{B}}$ is a closure of set $\mathcal{B}$.*

Proposition 4.2 (Artzner et al. [5, Proposition 2.2])**.** *If a risk measure ρ is coherent, then the acceptance set $\mathcal{A}_\rho$ is closed and satisfies Axioms A1, A2, A3, and A4. Moreover, $\rho = \rho_{\mathcal{A}_\rho,r}$.*

In the next subsection we introduce a particular coherent risk measure, which is in the focus of our further study.

4.2. Introduction of the Haezendonck–Goovaerts risk measure

In this subsection we introduce Haezendonck–Goovaerts (HG) risk measure, which has some connections with previously mentioned risk measures VaR and CVaR. HG risk measure was introduced by Haezendonck and Goovaerts [34] in 1982 and since then received much attention in insurance and finance. Original definition was extended in Goovaerts et al. [32], where the authors consider real-valued, but not necessarily nonnegative, random variables. In the latter article, this risk measure was called the Haezendonck risk measure in honor of the late J. Haezendonck. It was only later that it came to be called the Haezendonck–Goovaerts risk measure, so that contribution of both authors would be acknowledged. A useful alternative formulation of the definition was introduced by Bellini and Rosazza Gianin [6]. Many authors since then use their style – it is the case in this thesis, too. The proposal of this formulation was motivated by coherence problem. Goovaerts et al. [32] solved it only partially – they proved that HG risk measure satisfies all coherence axioms, except that subadditivity was proved only for "special" pairs of r.v.s (see [32, Theorem 3.2]). Even though restriction is very mild, the full treatment of the problem was still missing. Using an alternative formulation, Bellini and Rosazza Gianin [6] showed that subadditivity holds for any pair of random variables within the domain L^∞. Thus, the HG risk measure is coherent. Definition and respective proposition for coherence in Orlicz spaces was given by the same authors in 2012 (see [8]). It is natural to expect coherence and other "nice" properties from HG risk measure, as it is considered a generalisation of CVaR, which is known to have such properties since early works by Pflug [52], Acerbi et al. [1], Rockafellar and Uryasev [58]. We remark that Orlicz premium principle (risk measure), though closely related to HG risk measure, is not coherent. It doesn't satisfy translation invariance axiom (Axiom T). For definition, statement and comments, see the paper of Goovaerts et al. [31].

Stability properties of HG risk measure were investigated in a recent paper by Gao et al. [28]. They proved that it always satisfies the Fatou property and established necessary and sufficient condition so that a stronger Lebesgue property would be satisfied.

Possible economic interpretations of Haezendonck–Goovaerts risk measure are discussed in the works by Bellini and Rosazza Gianin [6], [7]. Several intersting relations between distortion, mean value and HG risk measures can be found in the paper by Goovaerts et al. [33].

Now, after introductory discussion, let's formally define Haezendonck–Goovaerts risk measure. We will follow the definitions given by Tang and Yang [63].

A function φ (defined on $\mathbb{R}$) is said to be a normalized Young function if φ is nonnegative, convex on the interval $[0, \infty)$ with $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(\infty) = \infty$. Orlicz space L^φ and the Orlicz heart L_0^φ of real-valued r.v.s X associated with function φ are defined by the following equalities:

$$L^{\varphi} = \{X : \mathbb{E}[\varphi(cX)] < \infty \text{ for some } c > 0\},$$
$$L_0^{\varphi} = \{X : \mathbb{E}[\varphi(cX)] < \infty \text{ for all } c > 0\}.$$

Let φ be a Young function and $X \in L_0^{\varphi}$. For $q \in (0,1)$, the Haezendonck–Goovaerts (HG) risk measure for variable X is defined as

$$H_q[X] = \inf_{x\in\mathbb{R}} \big(x + H_q[X,x]\big),$$

where $H_q[X,x]$ is a unique solution h of the equation

$$\mathbb{E}\,\varphi\left(\frac{(X-x)^+}{h}\right) = 1-q$$

if $\overline{F}_X(x) > 0$, and $H_q[X,x] = 0$ if $\overline{F}_X(x) = 0$.

It is mentioned in Tang and Yang [63] that an analytic expression for risk measure $H_q[X]$ is not possible in general. Although, in the case of power function $\varphi(t) = t^{\varkappa}$ with $\varkappa \geqslant 1$, the analytic expressions and asymptotic formulas can be derived for certain type of distributions. The following theorem is based on equality (1.3) and Theorem 2.1 from Tang and Yang [63]. It is later used to obtain asymptotic formulas for the HG risk measure of sums S_n^{ξ}.

Theorem 4.1 (Tang, Yang [63, equality (1.3) and Theorem 2.1])**.** *Consider the power Young function $\varphi(t) = t^{\varkappa}$ with $\varkappa \geqslant 1$.*
(i) *If $\varkappa = 1$ and r.v. X is such that $\mathbb{E}X^+ < \infty$, then*

$$H_q[X] = F_X^{\leftarrow}(q) + \frac{\mathbb{E}\big(X - F_X^{\leftarrow}(q)\big)^+}{1-q} = \mathrm{CVaR}_q[X],$$

where $F_X^{\leftarrow}(q) := \inf\{x \in \mathbb{R} : F_X(x) \geqslant q\}$ is the quantile function of r.v. X.
(ii) *If $\varkappa > 1$, $\mathbb{P}(X = F_X^{\leftarrow}(q)) = 0$ and $\mathbb{E}(X^+)^{\varkappa} < \infty$, then*

$$H_q[X] = x + \left(\frac{\mathbb{E}((X-x)^+)^{\varkappa}}{1-q}\right)^{1/\varkappa}, \quad q \in (0,1),$$

where $x = x(q) \in (-\infty, F_X^{\leftarrow}(1))$ is the unique solution to the equation

$$\frac{\big(\mathbb{E}((X-x)^+)^{\varkappa-1}\big)^{\varkappa}}{\big(\mathbb{E}((X-x)^+)^{\varkappa}\big)^{\varkappa-1}} = 1-q. \tag{4.2}$$

4.3. Asymptotic formulas for the HG risk measure. Pareto example

Let r.v.s $\xi_1, \dots, \xi_n$ be pQAI. Suppose that, for each k, r.v. ξ_k is distributed according to the Pareto law (see Section 1.5.1), i.e.,

$$F_{\xi_k}(x) = \left(1 - \left(\frac{k}{x}\right)^{\alpha}\right)\mathbb{1}_{[k,\infty)}(x), \quad k = 1, \dots, n$$

with shape parameter $\alpha > 1$.

In this subsection we derive asymptotic formulas for the HG risk measures $H_q\big[S_n^{\xi}\big]$, as $q \uparrow 1$, with the power generating functions $\varphi(t) = t^{\varkappa}$, $\varkappa \in \{1, 2, \dots\}$. To achieve this, we need the following lemma about the quantile function (see, for example, [18, Lemma 3] or [55, Proposition 0.8(vi)]).

Lemma 4.1 (De Haan [18, Lemma 3]; Resnick [55, Proposition 0.8(vi)]). *Let F and G be two d.f.s such that $G \in \mathcal{R}_{\alpha}$, $\alpha > 0$. Then $\overline{F}(x) \underset{x\to\infty}{\sim} c\,\overline{G}(x)$ for a positive c if and only if $F^{\leftarrow}(q) \underset{q\uparrow 1}{\sim} c^{1/\alpha} G^{\leftarrow}(q)$.*

Firstly, consider the case $\kappa = 1$. For $k \in \{1, \dots, n\}$ and $x > k$, we have

$$\mathbb{E}\big(\xi_k \mathbb{1}_{\{\xi_k > x\}}\big) = \int_x^{+\infty} u \mathrm{d}F_{\xi_k}(u) = \frac{\alpha}{\alpha - 1}\frac{k^{\alpha}}{x^{\alpha-1}}.$$

Since in Pareto case $L_{F_{\xi_k}} = 1$ for all $k \in \{1, \dots, n\}$, applying Theorem 3.4, we get

$$\mathbb{E}\big(S_n^{\xi}\mathbb{1}_{\{S_n^{\xi} > x\}}\big) \underset{x\to\infty}{\sim} \sum_{k=1}^{n} \mathbb{E}\big(\xi_k \mathbb{1}_{\{\xi_k > x\}}\big) \underset{x\to\infty}{\sim} \frac{1}{x^{\alpha-1}}\frac{\alpha}{\alpha-1}\sum_{k=1}^{n} k^{\alpha},$$
$$\mathbb{P}\big(S_n^{\xi} > x\big) \underset{x\to\infty}{\sim} \sum_{k=1}^{n} \mathbb{P}\big(\xi_k > x\big) \underset{x\to\infty}{\sim} \frac{1}{x^{\alpha}}\sum_{k=1}^{n} k^{\alpha}. \tag{4.3}$$

These two asymptotic relations imply that

$$\mathbb{E}\big(S_n^{\xi} - x\big)^{+} = \mathbb{E}\Big((S_n^{\xi} - x)\mathbb{1}_{\{S_n^{\xi} > x\}}\Big) = \mathbb{E}\Big(S_n^{\xi}\mathbb{1}_{\{S_n^{\xi} > x\}}\Big) - x\mathbb{P}\big(S_n^{\xi} > x\big)$$
$$\underset{x\to\infty}{\sim} \frac{1}{x^{\alpha-1}}\frac{1}{\alpha-1}\sum_{k=1}^{n} k^{\alpha}.$$

Hence, using the formula in Theorem 4.1(i), we get

$$H_q\big[S_n^\xi\big] = F^{\leftarrow}_{S_n^\xi}(q) + \frac{\mathbb{E}\Big(S_n^\xi - F^{\leftarrow}_{S_n^\xi}(q)\Big)^+}{1-q}$$

$$\underset{q\uparrow 1}{\sim} F^{\leftarrow}_{S_n^\xi}(q) + \frac{1}{1-q}\frac{1}{\alpha-1}\big(F^{\leftarrow}_{S_n^\xi}(q)\big)^{1-\alpha}\sum_{k=1}^{n} k^\alpha$$

because $F^{\leftarrow}_{S_n^\xi}(q) \to \infty$ if and only if $q \uparrow 1$. Here, by Lemma 4.1 and relation (4.3),

$$F^{\leftarrow}_{S_n^\xi}(q) \underset{q\uparrow 1}{\sim} \Big(\sum_{k=1}^{n} k^\alpha\Big)^{1/\alpha} F^{\leftarrow}_{\xi_1}(q) = \Big(\sum_{k=1}^{n} k^\alpha\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha}$$

and we get in the case $\varphi(t) = t$ that

$$H_q\big[S_n^\xi\big] \underset{q\uparrow 1}{\sim} \frac{\alpha}{\alpha-1}\Big(\sum_{k=1}^{n} k^\alpha\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha}. \tag{4.4}$$

Consider now the case $\varkappa \in \{2, 3, \dots\}$. By equation (2.2) in Lemma 2.1 and definition of Pareto distribution, we have for all $j = 0, 1, \dots, \varkappa$, $\varkappa < \alpha$ and $k = 1, \dots, n$ that

$$\mathbb{E}\left(\xi_k^j \mathbb{1}_{\{\xi_k > x\}}\right) = \frac{1}{x^{\alpha-j}}\frac{\alpha k^\alpha}{\alpha-j}, \quad x \geqslant k. \tag{4.5}$$

If $\alpha > \varkappa$, then the asymptotics of the HG risk measure can be derived using Theorem 3.4. The binomial theorem and properties of expectation imply that

$$\mathbb{E}\left(\left(S_n^\xi - x\right)^\varkappa \mathbb{1}_{\{S_n^\xi > x\}}\right) = \mathbb{E}\left(\sum_{j=0}^{\varkappa}\binom{\varkappa}{j}(-x)^{\varkappa-j}\left(S_n^\xi\right)^j \mathbb{1}_{\{S_n^\xi > x\}}\right)$$

$$= \sum_{j=0}^{\varkappa}\binom{\varkappa}{j}(-x)^{\varkappa-j}\mathbb{E}\left(\left(S_n^\xi\right)^j \mathbb{1}_{\{S_n^\xi > x\}}\right). \tag{4.6}$$

By Theorem 3.4, we have

$$\sum_{j=0}^{\varkappa}\binom{\varkappa}{j}(-x)^{\varkappa-j}\mathbb{E}\left(\left(S_n^\xi\right)^j \mathbb{1}_{\{S_n^\xi > x\}}\right)$$

$$\underset{x\to\infty}{\sim} \sum_{j=0}^{\varkappa}\binom{\varkappa}{j}(-x)^{\varkappa-j}\sum_{k=1}^{n}\mathbb{E}\left(\xi_k^j \mathbb{1}_{\{\xi_k > x\}}\right)$$

$$\underset{x\to\infty}{\sim} \frac{1}{x^{\alpha-\varkappa}}\sum_{k=1}^{n} k^\alpha \sum_{j=0}^{\varkappa}\binom{\varkappa}{j}(-1)^{\varkappa-j}\frac{\alpha}{\alpha-j}$$

$$= \frac{1}{x^{\alpha-\varkappa}}\varkappa \mathrm{B}(\varkappa, \alpha-\varkappa)\sum_{k=1}^{n} k^\alpha, \tag{4.7}$$

where we have also used (4.5) and then the identity

$$\sum_{j=0}^{\varkappa}\binom{\varkappa}{j}(-1)^{\varkappa-j}\frac{\alpha}{\alpha-j}=\varkappa\mathrm{B}(\varkappa,\alpha-\varkappa).$$

Here, $\mathrm{B}(u,v)$ denotes the Beta function. The last identity can be verified using equality

$$\varkappa\mathrm{B}(\varkappa,\alpha-\varkappa)=\alpha\mathrm{B}(\varkappa+1,\alpha-\varkappa)=\alpha\int_0^1(1-t)^{\varkappa}t^{\alpha-\varkappa-1}\mathrm{d}t. \tag{4.8}$$

Thus, from (4.6) and (4.7), we get

$$\mathbb{E}\left(\left(S_n^\xi-x\right)^{\varkappa}\mathbb{1}_{\{S_n^\xi>x\}}\right)\underset{x\to\infty}{\sim}\frac{1}{x^{\alpha-\varkappa}}\varkappa\mathrm{B}(\varkappa,\alpha-\varkappa)\sum_{k=1}^{n}k^{\alpha}. \tag{4.9}$$

In order to get asymptotic formula for $H_q\left[S_n^\xi\right]$, we have to approximate x that solves equation (4.2). Using (4.9), we obtain

$$\begin{aligned}
1-q&=\frac{\left(\mathbb{E}((S_n^\xi-x)^+)^{\varkappa-1}\right)^{\varkappa}}{\left(\mathbb{E}((S_n^\xi-x)^+)^{\varkappa}\right)^{\varkappa-1}}\\
&\underset{x\to\infty}{\sim}\frac{\left(\frac{1}{x^{\alpha-(\varkappa-1)}}(\varkappa-1)\mathrm{B}(\varkappa-1,\alpha-(\varkappa-1))\sum_{k=1}^{n}k^{\alpha}\right)^{\varkappa}}{\left(\frac{1}{x^{\alpha-\varkappa}}\varkappa\mathrm{B}(\varkappa,\alpha-\varkappa)\sum_{k=1}^{n}k^{\alpha}\right)^{\varkappa-1}}\\
&=\frac{1}{x^{\alpha}}\frac{(\alpha\mathrm{B}(\varkappa,\alpha-\varkappa+1))^{\varkappa}}{(\varkappa\mathrm{B}(\varkappa,\alpha-\varkappa))^{\varkappa-1}}\sum_{k=1}^{n}k^{\alpha}\\
&=\frac{1}{x^{\alpha}}\frac{(\mathrm{B}(\varkappa,\alpha-\varkappa)(\alpha-\varkappa))^{\varkappa}}{(\varkappa\mathrm{B}(\varkappa,\alpha-\varkappa))^{\varkappa-1}}\sum_{k=1}^{n}k^{\alpha}\\
&=\frac{1}{x^{\alpha}}\left(\frac{\alpha-\varkappa}{\varkappa}\right)^{\varkappa}\varkappa\mathrm{B}(\varkappa,\alpha-\varkappa)\sum_{k=1}^{n}k^{\alpha},
\end{aligned} \tag{4.10}$$

where we have applied (4.8) again (for $\varkappa-1$) and identity $\mathrm{B}(u,v+1)=\mathrm{B}(u,v)\frac{v}{u+v}$.

Since $x=x(q)\to\infty$ if and only if $q\uparrow 1$, from (4.10), we obtain that

$$x\underset{q\uparrow 1}{\sim}\left(\frac{\alpha-\varkappa}{\varkappa}\right)^{\varkappa/\alpha}\left(\varkappa\mathrm{B}(\varkappa,\alpha-\varkappa)\right)^{1/\alpha}\left(\sum_{k=1}^{n}k^{\alpha}\right)^{1/\alpha}\left(\frac{1}{1-q}\right)^{1/\alpha}.$$

By Theorem 4.1(ii) and relation (4.9),

$$
\begin{aligned}
H_q\big[S_n^\xi\big] &= x + \big(\mathbb{E}\,\big((S_n^\xi - x)^+\big)^{\varkappa}\big)^{1/\varkappa}\Big(\frac{1}{1-q}\Big)^{1/\varkappa} \\
&\underset{q\uparrow 1}{\sim} \Big(\frac{\alpha-\varkappa}{\varkappa}\Big)^{\varkappa/\alpha}\big(\varkappa \mathrm{B}(\varkappa,\alpha-\varkappa)\big)^{1/\alpha}\Big(\sum_{k=1}^{n} k^{\alpha}\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha} \\
&\quad + \frac{\big(\varkappa \mathrm{B}(\varkappa,\alpha-\varkappa)\big)^{1/\varkappa}\Big(\sum_{k=1}^{n} k^{\alpha}\Big)^{1/\varkappa}\Big(\frac{1}{1-q}\Big)^{1/\varkappa}}{\Big(\Big(\frac{\alpha-\varkappa}{\varkappa}\Big)^{\varkappa/\alpha}\big(\varkappa \mathrm{B}(\varkappa,\alpha-\varkappa)\big)^{1/\alpha}\Big(\sum_{k=1}^{n} k^{\alpha}\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha}\Big)^{\frac{\alpha-\varkappa}{\varkappa}}}.
\end{aligned}
$$

Thus, performing elementary mathematical operations, we have

$$
\begin{aligned}
H_q\big[S_n^\xi\big] &= \Big(\frac{\alpha-\varkappa}{\varkappa}\Big)^{\varkappa/\alpha}\big(\varkappa \mathrm{B}(\varkappa,\alpha-\varkappa)\big)^{1/\alpha}\Big(\sum_{k=1}^{n} k^{\alpha}\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha} \\
&\quad + \Big(\frac{\alpha-\varkappa}{\varkappa}\Big)^{\varkappa/\alpha-1}\big(\varkappa \mathrm{B}(\varkappa,\alpha-\varkappa)\big)^{1/\alpha}\Big(\sum_{k=1}^{n} k^{\alpha}\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha} \\
&= \Big(\frac{\alpha-\varkappa}{\varkappa}\Big)^{\varkappa/\alpha}\frac{\alpha}{\alpha-\varkappa}\big(\varkappa \mathrm{B}(\varkappa,\alpha-\varkappa)\big)^{1/\alpha}\Big(\sum_{k=1}^{n} k^{\alpha}\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha} \\
&= \frac{\alpha(\alpha-\varkappa)^{\varkappa/\alpha-1}}{\varkappa^{\varkappa/\alpha}}\big(\varkappa \mathrm{B}(\varkappa,\alpha-\varkappa)\big)^{1/\alpha}\Big(\sum_{k=1}^{n} k^{\alpha}\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha}.
\end{aligned}
$$

Therefore, the following asymptotic formula holds for generating function $\varphi(t) = t^{\varkappa}$, with $\varkappa \in \mathbb{N}$, $\varkappa < \alpha$:

$$
H_q\big[S_n^\xi\big] \underset{q\uparrow 1}{\sim} \frac{\alpha(\alpha-\varkappa)^{\varkappa/\alpha-1}}{\varkappa^{\varkappa/\alpha}}\big(\varkappa \mathrm{B}(\varkappa,\alpha-\varkappa)\big)^{1/\alpha}\Big(\sum_{k=1}^{n} k^{\alpha}\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha}. \tag{4.11}
$$

Notice that formula (4.11) covers the case $\varkappa = 1$ given by relation (4.4). Indeed, plugging in $\varkappa = 1$ and using identity $\mathrm{B}(1,x) = 1/x$, we have

$$
\begin{aligned}
H_q\big[S_n^\xi\big] &\underset{q\uparrow 1}{\sim} \alpha(\alpha-1)^{1/\alpha-1}\big(\mathrm{B}(1,\alpha-1)\big)^{1/\alpha}\Big(\sum_{k=1}^{n} k^{\alpha}\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha} \\
&= \alpha(\alpha-1)^{1/\alpha-1}\Big(\frac{1}{\alpha-1}\Big)^{1/\alpha}\Big(\sum_{k=1}^{n} k^{\alpha}\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha} \\
&= \frac{\alpha}{\alpha-1}\Big(\sum_{k=1}^{n} k^{\alpha}\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha}.
\end{aligned}
$$

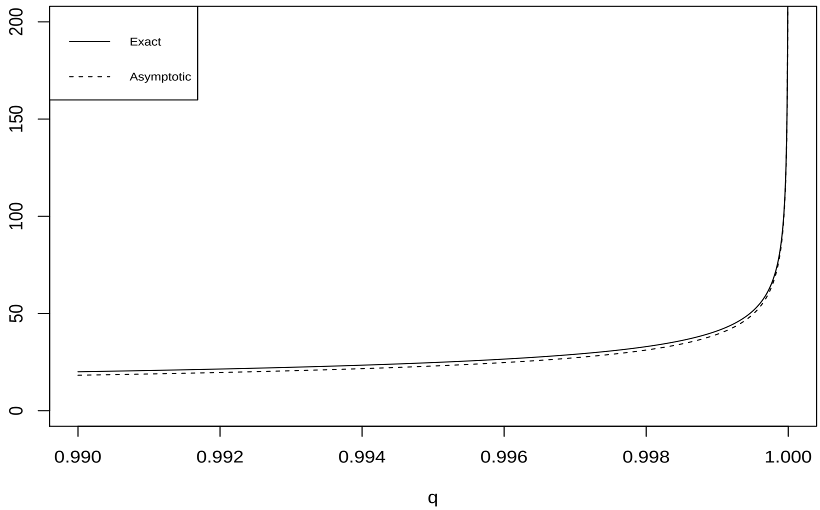

Figure 4.1: The exact and asymptotic values of the HG risk measure for sum S_n^ξ of independent r.v.s in the case $\alpha = 3$, $\varkappa = 2$ and $n = 2$.

In Figure 4.1, we compare the exact and asymptotic values of the HG risk measure for sum S_n^ξ of independent r.v.s in the case $\alpha = 3$, $\varkappa = 2$ and $n = 2$. The exact values are obtained by using Theorem 4.1. In our case, equation (4.2) has the form

$$\frac{\big(\mathbb{E}(S_2^\xi - x)^+\big)^2}{\mathbb{E}\big((S_2^\xi - x)^+\big)^2} = \frac{\left(\int_2^\infty \int_{\max\{1,x-y\}}^\infty (u+y-x)\,\frac{72}{u^4y^4}\,\mathrm{d}u\,\mathrm{d}y\right)^2}{\int_2^\infty \int_{\max\{1,x-y\}}^\infty (u+y-x)^2\,\frac{72}{u^4y^4}\,\mathrm{d}u\,\mathrm{d}y} = 1-q.$$

After calculations and some simplifications, we obtain the following equation:

$$\frac{3\Big(\frac{-3x^4-x^3-12x^2+104x-96}{2x^2(x-1)(2-x)} + \frac{16}{x^3}\big(\log|2-x| + \log\big|\frac{x-1}{2}\big|\big)\Big)^2}{3x^3+5x^2+12x+8\big(\log|2-x| + \log\big|\frac{x-1}{2}\big|\big)} = 1-q,$$

where $x \notin \{0,1,2\}$. To get exact values of HG risk measure, we solve this equation for different values of $q \in (0,1)$ by using computer software and substitute the obtained solutions $x = x(q)$ into expression of Theorem

4.1(ii), which in our case has the form

$$H_q\big[S_2^\xi\big] = x + \left(\frac{9x^2+15x+36}{x^3} + \frac{24}{x^4}\Big(\log|2-x| + \log\Big|\frac{x-1}{2}\Big|\Big)\right)^{1/2} \times \left(\frac{1}{1-q}\right)^{1/2}.$$

4.4. Asymptotic formulas for the HG risk measure. Peter and Paul example

Let r.v.s $\xi_1, \dots, \xi_n$ be pQAI. Suppose that, for each k, r.v. ξ_k is distributed according to the generalised Peter and Paul law (see Section 1.5.2), i.e.,

$$\mathbb{P}(\xi_k = a_k^{-j\beta}) = a_k^{j-1}(1-a_k),\ j = 1, 2, \dots,$$

where $a_k \in (0,1)$ and $\beta > 0$.

We will derive the asymptotic bounds for the HG risk measure $H_q[S_n^\xi]$ assuming the Young function $\varphi(t) = t^2$.

According to Theorem 4.1, the asymptotics of the risk measure $H_q[S_n^\xi]$ as $q \uparrow 1$ is related to the asymptotic behaviour of moments $\mathbb{E}(S_n^\xi - x)^+$ and $\mathbb{E}\big((S_n^\xi - x)^+\big)^2$ as $x \to \infty$. These moments admit the integral representation in Lemma 2.1 equation (2.3), i.e., depend on tail probabilities $\mathbb{P}(S_n^\xi > u)$, whose behaviour can be established applying Theorems 3.4 and 3.6.

For $x > 0$, we have

$$\mathbb{P}(\xi_k > x) = \sum_{j:\, a_k^{-j\beta} > x} a_k^{j-1}(1-a_k) \;=\; a_k^{\left\lfloor \frac{\log x}{\beta \log(1/a_k)} \right\rfloor}. \tag{4.12}$$

Hence, for each k, $L_{F_{\xi_k}} = a_k$ and the following inequalities are true:

$$\left(\frac{1}{x}\right)^{1/\beta} \leqslant a_k^{\left\lfloor \frac{\log x}{\beta \log(1/a_k)} \right\rfloor} < \frac{1}{a_k}\left(\frac{1}{x}\right)^{1/\beta}. \tag{4.13}$$

One can prove these inequalities in the following way. For any real number $z \geqslant 0$, it is true that $\lfloor z \rfloor = z - \{z\}$, where $\{z\}$ is a fractional part of z, satisfying $0 \leqslant \{z\} < 1$. Thus, $z - 1 < \lfloor z \rfloor \leqslant z$, and, since $a_k \in (0,1)$ for all $k \in \{1, \dots, n\}$,

$$a_k^{z-1} > a_k^{\lfloor z \rfloor} \geqslant a_k^z.$$

Applying these inequalities for the tail of generalised Peter and Paul distribution, we get

$$a_k^{\left\lfloor \frac{\log x}{\beta \log(1/a_k)} \right\rfloor} \geqslant a_k^{\frac{\log x}{\beta \log(1/a_k)}} = \mathrm{e}^{\log a_k^{\frac{\log x}{\beta \log(1/a_k)}}} = \mathrm{e}^{\frac{\log x}{\beta \log(1/a_k)} \log a_k}$$
$$= \mathrm{e}^{-\frac{\log x}{\beta}} = \left(\frac{1}{x}\right)^{1/\beta}.$$

Similarly,

$$a_k^{\left\lfloor \frac{\log x}{\beta \log(1/a_k)} \right\rfloor} < a_k^{\frac{\log x}{\beta \log(1/a_k)} - 1} = \frac{1}{a_k}\left(\frac{1}{x}\right)^{1/\beta}.$$

Relation (4.13) is proved.

If $0 < \beta < 1$, then $\mathbb{E}S_n^{\xi} = \sum_{k=1}^{n} \mathbb{E}\xi_k < \infty$ and, by Lemma 2.1 equation (2.3), we have that

$$\mathbb{E}\big(S_n^{\xi} - x\big)^{+} = \int_x^{\infty} \mathbb{P}\big(S_n^{\xi} > u\big)\mathrm{d}u. \tag{4.14}$$

Finishing the preliminaries for further arguments, from Theorems 3.4 and 3.6, we get that in case $\alpha = 0$,

$$\sum_{k=1}^{n} \mathbb{P}(\xi_k > x) \underset{x\to\infty}{\lesssim} \mathbb{P}(S_n^{\xi} > x) \underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbb{P}(\xi_k > x). \tag{4.15}$$

From (4.12), (4.13), (4.14) and (4.15), we have that with $0 < \beta < 1$ and sufficiently large x,

$$\mathbb{E}\big(S_n^{\xi} - x\big)^{+} = \int_x^{\infty} \mathbb{P}\big(S_n^{\xi} > u\big)\mathrm{d}u \leqslant \int_x^{\infty} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbb{P}(\xi_k > u)\mathrm{d}u$$
$$= \sum_{k=1}^{n} \frac{1}{a_k} \int_x^{\infty} a_k^{\left\lfloor \frac{\log u}{\beta \log(1/a_k)} \right\rfloor} \mathrm{d}u \leqslant \sum_{k=1}^{n} \frac{1}{a_k^2} \int_x^{\infty} \left(\frac{1}{u}\right)^{1/\beta} \mathrm{d}u$$
$$= \frac{\beta}{1-\beta} \frac{1}{x^{1/\beta - 1}} \sum_{k=1}^{n} \frac{1}{a_k^2}. \tag{4.16}$$

Similarly, evaluating from below, we have that with $0 < \beta < 1$ and sufficiently large x,

$$\mathbb{E}\big(S_n^{\xi} - x\big)^{+} = \int_x^{\infty} \mathbb{P}\big(S_n^{\xi} > u\big)\mathrm{d}u \geqslant \int_x^{\infty} \sum_{k=1}^{n} \mathbb{P}(\xi_k > u)\mathrm{d}u$$
$$= \sum_{k=1}^{n} \int_x^{\infty} a_k^{\left\lfloor \frac{\log u}{\beta \log(1/a_k)} \right\rfloor} \mathrm{d}u \geqslant n \int_x^{\infty} \left(\frac{1}{u}\right)^{1/\beta} \mathrm{d}u$$
$$= \frac{\beta}{1-\beta} \frac{n}{x^{1/\beta - 1}}. \tag{4.17}$$

Using inequalities (4.16) and (4.17), we get

$$\frac{\beta}{1-\beta}\frac{n}{x^{1/\beta-1}} \underset{x\to\infty}{\lesssim} \mathbb{E}\big(S_n^{\xi}-x\big)^{+} \underset{x\to\infty}{\lesssim} \frac{\beta}{1-\beta}\frac{1}{x^{1/\beta-1}}\sum_{k=1}^{n}\frac{1}{a_k^2}.$$

If $0<\beta<1/2$, then by equation (2.3) (in case p=2), we have that

$$\mathbb{E}\big((S_n^{\xi}-x)^{+}\big)^2 = 2\int_x^{\infty}(u-x)\mathbb{P}\big(S_n^{\xi}>u\big)\mathrm{d}u. \tag{4.18}$$

Similarly to the case of first order moment, from (4.12), (4.13), (4.15) and (4.18), we have that with $0<\beta<1/2$,

$$\frac{2\beta^2}{(1-2\beta)(1-\beta)}\frac{n}{x^{1/\beta-2}} \underset{x\to\infty}{\lesssim} \mathbb{E}\big((S_n^{\xi}-x)^{+}\big)^2 \underset{x\to\infty}{\lesssim} \frac{2\beta^2}{(1-2\beta)(1-\beta)}\frac{1}{x^{1/\beta-2}}\sum_{k=1}^{n}\frac{1}{a_k^2}. \tag{4.19}$$

In order to obtain the asymptotics of $H_q[S_n^{\xi}]$, we need to approximate solution x of equation (4.2) in the case $\varkappa=2$. Using derived asymptotic relations for the first and second order moments, we get that for sufficiently large x, the ratio in (4.2) satisfies the following inequalities:

$$\frac{\frac{\beta^2}{(1-\beta)^2}\frac{n^2}{x^{2/\beta-2}}}{\frac{2\beta^2}{(1-2\beta)(1-\beta)}\frac{1}{x^{1/\beta-2}}\sum_{k=1}^{n}\frac{1}{a_k^2}} \leqslant \frac{\big(\mathbb{E}(S_n^{\xi}-x)^{+}\big)^2}{\mathbb{E}\big((S_n^{\xi}-x)^{+}\big)^2} \leqslant \frac{\frac{\beta^2}{(1-\beta)^2}\frac{1}{x^{2/\beta-2}}\Big(\sum_{k=1}^{n}\frac{1}{a_k^2}\Big)^2}{\frac{2\beta^2}{(1-2\beta)(1-\beta)}\frac{n}{x^{1/\beta-2}}}.$$

After some simplifications and employing properties of limit superior, we prove that the following asymptic relation is true:

$$\frac{1}{2}\frac{1}{x^{1/\beta}}\frac{1-2\beta}{1-\beta}n^2\Big(\sum_{k=1}^{n}\frac{1}{a_k^2}\Big)^{-1} \underset{x\to\infty}{\lesssim} \frac{\big(\mathbb{E}(S_n^{\xi}-x)^{+}\big)^2}{\mathbb{E}\big((S_n^{\xi}-x)^{+}\big)^2} \underset{x\to\infty}{\lesssim} \frac{1}{2}\frac{1}{x^{1/\beta}}\frac{1-2\beta}{1-\beta}\frac{1}{n}\Big(\sum_{k=1}^{n}\frac{1}{a_k^2}\Big)^{2}.$$

Thus, solution of equation (4.2) satisfies the asymptotic inequality

$$\frac{1}{2^{\beta}}\Big(\frac{1-2\beta}{1-\beta}\Big)^{\beta}n^{2\beta}\Big(\sum_{k=1}^{n}\frac{1}{a_k^2}\Big)^{-\beta}\Big(\frac{1}{1-q}\Big)^{\beta} \underset{q\uparrow 1}{\lesssim} x \underset{q\uparrow 1}{\lesssim} \frac{1}{2^{\beta}}\Big(\frac{1-2\beta}{1-\beta}\Big)^{\beta}\frac{1}{n^{\beta}}\Big(\sum_{k=1}^{n}\frac{1}{a_k^2}\Big)^{2\beta}\Big(\frac{1}{1-q}\Big)^{\beta} \tag{4.20}$$

because $x = x(q) \to \infty$ if and only if $q \uparrow 1$.

Using expression from Theorem 4.1(ii) together with (4.19) and (4.20), we get that for q that is sufficiently close to 1,

$$\begin{aligned}
H_q[S_n^\xi] &= x(q) + \left(\frac{1}{1-q}\right)^{1/2} \big(\mathbb{E}\big((S_n^\xi - x(q))^+\big)^2\big)^{1/2} \\
&\leqslant x(q) + \left(\frac{1}{1-q}\right)^{1/2} \frac{2^{1/2}\beta}{(1-2\beta)^{1/2}(1-\beta)^{1/2}} x(q)^{-\frac{1}{2\beta}+1} \left(\sum_{k=1}^n \frac{1}{a_k^2}\right)^{1/2} \\
&\leqslant \frac{1}{2^\beta}\left(\frac{1-2\beta}{1-\beta}\right)^\beta \frac{1}{n^\beta}\left(\sum_{k=1}^n \frac{1}{a_k^2}\right)^{2\beta}\left(\frac{1}{1-q}\right)^\beta \\
&\quad + \left(\frac{1}{1-q}\right)^{1/2} \frac{2^{1/2}\beta\big(\sum_{k=1}^n \frac{1}{a_k^2}\big)^{1/2}}{(1-2\beta)^{1/2}(1-\beta)^{1/2}} \\
&\quad \times \left(\frac{1}{2^\beta}\left(\frac{1-2\beta}{1-\beta}\right)^\beta n^{2\beta}\left(\sum_{k=1}^n \frac{1}{a_k^2}\right)^{-\beta}\left(\frac{1}{1-q}\right)^\beta\right)^{-\frac{1}{2\beta}+1} \\
&= \frac{1}{2^\beta}\left(\frac{1}{1-q}\right)^\beta\left(\frac{1-2\beta}{1-\beta}\right)^\beta\Bigg[\frac{1}{n^\beta}\left(\sum_{k=1}^n \frac{1}{a_k^2}\right)^{2\beta} \\
&\quad + \frac{2\beta}{1-2\beta} n^{2\beta-1}\left(\sum_{k=1}^n \frac{1}{a_k^2}\right)^{1-\beta}\Bigg].
\end{aligned}$$

Similarly, for q that is sufficiently close to 1, we get the lower bound

$$\begin{aligned}
H_q[S_n^\xi] \geqslant \frac{1}{2^\beta}\left(\frac{1}{1-q}\right)^\beta\left(\frac{1-2\beta}{1-\beta}\right)^\beta\Bigg[& n^{2\beta}\left(\sum_{k=1}^n \frac{1}{a_k^2}\right)^{-\beta} \\
&+ \frac{2\beta}{1-2\beta} n^{1-\beta}\left(\sum_{k=1}^n \frac{1}{a_k^2}\right)^{2\beta-1}\Bigg].
\end{aligned}$$

Finally, we get that with generating function $\varphi(t) = t^2$ and $0 < \beta < 1/2$ the following asymptotic upper and lower bounds for the HG risk measure $H[S_n^\xi]$ are valid:

$$\begin{aligned}
H_q[S_n^\xi] \underset{q\uparrow 1}{\lesssim} \frac{1}{2^\beta}\left(\frac{1}{1-q}\right)^\beta\left(\frac{1-2\beta}{1-\beta}\right)^\beta\Bigg[& \frac{1}{n^\beta}\left(\sum_{k=1}^n \frac{1}{a_k^2}\right)^{2\beta} \\
&+ \frac{2\beta}{1-2\beta} n^{2\beta-1}\left(\sum_{k=1}^n \frac{1}{a_k^2}\right)^{1-\beta}\Bigg]
\end{aligned}$$

and

$$H_q[S_n^\xi] \underset{q\uparrow 1}{\gtrsim} \frac{1}{2^\beta}\left(\frac{1}{1-q}\right)^\beta \left(\frac{1-2\beta}{1-\beta}\right)^\beta \left[n^{2\beta}\left(\sum_{k=1}^n \frac{1}{a_k^2}\right)^{-\beta} + \frac{2\beta}{1-2\beta} n^{1-\beta}\left(\sum_{k=1}^n \frac{1}{a_k^2}\right)^{2\beta-1}\right].$$

In Figure 4.2, the values of risk measure $H_q[S_n^\xi]$ are presented together with its asymptotic bounds in the case $n = 3$, $a_1 = 0.35$, $a_2 = 0.45$, $a_3 = 0.55$ and $\kappa = 2$. These values are obtained using the standard Monte-Carlo method (with $2 \cdot 10^6$ simulations) by supposing that summands in S_n^ξ are independent.

5. Conclusions

Here we summarise the main results obtained in this dissertation:

• Let $\xi_1, \ldots, \xi_n$ be pQAI real-valued r.v.s. If $\mathbb{E}\,|\xi_k|^\alpha < \infty$, $F_{\xi_k} \in \mathcal{D}$ for all $k \in \{1, \ldots, n\}$ and some $\alpha \in [0, \infty)$, then

$$\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\{\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big((S_n^\xi)^\alpha \mathbb{1}_{\{S_n^\xi > x\}}\big) \underset{x\to\infty}{\lesssim} \sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\{\xi_k > x\}}\big). \tag{5.1}$$

If primary random variables are nonnegative, L-indices in the lower asymptotic bound can be omitted.

• If d.f.s of primary real-valued r.v.s are restricted to the class $\mathcal{C} \subset \mathcal{D}$, L-indices in equation (5.1) are equal to 1. Thus, we have the exact asymptotics

$$\mathbb{E}\big((S_n^\xi)^\alpha \mathbb{1}_{\{S_n^\xi > x\}}\big) \underset{x\to\infty}{\sim} \sum_{k=1}^n \mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\{\xi_k > x\}}\big).$$

• In essence, under additional requirement for existence of moments of random weights $\theta_1, \ldots, \theta_n$,

$$\max\{\mathbb{E}\theta_1^p, \ldots, \mathbb{E}\theta_n^p\} < \infty \text{ for some } p > \max\{J_{\xi_1}^+, \ldots, J_{\xi_n}^+\},$$

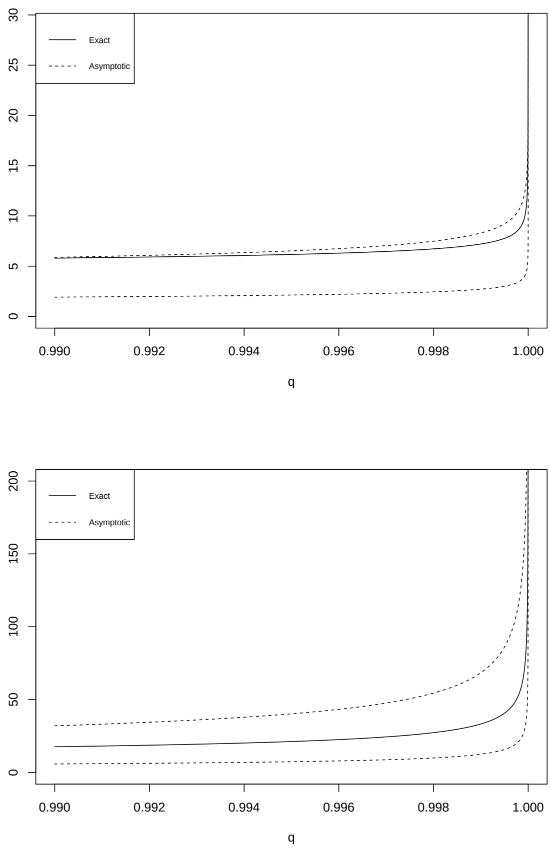

Figure 4.2: Values of HG risk measure and corresponding asymptotic bounds for $\beta = 0.15$ (top) and $\beta = 0.33$ (bottom). In both pictures $n = 3$, $a_1 = 0.35$, $a_2 = 0.45$, $a_3 = 0.55$ and $\kappa = 2$.

we get asymptotic bounds for the tail moment of the randomly weighted sum

$$\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}\big((\theta_k\xi_k)^{\alpha}\mathbb{1}_{\{\theta_k\xi_k>x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big(\big(S_n^{\theta\xi}\big)^{\alpha}\mathbb{1}_{\{S_n^{\theta\xi}>x\}}\big)$$
$$\underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}}\mathbb{E}\big((\theta_k\xi_k)^{\alpha}\mathbb{1}_{\{\theta_k\xi_k>x\}}\big).$$

• We demonstrated the way of combining our main results with Theorem 2.1 in the paper by Tang and Yang [63], so that asymptotic formulas for Haezendonck–Goovaerts risk measure would be achieved.

Illustrations, when primary r.v.s are distributed according to the Pareto law or the generalised Peter and Paul law, are given.

- We established new closure properties of the classes of heavy-tailed and related distributions. If the tail of a distribution has the property defining specific class, the tail of the moment has the same property. The converse statement is true in the class $\mathcal{R}$, but in other analysed classes the fact that the tail of the moment has class defining property doesn't imply that the tail of respective distribution has that property.

6. Santrauka

6.1. Įžanga

Pastaruosius kelis dešimtmečius stebime susidomėjimo sunkiauodegiais skirstiniais augimą. Priežasčių tam yra nemažai, tačiau tarp populiariausių paaiškinimų įvardijami informacinių ir komunikacijos technologijų plėtra, finansinių modelių, kurie geriau atspindėtų realias problemas, poreikis, bei gausėjantys įrodymai apie tokių skirstinių taikymo tinkamumą gamtos moksluose. Dažnai mokslininkai, kurie domisi finansais ir draudimu, domisi ir sunkiauodegiais skirstiniais - tyrimų apžvalgą galima rasti Embrechts, Klüppelberg, Mikosch [21] knygoje. Tarp klasikinių sunkiauodegių skirstinių pavyzdžių sutinkame Pareto, lognormalųjį ir Weibull skirstinius. Tarp mažiau žinomų, tačiau svarbių, pavyzdžių randame apibendrintą Petro ir Pauliaus, diskretųjį Weibull, Koši skirtinius. Disertacijoje pateikta iliustaracinė lentelė, kur įvairūs skirstinių pavyzdžiai priskiriami sunkiauodegių skirstinių ir susijusioms klasėms. Šiame darbe mus labiausiai domina skirstinių su dominuojamai kintančiomis uodegomis klasė $\mathcal{D}$. Tačiau atsakėme ir keletą klausimų apie susijusias klases, tarp kurių įvardintume klasę skirstinių su nuosaikiai kintančiomis uodegomis $\mathcal{C}$, klasę skirstinių su reguliariai kintančiomis uodegomis $\mathcal{R}$, ilgauodegių skirstinių klasę $\mathcal{L}$ ir subeksponentinių skirtinių klasę $\mathcal{S}$.

Egzistuoja daug praktinių modelių, įtraukiančių kumuliatyvius efektus, todėl yra svarbi atsitiktinių dydžių sumų, jų skirstinių, momentų ir kitų tikimybinių charakteristikų analizė. Tarkime, kad $n \in \mathbb{N} := \{1, 2, \ldots\}$ ir $\xi_1, \ldots, \xi_n$ yra galimai priklausomi, sunkiauodegiai, realiareikšmiai atsitiktiniai dydžiai, vadinami pagrindiniais atsitiktiniais dydžiais, o $\theta_1, \ldots, \theta_n$ yra neneigiami, neišsigimę nulyje atsitiktiniai dydžiai, vadinami atsitiktiniais svoriais. Mūsų pagrindinis tyrimų objektas yra momento uodega

$$\mathbb{E}((S_n^{\xi})^{\alpha} \mathbb{1}_{\{S_n^{\xi} > x\}}), \tag{6.1}$$

kur $\alpha \in [0, \infty)$ ir

$$S_n^{\xi} := \xi_1 + \xi_2 + \cdots + \xi_n.$$

Be to, suformulavome keletą rezultatų svoriniam momento uodegos (6.1) atitikmeniui:

$$\mathbb{E}((S_n^{\theta\xi})^{\alpha} \mathbb{1}_{\{S_n^{\theta\xi} > x\}}), \tag{6.2}$$

kur $\alpha \in [0, \infty)$ ir

$$S_n^{\theta\xi} := \theta_1\xi_1 + \theta_2\xi_2 + \cdots + \theta_n\xi_n. \tag{6.3}$$

Kartais laikysime, kad rodiklis $\alpha \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ ir tokiu atveju vizualinio atskyrimo tikslais vietoj raidės α rašysime m. Taip pat, atkreipiame dėmesį, kad literatūroje gana dažnai vietoj termino "momento uodega" (angl. "tail moment") yra naudojamas terminas "nupjautinis momentas" (angl. "truncated moment").

Įkvėpti Leipaus, Šiaulio, Vareikaitės [40] rezultato siekėme asimptotiškai aprėžti momentų uodegas (6.1) ir (6.2) atitinkamomis momentų uodegų

$$\mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\{\xi_k > x\}}\big) \quad \text{ir} \quad \mathbb{E}\big((\theta_k\xi_k)^\alpha \mathbb{1}_{\{\theta_k\xi_k > x\}}\big)$$

sumomis su tam tikromis koreguojančiomis konstantomis. Palyginus su ankstesniais rezultatais šioje disertacijoje yra gaunami tikslesni asimptotiniai rėžiai, parodant, kad kiekvienas dėmuo aproksimuojančiose sumose gali būti lydimas atitinkamos pasiskirstymo funkcijos L-indekso. Mūsų rezultatui naujumo suteikia ir tai, kad naudojame abstraktesnę priklausomybės struktūrą, bei tai, kad vietoj natūralaus momento uodegos rodiklio imamas bet koks neneigiamas realus skaičius. Tuo atveju, kai pagrindiniai atsitiktiniai dydžiai neneigiami, įrodėme, kad koreguojančios konstantos gali būti praleistos.

Yra parašyta nemažai straipsnių, kur nagrinėjami du atskiri atvejai $\alpha = 0$ ir $\alpha = 1$. Tarkime, kad $\alpha = 0$. Šiuo atveju momentų uodegos (6.1) ir (6.2) yra lygios skirstinių uodegoms, atitinkamai

$$\mathbb{P}\big(S_n^\xi > x\big) \quad \text{ir} \quad \mathbb{P}\big(S_n^{\theta\xi} > x\big).$$

Įvairiose studijose (trumpą apžvalgą pateikėme 3.1-iame disertacijos skyriuje) parodyta, kad šios skirstinių uodegos yra asimptotiškai lygios atitinkamoms atskirų atsitiktinių dydžių skirstinių uodegų sumoms:

$$\sum_{k=1}^n \mathbb{P}\big(\xi_k > x\big) \quad \text{ir} \quad \sum_{k=1}^n \mathbb{P}\big(\theta_k\xi_k > x\big).$$

Tarkime, kad $\alpha = 1$. Tada momentų uodegos (6.1) ir (6.2) yra lygios vidurkių uodegoms, atitinkamai

$$\mathbb{E}\big(S_n^\xi \mathbb{1}_{\{S_n^\xi > x\}}\big) \quad \text{ir} \quad \mathbb{E}\big(S_n^{\theta\xi} \mathbb{1}_{\{S_n^{\theta\xi} > x\}}\big).$$

Tie patys aukščiau paminėti klausimai tyrėjus domina ir šiuo atveju.

Atlikdami tyrimus pastebėjome tam tikrus momento uodegos elgesio reguliarumus lyginant su atitinkamo skirstinio uodega. Tai išaugo iki diskusijos apie sunkiauodegių ir susijusių skirstinių klasių uždarumo savybes momento skaičiavimo atžvilgiu. Mūsų tyrimai rodo, kad tam jog momento uodega turėtų savybę, apibrėžiančia konkrečią klasę, pakanka, kad atitinkamo skirstinio uodega turėtų tą pačią savybę, tačiau tai ne visada būtina.

Vienas iš būdų disertacijos rezultatus taikyti finansų ir draudimo srityje siejasi su aukščiau paminėtomis svorinėmis sumomis (6.3). Tarkime, kad pagrindiniai atsitiktiniai dydžiai ξ_k, $k = 1, \ldots, n$, reprezentuoja grynąjį draudimo kompanijos nuostolį (žalų sumos ir įmokų sumos skirtumą) periode $(k-1, k]$, suskaičiuotą laiko momentu k. Be to, tarkime, kad atsitiktiniai svoriai θ_k, $k = 1, \ldots, n$, reprezentuoja stochastinį diskonto faktorių iš laiko momento k į dabarties momentą 0. Tada suma $S_n^{\theta\xi}$ interpretuojama, kaip laiko periodo $(0, n]$ diskontuoti grynieji kompanijos nuostoliai.

Kiekybiškai vertindami bet kurio portfelio ar kompanijos atsitiktinius nuostolius galime pritaikyti įvairius rizikos matus, pavyzdžiui, “Value at Risk” (VaR), “Conditional Value at Risk” (CVaR), ar Haezendonck–Goovaerts (HG) rizikos matą. Disertacijos 4.1 ir 4.2 aprašėme minėtus rizikos matus bei trumpai pristatėme rizikos matavimo teoriją ir jos vystymąsi. Taip pat, disertacijoje pateikiame naujų taikymo pavyzdžių, t.y. radome Haezendonck–Goovaerts rizikos mato formules konkretiems skirstiniams kombinuodami mūsų tyrimų rezultatus su svarbia teorema iš Tang ir Yang [63, 2.1 Teorema] straipsnio.

Pagrindiniai rezultatai 2 ir 3.2 skyriuose yra nauji ir originalūs. Šie rezultatai yra gauti disertacijos autoriaus kartu su bendraautoriais. Iš esmės, 3.2 skyrius yra grindžiamas Leipaus, Paukščio, Šiaulio [38] ir Dirmos, Paukščio, Šiaulio [20] straipsniais. Naujos uždarumo savybės, kurios pateiktos 2 skyriuje, yra publikuotos Paukščio, Šiaulio, Leipaus [49] straipsnyje.

6.2. Apibrėžimai

6.2.1. Sunkiauodegiai ir susiję skirstiniai

Prieš pristatydami pagrindinius disertacijos rezultatus apibrėžkime pagrindines tam reikalingas sąvokas. Pirma, apibrėšime sunkiauodegių ir susijusių skirstinių klases, kurių elementams formuluoti mūsų rezultatai. Priminsime, kad sakome, jog pasiskirstymo funkcija F yra "virš $\mathbb{R}^+$", jeigu $F(-0) = 0$. Panašiai, sakome, kad pasiskirstymo funkcija yra "virš $\mathbb{R}$", jeigu sąlyga $F(-0) = 0$ gali būti netenkinama.

Pirmiausia formaliai apibrėškime sunkiauodegių skirstinių klasę $\mathcal{H}$.

Apibrėžimas 6.1. *Pasiskirstymo funkcija F virš $\mathbb{R}$ yra vadinama sunkiauodege, rašome $F \in \mathcal{H}$, jeigu kiekvienam $h > 0$ yra teisinga, kad*

$$\int_{-\infty}^{\infty} \mathrm{e}^{hx} \mathrm{d}F(x) = \infty.$$

Sakysime, kad skirstinys yra lengvauodegis, jeigu jis nėra sunkiauodegis.

Toliau apibrėžiame plačiai žinomą klasės $\mathcal{H}$ poklasį $\mathcal{L}$ taip, kaip apibrėžia Foss, Korshunov, Zachary [27].

Apibrėžimas 6.2. *Pasiskirstymo funkcija F virš $\mathbb{R}$ yra vadinama ilgauodege, rašome $F \in \mathcal{L}$, jeigu kiekvienam fiksuotam $y > 0$ yra teisinga, kad*

$$\overline{F}(x + y) \sim \overline{F}(x), \tag{6.4}$$

kai $x \to \infty$.

Tam, kad įsitikintume, jog pasiskirstymo funkcija yra ilgauodegė užtenka patikrinti, kad sąryšis (6.4) galioja kuriam nors vienam y nelygiam nuliui.

Panašiu būdu apibrėžiami ir eksponentinio tipo skirstiniai (skaitykite, pavyzdžiui, Ragulina, Šiaulys [54]) ir $\mathcal{O}$-eksponentiniai skirstiniai (skaitykite, pavyzdžiui, Xu, Foss, Wang [67]).

Apibrėžimas 6.3. *Sakome, kad pasiskirstymo funkcija F virš $\mathbb{R}$ yra su eksponentine uodega, rašome $F \in \mathcal{L}_\gamma$, $\gamma > 0$, jeigu kiekvienam $y > 0$ (ekvivalenčiai visiems $y \in \mathbb{R}$),*

$$\lim_{x\to\infty} \frac{\overline{F}(x + y)}{\overline{F}(x)} = \mathrm{e}^{-\gamma y}. \tag{6.5}$$

Apibrėžimas 6.4. *Pasiskirstymo funkcija F virš $\mathbb{R}$ yra vadinama $\mathcal{O}$-eksponentine, rašome $F \in \mathcal{OL}$, jeigu kiekvienam $y > 0$ yra teisinga, kad*

$$\limsup_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} < \infty.$$

Skirstinių su eksponentinėmis uodegomis klases $\mathcal{L}_\gamma$ pristatė Embrechts ir Goldie [22]. Klasę $\mathcal{OL}$ pirmieji pristatė Shimura ir Watanabe [60]. Pastebėkime, kad tuos pačius klasės $\mathcal{L}$ elementus galėtume gauti imdami $\gamma = 0$ lygybėje (6.5). Todėl galime rašyti $\mathcal{L}_0 = \mathcal{L}$. Be to, kaip pastebėta straipsnyje [67], klasė $\mathcal{OL}$ apima visas klases $\mathcal{L}_\gamma, \gamma \geqslant 0$.

Toliau apibrėžiame klasę $\mathcal{D}$, svarbų sunkiauodegių skirstinių poklasį. Šią klasę pristatė Feller [25]. Mūsų pagrindiniai rezultatai bus formuluojami būtent šiai skirstinių klasei.

Apibrėžimas 6.5. *Sakome, kad pasiskirstymo funkcija F virš $\mathbb{R}$ yra su dominuojamai kintančia uodega, rašome $F \in \mathcal{D}$, jeigu kiekvienam fiksuotam $0 < y < 1$,*

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

Įprastai klasė $\mathcal{S}$, viena svarbiausių sunkiauodegių skirstinių klasių, yra apibrėžiama tokiu būdu.

Apibrėžimas 6.6. *Pasiskirstymo funkcija F virš $\mathbb{R}^+$ yra vadinama sub-eksponentine, rašome $F \in \mathcal{S}$, jeigu*

$$\overline{F * F}(x) \sim 2\overline{F}(x) \tag{6.6}$$

kai $x \to \infty$.

Norint korektiškai apibrėžti subeksponentinių skirstinių klasę visiems skirstiniams virš $\mathbb{R}$, nepakanka, kad būtų tenkinama sąlyga (6.6).

Foss, Korshunov, Zachary [27, Example 3.3] pateikia pavyzdį skirstinio, kuriam

$$\overline{F * F}(x) \underset{x\to\infty}{\sim} 2\overline{F}(x),$$

bet atitinkama pasiskirstymo funkcija F nėra ilgauodegė ir net nėra sunkiauodegė. Taigi apibrėžiant pasiskirstymo funkcijos virš $\mathbb{R}$ subeksponentiškumą yra reikalaujama, kad būtų tenkinamos abi sąlygos - sąlyga (6.6) ir ilgauodegiškumas. Dažnai yra naudojamas ir toks apibrėžimas.

Apibrėžimas 6.7. *Reliareikšmio atsitiktinio dydžio* ξ *pasiskirstymo funkcija* F_ξ *virš* $\mathbb{R}$ *yra vadinama subeksponentine, rašome* $F \in \mathcal{S}$, *jeigu atsitiktinio dydžio* $\xi^+ = \max\{\xi, 0\}$ *pasiskirstymo funkcija* F_{ξ^+} *tenkina sąryšį* (6.6).

Apibrėžkime klasę $\mathcal{C}$, naudingą $\mathcal{L} \cap \mathcal{D}$ poklasį.

Apibrėžimas 6.8. *Sakome, kad pasiskirstymo funkcija* F *virš* $\mathbb{R}$ *yra su nuosaikiai kintančia uodega, rašome* $F \in \mathcal{C}$, *jeigu*

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

Ekvivalenčiai, $F \in \mathcal{C}$, jeigu

$$\lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

Svarbus klasės $\mathcal{C}$ poklasis yra gerai žinoma skirstinių su reguliariai kintančiomis uodegomis klasė.

Apibrėžimas 6.9. *Sakome, kad pasiskirstymo funkcija* F *virš* $\mathbb{R}$ *turi reguliariai kintančią uodegą su indeksu* $\alpha \geqslant 0$, *rašome* $F \in \mathcal{R}_\alpha$, *jeigu bet kuriam* $y > 0$ *galioja, kad*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}.$$

Rašome

$$\mathcal{R} := \bigcup_{\alpha\geqslant 0} \mathcal{R}_\alpha.$$

Klasės $\mathcal{R}$, $\mathcal{C}$ ir $\mathcal{D}$ gali būti apibūdinamos specifiniais indeksais. Pirmasis mus dominantis indeksas, vadinamas *viršutiniuoju Matuszewska indeksu*, apibrėžiamas taip:

$$J_F^+ := \inf_{y>1} \left\{ -\frac{1}{\log y} \log \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \right\}.$$

Antras indeksas, vadinamas *L-indeksu*, yra apibrėžiamas taip:

$$L_F := \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.$$

Yra gerai žinomi tokie sąryšiai:

$$L_F > 0 \;\Leftrightarrow\; F \in \mathcal{D} \;\Leftrightarrow\; J_F^+ < \infty; \qquad L_F = 1 \;\Leftrightarrow\; F \in \mathcal{C}.$$

Be to, jei pasiskirstymo funkcija $F \in \mathcal{R}_\alpha$, tai $J_F^+ = \alpha$ ir $L_F = 1$. Šias dvi lygybes nesunku patikrinti. Jeigu $F \in \mathcal{R}_\alpha$, tai

$$L_F = \lim_{y \downarrow 1} \liminf_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = \lim_{y \downarrow 1} y^{-\alpha} = 1$$

ir

$$J_F^+ = \inf_{y>1} \left\{ -\frac{1}{\log y} \log \liminf_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \right\} = \inf_{y>1} \left\{ -\frac{1}{\log y} \log y^{-\alpha} \right\} = \alpha.$$

Šį skyrių užbaigsime pateikdami žinomus ryšius tarp aukščiau apibrėžtų skirstinių klasių. Sunkiauodegių skirstinių klasėms teisinga:

$$\mathcal{R} \subsetneq \mathcal{C} \subsetneq \mathcal{L} \cap \mathcal{D} \subsetneq \mathcal{S} \subsetneq \mathcal{L} \subsetneq \mathcal{H}; \quad \mathcal{D} \subsetneq \mathcal{H}; \quad \mathcal{D} \not\subset \mathcal{S} \text{ and } \mathcal{S} \not\subset \mathcal{D}.$$

Nuorodas į straipsnius, kur įrodyti šie faktai, ar komentarus (anglų kalba) galima rasti šios disertacijos 1.4.1 skyrelyje.

Panašiai, yra gerai žinoma, kad klasei $\mathcal{OL}$ teisinga:

$$\bigcup_{\gamma \geqslant 0} \mathcal{L}_\gamma \cup \mathcal{D} \subsetneq \mathcal{OL}.$$

6.2.2. Priklausomumo struktūros

Jau yra parašyta daug straipsnių apie nepriklausomų atsitiktinių dydžių sumų skirstinių uodegų asimptotinį elgesį. Esant įvairioms papildomoms prielaidoms parodyta, kad ši uodega yra asimptotiškai lygi atskirų atsitiktinių dėmenų skirstinių uodegų sumai. Yra platus sutarimas, kad nepriklausomumo prielaida dažnai nerealistiška praktiniuose taikymuose, todėl mūsų tikslas yra gauti asimptotinius rezultatus abstraktesnėje priklausomumo klasėje.

Šiame darbe pagrindinė atsitiktinių dydžių $\xi_1, \ldots, \xi_n$ priklausomumo prielaida, kuri naudojama formuluojant pagrindinius rezultatus, yra poromis kvazi-asimptotinis nepriklausomumas (angl. "pairwise quasi-asymptotic independence"). Šią priklausomumo struktūrą pristatė mokslininkai Chen ir Yuen [13].

Apibrėžimas 6.10. *Realių reikšmių atsitiktiniai dydžiai $\xi_1, \ldots, \xi_n$ yra vadinami poromis kvazi-asimptotiškai nepriklausomais (pQAI), jei bet kuriems $k, l \in \{1, 2, \ldots, n\}$, $k \neq l$,*

$$\lim_{x \to \infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^+ > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} = \lim_{x \to \infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^- > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} = 0.$$

Kaip pažymi patys Chen ir Yuen, siūlydami QAI priklausomybės struktūrą jie pasiskolino terminą “asimptotinis nepriklausomumas” (angl. “asymptotic independence”) iš Resnick (žr. [55],[56]) ir pritaikė atitinkamą apibrėžimą nebūtinai vienodai pasiskirsčiusiems atsitiktiniams dydžiams. Čia apibrėžiame poromis asimptotinį nepriklausomumą tokiu būdu.

Apibrėžimas 6.11. *Atsitiktiniai dydžiai $\xi_1, \dots, \xi_n$ yra vadinami poromis asimptotiškai nepriklausomais (pAI), jei bet kuriems $k, l = 1, \dots, n$, $k \neq l$,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, \xi_l > x)}{\mathbb{P}(\xi_k > x)} = \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k > x, \xi_l > x)}{\mathbb{P}(\xi_l > x)} = 0.$$

Leipaus, Šiaulio ir Vareikaitės straipsnyje [40] randame pasiūlytą naują priklausomybės struktūrą:

PRIELAIDA $\mathcal{B}$. Atsitiktiniai dydžiai $\xi_1, \dots, \xi_n$ visiems $k, l = 1, \dots, n$, $k \neq l$, tenkina

$$\begin{aligned}\lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^+ > x \mid \xi_l^+ > u) &= \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^- > x \mid \xi_l^+ > u) \\ &= \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^+ > x \mid \xi_l^- > u) = 0.\end{aligned}$$

Prielaida $\mathcal{B}$ ir anksčiau apibrėžtas poromis kvazi-asimptotinis nepriklausomumas yra susiję. Galima parodyti, kad iš Prielaidos $\mathcal{B}$ išplaukia pQAI sąlygą. Tikrai, bet kuriems ξ_k, ξ_l, $1 \leqslant k \neq l \leqslant n$,

$$\begin{aligned}\lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^+ > x, \xi_l^+ > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} &\leqslant \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^+ > x \mid \xi_l^+ > u) = 0, \\ \lim_{x\to\infty} \frac{\mathbb{P}(\xi_k^- > x, \xi_l^+ > x)}{\mathbb{P}(\xi_k^+ > x) + \mathbb{P}(\xi_l^+ > x)} &\leqslant \lim_{x\to\infty} \sup_{u\geqslant x} \mathbb{P}(\xi_k^- > x \mid \xi_l^+ > u) = 0.\end{aligned}$$

Baigdami šį skyrelį užsiminsime, kad disertacijos 1.4.2 skyrelyje pateikėme du pavyzdžius, kai atsitiktinių dydžių rinkiniai turi pQAI struktūrą, generuotą konkrečiomis kopulomis. Pirmas pavyzdys konstruojamas pasitelkiant gerai žinomą Farlie–Gumbel–Morgenstein kopulą, antras – po-puliarią Ali–Mikhail–Haq kopulą.

6.3. Pagrindiniai rezultatai

Prieš pristatydami pagrindinius rezultatus suformuluosime porą rezultatų iš Leipaus, Šiaulio, Vareikaitės [40] straipsnio. Iš 3 ir 4 Teoremų straipsnyje [40] seka tokia teorema.

Teorema 6.1 (Leipus, Šiaulys, Vareikaitė [40, 3 ir 4 Teoremos]). *Tarkime, kad $\xi_1, \ldots, \xi_n$ yra realių reikšmių atsitiktiniai dydžiai. Jeigu galioja Prie-laida $\mathcal{B}$, $F_{\xi_1} \in \mathcal{D}$, $\overline{F}_{\xi_k}(x) \asymp \overline{F}_{\xi_1}(x)$, $\overline{F}_{\xi_k^-}(x) = O(\overline{F}_{\xi_1}(x))$ visiems $k = 1, \ldots, n$, ir $\mathbb{E}\,|\xi_1|^m < \infty$ kuriam nors $m \in \mathbb{N}$, tada*

$$L_{F_{\xi_k}} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{S_n^\xi > x\}}\big) \underset{x\to\infty}{\lesssim} \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big),$$

su $k = 1, \ldots, n$, ir

$$L_n^\xi \sum_{k=1}^{n} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big(\big(S_n^\xi\big)^m \mathbb{1}_{\{S_n^\xi > x\}}\big) \underset{x\to\infty}{\lesssim} \frac{1}{L_n^\xi} \sum_{k=1}^{n} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big),$$

kur $L_n^\xi := \min\big\{L_{F_{\xi_1}}, \ldots, L_{F_{\xi_n}}\big\}$.

Toliau cituojame 5 Teoremą iš [40] su nedideliais žymėjimų pakeitimais.

Teorema 6.2 (Leipus, Šiaulys, Vareikaitė [40, 5 Teorema]). *Tarkime, kad atsitiktiniai dydžiai $\xi_1, \ldots, \xi_n$ tenkina Prielaidą $\mathcal{B}$, o $F_{\xi_1} \in \mathcal{D}$ ir $\mathbb{E}\,|\xi_1|^m < \infty$ kuriam nors $m \in \mathbb{N}$. Be to, tarkime, kad $\theta_1, \ldots, \theta_n$ yra neneigiami, neišsigimę nulyje, aprėžti atsitiktiniai dydžiai nepriklausantys nuo $\xi_1, \ldots, \xi_n$. Jeigu $\overline{F}_{\theta_k\xi_k}(x) \asymp \overline{F}_{\theta_1\xi_1}(x)$, $\overline{F}_{\theta_k\xi_k^-}(x) = O(\overline{F}_{\theta_1\xi_1}(x))$ visiems $k = 2, \ldots, n$, tada*

$$\begin{aligned} L_n^\xi \sum_{k=1}^{n} \mathbb{E}\big((\theta_k\xi_k)^m \mathbb{1}_{\{\theta_k\xi_k > x\}}\big) &\underset{x\to\infty}{\lesssim} \mathbb{E}\big(\big(S_n^{\theta\xi}\big)^m \mathbb{1}_{\{S_n^{\theta\xi} > x\}}\big) \\ &\underset{x\to\infty}{\lesssim} \frac{1}{L_n^\xi} \sum_{k=1}^{n} \mathbb{E}\big((\theta_k\xi_k)^m \mathbb{1}_{\{\theta_k\xi_k > x\}}\big). \end{aligned}$$

Pirma, gavome rezultatą, kuris patikslina 6.1 Teoremą. Parodėme, kad kiekvienas dėmuo aproksimuojančiose sumose gali būti lydimas atskiro atitinkamos pasiskirstymo funkcijos L-indekso. Be to, papildomai įtraukėme atvejį $m = 0$.

Teorema 6.3. *Tarkime, kad $\xi_1, \ldots, \xi_n$ yra reliareikšmiai atsitiktiniai dydžiai tenkinantys 6.1 Teoremos reikalavimus, kur $\mathbb{E}\,|\xi_1|^m < \infty$ kuriam nors $m \in \mathbb{N}_0$. Tada*

$$\begin{aligned} \sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big) &\underset{x\to\infty}{\lesssim} \mathbb{E}\big(\big(S_n^\xi\big)^m \mathbb{1}_{\{S_n^\xi > x\}}\big) \\ &\underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\big(\xi_k^m \mathbb{1}_{\{\xi_k > x\}}\big). \end{aligned}$$

Vėlesnėje studijoje parodėme, kad galima apibendrinti pastarosios teoremos rezultatus susilpninant reikalavimus priklausomybės struktūrai ir momento rodikliui. Vietoj neneigiamo sveikojo laipsnio rodiklio imame bet kokį neneigiamą reliareikšmį rodiklį, o vietoj Prielaidos $\mathcal{B}$ leidžiame platesnę priklausomybės klasę pQAI.

Teorema 6.4. *Tarkime, kad $\xi_1, \ldots, \xi_n$ yra pQAI reliareikšmiai atsitiktiniai dydžiai. Jeigu $\mathbb{E}\,|\xi_k|^\alpha < \infty$, $F_{\xi_k} \in \mathcal{D}$ visiems $k \in \{1, \ldots, n\}$ ir kuriam nors $\alpha \in [0, \infty)$, tada*

$$\sum_{k=1}^{n} L_{F_{\xi_k}} \mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\{\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big(\big(S_n^\xi\big)^\alpha \mathbb{1}_{\{S_n^\xi > x\}}\big)$$
$$\underset{x\to\infty}{\lesssim} \sum_{k=1}^{n} \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\{\xi_k > x\}}\big). \tag{6.7}$$

Esant poromis kvazi-asimptotinio nepriklausomumo prielaidai iš 6.4 Teoremos seka štai toks rezultatas realių reikšmių atsitiktiniams dydžiams.

Teorema 6.5. *Tarkime, kad $\xi_1, \ldots, \xi_n$ yra pQAI reliareikšmiai atsitiktiniai dydžiai. Jeigu $F_{\xi_k} \in \mathcal{C}$ kiekvienam $k \in \{1, \ldots, n\}$, tada pasiskirstymo funkcija $F_{S_n^\xi} \in \mathcal{C}$ ir*

$$\overline{F}_{S_n^\xi}(x) \sim \sum_{k=1}^{n} \overline{F}_{\xi_k}(x).$$

Dabar nagrinėkime atvejį, kai ξ_k yra neneigiami atsitiktiniai dydžiai. Tokiu atveju 6.4 Teoremos rezultatas gali būti pagerintas praleidžiant atitinkamus L-indeksus apatiniame asimptotiniame rėžyje (6.7) formulėje.

Teorema 6.6. *Tarkime, kad $\xi_1, \ldots, \xi_n$ yra neneigiami pQAI atsitiktiniai dydžiai. Jeigu $F_{\xi_k} \in \mathcal{D}$, $\mathbb{E}\xi_k^\alpha < \infty$ visiems $k = 1, \ldots, n$ ir kuriam nors $\alpha \in [0, \infty)$, tada*

$$\sum_{k=1}^{n} \mathbb{E}\big(\xi_k^\alpha \mathbb{1}_{\{\xi_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big(\big(S_n^\xi\big)^\alpha \mathbb{1}_{\{S_n^\xi > x\}}\big).$$

Pagrindinių rezultatų skyrelį baigiame 6.4 Teoremos apibendrinimu atsitiktinei svorinei sumai $S_n^{\theta\xi}$. Į toliau pateikiamą rezultatą galime žiūrėti ir kaip į 6.2 Teoremos apibendrinimą, t.y. mes imame abstraktesnę priklausomybės struktūrą, leidžiame bet kokį neneigiamą realųjį momento rodiklį ir atsisakome ganėtinai apribojančios atsitiktinių svorių $\theta_1, \ldots, \theta_n$ aprėžtumo sąlygos.

Teorema 6.7. *Tarkime, kad $\xi_1, \ldots, \xi_n$ yra pQAI realiareikšmiai atsitiktiniai dydžiai tokie, kad $F_{\xi_k} \in \mathcal{D}$ visiems $k \in \{1, \ldots, n\}$, ir tarkime, kad $\theta_1, \ldots \theta_n$ yra kaip norima priklausomi, neneigiami, neišsigimę nulyje atsitiktiniai dydžiai su*

$$\max\{\mathbb{E}\theta_1^p, \ldots, \mathbb{E}\theta_n^p\} < \infty \text{ kuriam nors } p > \max\{J^+_{\xi_1}, \ldots J^+_{\xi_n}\}.$$

Jei rinkiniai $\{\xi_1, \ldots, \xi_n\}$ ir $\{\theta_1, \ldots, \theta_n\}$ yra nepriklausomi, bei

$$\mathbb{E}(\theta_k|\xi_k|)^\alpha < \infty \text{ visiems } k \in \{1, \ldots, n\}$$

ir kuriam nors $\alpha \in [0, \infty)$, tada

$$\begin{aligned}\sum_{k=1}^n L_{F_{\xi_k}} \mathbb{E}\big((\theta_k\xi_k)^\alpha \mathbb{1}_{\{\theta_k\xi_k > x\}}\big) &\underset{x\to\infty}{\lesssim} \mathbb{E}\big((S_n^{\theta\xi})^\alpha \mathbb{1}_{\{S_n^{\theta\xi} > x\}}\big) \\ &\underset{x\to\infty}{\lesssim} \sum_{k=1}^n \frac{1}{L_{F_{\xi_k}}} \mathbb{E}\big((\theta_k\xi_k)^\alpha \mathbb{1}_{\{\theta_k\xi_k > x\}}\big).\end{aligned}$$

6.4. Taikymai Haezendonck–Goovaerts rizikos matui

Šiame skyriuje pristatome Haezendonck–Goovaerts (HG) rizikos matą, kuris turi sąsajų su tokiais gerai žinomais rizikos matais kaip VaR ar CVaR. HG rizikos matą 1982-aisiais pristatė mokslininkai Haezendonck ir Goovaerts [34]. Nuo tada šis matas sulaukė daug dėmesio finansų ir draudimo matematikoje. Apibrėždami šį rizikos matą remsimės formuluotėmis, kurias siūlo Tang ir Yang [63].

Funkcija φ (apibrėžta virš $\mathbb{R}$) yra vadinama normuota Young funkcija, jeigu φ yra neneigiama, iškila į apačią intervale $[0, \infty)$, o $\varphi(0) = 0$, $\varphi(1) = 1$ ir $\varphi(\infty) = \infty$. Orlicz erdve L^φ ir Orlicz šerdimi L_0^φ vadinsime atsitiktinių dydžių X aibes, apibrėžtas lygybėmis:

$$\begin{aligned}L^\varphi &= \{X : \mathbb{E}[\varphi(cX)] < \infty \text{ kuriam nors } c > 0\}, \\ L_0^\varphi &= \{X : \mathbb{E}[\varphi(cX)] < \infty \text{ visiems } c > 0\}.\end{aligned}$$

Tarkime, kad φ yra Young funkcija ir $X \in L_0^{\varphi}$. Pasirinktam $q \in (0,1)$, atsitiktinio dydžio X Haezendonck–Goovaerts (HG) rizikos matas yra apibrėžiamas lygybe

$$H_q[X] = \inf_{x\in\mathbb{R}} \big(x + H_q[X,x]\big),$$

kur $H_q[X,x]$ yra vienintelis lygties

$$\mathbb{E}\,\varphi\bigg(\frac{(X-x)^+}{h}\bigg) = 1-q$$

sprendinys h, jeigu $\overline{F}_X(x) > 0$, ir $H_q[X,x] = 0$, jeigu $\overline{F}_X(x) = 0$.

Tang ir Yang [63] mini, kad analizinė rizikos mato $H_q[X]$ išraiška bendrai nėra galima, tačiau tuo atveju, kai $\varphi(t) = t^{\varkappa}$, $\varkappa \geqslant 1$, galima išvesti analizinę išraišką kai kurių tipų skirstiniams. Toliau pateikiame svarbią teoremą iš Tang ir Yang [63] straipsnio, kurią disertacijoje naudojame atsitiktinių dydžių sumos S_n^{ξ} HG rizikos mato asimptotinėms formulėms gauti.

Teorema 6.8 (Tang, Yang [63, (1.3) lygybė ir 2.1 Teorema])**.** *Nagrinėkime laipsninę Young funkciją $\varphi(t) = t^{\varkappa}$, $\varkappa \geqslant 1$.*

(i) *Jeigu $\varkappa = 1$, o atsitiktinis dydis X yra toks, kad $\mathbb{E}X^+ < \infty$, tada*

$$H_q[X] = F_X^{\leftarrow}(q) + \frac{\mathbb{E}\big(X - F_X^{\leftarrow}(q)\big)^+}{1-q} = \mathrm{CVaR}_q[X],$$

kur $F_X^{\leftarrow}(q) := \inf\{x \in \mathbb{R} : F_X(x) \geqslant q\}$ yra atsitiktinio dydžio X kvantilių funkcija.

(ii) *Jeigu $\varkappa > 1$, $\mathbb{P}(X = F_X^{\leftarrow}(q)) = 0$ ir $\mathbb{E}(X^+)^{\varkappa} < \infty$, tada*

$$H_q[X] = x + \left(\frac{\mathbb{E}((X-x)^+)^{\varkappa}}{1-q}\right)^{1/\varkappa}, \quad q \in (0,1),$$

kur $x = x(q) \in (-\infty, F_X^{\leftarrow}(1))$ yra vienintelis lygties

$$\frac{\big(\mathbb{E}((X-x)^+)^{\varkappa-1}\big)^{\varkappa}}{\big(\mathbb{E}((X-x)^+)^{\varkappa}\big)^{\varkappa-1}} = 1-q$$

sprendinys.

6.4.1. HG rizikos mato asimptotinės formulės. Pareto pavyzdys

Tarkime, kad atsitiktiniai dydžiai $\xi_1, \dots, \xi_n$ yra pQAI. Be to, tarkime, kad kiekvienam k atsitiktinis dydis ξ_k yra pasiskirstęs pagal Pareto dėsnį, t.y.

$$F_{\xi_k}(x) = \Big(1 - \Big(\frac{k}{x}\Big)^{\alpha}\Big)\mathbb{1}_{[k,\infty)}(x), \quad k = 1, \dots, n,$$

su formos parametru $\alpha > 1$.

Taikydami 6.4 ir 6.8 Teoremas disertacijoje parodome, kad tuo atveju, kai $\varphi(t) = t^{\varkappa}$, $\varkappa \in \mathbb{N}$, $\varkappa < \alpha$, galioja štai tokia asimptotinė formulė:

$$H_q\big[S_n^{\xi}\big] \underset{q\uparrow 1}{\sim} \frac{\alpha(\alpha-\varkappa)^{\varkappa/\alpha-1}}{\varkappa^{\varkappa/\alpha}}\big(\varkappa \mathrm{B}(\varkappa, \alpha-\varkappa)\big)^{1/\alpha}\Big(\sum_{k=1}^{n} k^{\alpha}\Big)^{1/\alpha}\Big(\frac{1}{1-q}\Big)^{1/\alpha};$$

čia $\mathrm{B}(u, v)$ žymi Beta funkciją.

6.4.2. HG rizikos mato asimptotinės formulės. Petro ir Pauliaus pavyzdys

Tarkime, kad atsitiktiniai dydžiai $\xi_1, \dots \xi_n$ yra pQAI. Be to, tarkime, kad kiekvienam k atsitiktiniai dydžiai ξ_k yra pasiskirstę pagal apibendrintą Petro ir Pauliaus dėsnį, t.y.

$$\mathbb{P}\big(\xi_k = a_k^{-j\beta}\big) = a_k^{j-1}(1-a_k), \quad j = 1, 2, \dots,$$

kur $a_k \in (0,1)$ ir $\beta > 0$.

Tuo atveju, kai generuojanti funkcija $\varphi(t) = t^2$, o $0 < \beta < 1/2$, taikant 6.4, 6.6 ir 6.8 Teoremas, disertacijoje gauti tokie viršutinis ir apatinis HG rizikos mato $H\big[S_n^{\xi}\big]$ rėžiai:

$$\begin{aligned}
H_q\big[S_n^{\xi}\big] &\underset{q\uparrow 1}{\lesssim} \frac{1}{2^{\beta}}\Big(\frac{1}{1-q}\Big)^{\beta}\Big(\frac{1-2\beta}{1-\beta}\Big)^{\beta}\Bigg[\frac{1}{n^{\beta}}\Big(\sum_{k=1}^{n}\frac{1}{a_k^2}\Big)^{2\beta} \\
&\qquad + \frac{2\beta}{1-2\beta}n^{2\beta-1}\Big(\sum_{k=1}^{n}\frac{1}{a_k^2}\Big)^{1-\beta}\Bigg], \\
H_q\big[S_n^{\xi}\big] &\underset{q\uparrow 1}{\gtrsim} \frac{1}{2^{\beta}}\Big(\frac{1}{1-q}\Big)^{\beta}\Big(\frac{1-2\beta}{1-\beta}\Big)^{\beta}\Bigg[n^{2\beta}\Big(\sum_{k=1}^{n}\frac{1}{a_k^2}\Big)^{-\beta} \\
&\qquad + \frac{2\beta}{1-2\beta}n^{1-\beta}\Big(\sum_{k=1}^{n}\frac{1}{a_k^2}\Big)^{2\beta-1}\Bigg].
\end{aligned}$$

6.5. Naujos skirstinių klasių uždarumo savybės

Šioje disertacijoje gavome ir naujas skirstinių klasių užarumo savybes, kurios siejasi su nupjautiniais momentais. Patogumo dėlei apibrėžkime naują pasiskirstymo funkciją.

Apibrėžimas 6.12. *Tarkime, kad ξ yra realiareikšmis atsitiktinis dydis, apibrėžtas tikimybinėje erdvėje $(\Omega, \mathcal{F}, \mathbb{P})$ ir turi pasiskirstymo funkciją F_ξ. Tarkime, kad momentas*

$$\mathbb{E}(\xi^+)^m = \int_{[0,\infty)} x^m \mathrm{d}F_\xi(x)$$

yra baigtinis, kai $m \geqslant 0$. Tokiu atveju funkcija

$$F_{\xi,m}(x) = \max\left\{0, 1 - \mathbb{E}(\xi^m \mathbb{1}_{\{\xi > x\}})\right\},\ x \geqslant 0,$$

yra nauja pasiskirstymo funkcija su uodega

$$\overline{F}_{\xi,m}(x) = \min\left\{1, \mathbb{E}(\xi^m \mathbb{1}_{\{\xi > x\}})\right\},\ x \geqslant 0.$$

Pirma, pristatome teoremą su pakankamomis sąlygomis, kad $F_{\xi,m}$ priklausytų konkrečiai skirstinių klasei.

Teorema 6.9. *Tarkime, kad ξ yra realių reikšmių atsitiktinis dydis su pasiskirstymo funkcija F_ξ ir baigtiniu momentu $\mathbb{E}(\xi^+)^m$ kuriam nors $m > 0$. Tada galioja tokie sąryšiai:*

(i) $F_\xi \in \mathcal{R}_\alpha,\ m < \alpha\ \Rightarrow\ F_{\xi,m} \in \mathcal{R}_{\alpha-m}$ $(F_\xi \in \mathcal{R}\ \Rightarrow\ F_{\xi,m} \in \mathcal{R})$,
(ii) $F_\xi \in \mathcal{C}\ \Rightarrow\ F_{\xi,m} \in \mathcal{C}$,
(iii) $F_\xi \in \mathcal{D}\ \Rightarrow\ F_{\xi,m} \in \mathcal{D}$,
(iv) $F_\xi \in \mathcal{L}\ \Rightarrow\ F_{\xi,m} \in \mathcal{L}$,
(v) $F_\xi \in \mathcal{L}_\gamma,\ \gamma > 0\ \Rightarrow\ F_{\xi,m} \in \mathcal{L}_\gamma$,
(vi) $F_\xi \in \mathcal{OL}\ \Rightarrow\ F_{\xi,m} \in \mathcal{OL}$.

Taigi įrodėme, kad norint, jog $F_{\xi,m}$ būtų klasėje $\mathcal{K}$ yra pakankama, kad pasiskirstymo funkcija F_ξ būtų kalsėje $\mathcal{K}$, tačiau kita teorema teigia, kad tai ne visada būtina. Pavyzdžiui, F_ξ, nepriklausanti klasei $\mathcal{D}$, gali generuoti $F_{\xi,m} \in \mathcal{D}$.

Teorema 6.10. *Tarkime, kad ξ yra realių reikšmių atsitiktinis dydis su pasiskirstymo funkcija F_ξ ir beigtiniu momentu $\mathbb{E}(\xi^+)^m$ kuriam nors $m > 0$. Tada, bendrai, galioja tokie sąryšiai:*

(i) $F_{\xi,m} \in \mathcal{R}_{\alpha-m},\ m < \alpha \ \Rightarrow\ F_\xi \in \mathcal{R}_\alpha\ (F_{\xi,m} \in \mathcal{R}\ \Rightarrow F_\xi \in \mathcal{R})$,

(ii) $F_{\xi,m} \in \mathcal{C} \ \not\Rightarrow\ F_\xi \in \mathcal{C}$,

(iii) $F_{\xi,m} \in \mathcal{D} \ \not\Rightarrow\ F_\xi \in \mathcal{D}$,

(iv) $F_{\xi,m} \in \mathcal{L} \ \not\Rightarrow\ F_\xi \in \mathcal{L}$,

(v) $F_{\xi,m} \in \mathcal{OL} \ \not\Rightarrow\ F_\xi \in \mathcal{OL}$.

6.6. Rezultatų sklaida

Disertacijos rezultatai publikuojami šiuose moksliniuose straipsniuose:

- R. Leipus, S. Paukštys, J. Šiaulys, *Tails of higher-order moments of sums with heavy-tailed increments and application to the Haezendonck–Goovaerts risk measure*, Statistics & Probability Letters 170 (2021), 1–12;

- M. Dirma, S. Paukštys, J. Šiaulys, *Tails of the Moments for Sums with Dominatedly Varying Random Summands*, Mathematics 9 (2021), 1–26;

- S. Paukštys, J. Šiaulys, R. Leipus, *Truncated Moments for Heavy-Tailed and Related Distribution Classes*, Mathematics 11 (2023), 1–15.

Disertacijos rezultatai buvo paskelbti šiuose moksliniuose renginiuose:

- S. Paukštys, *Atsitiktinių dydžių sumų momentų uodegų asimptotinis elgesys ir taikymai Haezendonck–Goovaerts rizikos matui*, Finansų ir draudimo matematikos seminaras, 2021 m. vasario 23 d., Vilnius;

- S. Paukštys, *Atsitiktinių dydžių sumų nupjautinių momentų ribinis elgesys*, Finansų ir draudimo matematikos seminaras, 2021 m. lapkričio 9d., Vilnius;

- S. Paukštys, *Atsitiktinių dydžių sumų nupjautinių momentų ribinio elgesio tyrimai ir taikymai rizikos teorijoje*, 10-asis Lietuvos jaunųjų matematikų susitikimas, 2021 m. gruodžio 28 d., Vilnius;

- S. Paukštys, *Tails of moments of sums with heavy-tailed summands and applications to the Haezendonck–Goovaerts risk measure*, European Actuarial Journal Conference 2022 Tartu, 2022 m. rugpjūčio 22 d., Tartu;

- S. Paukštys, *Sunkiauodegių skirstinių momentų uodegų savybės (disertacijos pristatymas)*, Finansų ir draudimo matematikos seminaras, 2023 m. kovo 13 d., Vilnius.

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NOTES

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