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Article

Randomly Stopped Sums with Generalized Subexponential Distribution

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Abstract: Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent possibly differently distributed random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution functions $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$. Let η be a counting random variable independent of sequence $\{\xi_1, \xi_2, \ldots\}$. In this paper, we find conditions under which the distribution function of randomly stopped sum $S_\eta = \xi_1 + \xi_2 + \ldots + \xi_\eta$ belongs to the class of generalized subexponential distributions.

Keywords: subexponentiality; generalized subexponentiality; heavy tail; randomly stopped sum

MSC: 60G50; 60G40; 60E05

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1. Introduction

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting random variable, that is, a nonnegative, nondegenerate at 0, and integer-valued r.v. In addition, we suppose that the r.v. η and the sequence $\{\xi_1, \xi_2, \ldots\}$ are independent.

Let $S_0 := 0$, $S_n := \xi_1 + \ldots + \xi_n$ for $n \in \mathbb{N}$, and let

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the randomly stopped sum of the r.v.s $\xi_1, \xi_2, \ldots$

By F_{S_η} we denote the d.f. of S_η, and, by $\overline{F}$, we denote the tail function (t.f.) of a d.f. F, that is, $\overline{F}(x) = 1 - F(x)$ for $x \in \mathbb{R}$. It is obvious that the following equalities hold for positive x:

$$F_{S_\eta}(x) = \mathbb{P}(\eta = 0) + \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n \leqslant x),$$

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n > x).$$

In this paper, we consider a sequence $\{\xi_1, \xi_2, \ldots\}$ of independent and possibly non-identically distributed r.v.s. We suppose that some of the d.f.s of these r.v.s belong to the class of generalized subexponential distributions $\mathcal{OS}$, and we find conditions under which d.f. F_{S_η} remains in this class.

We use the following notations for the asymptotic relations of arbitrary positive functions f and g: $f(x) = o(g(x))$ means that $\lim_{x\to\infty} f(x)/g(x) = 0$; $f(x) \underset{x\to\infty}{\sim} cg(x)$, c > 0, means that $\lim_{x\to\infty} f(x)/g(x) = c$; $f(x) = O\big(g(x)\big)$ means that $\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$; and $f(x) \underset{x\to\infty}{\asymp} g(x)$ means that

$$0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \leqslant \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty.$$

The rest of the paper is organized as follows. In Section 2, we describe a class of generalized subexponential distributions. Section 4 consists of some results on closure under randomly stopped sums for regularity classes related with generalized subexponential distributions. The main results of the paper are formulated in Section 3. The proofs of the main results are given in Sections 5 and 6. Finally, in Section 7, we provide two examples to expose the analytical usefulness of our results, and in section 8, we present short conclusions.

2. Generalized Subexponentiality

Let ξ be an r.v. defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with d.f. F_ξ.

- *A d.f. F_ξ of a real-valued r.v. is said to be generalized subexponential, denoted $F_\xi \in \mathcal{OS}$, if*

$$\limsup_{x\to\infty} \frac{\overline{F_\xi * F_\xi}(x)}{\overline{F}_\xi(x)} < \infty,$$

where $F_\xi * F_\xi$ denote the convolution of d.f. F_ξ with itself, i.e.,

$$F_\xi * F_\xi(x) = F_\xi^{*2}(x) := \int_{-\infty}^{\infty} F_\xi(x-y)\mathrm{d}F_\xi(y),\ x \in \mathbb{R}.$$

For distributions of nonnegative r.v.s, class $\mathcal{OS}$ was introduced by Klüppelberg [1] and later, for real-valued r.v.s, was studied by Shimura and Watanabe [2], Baltrūnas et al. [3], Watanabe and Yamamuro [4], Yu and Wang [5], Cheng and Wang [6], Lin and Wang [7], Konstantinides et al. [8], and Mikutavičius and Šiaulys [9], among others.

In [2], the class of distributions $\mathcal{OS}$ is considered together with other distribution regularity classes. In that paper, several closedness properties of the class $\mathcal{OS}$ were proven. For example, it is shown that the class $\mathcal{OS}$ is not closed under convolution roots. This means that there exists r.v. ξ such that n-fold convolution $F_\xi^{*n} \in \mathcal{OS}$ for all $n \geqslant 2$, but $F_\xi \notin \mathcal{OS}$. In [3], the simple conditions are provided under which a d.f. of the special form

$$F_\xi(x) = 1 - \exp\left\{ -\int\limits_0^x q(u)\mathrm{d}u \right\}$$

belongs to the class $\mathcal{OS}$, where q is some integrable hazard rate function. For distributions of class $\mathcal{OS}$, the closure under tail-equivalence and the closure under convolution are established in [4]. The detailed proofs of these closures for nonnegative r.v.s are presented in [1] and, for real-valued r.v.s, in [5]. The closure under convolution means that, in the case of independent r.v.s ξ_1, ξ_2, conditions $F_{\xi_1} \in \mathcal{OS}$, $F_{\xi_2} \in \mathcal{OS}$ imply that $F_{\xi_1} * F_{\xi_2} = F_{\xi_1+\xi_2} \in \mathcal{OS}$. The closure under tail-equivalence means that conditions $F_{\xi_1} \in \mathcal{OS}$, $\overline{F}_{\xi_1}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_2}(x)$ imply $F_{\xi_2} \in \mathcal{OS}$.

A counterexample, showing that $F_{\xi_1}, F_{\xi_2} \in \mathcal{OS}$ for independent r.v.s ξ_1, ξ_2 does not imply $F_{\xi_1 \vee \xi_2} \in \mathcal{OS}$, can be found in [7]. Moreover in that paper, the closure under minimum is established, which means that $F_{\xi_1}, F_{\xi_2} \in \mathcal{OS}$, for independent r.v.s ξ_1, ξ_2, imply $F_{\xi_1 \wedge \xi_2} \in \mathcal{OS}$. The authors of articles [8,9] consider when the distribution of the product of two independent random variables ξ, θ belongs to the class $\mathcal{OS}$. For instance, in [9], it is proven that d.f. $F_{\xi\theta}$ is generalized subexponential if $F_\xi \in \mathcal{OS}$ and θ is nonnegative and nondegenerate at point zero.

3. Main Results

In this section, we formulate two theorems which are the main assertions of this paper. The first theorem deals with the case when the counting r.v. has a finite support.

Theorem 1. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent r.v.s, and η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If η is bounded, $F_{\xi_1} \in \mathcal{OS}$, and, for other indices $k \geqslant 2$, either $F_{\xi_k} \in \mathcal{OS}$ or $\overline{F}_{\xi_k}(x) = O\big(\overline{F}_{\xi_1}(x)\big)$, then d.f. of randomly stopped sum F_{S_η} belongs to the class $\mathcal{OS}$.*

The case of unbounded support of counting r.v. is considered in the second theorem. In such a case, to be $F_{S_\eta} \in \mathcal{OS}$, we need the counting random variable to have a light tail.

Theorem 2. *Let $\{\eta, \xi_1, \xi_2, \ldots\}$ be independent r.v.s, where counting r.v. η is such that $\mathbb{E}e^{\lambda\eta} < \infty$ for all $\lambda > 0$. Then, $F_{S_\eta} \in OS$, if $F_{\xi_1} \in OS$ and one of the conditions below is satisfied:*

$$\text{(i)}\ \ \mathbb{P}(\eta = 1) > 0 \ \text{ and } \ \limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty;$$

$$\text{(ii)}\ 0 < \liminf_{x\to\infty} \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty.$$

We present the proofs of both theorems in Section 6. According to the statements of these theorems, many random variables with generalized subexponential distributions can be constructed. We demonstrate such constructions in Section 7.

4. Similar Results for Related Regularity Classes

In this section, we describe several classes of distributions related to the class $\mathcal{OS}$. For the described classes, we present some results on their closure with respect to a randomly stopped sum. We note that for some classes, the closedness of the randomly stopped sum is studied only in the case where the summands are identically distributed.

The class of generalized subexponential distributions is the direct generalization of

$$\widehat{\mathcal{S}} = \bigcup_{\gamma\geqslant 0} \mathcal{S}(\gamma),$$

where $\mathcal{S}(0) = \mathcal{S}$ is the class of the subexponential distributions and $\{\mathcal{S}(\gamma), \gamma > 0\}$ are the convolution equivalent distributions classes.

- *A d.f. F_ξ of a nonnegative r.v. ξ is said to be subexponential, denoted $F_\xi \in \mathcal{S}$, if*

  $$\overline{F_\xi * F_\xi}(x) \underset{x\to\infty}{\sim} 2\overline{F}_\xi(x).$$

  A d.f. F_ξ of a real-valued r.v. ξ is called subexponential if the positive part of d.f.

  $$F_\xi^+(x) = F_\xi(x)\mathbb{I}_{[0,\infty)}(x)$$

  belongs to the class $\mathcal{S}$.

The class of subexponential distributions was introduced by Chistyakov [10] and later considered by Athreya and Ney [11], Chover et al. [12,13], Embrechts and Goldie [14], Embrechts and Omey [15], Cline [16], and Cline and Samorodnitsky [17], among others.

- *A d.f. F_ξ of a real-valued r.v. ξ is said to be convolution equivalent with parameter $\gamma > 0$, denoted $F_\xi \in \mathcal{S}(\gamma)$, if the following requirements are satisfied:*

$$\text{(i)}\ \widehat{F}_\xi(\gamma) := \int_{-\infty}^{\infty} \mathrm{e}^{\gamma x} \mathrm{d}F_\xi(x) < \infty;$$

$$\text{(ii)}\ \lim_{x\to\infty} \frac{\overline{F}_\xi(x-y)}{\overline{F}_\xi(x)} = \mathrm{e}^{\gamma y} \ \text{ for all } \ y > 0;$$

$$\text{(iii)}\ \lim_{x\to\infty} \frac{\overline{F_\xi * F_\xi}(x)}{\overline{F}_\xi(x)} = 2c_\xi \ \text{ for some constant } c_\xi.$$

The study of class $\mathcal{S}(\gamma)$ goes back to Chover et al. [12,13], Embrechts and Goldie [14], and Klüppelberg [18]. It is well known that $F \in \mathcal{S}(\gamma)$ if and only if $F_\xi^+ \in \mathcal{S}(\gamma)$ (see Corollary 2.1(i) in [19]), and the constant c_ξ in the definition above is equal to $\widehat{F}_\xi(\gamma)$, (see [19–21]). For $\gamma > 0$, a standard example of d.f. in $\mathcal{S}(\gamma)$ is d.f. F satisfying

$$\overline{F}(x) \underset{x\to\infty}{\sim} c\,\mathrm{e}^{-\gamma x} x^{-\alpha}$$

with parameters $c > 0, \gamma > 0, \alpha > 1$ (see [22,23]).

For the class $\mathcal{S}$, the following result is obtained in Theorem 3.37 of [24] (see also [11,25–27]).

Theorem 3. *Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent real-valued r.v.s with common distribution $F_\xi \in \mathcal{S}$, and let η be independent of $\{\xi_1, \xi_2, \dots\}$ counting r.v. with expectation $\mathbb{E}\eta$, such that $\mathbb{E}(1+\varepsilon)^\eta < \infty$ for some $\varepsilon > 0$. Then,*

$$\overline{F}_{S_\eta}(x) \sim \mathbb{E}\eta \overline{F}_\eta(x),$$

and $F_{S_\eta} \in \mathcal{S}$.

For the class $\mathcal{S}(\gamma)$ with $\gamma > 0$, the following assertion is derived in Theorem C of [28] (see also [29–31] for related results).

Theorem 4. *Let $\{\xi_1, \xi_2, \dots\}$ be independent real-valued r.v.s with common distribution $F_\xi \in \mathcal{S}(\gamma)$, $\gamma > 0$, and let η be counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. If*

$$\sum_{n=0}^{\infty} \mathbb{P}(\eta = n) \max\left\{\left(\widehat{F}_\xi(\gamma) + \varepsilon\right)^n, 1\right\} < \infty$$

for some $\varepsilon > 0$, then $F_{S_\eta} \in \mathcal{S}(\gamma)$.

We note that, in Theorems 3 and 4, r.v.s in the sequences $\{\xi_1, \xi_2, \dots\}$ are identically distributed. However, there are related regularity classes for which similar results can be obtained in cases where r.v.s in $\{\xi_1, \xi_2, \dots\}$ are not necessarily identically distributed. Here, we discuss two such classes:

- *A d.f. F_ξ of a real-valued r.v. ξ is said to be dominatedly varying, denoted $F_\xi \in \mathcal{D}$, if*

$$\limsup_{x\to\infty} \frac{\overline{F_\xi}(yx)}{\overline{F}_\xi(x)} < \infty$$

  for all (or, equivalently, for some) $y \in (0,1)$;
- *A d.f. F_ξ of a real-valued r.v. ξ is said to be exponential-like-tailed, denoted $F_\xi \in \mathcal{L}(\gamma)$, if*

$$\lim_{x\to\infty} \frac{\overline{F_\xi}(x-y)}{\overline{F}_\xi(x)} = \mathrm{e}^{\gamma y}$$

for all $y > 0$.

- *A d.f.* F_ξ *of a real-valued r.v.* ξ *is said to be long-tailed, denoted* $F_\xi \in \mathcal{L}(0) = \mathcal{L}$, *if*

$$\lim_{x\to\infty} \frac{\overline{F_\xi}(x-y)}{\overline{F}_\xi(x)} = 1$$

for all (or,equivalently, for some) $y > 0$.

Class of dominatedly varying d.f.s $\mathcal{D}$ was introduced by Feller [32] and later considered in [4,33–38], among others. The class of long-tailed d.f.s $\mathcal{L}$ was introduced by Chistyakov [10] in the context of branching processes. The class $\mathcal{L}(\gamma)$ with $\gamma > 0$ was introduced by Chover et al. [12,13]. Later, the various properties of long-tailed and exponential-like-tailed d.f.s were considered in [1,19,24,28,37,39,40], for instance. Here, we recall only that $\mathcal{L} \cap \mathcal{D} \subset \mathcal{S}$ and $\mathcal{S}(\gamma) \subset \mathcal{L}(\gamma)$ for $\gamma \geqslant 0$.

The following assertion on $F_{S_\eta} \in \mathcal{D}$ is presented in Theorem 4 of [41].

Theorem 5. *Let* $\{\xi_1, \xi_2, \ldots\}$ *be a sequence of independent real-valued r.v.s with common d.f.* $F_\xi \in \mathcal{D}$, *and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$. *Then,* $F_{S_\eta} \in \mathcal{D}$ *if* $\mathbb{E}\eta^{p+1} < \infty$ *for some*

$$p > J^+_{F_\xi} := -\lim_{y\to\infty} \frac{1}{\log y} \log \liminf_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)}.$$

In the inhomogeneous case, when sumands are not necessarily identically distributed, the following statement is obtained in Theorem 2.1 of [42].

Theorem 6. *Let* $\{\xi_1, \xi_2, \ldots\}$ *be a sequence independent nonnegative r.v.s, and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$. *Then,* $F_{S_\eta} \in \mathcal{D}$ *if the following three conditions are satisfied:*

(i) $F_{\xi_\varkappa} \in \mathcal{D}$ *for some* $\varkappa \in \operatorname{supp}(\eta) := \{n \in \mathbb{N}_0 : \mathbb{P}(\eta = n) > 0\}$;

(ii) $\limsup_{x\to\infty} \sup_{n>\varkappa} \frac{1}{n\overline{F}_{\xi_\varkappa}(x)} \sum_{i=1}^{n} \overline{F}_{\xi_i}(x) < \infty$;

(iii) $\mathbb{E}\eta^{p+1} < \infty$ *for some* $p > J^+_{F_{\xi_\varkappa}}$.

Examples of conditions for the function F_{S_η} to belong to the class $\mathcal{L}(\gamma)$ are given in the theorems below. Theorem 7, proven in [41], presents conditions for the homogeneous case for class $\mathcal{L} = \mathcal{L}(0)$, while Theorem 8, proven in [43], gives conditions for the inhomogeneous case for class $\mathcal{L}(\gamma)$ with $\gamma \geqslant 0$.

Theorem 7. *Suppose that* $\{\xi_1, \xi_2, \ldots\}$ *are independent nonnegative r.v.s with common distribution* $F_\xi \in \mathcal{L}$, *and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$. *If*

$$\overline{F}_\eta(\delta x) = o\big(\sqrt{x}\,\overline{F}_\xi(x)\big)$$

for any $\delta \in (0,1)$, *then* $F_{S_\eta} \in \mathcal{L}$.

Theorem 8. *Let* $\{\xi_1, \xi_2, \ldots\}$ *be a sequence of independent r.v.s such that, for some* $\gamma \geqslant 0$,

$$\sup_{k\geqslant 1} \left| \frac{\overline{F}_{\xi_k}(x+y)}{\overline{F}_{\xi_k}(x)} - \mathrm{e}^{-\gamma y} \right| \underset{x\to\infty}{\to} 0$$

for each fixed $y > 0$, *and let* η *be a counting r.v. independent of* $\{\xi_1, \xi_2, \ldots\}$. *If*

$$\frac{\mathbb{P}(\eta = k+1)}{\mathbb{P}(\eta = k)} \underset{k\to\infty}{\to} 0,$$

then $F_{S_\eta} \in \mathcal{L}(\gamma)$.

In the context of the randomly stopped sums, the class $\mathcal{OS}$ was considered by Shimura and Watanabe [2]. In Proposition 3.1 of that paper, the following assertion is presented.

Theorem 9. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of nonnegative independent r.v.s with common d.f. F_ξ, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ such that*

$$\mathbb{P}(\eta > 1) > 0, \ \sup\left\{x \geqslant 1 : \sum_{k=0}^{\infty} \mathbb{P}(\eta = k)x^k < \infty\right\} = \infty.$$

Then, $F_\xi \in \mathcal{OS}$ if and only if $\overline{F}_{S_\eta}(x) \underset{x\to\infty}{\asymp} \overline{F}_\xi(x)$.

From the information presented, it can be seen that our main Theorems 1 and 2, in fact, are inhomogeneous versions of the formulated Theorem 9.

5. Auxiliary Lemmas

In this section, we present and prove some auxiliary lemmas that are then applied to the derivations of the main theorems, i.e., Theorems 1 and 2.

Lemma 1. *Let X and Y be two real-valued r.v.s with corresponding d.f.s F_X and F_Y. The following statements hold:*

(i) $F_X \in \mathcal{OS}$ if and only if $\sup\limits_{x\in\mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} < \infty$;

(ii) If $F_X \in \mathcal{OS}$ and $\overline{F}_Y(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$, then $F_Y \in \mathcal{OS}$;

(iii) If $F_X \in \mathcal{OS}$ and $F_Y \in \mathcal{OS}$, then $F_X * F_Y \in \mathcal{OS}$;

(iv) If $F_X \in \mathcal{OS}$, then $F_X \in \mathcal{OL}$ i.e., $\limsup\limits_{x\to\infty} \frac{\overline{F}_X(x-1)}{\overline{F}_X(x)} < \infty$;

(v) If $F_X \in \mathcal{OS}$ and $\overline{F}_Y(x) = O\big(\overline{F}_X(x)\big)$, then $F_X * F_Y \in \mathcal{OS}$ and $\overline{F_X * F_Y}(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$.

Proof. A large part of the properties of the class $\mathcal{OS}$ listed in Lemma 1 can be found, for instance, in [1,2,4,5]. However, for the sake of exposition completeness, we present the full proof of the formulated lemma.

Part (i). If $F_X \in OS$, then

$$\limsup_{x\to\infty} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} < \infty \tag{1}$$

according to the definition. This estimate implies that $\overline{F}_X(x) > 0$ for each $x \in \mathbb{R}$. In addition, the inequality (1) gives that

$$\frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} \leqslant M$$

if $x \geqslant x_M$ for some M and x_M.

If $x < x_M$, then, obviously, $\overline{F}_X(x) \geqslant \overline{F}_X(x_M)$ and $\overline{F_X * F_X}(x) \leqslant 1$.

Therefore, for each $x \in \mathbb{R}$, we obtain that

$$\frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} \leqslant \max\left\{M, \frac{1}{\overline{F}_X(x_M)}\right\} < \infty$$

because $\overline{F}_X(x_M) > 0$. The last estimate finishes the proof of part (i), because the condition

$$\sup_{x\in\mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} < \infty$$

implies (1), obviously.

Part (ii). The condition $\overline{F}_Y(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$ implies

$$\liminf_{x\to\infty} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)} > 0 \;\text{ and }\; \limsup_{x\to\infty} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)} < \infty. \tag{2}$$

It follows from this that

$$\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant N,\; x \geqslant x_N,$$

for some N and x_N. If $x < x_N$, then

$$\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant \frac{1}{\overline{F}_X(x_N)} < \infty$$

because $F_X \in \mathcal{OS}$. According to the derived estimates,

$$\sup_{x\in\mathbb{R}} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)} = \max\left\{N, \frac{1}{\overline{F}_X(x_N)}\right\} = C < \infty.$$

Therefore, for each $x \in \mathbb{R}$, we have

$$\begin{aligned}
\overline{F_Y * F_Y}(x) &= \int_{-\infty}^{\infty} \frac{\overline{F}_Y(x-y)}{\overline{F}_X(x-y)} \overline{F}_X(x-y)\mathrm{d}F_Y(y) \leqslant C \int_{-\infty}^{\infty} \overline{F}_X(x-y)\mathrm{d}F_Y(y) \\
&= C \int_{-\infty}^{\infty} \overline{F}_Y(x-y)\mathrm{d}F_X(y) = C \int_{-\infty}^{\infty} \frac{\overline{F}_Y(x-y)}{\overline{F}_X(x-y)} \overline{F}_X(x-y)\mathrm{d}F_X(y) \\
&\leqslant C^2 \int_{-\infty}^{\infty} \overline{F}_X(x-y)\mathrm{d}F_X(y) = C^2\, \overline{F_X * F_X}(x).
\end{aligned}$$

This estimate implies that

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\overline{F_Y * F_Y}(x)}{\overline{F}_Y(x)} &\leqslant C^2 \limsup_{x\to\infty} \frac{\overline{F_X * F_X}(x)}{\overline{F}_Y(x)} \\
&\leqslant C^2 \limsup_{x\to\infty} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} \frac{1}{\liminf\limits_{x\to\infty} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)}} < \infty
\end{aligned}$$

due to the assumption $F_X \in \mathcal{OS}$ and the first inequality in (2). The last estimate gives that d.f. F_Y belongs to the class $\mathcal{OS}$. Part (ii) of the lemma is proven.

Part (iii). According to part (i), we have that

$$\sup_{x\in\mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} = C_1 < \infty \;\text{ and }\; \sup_{x\in\mathbb{R}} \frac{\overline{F_Y * F_Y}(x)}{\overline{F}_Y(x)} = C_2 < \infty.$$

Let X_1, X_2, Y_1, Y_2 be independent r.v.s. Suppose that X_1, X_2 are distributed according to the d.f. F_X, and Y_1, Y_2 are distributed according to the d.f. F_Y. For each $x \in \mathbb{R}$, we obtain

$$\begin{aligned}
\overline{((F_X * F_Y)^*)^2}(x) &= \overline{(F_X * F_Y) * (F_X * F_Y)}(x) = \mathbb{P}(X_1 + Y_1 + X_2 + Y_2 > x) \\
&= \mathbb{P}(X_1 + X_2 + Y_1 + Y_2) > x) = \int_{-\infty}^{\infty} \mathbb{P}(X_1 + X_2 > x - y)\mathrm{d}\mathbb{P}(Y_1 + Y_2 \leqslant y) \\
&= \int_{-\infty}^{\infty} \frac{\overline{F_X * F_X}(x-y)}{\overline{F}_X(x-y)} \overline{F}_X(x-y)\mathrm{d}\mathbb{P}(Y_1 + Y_2 \leqslant y) \\
&\leqslant C_1 \int_{-\infty}^{\infty} \overline{F}_X(x-y)\mathrm{d}\mathbb{P}(Y_1 + Y_2 \leqslant y) = C_1\mathbb{P}(X_1 + Y_1 + Y_2 > x) \\
&= C_1 \int_{-\infty}^{\infty} \frac{\overline{F_Y * F_Y}(x-y)}{\overline{F}_Y(x-y)} \overline{F}_Y(x-y)\mathrm{d}\mathbb{P}(X_1 \leqslant y) \\
&\leqslant C_1 C_2 \int_{-\infty}^{\infty} \overline{F}_Y(x-y) dF_X(y) = C_1 C_2 \overline{F_X * F_Y}(x).
\end{aligned}$$

Hence,

$$\sup_{x \in \mathbb{R}} \frac{\overline{((F_X * F_Y)^*)^2}(x)}{\overline{F_X * F_Y}(x)} \leqslant C_1 C_2$$

implying that $F_X * F_Y \in \mathcal{OS}$ by part (i). Part (iii) of the lemma is proven.

Part (iv). Due to part (i),

$$\sup_{x \in \mathbb{R}} \frac{\overline{F_X * F_X}(x)}{\overline{F_X}(x)} = C_3 < \infty.$$

In addition, for $x > 2$, we obtain

$$\begin{aligned}
\overline{F_X * F_X}(x) &= \int_{-\infty}^{\infty} \overline{F}_X(x-t)\mathrm{d}F_X(t) \geqslant \int_{(1,x]} \overline{F}_X(x-t)\mathrm{d}F_X(t) \\
&\geqslant \overline{F}_X(x-1)(F_X(x) - F_X(1)).
\end{aligned}$$

When x is large enough, we have $F(x) - F(1) > 0$, and, therefore,

$$\frac{\overline{F}_X(x-1)}{\overline{F}_X(x)} \leqslant \frac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} \frac{1}{F_X(x) - F_X(1)}.$$

Hence,

$$\limsup_{x \to \infty} \frac{\overline{F}_X(x-1)}{\overline{F}_X(x)} \leqslant \frac{C_3}{\overline{F}_X(1)} < \infty,$$

and part (iv) of the lemma is proven.

Part (v). Since $\overline{F}_Y(x) = O(\overline{F}_X(x))$, we have

$$\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant Q, \ x \geqslant x_Q,$$

with certain constants Q and x_Q. If $x < x_Q$, then

$$\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant \frac{1}{\overline{F}_X(x_Q)} < \infty$$

because $F_X \in \mathcal{OS}$ implies $\overline{F}_X(x_Q) > 0$. From the above inequalities, it follows that

$$\sup_{x \in \mathbb{R}} \frac{\overline{F}_Y(x)}{\overline{F}_X(x)} \leqslant \max\left\{Q, \frac{1}{\overline{F}_X(x_Q)}\right\} = C_4.$$

Consequently, for $x \in \mathbb{R}$, we obtain

$$\begin{aligned} \overline{F_X * F_Y}(x) &= \int_{-\infty}^{\infty} \overline{F}_Y(x-y)\mathrm{d}F_X(y) \leqslant C_4 \int_{-\infty}^{\infty} \overline{F}_X(x-y)\mathrm{d}F_X(y) \\ &= C_4 \overline{F_X * F_X}(x) \leqslant C_5 \overline{F_X}(x) \end{aligned} \tag{3}$$

with some positive constant C_5, where the last step in the above derivation follows from part (i) of the lemma.

On the other hand, there exists a real $b \in \mathbb{R}$ for which

$$\overline{F}_Y(b) = 1 - F_Y(b) \geqslant \frac{1}{2}.$$

For this b, we obtain

$$\begin{aligned} \overline{F_X * F_Y}(x) &\geqslant \int_{(b,\infty)} \overline{F}_X(x-y)\mathrm{d}F_Y(y) \geqslant \overline{F}_X(x-b) \int_{(b,\infty)} \mathrm{d}F_Y(y) \\ &= \overline{F}_X(x-b)\overline{F}_Y(b) \geqslant \frac{1}{2}\overline{F}_X(x)\frac{\overline{F}_X(x-b)}{\overline{F}_X(x)}. \end{aligned}$$

Hence,

$$\liminf_{x \to \infty} \frac{\overline{F_X * F_Y}(x)}{\overline{F}_X(x)} \geqslant \frac{1}{2} \liminf_{x \to \infty} \frac{\overline{F}_X(x-b)}{\overline{F}_X(x)}. \tag{4}$$

In part (iv) of the lemma, we proved that $F_X \in \mathcal{OL}$. It is easy to verify that

$$F_X \in \mathcal{OL} \;\Leftrightarrow\; \limsup_{x \to \infty} \frac{\overline{F}_X(x-1)}{\overline{F}_X(x)} < \infty \;\Leftrightarrow\; \overline{F}_X(x-y) \underset{x\to\infty}{\asymp} \overline{F}_X(x) \text{ for each } y \in \mathbb{R}.$$

Therefore, the estimate (4) implies that

$$\liminf_{x \to \infty} \frac{\overline{F_X * F_Y}(x)}{\overline{F}_X(x)} > 0. \tag{5}$$

From inequalities (3) and (5), it follows that $\overline{F_X * F_Y}(x) \underset{x\to\infty}{\asymp} \overline{F_X}(x)$. Moreover, by part (ii) of the lemma, $F_X * F_Y \in \mathcal{OS}$. This finishes the proof of the last part of the lemma. □

Lemma 2. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent r.v.s, for which $F_{\xi_1} \in \mathcal{OS}$, and, for other indices $k \geqslant 2$, either $F_{\xi_k} \in \mathcal{OS}$ or $\overline{F}_{\xi_k}(x) = O(\overline{F}_{\xi_1}(x))$. Then, $F_{S_n} \in \mathcal{OS}$ for all $n \in \mathbb{N}$.*

Proof. If $n = 1$, then the statement is obvious because $S_1 = \xi_1$. If $n = 2$, then two options are possible: $F_{\xi_2} \in \mathcal{OS}$ or $\overline{F}_{\xi_2} = O(\overline{F}_{\xi_1}(x))$. In the first case, $F_{S_2} = F_{\xi_1} * F_{\xi_2} \in \mathcal{OS}$ according to part (iii) of Lemma 1. In the second case, $F_{S_2} \in \mathcal{OS}$ by part (v) of the same lemma.

Now, let $n > 2$, and denote

$$\mathcal{K} = \{k \in \{2, ..., n\} : \overline{F}_{\xi_k}(x) = O(\overline{F}_{\xi_1}(x))\}.$$

Initially, assume that the set $\mathcal{K}$ is empty. In such a case, $F_{\xi_k} \in \mathcal{OS}$ for all indices $k \in \mathcal{K}^c = \{1, 2, 3, \ldots, n\}$. By part (iii) of Lemma 1, we know that $F_{S_n} \in \mathcal{OS}$.

Now, let the index set $\mathcal{K} = \{k_1, k_2, \ldots, k_r\} \subset \{1, \ldots, n\}$ no longer be empty. Since

$$\overline{F}_{\xi_{k_1}}(x) = O(\overline{F}_{\xi_1}(x)),$$

part (v) of Lemma 1 implies that

$$F_{\xi_1} * F_{\xi_{k_1}} \in \mathcal{OS}, \tag{6}$$

and

$$\overline{F_{\xi_1} * F_{\xi_{k_1}}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x). \tag{7}$$

According to relation (7),

$$\limsup_{x\to\infty} \frac{\overline{F}_{\xi_{k_2}}(x)}{\overline{F_{\xi_1} * F_{\xi_{k_1}}}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}_{\xi_{k_2}}(x)}{\overline{F}_{\xi_1}(x)} \frac{1}{\liminf\limits_{x\to\infty} \frac{\overline{F_{\xi_1} * F_{\xi_{k_1}}}(x)}{\overline{F}_{\xi_1}(x)}} < \infty$$

because $\overline{F}_{\xi_{k_2}}(x) = O(\overline{F}_{\xi_1}(x))$. This means that

$$\overline{F}_{\xi_{k_2}}(x) = O(\overline{F_{\xi_1} * F_{\xi_{k_1}}}(x)).$$

Hence, according to (6) and part (v) of Lemma 1, we obtain

$$F_{\xi_1} * F_{\xi_{k_1}} * F_{\xi_{k_2}} = (F_{\xi_1} * F_{\xi_{k_1}}) * F_{\xi_{k_2}} \in \mathcal{OS},$$

and

$$\overline{F_{\xi_1} * F_{\xi_{k_1}} * F_{\xi_{k_2}}}(x) \underset{x\to\infty}{\asymp} \overline{F_{\xi_1} * F_{\xi_{k_1}}}(x).$$

Continuing the process we obtain

$$F_{\mathcal{K}} := F_{\xi_1} * \prod_{j=1}^{r}\!{}^{*}\, F_{\xi_{k_j}} = F_{\xi_1} * F_{\xi_{k_1}} * F_{\xi_{k_2}} * \ldots * F_{\xi_{k_r}} \in \mathcal{OS},$$

and

$$\overline{F_{\xi_1} * F_{\xi_{k_1}} * F_{\xi_{k_2}} * \ldots * F_{\xi_{k_r}}}(x) \underset{x\to\infty}{\asymp} \overline{F_{\xi_1} * F_{\xi_{k_1}} * F_{\xi_{k_2}} * \ldots * F_{\xi_{k_{r-1}}}}(x).$$

For the remaining indices $k \in \mathcal{K}^c = \{2, 3, \ldots, n\} \setminus \{k_1, k_2, \ldots, k_r\}$, d.f. F_{ξ_k} belongs to the class $\mathcal{OS}$. By part (iii) of Lemma 1, we obtain

$$F_{\mathcal{K}^c} := \prod_{k\in\mathcal{K}^c}\!{}^{*}\, F_{\xi_k} \in \mathcal{OS}.$$

Using part (iii) of Lemma 1 again, we derive that

$$F_{S_n} = F_{\mathcal{K}} * F_{\mathcal{K}^c} \in \mathcal{OS}.$$

This finishes the proof of Lemma 2. □

Lemma 3. *Let $\xi_1, \xi_2, \dots$ be a sequence of independent random variables, for which $F_{\xi_1} \in \mathcal{OS}$ and*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty \tag{8}$$

Then, there exists a constant $\widehat{C} \geqslant 1$ such that

$$\overline{F}_{S_n}(x) \leqslant \widehat{C}^{\,n-1}\overline{F}_{\xi_1}(x) \tag{9}$$

for all $x \in \mathbb{R}$ and for all $n \geqslant 2$.

Proof. The condition (8) implies that

$$\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant C_6$$

for all $x \geqslant A$, with some constants $C_6 \geqslant 1$ and $A > 0$. If $x < A$, then

$$\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \frac{1}{\overline{F}_{\xi_1}(x)} \leqslant \frac{1}{\overline{F}_{\xi_1}(A)} < \infty.$$

Therefore, for each $x \in \mathbb{R}$,

$$\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \max\left\{C_6, \frac{1}{\overline{F}_{\xi_1}(A)}\right\} := C_7. \tag{10}$$

In addition, part (i) of Lemma 1 gives that

$$\overline{F_{\xi_1} * F_{\xi_1}}(x) \leqslant C_8\overline{F}_{\xi_1}(x) \tag{11}$$

for all $x \in \mathbb{R}$ with some constant $C_8 \geqslant 1$.

We prove the inequality (9) with constant $\widehat{C} = C_7\, C_8$. If $n = 1$, the inequality (9) holds, evidently, because $\overline{F}_{S_1}(x) = \overline{F}_{\xi_1}(x)$. If $n = 2$, then, by (10) and (11), for $x \in \mathbb{R}$, we have

$$\begin{aligned}\overline{F}_{S_2}(x) &= \int_{-\infty}^{\infty} \overline{F}_{\xi_2}(x-y)\mathrm{d}F_{\xi_1}(y) \leqslant C_7 \int_{-\infty}^{\infty} \overline{F}_{\xi_1}(x-y)\mathrm{d}F_{\xi_1}(y)\\ &= C_7\overline{F_{\xi_1} * F_{\xi_1}}(x) \leqslant \widehat{C}\,\overline{F}_{\xi_1}(x).\end{aligned}$$

Suppose now that the inequality (9) holds for $n = m \geqslant 2$, i.e.,

$$\frac{\overline{F}_{S_m}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \widehat{C}^{\,m-1},\ x \in \mathbb{R}.$$

After choosing $n = m + 1$, from this assumption and from (10) and (11), we obtain

$$\begin{aligned}\overline{F}_{S_{m+1}}(x) &= \int_{-\infty}^{\infty} \overline{F}_{S_m}(x-y)\mathrm{d}F_{\xi_{m+1}}(y) \leqslant \widehat{C}^{\,m-1} \int_{-\infty}^{\infty} \overline{F}_{\xi_1}(x-y)\mathrm{d}F_{\xi_{m+1}}(y)\\ &= \widehat{C}^{\,m-1} \int_{-\infty}^{\infty} \overline{F}_{\xi_{m+1}}(x-y)\mathrm{d}F_{\xi_1}(y) \leqslant \widehat{C}^{\,m-1}C_7 \int_{-\infty}^{\infty} \overline{F}_{\xi_1}(x-y)\mathrm{d}F_{\xi_1}(y)\\ &= \widehat{C}^{\,m-1}C_7\,\overline{F_{\xi_1} * F_{\xi_1}}(x) \leqslant \widehat{C}^{\,m}\,\overline{F}_{\xi_1}(x),\ x \in \mathbb{R}.\end{aligned}$$

According to the induction principle, the inequality (9) holds for all $n \in \mathbb{N}$. Lemma 3 is proven. □

6. Proofs of the Main Results

In this section, we present proofs of the main results of the paper.

Proof of Theorem 1. According to conditions of the theorem, $\mathbb{P}(\eta \in \{0, 1, \ldots, L\}) = 1$ and $\mathbb{P}(\eta = L) > 0$ for some $L \in \mathbb{N}$. We have

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{L} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x),\, x > 0.$$

Hence, for each positive x,

$$\frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{S_L}(x)} \geqslant \frac{\mathbb{P}(\eta = L)\overline{F}_{S_L}(x)}{\overline{F}_{S_L}(x)} = \mathbb{P}(\eta = L) > 0. \tag{12}$$

On the other hand,

$$\overline{F}_{S_\eta}(x) = \sum_{k=0}^{L-1} \mathbb{P}(\eta = L-k)\mathbb{P}(S_{L-k} > x),\, x > 0. \tag{13}$$

For any random variable ξ_k, $k \in \{1, 2, \ldots, L\}$, there exists a negative number $-a_k$, for which $\mathbb{P}(\xi_k \geqslant -a_k) \geqslant 1/2$. We have

$$\begin{aligned}\mathbb{P}(S_{L-1} > x) &= \mathbb{P}(S_{L-1} - a_L > x - a_L, \xi_L \geqslant -a_L) + \mathbb{P}(S_{L-1} > x, \xi_L < -a_L)\\ &\leqslant \mathbb{P}(S_L > x - a_L) + \mathbb{P}(S_{L-1} > x)\mathbb{P}(\xi_L < -a_L).\end{aligned}$$

From this, we derive that

$$\mathbb{P}(S_{L-1} > x) \leqslant 2\mathbb{P}(S_L > x - a_L)$$

for each $x \in \mathbb{R}$. Similarly,

$$\mathbb{P}(S_{L-2} > x) \leqslant 2\mathbb{P}(S_{L-1} > x - a_{L-1}) \leqslant 4\mathbb{P}(S_L > x - a_{L-1} - a_L)$$

also for each real number x. Continuing the process, we obtain

$$\mathbb{P}(S_{L-k} > x) \leqslant 2^k \mathbb{P}\left(S_L > x - \sum_{j=0}^{k-1} a_{L-j}\right)$$

for all $x \in \mathbb{R}$ and for all $k = 1, 2, \ldots, L-1$. After inserting the derived estimates into inequality (13), we obtain that

$$\begin{aligned}\overline{F}_{S_\eta}(x) &\leqslant \sum_{k=0}^{L-1} \mathbb{P}(\eta = L-k)\, 2^k\, \mathbb{P}(S_L > x - \sum_{j=0}^{k-1} a_{L-j})\\ &\leqslant \mathbb{P}(S_L > x - a) \sum_{k=0}^{L-1} 2^k\, \mathbb{P}(\eta = L-k)\\ &= C^* \overline{F}_{S_L}(x - a),\end{aligned}$$

where

$$C^* = \sum_{k=0}^{L-1} 2^k\, \mathbb{P}(\eta = L-k), \text{ and } a = \sum_{j=1}^{L} a_j.$$

Consequently, for all positive x,

$$\frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{S_L}(x)} \leqslant \frac{C^*\overline{F}_{S_L}(x-a)}{\overline{F}_{S_L}(x)}.$$

By Lemma 2 and part (iv) of Lemma 1, we have that $F_{S_L} \in \mathcal{OS} \subset \mathcal{OL}$. Therefore,

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{S_{L^*}}(x)} < \infty. \tag{14}$$

By (12) and (14), we have that

$$\overline{F}_{S_\eta}(x) \underset{x\to\infty}{\asymp} \overline{F}_{S_L}(x).$$

Therefore, $F_{S_\eta} \in \mathcal{OS}$, together with F_{S_L} by part (ii) of Lemma 1. Theorem 1 is proven. □

Proof of Theorem 2. Part (i) Because

$$\overline{F}_{S_\eta}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)\overline{F}_{S_n}(x),\ x > 0,$$

by Lemma 3 for all positive numbers x, we obtain

$$\frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \frac{\sum_{n=1}^{\infty} \widehat{C}^{\,n-1}\mathbb{P}(\eta = n)\overline{F}_{\xi_1}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \mathbb{E}\, e^{\eta \log \widehat{C}} < \infty, \tag{15}$$

where $\widehat{C} > 1$ is some constant.

On the other hand,

$$\overline{F}_{S_\eta}(x) \geqslant \mathbb{P}(\eta = 1)\overline{F}_{\xi_1}(x).$$

Hence, under conditions of part (i), we have that $\overline{F}_{S_\eta}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$. Therefore, $F_{S_\eta} \in \mathcal{OS}$ according to part (ii) of Lemma 1. Part (i) of Theorem 2 is proven.

Part (ii). If $\mathbb{P}(\eta = 1) > 0$, then assertion of this part follows from the proven part (i). Hence, we can further suppose that $\mathbb{P}(\eta = 1) = 0$, implying that $\mathbb{P}(\eta \geqslant 2) > 0$. Since $\mathbb{E}e^{\lambda\eta} < \infty$ for each $\lambda > 0$, the inequality (15) implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{\xi_1}(x)} < \infty. \tag{16}$$

In addition, conditions of part (ii) of the theorem give that

$$\inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \geqslant \Delta$$

for all $x \geqslant x_\Delta$ and some positive Δ. If $x < x_\Delta$, then

$$\inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \geqslant \inf_{k\geqslant 1} \overline{F}_{\xi_k}(x_\Delta) \geqslant \inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x_\Delta)}{\overline{F}_{\xi_1}(x_\Delta)}\,\overline{F}_{\xi_1}(x_\Delta) \geqslant \Delta\,\overline{F}_{\xi_1}(x_\Delta) := \widetilde{C} > 0$$

due to the assumption $F_{\xi_1} \in \mathcal{OS}$. The derived inequalities imply that

$$\overline{F}_{\xi_k}(x) \geqslant \widetilde{C}\,\overline{F}_{\xi_1}(x)$$

for some positive constant $\widetilde{C}$, and for all $x \in \mathbb{R}$, $k \in \{1, 2, \ldots\}$.

Using the last estimate, we obtain

$$\overline{F}_{S_2}(x) = \int_{-\infty}^{\infty} \frac{\overline{F}_{\xi_2}(x-y)}{\overline{F}_{\xi_1}(x-y)} \, \overline{F}_{\xi_1}(x-y) \, \mathrm{d}F_{\xi_1}(y)$$
$$\geqslant \widetilde{C} \, \overline{F_{\xi_1} * F_{\xi_1}}(x) \geqslant \widetilde{C} \, \overline{F}_{\xi_1}(0)\overline{F}_{\xi_1}(x), \; x \in \mathbb{R}.$$

Similarly,

$$\overline{F}_{S_3}(x) = \int_{-\infty}^{\infty} \frac{\overline{F}_{S_2}(x-y)}{\overline{F}_{\xi_1}(x-y)} \, \overline{F}_{\xi_1}(x-y) \, \mathrm{d}F_{\xi_1}(y)$$
$$\geqslant \widetilde{C} \, \overline{F}_{\xi_1}(0)\overline{F_{\xi_1} * F_{\xi_1}}(x) \geqslant \widetilde{C} \, \left(\overline{F}_{\xi_1}(0)\right)^2 \overline{F}_{\xi_1}(x), \; x \in \mathbb{R}.$$

Continuing the process, we obtain

$$\overline{F}_{S_n}(x) \geqslant \widetilde{C} \left(\overline{F}_{\xi_1}(0)\right)^{n-1} \overline{F}_{\xi_1}(x)$$

for all $x \in \mathbb{R}$ and $n \in \{2, 3, \ldots\}$.

Therefore,

$$\liminf_{x\to\infty} \frac{\overline{F}_{S_\eta}(x)}{\overline{F}_{\xi_1}(x)} \geqslant \liminf_{x\to\infty} \frac{\mathbb{P}(\eta = \widetilde{L})\overline{F}_{S_{\widetilde{L}}}(x)}{\overline{F}_{\xi_1}(x)}$$
$$\geqslant \mathbb{P}(\eta = \widetilde{L}) \, \widetilde{C} \left(\overline{F}_{\xi_1}(0)\right)^{\widetilde{L}-1} > 0, \tag{17}$$

where $\widetilde{L} = \min\{n \geqslant 2 : \mathbb{P}(\eta = n) > 0\}$.

The derived inequalities (16) and (17) imply $\overline{F}_{S_\eta}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$. By part (ii) of Lemma 1, we have $F_{S_\eta} \in \mathcal{OS}$. Theorem 2 is proven. □

7. Illustration of the Results

In this section, we present two examples showing how, using Theorems 1 and 2, it is possible to construct distributions belonging to the class $\mathcal{OS}$. It is practically impossible to write the analytical expression of d.f F_{S_η} in the general case, but, according to Theorems 1 and 2, we can establish whether the constructed distributions are generalized subexponential.

Example 1. *Let ξ_1 be r.v. having the t.f.*

$$\overline{F}_{\xi_1}(x) = \mathbb{I}_{(-\infty,0)}(x) + \frac{\mathrm{e}^{-x}}{(1+x)^3}\left(1 + \frac{\sin x}{d}\right) \mathbb{I}_{[0,\infty)}(x),$$

where $d > 2$. According to the results of [44], the d.f. F_{ξ_1} belongs to class $\mathcal{OS}$. Therefore, Theorem 1 gives that d.f. F_{S_η} belongs to $\mathcal{OS}$ for each sequence of independent r.v.s $\{\xi_1, \xi_2, \ldots\}$ such that either $F_{\xi_k} \in \mathcal{OS}$ or

$$\overline{F}_{\xi_k}(x) = O\left(\frac{\mathrm{e}^{-x}}{(1+x)^3}\right)$$

when $k \in \{2, 3, \ldots\}$, and for each bounded counting r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$.

In particular, the d.f. with tail

$$\overline{F}_{S_\eta}(x) = \mathbb{I}_{(-\infty,0)}(x) + \frac{1}{3}\left(\overline{F}_{\xi_1}(x) + \overline{F_{\xi_1} * F_{\xi_2}}(x) + \overline{F_{\xi_1} * F_{\xi_2} * F_{\xi_3}}(x)\right) \mathbb{I}_{[0,\infty)}(x)$$

belongs to the class $\mathcal{OS}$ with

$$\overline{F}_{\xi_1}(x) = \mathbb{I}_{(-\infty,0)}(x) + \frac{e^{-x}}{(1+x)^3}\left(1+\frac{\sin x}{3}\right)\mathbb{I}_{[0,\infty)}(x),$$
$$\overline{F}_{\xi_2}(x) = \mathbb{I}_{(-\infty,0)}(x) + \frac{e^{-x}}{(1+x)^3}\left(1+\frac{\sin x}{4}\right)\mathbb{I}_{[0,\infty)}(x),$$
$$\overline{F}_{\xi_3}(x) = \mathbb{I}_{(-\infty,0)}(x) + \frac{e^{-x}}{(1+x)^3}\mathbb{I}_{[0,\infty)}(x).$$

Example 2. *Let $\{\eta, \xi_1, \xi_2, \ldots\}$ be independent r.v.s, where counting r.v. η is distributed according to the Poisson law with parameter $\mu > 0$, and*

$$\overline{F}_{\xi_k}(x) = \begin{cases} \mathbb{I}_{(-\infty,1)}(x) + e^{1-x}x^{-2}\,\mathbb{I}_{[1,\infty)}(x) & \text{if } k \in \{1,3,5,\ldots\}, \\ \mathbb{I}_{(-\infty,2)}(x) + 4\,e^{2-x}x^{-2}\,\mathbb{I}_{[2,\infty)}(x) & \text{if } k \in \{2,4,6,\ldots\}. \end{cases}$$

According to the results of [22], d.f. F_{ξ_1} belongs to the class $\mathcal{OS}$. In addition,

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} = 4\,e.$$

Hence, d.f. F_{S_η} with the t.f.

$$\overline{F}_{S_\eta}(x) = \mathbb{I}_{(-\infty,1)}(x) + e^{-\mu}\sum_{n=1}^{\infty}\frac{\mu^n}{n!}\,\overline{F_{\xi_1} * F_{\xi_2} * \ldots F_{\xi_n}}(x)\,\mathbb{I}_{[1,\infty)}(x)$$

is generalized subexponential, due to Theorem 2.

8. Concluding Remarks

One of the incentives to study the closure properties is the evolution of ruin probability in the insurance business. We recall that, in the renewal risk model (Sparre Andersen model), the risk process is

$$R_x(t) = x + pt - \sum_{i=1}^{N(t)} Z_i,\ t \geqslant 0,$$

where $x \geqslant 0$ is the initial capital, $p > 0$ is the premium rate, $\{Z_1, Z_2, \ldots\}$ is a sequence of nonnegative independent and identically distributed random claims, and

$$N(t) = \#\{n \geqslant 1 : \theta_1 + \theta_2 + \ldots + \theta_n \leqslant t\}$$

is a counting process generated by independent and identically distributed inter-arrival times $\{\theta_1, \theta_2, \ldots\}$. In addition, it is assumed that the sequences $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ are independent.

It is well known (see, for instance, [25,45–47]) that the model ruin probability

$$\psi(x) = \mathbb{P}\Big(\inf_{t\geqslant 0} R_x(t) < 0\Big) \underset{x\to\infty}{\sim} \frac{1}{\rho}\,\overline{F}_I(x)$$

in the case when $\rho = (p\,\mathbb{E}\theta_1/\mathbb{E}X_1) - 1 > 0$ and the integrated tail d.f.

$$F_I(x) = \frac{1}{\mathbb{E}X_1}\int_0^x \mathbb{P}(X_1 > y)\mathrm{d}y$$

belongs to the class $\mathcal{S}$. The similar results for the generalized subexponential distributions are presented in [48–50].

The results obtained in this work can be applied to the analysis of the compound renewal risk model, which is described in [51–53], for instance, and in which the claim amount has the form of a randomly stopped sum.

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