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Joint Approximation of Analytic Functions by Shifts of Lerch Zeta-Functions

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Abstract: In this paper, we consider the simultaneous approximation of tuples of analytic functions by tuples of shifts of Lerch zeta-functions with arbitrary parameters. We prove that there exists a closed set of tuples of functions analytic in the right-hand side of the critical strip, which is approximated by the above tuples of shifts. Further, a generalization for some compositions of tuples of Lerch zeta-functions is given.

Keywords: Lerch zeta-function; space of analytic functions; approximation of analytic functions; weak convergence

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1. Introduction

Let $s = \sigma + it$ be a complex variable, and $\lambda \in \mathbb{R}$ and $0 < \alpha \leqslant 1$ be fixed parameters. The Lerch zeta-function $L(\lambda, \alpha, s)$ (also called the Lerch–Hurwitz zeta-function because it was introduced independently in [1,2]) is defined, for $\sigma > 1$, by the Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s},$$

and has the meromorphic continuation to the whole complex plane $\mathbb{C}$. For integer λ, $L(\lambda, \alpha, s)$ coincides with the Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}, \quad \sigma > 1,$$

which, for $\alpha = 1$, becomes the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1.$$

Therefore, in this case, the function $L(\lambda, \alpha, s)$ has the unique simple pole at the point $s = 1$ with residue 1. For non-integer λ, $L(\lambda, \alpha, s)$ is an entire function. Thus, the Lerch zeta-function is a generalization of the classical zeta-functions $\zeta(s)$ and $\zeta(s, \alpha)$ and is widely cultivated not only in analytic number theory but also in the theory of special functions. Moreover, the function $L(\lambda, \alpha, s)$ appears in some problems of algebraic number theory connected to various types of dependence of numbers. The dependence of $L(\lambda, \alpha, s)$ on two parameters, λ and α, allows the study of certain classes of algebraic numbers. On the other hand, the parameters λ and α play an important role in the analytic behavior of the function $L(\lambda, \alpha, s)$. The analytic theory of $L(\lambda, \alpha, s)$ is given in [3].

We recall that the phenomenon of the universality of the function $\zeta(s)$ was discovered by S.M. Voronin in [4], see also [5–7], and, roughly speaking, means that any analytic non-vanishing function in the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ can be approximated by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. This surprising result is not only important in the approximation theory of analytic functions but also leads to solving some number-theoretical problems; for example, this was used to prove the functional independence of $\zeta(s)$ [8,9]. Moreover, the universality property of $\zeta(s)$ found applications in quantum mechanics; see, for example, [10,11], for the estimation of complicated integrals and description of the behavior of particles. Thus, universality is a useful property of $\zeta(s)$. Therefore, a problem rose to extend investigations of the universality of zeta-functions and study the related questions. More results and problems connected to the universality of zeta-functions can be found in an informative survey paper [12].

It is known that the function $L(\lambda, \alpha, s)$, for some classes of the parameters λ and α, is universal in the Voronin sense, i.e., its shifts $L(\lambda, \alpha, s + i\tau)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions defined in the strip D. We recall some results of such a type. We denote by $\mathcal{K}$ the class of compact subsets of D with connected complements and by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Moreover, let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then it is known [3,13] that, for transcendental α, every $K \in \mathcal{K}$, $f(s) \in H(K)$ and $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |L(\lambda, \alpha, s + i\tau) - f(s)| < \varepsilon \right\} > 0. \tag{1}$$

The latter inequality shows that there are infinitely many shifts $L(\lambda, \alpha, s + i\tau)$ that, with accuracy ε approximate a given function $f(s)$ of the class $H(K)$. Note that this result is stronger than Voronin's theorem [4] for $\zeta(s)$, which gives the existence of only one approximating shift $\zeta(s + i\tau)$, and K is a disc. A weighted version of the main theorem of [13] is given in [14]. We observe that the function $L(\lambda, \alpha, s)$, as an analytic object depending on two parameters having a rich set of values and closely connected to special functions, sometimes is more relevant than other zeta-functions for the investigation of the behavior of analytic functions. The situation requires paying attention to the approximating properties of $L(\lambda, \alpha, s)$ with different than in [13] classes of λ and α.

The hypothesis on the transcendence of α can be replaced by the requirement on the linear independence over the field of rational numbers $\mathbb{Q}$ for the set

$$\{\log(m + \alpha) : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}.$$

In view of the periodicity of $\mathrm{e}^{2\pi i \lambda m}$, it suffices to consider only a case $0 < \lambda \leqslant 1$. If the parameters λ and α are rational, say, $\lambda = a_1/q_1$ and $\alpha = a_2/q_2$, $a_1 < q_1$, $(a_1, q_1) = 1$, $a_2 < q_2$, $(a_2, q_2) = 1$, then the function $L(\lambda, \alpha, s)$ reduces to a linear combination of Hurwitz zeta-functions

$$\begin{aligned} L(\lambda, \alpha, s) &= \frac{1}{q_1^s} \sum_{l=0}^{q_1 - 1} \mathrm{e}^{2\pi i (a_1/q_1) l} \sum_{m=0}^{\infty} \frac{1}{(m + (l + a_2/q_2)/q_1)^s} \\ &= \frac{1}{q_1^s} \sum_{l=0}^{q_1 - 1} \mathrm{e}^{2\pi i (a_1/q_1) l} \zeta\left(s, \frac{lq_2 + a_2}{q_1 q_2}\right). \end{aligned}$$

Therefore, in this case, the universality of the function $L(\lambda, \alpha, s)$ reduces to the joint universality of Hurwitz zeta-functions with rational parameters.

The case of the algebraic irrational parameter α remains an open problem. For the Hurwitz zeta-function, the best result, in this case, is given in [15] and is based on the application of deep properties of algebraic numbers. There exists the Linnik–Ibragimov conjecture; see, for example, [16], that all functions defined by Dirichlet series and satisfying some natural growth conditions are universal in the Voronin sense. This conjecture also concerns the function $L(\lambda, \alpha, s)$. Therefore, it is important to develop the approximation theory for Lerch zeta-functions with arbitrary parameters λ and α.

In [17], the following theorem on approximation of analytic functions by shifts $L(\lambda, \alpha, s + i\tau)$ has been obtained. We denote by $H(D)$ the space of analytic on D functions endowed with the topology of uniform convergence on compacta. Then, in [17], it was proven that, for all λ and α, there exists a non-empty closed set $F_{\lambda,\alpha} \subset H(D)$ such that, for every compact set $K \subset D$, $f(s) \in F_{\lambda,\alpha}$ and $\varepsilon > 0$, the equality (1) is valid. Moreover, "lim inf" in (1) can be replaced by "lim" for all but at most countably many $\varepsilon > 0$.

Really, the Lerch zeta-function $L(\lambda, \alpha, s)$ is a class of functions depending on two parameters, λ and α. When λ and α vary, we obtain a collection of functions. Thus, a simultaneous approximation of a collection of analytic functions $(f_1(s), \dots, f_r(s))$ by shifts $(L(\lambda_1, \alpha_1, s + i\tau), \dots, L(\lambda_r, \alpha_r, s + i\tau))$ has a sense. Such an approximation of a wide class of collections of analytic functions is called joint universality. The first result in this direction is the following theorem obtained in [18]; see also [3]. Recall that the numbers $\alpha_1, \dots, \alpha_r$ are called algebraically independent of $\mathbb{Q}$, if for any polynomial $p(s_1, \dots, s_r) \not\equiv 0$ with rational coefficients, we have $p(\alpha_1, \dots, \alpha_r) \neq 0$.

Theorem 1 ([18])**.** *Suppose that $\alpha_1, \dots, \alpha_r$ are algebraically independent numbers over $\mathbb{Q}$, and, for $j = 1, \dots, r$, $\lambda_j = a_j/q_j$, $a_j \in \mathbb{Z}^+$, $(a_j, q_j) = 1$, $a_j < q_j$, where $q_1, \dots, q_r$ are distinct positive integers. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(\lambda_j, \alpha_j, s + i\tau) - f_j(s)| < \varepsilon \right\} > 0. \tag{2}$$

In [19], Theorem 1 was extended for arbitrary $0 < \lambda \leqslant 1$. Thus, the main result of [19] contains Theorem 1.

Let $k_1, \dots, k_r$ be positive integers, and $\lambda_{lj} \notin \mathbb{Z}$ arbitrary rational numbers, $l = 1, \dots, r$, $j = 1, \dots, k_r$. In [20,21], the approximation shifts by

$$\big(L(\lambda_{11}, \alpha_1, s + i\tau), \dots, L(\lambda_{1k_1}, \alpha_1, s + i\tau), \dots, L(\lambda_{r1}, \alpha_r, s + i\tau), \dots, L(\lambda_{rk_r}, \alpha_r, s + i\tau)\big)$$

was considered for algebraically independent numbers $\alpha_1, \dots, \alpha_r$ and numbers $e^{2\pi i \lambda_{lj}}$, $l = 1, \dots, r$, $j = 1, \dots, k_r$ satisfying a certain matrix rank condition. This also extends the region of joint approximation by shifts in Lerch zeta-functions.

In [22], H. Mishou conjectured that if $0 < \alpha < 1$ is a transcendental number and $\lambda_1, \dots, \lambda_r \in [0, 1)$ are distinct numbers, then the functions $L(\lambda_1, \alpha, s), \dots, L(\lambda_r, \alpha, s)$ are jointly universal, and prove the universality theorems supporting his conjecture. This conjecture was completely proven in [23]. We observe that an exceptional feature of Mishou's theorems is the simplicity of their statements. Ł. Pańkowski [24] proved the so-called joint hybrid universality theorem for Lerch zeta-functions. Denote by $\| \cdot \|$ the distance to the nearest integer.

Theorem 2 ([24])**.** *Suppose that $\alpha_1, \dots, \alpha_r$ are algebraically independent numbers, and $0 < \lambda_j \leqslant 1$, $j = 1, \dots, r$. Moreover, let $\beta_1, \dots, \beta_m$ be real numbers linearly independent over $\mathbb{Q}$, and $\theta_1, \dots, \theta_m$ any real numbers. Let $K \in \mathcal{K}$ and $f_j(s) \in H(K)$, $j = 1, \dots, r$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K} |L(\lambda_j, \alpha_j, s + i\tau) - f_j(s)| < \varepsilon, \right.$$
$$\left. \max_{1\leqslant j\leqslant m} \|\tau\beta_j - \theta_j\| < \varepsilon \right\} > 0.$$

Theorem 2 shows that the set of approximating shifts $L(\lambda_j, \alpha_j, s + i\tau)$, $j = 1, \dots, r$, is sufficiently wide, it has a positive lower density even under an additional condition.

All above-mentioned approximation theorems for Lerch zeta-functions use certain types of independence for the parameters λ_{jl} (matrix rank condition) and α_j (algebraic independence).

Our aim is motivated by the Linnik–Ibragimov conjecture, is devoted to an extension of approximating properties of Lerch zeta-functions, and consists of new joint theorems on a certain approximation by shifts in Lerch zeta-functions, which is valid for all parameters $0 < \lambda_j \leqslant 1$ and $0 < \alpha_j \leqslant 1$. In other words, we will give a joint generalization of the theorem from [17]. For brevity, let $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$ and $\underline{\alpha} = (\alpha_j, \dots, \alpha_r)$.

Theorem 3. *Suppose that $0 < \lambda_j \leqslant 1$ and $0 < \alpha_j \leqslant 1$ are arbitrary, $j = 1, \dots, r$. Then there exists a non-empty closed set $F_{\underline{\lambda},\underline{\alpha}} \subset H^r(D)$ such that, for compact sets $K_1, \dots, K_r$ of the strip D, $(f_1(s), \dots, f_r(s)) \in F_{\underline{\lambda},\underline{\alpha}}$, and $\varepsilon > 0$, inequality (2) is valid. Moreover, the limit*

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(\lambda_j, \alpha_j, s + i\tau) - f_j(s)| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Theorem 3 has a generalization for compositions $\Phi(L(\lambda_1, \alpha_1, s), \dots, L(\lambda_r, \alpha_r, s))$, where $\Phi : H^r(D) \to H(D)$ is a continuous operator. We give one theoretical example.

Theorem 4. *Suppose that $0 < \lambda_j \leqslant 1$ and $0 < \alpha_j \leqslant 1$, $j = 1, \dots, r$, are arbitrary. Then there exists a non-empty closed set $F_{\underline{\lambda},\underline{\alpha}} \subset H^r(D)$ such that if $\Phi : H^r(D) \to H(D)$ is a continuous operator such that, for every compact set $G \subset H(D)$, the set $(\Phi^{-1}G) \cap F_{\underline{\lambda},\underline{\alpha}}$ is not empty, then, for every compact set $K \subset D$, $f(s) \in \Phi(F_{\underline{\lambda},\underline{\alpha}})$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{ \tau \in [0, T] : \sup_{s\in K} |\Phi(L(\lambda_1, \alpha_1, s + i\tau), \dots, L(\lambda_r, \alpha_r, s + i\tau)) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, "$\liminf$" can be replaced by "$\lim$" for all but at most countably many $\varepsilon > 0$.

The novelty of Theorems 3 and 4 is a new type of joint approximation of analytic functions by shifts of Lerch zeta-functions with arbitrary parameters λ_j and α_j. These theorems are the first step to extending the joint universality property for Lerch zeta-functions with arbitrary parameters. Note that, differently from universality theorems, the requirement $K_j \in \mathcal{K}$ is not used. The theorems obtained, as the majority of results of the number theory, are of theoretical type; the main purpose is the development of new methods in the theory of approximation of analytic functions by using zeta-functions. The main shortcoming of Theorems 3 and 4, as in all universality theorems, is their non-effectivity. The most important task in the future is the description of the set $F_{\underline{\lambda},\underline{\alpha}}$. In our opinion, it is too early to speak about the practical applications of the above theorems. However, it can happen that physicists will observe them.

The proofs of Theorems 3 and 4 are probabilistic based on the weak convergence of probability measures in the space of analytic functions. Probabilistic limit theorems are the main ingredient for the proofs of Theorems 3 and 4.

2. Case of a Compact Group

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of a topological space $\mathbb{X}$. Let γ be the unit circle on the complex plane, i.e., $\gamma = \{s \in \mathbb{C} : |s| = 1\}$. Define the set

$$\Omega = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. With the product topology and operation of pointwise multiplication, the torus Ω, in view of the classical Tikhonov theorem [25] ("the product of any collection of compact topological spaces is compact with respect to the product topology"), is a compact topological group. Define one more set

$$\Omega^r = \prod_{j=1}^{r} \Omega_j,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$. Then, again, the Tikhonov theorem implies that Ω^r is a compact topological group. For $A \in \mathcal{B}(\Omega^r)$, define

$$Q_{T,\underline{\alpha}}(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \Big((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0\Big), \dots, \Big((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0\Big) \in A\Big\}.$$

Lemma 1. *Suppose that* $0 < \alpha_j \leqslant 1$, $j = 1, \dots, r$, *are arbitrary numbers. Then, on* $(\Omega^r, \mathcal{B}(\Omega^r))$, *there exists a probability measure* $Q_{\underline{\alpha}}$ *such that* $Q_{T,\underline{\alpha}}$ *converges weakly to* $Q_{\underline{\alpha}}$ *as* $T \to \infty$.

Proof. Denote by $\omega_j(m)$ the mth component of an element $\omega_j \in \Omega_j$, $m \in \mathbb{N}_0$, $j = 1, \dots, r$. We will prove that the Fourier transform $g_{T,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = (k_{jm} : k_{jm} \in \mathbb{Z}, m \in \mathbb{N}_0)$, $j = 1, \dots, r$, of $Q_{T,\underline{\alpha}}$, converges to a Fourier transform of a certain probability measure on $(\Omega^r, \mathcal{B}(\Omega^r))$ as $T \to \infty$. We have

$$\begin{aligned} g_{T,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) &= \int_{\Omega^r} \left(\prod_{j=1}^{r} \sideset{}{^*}\prod_{m \in \mathbb{N}_0} \omega_j^{k_{jm}}(m)\right) \mathrm{d}Q_{T,\underline{\alpha}} = \frac{1}{T}\int_0^T \left(\prod_{j=1}^{r} \sideset{}{^*}\prod_{m \in \mathbb{N}_0} (m+\alpha_j)^{-i\tau k_{jm}}\right) \mathrm{d}\tau \\ &= \frac{1}{T}\int_0^T \exp\left\{-i\tau \sum_{j=1}^{r} \sideset{}{^*}\sum_{m \in \mathbb{N}_0} k_{jm} \log(m+\alpha_j)\right\} \mathrm{d}\tau, \end{aligned} \tag{3}$$

where the star "$*$" shows that only a finite number of integers k_{jm} are distinct from zero. Define

$$A_{\underline{\alpha}} = \left\{(\underline{k}_1, \dots, \underline{k}_r) : \sum_{j=1}^{r} \sideset{}{^*}\sum_{m \in \mathbb{N}_0} k_{jm} \log(m+\alpha_j) = 0\right\}$$

and (4)

$$B_{\underline{\alpha}} = \left\{(\underline{k}_1, \dots, \underline{k}_r) : \sum_{j=1}^{r} \sideset{}{^*}\sum_{m \in \mathbb{N}_0} k_{jm} \log(m+\alpha_j) \neq 0\right\}.$$

Then, by (4), we have

$$g_{T,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) = 1$$

for $(\underline{k}_1, \dots, \underline{k}_r) \in A_{\underline{\alpha}}$, and

$$g_{T,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) = \frac{1 - \exp\Big\{-iT\sum_{j=1}^{r} \sum^*_{m \in \mathbb{N}_0} \log(m+\alpha_j)\Big\}}{iT\Big(1 - \exp\Big\{-i\sum_{j=1}^{r} \sum^*_{m \in \mathbb{N}_0} k_{jm} \log(m+\alpha_j)\Big\}\Big)}$$

for $(\underline{k}_1, \ldots, \underline{k}_r) \in B_{\underline{\alpha}}$. Therefore,

$$\lim_{T\to\infty} g_{T,\underline{\alpha}}(\underline{k}_1, \ldots, \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, \ldots, \underline{k}_r) \in A_{\underline{\alpha}}, \\ 0 & \text{if } (\underline{k}_1, \ldots, \underline{k}_r) \in B_{\underline{\alpha}}. \end{cases}$$

This shows that $Q_{T,\underline{\alpha}}$ converges weakly to the measure $Q_{\underline{\alpha}}$ with the Fourier transform

$$g_{\underline{\alpha}}(\underline{k}_1, \ldots, \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, \ldots, \underline{k}_r) \in A_{\underline{\alpha}}, \\ 0 & \text{if } (\underline{k}_1, \ldots, \underline{k}_r) \in B_{\underline{\alpha}}. \end{cases}$$

□

3. Case of Absolute Convergence

In this section, we apply Lemma 1 for the proof of a limit lemma in the space $H^r(D)$ for certain absolutely convergent Dirichlet series. For fixed $\theta > 0$, $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$, define

$$v_n(m, \alpha_j) = \exp\left\{ -\left(\frac{m+\alpha_j}{n} \right)^{\theta} \right\}, \quad j = 1, \ldots, r.$$

Then the series

$$L_n(\lambda_j, \alpha_j, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m} v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \ldots, r,$$

are absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0 because of the exponential decrease in $v_n(m, \alpha_j)$ with respect to m. For $A \in \mathcal{B}(H^r(D))$, define

$$P_{T,n,\underline{\lambda},\underline{\alpha}}(A) = \frac{1}{T} \text{meas}\{\tau \in [0, T] : \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + i\tau) \in A\},$$

where

$$\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s) = (L_n(\lambda_1, \alpha_1, s), \ldots, L_n(\lambda_r, \alpha_r, s)).$$

Lemma 2. *Suppose that $0 < \lambda_j \leqslant 1$ and $0 < \alpha_j \leqslant 1$, $j = 1, \ldots, r$, are arbitrary. Then, on $(H^r(D), \mathcal{B}(H^r(D)))$, there exists a probability measure $P_{n,\underline{\lambda},\underline{\alpha}}$ such that $P_{T,n,\underline{\lambda},\underline{\alpha}}$ converges weakly to $P_{n,\underline{\lambda},\underline{\alpha}}$ as $T \to \infty$.*

Proof. We will use a property of preservation of weak convergence under continuous mappings; see, for example, Theorem 5.1 of [26].

Denote by $\omega = (\omega_1, \ldots, \omega_r)$, $\omega_j \in \Omega_j$, $j = 1, \ldots, r$, the elements of Ω^r, and define

$$\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s, \omega) = (L_n(\lambda_1, \alpha_1, s, \omega_1), \ldots, L_n(\lambda_r, \alpha_r, s, \omega_r)),$$

where

$$L_n(\lambda_j, \alpha_j, s, \omega_j) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m} \omega_j(m) v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \ldots, r.$$

Clearly, the latter series, as $L_n(\lambda_j, \alpha_j, s)$, are also absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0. Let the mapping $u_{n,\underline{\lambda},\underline{\alpha}} : \Omega^r \to H^r(D)$ be given by

$$u_{n,\underline{\lambda},\underline{\alpha}}(\omega) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s, \omega).$$

In virtue of absolute convergence of the series $L_n(\lambda_j, \alpha_j, s, \omega_j)$, the mapping $u_{n,\underline{\lambda},\underline{\alpha}}$ is continuous. Moreover,

$$u_{n,\underline{\lambda},\underline{\alpha}}\Big(\big((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0 \big), \ldots, \big((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0 \big) \Big) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + i\tau),$$

thus, for all $A \in \mathcal{B}(H^r(D))$,

$$P_{T,n,\underline{\lambda},\underline{\alpha}}(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \Big(\Big((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0\Big), \ldots, \Big((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0\Big)\Big) \in u^{-1}_{n,\underline{\lambda},\underline{\alpha}} A\Big\} = Q_{T,\underline{\alpha}}(u^{-1}_{n,\underline{\lambda},\underline{\alpha}} A).$$

This shows that $P_{T,n,\underline{\lambda},\underline{\alpha}} = Q_{T,\underline{\alpha}} u^{-1}_{n,\underline{\lambda},\underline{\alpha}}$. Therefore, by Theorem 5.1 of [26] and Lemma 1, we find that $P_{T,n,\underline{\lambda},\underline{\alpha}}$ converges weakly to the measure $Q_{\underline{\alpha}} u^{-1}_{n,\underline{\lambda},\underline{\alpha}}$ as $T \to \infty$. □

Now we will deal with the measure $V_{n,\underline{\lambda},\underline{\alpha}} \overset{\text{def}}{=} Q_{\underline{\alpha}} u^{-1}_{n,\underline{\lambda},\underline{\alpha}}$. We have to show that the family of probability measures $\{V_{n,\underline{\lambda},\underline{\alpha}} : n \in \mathbb{N}\}$ is tight, i.e., that, for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H^r(D)$ such that $V_{n,\underline{\lambda},\underline{\alpha}}(K) > 1 - \varepsilon$ for all $n \in \mathbb{N}$. Let V_{n,λ_j,α_j} be marginal measures of $V_{n,\underline{\lambda},\underline{\alpha}}$. i.e., for $A \in \mathcal{B}(H(D))$,

$$V_{n,\lambda_j,\alpha_j}(A) = V_{n,\underline{\lambda},\underline{\alpha}}\Bigg(\underbrace{H(D) \times \cdots \times H(D)}_{j-1} \times A \times H(D) \times \cdots \times H(D)\Bigg), \quad j = 1, \ldots, r.$$

Lemma 3. *The family of probability measures $\{V_{n,\lambda_j,\alpha_j} : n \in \mathbb{N}\}$ is tight, $j = 1, \ldots, r$.*

Proof. We recall a metric in the space $H(D)$ inducing its topology of uniform convergence on compacta. There exists a sequence $\{K_l : l \in \mathbb{N}\} \subset D$ of embedded compact subsets such that D is the union of the sets K_l, and if $K \subset D$ is a compact set, then K lies in some K_l. Then

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(D),$$

is a desired metric in $H(D)$.

By the integral Cauchy formula we have, for $s \in K_l$,

$$L_n(\lambda_j, \alpha_j, s + i\tau) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}_l} \frac{L_n(\lambda_j, \alpha_j, z + i\tau)}{z - s}\, \mathrm{d}z, \quad j = 1, \ldots, r,$$

where $\mathcal{L}_l$ is a certain contour lying in D and containing K_l. Hence,

$$\frac{1}{T}\int_0^T \sup_{s \in K_l} |L_n(\lambda_j, \alpha_j, s + i\tau)|\, \mathrm{d}\tau \ll_l \int\limits_{\mathcal{L}_l} \frac{|\mathrm{d}z|}{|z - s|} \left(\frac{1}{T}\int_0^T |L_n(\lambda_j, \alpha_j, z + i\tau)|\, \mathrm{d}z\right).$$

Thus, with a certain $\sigma_l > 1/2$,

$$\begin{aligned}
&\sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \frac{1}{T}\int_0^T \sup_{s \in K_l} |L_n(\lambda_j, \alpha_j, s + i\tau)|\, \mathrm{d}\tau \\
&\quad \ll_l \sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \left(\frac{1}{T}\int_0^T \sup_{s \in K_l} |L_n(\lambda_j, \alpha_j, \sigma_l + i\tau)|^2\, \mathrm{d}\tau\right)^{1/2} \\
&\quad \ll_l \sup_{n \in \mathbb{N}} \left(\sum_{m=0}^{\infty} \frac{v_n^2(m, \alpha_j)}{(m + \alpha_j)^{2\sigma_l}}\right)^{1/2} \ll_l \left(\sum_{m=0}^{\infty} \frac{1}{(m + \alpha_j)^{2\sigma_l}}\right)^{1/2} \leqslant R_{l,\alpha_j} < \infty.
\end{aligned} \tag{5}$$

Here, and in what follows, the notation $a \ll_\beta b$, $b > 0$, means that there exists a constant $c = c(\beta) > 0$ such that $|a| \leqslant cb$.

Let the random variable θ_T be uniformly distributed in the interval $[0, T]$, and be defined on a certain probability space with the measure μ. On that probability space, define the $H^r(D)$-valued random element

$$\begin{aligned} X_{T,n,\underline{\lambda},\underline{\alpha}} &= (X_{T,n,\lambda_1,\alpha_1}, \dots, X_{T,n,\lambda_r,\alpha_r}) \\ &= X_{T,n,\underline{\lambda},\underline{\alpha}}(s) = (X_{T,n,\lambda_1,\alpha_1}(s), \dots, X_{T,n,\lambda_r,\alpha_r}(s)) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + i\theta_T). \end{aligned}$$

Moreover, let $X_{n,\underline{\lambda},\underline{\alpha}} = (X_{n,\lambda_1,\alpha_1}, \dots, X_{n,\lambda_r,\alpha_r})$ denote the $H^r(D)$-valued random element having the distribution $V_{n,\underline{\lambda},\underline{\alpha}}$, and by $\xrightarrow{\mathcal{D}}$ the convergence in distribution. Then Lemma 2 implies

$$X_{T,n,\underline{\lambda},\underline{\alpha}} \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,\underline{\lambda},\underline{\alpha}}. \tag{6}$$

Hence, we have

$$X_{T,n,\lambda_j,\alpha_j} \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,\lambda_j,\alpha_j}, \quad j = 1, \dots, r.$$

Therefore, for $M_l > 0$,

$$\limsup_{T\to\infty} \mu\left\{ \sup_{s\in K_l} |X_{T,n,\lambda_j,\alpha_j}(s)| > M_l \right\} = \mu\left\{ \sup_{s\in K_l} |X_{n,\lambda_j,\alpha_j}(s)| > M_l \right\}.$$

Thus, by the definition of $X_{T,n,\lambda_j,\alpha_j}$,

$$\begin{aligned} \mu\left\{ \sup_{s\in K_l} |X_{n,\lambda_j,\alpha_j}(s)| > M_l \right\} &\leqslant \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \mu\left\{ \sup_{s\in K_j} |X_{T,n,\lambda_j,\alpha_j}(s)| > M_l \right\} \\ &\leqslant \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{TM_l} \int_0^T \sup_{s\in K_l} |L_n(\lambda_j, \alpha_j, s + i\tau)| \, d\tau. \end{aligned}$$

Taking $M_l = M_{l,\alpha_j} = 2^l \varepsilon^{-1} R_{l,\alpha_j}$ with $\varepsilon > 0$, we obtain from this and (5)

$$\mu\left\{ \sup_{s\in K_l} |X_{n,\lambda_j,\alpha_j}(s)| > M_l \right\} \leqslant \frac{\varepsilon}{2^l} \tag{7}$$

for all $n, l \in \mathbb{N}$ and $j = 1, \dots, r$. Let

$$K = K(\varepsilon) = \left\{ g \in H(D) : \sup_{s\in K_l} |g(s)| \leqslant M_l, \, l \in \mathbb{N} \right\}.$$

Then the set K is compact in the space $H(D)$, and, by (7),

$$\mu\left\{ X_{n,\lambda_j,\alpha_j} \in K \right\} \geqslant 1 - \varepsilon \sum_{l=1}^{\infty} 2^{-l} = 1 - \varepsilon$$

for all $n \in \mathbb{N}$ and $j = 1, \dots, r$. The latter inequality shows that the family $\{V_{n,\lambda_j,\alpha_j} : n \in \mathbb{N}\}$ is tight for all $j = 1, \dots, r$. □

Lemma 4. *The family of probability measures $\{V_{n,\underline{\lambda},\underline{\alpha}} : n \in \mathbb{N}\}$ is tight.*

Proof. Fix $\varepsilon > 0$. By Lemma 3, there exist compact subsets $K_1, \dots, K_r$ in $H(D)$ such that

$$V_{n,\lambda_j,\alpha_j}(K_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, \dots, r, \tag{8}$$

for all $n \in \mathbb{N}$. Put $K = K_1 \times \cdots \times K_r$. Then K is a compact set in $H^r(D)$. Moreover, by (8),

$$\begin{aligned} V_{n,\underline{\lambda},\underline{\alpha}}(H^r(D) \setminus K) \\ &= V_{n,\underline{\lambda},\underline{\alpha}}\left(\bigcup_{j=1}^{r}\left(\underbrace{H(D) \times \cdots \times H(D)}_{j-1} \times (H(D) \setminus K_j) \times H(D) \times \cdots \times H(D)\right)\right) \\ &\leqslant \sum_{j=1}^{r} V_{n,\lambda_j,\alpha_j}(H(D) \setminus K_j) \leqslant r\frac{\varepsilon}{r} = \varepsilon \end{aligned}$$

for all $n \in \mathbb{N}$. Thus, $V_{n,\underline{\lambda},\underline{\alpha}}(K) > 1 - \varepsilon$ for all $n \in \mathbb{N}$, i.e., the family $\{V_{n,\underline{\lambda},\underline{\alpha}}\}$ is tight. □

4. Joint Limit Theorem

For brevity, let

$$\underline{L}(\underline{\lambda}, \underline{\alpha}, s) = (L(\lambda_1, \alpha_1, s), \ldots, L(\lambda_r, \alpha_r, s)).$$

In this section, we will consider the weak convergence for

$$P_{T,\underline{\lambda},\underline{\alpha}}(A) = \frac{1}{T}\text{meas}\{\tau \in [0, T] : \underline{L}(\underline{\lambda}, \underline{\alpha}, s + i\tau) \in A\}, \quad A \in \mathcal{B}(H^r(D)),$$

as $T \to \infty$.

Theorem 5. *Suppose that $0 < \lambda \leqslant 1$ and $0 < \alpha \leqslant 1$, $j = 1, \ldots, r$, are arbitrary numbers. Then, on $(H^r(D), \mathcal{B}(H^r(D)))$, there exists a probability measure $P_{\underline{\lambda},\underline{\alpha}}$ such that $P_{T,\underline{\lambda},\underline{\alpha}}$ converges weakly to $P_{\underline{\lambda},\underline{\alpha}}$ as $T \to \infty$.*

Before the proof of Theorem 5, we state one result on the distance between $\underline{L}(\underline{\lambda}, \underline{\alpha}, s)$ and $\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s)$. Let ρ be the metric in $H(D)$ defined in the previous section, and $\underline{g}_1 = (g_{11}, \ldots, g_{1r}), \underline{g}_2(g_{21}, \ldots, g_{2r}) \in H^r(D)$. Then

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} \rho(g_{1j}, g_{2j})$$

is the metric in $H^r(D)$ inducing the product topology.

Lemma 5. *Suppose that $0 < \lambda_j \leqslant 1$ and $0 < \alpha_j \leqslant 1$, $j = 1, \ldots, r$, are arbitrary numbers. Then the equality*

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \underline{\rho}(\underline{L}(\underline{\lambda}, \underline{\alpha}, s + i\tau), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + i\tau))\, d\tau = 0$$

holds.

Proof. In [17], Lemma 3, it was obtained that, for arbitrary $0 < \lambda \leqslant 1$ and $0 < \alpha \leqslant 1$,

$$\lim_{n \to \infty} \limsup_{TN \to \infty} \frac{1}{T} \int_0^T \rho(L(\lambda, \alpha, s + i\tau), L_n(\lambda, \alpha, s + i\tau))\, d\tau = 0.$$

Therefore, the lemma is a consequence of the definition of the metric $\underline{\rho}$. □

Proof of Theorem 5. We apply one theorem on convergence in distribution; see, for example, Theorem 4.2 of [26]. Let θ_T be the same random variable as in Section 3.

Since, by Lemma 4, the family of probability measures $\{V_{n,\underline{\alpha},\underline{\lambda}}\}$ is tight, it is relatively compact in view of the Prokhorov theorem, Theorem 6.1 of [26]. Therefore, there exists a probability measure $P_{\underline{\lambda},\underline{\alpha}}$ on $(H^r(D), \mathcal{B}(H^r(D)))$ and a sequence $V_{n_l,\underline{\lambda},\underline{\alpha}}$ such that $V_{n_l,\underline{\lambda},\underline{\alpha}}$ converges weakly to $P_{\underline{\lambda},\underline{\alpha}}$ as $l \to \infty$. Thus,

$$X_{n_l,\underline{\lambda},\underline{\alpha}} \xrightarrow[l\to\infty]{\mathcal{D}} P_{\underline{\lambda},\underline{\alpha}}. \tag{9}$$

We introduce one more $H^r(D)$-valued random element

$$Y_{T,\underline{\lambda},\underline{\alpha}} = Y_{T,\underline{\lambda},\underline{\alpha}}(s) = \underline{L}(\underline{\lambda}, \underline{\alpha}, s + i\theta_T).$$

Then an application of Lemma 5 gives, for $\varepsilon > 0$,

$$\begin{aligned}&\lim_{l\to\infty} \limsup_{T\to\infty} P\Big\{\underline{\rho}(Y_{T,\underline{\lambda},\underline{\alpha}}, X_{n_l,\underline{\lambda},\underline{\alpha}}) \geqslant \varepsilon\Big\} \\ &\qquad \leqslant \lim_{l\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_0^T \underline{\rho}\Big(\underline{L}(\underline{\lambda}, \underline{\alpha}, s + i\tau), \underline{L}_{n_l}(\underline{\lambda}, \underline{\alpha}, s + i\tau)\Big)\, \mathrm{d}\tau = 0.\end{aligned}$$

This equality and relations (6) and (9) show that the hypotheses of Theorem 4.2 of [26] are satisfied. Therefore,

$$Y_{T,\underline{\lambda},\underline{\alpha}} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\underline{\lambda},\underline{\alpha}}.$$

This means that $P_{T,\underline{\lambda},\underline{\alpha}}$ converges weakly to $P_{\underline{\lambda},\underline{\alpha}}$ as $T \to \infty$. □

For $A \in \mathcal{B}(H(D))$, define

$$P_{T,\Phi,\underline{\lambda},\underline{\alpha}}(A) = \frac{1}{T}\mathrm{meas}\{\tau \in [0, T] : \Phi(\underline{L}(\underline{\lambda}, \underline{\alpha}, s + i\tau)) \in A\},$$

where $\Phi : H^r(D) \to H(D)$ is a certain operator.

Corollary 1. *Suppose that $0 < \lambda_j \leqslant 1$ and $0 < \alpha_j \leqslant 1$, $j = 1, \dots, r$, are arbitrary numbers, and $\Phi : H^r(D) \to H(D)$ is a continuous operator. Then $P_{T,\Phi,\underline{\lambda},\underline{\alpha}}$ converges weakly to $P_{\underline{\lambda},\underline{\alpha}}\Phi^{-1}$ as $T \to \infty$.*

Proof. The corollary follows from Theorem 5, continuity of Φ, and Theorem 5.1 of [26] on the preservation of weak convergence under continuous mappings. □

5. Proof of Approximation Theorems

Theorems 3 and 4 are consequences of Theorem 5 and Corollary 1, respectively, and properties of weak convergence and supports of probability measures. We recall that the support of a probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, where $\mathbb{X}$ is a separable space, is a minimal closed set $S \subset \mathbb{X}$ such that $P(S) = 1$. The set S consists of all elements $x \in \mathbb{X}$ such that, for every open neighborhood G of x, the inequality $P(G) > 0$ holds.

Proof of Theorem 3. By Theorem 5, $P_{T,\underline{\lambda},\underline{\alpha}}$ converges weakly to the measure $P_{\underline{\lambda},\underline{\alpha}}$ as $T \to \infty$. Define the set

$$G_\varepsilon = \left\{(g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon\right\}.$$

Then the set G_ε is an open set in $H^r(D)$. Therefore, by the equivalent of weak convergence in terms of open sets, see Theorem 2.1 in [26],

$$\liminf_{T\to\infty} P_{T,\underline{\lambda},\underline{\alpha}}(G_\varepsilon) \geqslant P_{\underline{\lambda},\underline{\alpha}}(G_\varepsilon). \tag{10}$$

Let $F_{\underline{\lambda},\underline{\alpha}}$ be the support of the measure $P_{\underline{\lambda},\underline{\alpha}}$. Then, clearly, $F_{\underline{\lambda},\underline{\alpha}}$ is a non-empty closed set in $H^r(D)$. Since $(f_1, \dots, f_r) \in F_{\underline{\lambda},\underline{\alpha}}$, the set G_ε is an open neighborhood of an element of the support of the measure $P_{\underline{\lambda},\underline{\alpha}}$. Thus, by a property of the support, we have $P_{\underline{\lambda},\underline{\alpha}}(G_\varepsilon) > 0$. This inequality, (10) and the definitions of $P_{T,\underline{\lambda},\underline{\alpha}}$ and G_ε prove the first assertion of the theorem.

For the proof of the second assertion of the theorem, we apply the equivalent of weak convergence in terms of continuity sets of limit measures. The boundaries $\partial G_{\varepsilon_1}$ and $\partial G_{\varepsilon_2}$ of the set G_ε do not intersect for different positives ε_1 and ε_2. Therefore, $P_{\underline{\lambda},\underline{\alpha}}(\partial G_\varepsilon) > 0$ for at most countably many values of $\varepsilon > 0$, i.e., G_ε is a continuity set of the measure $P_{\underline{\lambda},\underline{\alpha}}$ for all but at most countably many $\varepsilon > 0$. Since, by the equivalent of weak convergence in terms of continuity sets, Theorem 2.1 of [26] and Theorem 5,

$$P_{T,\underline{\lambda},\underline{\alpha}}(G_\varepsilon) = P_{\underline{\lambda},\underline{\alpha}}(G_\varepsilon)$$

and $P_{\underline{\lambda},\underline{\alpha}}(G_\varepsilon) > 0$, the definitions of $P_{T,\underline{\lambda},\underline{\alpha}}$ and G_ε give the second assertion of the theorem. □

Proof of Theorem 4. We apply similar arguments as in the proof of Theorem 3 by using Corollary 1 in place of Theorem 5. It remains to find the support of the measure $P_{\underline{\lambda},\underline{\alpha}}\Phi^{-1}$. We will show that $\Phi(F_{\underline{\lambda},\underline{\alpha}})$ is the support of $P_{\underline{\lambda},\underline{\alpha}}\Phi^{-1}$.

We take an arbitrary element $x \in \Phi(F_{\underline{\lambda},\underline{\alpha}})$ and its open neighborhood G. Since Φ is continuous, the set $\Phi^{-1}G$ is open as well. Moreover, in view of $(\Phi^{-1}G) \cap F_{\underline{\lambda},\underline{\alpha}} \neq \emptyset$, the set $\Phi^{-1}G$ has a certain element of $F_{\underline{\lambda},\underline{\alpha}}$. Hence, by the support property,

$$P_{\underline{\lambda},\underline{\alpha}}(\Phi^{-1}G) > 0,$$

thus,

$$P_{\underline{\lambda},\underline{\alpha}}\Phi^{-1}(G) > 0.$$

Moreover,

$$P_{\underline{\lambda},\underline{\alpha}}\Phi^{-1}(\Phi(F_{\underline{\lambda},\underline{\alpha}})) = P_{\underline{\lambda},\underline{\alpha}}(F_{\underline{\lambda},\underline{\alpha}}) = 1.$$

Thus, the support of $P_{\underline{\lambda},\underline{\alpha}}\Phi^{-1}$ is the set $\Phi(F_{\underline{\lambda},\underline{\alpha}})$.

Define the set

$$\widehat{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}$$

with $f \in \Phi(F_{\underline{\lambda},\underline{\alpha}})$. Then $P_{\underline{\lambda},\underline{\alpha}}\Phi^{-1}(\widehat{G}_\varepsilon) > 0$, and, by Corollary 1,

$$\liminf_{T \to \infty} P_{T,\Phi,\underline{\lambda},\underline{\alpha}}(\widehat{G}_\varepsilon) \geqslant P_{\underline{\lambda},\underline{\alpha}}\Phi^{-1}(\widehat{G}_\varepsilon) > 0.$$

This gives the first assertion of the theorem.

The second assertion uses a remark that the set $\widehat{G}$ is a continuous set of the measure $P_{\underline{\lambda},\underline{\alpha}}\Phi^{-1}$ for all but at most countably many $\varepsilon > 0$, and the equivalent of weak convergence in terms of continuity sets. □

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