

On a Dirichlet Series Connected to a Periodic Hurwitz Zeta-Function with Transcendental and Rational Parameter

Aidas Balčiūnas^a, Antanas Laurinčikas^a and Mindaugas Stoncelis^b

^a*Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University*

Naugarduko g. 24, LT-03225 Vilnius, Lithuania

^b*Institute of Regional Development, Šiauliai Academy, Vilnius University*

P. Višinskio g. 25, LT-76351 Šiauliai, Lithuania

E-mail(*corresp.*): antanas.laurincikas@mif.vu.lt

E-mail: aidas.balciunas@mif.vu.lt

E-mail: mindaugas.stoncelis@sa.vu.lt

Received June 28, 2022; revised November 28, 2022; accepted November 28, 2022

Abstract. In the paper, we construct an absolutely convergent Dirichlet series which in the mean is close to the periodic Hurwitz zeta-function, and has the universality property on the approximation of a wide class of analytic functions.

Keywords: Haar measure, periodic Hurwitz zeta-function, space of analytic functions, universality, weak convergence.

AMS Subject Classification: 11M41.

1 Introduction

Let $\mathbf{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$, $0 < \alpha \leq 1$ fixed parameter, and $s = \sigma + it$ a complex variable. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathbf{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathbf{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s}.$$

If $a_m \equiv 1$, then we have the classical Hurwitz zeta-function $\zeta(s, \alpha)$ which has the meromorphic continuation to the whole complex plane with unique simple pole at the point $s = 1$ with residue 1. The periodicity of the sequence \mathbf{a} implies, for $\sigma > 1$, the equality

$$\zeta(s, \alpha; \mathbf{a}) = \frac{1}{q^s} \sum_{l=0}^{q-1} a_l \zeta\left(s, \frac{l + \alpha}{q}\right), \quad (1.1)$$

therefore, the function $\zeta(s, \alpha; \mathbf{a})$ also can be meromorphically continued to the whole complex plane with a simple pole at the point $s = 1$ with residue

$$\widehat{a} \stackrel{\text{def}}{=} \frac{1}{q} \sum_{l=0}^{q-1} a_l.$$

If $\widehat{a} = 0$, then $\zeta(s, \alpha; \mathbf{a})$ is an entire function. Clearly, $\zeta(s, 1; \{1\}) = \zeta(s)$ is the Riemann zeta-function. Thus, the periodic Hurwitz zeta-function is a generalization of the classical Hurwitz and Riemann zeta-functions. Analytical properties of $\zeta(s, \alpha; \mathbf{a})$ are governed by the sequence \mathbf{a} , and, in particular, by arithmetic of the parameter α .

The function $\zeta(s, \alpha; \mathbf{a})$, for some classes of the parameter α , as other zeta-functions, is universal, i. e., its shifts $\zeta(s + i\tau, \alpha; \mathbf{a})$, $\tau \in \mathbb{R}$, approximate all analytic functions defined in the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Note that the phenomenon of universality for zeta-functions was discovered by Voronin, in [6] he proved the universality of the Riemann zeta-function. Later, the universality of some other zeta-functions was obtained. By the Linnik-Ibragimov conjecture, all functions in some half-plane given by Dirichlet series, analytically continuable to the left of the absolute convergence half-plane and satisfying some natural growth conditions are universal in the Voronin sense. However, till now the universality of some zeta-functions are not known. The universality of the function $\zeta(s, \alpha; \mathbf{a})$ with transcendental α was considered in [3], and with rational α in [4]. Denote by \mathcal{K} the class of compact subsets of the strip D , and by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K . Let $\text{meas}A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the universality of the function $\zeta(s, \alpha; \mathbf{a})$ is described in the following theorem.

Theorem 1. *Suppose that the number α is transcendental, or $\alpha = a/b$, $a, b \in \mathbb{N}$, $a < b$, $(a, b) = 1$, $a/b \neq 1/2$ and $(lb + a, bq) = 1$ for all $l = 0, 1, \dots, q - 1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha; \mathbf{a}) - f(s)| < \varepsilon \right\} > 0.$$

It is easily seen that the transcendence of α can be replaced by the linear independence over the field of rational numbers \mathbb{Q} of the set

$$L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0\}.$$

The universality of the function $\zeta(s, \alpha; \mathbf{a})$ with algebraic irrational α is an open problem.

In the strip D , the function $\zeta(s, \alpha; \mathbf{a})$ is defined by using (1.1) and analytic continuation of the Hurwitz zeta-function. Therefore, it is not easy to derive an information on the function $f(s)$ from the inequality

$$\sup_{s \in K} |\zeta(s + i\tau, \alpha; \mathbf{a}) - f(s)| < \varepsilon.$$

It is more convenient to use an absolutely convergent Dirichlet series in place of $\zeta(s, \alpha; \mathbf{a})$. This paper is devoted to a realization of the mentioned idea.

Let $\theta > 1/2$ be a fixed number, and, for $m \in \mathbb{N}_0$ and $u > 0$,

$$v_u(m, \alpha) = \exp \left\{ -((m + \alpha)/u)^\theta \right\}.$$

Since $|a_m| \leq C$, $m \in \mathbb{N}_0$, with some $C < \infty$, the series

$$\zeta_u(s, \alpha; \mathbf{a}) = \sum_{m=0}^{\infty} \frac{a_m v_u(m, \alpha)}{(m + \alpha)^s}$$

is absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0 .

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ -field of the space \mathbb{X} , by $H(D)$ the space of analytic on D functions endowed with the topology of uniform convergence on compacta, and let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$. Define the set $\Omega = \prod_{m \in \mathbb{N}_0} \gamma_m$, where $\gamma_m = \gamma$ for all \mathbb{N}_0 . With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and we obtain a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by ω_m the m th component of an element $\omega \in \Omega$, $m \in \mathbb{N}_0$, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$ -valued random element

$$\zeta(s, \alpha, \omega; \mathbf{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m + \alpha)^s}.$$

The latter series, for almost all $\omega \in \Omega$ is uniformly convergent on compact subsets of the strip D . Let $P_{\zeta, \alpha, \mathbf{a}}$ be the distribution of the random element $\zeta(s, \alpha, \omega; \mathbf{a})$, i. e.,

$$P_{\zeta, \alpha, \mathbf{a}}(A) = m_H\{\omega \in \Omega : \zeta(s, \alpha, \omega; \mathbf{a}) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Theorem 2. *Suppose that the number α is transcendental, and $u_T \rightarrow \infty$, $u_T \ll T^2$ as $T \rightarrow \infty$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta_{u_T}(s + i\tau, \alpha; \mathbf{a}) - f(s)| < \varepsilon \right\} \\ & = m_H \left\{ \omega \in \Omega : \sup_{s \in K} |\zeta(s, \alpha, \omega; \mathbf{a}) - f(s)| < \varepsilon \right\} > 0 \end{aligned}$$

exists for all but at most countably many $\varepsilon > 0$.

For the case of rational α , define one more infinite-dimensional torus $\Omega_1 = \prod_{p \in \mathbb{P}} \gamma_p$, where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$, and \mathbb{P} is the set of all prime numbers. Analogically to Ω , we have the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$, where m_{1H} is the probability Haar measure on $(\Omega_1, \mathcal{B}(\Omega_1))$. Denote by $\omega_1(p)$ the p -th component of an element $\omega_1 \in \Omega_1$, $p \in \mathbb{P}$, and extend $\omega_1(p)$ to the set \mathbb{N} by the formula

$$\omega_1(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

Now, let $\alpha = a/b$, $a, b \in \mathbb{N}$, $a < b$, $(a, b) = 1$. Denote by χ Dirichlet characters, and by $L(s, \chi)$ the corresponding Dirichlet L -functions

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1.$$

The functions $L(s, \chi)$ have meromorphic continuation to the whole complex plane. Let $\varphi(m)$ be the totient Euler function. On the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$, define the $H(D)$ -valued random element

$$\zeta\left(s, \frac{a}{b}, \omega_1; \mathbf{a}\right) = \frac{b^s \omega_1(b)}{\varphi(bq)} \sum_{l=0}^{q-1} a_l \sum_{\chi \bmod bq} \bar{\chi}(a + bl) L(s, \omega_1, \chi),$$

where

$$L(s, \omega_1, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m) \omega_1(m)}{m^s}$$

and \bar{z} denotes the complex conjugate of $z \in \mathbb{C}$.

Theorem 3. *Suppose that $a, b \in \mathbb{N}$, $a < b$, $(a, b) = 1$, $a/b \neq 1/2$ and $(lb + a, q) = 1$ for all $l = 0, 1, \dots, q-1$, and $u_T \rightarrow \infty$, $u_T \ll T^2$ as $T \rightarrow \infty$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the limit*

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} \left| \zeta_{u_T} \left(s + i\tau, \frac{a}{b}; \mathbf{a} \right) - f(s) \right| < \varepsilon \right\} \\ & = m_{1H} \left\{ \omega_1 \in \Omega_1 : \sup_{s \in K} \left| \zeta \left(s, \frac{a}{b}, \omega_1; \mathbf{a} \right) - f(s) \right| < \varepsilon \right\} > 0 \end{aligned}$$

exists for all but at most countably many $\varepsilon > 0$.

We will derive Theorems 2 and 3 from Theorem 1 by using the approximation of $\zeta(s, \alpha; \mathbf{a})$ by $\zeta_{u_T}(s, \alpha; \mathbf{a})$ in the mean.

2 Some estimates

In this section, we prove the following equality.

Lemma 1. *Let K be a compact subset of the strip D , and $u_T \rightarrow \infty$ and $u_T \ll T^2$ as $T \rightarrow \infty$. Then*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sup_{s \in K} |\zeta(s + i\tau, \alpha; \mathbf{a}) - \zeta_{u_T}(s + i\tau, \alpha; \mathbf{a})| d\tau = 0.$$

Proof. From the Mellin formula

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) d^{-s} ds = e^{-d}, \quad c, d > 0,$$

where $\Gamma(s)$ is the Euler gamma-function, we have

$$v_{u_T}(m, \alpha) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \frac{1}{\theta} \Gamma\left(\frac{s}{\theta}\right) \left(\frac{m+\alpha}{u_T}\right)^{-s} ds.$$

Hence, denoting $l_{u_T}(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) u_T^s$, we obtain, for $s \in K$,

$$\begin{aligned} \zeta_{u_T}(s, \alpha; \mathbf{a}) &= \frac{1}{2\pi i} \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s} \int_{\theta-i\infty}^{\theta+i\infty} \frac{z}{\theta} \Gamma\left(\frac{z}{\theta}\right) \left(\frac{m+\alpha}{u_T}\right)^{-z} \frac{dz}{z} \\ &= \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \frac{l_{u_T}(z)}{z} \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^{s+z}} dz \\ &= \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z, \alpha; \mathbf{a}) l_{u_T}(z) \frac{dz}{z}. \end{aligned} \tag{2.1}$$

There exists $\varepsilon > 0$ such that $1/2 + 2\varepsilon \leq \sigma \leq 1 - \varepsilon$ for $s = \sigma + it \in K$. We take $\theta = \frac{1}{2} + \varepsilon$ and $\theta_1 = \frac{1}{2} + \varepsilon - \sigma$. Then $\theta_1 < 0$ for all $s \in K$. Therefore, the representation (2.1) and the residue theorem give

$$\zeta_{u_T}(s, \alpha; \mathbf{a}) - \zeta(s, \alpha; \mathbf{a}) = \frac{1}{2\pi i} \int_{\theta_1-i\infty}^{\theta_1+i\infty} \zeta(s+z, \alpha; \mathbf{a}) \frac{l_{u_T}(z)}{z} dz + \frac{\widehat{a} l_{u_T}(1-s)}{1-s}.$$

Thus, for all $s \in K$,

$$\begin{aligned} \zeta_{u_T}(s+i\tau, \alpha; \mathbf{a}) - \zeta(s+i\tau, \alpha; \mathbf{a}) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \zeta\left(\frac{1}{2} + \varepsilon + it + i\tau + iv, \alpha; \mathbf{a}\right) \\ &\quad \times \frac{l_{u_T}(1/2 + \varepsilon - \sigma + iv)}{1/2 + \varepsilon - \sigma + iv} dv + \frac{\widehat{a} l_{u_T}(1-s-i\tau)}{1-s-i\tau} \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \zeta\left(\frac{1}{2} + \varepsilon + i\tau + iv, \alpha; \mathbf{a}\right) \frac{l_{u_T}(1/2 + \varepsilon - s + iv)}{1/2 + \varepsilon - s + iv} dv \\ &\quad + \frac{\widehat{a} l_{u_T}(1-s-i\tau)}{1-s-i\tau} \ll \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + \varepsilon + i\tau + iv, \alpha; \mathbf{a}\right) \right| \\ &\quad \sup_{s \in K} \left| \frac{l_{u_T}(1/2 + \varepsilon - s + iv)}{1/2 + \varepsilon - s + iv} \right| dv + \widehat{a} \sup_{s \in K} \left| \frac{l_{u_T}(1-s-i\tau)}{1-s-i\tau} \right|. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{T} \int_0^T \sup_{s \in K} |\zeta(s+i\tau, \alpha; \mathbf{a}) - \zeta_{u_T}(s+i\tau, \alpha; \mathbf{a})| d\tau \tag{2.2} \\ &\ll \int_{-\infty}^{\infty} \left(\frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + \varepsilon + i\tau + iv, \alpha; \mathbf{a}\right) \right|^2 d\tau \right)^{1/2} \\ &\quad \times \sup_{s \in K} \left| \frac{l_{u_T}(1/2 + \varepsilon - s + iv)}{1/2 + \varepsilon - s + iv} \right| dv + \widehat{a} \frac{1}{T} \int_0^T \sup_{s \in K} \left| \frac{l_{u_T}(1-s-i\tau)}{1-s-i\tau} \right| d\tau \stackrel{def}{=} I_1 + I_2. \end{aligned}$$

It is well known that, for $1/2 < \sigma < 1$,

$$\int_0^T |\zeta(\sigma + it, \alpha)|^2 dt \ll_{\sigma, \alpha} T.$$

This and (1.1) show that, for $1/2 < \sigma < 1$,

$$\int_0^T |\zeta(\sigma + it, \alpha; \mathbf{a})|^2 dt \ll_{\sigma, \alpha, \mathbf{a}} T.$$

Hence, for the same σ and $v \in \mathbb{R}$,

$$\int_0^T |\zeta(\sigma + it + iv, \alpha; \mathbf{a})|^2 d\tau \ll_{\sigma, \alpha, \mathbf{a}} T(1 + |v|), \quad T \geq 1. \tag{2.3}$$

For the function $\Gamma(s)$, the estimate

$$\Gamma(\sigma + it) \ll \exp\{-c_1|t|\}$$

uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$ with arbitrary $\sigma_1 < \sigma_2$ is valid. Therefore, for all $s \in K$,

$$\begin{aligned} \frac{l_{u_T}(1/2 + \varepsilon - s + iv)}{1/2 + \varepsilon - s + iv} &\ll u_T^{1/2 + \varepsilon - \sigma} \left| \Gamma\left(\frac{1}{\theta}\left(\frac{1}{2} + \varepsilon - s + iv\right)\right) \right| \\ &\ll u_T^{-\varepsilon} \exp\left\{\frac{c_1}{\theta}|v - t|\right\} \ll_{\varepsilon, K} u_T^{-\varepsilon} \exp\{-c_2|v|\}, \quad c_2 > 0. \end{aligned}$$

Therefore, in view of (2.3),

$$I_1 \ll_{\varepsilon, \alpha, \mathbf{a}, K} u_T^{-\varepsilon} \int_{-\infty}^{\infty} (1 + |v|)^{1/2} \exp\{-c_2|v|\} dv \ll_{\varepsilon, \alpha, \mathbf{a}, K} u_T^{-\varepsilon}. \tag{2.4}$$

Similarly, we find that, for all $s \in K$,

$$\frac{l_{u_T}(1 - s - i\tau)}{1 - s - i\tau} \ll_{\varepsilon} u_T^{1 - \sigma} \exp\{-c_1|t + \tau|\} \ll_{\varepsilon, \mathbf{a}, K} u_T^{1/2 - 2\varepsilon} \exp\{-c_3|\tau|\}, \quad c_3 > 0.$$

Thus,

$$I_2 \ll_{\varepsilon, \mathbf{a}, K} u_T^{1/2 - 2\varepsilon} \frac{1}{T} \int_0^T \exp\{-c_3\tau\} d\tau \ll_{\varepsilon, \mathbf{a}, K} u_T^{1/2 - 2\varepsilon} / T.$$

This, (2.2) and (2.4) prove the lemma. \square

3 Limit theorems

Let P_n , $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Recall that P_n converges weakly to P as $n \rightarrow \infty$ if, for every real bounded continuous function g on \mathbb{X} ,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{X}} g dP_n = \int_{\mathbb{X}} g dP.$$

There are equivalents of weak convergence in terms of some classes of sets. We will use the following, see, for example, [1].

Lemma 2. P_n converges weakly to P as $n \rightarrow \infty$ if and only if, for every closed set $F \subset \mathbb{X}$,

$$\limsup_{n \rightarrow \infty} P_n(F) \leq P(F).$$

For $A \in \mathcal{B}(H(D))$, define

$$P_{T,\alpha,\mathbf{a}}(A) = \frac{1}{T} \text{meas}\{\tau \in [0, T] : \zeta(s + i\tau, \alpha; \mathbf{a}) \in A\}.$$

Lemma 3. Suppose that α is a transcendental number. Then $P_{T,\alpha,\mathbf{a}}$ converges weakly to

$$m_H\{\omega \in \Omega : \zeta(s, \omega, \alpha; \mathbf{a}) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

as $T \rightarrow \infty$.

The lemma was obtained in [2]. The transcendence of α can be replaced by the linear independence over \mathbb{Q} for the set $L(\alpha)$.

Let $V > 0$ be an arbitrary fixed number, $D_V = \{s \in \mathbb{C} : 1/2 < \sigma < 1, |t| < V\}$, and $H(D_V)$ the space of analytic on D_V functions. Denote by $P_{T,\alpha,\mathbf{a}}^1$ the analogue of $P_{T,\alpha,\mathbf{a}}$ for the space $H(D_V)$.

Lemma 4. Suppose that $\alpha = a/b$, $a, b \in \mathbb{N}$, $a < b$, $(a, b) = 1$, $a/b \neq 1/2$, and $(lb + a, bq) = 1$ for all $l = 0, 1, \dots, q - 1$. Then $P_{T,\alpha,\mathbf{a}}^1$ converges weakly to

$$m_{1H}\left\{\omega_1 \in \Omega_1 : \zeta\left(s, \frac{a}{b}, \omega_1; \mathbf{a}\right) \in A\right\} \stackrel{\text{def}}{=} P_{\zeta,\alpha,\mathbf{a}}^1(A), \quad A \in \mathcal{B}(H(D_V)),$$

as $T \rightarrow \infty$.

Proof of the lemma is given in [4], Theorem 3.

Now we will prove the analogues of Lemmas 3 and 4 for the function $\zeta_{u_T}(s, \alpha; \mathbf{a})$. For $A \in \mathcal{B}(H(D))$, define

$$Q_{T,\alpha,\mathbf{a}}(A) = \frac{1}{T} \text{meas}\{\tau \in [0, T] : \zeta_{u_T}(s + i\tau, \alpha; \mathbf{a}) \in A\}.$$

Lemma 5. Suppose that α is a transcendental number, and $u_T \rightarrow \infty$ and $u_T \ll T^2$ as $T \rightarrow \infty$. Then $Q_{T,\alpha,\mathbf{a}}$ converges weakly to $P_{\zeta,\alpha,\mathbf{a}}$ as $T \rightarrow \infty$.

Proof. For $g_1, g_2 \in H(D)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence of compact embedded subsets such that $D = \bigcup_{l=1}^{\infty} K_l$, and if $K \subset D$ is a compact set, then K lies in some K_l . Then ρ is a metric in the space $H(D)$ inducing its topology of uniform convergence on compacta.

Let ξ_T be the random variable defined on a certain probability space $(\widehat{\Omega}, \mathcal{A}, \mu)$ and uniformly distributed in the interval $[0, T]$. Define the $H(D)$ -valued random elements

$$\begin{aligned} X_{T,\alpha,\mathbf{a}} &= X_{T,\alpha,\mathbf{a}}(s) = \zeta(s + i\xi_T, \alpha; \mathbf{a}), \\ Y_{T,\alpha,\mathbf{a}} &= Y_{T,\alpha,\mathbf{a}}(s) = \zeta_{u_T}(s + i\xi_T, \alpha; \mathbf{a}). \end{aligned}$$

Let F be an arbitrary closed set of the space $H(D)$ and $\varepsilon > 0$. Then the set

$$F_\varepsilon = \{g : \rho(g, F) \leq \varepsilon\}$$

is closed as well. Therefore, by Lemmas 3 and 2,

$$\limsup_{T \rightarrow \infty} P_{T,\alpha,\mathbf{a}}(F_\varepsilon) \leq P_{\zeta,\alpha,\mathbf{a}}(F_\varepsilon).$$

Using the inclusion

$$\{Y_{T,\alpha,\mathbf{a}} \in F\} \subset \{X_{T,\alpha,\mathbf{a}} \in F_\varepsilon\} \cup \{\rho(X_{T,\alpha,\mathbf{a}}, Y_{T,\alpha,\mathbf{a}}) \geq \varepsilon\},$$

we find

$$\mu\{Y_{T,\alpha,\mathbf{a}} \in F\} \leq \mu\{X_{T,\alpha,\mathbf{a}} \in F_\varepsilon\} + \mu\{\rho(X_{T,\alpha,\mathbf{a}}, Y_{T,\alpha,\mathbf{a}}) \geq \varepsilon\}. \quad (3.1)$$

Since the density of the random variable ξ_T is $1/T$ on $[0, T]$, and 0 elsewhere, for every measurable function $h : \widehat{\Omega} \rightarrow \mathbb{X}$, we have

$$\mu\{h(\xi_T) \in A\} = \frac{1}{T} \int_0^T \mathbf{1}_{A \circ h^{-1}}(\tau) d\tau, \quad A \in \mathcal{B}(\mathbb{X}).$$

Therefore, by the definitions of $X_{T,\alpha,\mathbf{a}}$ and $Y_{T,\alpha,\mathbf{a}}$,

$$\mu\{X_{T,\alpha,\mathbf{a}} \in F_\varepsilon\} = \frac{1}{T} \text{meas}\{\tau \in [0, T] : \zeta(s + i\tau, \alpha; \mathbf{a}) \in F_\varepsilon\} = P_{T,\alpha,\mathbf{a}}(F_\varepsilon),$$

and similarly

$$\mu\{Y_{T,\alpha,\mathbf{a}} \in F\} = Q_{T,\alpha,\mathbf{a}}(F).$$

Moreover, in view of Lemma 1 and the definition of the metric ρ , for every $\varepsilon > 0$,

$$\begin{aligned} &\mu\{\rho(X_{T,\alpha,\mathbf{a}}, Y_{T,\alpha,\mathbf{a}}) \geq \varepsilon\} \\ &= \frac{1}{T} \text{meas}\{\tau \in [0, T] : \rho(\zeta(s + i\tau, \alpha, \mathbf{a}), \zeta_{u_T}(s + i\tau, \alpha, \mathbf{a})) \geq \varepsilon\} \\ &\leq \frac{1}{T\varepsilon} \int_0^T \rho(\zeta(s + i\tau, \alpha, \mathbf{a}), \zeta_{u_T}(s + i\tau, \alpha, \mathbf{a})) d\tau = 0. \end{aligned}$$

These three equalities and (3.1) give

$$\liminf_{T \rightarrow \infty} Q_{T,\alpha,\mathbf{a}}(F) \leq P_{\zeta,\alpha,\mathbf{a}}(F_\varepsilon).$$

Letting $\varepsilon \rightarrow +0$ together with Lemma 2 proves the lemma. \square

For $A \in \mathcal{B}(H(D_V))$, let $Q_{T,\alpha,\mathbf{a}}^1(A)$ be an analogue of $Q_{T,\alpha,\mathbf{a}}$ for the space $H(D_V)$. Using Lemma 4 and repeating a proof of Lemma 5 lead to the following assertion.

Lemma 6. *Suppose that the hypotheses of Lemma 4 on the parameter α and the sequence \mathbf{a} are satisfied, and $u_T \rightarrow \infty$ and $u_T \ll T^2$ as $T \rightarrow \infty$. Then $Q_{T,\alpha,\mathbf{a}}^1$ converges weakly to $P_{\zeta,\alpha,\mathbf{a}}^1$ as $T \rightarrow \infty$.*

Lemmas 5 and 6 imply the weak convergence for the corresponding measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. To see this, recall a preservation property of weak convergence. Let P be a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, and $h : \mathbb{X} \rightarrow \mathbb{X}_1$ be a $(\mathcal{B}(\mathbb{X}), \mathcal{B}(\mathbb{X}_1))$ -measurable mapping. Then the measure P defines the unique probability measure Ph^{-1} on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$ by the formula

$$Ph^{-1}(A) = P(h^{-1}A), \quad A \in \mathcal{B}(\mathbb{X}_1).$$

Lemma 7. *Suppose that $P_n, n \in \mathbb{N}$, and P are probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, $h : \mathbb{X} \rightarrow \mathbb{X}_1$ a continuous mapping, and P_n converges weakly to P as $n \rightarrow \infty$. Then also $P_n h^{-1}$ converges weakly to Ph^{-1} as $n \rightarrow \infty$.*

Proof of the lemma can be found in [1].

For $A \in \mathcal{B}(\mathbb{R})$, define

$$\widehat{Q}_{T,\alpha,\mathbf{a}}(A) = \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta_{u_T}(s + i\tau, \alpha; \mathbf{a}) - f(s)| \in A \right\},$$

where K and $f(s)$ are from Theorems 2 and 3.

Lemma 8. *Suppose that α is a transcendental number, $u_T \rightarrow \infty$ and $u_T \ll T^2$ as $T \rightarrow \infty$, and $K \subset D$ is a compact set. Then $\widehat{Q}_{T,\alpha,\mathbf{a}}$ converges weakly to*

$$m_H \left\{ \omega \in \Omega : \sup_{s \in K} |\zeta(s, \alpha, \omega; \mathbf{a}) - f(s)| \in A \right\}, \quad A \in \mathcal{B}(\mathbb{R}),$$

as $T \rightarrow \infty$.

Proof. The mapping $h : H(D) \rightarrow \mathbb{R}$ given by $h(g) = \sup_{s \in K} |g(s) - f(s)|$ is continuous. Therefore, the lemma follows from Lemmas 5 and 7. \square

Similarly, Lemmas 6 and 7 imply the following statement.

Lemma 9. *Let $K \subset D$ be a compact set, $u_T \rightarrow \infty$ and $u_T \ll T^2$ as $T \rightarrow \infty$. Then, under hypotheses of Lemma 4 for the parameter α and sequence \mathbf{a} , $\widehat{Q}_{T,\alpha,\mathbf{a}}$ converges weakly to*

$$m_{1H} \left\{ \omega_1 \in \Omega_1 : \sup_{s \in K} \left| \zeta \left(s, \frac{a}{b}, \omega_1; \mathbf{a} \right) - f(s) \right| \in A \right\}, \quad A \in \mathcal{B}(\mathbb{R}),$$

as $T \rightarrow \infty$.

4 Proof of universality

Let $G_n(x)$, $n \in \mathbb{N}$ and $G(x)$ be the distribution functions. We recall that G_n converges weakly to G if

$$\lim_{n \rightarrow \infty} G_n(x) = G(x)$$

for every continuity point x of $G(x)$. Moreover, every distribution function has no more than a countable set of discontinuity points.

It is well known that the weak convergence of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is equivalent to that of the corresponding distribution functions.

Proof. (*Proof of Theorem 2*). In [3], it is obtained that the support of the measure $P_{\zeta, \alpha, \mathbf{a}}$ is the space $H(D)$. Therefore,

$$P_{\zeta, \alpha, \mathbf{a}}(\mathcal{G}_\varepsilon) > 0, \quad (4.1)$$

where

$$\mathcal{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\},$$

and $p(s)$ is a polynomial. In view of the Mergelyan theorem on the approximation of analytic functions by polynomials [5], there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}.$$

This shows that

$$\mathcal{G}_\varepsilon \subset \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\} \stackrel{\text{def}}{=} \widehat{\mathcal{G}}_\varepsilon.$$

Thus, in view of (4.1), we have $P_{\zeta, \alpha, \mathbf{a}}(\widehat{\mathcal{G}}_\varepsilon) > 0$, i. e.,

$$m_H \left\{ \omega \in \Omega : \sup_{s \in K} |\zeta(s, \alpha, \omega; \mathbf{a}) - f(s)| < \varepsilon \right\} > 0.$$

Define the distribution function

$$G_{T, \alpha, \mathbf{a}}(\varepsilon) = \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta_{u_T}(s + i\tau, \alpha; \mathbf{a}) - f(s)| < \varepsilon \right\}.$$

Then Lemma 8 implies that

$$\lim_{T \rightarrow \infty} G_{T, \alpha, \mathbf{a}}(\varepsilon) = m_H \left\{ \omega \in \Omega : \sup_{s \in K} |\zeta(s, \alpha, \omega; \mathbf{a}) - f(s)| < \varepsilon \right\}$$

for all continuity points ε of the latter limit distribution function, i. e., for all but at most countably many $\varepsilon > 0$. \square

Proof. (*Proof of Theorem 3*). Let V be such that $K \subset D_V$. The support of the measure $P_{\zeta, \alpha, \mathbf{a}}^1$ is the set $H(D_V)$ [4]. Therefore, the analogue of the set \mathcal{G}_ε , the set

$$\mathcal{G}_{\varepsilon, V} = \left\{ g \in H(D_V) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\},$$

is an open neighbourhood of an element of the support of the measure $P_{\zeta, \alpha, a}^1$. Thus $P_{\zeta, \alpha, a}^1(\mathcal{G}_{\varepsilon, V}) > 0$, and the further proof runs in the same way as that of Theorem 2 by using Lemma 9. \square

References

- [1] P. Billingsley. *Convergence of Probability Measures*. Wiley, New York, 1968.
- [2] A. Javtokas and A. Laurinčikas. On the periodic Hurwitz zeta-function. *Hardy-Ramanujan Journal*, **29**:18–36, 2006. <https://doi.org/10.46298/hrj.2006.154>.
- [3] A. Javtokas and A. Laurinčikas. Universality of the periodic Hurwitz zeta-function. *Integral Transforms and Special Functions*, **17**(10):711–722, 2006. <https://doi.org/10.1080/10652460600856484>.
- [4] A. Laurinčikas, R. Macaitienė, D. Mochov and D. Šiaučiušas. Universality of the periodic Hurwitz zeta-function with rational parameter. *Siberian Mathematical Journal*, **59**(5):894–900, 2018. <https://doi.org/10.1134/S0037446618050130>.
- [5] S.N. Mergelyan. Uniform approximations to functions of complex variable. *Uspekhi Mat. Nauk*, **7**(2):31–122, 1952. (in Russian).
- [6] S.M. Voronin. Theorem on the “universality” of the Riemann zeta-function. *Mathematics of the USSR-Izvestiya, Ser. Matem.*, **9**(3):475–486, 1975. <https://doi.org/10.1070/IM1975v009n03ABEH001485>. (in Russian).