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Joint Discrete Approximation of Analytic Functions by Shifts of the Riemann Zeta-Function Twisted by Gram Points

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Abstract: Let $\theta(t)$ denote the increment of the argument of the product $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$, and t_n denote the solution of the equation $\theta(t) = (n-1)\pi$, $n = 0, 1, \ldots$. The numbers t_n are called the Gram points. In this paper, we consider the approximation of a collection of analytic functions by shifts in the Riemann zeta-function $(\zeta(s+it_k^{\alpha_1}), \ldots, \zeta(s+it_k^{\alpha_r}))$, $k = 0, 1, \ldots$, where $\alpha_1, \ldots, \alpha_r$ are different positive numbers not exceeding 1. We prove that the set of such shifts approximating a given collection of analytic functions has a positive lower density. For the proof, a discrete limit theorem on weak convergence of probability measures in the space of analytic functions is applied.

Keywords: Gram points; joint universality; weak convergence

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1. Introduction

Let $s = \sigma + it$ be a complex variable, and $\mathbb{P}$ be the set of all prime numbers. The Riemann zeta-function is defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The function $\zeta(s)$ with real s was already known to L. Euler. B. Riemann began to study $\zeta(s)$ with complex s and applied it to the investigation of the distribution of prime numbers in the set $\mathbb{N}$. The function $\zeta(s)$ has analytic continuation to the whole complex plane, except for the point $s = 1$, which is a simple pole with residue 1. Riemann's ideas were successfully realized by J. Hadamard and C.J. de la Valée Poussin at the end of the 19th century. Riemann proved for $\zeta(s)$ the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \quad s \in \mathbb{C}, \tag{1}$$

where $\Gamma(s)$ denotes the Euler gamma function. Moreover, Riemann stated some hypotheses on the zero-distribution of the function $\zeta(s)$. From Equation (1), it follows that $\zeta(-2k) = 0$, $k \in \mathbb{N}$, and the points $s = -2k$ are called trivial zeros of $\zeta(s)$. Moreover, it is known that the function $\zeta(s)$ has infinitely many of the so-called non-trivial zeros that are complex and lie in the strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$. The famous Riemann hypothesis asserts that all non-trivial zeros of $\zeta(s)$ are located on the critical line $\sigma = 1/2$. It is known that more than 41 percent of non-trivial zeros in the sense of density lie on the critical line [1]. Recently, in [2], this was improved to more than 41.7293 percent. There are also other important hypotheses on the value distribution of the function $\zeta(s)$. For example, by the Lindelöf hypothesis, for every $\varepsilon > 0$,

$$\zeta\left(\frac{1}{2} + it\right) \ll_\varepsilon t^\varepsilon, \quad t \geqslant t_0.$$

Recall that the notation $f \ll_\theta g$, $g > 0$, means that there exists a constant $c = c(\theta) > 0$ such that $|f| \leqslant cg$. On the other hand, the theory of the function $\zeta(s)$ is sufficiently rich in the final results. One of them is the universality property discovered by S.M. Voronin [3], which means that a wide class of analytic functions defined in the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ can be approximated by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. More precisely, we denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then the improved Voronin universality theorem says [4] that for every $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right\} > 0. \tag{2}$$

The latter inequality shows that there exists a constant $c > 0$ such that, for sufficiently large T, the Lebesgue measure of the set

$$\left\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right\}$$

is greater than cT. Thus, there are infinitely many shifts in $\zeta(s + i\tau)$ approximating a given function from the class $H_0(K)$. Obviously, the above theorem is useful in the approximation theory of analytic functions, but also has applications in the theory of the function $\zeta(s)$ (functional independence, zero distribution, moment problem); see, for example, [5], [6] and [7], respectively, and an informative survey paper [8].

The above universality theorem has a discrete version [9]. Denote by $\#A$ the cardinality of set A. Then, for the same K and $f(s)$ as in (2), and every $h > 0$ and $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N+1} \#\left\{0 \leqslant k \leqslant N : \sup_{s \in K} |\zeta(s + ikh) - f(s)| < \varepsilon\right\} > 0.$$

Here N runs over the set of non-negative integers.

Universality theorems for the function $\zeta(s)$ also have their joint versions. In this case, a collection of functions from the class $H_0(K)$ is simultaneously approximated by a collection of different shifts in $\zeta(s)$, for example, by $(\zeta(s + ikh_1), \dots, \zeta(s + ikh_r))$, where $h_1, \dots, h_r$ satisfy a certain independence condition. In place of traditional shifts $\zeta(s + i\tau)$ and $\zeta(s + ikh)$, generalized shifts $\zeta(s + i\varphi(\tau))$ and $\zeta(s + i\varphi_1(k))$ are possible with certain functions $\varphi(\tau)$ and $\varphi_1(k)$.

The function $\pi^{-s/2}\Gamma(s/2)$, as the main ingredient of the functional Equation (1), plays an important role in the theory of $\zeta(s)$. This was observed once more by J.-P. Gram in [10]. Denote by $\theta(t)$, $t > 0$, the increment of the argument of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting the points $1/2$ and $1/2 + it$. The function $\theta(t)$ is increasing and unbounded from above for $t \geqslant t^* = 6.289\dots$, therefore, the equation

$$\theta(t) = (n-1)\pi, \quad n = 0, 1, \dots, \tag{3}$$

has the unique solution t_n for $t \geqslant t^*$. Gram considered the points t_n in connection with zeros $\widehat{\gamma}_n$ of $\zeta(1/2 + it)$. He observed that each interval $(t_{n-1}, t_n]$, $n = 1, \dots, 15$, contains $\widehat{\gamma}_n$ such that $1/2 + i\widehat{\gamma}_n$ is a zero of $\zeta(s)$, and conjectured that this is impossible for $n > 15$.

The Gram conjecture was later confirmed by other authors. Moreover, the Riemann-von Mangoldt formula

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \quad T \to \infty,$$

where $N(T)$ is the number of zeros of $\zeta(s)$ counted the according multiplicities in the region $\{s \in \mathbb{C} : 0 < t < T\}$, implies that $t_n \sim \gamma_n$ as $n \to \infty$, where γ_n are imaginary parts of non-trivial zeros of $\zeta(s)$. Thus, the sequence $\{t_n\}$ of the Gram points is quite intriguing. A wide survey of the results on the Gram points is given in [11]. Equation (3) also offers a unique solution with arbitrary $\tau \geqslant 0$ in place of n, and this solution is called the Gram function.

In [12], a joint universality theorem for the Riemann zeta function with shifts involving the powers of the Gram function has been obtained.

Theorem 1 ([12])**.** *Suppose that $\alpha_1, \ldots, \alpha_r$ are fixed different positive numbers. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |\zeta(s + it_\tau^{a_j}) - f_j(s)| < \varepsilon\right\} > 0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many $\varepsilon > 0$.*

The aim of this paper is to obtain a discrete version of Theorem 1, i.e., the joint approximation of analytic functions by using shifts involving the Gram points. It turns out that the discrete case is more complicated, and we have to add the restriction $0 < \alpha_j \leqslant 1$, $j = 1, \ldots, r$.

Theorem 2. *Suppose that $\alpha_1, \ldots, \alpha_r$ are different fixed positive numbers not exceeding 1. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |\zeta(s + it_k^{a_j}) - f_j(s)| < \varepsilon\right\} > 0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many $\varepsilon > 0$.*

Theorem 2 is weaker than Theorem 1 with respect to numbers $\alpha_1, \ldots, \alpha_r$. However, discrete universality theorems are sometimes more convenient for applications because of the easier detection of approximating shifts. This is our motivation to consider a discrete version of Theorem 1.

2. Discrete Mean Square Estimates

We start with recalling the Gallagher lemma with discrete and continuous connections to mean squares of some functions; see, for example, Lemma 1.4 in [13].

Lemma 1. *Suppose that $T_0 \geqslant \delta > 0$, $T \geqslant \delta$, $\mathcal{T}$ is a finite non-empty set in the interval $[T_0 + \delta/2, T_0 + T - \delta/2]$, and*

$$N_\delta(\tau) = \sum_{\substack{t\in\mathcal{T} \\ |t-\tau|<\delta}} 1, \quad \tau \in \mathcal{T}.$$

Let a complex valued function $S(t)$ be continuous on $[T_0, T_0+T]$ and have a continuous derivative on (T_0, T_0+T). Then

$$\sum_{t\in\mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leqslant \frac{1}{\delta}\int_{T_0}^{T_0+T}|S(t)|^2\,\mathrm{d}t + \left(\int_{T_0}^{T_0+T}|S(t)|^2\,\mathrm{d}t \int_{T_0}^{T_0+T}|S'(t)|^2\,\mathrm{d}t\right)^{1/2}.$$

The next lemma is Lemma 2.2 from [12].

Lemma 2. *Suppose that $1/2 < \sigma < 1$ and $\alpha > 0$ are fixed. Then, for fixed $t \in \mathbb{R}$,*

$$\int_0^T \left|\zeta(\sigma + it_\tau^\alpha + it)\right|^2 \mathrm{d}\tau \ll_{\sigma,\alpha} T(1+|t|), \quad T\to\infty.$$

For the Gram function t_τ, the following asymptotics is known [11], Lemma 1.1.

Lemma 3. *Suppose that t_τ, $\tau \geqslant 0$, is the unique solution of equation $\theta(t) = (\tau-1)\pi$ such that the derivative $\theta'(t_\tau) > 0$, and that $\tau\to\infty$. Then*

$$t_\tau = \frac{2\pi\tau}{\log\tau}(1+o(1))$$

and

$$t'_\tau = \frac{2\pi}{\log\tau}(1+o(1)),$$

where $t'_\tau = \frac{\partial t_\tau}{\partial\tau}$.

Lemma 4. *Suppose that $1/2 < \sigma < 1$ and $\alpha > 0$ are fixed. Then, for fixed $t \in \mathbb{R}$,*

$$\int_0^T \left|\zeta'(\sigma + it_\tau^\alpha + it)\right|^2 \mathrm{d}\tau \ll_{\sigma,\alpha} T(1+|t|\log^2|t|).$$

Proof. For fixed $1/2 < \sigma < 1$, the estimate

$$\int_{-T}^T \left|\zeta'(\sigma+it)\right|^2 \mathrm{d}t \ll_\sigma T, \quad T\to\infty, \tag{4}$$

is valid. Let $X \geqslant \log T$. Define $g(\tau) = t_\tau^\alpha + t$. Then, in view of Lemma 3,

$$\begin{aligned} I_{\sigma,\alpha}(X,t) &\overset{\text{def}}{=} \int_X^{2X} \left|\zeta'(\sigma + it_\tau^\alpha + it)\right|^2 \mathrm{d}\tau = \int_X^{2X} \frac{g'(\tau)}{g'(\tau)}|\zeta'(\sigma+ig(\tau))|^2\,\mathrm{d}\tau \\ &\ll_\alpha \frac{(\log X)^\alpha}{X^{\alpha-1}} \int_{g(X)}^{g(2X)} |\zeta'(\sigma+iu)|^2\,\mathrm{d}u. \end{aligned}$$

Therefore, estimate (4) and Lemma 3 imply

$$I_{\sigma,\alpha}(X,t) \ll_{\sigma,\alpha} \frac{(\log X)^\alpha}{X^{\alpha-1}}\left(\frac{X^\alpha}{(\log X)^\alpha} + |t|\right) \ll_{\sigma,\alpha} X(1+|t|).$$

This shows that

$$\int_{\log T}^{T} \left|\zeta'(\sigma + it_\tau^\alpha + it)\right|^2 \mathrm{d}\tau \ll_{\sigma,\alpha} T(1 + |t|). \tag{5}$$

It is known (see, for example, [14], p. 55) that, for $t \geqslant t_0$ and $\sigma > 0$,

$$\zeta'(\sigma + it) + \sum_{m \leqslant t} \frac{\log m}{m^{\sigma+it}} \ll_\sigma t^{-\sigma} \log t.$$

Thus, for fixed $1/2 < \sigma < 1$,

$$\zeta'(\sigma + it) \ll_\sigma t^{1/2} \log t, \quad t \geqslant t_0.$$

Therefore, Lemma 3 gives

$$\int_{0}^{\log T} \left|\zeta'(\sigma + it_\tau^\alpha + it)\right|^2 \mathrm{d}\tau \ll_\sigma (\log T)^{\alpha+2}(1 + |t| \log^2 |t|).$$

This and (5) prove the lemma. □

Now we are ready to estimate mean square of $\zeta(s)$ involving the points t_k^α.

Lemma 5. *Suppose that* $1/2 < \sigma < 1$ *and* $0 < \alpha \leqslant 1$ *are fixed. Then, for fixed* $t \in \mathbb{R}$,

$$\sum_{k=0}^{N} |\zeta(\sigma + it_k^\alpha + it)|^2 \ll_{\sigma,\alpha} N(1 + |t| \log^2 |t|).$$

Proof. We apply Lemma 1 with $\delta = 1$, $T_0 = 1$, $T = N$ and $\mathcal{T} = \{3/2, 2, 3, \ldots, N, N + 1/2\}$. Then $N_\delta(x) = 1$, and, by Lemmas 1 and 2,

$$\begin{aligned}
&\sum_{k=2}^{N} |\zeta(\sigma + it_k^\alpha + it)|^2 \\
&\leqslant \int_{1}^{N+1} |\zeta(\sigma + it_\tau^\alpha + it)|^2 \mathrm{d}\tau + \left(\int_{1}^{N+1} |\zeta(\sigma + it_\tau^\alpha + it)|^2 \mathrm{d}\tau \int_{1}^{N+1} \left|\zeta'(\sigma + it_\tau^\alpha + it)\right|^2 \mathrm{d}\tau \right)^{1/2} \\
&\ll_{\sigma,\alpha} N(1 + |t|) + N^{1/2}(1 + |t|)^{1/2} \left(\int_{1}^{N+1} \left|\zeta'(\sigma + it_\tau^\alpha + it)\right|^2 ((t_\tau^\alpha)')^2 \mathrm{d}\tau \right)^{1/2}.
\end{aligned}$$

Since $0 < \alpha \leqslant 1$, the last integral, in view of Lemmas 3 and 4, is estimated as $N(1 + |t| \log^2 |t|)$. Thus,

$$\sum_{k=2}^{N} |\zeta(\sigma + it_k^\alpha + it)|^2 \ll_{\sigma,\alpha} N(1 + |t| \log^2 |t|). \tag{6}$$

Since $\zeta(\sigma + it) \ll_\sigma 1 + |t|^{1/6}$, we have

$$\sum_{k=0}^{1} |\zeta(\sigma + it_k^\alpha + it)|^2 \ll_{\sigma,\alpha} 1 + |t|,$$

and this together with (6) proves the lemma. □

In the sequel, we will need the approximation of $\zeta(s+it_k^\alpha)$ by a certain absolutely convergent Dirichlet series. Let $\theta > 1/2$ be a fixed number,

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^\theta\right\}, \quad m, n \in \mathbb{N},$$

and

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s}.$$

Since $v_n(m)$ with respect to m decreases exponentially, the series for $\zeta_n(s)$ is absolutely convergent for all $s \in \mathbb{C}$.

Lemma 6. *Suppose that $0 < \alpha \leqslant 1$ is fixed. Then, for every compact subset K of the strip D,*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |\zeta(s+it_k^\alpha) - \zeta_n(s+it_k^\alpha)| = 0.$$

Proof. For brevity, define

$$l_n(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)n^s.$$

Then the Mellin formula

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(s) a^{-s}\, \mathrm{d}s = \mathrm{e}^{-a}, \quad a, b > 0,$$

and the definition of $v_n(m)$ lead to the integral representation (see, for example, [4], for proof of Theorem 5.4.2)

$$\zeta_n(s) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z) l_n(z) \frac{\mathrm{d}z}{z}, \quad s \in D.$$

There exists $\varepsilon > 0$ such that $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for all $s = \sigma + it \in K$. Now, for such ε and σ, put

$$\theta = \frac{1}{2} + \varepsilon, \qquad \kappa = \sigma - \frac{1}{2} - \varepsilon > 0.$$

The integrand in the latter integral, for $-\kappa \leqslant \mathrm{Re} z \leqslant \theta$, has simple poles at the points $z = 0$ and $z = 1 - s$. Therefore, the residue theorem gives

$$\zeta_n(s) - \zeta(s) = \frac{1}{2\pi i} \int_{-\kappa-i\infty}^{-\kappa+i\infty} \zeta(s+z) l_n(z) \frac{\mathrm{d}z}{z} + \frac{l_n(1-s)}{1-s}. \tag{7}$$

Then, by (7), for $s \in K$, we have

$$\begin{aligned}
\zeta_n(s+it_k^\alpha) - \zeta(s+it_k^\alpha) &= \frac{1}{2\pi i} \int_{-\kappa-i\infty}^{-\kappa+i\infty} \zeta(s+it_k^\alpha+z) l_n(z) \frac{\mathrm{d}z}{z} + \frac{l_n(1-s-it_k^\alpha)}{1-s-it_k^\alpha} \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \zeta\left(\frac{1}{2} + \varepsilon + it_k^\alpha + it + iv\right) \frac{l_n(1/2+\varepsilon-\sigma+iv)}{1/2+\varepsilon-\sigma+iv}\, \mathrm{d}v + \frac{l_n(1-s-it_k^\alpha)}{1-s-it_k^\alpha} \\
&\ll \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + \varepsilon + it_k^\alpha + iv\right)\right| \sup_{s\in K} \left|\frac{l_n(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\right| \mathrm{d}v + \sup_{s\in K} \left|\frac{l_n(1-s-it_k^\alpha)}{1-s-it_k^\alpha}\right|.
\end{aligned}$$

Here we made a shift $t + v \to v$. Hence,

$$\begin{aligned} &\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta(s+it_k^{\alpha})-\zeta_n(s+it_k^{\alpha})| \\ &\quad \ll \int_{-\infty}^{\infty}\left(\frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+it_k^{\alpha}+iv\right)\right|\right)\sup_{s\in K}\left|\frac{l_n(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\right| dv \\ &\quad + \frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}\left|\frac{l_n(1-s-it_k^{\alpha})}{1-s-it_k^{\alpha}}\right| \stackrel{\text{def}}{=} I_1 + I_2. \end{aligned} \tag{8}$$

Lemma 5 shows that

$$\begin{aligned} \frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+it_k^{\alpha}+iv\right)\right| &\leqslant \left(\frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+it_k^{\alpha}+iv\right)\right|^2\right)^{1/2} \\ &\ll_{\varepsilon,\alpha} (1+|v|\log^2|v|)^{1/2} \ll_{\varepsilon,\alpha} 1+|v|\log^2|v|. \end{aligned} \tag{9}$$

For the function $\Gamma(s)$ the estimate, for large $|t|$,

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad c > 0, \tag{10}$$

is valid uniformly in any interval $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $\sigma_1 < \sigma_2$. Therefore, for all $s \in K$,

$$\frac{l_n(1/2+\varepsilon-\sigma+iv)}{1/2+\varepsilon-\sigma+iv} \ll_\theta n^{1/2+\varepsilon-\sigma}\exp\left\{-\frac{c}{\theta}|v-t|\right\} \ll_{\theta,K} n^{-\varepsilon}\exp\{-c_1|v|\}, \quad c_1 > 0.$$

This and (9) give the bound

$$I_1 \ll_{\varepsilon,\alpha,\theta,K} n^{-\varepsilon}\int_{-\infty}^{\infty}(1+|v|\log^2|v|)\exp\{-c_1|v|\}\, dv \ll_{\varepsilon,\alpha,\theta,K} n^{-\varepsilon}. \tag{11}$$

By (10) again, for $s \in K$,

$$\frac{l_n(1-s-it_k^{\alpha})}{1-s-it_k^{\alpha}} \ll_\theta n^{1-\sigma}\exp\left\{-\frac{c}{\theta}|t_k^{\alpha}+t|\right\} \ll_{\theta,K} n^{1/2-2\varepsilon}\exp\{-c_2 t_k^{\alpha}\}, \quad c_2 > 0.$$

Thus, in view of Lemma 3,

$$\begin{aligned} I_2 &\ll_{\theta,K} n^{1/2-2\varepsilon}\frac{1}{N}\sum_{k=0}^{N}\exp\{-c_2 t_k^{\alpha}\} \\ &\ll_{\theta,K} n^{1/2-2\varepsilon}\frac{1}{N}\sum_{k=\log N}^{N}\exp\left\{-c_3\left(\frac{k}{\log k}\right)^{\alpha}\right\} + n^{1/2-2\varepsilon}\frac{\log N}{N} \\ &\ll_{\theta,K} n^{1/2-2\varepsilon}\frac{1}{N}\exp\left\{-c_4\left(\frac{\log N}{\log\log N}\right)^{\alpha}\right\}, \quad c_3, c_4 > 0. \end{aligned}$$

Therefore, (11) and (9) show that

$$\begin{aligned} &\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta(s+it_k^{\alpha})-\zeta_n(s+it_k^{\alpha})| \\ &\quad \ll_{\varepsilon,\alpha,\theta,K} n^{-\varepsilon} + n^{1/2-2\varepsilon}\left(\frac{\log N}{N}+\frac{1}{N}\exp\left\{-c_4\left(\frac{\log N}{\log\log N}\right)^{\alpha}\right\}\right). \end{aligned}$$

Letting $N \to \infty$, and then $n \to \infty$, proves the lemma. □

Lemma 6 is important for the proof of a discrete limit theorem for $\zeta(s)$.

3. Limit Theorems

Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta, and put

$$H^r(D) = \underbrace{H(D) \times \cdots \times H(D)}_{r}.$$

Let $\mathcal{B}(\mathbb{X})$ be the Borel σ-field of a topological space $\mathbb{X}$. Define the set

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all $p \in \mathbb{P}$. The infinite-dimensional torus Ω, with the product topology and pointwise multiplication, by the classical Tikhonov theorem, is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure exists. Let

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$. Then, again, Ω^r is a compact topological Abelian group, and, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H can be defined. This gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$. Note that m_H is the product of the Haar measures m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \dots, r$. Denote by $\omega_j(p)$ the pth component, $p \in \mathbb{P}$, of an element of $\omega_j \in \Omega_j$, $j = 1, \dots, r$, and by $\omega = (\omega_1, \dots, \omega_r)$ the elements of Ω^r. On the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$, define the $H^r(D)$-valued random element

$$\underline{\zeta}(s, \omega) = (\zeta(s, \omega_1), \dots, \zeta(s, \omega_r)),$$

where

$$\zeta(s, \omega_j) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\omega_j(p)}{p^s}\right)^{-1}, \quad j = 1, \dots, r.$$

Note that the latter products, for almost all $\omega_j \in \Omega_j$, are uniformly convergent on a compact subset of the strip D, see, for example, Theorem 5.1.7 of [4], or Lemma 4 of [15]. Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(s, \omega)$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H\Big\{\omega \in \Omega^r : \underline{\zeta}(s, \omega) \in A\Big\}, \quad A \in \mathcal{B}(H^r(D)).$$

For brevity, we set $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $t_k^{\underline{\alpha}} = (t_k^{\alpha_1}, \dots, t_k^{\alpha_r})$,

$$\underline{\zeta}(s + it_k^{\underline{\alpha}}) = \big(\zeta(s + it_k^{\alpha_1}), \dots, \zeta(s + it_k^{\alpha_r})\big),$$

and, for $A \in \mathcal{B}(H^r(D))$, define

$$P_{N,\underline{\alpha}}(A) = \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \underline{\zeta}(s + it_k^{\underline{\alpha}}) \in A\Big\}.$$

This section is devoted to weak convergence for $P_{N,\underline{\alpha}}$ as $N \to \infty$.

Theorem 3. *Suppose that $\alpha_1, \dots, \alpha_r$ are different fixed positive numbers not exceeding* 1. *Then $P_{N,\underline{\alpha}}$ converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$.*

We divide the proof of Theorem 3 into lemmas. The first of them deals with probability measures on $(\Omega^r, \mathcal{B}(\Omega^r))$. For $A \in \mathcal{B}(\Omega^r)$, define

$$Q_{N,\underline{\alpha}}(A) = \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \Big((p^{-it_k^{\alpha_1}} : p \in \mathbb{P}), \dots, (p^{-it_k^{\alpha_r}} : p \in \mathbb{P})\Big) \in A\Big\}.$$

For the proof of weak convergence for $Q_{N,\underline{\alpha}}$, we will apply a notion of uniform distribution modulo 1. Recall that a sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1 if, for every subinterval $(a,b] \subset (0,1]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I_{(a,b]}(\{x_k\}) = b - a,$$

where $I_{(a,b]}$ is the indicator function of $(a,b]$, and $\{u\}$ denotes the fractional part of $u \in \mathbb{R}$.

We will use the Weil criterion on the uniform distribution modulo 1; see, for example, [16].

Lemma 7. *A sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if and only if, for every $m \in \mathbb{Z} \setminus \{0\}$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathrm{e}^{2\pi i m x_k} = 0.$$

The next lemma gives sufficient conditions for uniform distribution modulo 1; see, for example, [16], Theorem 3.5.

Lemma 8. *Let $g(u)$ be a function defined for $u \geqslant 1$ that is l-times differentiable for $u > u_0$. If $g^{(l)}(u)$ tends monotonically to zero as $u \to \infty$ and if $\lim_{u\to\infty} u|g^{(l)}(u)| = +\infty$, then the sequence $\{g(k) : k \in \mathbb{N}\}$ is uniformly distributed modulo 1.*

Lemma 9. *Suppose that $\alpha_1, \dots, \alpha_r$ are different fixed positive numbers not exceeding 1. Then $Q_{N,\underline{\alpha}}$ converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We apply the Fourier transform method. Denote by $g_{N,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z},\ p \in \mathbb{P})$, $j = 1, \dots, r$, the Fourier transform of $Q_{N,\underline{\alpha}}$, i.e.,

$$g_{N,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) = \int\limits_{\Omega^r} \prod_{j=1}^{r} \prod_{p\in\mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p)\, \mathrm{d}Q_{N,\underline{\alpha}},$$

where the sign " $*$ " means that only a finite number of integers k_{jp} are distinct from zero. The definition of $Q_{N,\underline{\alpha}}$ gives

$$\begin{aligned} g_{N,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) &= \frac{1}{N+1} \sum_{k=0}^{N} \prod_{j=1}^{r} \prod_{p\in\mathbb{P}}{}^{*} p^{-ik_{jp}t_k^{\alpha_j}} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \exp\left\{ -i \sum_{j=1}^{r} t_k^{\alpha_j} \sum_{p\in\mathbb{P}}{}^{*} k_{jp} \log p \right\}. \end{aligned} \tag{12}$$

Obviously,

$$g_{N,\underline{\alpha}}(\underline{0}, \dots, \underline{0}) = 1, \tag{13}$$

where $\underline{0} = (k_{jp} : k_{jp} = 0,\ p \in \mathbb{P})$. Thus, it remains to consider the case $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$. In this case, there exists $j \in \{1, \dots, r\}$ such that

$$a_j \overset{\mathrm{def}}{=} \sum_{p\in\mathbb{P}}{}^{*} k_{jp} \log p \neq 0$$

because the set of logarithms of all prime numbers is linearly independent over the field of rational numbers. Without a loss of generality, we suppose that $\alpha_1 < \dots < \alpha_r$, and $j_0 = \max\{j \leqslant r : a_j \neq 0\}$. Then

$$b_{\underline{\alpha}}(\tau) \overset{\mathrm{def}}{=} \sum_{j=1}^{r} a_j t_\tau^{\alpha_j} = \sum_{j=1}^{j_0} a_j t_\tau^{\alpha_j}.$$

Thus, by Lemma 3,

$$b'_{\underline{\alpha}}(\tau) = \left(\sum_{j=1}^{j_0} a_j t_\tau^{\alpha_j} \right)' = a_{j_0} \alpha_{j_0} t_\tau^{\alpha_{j_0}-1} t'_\tau (1 + o(1)) = 2\pi a_{j_0} \alpha_{j_0} \frac{(2\pi\tau)^{\alpha_{j_0}-1}}{(\log \tau)^{\alpha_{j_0}}} (1 + o(1))$$

as $\tau \to \infty$. This shows that $b'_{\underline{\alpha}}(\tau)$ tends monotonically to zero as $\tau \to \infty$, and

$$\lim_{\tau \to \infty} \tau |b'_{\underline{\alpha}}(\tau)| = \infty.$$

Therefore, Lemma 8 implies that the sequence

$$x_k \stackrel{\text{def}}{=} \left\{ -\frac{1}{2\pi} b_{\underline{\alpha}}(k) \right\}$$

is uniformly distributed modulo 1. Hence, by Lemma 7 and (12)

$$g_{N,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) = \frac{1}{N+1} \sum_{k=0}^{N} \exp\{-i b_{\underline{\alpha}}(k)\} = \frac{1}{N+1} \sum_{k=0}^{N} \mathrm{e}^{-2\pi i x_k} = o(1)$$

as $N \to \infty$. This and (13) show that

$$\lim_{N \to \infty} g_{N,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) = \begin{cases} 1 & \text{if} \quad (\underline{k}_1, \dots, \underline{k}_r) = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{if} \quad (\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the measure m_H, the lemma is proven. □

Let

$$\underline{\zeta}_n(s + i t_k^{\underline{\alpha}}) = \left(\zeta_n(s + i t_k^{\alpha_1}), \dots, \zeta_n(s + i t_k^{\alpha_r}) \right).$$

The next step of the proof of Theorem 3 is a limit lemma for

$$P_{N,n,\underline{\alpha}}(A) \stackrel{\text{def}}{=} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \underline{\zeta}_n(s + i t_k^{\underline{\alpha}}) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)).$$

Before that, we recall one assertion on the preservation of weak convergence under certain mappings. Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be two spaces, and $h : \mathbb{X}_1 \to \mathbb{X}_2$ a $(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2))$-measurable mapping, i.e., for every $A \in \mathcal{B}(\mathbb{X}_2)$,

$$h^{-1} A \in \mathcal{B}(\mathbb{X}_1).$$

Then every probability measure P on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$ defines the unique probability measure Ph^{-1} by

$$Ph^{-1}(A) = P(h^{-1} A), \quad A \in \mathcal{B}(\mathbb{X}_2).$$

It is well known that every continuous mapping h is $(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2))$-measurable, and the following useful statement is valid; see, for example, [17], Theorem 5.1.

Lemma 10. *Suppose that P_n, $n \in \mathbb{N}$, and P are probability measures on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$, $h : \mathbb{X}_1 \to \mathbb{X}_2$ a continuous mapping, and P_n converges weakly to P as $n \to \infty$. Then $P_n h^{-1}$ converges weakly to Ph^{-1} as $n \to \infty$.*

Let, for $m \in \mathbb{N}$,

$$\omega_j(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad j = 1, \dots, r,$$

and

$$\underline{\zeta}_n(s,\omega) = (\zeta_n(s,\omega_1),\dots,\zeta_n(s,\omega_r)),$$

where

$$\zeta_n(s,\omega_j) = \sum_{m=1}^{\infty} \frac{\omega_j(m)v_n(m)}{m^s}, \quad j=1,\dots,r.$$

Since $|\omega_j(m)| = 1$, the latter series are absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0.

Consider the mapping $h_n : \Omega^r \to H^r(D)$ given by

$$h_n(\omega) = \underline{\zeta}_n(s,\omega).$$

Let $V_n = m_H h_n^{-1}$. Then the following statement is valid.

Lemma 11. *Suppose that $\alpha_1,\dots,\alpha_r$ are different fixed positive numbers. Then $P_{N,n,\underline{\alpha}}$ converges weakly to V_n as $N \to \infty$.*

Proof. By the definition of h_n, we have

$$h_n\Big((p^{-it_k^{\alpha_1}} : p \in \mathbb{P}),\dots,(p^{-it_k^{\alpha_r}} : p \in \mathbb{P})\Big) = \underline{\zeta}_n(s + i\underline{t}_k^{\underline{\alpha}}).$$

Therefore, for $A \in \mathcal{B}(H^r(D))$,

$$\begin{aligned} P_{N,n,\underline{\alpha}}(A) &= \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N : \Big((p^{-it_k^{\alpha_1}} : p \in \mathbb{P}),\dots,(p^{-it_k^{\alpha_r}} : p \in \mathbb{P})\Big) \in h_n^{-1}A\Big\} \\ &= Q_{N,\underline{\alpha}}(h^{-1}A) = Q_{N,\underline{\alpha}}h^{-1}(A). \end{aligned}$$

Thus,

$$P_{N,n,\underline{\alpha}} = Q_{N,\underline{\alpha}}h_n^{-1}. \tag{14}$$

Since the series for $\zeta_n(s,\omega_j)$, $j = 1,\dots,r$, are absolutely convergent, the mapping h_n is continuous. Therefore, (14) and Lemmas 9 and 10 prove the lemma. □

The measure V_n appears in all joint limit theorems for $\zeta(s)$ and other Dirichlet series. The following lemma is known; see, for example, the proof of Theorem 5.4 in [12].

Lemma 12. *V_n converges weakly to $P_{\underline{\zeta}}$ as $n \to \infty$.*

Recall one lemma on convergence in distribution ($\xrightarrow{\mathcal{D}}$) of random elements; see, for example, Theorem 4.2 of [17].

Lemma 13. *Suppose that the space $(\mathbb{X}, d)$ is separable, and the $\mathbb{X}$-valued random elements Y_n and X_{kn}, $k,n \in \mathbb{N}$, are defined on the same probability space with measure μ. Moreover,*

$$X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k,$$
$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X,$$

and, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\{d(X_{kn}, Y_n) \geqslant \varepsilon\} = 0.$$

Then

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X.$$

Proof of Theorem 3. Let θ_N be a random variable defined on a certain probability space with measure μ and having a distribution

$$\mu\{\theta_N = k\} = \frac{1}{N+1}, \quad k = 0, 1, \dots, N.$$

Define two $H^r(D)$-valued random elements

$$X_{N,n,\underline{\alpha}} = X_{N,n,\underline{\alpha}}(s) = \underline{\zeta}_n(s + i\underline{t}^{\alpha}_{\theta_N})$$

and

$$X_{N,\underline{\alpha}} = X_{N,\underline{\alpha}}(s) = \underline{\zeta}(s + i\underline{t}^{\alpha}_{\theta_N}),$$

and denote by X_n the $H^r(D)$-valued random element with distribution V_n. Then the assertion of Lemma 12 can be written in the form

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P_{\underline{\zeta}}, \tag{15}$$

and, in view of Lemma 11,

$$X_{N,n,\underline{\alpha}} \xrightarrow[N\to\infty]{\mathcal{D}} X_n. \tag{16}$$

Next we need a metric in the space $H^r(D)$. Suppose that $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence of embedded compact subsets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

and every compact set $K \subset D$ lies in some K_l. Such a sequence exists, for example, we can take a sequence of closed rectangles. Then setting

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(D),$$

gives a metric in $H(D)$ inducing the topology of uniform convergence on compacta, and

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1\leqslant j\leqslant r} \rho(g_{1j}, g_{2j}), \quad \underline{g}_k = (g_{k1}, \dots, g_{kr}) \in H^r(D), \ k = 1, 2,$$

defines a metric in $H^r(D)$ inducing the product topology.

Now, Lemma 6, together with definitions of the metrics ρ and $\underline{\rho}$, yields the equality

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \underline{\rho}\Big(\underline{\zeta}(s + i\underline{t}^{\alpha}_k), \underline{\zeta}_n(s + i\underline{t}^{\alpha}_k)\Big) = 0.$$

Therefore, the definitions of random elements $X_{N,\underline{\alpha}}$ and $X_{N,n,\underline{\alpha}}$ show that, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} \limsup_{N\to\infty} \mu\Big\{\underline{\rho}(X_{N,\underline{\alpha}}, X_{N,n,\underline{\alpha}}) \geqslant \varepsilon\Big\} \\ &\quad \leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{k=0}^{N} \underline{\rho}(\underline{\zeta}(s + i\underline{t}^{\alpha}_k), \underline{\zeta}_n(s + i\underline{t}^{\alpha}_k)) = 0. \end{aligned}$$

This equality and relations (15) and (16) allow applying Lemma 13 for the random elements $X_{N,\underline{\alpha}}$, $X_{N,n,\underline{\alpha}}$ and X_n. Thus, we obtain the relation

$$X_{N,\underline{\alpha}} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\underline{\zeta}},$$

and the theorem is proven. □

4. Proof of Theorem 2

Before the proof of Theorem 2, we recall two equivalents of weak convergence of probability measures; see, for example, [17]. Recall that A is a continuity set of the measure P if $P(\partial A) = 0$, where ∂A is the boundary of A.

Lemma 14. *Suppose that P_n, $n \in \mathbb{N}$, and P are probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then the following statements are equivalent:*

1° *P_n converges weakly to P as $n \to \infty$;*

2° *For every open set $G \subset \mathbb{X}$,*

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$

3° *For every continuity set A of P,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

One of the ingredients of the proof of Theorem 2 is the Mergelyan theorem on approximation of analytic functions by polynomials, see [18].

Lemma 15. *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complements, and $f(s)$ is a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |g(s) - p(s)| < \varepsilon.$$

Recall that the support of the measure $P_{\underline{\zeta}}$ is a minimal closed set $S_P \subset H^r(D)$ such that $P_{\underline{\zeta}}(S_{P_{\underline{\zeta}}}) = 1$. The set $S_{P_{\underline{\zeta}}}$ consists of all elements $\underline{g} \in H^r(D)$ such that, for every open neighborhood G of $\underline{g}$, the inequality $P_{\underline{\zeta}}(G) > 0$ is satisfied.

Let $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$. The following lemma is well known; see, for example, [12], Lemma 6.8.

Lemma 16. *The support of the measure $P_{\underline{\zeta}}$ is the set S^r.*

Proof of Theorem 2. By Lemma 15, there exist polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| f_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2}. \tag{17}$$

The latter inequality is a simple application of Lemma 15 for $\log f(s)$; the details can be found in [19], proof of Theorem 2. Let

$$G_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} \right\}.$$

Then, in view of Lemma 16, the set G_ε is an open neighborhood of an element $(\mathrm{e}^{p_1(s)}, \dots, \mathrm{e}^{p_r(s)})$ of the support of the measure $P_{\underline{\zeta}}$. Therefore,

$$P_{\underline{\zeta}}(G_\varepsilon) > 0. \tag{18}$$

Hence, Theorem 3, and 1° and 2° of Lemma 14 imply

$$\liminf_{N\to\infty} P_{N,\underline{\alpha}}(G_\varepsilon) \geqslant P_{\underline{\zeta}}(G_\varepsilon) > 0.$$

This inequality, (17) and the definitions of $P_{N,\underline{\alpha}}$ and G_ε prove the first assertion of the theorem.

Let

$$\widehat{G}_\varepsilon = \left\{(g_1,\ldots,g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r}\sup_{s\in K}|g_j(s)-f_j(s)| < \varepsilon\right\}.$$

Then (17) implies the inclusion $G_\varepsilon \subset \widehat{G}_\varepsilon$. Thus, by (18),

$$P_{\underline{\zeta}}(\widehat{G}_\varepsilon) > 0. \tag{19}$$

Moreover, the boundaries $\partial\widehat{G}_{\varepsilon_1}$ and $\partial\widehat{G}_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. From this, it follows that the set $\widehat{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 3, 1° and 3° of Lemma 14, and (19) give the inequality

$$\lim_{N\to\infty} P_{N,\underline{\alpha}}(\widehat{G}_\varepsilon) \geqslant P_{\underline{\zeta}}(\widehat{G}_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. This proves the second assertion of the theorem. □

5. Conclusions

Let $\{t_k\}$ be a sequence of Gram points, $\zeta(s)$ be the Riemann zeta function, for $j = 1,\ldots,r$, $0 < \alpha_j \leqslant 1$, $\alpha_j \neq \alpha_k$ for $j \neq k$, K_j compact subset of the strip D with connected complement, and $f_j(s)$ be a continuous non-vanishing function on K_j and analytic in the interior of K_j. In this paper, it is obtained that, for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+it_k^{\alpha_j})-f_j(s)| < \varepsilon\right\} > 0,$$

and that the limit

$$\lim_{N\to\infty} \frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+it_k^{\alpha_j})-f_j(s)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Problem. Does the above theorem remain valid without a restriction $0 < \alpha_j \leqslant 1$, $j = 1,\ldots,r$?

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