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Article

On the Discrete Approximation by the Mellin Transform of the Riemann Zeta-Function

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Abstract: In the paper, it is obtained that there are infinite discrete shifts $\Xi(s+ikh)$, $h>0$, $k\in\mathbb{N}_0$ of the Mellin transform $\Xi(s)$ of the square of the Riemann zeta-function, approximating a certain class of analytic functions. For the proof, a probabilistic approach based on weak convergence of probability measures in the space of analytic functions is applied.

Keywords: discrete limit theorem; Mellin transform; Riemann zeta-function; weak convergence

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1. Introduction

As usual, $\zeta(s)$ is denoted by $s=\sigma+it$, the Riemann zeta-function, which, for $\sigma>1$, is defined by

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},$$

and has the meromorphic continuation of the whole complex plane with a unique simple pole at the point of $s=1$ with a residue of 1. In the theory of the function of $\zeta(s)$, the modified Mellin transforms $\Xi_k(s)$ play an important role. For $k\geqslant 0$ and $\sigma>\sigma(k)>1$, the functions $\Xi_k(s)$ are defined by

$$\Xi_k(s)=\int_1^{\infty}\left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k}x^{-s}\,\mathrm{d}x.$$

The functions $\Xi_k(s)$ were introduced in [1,2] and are applied for the investigation of the moments

$$\int_1^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}\mathrm{d}t.$$

In general, $\Xi_k(s)$ are attractive analytic functions and are widely studied; see, for example, [3–6].

In [7], the approximation properties of the function $\Xi_1(s)$ were studied. Let $G=\{s\in\mathbb{C}:1/2<\sigma<1\}$. $\mathcal{H}(G)$ is denoted by the space of analytic functions on G endowed with the topology of uniform convergence on compacta, and by meas$A$ the Lebesgue measure of a measurable set $A\subset\mathbb{R}$. Then, in [7], the following theorem is proven.

Theorem 1. *There exists a closed, non-empty set $F \subset \mathcal{H}(G)$, such that, for every compact set $K \subset G$, function $f(s) \in F$, and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|\Xi_1(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|\Xi_1(s+i\tau) - f(s)| < \varepsilon\right\}$$

exists and is positive for all but, at most, is a countable number $\varepsilon > 0$.

Theorem 1 is of continuous type, τ in the shifts $\Xi_1(s+i\tau)$ takes arbitrary real values. The aim of this paper is to obtain a discrete version of Theorem 1 with shifts $\Xi_1(s+ikh)$, where $h > 0$ is a fixed number and $k \in \mathbb{N} \cup \{0\} \stackrel{def}{=} \mathbb{N}_0$.

$\#A$ denotes the cardinality of a set $A \subset \mathbb{R}$. For brevity, we write $\Xi(s)$ in place of $\Xi_1(s)$. Let N run over the set $\mathbb{N}_0$.

Theorem 2. *For every $h > 0$, there exists a closed non-empty set $F_h \subset \mathcal{H}(G)$ such that, for every compact set $K \subset G$, function $f(s) \in F_h$, and $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \sup_{s\in K}|\Xi(s+ikh) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, the limit

$$\lim_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \sup_{s\in K}|\Xi(s+ikh) - f(s)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Theorem 2 shows that the set of discrete shifts $\Xi(s+ikh)$ approximating with a given accuracy the function $f(s) \in F_h$ is infinite.

We note that Theorem 2 has a certain advantage against Theorem 1 because it is easier to detect discrete approximating shifts.

Unfortunately, the sets F and F_h in Theorems 1 and 2, respectively, are not identified; however, Theorems 1 and 2 show good approximation properties of the function $\Xi(s)$. In some sense, Theorems 1 and 2 recall universality theorems for the function $\zeta(s)$. In this case, F and F_h are sets of non-vanishing analytic functions on G; see, for example, [8,9].

Here, we prove that the set F_h is a support of a certain $\mathcal{H}(G)$-valued random element defined in terms of $\Xi(s)$. The distribution of that random element is the limit measure in a probabilistic discrete limit theorem for the function $\Xi(s)$. $\mathcal{B}(\mathcal{X})$ denotes the Borel σ-field of the space $\mathcal{X}$, by $\xrightarrow{\text{W}}$ the weak convergence of probability measures, and, for $A \in \mathcal{B}(\mathcal{H}(G))$, define

$$P_{N,h}(A) = \frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \Xi(s+ikh) \in A\}.$$

Theorem 3. *For every fixed $h > 0$, on $(\mathcal{H}(G), \mathcal{B}(\mathcal{H}(G)))$, there exists a probability measure P_h such that $P_{N,h} \xrightarrow[N\to\infty]{\text{W}} P_h$.*

2. Some Lemmas

Let $a > 1$ be a fixed number. Define the set

$$\Omega_a = \prod_{u \in [1,a]} \gamma_u,$$

where $\gamma_u = \{s \in \mathbb{C} : |s| = 1\}$ for all $u \in [1,a]$. As a Cartesian product of compact sets, the torus Ω_a is a compact topological Abelian group. Let $\omega = \{\omega_u : u \in [1,a]\}$ be elements of Ω_a.

For $A \in \mathcal{B}(\Omega_a)$ and $h > 0$, define

$$Q_{N,a,h}(A) = \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \Big(u^{-ikh} : u \in [1,a]\Big) \in A\Big\}.$$

Lemma 1. *On $(\Omega_a, \mathcal{B}(\Omega_a))$, there exists a probability measure $Q_{a,h}$ such that $Q_{N,a,h} \xrightarrow[N\to\infty]{\mathrm{W}} Q_{a,h}$.*

Proof. We apply the Fourier transform method. Let $F_{Q_{N,a,h}}(\underline{k})$, $\underline{k} = (k_u : k_u \in \mathbb{Z}, u \in [1,a])$, be the Fourier transform of $Q_{N,a,h}$, i.e.,

$$F_{Q_{N,a,h}}(\underline{k}) = \int_{\Omega_a} \prod_{u \in [1,a]}^{*} \omega_u^{k_u} \, \mathrm{d}Q_{N,a,h},$$

where "$*$" shows that only a finite number of integers k_u are non-zero. Thus, by the definition of $Q_{N,a,h}$,

$$F_{Q_{N,a,h}}(\underline{k}) = \frac{1}{N+1} \sum_{k=0}^{N} \prod_{u \in [1,a]}^{*} u^{-ikhk_u} = \frac{1}{N+1} \sum_{k=0}^{N} \exp\Bigg\{-ikh \sum_{u \in [1,a]}^{*} k_u \log u\Bigg\}. \tag{1}$$

If

$$\sum_{u \in [1,a]}^{*} k_u \log u = \frac{2\pi r}{h}, \quad r \in \mathbb{Z}, \tag{2}$$

then

$$F_{Q_{N,a,h}}(\underline{k}) = 1. \tag{3}$$

If $\underline{k} = (k_u : u \in [1,a])$ does not satisfy (2), then using the formula of geometric progression gives

$$F_{Q_{N,a,h}}(\underline{k}) = \frac{1}{N+1} \frac{1 - A^{N+1}(\underline{k}, h)}{1 - A(\underline{k}, h)},$$

where

$$A(\underline{k}, h) = \exp\Bigg\{-ih \sum_{u \in [1,a]}^{*} k_u \log u\Bigg\}.$$

Therefore, by (3),

$$\lim_{N\to\infty} F_{Q_{N,a,h}}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} \text{ satisfies (2)}, \\ 0 & \text{otherwise}. \end{cases} \tag{4}$$

This shows that $Q_{N,a,h} \xrightarrow[N\to\infty]{\mathrm{W}} Q_{a,h}$, where the Fourier transform of $Q_{a,h}$ is the right-hand side of (4). □

We apply Lemma 1 for the proof of a limit theorem for one integral sum. For $x, y \in [1, \infty]$ and fixed $\theta > 1/2$, define

$$v(x,y) = \exp\left\{-\left(\frac{x}{y}\right)^{\theta}\right\},$$

and

$$\Xi_{a,y}(s) = \int_1^a g(x,y)x^{-s}\,\mathrm{d}x,$$

where

$$g(x,y) = \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 v(x,y).$$

$\mathcal{Z}_{n,a,y}(s)$ denotes the integral sum of the function $g(x,y)x^{-s}$ over the interval $[1,a]$, i.e.,

$$\mathcal{Z}_{n,a,y}(s) = \frac{a-1}{n}\sum_{l=1}^{n} g(\xi_l,y)\xi_l^{-s},$$

where $\xi_l \in [x_{l-1}, x_l]$ and $x_l = 1 + ((a-1)/n)l$, $n \in \mathbb{N}$. For $A \in \mathcal{B}(\mathcal{H}(G))$, define

$$P_{N,n,a,y,h}(A) = \frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \mathcal{Z}_{n,a,y}(s+ikh) \in A\}.$$

Lemma 2. *On $(\mathcal{H}(G),\mathcal{B}(\mathcal{H}(G)))$, there exists a probability measure $P_{n,a,y,h}$ such that $P_{N,n,a,y,h} \xrightarrow[N\to\infty]{\mathrm{W}} P_{n,a,y,h}$.*

Proof. We apply the following simple remark on the preservation of weak convergence under continuous mappings. Let $w : \mathcal{X}_1 \to \mathcal{X}_2$ be a $(\mathcal{B}(\mathcal{X}_1),\mathcal{B}(\mathcal{X}_2))$-measurable mapping. Then, every probability measure P on $(\mathcal{X}_1,\mathcal{B}(\mathcal{X}_1))$ induces the unique probability measure Pw^{-1} on $(\mathcal{X}_2,\mathcal{B}(\mathcal{X}_2))$ defined by $Pw^{-1}(A) = P(w^{-1}A)$, $A \in \mathcal{B}(\mathcal{X}_2)$. If the mapping w is continuous, then the weak convergence is preserved. Thus, if $P_n \xrightarrow[n\to\infty]{\mathrm{W}} P$ in the space $\mathcal{X}_1$, then $P_n w^{-1} \xrightarrow[n\to\infty]{\mathrm{W}} Pw^{-1}$ in the space $\mathcal{X}_2$ as well [10].

Define the mapping $w_{n,a,y} : \Omega_a \to \mathcal{H}(G)$ by the formula

$$w_{n,a,y}(\omega) = \frac{a-1}{n}\sum_{l=1}^{n} g(\xi_l,y)\xi_l^{-s}\omega_{\xi_l}.$$

Since the above sum is finite, the mapping $w_{n,a}$ is continuous in the product topology. Moreover,

$$w_{n,a,y}\left(u^{-ikh} : u \in [1,a]\right) = \frac{a-1}{n}\sum_{l=1}^{n} g(\xi_l,y)\xi_l^{-s-ikh} = \mathcal{Z}_{n,a,y}(s+ikh).$$

Hence, $P_{N,n,a,y,h} = Q_{N,a,h}w_{n,a,y}^{-1}$. Therefore, the above remark, continuity of $w_{n,a,y}$ and Lemma 1 show that $P_{N,n,a,y,h} \xrightarrow[N\to\infty]{\mathrm{W}} P_{n,a,y,h} = Q_{a,h}w_{n,a,y}^{-1}$. □

The next step consists of the passage from $\mathcal{Z}_{n,a,y}(s)$ to $\Xi_{a,y}(s)$ in Lemma 2. For this, one statement on convergence in distribution ($\xrightarrow{\mathcal{D}}$) of $\mathcal{H}(G)$-valued random elements is useful, and we recall it. There exists a sequence $\{K_l : l \in \mathbb{N}\} \subset G$ of compact embedded sets such that G is union of sets K_l, and every compact $K \subset G$ lies in some set K_l. Then

$$\rho(g_1,g_2) = \sum_{l=1}^{\infty} 2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|}, \quad g_1,g_2 \in \mathcal{H}(G),$$

is a metric in $\mathcal{H}(G)$ which induces the topology of uniform convergence on compacta.

Lemma 3. *Suppose that X, Y_N and Y_{Nl} are $\mathcal{H}(G)$-valued random elements defined on the same probability space with measure P such that, for $l \in \mathbb{N}$,*

$$X_{Nl} \xrightarrow[N\to\infty]{\mathcal{D}} X_l,$$

and

$$X_l \xrightarrow[l\to\infty]{\mathcal{D}} X.$$

Moreover, let, for every $\varepsilon > 0$,

$$\lim_{l\to\infty} \limsup_{N\to\infty} P\{\rho(X_{Nl}, Y_N) \geqslant \varepsilon\} = 0.$$

Then, $Y_N \xrightarrow[N\to\infty]{\mathcal{D}} X$.

Proof. Since the space $\mathcal{H}(G)$ is separable, the lemma is a particular case of a general theorem on convergence in distribution; see, for example, Theorem 4.2 of [10]. □

An application of Lemma 3 requires the following statement:

Lemma 4. *The equality*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho\big(\mathcal{Z}_{n,a,y}(s+ikh), \Xi_{a,y}(s+ikh)\big) = 0$$

holds for every fixed $h > 0$.

Proof. In view of the definition of the metric ρ, it is suffice to show that, for arbitrary compact set $K \subset G$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} \big|\mathcal{Z}_{n,a,y}(s+ikh) - \Xi_{a,y}(s+ikh)\big| = 0. \tag{5}$$

Let L be a simple closed contour lying in G and enclosing a compact set $K \subset G$. Then, by the integral Cauchy formula,

$$\sup_{s\in K} \big|\mathcal{Z}_{n,a,y}(s+ikh) - \Xi_{a,y}(s+ikh)\big| \ll_L \int_L \big|\mathcal{Z}_{n,a,y}(z+ikh) - \Xi_{a,y}(z+ikh)\big| |\mathrm{d}z|,$$

where $a \ll_\xi b$, $b > 0$, means that there exists a constant $c = c(\xi) > 0$ such that $|a| \leqslant cb$. Hence,

$$\begin{aligned} &\frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} \big|\mathcal{Z}_{n,a,y}(s+ikh) - \Xi_{a,y}(s+ikh)\big| \\ &\quad \ll_L \int_L |\mathrm{d}z| \left(\frac{1}{N+1} \sum_{k=0}^{N} \big|\mathcal{Z}_{n,a,y}(z+ikh) - \Xi_{a,y}(z+ikh)\big| \right). \end{aligned} \tag{6}$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned} &\frac{1}{N+1} \sum_{k=0}^{N} \big|\mathcal{Z}_{n,a,y}(z+ikh) - \Xi_{a,y}(z+ikh)\big| \\ &\quad \leqslant \left(\frac{1}{N+1} \sum_{k=0}^{N} \big|\mathcal{Z}_{n,a,y}(z+ikh) - \Xi_{a,y}(z+ikh)\big|^2 \right)^{1/2}. \end{aligned} \tag{7}$$

Obviously,

$$\begin{aligned}\left|\mathcal{Z}_{n,a,y}(z+ikh)-\Xi_{a,y}(z+ikh)\right|^2 =&\mathcal{Z}_{n,a,y}(z+ikh)\overline{\mathcal{Z}_{n,a,y}(z+ikh)}\\ &-\mathcal{Z}_{n,a,y}(z+ikh)\overline{\Xi_{a,y}(z+ikh)}\\ &-\overline{\mathcal{Z}_{n,a,y}(z+ikh)}\Xi_{a,y}(z+ikh)\\ &+\Xi_{a,y}(z+ikh)\overline{\Xi_{a,y}(z+ikh)},\end{aligned} \tag{8}$$

where $\overline{z}$ denotes the complex conjugate of $z\in\mathbb{C}$. By the definition of $\mathcal{Z}_{n,a,y}(s)$,

$$\begin{aligned}&\mathcal{Z}_{n,a,y}(z+ikh)\overline{\mathcal{Z}_{n,a,y}(z+ikh)}\\ &=\left(\frac{a-1}{n}\right)^2\sum_{\substack{l_1=1\\ \log(\xi_{l_1}/\xi_{l_2})=2\pi r/h}}^{n}\sum_{l_2=1}^{n} g(\xi_{l_1},y)g(\xi_{l_2},y)\xi_{l_1}^{-z}\xi_{l_2}^{-\overline{z}}\\ &\quad+\left(\frac{a-1}{n}\right)^2\sum_{\substack{l_1=1\\ \log(\xi_{l_1}/\xi_{l_2})\neq2\pi r/h}}^{n}\sum_{l_2=1}^{n} g(\xi_{l_1},y)g(\xi_{l_2},y)\xi_{l_1}^{-z}\xi_{l_2}^{-\overline{z}}\left(\frac{\xi_{l_1}}{\xi_{l_2}}\right)^{-ikh},\end{aligned}$$

where $r\in\mathbb{Z}$ is arbitrary. Therefore,

$$\begin{aligned}&\frac{1}{N+1}\sum_{k=0}^{N}\mathcal{Z}_{n,a,y}(z+ikh)\overline{\mathcal{Z}_{n,a,y}(z+ikh)}\\ &=\left(\frac{a-1}{n}\right)^2\sum_{\substack{l_1=1\\ \log(\xi_{l_1}/\xi_{l_2})=2\pi r/h}}^{n}\sum_{l_2=1}^{n} g(\xi_{l_1},y)g(\xi_{l_2},y)\xi_{l_1}^{-z}\xi_{l_2}^{-\overline{z}}\\ &\quad+O\left(\left(\frac{a-1}{n}\right)^2\frac{1}{N}\sum_{\substack{l_1=1\\ \log(\xi_{l_1}/\xi_{l_2})\neq2\pi r/h}}^{n}\sum_{l_2=1}^{n} g(\xi_{l_1},y)g(\xi_{l2},y)\xi_{l_1}^{-\mathrm{Re}z}\xi_{l_2}^{-\mathrm{Re}z}\left|1-\left(\frac{\xi_{l_1}}{\xi_{l_2}}\right)^{-ih}\right|^{-1}\right).\end{aligned}$$

Since

$$\begin{aligned}&\lim_{n\to\infty}\left(\frac{a-1}{n}\right)^2\sum_{\substack{l_1=1\\ \log(\xi_{l_1}/\xi_{l_2})=2\pi r/h}}^{n}\sum_{l_2=1}^{n} g(\xi_{l_1},y)g(\xi_{l_2},y)\xi_{l_1}^{-z}\xi_{l_2}^{-\overline{z}}\\ &\qquad=\int_1^a\int_{\substack{1\\ \log(x_1/x_2)=2\pi r/h}}^{a} g(x_1,y)g(x_2,y)x_1^{-z}x_2^{-\overline{z}}\,\mathrm{d}x_1\,\mathrm{d}x_2=0,\end{aligned}$$

from this we obtain that, for all $z\in L$,

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\mathcal{Z}_{n,a,y}(z+ikh)\overline{\mathcal{Z}_{n,a,y}(z+ikh)}=0. \tag{9}$$

By the definition of $\Xi_{a,y}(s)$, for all $z \in L$, we have

$$\begin{aligned}
&\frac{1}{N+1}\sum_{k=0}^{N}\Xi_{a,y}(z+ikh)\overline{\Xi_{a,y}(z+ikh)} \\
&= \frac{1}{N+1}\sum_{k=0}^{N}\int_1^a\int_1^a g(x_1,y)g(x_2,y)x_1^{-z-ikh}x_2^{-\overline{z}+ikh}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \frac{1}{N+1}\sum_{k=0}^{N}\left(\underset{\log(x_1/x_2)=2\pi r/h}{\int_1^a\int_1^a} + \underset{\log(x_1/x_2)\neq 2\pi r/h}{\int_1^a\int_1^a}\right)g(x_1,y)g(x_2,y)x_1^{-z-ikh}x_2^{-\overline{z}+ikh}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \frac{1}{N+1}\underset{\log(x_1/x_2)\neq 2\pi r/h}{\int_1^a\int_1^a} g(x_1,y)g(x_2,y)x_1^{-z}x_2^{-\overline{z}}\left(1-\left(\frac{x_1}{x_2}\right)^{-i(N+1)h}\right) \\
&\quad\times\left(1-\left(\frac{x_1}{x_2}\right)^{-ih}\right)^{-1}\frac{1}{i}\,\mathrm{d}x_1\,\mathrm{d}x_2,
\end{aligned}$$

where $r \in \mathbb{Z}$. Therefore,

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\Xi_{a,y}(z+ikh)\overline{\Xi_{a,y}(z+ikh)} = 0. \tag{10}$$

Since the sum of the last two terms in (7) is estimated as

$$\ll \left(\sum_{k=0}^{N}\left|\mathcal{Z}_{n,a,y}(z+ikh)\right|^2\sum_{k=0}^{N}\left|\Xi_{a,y}(z+ikh)\right|^2\right)^{1/2},$$

equality (5) follows from (6)–(10). □

For $A \in \mathcal{B}(\mathcal{H}(G))$, define

$$P_{N,a,y,h}(A) = \frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \Xi_{a,y}(s+ikh) \in A\}.$$

Lemma 5. *For every fixed $h > 0$, on $(\mathcal{H}(G), \mathcal{B}(\mathcal{H}(G)))$, there exists a probability measure $P_{a,y,h}$ such that $P_{N,a,y,h} \xrightarrow[N\to\infty]{\mathrm{W}} P_{a,y,h}$.*

Proof. Let $\theta_{N,h}$ be a random variable defined on a certain probability space with measure P, and having the distribution

$$P\{\theta_{N,h} = kh\} = \frac{1}{N+1}, \quad k = 0, 1, \ldots, N.$$

$X_{n,a,y,h}$ denotes the $\mathcal{H}(G)$-valued random element with the distribution $P_{n,a,y,h}$, where $P_{n,a,y,h}$ is the measure from Lemma 2, and define the $\mathcal{H}(G)$-valued random element

$$X_{N,n,a,y,h} = X_{N,n,a,y,h}(s) = \mathcal{Z}_{n,a,y}(s+i\theta_{N,h}).$$

Then, in view of Lemma 2, we have

$$X_{N,n,a,y,h} \xrightarrow[N\to\infty]{\mathcal{D}} X_{n,a,y,h}. \tag{11}$$

Consider the sequence $\{P_{n,a,y,h} : n \in \mathbb{N}\}$. Let K_l be the sets from the definition of the metric ρ. Then, applying the integral Cauchy formula and (9), we find that

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K_l} \left| \mathcal{Z}_{n,a,y}(s+ikh) \right| \leqslant C_{l,a,y,h} < \infty.$$

Fix $\varepsilon > 0$ and define $V_l = V_{l,a,y,h} = 2^l \varepsilon^{-1} C_{l,a,y,h}$. Then, using (11),

$$P\left\{ \sup_{s\in K_l} \left| X_{n,a,y,h}(s) \right| \geqslant V_l \right\} = \limsup_{N\to\infty} P\left\{ \sup_{s\in K_l} \left| X_{N,n,a,y,h}(s) \right| \geqslant V_l \right\}$$
$$\leqslant \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{V_l(N+1)} \sum_{k=0}^{N} \sup_{s\in K_l} \left| \mathcal{Z}_{n,a,y}(s+ikh) \right| \leqslant \frac{\varepsilon}{2^l}$$

for all $n, l \in \mathbb{N}$. Hence, taking

$$K = K(\varepsilon) = \left\{ g \in \mathcal{H}(G) : \sup_{s\in K_l} |g(s)| \leqslant V_l,\ l \in \mathbb{N} \right\},$$

we have

$$P\left\{ X_{n,a,y,h} \in K \right\} = 1 - P\left\{ X_{n,a,y,h} \notin K \right\} > 1 - \varepsilon \sum_{l=1}^{\infty} 2^{-l} = 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Since the set K is compact in the space $\mathcal{H}(G)$, this shows that the sequence $\{P_{n,a,y,h}\}$ is tight. Therefore, by the Prokhorov theorem; see, for example, [10], the sequence $\{P_{n,a,y,h}\}$ is relatively compact. Thus, there exists a subsequence $\{P_{n_r,a,y,h}\}$ weakly convergent to a certain probability measure $P_{a,y,h}$ on $(\mathcal{H}(G), \mathcal{B}(\mathcal{H}(G)))$ as $r \to \infty$. In other words,

$$X_{n_r,a,y,h} \xrightarrow[r\to\infty]{\mathcal{D}} P_{a,y,h}. \tag{12}$$

Define one more $\mathcal{H}(G)$-valued random element

$$Y_{N,a,y,h} = Y_{N,a,y,h}(s) = \Xi_{a,y}(s + i\theta_{N,h}).$$

Then, Lemma 4 implies that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} P\left\{ \rho\left(Y_{N,a,y,h}, X_{n_r,a,y,h} \right) \geqslant \varepsilon \right\}$$
$$\leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{k=0}^{N} \rho\big(\mathcal{Z}_{n,a,y}(s+ikh), \Xi_{a,y}(s+ikh) \big) = 0. \tag{13}$$

Now, in view of (11)–(13), we may apply Lemma 3 for the random elements $Y_{N,a,y,h}$, $X_{N,n_r,a,y,h}$ and $X_{n_r,a,y,h}$. Then, we have the relation

$$Y_{N,a,y,h} \xrightarrow[N\to\infty]{\mathcal{D}} P_{a,y,h},$$

i.e., $P_{N,a,y,h} \xrightarrow[N\to\infty]{W} P_{a,y,h}$. □

Now, we are ready to prove a discrete limit lemma for the function

$$\Xi_y(s) = \int_1^\infty g(x,y) x^{-s}\, dx.$$

Since $\zeta(1/2 + it) \ll t^{1/6}$, $t \geqslant 1$, and $v(x,y)$ decreases exponentially, the integral for $\Xi_y(s)$ is absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0.

For $A \in \mathcal{B}(\mathcal{H}(G))$, define

$$P_{N,y,h}(A) = \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \Xi_y(s+ikh) \in A\right\}.$$

Lemma 6. *For every fixed $h > 0$, on $(\mathcal{H}(G), \mathcal{B}(\mathcal{H}(G)))$, there exists a probability measure $P_{y,h}$ such that $P_{N,y,h} \xrightarrow[N\to\infty]{\mathrm{W}} P_{y,h}$.*

Proof. Let $\theta_{N,h}$ be the same as in the proof of Lemma 5. Define

$$Y_{N,y,h} = Y_{N,y,h}(s) = \Xi_y(s + i\theta_{N,h}),$$

and $X_{a,y,h}$ denotes the $\mathcal{H}(G)$-valued random element with distribution $P_{a,y,h}$. Then, by Lemma 5,

$$Y_{N,a,y,h} \xrightarrow[N\to\infty]{\mathcal{D}} X_{a,y,h}. \tag{14}$$

The integral Cauchy formula and (10) lead to

$$\sup_{a\geqslant 1}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K_l}\left|\Xi_{a,y}(s+ikh)\right| \leqslant C_{l,y,h} < \infty.$$

Therefore, taking $V_l = V_{l,y,h} = 2^l\varepsilon^{-1}C_{l,y,h}$, we find by (14) that

$$P\left\{\sup_{s\in K_l}\left|X_{a,y,h}(s)\right| \geqslant V_l\right\} < \sup_{a\geqslant 1}\frac{1}{V_l(N+1)}\sum_{k=0}^{N}\sup_{s\in K_l}\left|\Xi_{a,y}(s+ikh)\right| \leqslant \frac{\varepsilon}{2^l}$$

for all $a \geqslant 1$ and $l \in \mathbb{N}$. This shows that, for $a \geqslant 1$,

$$P\left\{X_{a,y,h} \in K\right\} \geqslant 1 - \varepsilon,$$

where

$$K = \left\{g \in \mathcal{H}(G) : \sup_{s\in K_l}|g(s)| \leqslant V_{l,y,h},\ l \in \mathbb{N}\right\}.$$

This means that the family of probability measures $\{P_{a,y,h} : a \geqslant 1\}$ is tight. Hence, there exists a sequence $\{P_{a_r,y,h}\} \subset \{P_{a,y,h}\}$ weakly convergent to a certain probability measure $P_{y,h}$ as $r \to \infty$. Thus,

$$X_{a_r,y,h} \xrightarrow[r\to\infty]{\mathcal{D}} P_{y,h}. \tag{15}$$

It remains to show the nearestness in the mean of $\Xi_{a,y}(s)$ and $\Xi_y(s)$. We have that, for a compact set $K \subset D$ and fixed $y > 0$, $h > 0$,

$$\Xi_y(s+ikh) - \Xi_{a,y}(s+ikh) = \int_a^\infty g(x,y)x^{-s-ikh}\,\mathrm{d}x \ll_y \int_a^\infty g(x,y)x^{-1/2}\,\mathrm{d}x = o_y(1)$$

as $a \to \infty$. From this, we have

$$\lim_{a\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}\left|\Xi_y(s+ikh) - \Xi_{a,y}(s+ikh)\right| = 0,$$

and

$$\lim_{a\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\rho\left(\Xi_y(s+ikh), \Xi_{a,y}(s+ikh)\right) = 0.$$

The latter equality, relations (14) and (15) together with Lemma 3 prove the lemma. □

To obtain a limit theorem for the function $\Xi(s)$, we use the integral representation for the function $\Xi_y(s)$. Define

$$a_y(s) = \frac{s}{\theta}\Gamma\Big(\frac{s}{\theta}\Big)y^s,$$

where $\Gamma(s)$ is the Euler gamma-function, and θ is from the definition of $v(x,y)$.

Lemma 7. *For $s \in D$, the integral representation*

$$\Xi_y(s) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \Xi(s+z)a_y(z)\frac{\mathrm{d}z}{z}$$

is valid.

Proof. The lemma is Lemma 9 proved in [7]. □

In addition, we need a discrete mean square estimate for $\Xi(s)$.

Lemma 8. *Suppose that σ, $1/2 < \sigma < 1$, and $h > 0$ are fixed, and $\tau \in \mathbb{R}$. Then, for every $\varepsilon_1 > 0$,*

$$\sum_{k=0}^{N} |\Xi(\sigma + ikh + i\tau)|^2 \ll_{\sigma,h,\varepsilon_1} (N(1+|\tau|))^{2-2\sigma+\varepsilon_1}.$$

Proof. It is well known [4] that, for fixed $1/2 < \sigma < 1$, and any $\varepsilon_1 > 0$,

$$\int_0^T |\Xi(\sigma+it)|^2\,\mathrm{d}t \ll_{\sigma,\varepsilon_1} T^{2-2\sigma+\varepsilon_1}.$$

From this, we find

$$\int_0^T |\Xi(\sigma+it+i\tau)|^2\,\mathrm{d}t = \int_\tau^{T+\tau} |\Xi(\sigma+it)|^2\,\mathrm{d}t \leqslant 2\int_0^{T+|\tau|} |\Xi(\sigma+it)|^2\,\mathrm{d}t$$
$$\ll_{\sigma,\varepsilon_1} (T+|\tau|)^{2-2\sigma+\varepsilon_1}. \tag{16}$$

The latter estimate together with integral Cauchy formula gives

$$\int_0^T \left|\Xi'(\sigma+it+i\tau)\right|^2\mathrm{d}t \ll_{\sigma,\varepsilon_1} (T+|\tau|)^{2-2\sigma+\varepsilon_1}. \tag{17}$$

Now, we apply the Gallagher lemma; see, for example, Lemma 1.4 of [11], connecting continuous and discrete mean squares of certain functions. Thus, by (16) and (17),

$$\sum_{k=2}^{N} |\Xi(\sigma+ikh+i\tau)|^2 \ll_h \int_0^{Nh} |\Xi(\sigma+it+i\tau)|^2\,\mathrm{d}t$$
$$+\left(\int_0^{Nh} |\Xi(\sigma+it+i\tau)|^2\,\mathrm{d}t \int_0^{Nh} \left|\Xi'(\sigma+it+i\tau)\right|^2\mathrm{d}t\right)^{1/2} \ll_{\sigma,h,\varepsilon_1} (N(1+|\tau|))^{2-2\sigma+\varepsilon_1}. \tag{18}$$

Since [4]

$$\Xi(\sigma+it) \ll_{\varepsilon_1} |t|^{1-\sigma+\varepsilon_1},$$

for $0 \leqslant \sigma \leqslant 1$, $|t| \geqslant t_0$, and $\varepsilon_1 > 0$,

$$\sum_{k=0}^{1} |\Xi(\sigma+ikh+i\tau)|^2 \ll_{\sigma,h,\varepsilon_1} (1+|\tau|)^{2-2\sigma+\varepsilon_1}.$$

Therefore, in view of (18),

$$\sum_{k=0}^{N} |\Xi(\sigma+ikh+i\tau)|^2 \ll_{\sigma,h,\varepsilon_1} (N(1+|\tau|))^{2-2\sigma+\varepsilon_1}. \tag{19}$$

□

The next lemma gives an approximation of $\Xi(s)$ by $\Xi_y(s)$.

Lemma 9. *The equality*

$$\lim_{y\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\rho\big(\Xi(s+ikh),\Xi_y(s+ikh)\big)=0$$

holds for all $h>0$.

Proof. It is suffice to show that, for compact sets $K\subset G$,

$$\lim_{y\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}\left|\Xi(s+ikh)-\Xi_y(s+ikh)\right|=0.$$

Let $K\subset G$ be an arbitrary fixed compact set. Fix $\varepsilon>0$ such that, for all $s=\sigma+it\in K$, the inequalities $1/2+2\varepsilon\leqslant\sigma\leqslant 1-\varepsilon$ would be satisfied. Then, for such σ,

$$\theta_1\stackrel{def}{=}\sigma-\varepsilon-\frac{1}{2}>0.$$

Let $\theta=1/2+\varepsilon$ in Lemma 7. The point $z=1-s$ is a double pole, and $z=0$ is a simple pole of the function

$$\Xi(s+z)\widehat{a}(z),\qquad \widehat{a}(z)=\frac{a(z)}{z};$$

therefore, Lemma 7 and the residue theorem give

$$\Xi_y(s)-\Xi(s)=\frac{1}{2\pi i}\int_{-\theta_2-i\infty}^{-\theta_2+i\infty}\Xi(s+z)\widehat{a}_y(z)\,\mathrm{d}z+r_y(s) \tag{20}$$

where

$$r_y(s)=\operatorname*{Res}_{z=1-s}\Xi(s+z)\widehat{a}(z).$$

It is known [4] that, for $\sigma>-3/4$,

$$\Xi(s)=\frac{1}{(s-1)^2}+\frac{a_1}{s-1}+E(1)\pi(s-1)+s(s+1)(s+2)\int_1^\infty G_1(x)x^{-s-3}\,\mathrm{d}x,$$

where $a_1=2\gamma_0-\log 2\pi$, γ_0 is the Euler constant, $E(T)$ is defined by

$$\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t=T\log\frac{T}{2\pi}+(2\gamma_0-1)T+E(T),$$

$$G_1(T)=\int_1^T G(T)\,\mathrm{d}t,\qquad G(T)=\int_1^T E(T)\,\mathrm{d}t-\pi T.$$

Therefore,

$$r_y(s)=(\widehat{a}(z))'\Big|_{z=1-s}+a_1\widehat{a}(1-s). \tag{21}$$

Equality (20), for all $s \in K$ and $h > 0$, gives

$$\begin{aligned}
&\Xi_y(s+ikh) - \Xi(s+ikh) \\
&\quad = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Xi\left(\sigma + it - \sigma + \frac{1}{2} + \varepsilon + ikh + i\tau\right) \widehat{a}\left(\frac{1}{2} + \varepsilon - \sigma + i\tau\right) \mathrm{d}\tau + r_y(s+ikh) \\
&\quad = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Xi\left(\frac{1}{2} + \varepsilon + ikh + i\tau\right) \widehat{a}\left(\frac{1}{2} + \varepsilon - s + i\tau\right)) \mathrm{d}\tau + r_y(s+ikh) \\
&\quad \ll \int_{-\infty}^{\infty} \left|\Xi\left(\frac{1}{2} + \varepsilon + ikh + i\tau\right)\right| \sup_{s \in K} \left|\widehat{a}\left(\frac{1}{2} + \varepsilon - s + i\tau\right)\right| \mathrm{d}\tau + \sup_{s \in K} |r_y(s+ikh)|
\end{aligned}$$

after writing τ in place of $t + \tau$. Hence,

$$\begin{aligned}
&\frac{1}{N+1} \sum_{k=0}^{N} \sup_{s \in K} |\Xi(s+ikh) - \Xi_y(s+ikh)| \\
&\qquad \ll \int_{-\infty}^{\infty} \left(\frac{1}{N+1} \sum_{k=0}^{N} \Xi\left(\frac{1}{2} + \varepsilon + ikh + i\tau\right)\right) \sup_{s \in K} \left|\widehat{a}\left(\frac{1}{2} + \varepsilon - s + i\tau\right)\right| \mathrm{d}\tau \\
&\qquad + \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s \in K} |r_y(s+ikh)| \overset{def}{=} I_1 + I_2.
\end{aligned} \tag{22}$$

The classical estimate

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0, \tag{23}$$

which is uniform in any fixed strip $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ is well-known. Thus, for $s = \sigma + it \in K$,the definition of $\widehat{a}_y(s)$ implies

$$\begin{aligned}
\widehat{a}\left(\frac{1}{2} + \varepsilon - s + i\tau\right) &\ll_\theta y^{1/2+\varepsilon-\sigma} \left|\Gamma\left(\frac{1}{\theta}\left(\frac{1}{2} + \varepsilon - \sigma - it + i\tau\right)\right)\right| \ll_\theta y^{-\varepsilon} \exp\left\{-\frac{c}{\theta}|t - \tau|\right\} \\
&\ll_\theta y^{-\varepsilon} \exp\left\{\frac{c}{\theta}|t|\right\} \exp\left\{-\frac{c}{\theta}|\tau|\right\} \ll_{\theta,K} y^{-\varepsilon} \exp\{-c_1|\tau|\}, \quad c_1 > 0.
\end{aligned}$$

Therefore, using Lemma 8, we obtain with $\varepsilon_1 = 2\varepsilon$

$$I_1 \ll_{\theta,K,h,\varepsilon,\varepsilon_1} y^{-\varepsilon} N^{-\varepsilon+\varepsilon_1/2} \int_{-\infty}^{\infty} \exp\{-c_1|\tau|\} (1 + |\tau|)^{(2-2\sigma+\varepsilon_1)/2} \mathrm{d}\tau \ll_{\theta,K,h,\varepsilon} y^{-\varepsilon}. \tag{24}$$

To estimate I_2, first we evaluate $r_y(s)$. By (21),

$$r_y(s) = \frac{y^{1-s}}{\theta} \Gamma\left(\frac{1-s}{\theta}\right) \left(\frac{1}{\theta} \frac{\Gamma'((1-s)/\theta)}{\Gamma((1-s)/\theta)} + \log y + a_1\right).$$

Hence, in virtue of (23) and the estimate $\Gamma'(s)/\Gamma(s) \ll \log|s|$,

$$\begin{aligned}
r_y(s) &\ll_\theta y^{1-\sigma} \left|\Gamma\left(\frac{1-\sigma}{\theta} + i\frac{t+kh}{\theta}\right)\right| \left(\frac{1}{\theta} \left|\frac{\Gamma'((1-\sigma)/\theta - i(t+kh)/\theta)}{\Gamma((1-\sigma)/\theta - i(t+kh)/\theta)}\right| + \log y + 1\right) \\
&\ll_\theta y^{1-\sigma} \exp\left\{-\frac{c}{\theta}|t+kh|\right\} \left(\log\left|\frac{t+ikh}{\theta}\right| + \log y + 1\right) \\
&\ll_{\theta,K,\varepsilon} y^{1/2-\varepsilon} \exp\{-c_2 kh\}, \quad c_2 > 0.
\end{aligned}$$

This shows that

$$I_2 \ll_{\theta,K,\varepsilon} \frac{y^{1/2-\varepsilon}}{N} \sum_{k=0}^{N} \exp\{-c_2 kh\} \ll_{\theta,K,\varepsilon,h} y^{1/2-\varepsilon} N^{-1} \log N.$$

Therefore, in view of (22) and (24),

$$\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}\left|\Xi(s+ikh)-\Xi_y(s+ikh)\right| \ll_{\theta,K,\varepsilon,h} y^{-\varepsilon}+y^{1/2-\varepsilon}N^{-1}\log N.$$

From this, we find that

$$\lim_{y\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}\left|\Xi(s+ikh)-\Xi_y(s+ikh)\right|=0,$$

and the lemma is proved. □

Recall that $P_{y,h}$ is the limit measure in Lemma 6.

Lemma 10. *The family of probability measures $\{P_{y,h} : y \geqslant 1\}$ is tight.*

Proof. Let $K_l \subset G$ be a arbitrary compact set from the definition of metric in $\mathcal{H}(G)$. Then, for every fixed $h > 0$,

$$\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K_l}\left|\Xi_y(s+ikh)\right| \leqslant \frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K_l}\left|\Xi(s+ikh)-\Xi_y(s+ikh)\right| + \frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K_l}|\Xi(s+ikh)|. \tag{25}$$

Estimate (19), for fixed $1/2 < \sigma < 1$, gives

$$\frac{1}{N+1}\sum_{k=0}^{N}|\Xi(s+ikh)|^2 \ll_{\sigma,\varepsilon_1,h} N^{1-2\sigma+\varepsilon_1}.$$

This and the integral Cauchy formula lead to

$$\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K_l}|\Xi(s+ikh)| \leqslant C_{l,h} < \infty.$$

Therefore, by (25) and the proof of Lemma 9,

$$\sup_{y\geqslant 1}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K_l}\left|\Xi_y(s+ikh)\right| \leqslant C_{l,h} < \infty.$$

Fix $\varepsilon > 0$ and take $V_l = V_{l,h} = 2^l\varepsilon^{-1}C_{l,h}$. Moreover, let $Y_{y,h}$ be the $\mathcal{H}(G)$-valued random element having the distribution $P_{y,h}$. Then, by Lemma 6,

$$P\left\{\sup_{s\in K_l}\left|Y_{y,h}(s)\right| \geqslant V_l\right\} = \limsup_{N\to\infty} P\left\{\sup_{s\in K_l}\left|Y_{N,y,h}(s)\right| \geqslant V_l\right\} < \sup_{y\geqslant 1}\limsup_{N\to\infty}\frac{1}{(N+1)V_l}\sum_{k=0}^{N}\sup_{s\in K_l}|\Xi(s+ikh)| \leqslant \frac{\varepsilon}{2^l}.$$

Hence, for all $y \geqslant 1$,

$$P\left\{Y_{y,h} \in K_l\right\} \geqslant 1-\varepsilon,$$

where

$$K = \left\{g \in \mathcal{H}(G) : \sup_{s\in K_l}|g(s)| \leqslant V_l,\ l \in \mathbb{N}\right\},$$

and the lemma is proved. □

3. Proofs of Theorems

Proof of Theorem 3. Lemma 10 and Prokhorov's theorem imply the relative compactness of the family $\{P_{y,h} : y \geqslant 1\}$. Thus, there exists a sequence $\{P_{y_r,h}\} \subset \{P_{y,h}\}$, such that $P_{y_r,h} \xrightarrow[r\to\infty]{\mathrm{W}} P_h$, where P_h is a certain probability measure on $(\mathcal{H}(G), \mathcal{B}(\mathcal{H}(G)))$. Thus, in the above notation,

$$Y_{y_r,h} \xrightarrow[r\to\infty]{\mathcal{D}} P_h. \tag{26}$$

Define the $\mathcal{H}(G)$-valued random element

$$\Xi_{N,h} = \Xi_{N,h}(s) = \Xi(s + i\theta_{N,h}).$$

Then, for every $\varepsilon > 0$ and $y \geqslant 1$,

$$0 \leqslant \limsup_{N\to\infty} P\Big\{\rho\Big(\Xi_{N,h}, \Xi_{N,y,h}\Big) \geqslant \varepsilon\Big\} \leqslant \limsup_{N\to\infty} \frac{1}{(N+1)\varepsilon} \sum_{k=0}^{N} \rho\big(\Xi(s+ikh), \Xi_y(s+ikh)\big).$$

Thus, Lemma 9 shows that

$$\lim_{y\to\infty} \limsup_{N\to\infty} P\Big\{\rho\Big(\Xi_{N,h}, \Xi_{N,y,h}\Big) \geqslant \varepsilon\Big\} = 0.$$

This equality, (26) and Lemmas 6 and 3 prove that

$$\Xi_{N,h} \xrightarrow[N\to\infty]{\mathcal{D}} P_h.$$

The theorem is proved. □

Proof of Theorem 2. Let F_h denote the support of the limit measure P_h in Theorem 3, i.e., F_h is the minimal closed subset of the space $\mathcal{H}(G)$ such that $P_h(F_h) = 1$. For every element $f \in F_h$ and every open neighbourhood D of f, we have $P_h(D) > 0$. Clearly, $F_h \neq \varnothing$.

For $f \in F_h$, let

$$G_\varepsilon = \left\{ g \in \mathcal{H}(G) : \sup_{s\in K} |g(s) - f(s) < \varepsilon \right\}.$$

Then, by the above mentioned property of the support,

$$P_h(G_\varepsilon) > 0. \tag{27}$$

Therefore, Theorem 3 and the equivalent of weak convergence in terms of open sets; see, for example, Theorem 2.1 of [10], give

$$\liminf_{N\to\infty} P_{N,h}(G_\varepsilon) \geqslant P_h(G_\varepsilon) > 0.$$

This, the definitions of $P_{N,h}$ and G_ε prove the first inequality of theorem.

Since the boundary ∂G_ε of the set G_ε lies in the set

$$\left\{ g \in \mathcal{H}(G) : \sup_{s\in K} |g(s) - f(s) = \varepsilon \right\},$$

we have $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \varnothing$ for different positive ε_1 and ε_2. Thus, $P_h(\partial G_\varepsilon) > 0$ for all but at most countably many $\varepsilon > 0$, i.e., G_ε is a continuity set of the measure P_h for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 3 and the equivalent of weak convergence in terms of continuity sets [10] and (27) show that

$$\lim_{N \to \infty} P_{N,h}(G_\varepsilon) = P_h(G_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$, and the definitions of $P_{N,h}$ and G_ε prove the second inequality of the theorem. □

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