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Article

On the Mishou Theorem for Zeta-Functions with Periodic Coefficients

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Abstract: Let $\mathfrak{a} = \{a_m\}$ and $\mathfrak{b} = \{b_m\}$ be two periodic sequences of complex numbers, and, additionally, $\mathfrak{a}$ is multiplicative. In this paper, the joint approximation of a pair of analytic functions by shifts $(\zeta_{n_T}(s+i\tau;\mathfrak{a}), \zeta_{n_T}(s+i\tau,\alpha;\mathfrak{b}))$ of absolutely convergent Dirichlet series $\zeta_{n_T}(s;\mathfrak{a})$ and $\zeta_{n_T}(s,\alpha;\mathfrak{b})$ involving the sequences $\mathfrak{a}$ and $\mathfrak{b}$ is considered. Here, $n_T \to \infty$ and $n_T \ll T^2$ as $T \to \infty$. The coefficients of these series tend to a_m and b_m, respectively. It is proved that the set of the above shifts in the interval $[0, T]$ has a positive density. This generalizes and extends the Mishou joint universality theorem for the Riemann and Hurwitz zeta-functions.

Keywords: Hurwitz zeta-function; joint universality; periodic Hurwitz zeta-function; periodic zeta-function; universality

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1. Introduction

Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ and $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be two periodic sequences of complex numbers with minimal periods $q_1 \in \mathbb{N}$ and $q_2 \in \mathbb{N}_0$, respectively, $0 < \alpha \leqslant 1$ a fixed parameter, and $s = \sigma + it$ a complex variable. The periodic $\zeta(s;\mathfrak{a})$ and periodic Hurwitz $\zeta(s,\alpha;\mathfrak{b})$ zeta-functions are defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \qquad \text{and} \qquad \zeta(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}.$$

When $a_m \equiv 1$ and $b_m \equiv 1$, the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ reduce to the classical Riemann zeta-function $\zeta(s)$ and Hurwitz zeta-function $\zeta(s,\alpha)$, respectively. In view of the periodicity of the sequences $\mathfrak{a}$ and $\mathfrak{b}$, for $\sigma > 1$, it follows that

$$\zeta(s;\mathfrak{a}) = \frac{1}{q_1^s} \sum_{l=1}^{q_1} a_l \zeta\left(s, \frac{l}{q_1}\right) \qquad \text{and} \qquad \zeta(s,\alpha;\mathfrak{b}) = \frac{1}{q_2^s} \sum_{l=0}^{q_2-1} b_l \zeta\left(s, \frac{l+\alpha}{q_2}\right).$$

Thus, the properties of the function $\zeta(s,\alpha)$ imply the analytic continuation for the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ to the whole complex plane, except the point $s = 1$ which is a simple pole with residues

$$a \stackrel{\text{def}}{=} \frac{1}{q_1} \sum_{l=1}^{q_1} a_l \qquad \text{and} \qquad b \stackrel{\text{def}}{=} \frac{1}{q_2} \sum_{l=0}^{q_2-1} b_l,$$

respectively. If $a = 0$, then the function $\zeta(s;\mathfrak{a})$ is entire, and if $b = 0$, then the function $\zeta(s,\alpha;\mathfrak{b})$ is entire.

Examples of the function $\zeta(s;\mathfrak{a})$ are Dirichlet L-functions

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s},\quad \sigma>1,$$

and of the function $\zeta(s,\alpha;\mathfrak{b})$ they are Lerch zeta-functions

$$L(\lambda,\alpha,s)=\sum_{m=0}^{\infty}\frac{\mathrm{e}^{2\pi i\lambda m}}{(m+\alpha)^s},\quad \sigma>1,$$

with rational parameter λ.

The analytic properties of the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$, including the universality property of the approximation of the analytic functions by shifts $\zeta(s+i\tau;\mathfrak{a})$ and $\zeta(s+i\tau,\alpha;\mathfrak{b})$, $\tau\in\mathbb{R}$, are closely connected to the sequence $\mathfrak{a}$, the sequence $\mathfrak{b}$, and parameter α, respectively.

Let $\Delta=\{s\in\mathbb{C}:1/2<\sigma<1\}$. Denote by $\mathfrak{K}$ the class of compact sets of the strip Δ with connected complements, by $\mathcal{H}(K)$ with $K\in\mathfrak{K}$ the class of continuous functions on K that are analytic in the interior of K, and by $\mathcal{H}_0(K)$ the subclass of $\mathcal{H}(K)$ of non-vanishing on K functions. Let $\mathfrak{M}A$ stand for the Lebesgue measure of a measurable set $A\subset\mathbb{R}$.

The first universality results for the function $\zeta(s;\mathfrak{a})$ were obtained by J. Steuding. In [1], he proved that if $\mathfrak{a}$ is not a multiple of the Dirichlet character modulo q_1, and $a_m=0$ for $(m,q_1)>1$, then for $K\in\mathfrak{K}$, $f(s)\in\mathcal{H}(K)$ and all $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\mathfrak{M}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a})-f(s)|<\varepsilon\right\}>0. \tag{1}$$

Under the above conditions on the sequence $\mathfrak{a}$, this sequence is not multiplicative. We recall that the sequence $\mathfrak{a}$ is multiplicative, if $a_1=1$ and $a_{mn}=a_m a_n$ for all $m,n\in\mathbb{N}$, $(m,n)=1$. The universality of the function $\zeta(s;\mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$ was proved in [2]. In [3], it was obtained that there exists a constant $c_0=c_0(\mathfrak{a})$ such that, for $K\in\mathfrak{K}$, $\max_{s\in K}\operatorname{Im}s-\min_{s\in K}\operatorname{Im}s\leqslant c_0$, $f(s)\in H_0(K)$ and $\varepsilon>0$, equality (1) holds.

The universality properties of the function $\zeta(s,\alpha)$ are included in the following theorem [4–6]. *Suppose that α is transcendental or rational, not equal to* 1 *or* 1/2. *Let $K\in\mathfrak{K}$ and $f(s)\in\mathcal{H}(K)$. Then, for all $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\mathfrak{M}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon\right\}>0.$$

The universality of $\zeta(s,\alpha)$ with algebraic irrational α remains an open problem up to our days. A certain approximation to this problem is given in [7], and see also [8]. The best result in this direction was obtained in [9]. The universality property of the function $\zeta(s,\alpha;\mathfrak{b})$ was first studied in [10], and similar theorems to those for $\zeta(s,\alpha)$ with transcendental and algebraic irrational α were obtained in [11] and [12]. The case of rational α is studied in [13]. In this case, some hypotheses for the sequence $\mathfrak{b}$ are also involved.

The aim of this paper is the joint universality of certain Dirichlet series connected to the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$. Recall that the first joint universality theorem for the functions $\zeta(s)$ and $\zeta(s,\alpha)$ with transcendental α was obtained by H. Mishou in [14].

Suppose that $K_1, K_2 \in \mathfrak{K}$ and $f_1(s) \in \mathcal{H}_0(K_1)$, $f_2(s) \in \mathcal{H}(K_2)$. Then, he proved that, for all $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathfrak{M}\Bigg\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon, \\ \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon \Bigg\} > 0.$$

A similar result for the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ was given in [15]. The approximation problem of a pair of analytic functions by shifts $(\zeta(s+i\tau;\mathfrak{a}), \zeta(s+i\tau,\alpha;\mathfrak{b}))$ with algebraic irrational α was considered in [16]. More general joint universality results for periodic and periodic Hurwitz zeta-functions can be found in [17–20]. A weighted generalization of the Mishou theorem was obtained in [21].

The abovementioned universality results are of a continuous type because τ in shifts takes arbitrary real values. Moreover, there are results of a discrete type when τ takes values in a certain discrete set, see, for example, [22–30].

Let $\theta > 1/2$ be a fixed number, $u > 0$, and

$$v_u(m;\theta) = \exp\left\{-\left(\frac{m}{u}\right)^\theta\right\}, \qquad v_u(m,\alpha;\theta) = \exp\left\{-\left(\frac{m+\alpha}{u}\right)^\theta\right\}.$$

Define the series

$$\zeta_u(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_u(m;\theta)}{m^s}, \qquad \zeta_u(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m v_u(m,\alpha;\theta)}{(m+\alpha)^s}.$$

Then, the latter series are absolutely convergent for $\sigma > 1/2$. Really, in view of the exponential decreasing of $v_u(m;\theta)$ and $v_u(m,\alpha;\theta)$, these series are absolutely convergent for $\sigma > \sigma_0$ for all finite σ_0. We will consider the approximation of pairs of analytic functions by shifts $(\zeta_{n_T}(s+i\tau;\mathfrak{a}), \zeta_{n_T}(s+i\tau,\alpha;\mathfrak{b}))$, where $n_T \to \infty$ as $T \to \infty$. For the statement of a theorem, we need some definitions. Denote by η the unit circle on the complex plane, and by $\mathcal{B}(\mathcal{X})$ the Borel σ-field of the space $\mathcal{X}$. Define two tori

$$\Omega_1 = \prod_{p\in\mathbb{P}} \eta_p \qquad \text{and} \qquad \Omega_2 = \prod_{m\in\mathbb{N}_0} \eta_m,$$

where $\eta_p = \eta$ for all $p \in \mathbb{P}$ ($\mathbb{P}$ is the set of all prime numbers), and $\eta_m = \eta$ for all $m \in \mathbb{N}_0$. With the product topology and pointwise multiplication, the tori Ω_1 and Ω_2 are compact topological Abelian groups. Therefore, by the Tikhonov theorem [31],

$$\Omega = \Omega_1 \times \Omega_2$$

also is a compact topological group. Thus, on $(\Omega, \mathcal{B}(\Omega))$, we can define the probability Haar measure μ_H, and we have the probability space $(\Omega, \mathcal{B}(\Omega), \mu_H)$. Denote by $\omega(p)$ the pth component of an element $\omega_1 \in \Omega_1$, $p \in \mathbb{P}$, and by $\omega_2(m)$ the mth component of an element $\omega_2 \in \Omega_2$, $m \in \mathbb{N}_0$. Extend the functions $\omega_1(p)$ to the set $\mathbb{N}$ by the formula

$$\omega_1(m) = \prod_{\substack{p^l | m \\ p^{l+1}\nmid m}} \omega_1^l(p), \quad m \in \mathbb{N}.$$

Denote by $\mathcal{H}(\Delta)$ the space of analytic functions on Δ equipped with the topology of uniform convergence on compact sets, let $\mathcal{H}^2(\Delta) = \mathcal{H}(\Delta) \times \mathcal{H}(\Delta)$, and, on the probability space $(\Omega, \mathcal{B}(\Omega), \mu_H)$, define the $\mathcal{H}^2(\Delta)$-valued random element

$$\underline{\zeta}(s,\alpha,\omega_1,\omega_2;\mathfrak{a},\mathfrak{b}) = (\zeta(s,\omega_1;\mathfrak{a}), \zeta(s,\alpha,\omega_2;\mathfrak{b})),$$

where

$$\zeta(s,\omega_1;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m\omega_1(m)}{m^s} \qquad \text{and} \qquad \zeta(s,\alpha,\omega_2;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m\omega_2(m)}{(m+\alpha)^s}.$$

Note that the latter series are uniformly convergent on compact subsets of the strip Δ for almost all ω_1 and ω_2 with respect to the Haar measures μ_{1H} on $(\Omega_1, \mathcal{B}(\Omega_1))$ and μ_{2H} on $(\Omega_2, \mathcal{B}(\Omega_2))$, respectively. The notation $x \ll_\xi y$, $y > 0$, means that there exists a constant $c = c(\xi) > 0$ such that $|x| \leqslant cy$.

Theorem 1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, α is transcendental, and $n_T \to \infty$ and $n_T \ll T^2$ as $T \to \infty$. Let $K_1, K_2 \in \mathfrak{K}$, and $f_1(s) \in \mathcal{H}_0(K_1)$, $f_2(s) \in \mathcal{H}(K_2)$. Then, the limit*

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\mathfrak{M}\bigg\{ &\tau \in [0,T] : \sup_{s\in K_1} |\zeta_{n_T}(s+i\tau;\mathfrak{a}) - f_1(s)| < \varepsilon_1, \\ &\sup_{s\in K_2} |\zeta_{n_T}(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon_2 \bigg\} \\ = \mu_H\bigg\{ &(\omega_1,\omega_2) \in \Omega : \sup_{s\in K_1} |\zeta(s,\omega_1;\mathfrak{a}) - f_1(s)| < \varepsilon_1, \\ &\sup_{s\in K_2} |\zeta(s,\alpha,\omega_2;\mathfrak{b}) - f_2(s)| < \varepsilon_2 \bigg\}\end{aligned}$$

exists and is positive for all but at most countably many $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

The first result on the approximation of the analytic functions by shifts of the absolutely convergent Dirichlet series was obtained in [32] and generalized in [33]. Discrete versions of the latter results are given in [34] and [35].

Theorem 1 extends the previous results on the universality of the Dirichlet series involving periodic sequences in two directions. Firstly, Theorem 1 is a joint universality on the simultaneous approximation of a pair of analytic functions. Secondly, the analytic functions are approximated by shifts of absolutely convergent series. This moment is a certain advantage in the estimation of approximated functions.

2. Approximation in the Mean

Recall the metric in the space $H^2(\Delta)$. There exists a sequence of compact sets $\{K_l : l \in \mathbb{N}\} \subset \Delta$ satisfying the requirements:

1. Δ is the union of the sets K_l;
2. $K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$;
3. For every compact set $K \subset \Delta$, there exists K_l such that $K \subset K_l$.

Then,

$$\rho(F_1,F_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |F_1(s) - F_2(s)|}{1 + \sup_{s\in K_l} |F_1(s) - F_2(s)|}, \quad F_1, F_2 \in \mathcal{H}(\Delta),$$

is a metric in $\mathcal{H}(\Delta)$ inducing its topology of uniform convergence on compacta. Putting, for $\underline{F}_1 = (F_{11}, F_{12})$, $\underline{F}_2 = (F_{21}, F_{22}) \in \mathcal{H}^2(\Delta)$,

$$\rho_2(\underline{F}_1, \underline{F}_2) = \max_{j=1,2} \rho(F_{1j}, F_{2j})$$

gives a metric in $\mathcal{H}^2(\Delta)$ inducing the product topology.

Lemma 1. *Suppose that $n_T \to \infty$ and $n_T \ll T^2$ as $T \to \infty$. Let*

$$\underline{\zeta}(s,\alpha;\mathfrak{a},\mathfrak{b}) = (\zeta(s;\mathfrak{a}), \zeta(s,\alpha;\mathfrak{b}))$$

and

$$\underline{\zeta}_{n_T}(s,\alpha;\mathfrak{a},\mathfrak{b}) = (\zeta_{n_T}(s;\mathfrak{a}), \zeta_{n_T}(s,\alpha;\mathfrak{b})).$$

Then,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \rho_2\Big(\underline{\zeta}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b}), \underline{\zeta}_{n_T}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b})\Big)\,\mathrm{d}\tau = 0.$$

Proof. By the definition of the metric ρ_2, it suffices to show that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \rho(\zeta(s+i\tau;\mathfrak{a}), \zeta_{n_T}(s+i\tau;\mathfrak{a}))\,\mathrm{d}\tau = 0$$

and

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \rho(\zeta(s+i\tau,\alpha;\mathfrak{b}), \zeta_{n_T}(s+i\tau,\alpha;\mathfrak{b}))\,\mathrm{d}\tau = 0.$$

The first of these equalities follows from Lemma 2 of [33] which states that, for every compact set $K\subset\Delta$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a}) - \zeta_{n_T}(s+i\tau;\mathfrak{a})|\,\mathrm{d}\tau = 0,$$

and from the definition of the metric ρ. The second equality is obtained similarly using the representation

$$\zeta_{n_T}(s,\alpha;\mathfrak{b}) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z,\alpha;\mathfrak{b}) l_{n_T}(z;\theta)\,\mathrm{d}z,$$

where $s\in\Delta$, $\Gamma(s)$ is the Euler gamma-function, and

$$l_{n_T}(s;\theta) = \frac{1}{\theta}\Gamma\Big(\frac{s}{\theta}\Big) n_T^s.$$

□

3. Limit Theorem

We will apply a limit theorem in the space $\mathcal{H}^2(\Delta)$ obtained in [15]. For $A\in\mathcal{B}(\mathcal{H}^2(\Delta))$, define

$$P_{T,\alpha,\mathfrak{a},\mathfrak{b}}(A) = \frac{1}{T}\mathfrak{M}\Big\{\tau\in[0,T] : \underline{\zeta}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b})\in A\Big\}.$$

Moreover, let $P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}$ be the distribution of the random element $\underline{\zeta}(s+i\tau,\alpha,\omega_1,\omega_2;\mathfrak{a},\mathfrak{b})$, i.e.,

$$P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(A) = \mu_H\{(\omega_1,\omega_2)\in\Omega : \underline{\zeta}(s+i\tau,\alpha,\omega_1,\omega_2;\mathfrak{a},\mathfrak{b})\in A\}.$$

Lemma 2. *Suppose that the sequence $\mathfrak{a}$ is multiplicative and the parameter α is transcendental. Then, $P_{T,\alpha,\mathfrak{a},\mathfrak{b}}$ converges weakly to $P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}$ as $T\to\infty$. Moreover, the support of the measure $P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}$ is the set*

$$\{g\in\mathcal{H}(\Delta) : \text{either } g(s)\neq 0 \text{ on } \Delta, \text{ or } g(s)\equiv 0\}\times\mathcal{H}(\Delta).$$

Proof. The lemma is the union of Theorem 6 and Lemma 12 from [15]. □

Now, we consider a limit theorem for $\underline{\zeta}_{n_T}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b})$. For $A\in\mathcal{B}(\mathcal{H}^2(\Delta))$, define

$$\widehat{P}_{T,\alpha,\mathfrak{a},\mathfrak{b}}(A) = \frac{1}{T}\mathfrak{M}\Big\{\tau\in[0,T] : \underline{\zeta}_{n_T}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b})\in A\Big\}.$$

Theorem 2. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, the parameter α is transcendental, and $n_T\to\infty$ and $n_T\ll T^2$ as $T\to\infty$. Then, $\widehat{P}_{T,\alpha,\mathfrak{a},\mathfrak{b}}$ converges weakly to $P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}$ as $T\to\infty$.*

Proof. Let θ_T be a random variable defined on a certain probability space $(\widehat{\Omega}, \mathcal{A}, P)$ and uniformly distributed on the segment $[0, T]$. Define the $\mathcal{H}^2(\Delta)$-valued random elements

$$\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}} = \underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}}(s) = (X_{T,\mathfrak{a}}(s), X_{T,\alpha,\mathfrak{b}}(s)),$$

where

$$X_{T,\mathfrak{a}}(s) = \zeta(s + i\theta_T; \mathfrak{a}), \qquad X_{T,\alpha,\mathfrak{b}}(s) = \zeta(s + i\theta_T, \alpha; \mathfrak{b}),$$

and

$$\widehat{\underline{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}} = \widehat{\underline{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}}(s) = \Big(\widehat{X}_{T,\mathfrak{a}}(s), \widehat{X}_{T,\alpha,\mathfrak{b}}(s)\Big),$$

where

$$\widehat{X}_{T,\mathfrak{a}}(s) = \zeta_{n_T}(s + i\theta_T; \mathfrak{a}), \qquad \widehat{X}_{T,\alpha,\mathfrak{b}}(s) = \zeta_{n_T}(s + i\theta_T, \alpha; \mathfrak{b}).$$

By the definitions of θ_T, $\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}}$ and $\widehat{\underline{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}}$, for $A \in \mathcal{B}(\mathcal{H}^2(\Delta))$, we have

$$P\{\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in A\} = P_{T,\alpha,\mathfrak{a},\mathfrak{b}}(A) \tag{2}$$

and

$$P\Big\{\widehat{\underline{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in A\Big\} = \widehat{P}_{T,\alpha,\mathfrak{a},\mathfrak{b}}(A). \tag{3}$$

Fix $\varepsilon > 0$, a closed set $F \subset \mathcal{H}^2(\Delta)$, and define

$$F_\varepsilon = \Big\{\underline{F} \in \mathcal{H}^2(\Delta) : \rho_2(\underline{F}, F) \leqslant \varepsilon\Big\},$$

where $\rho_2(\underline{F}, F) = \inf_{\underline{\hat{F}} \in F} \rho_2(\underline{F}, \underline{\hat{F}})$. Then, Lemma 2, equality (2), and the equivalent of weak convergence in terms of closed sets [36] show that

$$\limsup_{T\to\infty} P_{T,\alpha,\mathfrak{a},\mathfrak{b}}(F_\varepsilon) = \limsup_{T\to\infty} P\{X_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in F_\varepsilon\} \leqslant P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(F_\varepsilon). \tag{4}$$

It is easily seen that

$$\Big\{\widehat{\underline{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in F\Big\} \subset \{\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in F_\varepsilon\} \cup \Big\{\rho_2(\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}}, \widehat{\underline{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}}) \geqslant \varepsilon\Big\}.$$

Note that $\rho_2(\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}}, \widehat{\underline{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}})$ is a random variable, and, by the definition of θ_T, its expectation is

$$\frac{1}{T}\int_0^T \rho_2\Big(\underline{\zeta}(s + i\tau, \alpha; \mathfrak{a}, \mathfrak{b}), \underline{\zeta}_{n_T}(s + i\tau, \alpha; \mathfrak{a}, \mathfrak{b})\Big)\, d\tau.$$

Thus,

$$P\Big\{\widehat{\underline{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in F\Big\} \leqslant P\{\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in F_\varepsilon\} + P\Big\{\rho_2(\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}}, \widehat{\underline{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}}) \geqslant \varepsilon\Big\}, \tag{5}$$

and Lemma 1 together with Chebyshev's type inequality

$$\begin{aligned}&\mathfrak{M}\Big\{\tau \in [0, T] : \rho_2\Big(\underline{\zeta}(s + i\tau, \alpha; \mathfrak{a}, \mathfrak{b}), \underline{\zeta}_{n_T}(s + i\tau, \alpha; \mathfrak{a}, \mathfrak{b})\Big) \geqslant \varepsilon\Big\}\\ &\quad\leqslant \frac{1}{\varepsilon}\int_0^T \rho_2\Big(\underline{\zeta}(s + i\tau, \alpha; \mathfrak{a}, \mathfrak{b}), \underline{\zeta}_{n_T}(s + i\tau, \alpha; \mathfrak{a}, \mathfrak{b})\Big)\, d\tau\end{aligned}$$

implies that

$$\begin{aligned}P\Big\{\rho_2\Big(\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}}, \widehat{\underline{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}}\Big) \geqslant \varepsilon\Big\} &\leqslant \frac{1}{\varepsilon T}\int_0^T \rho_2\Big(\underline{\zeta}(s + i\tau, \alpha; \mathfrak{a}, \mathfrak{b}), \underline{\zeta}_{n_T}(s + i\tau, \alpha; \mathfrak{a}, \mathfrak{b})\Big)\, d\tau\\ &= 0.\end{aligned} \tag{6}$$

Therefore, in view of (5) and (6),

$$\limsup_{T\to\infty} P\Big\{\widehat{\underline{X}}_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in F\Big\} \leqslant \limsup_{T\to\infty} P\{\underline{X}_{T,\alpha,\mathfrak{a},\mathfrak{b}} \in F_\varepsilon\},$$

and, by (2), (3), and (4),

$$\limsup_{T\to\infty} \widehat{P}_{T,\alpha,\mathfrak{a},\mathfrak{b}}(F) \leqslant P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(F_\varepsilon).$$

Because $F_\varepsilon \to F$ as $\varepsilon \to +0$, this gives

$$\limsup_{T\to\infty} \widehat{P}_{T,\alpha,\mathfrak{a},\mathfrak{b}}(F) \leqslant P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(F),$$

and the equivalent of weak convergence in terms of closed sets proves the theorem. □

Let K_1, K_2, and $f_1(s)$, $f_2(s)$ be as in Theorem 1. For $A \in \mathcal{B}(\mathbb{R}^2)$, define

$$Q_{T,\alpha,\mathfrak{a},\mathfrak{b}}(A) = \frac{1}{T}\mathfrak{M}\Bigg\{\tau \in [0,T] : \bigg(\sup_{s\in K_1} |\zeta_{n_T}(s+i\tau;\mathfrak{a}) - f_1(s)| \\ \sup_{s\in K_2} |\zeta_{n_T}(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)|\bigg) \in A\Bigg\}.$$

Corollary 1. *Under hypotheses of Theorem 2, $Q_{T,\alpha,\mathfrak{a},\mathfrak{b}}$ converges weakly to the measure*

$$\mu_H\Bigg\{(\omega_1,\omega_2) \in \Omega : \bigg(\sup_{s\in K_1} |\zeta_{n_T}(s,\omega_1;\mathfrak{a}) - f_1(s)|, \\ \sup_{s\in K_2} |\zeta_{n_T}(s,\alpha,\omega_2;\mathfrak{a}) - f_2(s)|\bigg) \in A\Bigg\}, \quad A \in \mathcal{B}(\mathbb{R}^2),$$

as $T \to \infty$.

Proof. Define the function $h : \mathcal{H}^2(\Delta) \to \mathbb{R}^2$ by the formula

$$h(F_1,F_2) = \bigg(\sup_{s\in K_1} |F_1(s) - f_1(s)|, \sup_{s\in K_2} |F_2(s) - f_2(s)|\bigg).$$

Because the space $\mathcal{H}(\Delta)$ is equipped with the topology of the uniform convergence on compacta, the function h is continuous. Therefore, using a property of weak convergence preservation under continuous mappings [36], by Theorem 2, we have that $\widehat{P}_{T,\alpha,\mathfrak{a},\mathfrak{b}}h^{-1}$ converges weakly to $P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}h^{-1}$ as $T \to \infty$. However,

$$\widehat{P}_{T,\alpha,\mathfrak{a},\mathfrak{b}}h^{-1}(A) = \widehat{P}_{T,\alpha,\mathfrak{a},\mathfrak{b}}(h^{-1}A) = \frac{1}{T}\mathfrak{M}\{\tau \in [0,T] : \underline{\zeta}_{n_T}(s+i\tau,\alpha;\mathfrak{a},\mathfrak{b}) \in h^{-1}A\} \\ = Q_{T,\alpha,\mathfrak{a},\mathfrak{b}}(A)$$

and

$$P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}h^{-1}(A) = P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(h^{-1}A) \\ = \mu_H\Bigg\{(\omega_1,\omega_2) \in \Omega : \bigg(\sup_{s\in K_1} |\zeta(s,\omega_1;\mathfrak{a}) - f_1(s)|, \sup_{s\in K_2} |\zeta(s,\alpha,\omega_2;\mathfrak{a}) - f_2(s)|\bigg) \in A\Bigg\}.$$

This proves the corollary. □

Taking $A = (-\infty,\varepsilon_1) \times (-\infty,\varepsilon_2)$ in the definition of $Q_{T,\alpha,\mathfrak{a},\mathfrak{b}}$ and its limit measure, we obtain the distribution functions

$$F_{T,\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2) = \frac{1}{T}\mathfrak{M}\Bigg\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta_{n_T}(s+i\tau;\mathfrak{a}) - f_1(s)| < \varepsilon_1, \\ \sup_{s\in K_2} |\zeta_{n_T}(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon_2\Bigg\}$$

and

$$F_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2) = \mu_H\Big\{(\omega_1,\omega_2) \in \Omega : \sup_{s\in K_1} |\zeta(s,\omega_1;\mathfrak{a}) - f_1(s)| < \varepsilon_1, \\ \sup_{s\in K_2} |\zeta(s,\alpha,\omega_2;\mathfrak{b}) - f_2(s)| < \varepsilon_2\Big\}.$$

It is well-known that the weak convergence of probability measures on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ is equivalent to that of the corresponding distribution functions. Recall that $F_{T,\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)$ converges weakly to $F_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)$ if

$$\lim_{T\to\infty} F_{T,\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2) = F_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)$$

for all $(\varepsilon_1,\varepsilon_2)$ such that ε_1 and ε_2 are continuity points of the functions $F_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,+\infty)$ and $F_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(+\infty,\varepsilon_2)$, respectively. Thus, Corollary 1 implies the following:

Corollary 2. *Under hypotheses of Theorem 2, the distribution function $F_{T,\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)$ converges weakly to the distribution function $F_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)$ as $T\to\infty$.*

4. Proof of Theorem 1

Proof of Theorem 1. Because the set of the discontinuity points of the distribution function is at most countable, by Corollary 2, the limit

$$\lim_{T\to\infty} F_{T,\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2) = F_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)$$

exists for all but at most countably many $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Thus, it remains to prove the positivity of $F_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2)$.

In view of the Mergelyan theorem on the approximation of analytic functions by polynomials [37], there exist polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s\in K_1}\left|f_1(s) - \mathrm{e}^{p_1(s)}\right| < \frac{\varepsilon_1}{2} \quad \text{and} \quad \sup_{s\in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon_2}{2}. \tag{7}$$

By Lemma 2, the support S of the measure $P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}$ is the set $\{g \in \mathcal{H}(\Delta) : \text{either}\, g(s) \neq 0 \text{ on } D, \text{ or } g(s) \equiv 0\}$. Therefore, $\left(\mathrm{e}^{p_1(s)}, p_2(s)\right)$ is an element of S. Hence,

$$P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(G_{\varepsilon_1,\varepsilon_2}) > 0, \tag{8}$$

where

$$G_{\varepsilon_1,\varepsilon_2} = \left\{(F_1,F_2) \in \mathcal{H}^2(\Delta) : \sup_{s\in K_1}\left|F_1(s) - \mathrm{e}^{p_1(s)}\right| < \frac{\varepsilon_1}{2}, \sup_{s\in K_2} |F_2(s) - p_2(s)| < \frac{\varepsilon_2}{2}\right\}.$$

Define one more set

$$\widehat{G}_{\varepsilon_1,\varepsilon_2} = \left\{(F_1,F_2) \in \mathcal{H}^2(\Delta) \sup_{s\in K_1}|F_1(s) - f_1(s)| < \varepsilon_1, \sup_{s\in K_2} |F_2(s) - f_2(s)| < \varepsilon_2\right\}.$$

The inequalities (7) show that if $(F_1,F_2) \in G_{\varepsilon_1,\varepsilon_2}$, then $(F_1,F_2) \in \widehat{G}_{\varepsilon_1,\varepsilon_2}$. Thus, $G_{\varepsilon_1,\varepsilon_2} \subset \widehat{G}_{\varepsilon_1,\varepsilon_2}$. Therefore, in virtue of (8), $P_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(\widehat{G}_{\varepsilon_1,\varepsilon_2}) > 0$, i.e., $F_{\underline{\zeta},\alpha,\mathfrak{a},\mathfrak{b}}(\varepsilon_1,\varepsilon_2) > 0$. The theorem is proved. □

5. Conclusions

In this paper, the joint approximation of a pair of analytic functions by shifts of absolutely convergent Dirichlet series

$$\zeta_{n_T}(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_{n_T}(m;\theta)}{m^s} \qquad \text{and} \qquad \zeta_{n_T}(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m v_{n_T}(m,\alpha;\theta)}{(m+\alpha)^s}$$

with periodic sequences $\{a_m\}$ and $\{b_m\}$, and exponentially decreasing sequences $\{v_{n_T}(m;\theta)\}$ and $\{v_{n_T}(m,\alpha;\theta)\}$, is obtained. It is proved that if $n_T \to \infty$ and $n_T \ll T^2$ as $T \to \infty$, then the set of approximating shifts $(\zeta_{n_T}(s+i\tau;\mathfrak{a}), \zeta_{n_T}(s+i\tau,\alpha;\mathfrak{b}))$ has an explicitly given density on the interval $[0, T]$.

A possible improvement to the main theorem is an extension of the class of functions n_T. Moreover, we are planning to invite experts in numerical methods and IT into our group to obtain some numerical calculations of concrete examples. This is a very difficult problem closely connected to the effectivization of universality theorems for zeta-functions.

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