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# Closure Properties for Heavy-Tailed and Related Distributions

An Overview

 Springer

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# Preface

*... then, prudence cannot be science. Nor can it be skill. It is not science because the matters of conduct may vary; and it is not skill because the genus of action is different from that of production. It remains, therefore, that prudence is a disposition with truth, involving reason and concerned with action about things that are good or bad for a human being.*

*Aristotle, Nicomachean Ethics, VI.5, 1140b*

The classification of scientific subjects has its origin in early antiquity and up to now continues to participate in active research. In fact, it provides the correct framework in which the research activity becomes exact science. Recently, even the social sciences take the luxury of rigorous methods of testing assumptions and employ accurate quantitative expressions to go deeper into the nature of the human beings.

The closure properties in probability theory have a long history, back from the middle of the previous century. They appear as substantial supports in reliability theory, queueing theory, branching processes, risk theory, stochastic control, asset pricing, and others. Especially in the context of heavy-tailed distributions, they play a crucial role in the classification of the random variables and eventually of various risks. Therefore, a systematic presentation of the subject has already been mature.

The lack of closure properties is a strong motivation for enlarging the class of distributions. In this way, some new classes of theoretical or practical interest may appear. On the other hand, the presence of closure properties can bring up other possibilities for the investigation of the origin of the class. In both cases, the study of the closure properties can be instrumental for a deeper understanding of the class.

We find it necessary to restrict our overview in the space of *heavy-tailed* and related distributions, as they present special interest in applications, like in actuarial and financial issues, where economical stability is under threat. For this purpose, we take advantage of the recent accumulation of a vast amount of results on this topic. Eventually, the modelling of extremal events can serve as a tool for investigation on other subjects, for example, physics, chemistry, biology, or geology.

This book is written as a consequence of rather intensive research work of the authors in ruin theory and other problems of insurance mathematics, which is a rapidly developing area. The recent project started about 5 years ago and was

intended as a survey paper. However, later it was decided to write a book devoted to the overview of closure properties, which although mostly known for several decades and are reported in the papers and books but were not presented in a unified manner and systematically classified. In a sense, this book can serve as a handbook.

The reader is supposed to have advanced knowledge of calculus and probability theory, as also some familiarity with stochastic processes. In the references are found most of the classical works related to the topic, which can be used for deeper study. Most statements are without proofs but there are clear indications of where these proofs can be found. From numerous examples and counterexamples, we kept only the most representative, which provide substantial insight into the theory. This short book is written primarily to help graduate students and young researchers to enter quickly into the subject. Furthermore, it can be used by applied scientists and industrial scholars, or people in the market, to optimize their preferences and decisions. Any comments or criticism are encouraged and welcomed, as they are helpful for the further development of the state-of-the-art. Practical implementation of this theory is of paramount importance and much anticipated.

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# Contents

<b>1</b>	<b>Introduction</b> .....	1
1.1	An Overview of the Book .....	2
1.2	Notations and Definitions .....	3
<b>2</b>	<b>Heavy-Tailed and Related Classes of Distributions</b> .....	7
2.1	Heavy-Tailed Distributions .....	7
2.2	Regularly Varying Distributions .....	9
2.3	Consistently Varying Distributions .....	10
2.4	Dominatedly Varying Distributions .....	11
2.5	Long-Tailed Distributions .....	14
2.6	Exponential-Like-Tailed Distributions .....	15
2.7	Generalized Long-Tailed Distributions .....	17
2.8	Subexponential Distributions .....	18
2.9	Strong Subexponential Distributions .....	23
2.10	Convolution Equivalent Distributions .....	25
2.11	Generalized Subexponential Distributions .....	27
2.12	Bibliographical Notes .....	28
<b>3</b>	<b>Closure Properties Under Tail-Equivalence, Convolution, Finite Mixing, Maximum, and Minimum</b> .....	31
3.1	Ruin Probability in the Cramér-Lundberg Risk Model in the Case of Heavy-Tailed Claims .....	31
3.2	Convolution Closure and Max-Sum Equivalence .....	33
3.3	Closure Properties for Heavy-Tailed Class of Distributions .....	35
3.4	Closure Properties for Regularly Varying Class of Distributions ....	38
3.5	Closure Properties for Consistently Varying Class of Distributions .....	40
3.6	Closure Properties for Dominatedly Varying Class of Distributions .....	41
3.7	Closure Properties for Long-Tailed Class of Distributions .....	43
3.8	Closure Properties for Exponential-Like-Tailed Class of Distributions .....	45

- 3.9 Closure Properties for Generalized Long-Tailed Class of Distributions ..... 46
- 3.10 Closure Properties for Subexponential Class of Distributions ..... 47
- 3.11 Closure Properties for Strong Subexponential Class of Distributions ..... 50
- 3.12 Closure Properties for Convolution Equivalent Class of Distributions ..... 52
- 3.13 Closure Properties for Generalized Subexponential Class of Distributions ..... 54
- 3.14 Bibliographical Notes ..... 55
- 4 Convolution-Root Closure ..... 57**
  - 4.1 Distribution Classes Closed Under Convolution Roots ..... 57
  - 4.2 Distribution Classes Not Closed Under Convolution Roots ..... 58
  - 4.3 Bibliographical Notes ..... 59
- 5 Product-Convolution of Heavy-Tailed and Related Distributions ..... 61**
  - 5.1 Product-Convolution ..... 61
  - 5.2 From Light Tails to Heavy Tails Through Product-Convolution ..... 62
  - 5.3 Product-Convolution Closure Properties for Heavy-Tailed Class of Distributions ..... 66
  - 5.4 Product-Convolution Closure Properties for Regularly Varying Class of Distributions ..... 67
  - 5.5 Product-Convolution Closure Properties for Consistently Varying Class of Distributions ..... 68
  - 5.6 Product-Convolution Closure Properties for Dominatedly Varying Class of Distributions ..... 68
  - 5.7 Product-Convolution Closure Properties for Exponential-Like-Tailed Distributions ..... 69
  - 5.8 Product-Convolution Closure Properties for Generalized Long-Tailed Class of Distributions ..... 71
  - 5.9 Product-Convolution Closure Properties for Convolution Equivalent Class of Distributions ..... 72
  - 5.10 Product-Convolution Closure Properties for Generalized Subexponential Class of Distributions ..... 75
  - 5.11 Some Extensions ..... 76
  - 5.12 Bibliographical Notes ..... 77
- 6 Summary of Closure Properties ..... 79**
- References ..... 81**
- Index ..... 91**



# Acronyms

Distribution classes:

- Heavy-tailed distributions:  $\mathcal{H}$ .
- Strongly heavy-tailed distributions:  $\mathcal{H}^*$ .
- Regularly varying distributions:  $\mathcal{R}(\alpha)$ .
- Extended regularly varying distributions:  $\mathcal{ERV}(\alpha, \beta)$ .
- Rapidly varying distributions:  $\mathcal{R}(\infty)$ .
- Consistently varying distributions:  $\mathcal{C}$ .
- Dominatedly varying distributions:  $\mathcal{D}$ .
- Positively decreasing-tailed distributions:  $\mathcal{PD}$ .
- Long-tailed distributions:  $\mathcal{L}$ .
- Exponential-like-tailed distributions:  $\mathcal{L}(\gamma)$ .
- Generalized long-tailed distributions:  $\mathcal{OL}$ .
- Subexponential distributions:  $\mathcal{S}$ .
- Strong subexponential distributions:  $\mathcal{S}^*$ .
- Strongly subexponential distributions:  $\mathcal{S}_*$ .
- Convolution equivalent distributions:  $\mathcal{S}(\gamma)$ .
- Generalized subexponential distributions:  $\mathcal{OS}$ .
- Generalized strong subexponential distributions:  $\mathcal{OS}^*$ .
- Subexponential positively decreasing-tailed distributions:  $\mathcal{A}$ .
- Generalized subexponential positively decreasing-tailed distributions:  $\mathcal{OA}$ .

Methods:

- Strong tail-equivalence principle: STEP.
- Weak tail-equivalence principle: WTEP.