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Article

On the Approximation by Mellin Transform of the Riemann Zeta-Function

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Abstract: This paper is devoted to the approximation of a certain class of analytic functions by shifts $\mathcal{Z}(s+i\tau)$, $\tau\in\mathbb{R}$, of the modified Mellin transform $\mathcal{Z}(s)$ of the square of the Riemann zeta-function $\zeta(1/2+it)$. More precisely, we prove the existence of a closed non-empty set F such that there are infinitely many shifts $\mathcal{Z}(s+i\tau)$, which approximate a given analytic function from F with a given accuracy. In the proof, the weak convergence of measures in the space of analytic functions is applied. Then, the set F coincides with the support of a limit measure.

Keywords: limit theorem; Mellin transform; Riemann zeta-function; weak convergence

MSC: 11M06

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1. Introduction

Recall that the Riemann zeta-function $\zeta(s)$, $s=\sigma+it$, is defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},$$

and is analytically continuable to the whole complex plane, except for the point $s=1$, which is a simple pole with residue 1. It is well known that the function $\zeta(s)$ has good approximation properties, its shifts $\zeta(s+i\tau)$, $\tau\in\mathbb{R}$, approximate every non-vanishing analytic function defined on the strip $\{s\in\mathbb{C}:1/2<\sigma<1\}$. On the other hand, in the theory of the function $\zeta(s)$, there exists several important unsolved problems. One of them is the moment problem on the asymptotic behavior as $T\to\infty$ for

$$\int_0^T|\zeta(\sigma+it)|^{2k}\,\mathrm{d}t,\quad \sigma\geqslant\frac{1}{2},\ k>0.$$

Y. Motohashi introduced [1], see also [2], the modified Mellin transforms

$$\mathcal{Z}_k(s)=\int_1^{\infty}\left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k}x^{-s}\,\mathrm{d}x,\quad k\in\mathbb{N},$$

and applied them for investigation of the latter problem. He first considered the case $k=2$. The integral for $\mathcal{Z}_k(s)$ [3] is absolutely convergent for $\sigma>1$ if $0\leqslant k\leqslant 2$, and for $\sigma>(k+2)/4$ if $2\leqslant k\leqslant 6$. Hence, the function $\mathcal{Z}_k(s)$ is analytic in the corresponding half-planes. Later, the Mellin transforms $\mathcal{Z}_k(s)$ with applications were studied in [4–6].

We give one example from [4]. Define $E_2(T)$ by

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t = TP_4(\log T) + E_2(T)$$

with

$$P_4(x) = \sum_{j=0}^{4} a_j x^j, \quad a_4 = \frac{1}{2\pi^2}.$$

There exists a problem to estimate $E_2(T)$. In [4], using the mean square estimates for $\mathcal{Z}(s)$, it was obtained that, for $\varepsilon > 0$,

$$E_2(T) \ll_\varepsilon T^{2/3+\varepsilon},$$

and this estimate is the best up to ε. The notation $a \ll_\varepsilon b$, $a \in \mathbb{C}$, $b > 0$, means that there exists a positive constant $c = c(\varepsilon)$ such that $|a| \leqslant cb$.

In function theory, much attention is devoted to the approximation of analytic functions. We recall some results related to number theory. S.N. Mergelyan obtained [7] a very deep result connected to polynomials. Suppose that K is a compact set with a connected complement, and $f(s)$ a continuous function on K, which is analytic inside of K. Mergelyan proved [7] the existence of a polynomial sequence uniformly convergent on K to the function $f(s)$. From this, it follows that, for any $\varepsilon > 0$, we can find a polynomial $p_{f,\varepsilon}(s)$ satisfying

$$\sup_{s\in K}\left|f(s) - p_{f,\varepsilon}(s)\right| < \varepsilon.$$

Thus, a function satisfying the above hypotheses can be approximated by a polynomial.

In 1975, it turned out that there exist functions that approximate a whole class of analytic functions. The first example of such functions is the Riemann zeta-function. S.M. Voronin proved [8] that if $0 < r < 1/4$, the function $f(s) \neq 0$ is continuous on the disc $|s| \leqslant r$, and analytic inside that disc, then, for any $\varepsilon > 0$, there is a real number $\tau = \tau(\varepsilon)$ satisfying the inequality

$$\max_{|s|\leqslant r}\left|f(s) - \zeta\left(s + \frac{3}{4} + i\tau\right)\right| < \varepsilon.$$

This shows that a set of non-vanishing analytic functions defined in the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ is approximated by shifts $\zeta(s + i\tau)$ of one and the same function. In other words, $\zeta(s)$ is universal with respect to the approximation of analytic functions. The Voronin universality theorem was reinforced and extended for other zeta-functions. We recall its last form, see [9–12]. Suppose that $K \subset D$ is a compact set having a connected complement, $f(s)$ is continuous, having no zeros on K and analytic in inside of K function. Then, for any positive ε,

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Here, μ stands for the Lebesgue measure on the line $\mathbb{R}$.

The proof of the Voronin theorem in [8] is based on the rearrangement theorem for series in Hilbert space. B. Bagchi proposed [12] a new original probabilistic method that uses weak convergence of measures in the space of analytic functions. The Bagchi method was developed in [9,10]. Other results on the universality of zeta-functions are discussed in a survey paper [13]. We notice that an idea of application probabilistic methods in the theory of $\zeta(s)$ was proposed by H. Bohr and B. Jessen. In [14,15], they obtained the existence of the limit

$$\lim_{T\to\infty} \frac{1}{T} \mathrm{J}\{t \in [0, T] : \zeta(\sigma + it) \in \mathrm{R}\}$$

for every rectangle $\mathrm{R} \subset \mathbb{C}$ with edges parallel to the axis and $\sigma > 1/2$. Here, J denotes the Jordan measure on $\mathbb{R}$. A modern version of the Bohr–Jessen theorem in terms of weak convergence is presented in [9].

In general, for description value distribution of $\zeta(s)$, various methods and terms are used. For example, it was observed in [16] that the distribution of a-points of $\zeta(s)$, $a \neq 0$ (the solution of $\zeta(s) = a$), has a certain relation to a Julia line [17] with respect to the essential singularity of $\zeta(s)$ at infinity.

In the present paper, we are connected to a new problem—the approximation of analytic functions by the function $\mathcal{Z}(s) \overset{def}{=} \mathcal{Z}_1(s)$. We need some results from [3]. The function $\mathcal{Z}(s)$ is analytic in the half-plane $\sigma > -3/4$, except for a double pole at the point $s = 1$, and has simple poles at the points $s = -(2k-1)$, $k \in \mathbb{N}$. Let γ_0 be the Euler constant, $E(T)$ be defined by

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t = T\log\frac{T}{2\pi} + (2\gamma_0 - 1)T + E(t),$$

and

$$G(T) = \int_1^T E(t)\,\mathrm{d}t - \pi T, \qquad G_1(T) = \int_1^T G(t)\,\mathrm{d}t.$$

Then, it was obtained that

$$\begin{aligned}\mathcal{Z}(s) =& \frac{1}{(s-1)^2} + \frac{2\gamma_0 - \log 2\pi}{s-1} - E(1) + \pi(s+1) \\ &+ s(s+1)(s+2)\int_1^\infty G_1(x)x^{-s-3}\,\mathrm{d}x, \quad \sigma > -\frac{3}{4}. \end{aligned} \tag{1}$$

Moreover, for $0 \leqslant \sigma \leqslant 1$, $t \geqslant t_0 > 0$, and fixed $\varepsilon > 0$,

$$\mathcal{Z}(\sigma+it) \ll_\varepsilon t^{1-\sigma+\varepsilon},$$

and

$$\int_1^T |\mathcal{Z}(\sigma+it)|^2\,\mathrm{d}t \ll_\varepsilon \begin{cases} T^{3-4\sigma+\varepsilon} & \text{if } 0 \leqslant \sigma \leqslant \frac{1}{2}, \\ T^{2-2\sigma+\varepsilon} & \text{if } \frac{1}{2} \leqslant \sigma \leqslant 1. \end{cases} \tag{2}$$

Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, and $H(D)$ be the space of analytic on D functions equipped with the topology of uniform convergence on compacts. The main result of the paper is the following theorem.

Theorem 1. *There is a non-empty closed subset $F \subset H(D)$ such that, for arbitrary compact set $K \subset D$, $f(s) \in F$, and every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \sup_{s\in K}|\mathcal{Z}(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

*Moreover, except for at most a countably set of values of $\varepsilon > 0$, "*lim inf*" can be replaced by "*lim*".*

Theorem 1 implies that there are infinitely many shifts $\mathcal{Z}(s+i\tau)$ approximating a function from the set F. Theorem 1 is a certain version of the modern form of the Voronin universality theorem [8] for the Riemann zeta-function, see, for example, [9,10]. In the case of $\zeta(s)$, $F = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

Unfortunately, in the case of Theorem 1, the set F cannot be explicit described. Let $\mathcal{B}(\mathcal{X})$ stand for Borel σ-field of the space $\mathcal{X}$. We will show that F is the support of a probability measure on $(H(D), \mathcal{B}(H(D)))$. Theorem 1 is a corollary of a limit theorem for weakly convergent measures in the space $(H(D), \mathcal{B}(H(D)))$. For $A \in \mathcal{B}(H(D))$, set

$$P_T(A) = \frac{1}{T}\,\mu\{\tau \in [0,T] : \mathcal{Z}(s+i\tau) \in A\}.$$

Denote by $\xrightarrow{W}$ the weak convergence.

Theorem 2. *On the space $(H(D), \mathcal{B}(H(D)))$, there is a probability measure P such that $P_T \xrightarrow[T\to\infty]{W} P$.*

For the proof of Theorem 2, the auxiliary function

$$\mathcal{Z}_y(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 v(x,y)x^{-s}\,\mathrm{d}x,$$

where

$$v(x,y) = \exp\left\{-\left(\frac{x}{y}\right)^{\sigma_0}\right\}, \quad x,y \in (1,\infty),$$

with a fixed $\sigma_0 > 0$, will be useful.

2. Case of Finite Interval

Let $a > 1$, and

$$\mathcal{Z}_{a,y}(s) = \int_1^a \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 v(x,y)x^{-s}\,\mathrm{d}x,$$

For $A \in \mathcal{B}(H(D))$, set

$$P_{T,a,y}(A) = \frac{1}{T}\,\mu\big\{\tau \in [0,T] : \mathcal{Z}_{a,y}(s+i\tau) \in A\big\}.$$

Lemma 1. *On the space $(H(D), \mathcal{B}(H(D)))$, there is a probability measure $P_{a,y}$ such that $P_{T,a,y} \xrightarrow[T\to\infty]{W} P_{a,y}$.*

The proof of Lemma 1 is divided into parts. In the first part, we will deal with weak convergence on a certain Cartesian product. Let γ be the unit circle on $\mathbb{C}$, and

$$\Omega_a = \prod_{u\in[1,a]} \gamma.$$

In virtue of the Tikhonov theorem, the set Ω_a with the product topology is a compact topological Abeliam group. For $A \in \mathcal{B}(\Omega_a)$, set

$$Q_{T,a}(A) = \frac{1}{T}\,\mu\Big\{\tau \in [0,T] : \Big(u^{-i\tau} : u \in [1,a]\Big) \in A\Big\}.$$

Lemma 2. *On the space $(\Omega_a, \mathcal{B}(\Omega_a))$, there is a probability measure Q_a such that $Q_{T,a} \xrightarrow[T\to\infty]{W} Q_a$.*

Proof. The character group of Ω_a is isomorphic to $\bigoplus_{u\in[1,a]} \mathbb{Z}_u$, where $\mathbb{Z}_u = \mathbb{Z}$ for all $u \in [1,a]$. Therefore, the Fourier transform $g_{T,a}(k_u : u \in [1,a])$ of $Q_{T,a}$ is given by

$$g_{T,a}(k_u : u \in [1,a]) = \int_{\Omega_a} \prod_{u\in[1,a]} x_u^{k_u}\,\mathrm{d}Q_{T,a},$$

where $x_u \in \gamma$, $k_u \in \mathbb{Z}$, and only a finite number of k_u are not zeros. Thus,

$$g_{T,a}(k_u : u \in [1,a]) = \frac{1}{T}\int_0^T \exp\left\{-i\tau \sum_{u\in[1,a]} k_u \log u\right\} d\tau$$
$$= \begin{cases} 1 & \text{if } \sum\limits_{u\in[1,a]} k_u \log u = 0, \\ \frac{1-\exp\{-i\tau\sum_{u\in[1,a]} k_u \log u\}}{iT\sum_{u\in[1,a]} k_u \log u} & \text{if } \sum\limits_{u\in[1,a]} k_u \log u \neq 0. \end{cases}$$

Therefore, we have

$$\lim_{T\to\infty} g_{T,a}(k_u : u \in [1,a]) \overset{def}{=} g_a(k_u : u \in [1,a]) = \begin{cases} 1 & \text{if } \sum\limits_{u\in[1,a]} k_u \log u = 0, \\ 0 & \text{if } \sum\limits_{u\in[1,a]} k_u \log u \neq 0. \end{cases}$$

This shows that $Q_{T,a} \xrightarrow[T\to\infty]{W} Q_a$ with Q_a on $(\Omega_a, \mathcal{B}(\Omega_a))$ with the Fourier transform $g_a(k_u : u \in [1,a])$. □

Lemma 2 implies a certain limit lemma in the space $H(D)$. We recall that if $h : \mathcal{X} \to \mathcal{X}_1$ is a $(\mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{X}_1))$-measurable mapping, then a probability measure P on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ defines the unique probability measure Ph^{-1} on $(\mathcal{X}_1, \mathcal{B}(\mathcal{X}_1))$ defined by $Ph^{-1}(A) = P(h^{-1}A)$, $A \in \mathcal{X}_1$. Moreover, if the mapping h is continuous, then the weak convergence is preserved, i.e., if $P_n \xrightarrow[n\to\infty]{W} P$ in $\mathcal{X}$, then also $P_n h^{-1} \xrightarrow[n\to\infty]{W} Ph^{-1}$ in $\mathcal{X}_1$. The latter remark is sometimes very useful.

Let

$$S_{n,a,y}(s) = \frac{a-1}{n}\sum_{k=1}^{n}\left|\zeta\left(\frac{1}{2}+i\xi_k\right)\right|^2 v(\xi_k, y)\xi_k^{-s},$$

where $\xi_k \in [x_{k-1}, x_k]$ and $x_k = 1 + ((a-1)/n)k$. For $A \in \mathcal{B}(H(D))$, let

$$P_{T,n,a,y}(A) = \frac{1}{T}\mu\{\tau \in [0,T] : S_{n,a,y}(s+i\tau) \in A\}.$$

Lemma 3. *On the space $(H(D), \mathcal{B}(H(D)))$, there is a probability measure $P_{n,a,y}$ such that $P_{T,n,a,y} \xrightarrow[T\to\infty]{W} P_{n,a,y}$.*

Proof. Let the mapping $h_{n,a} : \Omega_a \to H(D)$ be given by the formula

$$h_{n,a}(\underline{y}) = \frac{a-1}{n}\sum_{k=1}^{n}\left|\zeta\left(\frac{1}{2}+i\xi_k\right)\right|^2 v(\xi_k, y)\xi_k^{-s} y_{\xi_k}, \quad \underline{y} = \{y_u \in \gamma : u \in [1,a]\}.$$

Then, $h_{n,a}$ is a continuous in the product topology, and $h_{n,a}(\{u^{-i\tau} : u \in [1,a]\}) = S_{n,a,y}(s + i\tau)$. Thus, $P_{T,n,a,y} = Q_{T,a}h_{n,a}^{-1}$, where $Q_{T,a}$ comes from Lemma 2. This equality, the continuity of $h_{n,a}$, Lemma 2 and the above remark on the preservation of weak convergence show that $P_{T,n,a,y}$ converges weakly to $P_{n,a,y} = Q_a h_{n,a}^{-1}$ as $T \to \infty$ with Q_a defined in Lemma 2. □

In the sequel, we will use one lemma on the convergence in distribution ($\xrightarrow{\mathcal{D}}$). Recall that the random element X_n converges in distribution to X as $n \to \infty$, if the distribution P_n of X_n converges weakly to that P of X as $n \to \infty$. In this case, we use the notation $X_n \xrightarrow[n\to\infty]{\mathcal{D}} P$ as well.

Suppose that the metric space $(\mathcal{X}, d)$ is separable and the $\mathcal{X}$-valued random elements X, Y_n and X_{nk} are defined on the same probability space with measure $\mathbb{P}$.

Lemma 4. *For $k \in \mathbb{N}$, let*

$$X_{nk} \xrightarrow[n\to\infty]{\mathcal{D}} X_k$$

and

$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X.$$

If, for any $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}\{d(X_{nk}, Y_n) \geqslant \varepsilon\} = 0,$$

then $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

The lemma is proved, for example, in [18], Theorem 4.2.

We note that the space $H(D)$ is separable and metrizable. It is known that there is a sequence $\{K_l : l \in \mathbb{N}\} \subset D$ of compact embedded sets such that D is the union of K_l, and every set $K \subset D$ lies in some set K_l. Then,

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(D),$$

is a metric in $H(D)$ which induces its topology of uniform convergence on compacts.

Lemma 5. *The equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\big(S_{n,a,y}(s+i\tau), \mathcal{Z}_{a,y}(s+i\tau)\big)\, \mathrm{d}\tau = 0$$

holds.

Proof. The definition of the metric ρ implies that it suffices to show that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} \big|S_{n,a,y}(s+i\tau) - \mathcal{Z}_{a,y}(s+i\tau)\big|\, \mathrm{d}\tau = 0 \tag{3}$$

for every compact set $K \subset D$. Let L be a simple closed contour lying in D and enclosing a compact set K; suppose also that $\inf_{s\in K} \inf_{z\in L} |s - z| \geqslant c(L) > 0$. Then, by the integral Cauchy formula, we have

$$\sup_{s\in K} \big|S_{n,a,y}(s+i\tau) - \mathcal{Z}_{a,y}(s+i\tau)\big| \ll_L \int_L \big|S_{n,a,y}(z+i\tau) - \mathcal{Z}_{a,y}(z+i\tau)\big| |\mathrm{d}z|.$$

Therefore,

$$\begin{aligned} &\frac{1}{T} \int_0^T \sup_{s\in K} \big|S_{n,a,y}(s+i\tau) - \mathcal{Z}_{a,y}(s+i\tau)\big|\, \mathrm{d}\tau \\ &\qquad \ll_L \int_L |\mathrm{d}z| \left(\frac{1}{T} \int_0^T \big|S_{n,a,y}(z+i\tau) - \mathcal{Z}_{a,y}(z+i\tau)\big| \mathrm{d}\tau \right). \end{aligned} \tag{4}$$

Clearly,

$$
\frac{1}{T}\int_0^T \left|S_{n,a,y}(z+i\tau)-\mathcal{Z}_{a,y}(z+i\tau)\right| \mathrm{d}\tau
\leqslant \left(\frac{1}{T}\int_0^T \left|S_{n,a,y}(z+i\tau)-\mathcal{Z}_{a,y}(z+i\tau)\right|^2 \mathrm{d}\tau\right)^{1/2} \tag{5}
$$

We have

$$
\begin{aligned}
&\left|S_{n,a,y}(z+i\tau)-\mathcal{Z}_{a,y}(z+i\tau)\right|^2\\
&=\left(S_{n,a,y}(z+i\tau)-\mathcal{Z}_{a,y}(z+i\tau)\right)\overline{\left(S_{n,a,y}(z+i\tau)-\mathcal{Z}_{a,y}(z+i\tau)\right)}\\
&=S_{n,a,y}(z+i\tau)\overline{S_{n,a,y}(z+i\tau)}-S_{n,a,y}(z+i\tau)\overline{\mathcal{Z}_{a,y}(z+i\tau)}\\
&\quad-\overline{S_{n,a,y}(z+i\tau)}\mathcal{Z}_{a,y}(z+i\tau)+\mathcal{Z}_{a,y}(z+i\tau)\overline{\mathcal{Z}_{a,y}(z+i\tau)},
\end{aligned} \tag{6}
$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. By the definition of $S_{n,a,y}(s)$,

$$
\begin{aligned}
S_{n,a,y}(z+i\tau)\overline{S_{n,a,y}(z+i\tau)} &= \left(\frac{a-1}{n}\right)^2 \sum_{k=1}^{n}\left|\zeta\left(\frac{1}{2}+i\xi_k\right)\right|^4 v^2(\xi_k,y)\xi_k^{-2\mathrm{Re}z}\\
&+\left(\frac{a-1}{n}\right)^2 \sum_{\substack{k_1=1\\ k_1\neq k_2}}^{n}\sum_{k_2=1}^{n}\left|\zeta\left(\frac{1}{2}+i\xi_{k_1}\right)\right|^2\left|\zeta\left(\frac{1}{2}+i\xi_{k_2}\right)\right|^2\\
&\times v(\xi_{k_1},y)v(\xi_{k_2},y)\xi_{k_1}^{-z}\xi_{k_2}^{-\overline{z}}\left(\frac{\xi_{k_1}}{\xi_{k_2}}\right)^{-i\tau}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{T}\int_0^T S_{n,a,y}(z+i\tau)\overline{S_{n,a,y}(z+i\tau)}\,\mathrm{d}\tau\\
&\quad=\left(\frac{a-1}{n}\right)^2 \sum_{k=1}^{n}\left|\zeta\left(\frac{1}{2}+i\xi_k\right)\right|^4 v^2(\xi_k,y)\xi_k^{-2\mathrm{Re}z}\\
&\quad+O\Bigg(\left(\frac{a-1}{n}\right)^2 \frac{1}{T}\sum_{\substack{k_1=1\\ k_1\neq k_2}}^{n}\sum_{k_2=1}^{n}\left|\zeta\left(\frac{1}{2}+i\xi_{k_1}\right)\right|^2\left|\zeta\left(\frac{1}{2}+i\xi_{k_2}\right)\right|^2\\
&\quad\times v(\xi_{k_1},y)v(\xi_{k_2},y)\xi_{k_1}^{-\mathrm{Re}z}\xi_{k_2}^{\mathrm{Re}z}\left|\log\frac{\xi_{k_1}}{\xi_{k_2}}\right|^{-1}\Bigg).
\end{aligned}
$$

Since

$$
\lim_{n\to\infty}\frac{a-1}{n}\sum_{k=1}^{n}\left|\zeta\left(\frac{1}{2}+i\xi_k\right)\right|^4 v^2(\xi_k,y)\xi_k^{-2\mathrm{Re}z}=\int_1^a\left|\zeta\left(\frac{1}{2}+ix\right)\right|^4 v^2(x,y)x^{-2\mathrm{Re}z}\,\mathrm{d}x,
$$

hence we obtain, for all $z \in L$,

$$
\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T S_{n,a,y}(z+i\tau)\overline{S_{n,a,y}(z+i\tau)}\,\mathrm{d}\tau=0. \tag{7}
$$

Similarly, by the definition of $\mathcal{Z}_{a,y}(s)$, for all $z \in L$,

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathcal{Z}_{a,y}(z+i\tau)\overline{\mathcal{Z}_{a,y}(z+i\tau)}\,\mathrm{d}\tau \\
&= \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left(\int_1^a\int_1^a \left|\zeta\left(\frac{1}{2}+ix_1\right)\right|^2 \left|\zeta\left(\frac{1}{2}+ix_2\right)\right|^2 \right. \\
&\quad \left. \times v(x_1,y)v(x_2,y)x_1^{-z-i\tau}x_2^{-\overline{z}+i\tau}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)\mathrm{d}\tau \\
&= \limsup_{T\to\infty} \frac{1}{T}\underset{x_1\neq x_2}{\int_1^a\int_1^a} \left|\zeta\left(\frac{1}{2}+ix_1\right)\right|^2 \left|\zeta\left(\frac{1}{2}+ix_2\right)\right|^2 \\
&\quad \times v(x_1,y)v(x_2,y)x_1^{-z}x_2^{-\overline{z}}\left(e^{-iT\log(x_1/x_2)}-1\right)\left(\log\frac{x_1}{x_2}\right)^{-1}\mathrm{d}x_1\,\mathrm{d}x_2 = 0. \qquad (8)
\end{aligned}$$

The latter equality suggests that the set of values of the function $\mathcal{Z}_{a,y}(s)$ is not dense. On the other hand, this case is convenient for our investigations. Moreover,

$$\begin{aligned}
&\left|\frac{1}{T}\int_0^T S_{n,a,y}(z+i\tau)\overline{\mathcal{Z}_{a,y}(z+i\tau)}\,\mathrm{d}\tau\right| \\
&\leqslant \left(\frac{1}{T}\int_0^T \left|S_{n,a,y}(z+i\tau)\right|^2\mathrm{d}\tau\right)^{1/2}\left(\frac{1}{T}\int_0^T \left|\mathcal{Z}_{a,y}(z+i\tau)\right|^2\mathrm{d}\tau\right)^{1/2},
\end{aligned}$$

and this is true for the integral of $\overline{S_{n,a,y}(z+i\tau)}\mathcal{Z}_{a,y}(z+i\tau)$. This and (4)–(8) show that

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}\left|S_{n,a,y}(s+i\tau)-\mathcal{Z}_{a,y}(s+i\tau)\right|\mathrm{d}\tau = 0.$$

□

Proof of Lemma 1. Suppose that θ_T is a random variable uniformly distributed on $[0,T]$ and defined on a certain probability space with measure $\mathbb{P}$. Define the $H(D)$-valued random element

$$X_{T,n,a,y}(s) = S_{n,a,y}(s+i\theta_T).$$

In view of Lemma 3, we have

$$X_{T,n,a,y} \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,a,y}, \qquad (9)$$

where $X_{n,a,y}$ is the $H(D)$-valued random element with the distribution $P_{n,a,y}$.

Now, we will prove that the sequence $\{P_{n,a,y} : n\in\mathbb{N}\}$ is tight, i.e., that, for every $\varepsilon>0$, there exists a compact set $K=K(\varepsilon)\subset H(D)$ such that

$$P_{n,a,y}(K) > 1-\varepsilon$$

for all $n\in\mathbb{N}$. Let K_l be a compact set in the definition of the metric ρ. Then, (7) and the integral Cauchy formula imply

$$\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K_l}\left|S_{n,a,y}(s+i\tau)\right|\mathrm{d}\tau \leqslant R_{l,a,y} < \infty.$$

Let $\varepsilon>0$ be a fixed, and $M_l = M_{l,a,y} = R_{l,a,y}2^l\varepsilon^{-1}$. Then, in view of (9),

$$\mathbb{P}\left(\sup_{s\in K_l}\left|X_{n,a,y}(s)\right| > M_l\right) \leqslant \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{M_l T}\int_0^T \sup_{s\in K_l}\left|S_{n,a,y}(s+i\tau)\right| \mathrm{d}\tau \leqslant \frac{\varepsilon}{2^l} \tag{10}$$

for all n and $l \in \mathbb{N}$. Let

$$K = K(\varepsilon) = \left\{g \in H(D) : \sup_{s\in K_l} |g(s)| \leqslant M_l,\ l \in \mathbb{N}\right\}.$$

Then, the set K is compact in the space $H(D)$, and, by (10),

$$\mathbb{P}\big(X_{n,a,y} \in K\big) = 1 - \mathbb{P}\big(X_{n,a,y} \notin K\big) > 1 - \varepsilon\sum_{l=1}^{\infty} 2^{-l} = 1 - \varepsilon$$

for all $n \in \mathbb{N}$. This and the definition of $P_{n,a,y}$ prove the tightness of the sequence $\{P_{n,a,y} : n \in \mathbb{N}\}$. In the theory of weak convergence of probability measures, the Prokhorov theorem, see, for example, [18], occupies an important place. Let $\{P\}$ be a family of probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. The Prokhorov theorem connects the tightness and relative compactness of $\{P\}$; namely, if the family $\{P\}$ is tight, then it is relatively compact.

Since the sequence $\{P_{n,a,y}\}$ is tight, by the Prokhorov theorem [18], it is relatively compact, i.e., every subsequence $\{P_{n_k,a,y}\}$ contains a subsequence weakly convergent to a certain probability measure on $(H(D), \mathcal{B}(H(D)))$. Thus, there exists a probability measure $P_{a,y}$ on $(H(D), \mathcal{B}(H(D)))$ and a sequence $\{P_{n_r,a,y}\}$ such that $P_{n_r,a,y}$ converges weakly to $P_{a,y}$ as $r \to \infty$. In other words,

$$X_{n_r,a,y} \xrightarrow[r\to\infty]{\mathcal{D}} P_{a,y}. \tag{11}$$

Now, we are in position to apply Lemma 4 for the random elements

$$Y_{T,a,y}(s) = \mathcal{Z}_{a,y}(s + i\theta_T),$$

$X_{n_r,a,y}$ and $X_{a,y}$, where $X_{a,y}$ has the distribution $P_{a,y}$. By Lemma 5, we have, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\big(\rho(Y_{T,a,y}, X_{n_r,a,y}) \geqslant \varepsilon\big) \\
&\quad\leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon}\int_0^T \rho\big(S_{n_r,a,y}(s+i\tau), \mathcal{Z}_{a,y}(s+i\tau)\big)\,\mathrm{d}\tau = 0.
\end{aligned}$$

This, (9) and (11) together with Lemma 5 prove Lemma 1, i.e., $P_{T,a,y}$ converges weakly to $P_{a,y}$ as $T \to \infty$. □

3. Case of Infinite Interval

In this section, we will prove a limit theorem for the function $\mathcal{Z}_y(s)$. Since $\zeta(1/2 + it) \ll t^{1/6}$ as $t \to \infty$, and $v(x, y)$ with respect to x is decreasing exponentially, the integral for $\mathcal{Z}_y(s)$ is absolutely convergent for $\sigma > \sigma_0$ with every fixed σ_0 and $y > 0$.

For $A \in \mathcal{B}(H(D))$, define

$$P_{T,y}(A) = \frac{1}{T}\mu\big\{\tau \in [0, T] : \mathcal{Z}_y(s + i\tau) \in A\big\}.$$

Lemma 6. *On $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure P_y such that* $P_{T,y} \xrightarrow[T\to\infty]{W} P_y$.

Proof. First, we observe that the equality

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho\big(\mathcal{Z}_y(s+i\tau),\mathcal{Z}_{a,y}(s+i\tau)\big)\,\mathrm{d}\tau=0 \tag{12}$$

holds. As in the case of Lemma 5, it suffices to show that, for every compact set $K\subset D$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}\big|\mathcal{Z}_y(s+i\tau)-\mathcal{Z}_{a,y}(s+i\tau)\big|\,\mathrm{d}\tau=0. \tag{13}$$

It is easily seen that, for every fixed $y>0$ and $s\in K$,

$$\begin{aligned}\mathcal{Z}_y(s+i\tau)-\mathcal{Z}_{a,y}(s+i\tau)&=\int_a^\infty\left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 v(x,y)x^{-s-i\tau}\,\mathrm{d}x\\&\ll_y\int_a^\infty\left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 v(x,y)x^{-1/2}\,\mathrm{d}x=o_y(1)\end{aligned}$$

as $a\to\infty$ in view of convergence of the integral. From this, equality (13) follows.

Let θ_T be the same random variable as in the proof of Lemma 1. Define

$$Y_{T,y}(s)=\mathcal{Z}_y(s+i\theta_T),$$

and denote by $Y_{a,y}$ the $H(D)$-valued random element with the distribution $P_{a,y}$. Then, Lemma 1 implies the relation

$$Y_{T,a,y}\xrightarrow[T\to\infty]{\mathcal{D}}Y_{a,y}. \tag{14}$$

Let K_l, $l\in\mathbb{N}$ be a compact set from the definition of metric ρ. Then, (8) and the integral Cauchy formula give

$$\sup_{a\geqslant1}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K_l}\big|\mathcal{Z}_{a,y}(s+i\tau)\big|\,\mathrm{d}\tau\leqslant R_{l,y}<\infty.$$

Thus, taking $\widehat{M}_l=\widehat{M}_{l,y}=R_{l,y}2^l\varepsilon^{-1}$, we find by (14)

$$\mathbb{P}\left(\sup_{s\in K_l}\big|Y_{a,y}(s)\big|>\widehat{M}_l\right)\leqslant\sup_{a\geqslant1}\limsup_{T\to\infty}\frac{1}{\widehat{M}_lT}\int_0^T\sup_{s\in K_l}\big|\mathcal{Z}_{a,y}(s+i\tau)\big|\,\mathrm{d}\tau\leqslant\frac{\varepsilon}{2^l}.$$

This shows that

$$\mathbb{P}\big(Y_{a,y}\in K\big)>1-\varepsilon,$$

for all $a\geqslant1$, where $K=\{g\in H(D):\sup_{s\in K_l}|g(s)|\leqslant\widehat{M}_l,\ l\in\mathbb{N}\}$. Therefore, the family of probability measures $\{P_{a,y}:a\geqslant1\}$ is tight. Thus, there exists a sequence $P_{a_r,y}$ weakly convergent to a certain probability measure P_y as $r\to\infty$, i.e.,

$$Y_{a_r,y}\xrightarrow[r\to\infty]{\mathcal{D}}P_y.$$

This, (12), (14) and Lemma 4 prove that

$$Y_{T,y}\xrightarrow[T\to\infty]{\mathcal{D}}P_y, \tag{15}$$

and the lemma is proved. □

4. Formula for $\mathcal{Z}_y(s)$

As usual, denote by $\Gamma(s)$ the Euler gamma-function, and define

$$l_y(s) = \frac{s}{\sigma_0}\Gamma\left(\frac{s}{\sigma_0}\right)y^s,$$

where σ_0 is from definition of $v(x,y)$.

Lemma 7. *The integral representation, for $s \in D$,*

$$\mathcal{Z}_y(s) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\mathcal{Z}(s+z)l_y(z)\frac{\mathrm{d}z}{z}$$

is valid.

Proof. We will apply the classical Mellin formula

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(s)b^{-s}\,\mathrm{d}s = e^{-b}, \quad a,b>0,$$

and obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}l_y(z)x^{-z}\frac{\mathrm{d}z}{z} &= \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{1}{\sigma_0}\Gamma\left(\frac{z}{\sigma_0}\right)\left(\frac{x}{y}\right)^{-z}\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\Gamma(z)\left(\frac{x}{y}\right)^{-\sigma_0 z}\mathrm{d}z \\ &= \exp\left\{-\left(\frac{x}{y}\right)^{\sigma_0}\right\} = v(x,y). \end{aligned} \tag{16}$$

Setting, for brevity,

$$f(x,t) = \frac{1}{2\pi i}\frac{l_y(\sigma_0+it)}{\sigma_0+it}\left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 x^{-s-\sigma_0-it}$$

and applying theorem from ([19], §1.84), we obtain

$$\int_{-T}^{T}\mathrm{d}t\int_{1}^{X}f(x,t)\,\mathrm{d}x = \int_{1}^{X}\mathrm{d}x\int_{-T}^{T}f(x,t)\,\mathrm{d}t, \tag{17}$$

for any $X,T>1$. Next, the well-known estimate

$$\Gamma(\sigma+it) \ll \exp\left(-c|t|\right), \quad c>0,$$

which is uniform in any fixed strip $\sigma_1<\sigma<\sigma_2$, together with the inequality

$$\int_1^X\left|\zeta\left(\frac{1}{2}+ix\right)\right|^2\mathrm{d}x \ll X(\log X)$$

imply

$$\int_T^{+\infty} \mathrm{d}t \int_1^X (|f(x,t)| + |f(x,-t)|)\,\mathrm{d}x, \quad \int_1^X \mathrm{d}x \int_T^{+\infty} (|f(x,t)| + |f(x,-t)|)\,\mathrm{d}t \ll R,$$

where

$$\begin{aligned} R = R(X,T) &= y^{\sigma_0} \int_T^{+\infty} e^{-ct/\sigma_0}\,\mathrm{d}t \int_1^X \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 x^{-\sigma-\sigma_0}\,\mathrm{d}x \\ &\ll y^{\sigma_0} \exp\left\{-\frac{c}{\sigma_0}T\right\}\left(1+X^{1-\sigma-\sigma_0}\right)(\log X)^2. \end{aligned} \tag{18}$$

Hence, using (17) and (18), we find that

$$\begin{aligned} \int_{-\infty}^{+\infty} \mathrm{d}t \int_1^X f(x,t)\,\mathrm{d}x &= \left(\int_{-T}^{T} + \int_T^{+\infty} + \int_{-\infty}^{-T}\right) \mathrm{d}t \int_1^X f(x,t)\,\mathrm{d}x \\ &= \int_1^X \mathrm{d}x \int_{-T}^{T} f(x,t)\,\mathrm{d}t + O(R) \\ &= \int_1^X \mathrm{d}x \left(\int_{-\infty}^{+\infty} - \int_T^{+\infty} - \int_{-\infty}^{-T}\right) f(x,t)\,\mathrm{d}t + O(R) \\ &= \int_1^X \mathrm{d}x \int_{-\infty}^{+\infty} f(x,t)\,\mathrm{d}t + O(R). \end{aligned}$$

Tending $T \to \infty$, we obtain

$$\int_{-\infty}^{+\infty} \mathrm{d}t \int_1^X f(x,t)\,\mathrm{d}x = \int_1^X \mathrm{d}x \int_{-\infty}^{+\infty} f(x,t)\,\mathrm{d}t$$

for any $X > 1$. Therefore, the application of a theorem from ([19], §1.84) together with (16) yields

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \mathcal{Z}(s+z) l_y(z) \frac{\mathrm{d}z}{z} \\ &= \int_{-\infty}^{+\infty} \int_1^{+\infty} \frac{1}{2\pi i} \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 \frac{l_y(\sigma_0+it)}{\sigma_0+it}\, x^{-s-\sigma_0-it} \mathrm{d}x \\ &= \int_{-\infty}^{+\infty} \mathrm{d}t \int_1^{+\infty} f(x,t)\,\mathrm{d}x = \int_1^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} f(x,t)\mathrm{d}t \\ &= \int_1^{+\infty} \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 x^{-s} \frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} l_y(z) x^{-z} \frac{\mathrm{d}z}{z}\,\mathrm{d}x \\ &= \int_1^{+\infty} \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 x^{-s} v(x,y)\,\mathrm{d}x = \mathcal{Z}_y(s). \end{aligned}$$

□

5. Approximation of $\mathcal{Z}(s)$ by $\mathcal{Z}_y(s)$

Lemma 8. *The equality*

$$\lim_{y\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho\big(\mathcal{Z}(s+i\tau),\mathcal{Z}_y(s+i\tau)\big)\,\mathrm{d}\tau=0$$

holds.

Proof. Let K be an arbitrary fixed compact set of the strip D. Then, there exists a number $\varepsilon>0$ such that, for all $s=\sigma+it\in K$, $1/2+2\varepsilon\leqslant\sigma\leqslant 1-\varepsilon$. Denote

$$\sigma_1=\sigma-\varepsilon-\frac{1}{2},\qquad \sigma_0=\frac{1}{2}+\varepsilon.$$

Then, $\sigma_1>0$ for all $s\in K$. Since the point $z=1-s$ is a double pole, and $z=0$ is a simple pole of the function

$$\mathcal{Z}(s+z)\frac{l_y(z)}{z},$$

Lemma 7 and the residue theorem imply

$$\mathcal{Z}_y(s)-\mathcal{Z}(s)=\frac{1}{2\pi i}\int_{-\sigma_1-i\infty}^{-\sigma_1+i\infty}\mathcal{Z}(s+z)l_y(z)\frac{\mathrm{d}z}{z}+R_y(s), \tag{19}$$

where

$$R_y(s)=\operatorname*{Res}_{z=1-s}\mathcal{Z}(s+z)\frac{l_y(z)}{z}.$$

Let $a_0=2\gamma_0-\log 2\pi$. Then, in view of (1),

$$R_y(s)=\left(\frac{l_y(z)}{z}\right)'\bigg|_{z=1-s}+a_0\frac{l_y(1-s)}{1-s}. \tag{20}$$

By (19), for all $s\in K$, we have

$$\begin{aligned}\mathcal{Z}_y(s+i\tau)-\mathcal{Z}(s+i\tau)=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{Z}\left(\sigma+it-\sigma+\frac{1}{2}+\varepsilon+i\tau+iv\right)\\ &\times\frac{l_y(1/2+\varepsilon-\sigma+iv)}{1/2+\varepsilon-\sigma+iv}\,\mathrm{d}v+R_y(s+i\tau).\end{aligned}$$

Hence, writing v in place of $t+v$, gives, for $s\in K$,

$$\begin{aligned}\mathcal{Z}_y(s+i\tau)-\mathcal{Z}(s+i\tau)=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{Z}\left(\frac{1}{2}+\varepsilon+i\tau+iv\right)\frac{l_y(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\,\mathrm{d}v\\ &+R_y(s+i\tau)\\ \ll&\int_{-\infty}^{\infty}\left|\mathcal{Z}\left(\frac{1}{2}+\varepsilon+i\tau+iv\right)\right|\sup_{s\in K}\left|\frac{l_y(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\right|\mathrm{d}v\\ &+\sup_{s\in K}\left|R_y(s+i\tau)\right|.\end{aligned}$$

Thus,

$$\frac{1}{T}\int_0^T \sup_{s\in K}\left|\mathcal{Z}_y(s+i\tau)-\mathcal{Z}(s+i\tau)\right| \mathrm{d}\tau$$
$$\ll \int_{-\infty}^{\infty}\left(\frac{1}{T}\int_0^T\left|\mathcal{Z}\left(\frac{1}{2}+\varepsilon+i\tau+iv\right)\right|\mathrm{d}\tau\right)\sup_{s\in K}\left|\frac{l_y(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\right|\mathrm{d}v$$
$$+\frac{1}{T}\int_0^T \sup_{s\in K}\left|R_y(s+i\tau)\right|\mathrm{d}\tau \overset{def}{=} I_1+I_2. \tag{21}$$

First we estimate I_1. The definition of $l_y(s)$ and (17) imply that, for $s=\sigma+it\in K$,

$$\begin{aligned}\frac{l_y(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv} &\ll_{\sigma_0} y^{1/2+\varepsilon-\sigma}\left|\Gamma\left(\frac{1}{\sigma_0}\left(\frac{1}{2}+\varepsilon-\sigma-it+iv\right)\right)\right|\\ &\ll_{\sigma_0} y^{1/2+\varepsilon-\sigma}\exp\left\{-\frac{c}{\sigma_0}|t-v|\right\}\\ &\ll_{\sigma_0} y^{1/2+\varepsilon-\sigma}\exp\left\{-\frac{c}{\sigma_0}|v-t|\right\}\\ &\ll_{\sigma_0} y^{1/2+\varepsilon-\sigma}\exp\left\{\frac{c}{\sigma_0}|t|\right\}\exp\left\{-\frac{c}{\sigma_0}|v|\right\}.\end{aligned} \tag{22}$$

Since $\sigma \geqslant 1/2+2\varepsilon$, then $1/2+\varepsilon-\sigma\leqslant -\varepsilon$ for any $s\in K$. Therefore,

$$\sup_{s\in K}\left|\frac{l_y(1/2+\varepsilon+iv-s)}{1/2+\varepsilon+iv-s}\right| \ll_{\sigma_0,K} y^{-\varepsilon}\exp\{-c_1|v|\}.$$

Thus, we obtain

$$I_1 \ll_{\sigma_0,K} y^{-\varepsilon}\int_{-\infty}^{+\infty}\exp\left\{-\frac{c}{\sigma_0}|v|\right\}\frac{1}{T}\int_0^T\left|\mathcal{Z}\left(\frac{1}{2}+\varepsilon+i\tau+iv\right)\right|\mathrm{d}\tau\,\mathrm{d}v.$$

Given $0<\varepsilon_1<\varepsilon$, Cauchy inequality together with (2) imply

$$\begin{aligned}\frac{1}{T}\int_0^T\left|\mathcal{Z}\left(\frac{1}{2}+\varepsilon+i\tau+iv\right)\right|\mathrm{d}\tau &\leqslant \left(\frac{1}{T}\int_0^T\left|\mathcal{Z}\left(\frac{1}{2}+\varepsilon+i\tau+iv\right)\right|^2\mathrm{d}\tau\right)^{1/2}\\ &= \left(\frac{1}{T}\int_v^{T+v}\left|\mathcal{Z}\left(\frac{1}{2}+\varepsilon+i\tau\right)\right|^2\mathrm{d}\tau\right)^{1/2}\\ &\leqslant \left(\frac{2}{T}\int_0^{T+|v|}\left|\mathcal{Z}\left(\frac{1}{2}+\varepsilon+i\tau\right)\right|^2\mathrm{d}\tau\right)^{1/2}\\ &\ll_\varepsilon \left(\frac{1}{T}(T+|v|)^{2-2(1/2+\varepsilon)+\varepsilon_1}\right)^{1/2}\\ &\ll_\varepsilon T^{-\varepsilon+\varepsilon_1/2}+\frac{1}{\sqrt{T}}|v|^{1/2-\varepsilon+\varepsilon_1/2}\\ &\ll_\varepsilon T^{-\varepsilon/2}+\frac{|v|^{1/2}}{\sqrt{T}}.\end{aligned}$$

Hence,

$$I_1 \ll_{\sigma_0,K,\varepsilon} y^{-\varepsilon} \int_{-\infty}^{+\infty} e^{c_1|v|}\left(T^{-\varepsilon/2} + \frac{|v|^{1/2}}{\sqrt{T}}\right) \mathrm{d}v \ll_{\sigma_0,K,\varepsilon} y^{-\varepsilon}\left(T^{-\varepsilon/2} + T^{-1/2}\right)$$
$$\ll_{\sigma_0,K,\varepsilon} y^{-\varepsilon} T^{-\varepsilon/2}.$$

Passing to the estimate of I_2, and setting $g(z) = l_y(z)/z$, we obtain

$$g'(z) = \frac{y^z}{\sigma_0}\Gamma\left(\frac{z}{\sigma_0}\right)\left(\frac{1}{\sigma_0}\psi\left(\frac{z}{\sigma_0}\right) + \log y\right)$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ denotes the digamma-function. Hence,

$$\begin{aligned} R_y(s) &= g'(1-s) + (2\gamma_0 - \log 2\pi) g(1-s) \\ &= \frac{y^{1-s}}{\sigma_0}\Gamma\left(\frac{1-s}{\sigma_0}\right)\left(\frac{1}{\sigma_0}\psi\left(\frac{1-s}{\sigma_0}\right) + \log y + 2\gamma_0 - \log 2\pi\right). \end{aligned}$$

Thus, we find

$$\begin{aligned} R_y(s+i\tau) &\ll y^{1-\sigma}\left|\Gamma\left(\frac{1-\sigma}{\sigma_0} + i\frac{t+\tau}{\sigma_0}\right)\right|\left(\left|\psi\left(\frac{1-\sigma}{\sigma_0} + i\frac{t+\tau}{\sigma_0}\right)\right| + \log y + 1\right) \\ &\ll_{\sigma_0} y^{1-\sigma}\exp\left\{-\frac{c}{\sigma_0}|t+\tau|\right\}\left(\log\left|\frac{t+\tau}{\sigma_0}\right| + \log y + 1\right) \end{aligned}$$

and therefore

$$\begin{aligned} \sup_{s\in K}|R_y(s+i\tau)| &\ll_{\sigma_0,K} y^{1-(1/2+2\varepsilon)}\exp\left\{-\frac{c}{\sigma_0}\tau\right\}(\log(\tau+2) + \log y) \\ &\ll_{\sigma_0,K,\varepsilon} y^{1/2-\varepsilon}\exp\{-c_2|\tau|\}. \end{aligned}$$

Thus, we obtain

$$I_2 = \frac{1}{T}\int_0^T \sup_{s\in K}|R_y(s+i\tau)|\,\mathrm{d}\tau \ll_{\sigma_0,K,\varepsilon} \frac{y^{1/2-\varepsilon}}{T}\int_0^T \exp\{-c_2\tau\}\,\mathrm{d}\tau \ll_{\sigma_0,K,\varepsilon} \frac{y^{1/2-\varepsilon}}{T}.$$

Consequently,

$$\frac{1}{T}\int_0^T \sup_{s\in K}\left|\mathcal{Z}_y(s+i\tau) - \mathcal{Z}(s+i\tau)\right|\mathrm{d}\tau = I_1 + I_2 \ll_{\sigma_0,K,\varepsilon} y^{-\varepsilon}T^{-\varepsilon/2} + \frac{y^{1/2-\varepsilon}}{T}.$$

Tending $T \to \infty$, we find

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}\left|\mathcal{Z}_y(s+i\tau) - \mathcal{Z}(s+i\tau)\right|\mathrm{d}\tau = 0.$$

Then, the desired assertion follows. □

6. Proof of Theorem 2

We return to the limit measure P_y in Lemma 6. We recall that $P_{T,y} \xrightarrow[T\to\infty]{W} P_y$.

Lemma 9. *The family of probability measures $\{P_y : y > 1\}$ is tight.*

Proof. Let $K \subset D$ be a compact set. Then, we have

$$\frac{1}{T}\int_0^T \sup_{s\in K}\left|\mathcal{Z}_y(s+i\tau)\right| \mathrm{d}\tau \leqslant \frac{1}{T}\int_0^T \sup_{s\in K}\left|\mathcal{Z}(s+i\tau)-\mathcal{Z}_y(s+i\tau)\right| \mathrm{d}t + \frac{1}{T}\int_0^T \sup_{s\in K}\left|\mathcal{Z}(s+i\tau)\right| \mathrm{d}\tau. \tag{23}$$

Let L be a simple closed contour lying in D, enclosing a compact K and such that

$$\inf_{z\in L}\inf_{s\in K}|z-s| \geqslant c(L) > 0.$$

Then, the application of the integral Cauchy formula gives

$$|\mathcal{Z}(s+i\tau)| \leqslant \frac{1}{2\pi}\int_L \frac{|\mathcal{Z}(z+i\tau)|}{|z-s|}|\mathrm{d}z|,$$

and

$$\sup_{s\in K}|\mathcal{Z}(s+i\tau)| \leqslant \frac{1}{2\pi c}\int_L |\mathcal{Z}(z+i\tau)|\,|\mathrm{d}z| \ll_L \int_L |\mathcal{Z}(z+i\tau)|\,|\mathrm{d}z|.$$

Setting $z=\sigma+it$ for $z\in L$, and using the inequality (2), we find

$$\begin{aligned}\frac{1}{T}\int_0^T \sup_{s\in K}|\mathcal{Z}(s+i\tau)|\,\mathrm{d}\tau &\ll_L \int_L \frac{1}{T}\int_0^T |\mathcal{Z}(z+i\tau)|\,\mathrm{d}\tau\,|\mathrm{d}z| \\ &\ll_L \int_L \left(\frac{1}{T}\int_0^T |\mathcal{Z}(z+i\tau)|^2\,\mathrm{d}\tau\right)^{1/2} |\mathrm{d}z| \\ &\ll_L \int_L \left(\frac{1}{T}\int_0^{T+|t|+1} |\mathcal{Z}(\sigma+i\tau)|^2\,\mathrm{d}\tau\right)^{1/2} |\mathrm{d}z| \\ &\ll_L \int_L \left(\frac{1}{T}(T+|t|)^{2-2\sigma+\varepsilon_1}\right)^{1/2} |\mathrm{d}z| \ll_L T^{-\varepsilon/2}.\end{aligned}$$

Thus,

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}|\mathcal{Z}(s+i\tau)|\,\mathrm{d}t \leqslant C < \infty.$$

Therefore, by (23) and Lemma 8,

$$\sup_{y\geqslant 1}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}\left|\mathcal{Z}_y(s+i\tau)\right|\mathrm{d}t \leqslant R < \infty.$$

Fix $\varepsilon > 0$ and put $M_l = R2^l\varepsilon^{-1}$, $l\in\mathbb{N}$. Let $Y_y(s)$ be a $H(D)$-valued random element with the distribution P_y. Then,

$$\mathbb{P}\left(\sup_{s\in K_l}|Y_y(s)| > M_l\right) \leqslant \sup_{y\geqslant 1}\limsup_{T\to\infty}\frac{1}{TM_l}\int_0^T \sup_{s\in K_l}\left|\mathcal{Z}_y(s+i\tau)\right|\mathrm{d}\tau \leqslant \frac{\varepsilon}{2^l}.$$

Hence, we have

$$\mathbb{P}(Y_y(s) \in K) \geqslant 1 - \varepsilon$$

for all $y \geqslant 1$, where

$$K = \left\{ g \in H(D) : \sup_{s \in K_l} |g(s)| \leqslant M_l,\ l \in \mathbb{N} \right\},$$

and the lemma is proved. □

Proof of Theorem 2. By Lemma 9 and the Prokhorov theorem, the family $\{P_y : y \geqslant 1\}$ is relatively compact. Therefore, there exists a sequence $\{y_k\}$, $y_k \to \infty$ as $k \to \infty$ such that $\{P_{y_k}\}$ converges weakly to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $k \to \infty$. Since the distribution of Y_{y_k} is P_{y_k}, we have

$$Y_{y_k} \xrightarrow[k \to \infty]{\mathcal{D}} P. \tag{24}$$

Define

$$Z_T(s) = \mathcal{Z}(s + i\theta_T).$$

Then, in view of Lemma 8, for every $\varepsilon > 0$,

$$\lim_{y \to \infty} \limsup_{T \to \infty} \mathbb{P}(\rho(Z(s), Y_{T,Y}(s)) \geqslant \varepsilon)$$
$$\leqslant \lim_{y \to \infty} \limsup_{T \to \infty} \frac{1}{T\varepsilon} \int_0^T \rho(\mathcal{Z}(s + i\tau), \mathcal{Z}_y(s + i\tau))\, \mathrm{d}\tau = 0.$$

The latter equality, relations (15) and (24) show that all hypotheses of Lemma 4 are satisfied. Therefore, we obtain the relation

$$Z_T \xrightarrow[T \to \infty]{\mathcal{D}} P,$$

and this is equivalent to the assertion of the theorem. □

7. Proof of Theorem 1

We derive Theorem 1 from Theorem 2 by applying properties of weak convergence of probability measures. We will approximate functions from the support of the limit measure P of P_T in Theorem 2. The application of Theorem 2 is based on the following equivalents of weak convergence, see, for example, [18].

Lemma 10. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Then, the following statements are equivalent.*

1° $P_n \xrightarrow[n \to \infty]{W} P$.

2° *For every open set $G \subset \mathcal{X}$,*
$$\liminf_{n \to \infty} P_n(G) \geqslant P(G).$$

3° *For every continuity set A of the measure P (A is a continuity set of P if $P(\partial A) = 0$, where ∂A is a boundary of A),*
$$\lim_{n \to \infty} P_n(A) = P(A).$$

Proof of Theorem 1. Denote by F the support of the limit measure P in Theorem 2. The set F is a minimal closed subset of $H(D)$ such that $P(F) = 1$. The set F consists of all elements $f \in H(D)$ such that, for every open neighborhood G of f, the inequality $P(G) > 0$ is satisfied. Obviously, $F \neq \varnothing$.

For $f \in F$, define

$$G_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then, G_ε is an open neighborhood of the element f of the support of P. Therefore,

$$P(G_\varepsilon) > 0. \tag{25}$$

Thus, by Theorem 2, and 1° and 2° of Lemma 10,

$$\liminf_{T \to \infty} \frac{1}{T} \mu \left\{ \tau \in [0, T] : \sup_{s \in K} |\mathcal{Z}(s + i\tau) - f(s)| < \varepsilon \right\} \geqslant P(G_\varepsilon) > 0.$$

To prove the second assertion of the theorem, we notice that the boundary ∂G_ε lies in

$$\left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\}.$$

Hence, $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \varnothing$ for $\varepsilon_1 \neq \varepsilon_2$. Thus, $P(\partial G_\varepsilon)$ can be positive for at most countably many positive ε, in other words, G_ε is a continuity set of P, except for all but at most a countable set of values $\varepsilon > 0$. Therefore, Theorem 2, 1° and 3° of Lemma 10, and (25) show that the limit

$$\lim_{T \to \infty} \frac{1}{T} \mu \left\{ \tau \in [0, T] : \sup_{s \in K} |\mathcal{Z}(s + i\tau) - f(s)| < \varepsilon \right\} = P(G_\varepsilon) > 0.$$

exists for all but at most countably many $\varepsilon > 0$. The theorem is proved. □

8. Conclusions

In this paper, we found that the shifts of the Mellin transform of the square of the Riemann zeta-function $\zeta(s)$

$$\mathcal{Z}(s) = \int_1^\infty \left| \zeta\left(\frac{1}{2} + ix \right) \right|^2 x^{-s} \, dx$$

approximate a certain class F of analytic functions defined in the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. The main ingredient of the proof of the above result is a limit theorem for the function $\mathcal{Z}(s)$ in the space of analytic functions. Note that a problem of approximation of analytic functions by shifts of the function $\mathcal{Z}(s)$ is new, and it is discussed for the first time. The main result and method are inspired by universality theorems for $\zeta(s)$. Unfortunately, the set F is not explicitly given. This is a complicated future problem. Additionally, we are planning to extend the results of the paper for Mellin transforms of other powers of $\zeta(s)$.

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