Nonlinear Analysis: Modelling and Control, Online First, 1–19
https://doi.org/10.15388/namc.2024.29.33535

Logarithm of multivector in real 3D Clifford algebras

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Received: May 12, 2023 / **Revised:** September 27, 2023 / **Published online:** November 7, 2023

Abstract. Closed form expressions for a logarithm of general multivector (MV) in basis-free form in real geometric algebras (GAs) $Cl_{p,q}$ are presented for all $n = p+q = 3$. In contrast to logarithm of complex numbers (isomorphic to $Cl_{0,1}$), 3D logarithmic functions, due to appearance of two double angle arc tangent functions, allow to include *two sets of sheets* characterized by discrete coefficients. Formulas for generic and special cases of individual blades and their combinations are provided.

Keywords: Clifford (geometric) algebra, multivector logarithm, computer-aided theory.

1 Introduction

Logarithm properties are well known for real and complex numbers. Except the Hamilton quaternions which are isomorphic to $Cl_{0,2}$, the properties of logarithm in other 2D algebras (some partial formulas for 2D GAs are provided in [7, 9, 13]) and higher-dimensional Clifford algebras remain uninvestigated as yet. In general, GA logarithm properties are simplest for anti-Euclidean algebras $Cl_{0,n}$. As in the complex algebra case, we expect at least to have a principal logarithm and a part that makes the GA logarithm a multivalued function.

Recently, in papers [2, 8], which will be the starting point for the present article, we have performed a detailed investigation of 3D exponential functions in real GAs. However, the GA logarithm is more difficult to analyze since one must take into account the multivaluedness and the fact that in 3D algebras (except $Cl_{0,3}$) the logarithm may not exist for all MVs. Here we have treated the logarithm as an inverse problem using for this purpose the *Mathematica* symbolic package, more precisely, as an inverse GA function to exponential in separate 3D algebras $Cl_{0,3}$, $Cl_{3,0}$, $Cl_{1,2}$, and $Cl_{2,1}$. The final

GA logarithm formulas were checked symbolically as well as numerically. They are in a complete agreement with more general formulas [3, 4] suitable for computation on any function of diagonalizable multivector (MV). The exact logarithm formulas also have been applied to study convergence of series expansion of MV logarithms.

In Section 2 the notation is introduced. Since GA logarithm is connected with a two argument arc tangent function $\arctan(x, y)$, its properties are mentioned briefly in this section as well. In Section 3 the GA logarithm of the simplest, namely, $Cl_{0,3}$ algebra is examined. The general and special cases are presented in a form handy for the programming. Since algebras $Cl_{3,0}$ and $Cl_{1,2}$ are isomorphic, in Section 4 the logarithms of both algebras are investigated simultaneously. In Section 5 the most difficult logarithm of $Cl_{2,1}$ algebra is presented. In Section 6 the relations of the logarithm to GA inverse trigonometric and hyperbolic functions are discussed briefly. Finally, in Section 7, we summarize the obtained results. The analysis and examples of GA inverse trigonometric and hyperbolic functions, including two argument arc tangent function and their relations to a GA multiple square roots [1, 9] and logarithm, can be found in our extended preprint [6].

2 Notation and general properties of GA logarithm

A general MV is expanded in the orthonormal basis in inverse degree lexicographic ordering: $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123} \equiv I\}$, where $\mathbf{e}_i$ are basis vectors, $\mathbf{e}_{ij}$ are the bivectors, and I is the pseudoscalar. The number of subscripts indicates the grade. In the orthonormalized basis the geometric products of basis vectors satisfy the anticommutation relation, $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = \pm 2\delta_{ij}$. For $Cl_{3,0}$ and $Cl_{0,3}$ algebras, the squares of all basis vectors, correspondingly, are $\mathbf{e}_i^2 = +1$ and $\mathbf{e}_i^2 = -1$, where $i = 1, 2, 3$. For mixed signature algebras such as $Cl_{2,1}$ and $Cl_{1,2}$, the squares are $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$, $\mathbf{e}_3^2 = -1$ and $\mathbf{e}_1^2 = 1$, $\mathbf{e}_2^2 = \mathbf{e}_3^2 = -1$, respectively.

A general MV in real Clifford algebras $Cl_{p,q}$ when $n = p + q = 3$ is

$$\begin{aligned} \mathsf{A} &= a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{12}\mathbf{e}_{12} + a_{23}\mathbf{e}_{23} + a_{13}\mathbf{e}_{13} + a_{123}I \\ &\equiv a_0 + \mathbf{a} + \mathcal{A} + a_{123}I = a_0 + \mathsf{A}_{1,2} + a_{123}I = \mathsf{A}_{0,1,2,3}, \end{aligned}$$

where a_i, a_{ij} and a_{123} are real coefficients, and $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathcal{A} = a_{12}\mathbf{e}_{12} + a_{23}\mathbf{e}_{23} + a_{13}\mathbf{e}_{13}$ is, respectively, the vector and the bivector. I is the pseudoscalar, $I = \mathbf{e}_{123}$. A comma-separated multiple index in MV $\mathsf{A}_{\mathsf{i},\mathsf{j},\ldots}$ indicates a sum of MVs of the grades $i, j, \ldots$, i.e., $\mathsf{A}_{\mathsf{i},\mathsf{j},\ldots} = \langle\mathsf{A}\rangle_i + \langle\mathsf{A}\rangle_j + \langle\mathsf{A}\rangle_{\ldots}$.

The main involutions, namely, the reversion, the grade inversion and Clifford conjugation are denoted, respectively, by tilde, circumflex and their combination:

$$\begin{gathered} \widetilde{\mathsf{A}} = a_0 + \mathbf{a} - \mathcal{A} - a_{123}I, \quad \widehat{\mathsf{A}} = a_0 - \mathbf{a} + \mathcal{A} - a_{123}I, \\ \widetilde{\widehat{\mathsf{A}}} = a_0 - \mathbf{a} - \mathcal{A} + a_{123}I. \end{gathered}$$

2.1 General properties of GA logarithm. Determinant norm

The logarithm of MV is another MV that belongs to the same geometric algebra. The defining equation is $\log(\mathrm{e}^{\mathsf{A}}) = \mathsf{A}$, where $\mathsf{A} \in Cl_{p,q}$. The GA logarithm is a multivalued function with the properties:

$$\log(\mathsf{AB}) = \log\mathsf{A} + \log\mathsf{B} \quad \text{if } \mathsf{AB} = \mathsf{BA},$$
$$\mathrm{e}^{\log\mathsf{A}} = \mathsf{A}, \qquad \mathrm{e}^{-\log\mathsf{A}} = \mathsf{A}^{-1},$$
$$\widetilde{\log\mathsf{A}} = \log\widetilde{\mathsf{A}}, \qquad \widehat{\log\mathsf{A}} = \log\widehat{\mathsf{A}}, \qquad \widetilde{\widehat{\log\mathsf{A}}} = \log\widetilde{\widehat{\mathsf{A}}},$$
$$\mathsf{V}(\log\mathsf{A})\mathsf{V}^{-1} = \log\bigl(\mathsf{VAV}^{-1}\bigr).$$

The last expression shows that the transformation V, for example, the rotor, can be pushed inside the logarithm.

GA logarithm is a multivalued function with period 2π. To account for quadrant sign in GA (in complex plane) properly, we shall need the double argument arc tangent function $\arctan(x, y)$, $x, y \in \mathbb{R}$, as given in the *Mathematica*, the properties of which are briefly mentioned below. The double argument arc tangent principal values lie in interval $\theta = [-\pi, \pi)$. Thus, one can write $\arctan(x, y) = \arctan(\cos\theta, \sin\theta)$, where x and y are orthogonal axes, and θ is an angle between x-axis and vector attached to the coordinate center. If the vector is rotated from x-axis anticlockwise, then $\theta = (0, \dots, \pi]$. So a jump in the double arc tangent value and associated branching occurs on the negative side of x-axis rather than on y-axis as is in the standard single argument case. When x, y are replaced by real numbers, *Mathematica* automatically switches to a single argument arc tangent in the first quadrant (and respective principal value). For example, $\arctan(17, 10) = \arctan(10/17)$, $\arctan(-17, 10) = \pi - \arctan(10/17)$, $\arctan(17, -10) = -\arctan(10/17)$, $\arctan(-17, -10) = -\pi + \arctan(10/17)$.

As we shall see, in special cases the logarithm is controlled by determinant norm of MV B. It is defined as an absolute value of MV determinant [14] $\operatorname{Det}\mathsf{B}$ raised to fractional power $1/k$, where $k = 2^{\lceil n/2\rceil}$, i.e., $(\operatorname{Det}\mathsf{B})^{1/k} \equiv |\mathsf{B}| > 0$. For algebras having negative determinant, instead a seminorm is introduced $|\mathsf{B}| = (\operatorname{abs}(\operatorname{Det}\mathsf{B}))^{1/k} \geqslant 0$, where the equality sign means that in case of seminorm the determinant may be zero although $\mathsf{B} \neq 0$. In the following the same symbol will be used for both the norm and seminorm. The (semi)norm power k can be interpreted as a number of multipliers needed to define $\operatorname{Det}\mathsf{B}$. In 3D algebras ($n = 3$), we have $k = 2^{\lceil 3/2\rceil} = 2^2 = 4$. The integer k coincides with the number of multipliers in the 3D determinant: $\operatorname{Det}\mathsf{B} = \mathsf{B}\widetilde{\mathsf{B}}\widehat{\mathsf{B}}\widetilde{\widehat{\mathsf{B}}}$. The determinant norm for MV B in 3D algebras, therefore, is $|\mathsf{B}| = \sqrt[4]{\operatorname{abs}(\operatorname{Det}\mathsf{B})}$. It can be shown that for arbitrary GA, which holds a basis element with property $\mathbf{e}_i^2 = -1$, by adding a scalar one can construct a MV the norm of which may be identified with a module of a complex number. For example, in $Cl_{3,0}$ the norm of $\mathsf{B} = 1 + \mathbf{e}_{12}$ is $\sqrt{(1 + \mathbf{e}_{12})(1 - \mathbf{e}_{12})} = \sqrt{2}$, which coincides with $|\mathsf{B}| = \sqrt[4]{\operatorname{abs}(\operatorname{Det}\mathsf{B})} = \sqrt{2}$.

3 MV logarithms in $Cl_{0,3}$

3.1 Logarithm formula for generic MV

The term "generic" here will be understood as a typical case where MV coefficients are not causing the problems [10]. If for a given set of MV coefficients, the generic formula is not applicable, for example, due to zero denominator or appearance of an undefined subexpression like $\arctan(0,0)$, we will refer to as "a special or specific case". Special cases will be covered by more elaborate formulas later.

Theorem 1 [Logarithm of multivector in $Cl_{0,3}$]. *The generic logarithm of MV* $\mathsf{A} = a_0 + \mathbf{a} + \mathcal{A} + a_{123}I$ *is the MV given by*

$$\log \mathsf{A} = \frac{1}{2}\big(\mathsf{A}_{0_+} + \mathsf{A}_{0_-} + \mathsf{A}_{1,2_+} + \mathsf{A}_{1,2_-} + (\mathsf{A}_{0_+} - \mathsf{A}_{0_-})I\big) \tag{1}$$

with

$$\mathsf{A}_{0_+} = \frac{1}{2}\log\big((a_0 + a_{123})^2 + a_+^2\big), \quad a_+ \neq 0, \tag{2}$$

$$\mathsf{A}_{0_-} = \frac{1}{2}\log\big((a_0 - a_{123})^2 + a_-^2\big), \quad a_- \neq 0, \tag{3}$$

$$\mathsf{A}_{1,2_+} = \frac{1}{a_+}\big(\arctan(a_0 + a_{123},\, a_+) + 2\pi c_{1+}\big)(1+I)(\mathbf{a} + \mathcal{A}), \quad a_+ \neq 0, \tag{4}$$

$$\mathsf{A}_{1,2_-} = \frac{1}{a_-}\big(\arctan(a_0 - a_{123},\, a_-) + 2\pi c_{1-}\big)(1-I)(\mathbf{a} + \mathcal{A}), \quad a_- \neq 0. \tag{5}$$

The MVs $\mathsf{A}_{0_\pm}$, $\mathsf{A}_{1,2_\pm}$ *and* $\mathsf{A}_{0_\pm}I$ *denote, respectively, the scalar, vector* $\pm$ *bivector and the pseudoscalar components.* $c_{1_\pm}, c_{2_\pm} \in \mathbb{Z}$ *are arbitrary integers. The scalars* $a_+ \geqslant 0$ *and* $a_- \geqslant 0$ *are given by the following expressions* [2, 8]:

$$\begin{aligned} a_- &= \sqrt{-(\mathbf{a}\cdot\mathbf{a} + \mathcal{A}\cdot\mathcal{A}) + 2I\mathbf{a}\wedge\mathcal{A}} \\ &= \sqrt{(a_3 + a_{12})^2 + (a_2 - a_{13})^2 + (a_1 + a_{23})^2}, \end{aligned} \tag{6}$$

$$\begin{aligned} a_+ &= \sqrt{-(\mathbf{a}\cdot\mathbf{a} + \mathcal{A}\cdot\mathcal{A}) - 2I\mathbf{a}\wedge\mathcal{A}} \\ &= \sqrt{(a_3 - a_{12})^2 + (a_2 + a_{13})^2 + (a_1 - a_{23})^2}. \end{aligned} \tag{7}$$

Proof. It is enough to check that after substitution of (1) into generic GA exponential formula (1) in [8], one gets the initial MV A. □

Theorem 1 presents the GA logarithm in a basis-free form. However, the derivation of the generic logarithm formula at first was performed in a coordinate form [6], from which, after collection and simplification of coefficients at different grades, Theorem 1 comes next (for more details, see the preprint [6]).

Theorem 1 ensures the existence of GA logarithm for MVs with real and nonzero coefficients in $Cl_{0,3}$ because in the mentioned algebra the zero determinant of MV ($\operatorname{Det}\mathsf{A} = 0$) occurs only if $\mathsf{A} = 0$. As we shall see, this property does not hold for remaining algebras.

3.2 Special cases

In Theorem 1, it is presumed that the both scalars a_- and a_+ do not vanish. This assumption is equivalent to the condition that a sum of vector and bivector must have nonzero determinant, $\operatorname{Det}(\mathbf{a} + \mathcal{A}) = a_+^2 a_-^2 \neq 0$. If either of the scalars is zero, then we have a special case. This situation is rare, for instance[1], when $a_1 = a_{23}$, $a_2 = -a_{13}$, $a_3 = a_{12}$. In such and similar cases the MVs $\mathsf{A}_{0_\pm}$, $\mathsf{A}_{1,2_\pm}$ in Theorem 1 must be supplemented by additional conditions:

$$\mathsf{A}_{0_+} = \begin{cases} \log(a_0 + a_{123}) + 2\pi c_{2_+}\hat{\mathcal{U}}, & (a_+ = 0) \wedge (a_0 + a_{123} > 0), \\ \log(0_+), & (a_+ = 0) \wedge (a_0 + a_{123} = 0), \\ \log(-(a_0 + a_{123}) + (\pi + 2\pi c_{2_+})\hat{\mathbf{u}}, & (a_+ = 0) \wedge (a_0 + a_{123} < 0), \end{cases} \tag{8}$$

$$\mathsf{A}_{0_-} = \begin{cases} \log(a_0 - a_{123}) + 2\pi c_{2_-}\hat{\mathcal{U}}, & (a_- = 0) \wedge (a_0 - a_{123} > 0), \\ \log(0_+), & (a_- = 0) \wedge (a_0 - a_{123} = 0), \\ \log(-(a_0 - a_{123})) + (\pi + 2\pi c_{2_-})\hat{\mathbf{u}}, & (a_- = 0) \wedge (a_0 - a_{123} < 0), \end{cases} \tag{9}$$

$$\mathsf{A}_{1,2_+} = \begin{cases} (\frac{1}{a_0 + a_{123}} + 2\pi c_{1_+})(1 + I)(\mathbf{a} + \mathcal{A}), & (a_+ = 0) \wedge (a_0 + a_{123} > 0), \\ 0, & (a_+ = 0) \wedge (a_0 + a_{123} = 0), \\ (\pi + 2\pi c_{1_+})(1 + I)(\mathbf{a} + \mathcal{A}), & (a_+ = 0) \wedge (a_0 + a_{123} < 0), \end{cases} \tag{10}$$

$$\mathsf{A}_{1,2_-} = \begin{cases} (\frac{1}{a_0 - a_{123}} + 2\pi c_{1_-})(1 - I)(\mathbf{a} + \mathcal{A}), & (a_- = 0) \wedge (a_0 - a_{123} > 0), \\ 0, & (a_- = 0) \wedge (a_0 - a_{123} = 0), \\ (\pi + 2\pi c_{1_-})(1 - I)(\mathbf{a} + \mathcal{A}), & (a_- = 0) \wedge (a_0 - a_{123} < 0). \end{cases} \tag{11}$$

Here $c_{1_\pm}, c_{2_\pm} \in \mathbb{Z}$ are the arbitrary integers. The conditions for $(a_0 \pm a_{123})$ on the right-hand side take into account the case $\mathrm{Det}(\mathbf{a} + \mathcal{A}) = 0$. In scalars[2] A_{0_+} and A_{0_-}, the symbols $\hat{\mathbf{u}}$ and $\hat{\mathcal{U}}$ represent any free unit vector or bivector, respectively, $\hat{\mathbf{u}}^2 = \hat{\mathcal{U}}^2 = -1$, for example, $\hat{\mathbf{u}} = (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)/\sqrt{u_1^2 + u_2^2 + u_2^3}$. It should be noted that the term $1/(a_0 \pm a_{123})$ in Eqs. (10) and (11) represents the limit $\lim_{a_\pm \to 0} \arctan(a_0 \pm a_{123},\, a_\pm)/a_\pm = 1/(a_0 \pm a_{123})$, which is valid only when $a_0 \pm a_{123} > 0$. The notation of $\log(0_+)$ in formulas for $\mathsf{A}_{0_\pm}$ is explained in Example 4.

Interpretation of special conditions (8)–(11) in terms of the MV determinant [11, 14] becomes more evident if one remembers that the determinant of MV A in $Cl_{0,3}$ can be expressed in a form $\mathrm{Det}\,\mathsf{A} = (a_-^2 + (a_0 - a_{123})^2)(a_+^2 + (a_0 + a_{123})^2)$, whereas the condition $a_\pm = 0$ is equivalent to $\mathrm{Det}\,\mathsf{A}_{1,2} = \mathrm{Det}(\mathbf{a} + \mathcal{A}) = a_+^2 a_-^2$. All special cases therefore occur if $\mathrm{Det}(\mathbf{a} + \mathcal{A}) = 0$ and the conditions are expressed by $a_0 \pm a_{123} \lesseqqgtr 0$. In conclusion, the expression for logarithm has three special pieces (branches) $a_0 \pm a_{123} \lesseqqgtr 0$, provided that the condition $a_\pm = 0$ is satisfied. The generic piece is characterized by $a_\pm \neq 0$.

3.3 Multivaluedness and free multivector

To include a multivaluedness of GA logarithm, we introduce a free multivector F by defining equation [9]

$$\mathrm{e}^{\log \mathsf{A} + \mathsf{F}} = \mathrm{e}^{\log \mathsf{A}}\mathrm{e}^{\mathsf{F}} = \mathrm{e}^{\log \mathsf{A}},$$

[1]Minus sign in $a_2 = -a_{13}$ comes from a strict ordering of indices of basis elements [8].

[2]The incorporation of free vector/bivector breaks the convention of grade arrangement in generic terms (2) and (3). This choice, however, results in a more simple final expression.

which implies two conditions that the MV F must satisfy: the commutator $[\log \mathsf{A}, \mathsf{F}] = 0$ and $\mathrm{e}^{\mathsf{F}} = 1$. As we shall see, for remaining $n = 3$ algebras, the free MV F will play a similar role. One can check that the expression

$$\mathsf{F} = \frac{\pi c_{1_+}}{a_+}(1 + I)(\mathbf{a} + \mathcal{A}) + \frac{\pi c_{1_-}}{a_-}(1 - I)(\mathbf{a} + \mathcal{A}) \tag{12}$$

satisfies $\mathrm{e}^{\mathsf{F}} = 1$, and that for a generic MV A, Eq. (1), the free term (12) commutes with $\log \mathsf{A}$, i.e., $[\log \mathsf{A}, \mathsf{F}] = 0$. The constants $c_{1_+}, c_{1_-} \in \mathbb{Z}$ in Eq. (12) add two free (discrete) parameters that may be used to shift the coefficients of vector and bivector in $\log \mathsf{A}$ by some multiple of π. The sum $(\mathbf{a} + \mathcal{A})$ in (12) constitute a vector + bivector part[3] of the original MV A, therefore $(\mathbf{a} + \mathcal{A})$ automatically commutes with A. As a result, only discrete free coefficients are possible in the logarithm generic formula. In special cases the free MV F may also contain arbitrary unit vector $\hat{\mathbf{u}}$ and/or unit bivector $\hat{\mathcal{U}}$. In such cases, one can include two additional continuous parameters interpreted as directions of $\hat{\mathbf{u}}$ or $\hat{\mathcal{U}}$.

Since $\arctan(x, y)$ has been defined in the range $(-\pi, \pi]$, we can add any multiple of 2π to it. Therefore, the plus/minus instances of $\arctan(a_0 \pm a_{123},\ a_\pm)$ were replaced by more general expressions $\arctan(a_0 + a_{123},\ a_+) + 2\pi c_{1_+}$ and $\arctan(a_0 - a_{123},\ a_-) + 2\pi c_{1_-}$ in Eqs. (4) and (5), respectively, which take into account the multivaluedness of the argument. This explains the rationale behind the construction of the free MVs for GA logarithm.

In [12] the notion of principal logarithm (also called the principal value of logarithm) in case of matrices was introduced. In [7], it was proposed that the "logarithm principal value in GA can be defined as the MV $\mathsf{M} = \log \mathsf{Y}$ with the smallest norm". Formulas (2)–(5) and (8)–(11) might suggest that we could obtain the principal logarithm after equating discrete free constants $c_{1_\pm}$, $c_{2_\pm}$ to zero. Unfortunately, our extensive numerical checks revealed that in rare cases the suggestion is violated.

Example 1 [*Logarithm of generic MV in $Cl_{0,3}$*]. Let us compute the logarithm of $\mathsf{A} = -8 - 6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23} - 4\mathbf{e}_{123}$. Then $a_-^2 = 53$ and $a_+^2 = 353$. Equations (2)–(5) give $\mathsf{A}_{0_+} = \log(497)/2$, $\mathsf{A}_{0_-} = \log(69)/2$, $\mathsf{A}_{1,2_+} = (353)^{-1/2} \times (\pi - \arctan(\sqrt{353}/12) + 2\pi c_{1_+})(1 + I)(-6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23})$ and $\mathsf{A}_{1,2_-} = (53)^{-1/2}(\pi - \arctan(\sqrt{53}/4) + 2\pi c_{1_-})(1 - I)(-6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23})$, where the free term F, Eq. (12), has been included via $c_{1_\pm}$. The logarithm is the sum of all above listed MVs: $\log \mathsf{A} = (\mathsf{A}_{0_+} + \mathsf{A}_{0_-} + \mathsf{A}_{1,2_+} + \mathsf{A}_{1,2_-} + (\mathsf{A}_{0_+} - \mathsf{A}_{0_-})I)/2$. Using the exponential [8], one can check that the logarithm $\log \mathsf{A}$ indeed yields the initial MV A for arbitrary integer constants $c_{1_\pm}$.

Example 2 [*Logarithm of MV when $a_+ = 0$ and $a_0 + a_{123} > 0$*]. The MV that satisfies these conditions is $\mathsf{A} = 1 + (3\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3) + (\mathbf{e}_{12} + 2\mathbf{e}_{13} + 3\mathbf{e}_{23}) + 7\mathbf{e}_{123} = 1 + \mathbf{a} + \mathcal{A} + 7\mathbf{e}_{123}$. Equation (6) gives $a_- = \sqrt{56} = 2\sqrt{14}$ and $a_0 - a_{123} = -6$. Equations (8), (10) give $\mathsf{A}_{0_+} = \log 8 + 2\pi c_{2_+}\hat{\mathcal{U}}$, $\mathsf{A}_{1,2_+} = (1/8 + 2\pi c_{1_+})(1 + I)(\mathbf{a} + \mathbf{e}_3 + \mathcal{A}) = 0$. Then

[3] In 3D algebras the scalar and pseudoscalar belong to algebra center, i.e., they commute with all MV elements.

from (3) and (5) we have $\mathsf{A}_{0_-} = \log(92)/2$ and $\mathsf{A}_{1,2_-} = (\pi - \arctan(\sqrt{14}/3) + 2\pi c_{1_-}) \times (1 - I)(\mathbf{a} + \mathcal{A})/(2\sqrt{14})$. Finally, from Eq. (1) $\log \mathsf{A} = (7(\log 5888 - \log(23/16)\mathbf{e}_{123}) + \sqrt{14}((2c_{1_-} + 1)\pi - \arctan(\sqrt{14}/3)(\mathbf{a} + \mathcal{A}))/28 + (1 + I)\pi c_{2_+}\hat{\mathcal{U}}$. After exponentiation of A, the constants c_{1_-} and c_{2_+} and bivector $\hat{\mathcal{U}}$ simplify out.

Example 3 [*Logarithm of MV when $a_- = 0$ and $a_0 - a_{123} < 0$*]. These conditions are satisfied by $\mathsf{A} = 1 + (-3\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3) + (\mathbf{e}_{12} + 2\mathbf{e}_{13} + 3\mathbf{e}_{23}) + 7\mathbf{e}_{123} = 1 + \mathbf{a} + \mathcal{A} + 7\mathbf{e}_{123}$. We have $a_+^2 = 56$, $a_0 + a_{123} = 9$ and $a_0 - a_{123} = -6$. Then Eq. (9) gives $\mathsf{A}_{0_-} = \log 6 + (\pi + 2\pi c_{2_-})\hat{\mathbf{u}}$. Equations (2) and (4) give $\mathsf{A}_{0_+} = \log(120)/2$, $\mathsf{A}_{1,2_+} = (\arctan(\sqrt{7/2}/2) + 2\pi c_{1_+})(1 + I)(\mathbf{a} + \mathcal{A})/(2\sqrt{14})$; and Eq. (11) gives $\mathsf{A}_{1,2_-} = (\pi + 2\pi c_{1_-})(1 - I)(\mathbf{a} + \mathcal{A}) = 0$. Finally, $\log \mathsf{A} = (\mathsf{A}_{0_+} + \mathsf{A}_{0_-} + \mathsf{A}_{1,2_+} + (\mathsf{A}_{0_+} - \mathsf{A}_{0_-})I)/2$.

Example 4 [*Logarithm with infinite subparts: the case $a_+ = 0$ and $a_0 + a_{123} = 0$*]. This example exhibits unusual and the most interesting instance. In $Cl_{0,3}$, let us compute GA logarithm of $\mathsf{A} = 1 + (-2\mathbf{e}_1 - 3\mathbf{e}_2 + 5\mathbf{e}_3) + (5\mathbf{e}_{12} + 3\mathbf{e}_{13} - 2\mathbf{e}_{23}) - \mathbf{e}_{123} = 1 + \mathbf{a} + \mathcal{A} - \mathbf{e}_{123}$. The remaining scalars are $a_- = 2\sqrt{38}$, $(a_0 - a_{123}) = 2$. Then Eq. (8) gives $\mathsf{A}_{0_+} = \log(0_+)$; Eq. (3) gives $\mathsf{A}_{0_-} = \log(156)/2$; Eq. (10) gives $\mathsf{A}_{1,2_+} = 0$; Eq. (5) gives $\mathsf{A}_{1,2_-} = (\arctan(\sqrt{38}) + 2\pi c_{1_-}(\mathbf{a} + \mathcal{A})/\sqrt{38}$. Finally, the logarithm of A is

$$\log \mathsf{A} = \frac{\arctan(\sqrt{38}) + 2\pi c_{1_-}}{2\sqrt{38}}(\mathbf{a} + \mathcal{A}) + \frac{1}{2}\Big(\log(0_+)(1 + \mathbf{e}_{123}) + \frac{1}{2}\log(156)(1 - \mathbf{e}_{123})\Big).$$

Note the factor $\log(0_+)$ in front of $(1 + \mathbf{e}_{123})$. If logarithm in this form is inserted into coordinate-free exponential [2], we will get

$$\Big(\frac{1}{2}\mathrm{e}^{\log(0_+)} + 1\Big) + \mathbf{a} + \mathcal{A} + \Big(\frac{1}{2}\mathrm{e}^{\log(0_+)} - 1\Big)\mathbf{e}_{123},$$

which coincides with the initial MV if we assume that[4] $\log(0_+) = -\infty$.

3.4 GA logarithm of blades and their combinations in $Cl_{0,3}$

In this subsection the logarithms for individual blades and their combinations that follow from generic logarithm (Theorem 1) and may be useful in practice are collected. The norms listed below are positive scalars.

Vector norm: $|\mathbf{a}| = \sqrt{\mathbf{a}\widehat{\mathbf{a}}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
Paravector norm: $|a_0 + \mathbf{a}| = |\mathsf{A}_{0,1}| = (\mathsf{A}_{0,1}\widehat{\mathsf{A}}_{0,1})^{1/2} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$.
Bivector norm: $|\mathcal{A}| = (\mathcal{A}\widetilde{\mathcal{A}})^{1/2} = \sqrt{a_{12}^2 + a_{13}^2 + a_{23}^2}$.
Rotor norm: $|a_0 + \mathcal{A}| = |\mathsf{A}_{0,2}| = (\mathsf{A}_{0,2}\widetilde{\mathsf{A}}_{0,2})^{1/2} = \sqrt{a_0^2 + a_{12}^2 + a_{13}^2 + a_{23}^2}$.

[4]The statement can be made strict by considering the limit $\lim_{x\to 0_+} \exp(\log x) = 0$, where $x \to 0_+$ indicates that the limit is taken for $x > 0$.

Logarithms of blades and their combinations.

Logarithm of vector $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $c_i \in \mathbb{Z}$,

$$\log \mathbf{a} = \frac{1}{2}\log\bigl(|\mathbf{a}|^2\bigr) + \pi\frac{\mathbf{a}}{|\mathbf{a}|}\biggl(\frac{1}{2} + c_1(1+I) + c_2(1-I)\biggr), \quad |\mathbf{a}| \neq 0.$$

Logarithm of paravector $\mathsf{A}_{0,1} = a_0 + \mathbf{a}$, $c_i \in \mathbb{Z}$,

$$\begin{aligned}\log \mathsf{A}_{0,1} = {}& \frac{1}{2}\log\bigl(|\mathsf{A}_{0,1}|^2\bigr) \\ &+ \frac{\mathbf{a}}{|\mathbf{a}|}\bigl(\arctan(a_0, |\mathbf{a}|) + \pi\bigl(c_1(1+I) + c_2(1-I)\bigr)\bigr), \quad |\mathbf{a}| \neq 0.\end{aligned}$$

Logarithm of bivector $\mathcal{A} = a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23}$, $c_i \in \mathbb{Z}$,

$$\log \mathcal{A} = \frac{1}{2}\log\bigl(|\mathcal{A}|^2\bigr) + \pi\frac{\mathcal{A}}{|\mathcal{A}|}\biggl(\frac{1}{2} + c_1(1+I) + c_2(1-I)\biggr), \quad |\mathcal{A}| \neq 0.$$

Logarithm of parabivector and rotor $\mathsf{A}_{0,2} = a_0 + \mathcal{A}$ with $a_0 \neq 0$,

$$\begin{aligned}\log \mathsf{A}_{0,2} = {}& \frac{1}{2}\log\bigl(|\mathsf{A}_{0,2}|^2\bigr) \\ &+ \frac{\mathcal{A}}{|\mathcal{A}|}\bigl(\arctan(a_0, |\mathcal{A}|) + \pi(c_1(1+I) + c_2(1-I))\bigr), \quad |\mathcal{A}| \neq 0.\end{aligned}$$

Logarithm of center $\mathsf{A}_{0,3} = a_0 + a_{123}I$,

$$\log \mathsf{A}_{0,3} = \begin{cases} (\frac{1}{2}\log(a_0 - a_{123}) + \pi c_1\hat{\mathcal{U}}_1)(1-I) \\ \quad +(\frac{1}{2}\log(a_0 + a_{123}) + \pi c_2\hat{\mathcal{U}}_2)(1+I), \\ (a_0 - a_{123} > 0) \wedge (a_0 + a_{123} > 0); \\ (\frac{1}{2}\log(a_0 - a_{123}) + \pi c_1\hat{\mathcal{U}}_1)(1-I) \\ \quad +(\frac{1}{2}\log(-a_0 - a_{123}) + \pi(c_2 + \frac{1}{2})\hat{\mathbf{u}}_2)(1+I), \\ (a_0 - a_{123} > 0) \wedge (a_0 + a_{123} < 0); \\ (\frac{1}{2}\log(-a_0 + a_{123}) + \pi(c_1 + \frac{1}{2})\hat{\mathbf{u}}_1)(1-I) \\ \quad +(\frac{1}{2}\log(a_0 + a_{123}) + \pi c_2\hat{\mathcal{U}}_2)(1+I), \\ (a_0 - a_{123} < 0) \wedge (a_0 + a_{123} > 0); \\ (\frac{1}{2}\log(-a_0 + a_{123}) + \pi(c_1 + \frac{1}{2})\hat{\mathbf{u}}_1)(1-I) \\ \quad +(\frac{1}{2}\log(-a_0 - a_{123}) + \pi(c_2 + \frac{1}{2})\hat{\mathbf{u}}_2)(1+I), \\ (a_0 - a_{123} < 0) \wedge (a_0 + a_{123} < 0), \end{cases}$$

where $\hat{\mathbf{u}}_i$ and $\hat{\mathcal{U}}_j$ are arbitrary noncommuting unit vector $\hat{\mathbf{u}}^2 = -1$ and bivector $\hat{\mathcal{U}}^2 = -1$ respectively. If $(a_0 - a_{123}) = 0$ or $(a_0 + a_{123}) = 0$, some of subparts give $\log(0_+)$.

4 MV logarithms in $Cl_{3,0}$ and $Cl_{1,2}$

$Cl_{3,0}$ and $Cl_{1,2}$ algebras are isomorphic. Their multiplication tables coincide, for example, if basis elements are swapped by pairs:

$$\begin{array}{ll} Cl_{3,0} & \{1,\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_{12},\ \mathbf{e}_{13},\ \mathbf{e}_{23},\ \mathbf{e}_{123}\}\downarrow \\ Cl_{1,2} & \{1,-\mathbf{e}_1,-\mathbf{e}_{12},-\mathbf{e}_{13},-\mathbf{e}_2,\ -\mathbf{e}_3,\ \mathbf{e}_{23},-\mathbf{e}_{123}\}. \end{array}$$

To find formulas for logarithm in coordinate and coordinate-free forms, the same inverse solution method was used as for $Cl_{0,3}$ algebra [6]. The logarithm in $Cl_{3,0}$ and $Cl_{1,2}$ exists for all MVs except for nonzero MVs of the form $\mathsf{A}_{1,2} = \mathbf{a} + \mathcal{A}$ that satisfy the condition $\operatorname{Det} \mathsf{A}_{1,2} = (a_+^2 + a_-^2)^2 = 0$, i.e., for MVs that are the sums of vector and bivector and the determinant is equal to zero. These restrictions are the same as those for GA square root to exist (see [1] and Example 3 therein in case $s = S = 0$).

4.1 Logarithm formula for generic MV

Theorem 2 [Logarithm of multivector in $Cl_{3,0}$ and $Cl_{1,2}$]. *The logarithm of generic MV $\mathsf{A} = a_0 + \mathbf{a} + \mathcal{A} + a_{123}I$ is another MV*

$$\log \mathsf{A} = \mathsf{A}_0 + \mathsf{A}_{1,2_{\log}} + \mathsf{A}_{1,2_{\arctan}} + \mathsf{A}_I, \tag{13}$$

where

$$\mathsf{A}_0 = \frac{1}{2}(\log k_+ + \log k_-) \tag{14}$$

when $a_+^2 + a_-^2 \neq 0$,

$$\mathsf{A}_{1,2_{\log}} = \frac{1}{2}\frac{a_+ - a_- I}{a_-^2 + a_+^2}(\log k_+ - \log k_-)(\mathbf{a} + \mathcal{A}) \tag{15}$$

when $a_+^2 + a_-^2 \neq 0$,

$$\begin{aligned} \mathsf{A}_{1,2_{\arctan}} = {} & I\frac{a_+ - a_- I}{a_-^2 + a_+^2}(\mathbf{a} + \mathcal{A}) \\ & \times \biggl(\frac{1}{2}\arctan\bigl(-\bigl(a_+^2 - a_0^2\bigr) - \bigl(a_-^2 - a_{123}^2\bigr), \\ & \quad (a_+ - a_0)(a_- + a_{123}) - (a_+ + a_0)(a_- - a_{123})\bigr) + 2\pi c_1\biggr) \end{aligned} \tag{16}$$

when $a_+^2 + a_-^2 \neq 0$ and $k_- k_+ \neq 0$,

$$\begin{aligned} \mathsf{A}_I = {} & I\arctan\bigl((a_+ + a_0)k_- - (a_+ - a_0)k_+, \\ & \qquad (a_- + a_{123})k_- - (a_- - a_{123})k_+\bigr) \\ & + 2\pi c_2 I \end{aligned} \tag{17}$$

when $a_+^2 + a_-^2 \neq 0$ *and either* $(a_+ + a_0)k_- - (a_+ - a_0)k_+ \neq 0$ *or* $(a_- + a_{123})k_- - (a_- - a_{123})k_+ \neq 0$*, where scalar coefficients are*

$$k_-^2 = (a_+ - a_0)^2 + (a_- - a_{123})^2, \qquad k_+^2 = (a_+ + a_0)^2 + (a_- + a_{123})^2, \tag{18}$$

and

$$\begin{aligned} a_- &= \frac{-2I\mathbf{a}\wedge\mathcal{A}}{\sqrt{2}\sqrt{\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A}+\sqrt{(\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A})^2-4(\mathbf{a}\wedge\mathcal{A})^2}}},\\ a_+ &= \frac{\sqrt{\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A}+\sqrt{(\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A})^2-4(\mathbf{a}\wedge\mathcal{A})^2}}}{\sqrt{2}} \end{aligned} \tag{19}$$

for $\mathbf{a}\wedge\mathcal{A}\neq 0$ *and*

$$\begin{aligned} &a_+ = \sqrt{\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A}}, \quad a_- = 0 && \textit{if } \mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A}\geqslant 0,\\ &a_+ = 0, \quad a_- = \sqrt{-(\mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A})} && \textit{if } \mathbf{a}\cdot\mathbf{a}+\mathcal{A}\cdot\mathcal{A}< 0 \end{aligned}$$

when $\mathbf{a}\wedge\mathcal{A} = 0$*. The constants* c_1*,* c_2 *are arbitrary integers.*

Proof. It is enough to check that after substitution of $\log \mathsf{A}$ expressions into exponential formula presented in [8], one gets the initial MV A. The factor $(a_+ - a_-I)/(a_-^2 + a_+^2)$ in the above formulas alternatively may be written as $(a_+ + a_-I)^{-1}$. □

4.2 Special cases

When the conditions listed in Eqs. (14)–(17) are not satisfied, we have special cases. In particular, the condition $k_\pm = 0$ means that the MV determinant is zero, $\operatorname{Det}\mathsf{A} = k_-^2 k_+^2 = 0$. Similarly, the condition $a_+^2 + a_-^2 = 0$ implies that determinant of vector + bivector part vanishes, $\operatorname{Det}\mathsf{A}_{1,2} = (a_+^2 + a_-^2)^2 = 0$. The specific relations $(a_+ + a_0)k_- - (a_+ - a_0)k_+ \neq 0$ and $(a_- + a_{123})k_- - (a_- - a_{123})k_+ \neq 0$ in Eq. (17), as well as the relation $k_-k_+ \neq 0$ in Eq. (16), ensure that both arguments of $\arctan(x, y)$ do not nullify simultaneously.

When the generic formula is not applicable, the expressions for A_0, $\mathsf{A}_{1,2_{\log}}$, $\mathsf{A}_{1,2_{\arctan}}$ and A_I in Theorem 2 must be supplemented by the following formulas:

$$\mathsf{A}_0 = \begin{cases} \frac{1}{2}\log(a_0^2 + a_{123}^2), & (a_+^2 + a_-^2 = 0)\wedge(a_0^2 + a_{123}^2 \neq 0),\\ \emptyset, & (a_+^2 + a_-^2 = 0)\wedge(a_0^2 + a_{123}^2 = 0), \end{cases} \tag{20}$$

$$\mathsf{A}_{1,2_{\log}} = \begin{cases} 0, & (a_+^2 + a_-^2 = 0)\wedge(a_0^2 + a_{123}^2 \neq 0),\\ \emptyset, & (a_+^2 + a_-^2 = 0)\wedge(a_0^2 + a_{123}^2 = 0), \end{cases} \tag{21}$$

$$\mathsf{A}_{1,2_{\arctan}} = \begin{cases} \pi(\frac{1}{2} + 2c_1)I\frac{a_+ - a_-I}{a_-^2 + a_+^2}(\mathbf{a}+\mathcal{A}), & (a_+^2 + a_-^2 \neq 0)\wedge(k_-k_+ = 0),\\ \frac{a_0 - a_{123}I}{a_0^2 + a_{123}^2}(\mathbf{a}+\mathcal{A}) + \hat{\mathcal{F}}, & (a_+^2 + a_-^2 = 0)\wedge(a_0^2 + a_{123}^2 \neq 0),\\ \emptyset, & (a_+^2 + a_-^2 = 0)\wedge(a_0^2 + a_{123}^2 = 0), \end{cases} \tag{22}$$

$$\mathsf{A}_I = \begin{cases} I(\arctan(-a_-, a_+) + 2\pi c_2), & (a_+^2 + a_-^2 \neq 0) \\ & \wedge((a_+ + a_0)k_- - (a_+ - a_0)k_+ = 0) \\ & \wedge((a_- + a_{123})k_- - (a_- - a_{123})k_+ = 0), \\ I(\arctan(a_0, a_{123}) + 2\pi c_2), & (a_+^2 + a_-^2 = 0) \wedge (a_0^2 + a_{123}^2 \neq 0), \\ \emptyset, & (a_+^2 + a_-^2 = 0) \wedge (a_0^2 + a_{123}^2 = 0). \end{cases} \tag{23}$$

Here the symbol $\wedge$ in the conditions represents logical conjunction. The free unit MV $\hat{\mathcal{U}}$ in

$$\hat{\mathcal{F}} = \begin{cases} 2\pi c_1 \hat{\mathcal{U}} & \text{if } \mathbf{a} + \mathcal{A} = 0, \\ 0 & \text{if } \mathbf{a} + \mathcal{A} \neq 0 \end{cases}$$

must satisfy $\hat{\mathcal{U}}^2 = -1$. After exponentiation, it gives $\exp \hat{\mathcal{U}} = 1$ and represents a continuous degree of freedom (direction) in (22), and it can be parameterized as

$$\hat{\mathcal{U}} = \begin{cases} \frac{d_{12}\mathbf{e}_{12} + d_{13}\mathbf{e}_{13} + d_{23}\mathbf{e}_{23}}{\sqrt{d_{12}^2 + d_{13}^2 + d_{23}^2}} & \text{for } Cl_{3,0}, \\ \frac{d_{12}\mathbf{e}_{12} + d_{13}\mathbf{e}_{13} + d_{23}\mathbf{e}_{23}}{\sqrt{-d_{12}^2 - d_{13}^2 + d_{23}^2}} & \text{for } Cl_{1,2} \text{ when } -d_{12}^2 - d_{13}^2 + d_{23}^2 > 0. \end{cases}$$

The case $k_\pm = 0$ that represent MV with a vanishing determinant, $\operatorname{Det} \mathsf{A} = k_-^2 k_+^2 = 0$, yields MVs with infinite coefficients (see Example 7 for details).

4.3 Multiveluedness and free multivector

In Eqs. (16) and (17), we may add any multiple of 2π to both arc tangent functions, i.e., $\arctan(y_1, y_2) \to \arctan(y_1, y_2) + 2\pi c_i$. After collecting terms in front of free coefficients $c_1, c_2 \in \mathbb{Z}$, we obtain a free MV F that satisfies $\exp \mathsf{F} = 1$, and where a_+ and a_- are given by Eq. (19),

$$\mathsf{F} = \frac{2\pi c_1}{(a_-^2 + a_+^2)} \big(a_-(\mathbf{a} + \mathcal{A}) + a_+(\mathbf{a} + \mathcal{A})I\big) + 2\pi c_2 I. \tag{24}$$

Example 5 [*Logarithm of generic MV in* $Cl_{3,0}$]. Let us take simple but representative MV: $\mathsf{A} = -2 + \mathbf{e}_1 + \mathbf{e}_{23} - 3\mathbf{e}_{123}$. From Eqs. (18) and (19) we have $k_+^2 = 5$, $k_-^2 = 25$ and $a_+ = a_- = 1$. Then (14) and (15) yield $\mathsf{A}_0 = 3\log(5)/4$ and $\mathsf{A}_{1,2_{\log}} = -\log(5)(\mathbf{e}_1 + \mathbf{e}_{23}) \times (1-I)/8$. Next, Eqs. (16) and (17) give $\mathsf{A}_{1,2_{\arctan}} = -(-\arctan(2/11) + 4\pi c_2)(\mathbf{e}_1 + \mathbf{e}_{23}) \times (1+I)/4$ and $\mathsf{A}_I = (-\pi + \arctan((-10-4\sqrt{5})/(-5-3\sqrt{5})) + 2\pi c_1)\mathbf{e}_{123}$. Finally, after summation of all terms in (13), we obtain $\log \mathsf{A} = \log(5)(3-\mathbf{e}_1)/4 + \arctan(2/11)\mathbf{e}_{23}/2 + (-\pi + \arctan((1+\sqrt{5})/2)\mathbf{e}_{123} + \mathsf{F}$, where the free MV is $\mathsf{F} = 2\pi(c_1\mathbf{e}_{123} - c_2\mathbf{e}_{23})$. The coefficients $c_1, c_2 \in \mathbb{Z}$ come from $\mathsf{A}_{1,2_{\arctan}}$ and A_I terms, respectively. Substitution of this result into exponential $\exp(\log \mathsf{A})$ returns the initial MV.

Example 6 [*Logarithm of center of* $Cl_{3,0}$]. $\mathsf{A} = 1 - 2\mathbf{e}_{123}$. Since $\mathbf{e}_{123}^2 = -1$, the MV is a counterpart of complex number logarithm. Equations (18) and (19) give $a_+ = a_- = 0$ and $k_+^2 = k_-^2 = 5$. Then Eq. (20) gives $\mathsf{A}_0 = \log(5)/2$; Eq. (21) gives $\mathsf{A}_{1,2_{\log}} = 0$; Eq. (22) gives $\mathsf{A}_{1,2_{\arctan}} = 2\pi c_1 \hat{\mathcal{U}}$; Eq. (23) gives $\mathsf{A}_I = (-\arctan 2 + 2\pi c_2)\mathbf{e}_{123}$. After summation of terms in (13), the final answer is $\log \mathsf{A} = \log(5)/2 + (-\arctan 2 + 2\pi c_2)\mathbf{e}_{123} + 2\pi c_1 \hat{\mathcal{U}}$. On the other hand, the complex number $1 - 2\mathrm{i}$ gives $\log(1 - 2\mathrm{i}) = \log(5)/2 - \arctan 2$, which coincides with $Cl_{3,0}$ algebra result if $c_1 = c_2 = 0$.

Example 7 [*Logarithm of singular MV when* $\operatorname{Det}\mathsf{A} = 0$]. This is the most intriguing and complicated case in Euclidean algebra $Cl_{3,0}$. Since $\operatorname{Det}(\mathsf{A}) = k_-^2 k_+^2$, we may have either $k_-^2 = 0$ or $k_+^2 = 0$. The case when $k_-^2 = k_+^2 = 0$ is trivial since it requires all MV components to vanish. Let us analyze the case when $k_+^2 \neq 0$ and $k_-^2 = 0$. It is represented, for example, by $\mathsf{A} = 6 + (-8\mathbf{e}_1 - 2\mathbf{e}_3) + (-\mathbf{e}_{12} + 10\mathbf{e}_{13} + 10\mathbf{e}_{23}) - 13\mathbf{e}_{123} = 6 + \mathbf{a} + \mathcal{A} - 13\mathbf{e}_{123}$. From Eq. (19) we find $a_+ = 6$, $a_- = -13$, and from (18) $k_+^2 = 820$, $k_-^2 = 0$. Then Eq. (14) gives $\mathsf{A}_0 = (\log(2\sqrt{205}) + \log(0_+))/2$; Eq. (15) gives $\mathsf{A}_{1,2_{\log}} = (\log(2\sqrt{205}) - \log(0_+))(6 + 13\mathbf{e}_{123})(\mathbf{a} + \mathcal{A})/410$; Eq. (22) gives $\mathsf{A}_{1,2_{\arctan}} = \pi(1/2 + 2c_1)(-6 + 13\mathbf{e}_{123})(\mathbf{a} + \mathcal{A})/205$; Eq. (23) gives $\mathsf{A}_I = (\arctan(6/13) + 2\pi c_2)\mathbf{e}_{123}$. Summing up all terms, we obtain the answer:

$$\log \mathsf{A} = \frac{1}{2}\big(\log(2\sqrt{205}) + \log(0_+)\big) + \bigg(\frac{1}{410}\big(\log(2\sqrt{205}) - \log(0_+)\big)(6 + 13\mathbf{e}_{123}) + \frac{\pi}{205}\Big(\frac{1}{2} + 2c_1\Big)(-6 + 13\mathbf{e}_{123})\bigg)(\mathbf{a} + \mathcal{A}) + \Big(\arctan\frac{6}{13} + 2\pi c_2\Big)\mathbf{e}_{123}.$$

The result can be checked after replacement of $\log(0_+)$ by $\log x$ and substitution into exponential formula (4.1) of the paper [8]. After simplification, one can take the limit $\lim_{x\to 0_+} \exp(\log \mathsf{A})$, which returns the initial MV. This example demonstrates that the logarithm of MV with specific finite coefficients may yield MV with some of coefficients in the answer being infinite and which have to be understood as the limit $\lim_{x\to 0_+} \log x$. The answer, nevertheless, is meaningful since the substitution of the answer back into exponential formula and computation of the limit reproduces the initial MV.

4.4 Logarithms of individual blades and their combinations

Below we use different norms for individual blades of $Cl_{3,0}$ since a positive scalar for vectors and bivectors is calculated differently. For a vector, we will use

$$|\mathbf{a}| = \sqrt{\mathbf{a}\mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2};$$

for a bivector,

$$|\mathcal{A}| = (\mathcal{A}\widetilde{\mathcal{A}})^{1/2} = \sqrt{a_{12}^2 + a_{13}^2 + a_{23}^2};$$

for a rotor,

$$|a_0 + \mathcal{A}| = |\mathsf{A}_{0,2}| = (\mathsf{A}_{0,2}\widetilde{\mathsf{A}}_{0,2})^{1/2} = \sqrt{a_0^2 + a_{12}^2 + a_{13}^2 + a_{23}^2};$$

and for an element of center,

$$|a_0 + a_{123}I| = |\mathsf{A}_{0,3}| = (\mathsf{A}_{0,3}\widehat{\mathsf{A}}_{0,3})^{1/2} = \sqrt{a_0^2 + a_{123}^2}.$$

Logarithm of *vector*: $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$,

$$\log \mathbf{a} = \frac{1}{2}\log\big(|\mathbf{a}|^2\big) - \pi\Big(\frac{1}{2} + 2c_2\Big)\frac{\mathbf{a}}{|\mathbf{a}|}I + \pi\Big(\frac{1}{2} + 2c_1\Big)I, \quad |\mathbf{a}| \neq 0.$$

Logarithm of *bivector*: $\mathcal{A} = a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23}$,

$$\log\mathcal{A} = \frac{1}{2}\log\bigl(|\mathcal{A}|^2\bigr) - \pi\left(\frac{1}{2} + 2c_2\right)\frac{\mathcal{A}}{|\mathcal{A}|} + \pi(1+2c_1)I, \quad |\mathcal{A}| \neq 0. \tag{25}$$

Logarithm of *rotor*: $\mathsf{A}_{0,2} = a_0 + \mathcal{A}$,

$$\log\mathsf{A}_{0,2} = \begin{cases} \frac{1}{2}\log(|\mathsf{A}_{0,2}|^2) + (\arctan\bigl(a_0, 0\bigr) + 2\pi c_1)I \\ \quad + \frac{\mathcal{A}}{|\mathcal{A}|}(2\pi c_2 - \frac{1}{2}\arctan(a_0^2 - |\mathcal{A}|^2, -2a_0|\mathcal{A}|)), & |\mathsf{A}_{0,2}| \neq 0, \\ \log a_0 + 2\pi c_1 I, & (|\mathcal{A}| = 0) \wedge (a_0 \geqslant 0), \\ \log(-a_0) + 2\pi(c_1+1)I, & (|\mathcal{A}| = 0) \wedge (a_0 < 0), \\ \text{see bivector formula (25)}, & (|\mathcal{A}| \neq 0) \wedge (a_0 = 0). \end{cases}$$

Logarithm of *center*: $\mathsf{A}_{0,3} = a_0 + a_{123}\mathbf{e}_{123} = a_0 + a_{123}I$,

$$\log\mathsf{A}_{0,3} = \begin{cases} \frac{1}{2}\log(|\mathsf{A}_{0,3}|^2) + 2\pi c_2\hat{\mathcal{U}} + (\arctan(a_0, a_{123}) + 4\pi c_1)I, & |\mathsf{A}_{0,3}| \neq 0, \\ \log(0_+) + 2\pi c_2\hat{\mathcal{U}}, & |\mathsf{A}_{0,3}| = 0. \end{cases}$$

The paravector $\mathsf{A}_{0,1} = a_0 + \mathbf{a}$ norm $|a_0 + \mathbf{a}|^2 \equiv |\mathsf{A}_{0,1}|^2 = \mathsf{A}_{0,1}\widehat{\mathsf{A}}_{0,1} = a_0^2 - a_1^2 - a_2^2 - a_3^2$ contains coefficients with opposite signs. The logarithm formula, therefore, splits into many subcases and is impractical.

5 MV logarithms in $Cl_{2,1}$

In $Cl_{0,3}$ algebra the logarithm exists for all MVs. The logarithm in $Cl_{3,0}$ and $Cl_{1,2}$ algebras exist for almost all MVs except very small specific class $\mathbf{a}+\mathcal{A} \neq 0$ having a vanishing determinant, $\mathrm{Det}(\mathbf{a} + \mathcal{A}) = 0$. In $Cl_{2,1}$ algebra the logarithm does not exist for a large class of MVs. Of all three algebras, the logarithm of $Cl_{2,1}$ appeared technically the most hard to recover.

Theorem 3 [Logarithm of multivector in $Cl_{2,1}$]. *The logarithm of MV*

$$\begin{aligned} \mathsf{A} &= a_0 + (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) + (a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23}) + a_{123}I \\ &= a_0 + \mathbf{a} + \mathcal{A} + a_{123}I \end{aligned}$$

is

$$\log\mathsf{A} = \begin{cases} \frac{1}{2}(\mathsf{A}_{0_+} + \mathsf{A}_{0_-} + \mathsf{A}_{1,2_+} + \mathsf{A}_{1,2_-} + (\mathsf{A}_{0_+} - \mathsf{A}_{0_-})I), & f_\pm \geqslant 0, \\ \emptyset, & f_\pm < 0, \end{cases} \tag{26}$$

where

$$\begin{aligned} f_\pm &= (a_0 \pm a_{123})^2 + a_\pm^2, \quad f_\pm \lesseqqgtr 0, \\ a_-^{(2)} &= -(\mathbf{a}\cdot\mathbf{a} + \mathcal{A}\cdot\mathcal{A}) + 2I\mathbf{a}\wedge\mathcal{A}, \quad a_-^{(2)} \lesseqqgtr 0, \\ a_+^{(2)} &= -(\mathbf{a}\cdot\mathbf{a} + \mathcal{A}\cdot\mathcal{A}) - 2I\mathbf{a}\wedge\mathcal{A}, \quad a_+^{(2)} \lesseqqgtr 0, \end{aligned} \tag{27}$$

$$\mathsf{A}_{0_\pm} = \begin{cases} \frac{1}{2}\log(f_\pm), & (a_\pm^{(2)} > 0), \\ \frac{1}{2}\log(a_0 \pm a_{123} + \sqrt{-a_\pm^{(2)}}) \\ \quad + \frac{1}{2}\log(a_0 \pm a_{123} - \sqrt{-a_\pm^{(2)}}), & (a_\pm^{(2)} < 0) \wedge (a_0 \pm a_{123} > 0), \\ \log(a_0 \pm a_{123}) + 2\pi c_{2\pm}\hat{\mathcal{F}}, & (a_\pm^{(2)} = 0) \wedge (a_0 \pm a_{123} > 0), \\ \log(-(a_0 \pm a_{123})) + (\pi + 2\pi c_{2\pm})\hat{\mathcal{U}}, & (a_\pm^{(2)} = 0) \wedge (a_0 \pm a_{123} \leqslant 0) \\ & \quad \wedge(\mathfrak{D} = \text{True}), \\ \emptyset, & ((a_\pm^{(2)} < 0) \wedge (a_0 \pm a_{123} < 0)) \\ & \quad \vee((a_\pm^{(2)} = 0) \wedge (a_0 \pm a_{123} \leqslant 0) \\ & \quad \wedge(\mathfrak{D} = \text{False})), \end{cases} \tag{28}$$

$$\mathsf{A}_{1,2_\pm} = \begin{cases} \frac{1}{\sqrt{a_\pm^{(2)}}}(\arctan(a_0 \pm a_{123},\ a_\pm) \\ \quad + 2\pi c_{1\pm})(1 \pm I)(\mathbf{a} + \mathcal{A}), & (a_\pm^{(2)} > 0), \\ \frac{1}{\sqrt{-a_\pm^{(2)}}}\operatorname{artanh}(\frac{\sqrt{-a_\pm^{(2)}}}{a_0 \pm a_{123}})(1 \pm I) & (a_\pm^{(2)} < 0) \ \wedge (a_0 \pm a_{123} > 0) \\ \quad \times(\mathbf{a} + \mathcal{A}), & \quad \wedge(-a_\pm^{(2)} \neq (a_0 \pm a_{123})), \\ \frac{1}{2}(\log(a_0 \pm a_{123} + \sqrt{-a_\pm^{(2)}}) \\ \quad - \log(a_0 \pm a_{123} - \sqrt{-a_\pm^{(2)}})) & (a_\pm^{(2)} < 0) \wedge (a_0 \pm a_{123} > 0) \\ \quad \times \frac{1}{\sqrt{-a_\pm^{(2)}}}(1 \pm I)(\mathbf{a} + \mathcal{A}), & \quad \wedge(-a_\pm^{(2)} = (a_0 \pm a_{123})), \\ \frac{1}{a_0 \pm a_{123}}(1 \pm I)(\mathbf{a} + \mathcal{A}), & (a_\pm^{(2)} = 0) \wedge (a_0 \pm a_{123} > 0), \\ 0, & (a_\pm^{(2)} = 0) \wedge (a_0 \pm a_{123} \leqslant 0) \\ & \quad \wedge(\mathfrak{D} = \text{True}), \\ \emptyset, & ((a_\pm^{(2)} < 0) \wedge (a_0 \pm a_{123} < 0)) \\ & \quad \vee((a_\pm^{(2)} = 0) \wedge (a_0 \pm a_{123} \leqslant 0) \\ & \quad \wedge(\mathfrak{D} = \text{False})), \end{cases} \tag{29}$$

where the upper symbol in $a_\pm^{(2)}$ *indicates that* $a_\pm^{(2)}$ *consists of the squared coefficients* a_i^2, a_{ij}^2 *and* $a_i a_{ij}$,

$$\hat{\mathcal{F}} = \begin{cases} \hat{\mathcal{U}} & \text{if } \mathfrak{D} = \text{True}, \\ 0 & \text{if } \mathfrak{D} = \text{False}. \end{cases}$$

The logical condition $\mathfrak{D}$ *is a conjunction of outcomes of three comparisons*

$$\begin{aligned} \mathfrak{D} &= (a_1 = \pm a_{23}) \wedge (a_2 = \mp a_{13}) \wedge (a_3 = \mp a_{12}) \\ &\equiv \big((a_1 = a_{23}) \wedge (a_2 = -a_{13}) \wedge (a_3 = -a_{12})\big) \\ &\quad \vee \big((a_1 = -a_{23}) \wedge (a_2 = a_{13}) \wedge (a_3 = a_{12})\big) \end{aligned}$$

that should be applied to $\mathsf{A}_{0\pm}$ *and* $\mathsf{A}_{1,2\pm}$ *terms without paying attention to* $\pm$ *signs in their subscripts.* $\vee$ *denotes logical disjunction operation. Unit bivector in* $\mathsf{A}_{0\pm}$ *may be parameterized as* $\hat{\mathcal{U}} = (d_{12}\mathbf{e}_{12} + d_{13}\mathbf{e}_{13} + d_{23}\mathbf{e}_{23})/\sqrt{d_{12}^2 - d_{13}^2 - d_{23}^2}$ *for* $d_{12}^2 - d_{13}^2 - d_{23}^2 > 0$*. The symbol* $\emptyset$ *means that the solution set is empty. In all formulas the indices and conditions (except* $\mathfrak{D}$ *as stated explicitly) must be included with either all upper or with all lower signs.*

When $f_\pm \neq 0$ and $a_\pm^{(2)} > 0$, we have a generic case. When either $f_\pm = 0$ or $a_\pm^{(2)} \leqslant 0$, we have special cases. Note that in Eq. (27) the condition $f_\pm = 0$ implies $a_\pm^{(2)} \leqslant 0$. Also, observe that the condition $f_\pm \geqslant 0$ ensures that a less restrictive requirement $\operatorname{Det}\mathsf{A} = f_- f_+ \geqslant 0$ is fulfilled automatically.

Equations (28)–(29) are similar to Eqs. (8)–(11) for $Cl_{0,3}$ (see Section 3.2). Also, in (27) the expressions for scalar coefficients

$$a_\pm = \begin{cases} \sqrt{a_\pm^{(2)}}, & a_\pm^{(2)} \geqslant 0, \\ \sqrt{-a_\pm^{(2)}}, & a_\pm^{(2)} < 0 \end{cases}$$

are similar to Eqs. (6) and (7). The differences mainly arise at the parameter boundaries that define the existence of MV logarithm for $Cl_{2,1}$.

From our earlier calculations [1] we know the algebraic conditions that ensure an existence of MV square roots in $Cl_{2,1}$ algebra. Thus, we can rewrite and use here these conditions, which at the same time limit the extent of the logarithm in Theorem 3. It appears that quantities b_S and b_I introduced in [1] may be expressed in terms of multipliers f_+ and f_- in the determinant $D = \operatorname{Det}\mathsf{A} = f_- f_+$ as $b_I = (f_+ - f_-)/2$ and $b_S = (f_+ + f_-)/2$, where $f_\pm = (a_0 \pm a_{123})^2 + a_\pm^{(2)}$. Now, note that $f_\pm$ enter as arguments in log-functions of Theorem 3, Eq. (29). Therefore, the square root existence condition $b_S - \sqrt{D} \geqslant 0$ in [1], in terms of the logarithm problem, can be rewritten as a difference of the determinant factors, namely, $b_S - \sqrt{D} \Leftrightarrow (\sqrt{f_-} - \sqrt{f_+})^2/2$. Now it becomes clear that this condition is always satisfied and therefore can be ignored, once we assume that the both factors satisfy $f_- \geqslant 0$ and $f_+ \geqslant 0$. From all this we conclude that the requirement $f_\pm > 0$ constitutes one of the existence conditions of the logarithm in Theorem 3. Also, $b_S - \sqrt{D} = 0$ is equivalent to $f_+ = f_-$. This restricts the maximal possible value of $a_\pm^{(2)}$. In particular, $|a_\pm^{(2)}| \leqslant (a_0 \pm a_{123})^2$. Remember that (instead of $a_\pm$) the notation $a_\pm^{(2)}$ was introduced to keep an analogy with $Cl_{0,3}$ case. It may be negative $a_\pm^{(2)} < 0$ (see definition (27)), and therefore the notation, in general, cannot be interpreted as a square of scalar unless $a_\pm^{(2)} \geqslant 0$. When $a_\pm^{(2)} = 0$, an additional condition $a_0 \pm a_{123} \geqslant 0$ is required for logarithm to exist.

Since $Cl_{2,1}$ algebra is rarely used, we will not provide explicit formulas for pure blades (they can be found in the notebook `ElementaryFunctions.nb` in [5]). Also, because generic formulas are similar to those in $Cl_{0,3}$, the examples below are restricted to special cases only.

Example 8 [*Logarithm in* $Cl_{2,1}$ *when* $a_\pm^{(2)} = 0$ *and* $a_0 \pm a_{123} > 0$]. Let the MV be $\mathsf{A} = 7 + (2\mathbf{e}_1 + \mathbf{e}_2 + 3\mathbf{e}_3) + (2\mathbf{e}_{12} + 2\mathbf{e}_{13} - 2\mathbf{e}_{23}) + 5I = 7 + \mathbf{a} + \mathcal{A} + 5I$. From (27) we find $a_+^{(2)} = 0$, $f_+ = 144$ and $a_-^{(2)} = 0$, $f_- = 144$. Since $a_0 \pm a_{123} = 7 \pm 5 > 0$, from (28) we have $\mathsf{A}_{0_-} = \log 2$, $\mathsf{A}_{0_+} = \log 12$, and from (29) $\mathsf{A}_{1,2_-} = (1-I)(\mathbf{a}+\mathcal{A})/2$, $\mathsf{A}_{1,2_+} = (1+I)(\mathbf{a}+\mathcal{A})/12$. Finally, $\log \mathsf{A} = (12\log 24 + 24\mathbf{e}_1 + 17\mathbf{e}_2 + 31\mathbf{e}_3 + 29\mathbf{e}_{12} + 19\mathbf{e}_{13} - 24\mathbf{e}_{23} + 12\log(6)I)/24$. Note, because MV coefficients $a_3 \neq \pm a_{12}$, the condition $\mathfrak{D}$ is False, therefore the free MV in (28) is absent, $\hat{\mathcal{F}} = 0$.

Example 9 [*Logarithm when* $a_-^{(2)} = 0$, $a_0 - a_{123} = 0$ *and* $a_+^{(2)} > 0$, $a_0 + a_{123} < 0$]. In $Cl_{2,1}$, these properties are satisfied by MV $\mathsf{A} = -2 + (7\mathbf{e}_1 + 4\mathbf{e}_2 + 10\mathbf{e}_3) + (-10\mathbf{e}_{12} - 4\mathbf{e}_{13} + 7\mathbf{e}_{23}) - 2I = -2 + \mathbf{a} + \mathcal{A} - 2I$. From (27) we find $a_+^{(2)} = 140$, $f_+ = 156$ and $a_-^{(2)} = 0$, $f_- = 0$. Then, because $a_0 - a_{123} = -2 - (-2) = 0$ and $a_0 + a_{123} = -2 - 2 < 0$, from (28) we have $\mathsf{A}_{0_-} = \log(0_+) + (\pi + 2\pi c_{2_-})\hat{\mathcal{U}}$, $\mathsf{A}_{0_+} = \log(156)/2$. From (29) $\mathsf{A}_{1,2_-} = 0$, $\mathsf{A}_{1,2_+} = 2(\pi + 2\pi c_{1_+} - \arctan(\sqrt{35}/2))(1+I)(\mathbf{a}+\mathcal{A})/\sqrt{35}$. Then, using (26), we obtain the final answer $\log \mathsf{A} = \alpha_0 + \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 + \alpha_{12}\mathbf{e}_{12} + \alpha_{13}\mathbf{e}_{13} + \alpha_{23}\mathbf{e}_{23} + \alpha_{123}I$, where $\beta = \arctan(\sqrt{35}/2)$ and

$$\alpha_0 = \frac{1}{2}(\log 0_+ + \log\sqrt{156}), \qquad \alpha_{123} = -\frac{1}{4}(2\log 0_+ - \log 156),$$
$$\alpha_1 = -\frac{1}{20}\big(5\sqrt{3}\,\pi - 3\sqrt{35}\,\pi + 2\sqrt{35}\,\beta\big), \qquad \alpha_2 = \left(\frac{\pi}{2\sqrt{3}} + \frac{2}{\sqrt{35}}\pi - \frac{2}{\sqrt{35}}\beta\right),$$
$$\alpha_3 = \left(\sqrt{\frac{5}{7}}\,\pi + \frac{5\pi}{4\sqrt{3}} - \sqrt{\frac{5}{7}}\,\beta\right), \qquad \alpha_{12} = \left(-\sqrt{\frac{5}{7}}\,\pi + \frac{5\pi}{4\sqrt{3}} + \sqrt{\frac{5}{7}}\,\beta\right),$$
$$\alpha_{13} = \left(\frac{\pi}{2\sqrt{3}} - \frac{2}{\sqrt{35}}\pi + \frac{2}{\sqrt{35}}\beta\right), \qquad \alpha_{23} = \frac{1}{20}\big(5\sqrt{3}\,\pi + 2\sqrt{35}\,\pi - 2\sqrt{35}\,\beta\big).$$

For simplicity, the constants $c_{i\pm}$ and $\hat{\mathcal{U}}$ were equated to zero. One can check that after replacement of $\log(0_+)$ by $\log x$ and substituting the final result into exponential formula (23) in [2] and then computing the limit $x \to 0$, we recover the initial MV. To make the verification simple when $c_{i\pm}$ and $\hat{\mathcal{U}}$ are included, one may choose concrete values for arbitrary free constants $c_{i\pm}$ and arbitrary unit bivector $\hat{\mathcal{U}}^2 = -1$.

6 Roots of MV. Relations of the logarithm to GA inverse trigonometric and hyperbolic functions

When the GA logarithm is known, the powers of a MV may be found with $\mathsf{A}^r = \exp(r\log\mathsf{A})$, i.e., by multiplying logarithm by r, which may be either integer or rational number, and then computing the exponential. In the preprint [1], we have provided the algorithm how to obtain all possible square roots ($r = 1/2$) of numerical MV for all $n = 3$ Clifford algebras. Here we want to show that the roots presented in [1] are consistent with the mentioned exp-log formula. Thus, we will do a cross check of 3D GA logarithm formulas by different methods. However, it must be stressed that the logarithm formula is limited since it allows to find only a single[5] square root, although, as shown in [1], there may exist many (up to 16 in case of $Cl_{2,1}$ algebra) roots. Thus, the GA logarithm function

[5] More precisely, two (plus/minus) roots.

is not universal enough, although sometimes it may be useful if only a single fractional root ($r = 1/n$ and $n \in \mathbb{N}$) is needed. For example, the logarithm of basis vector $\mathbf{e}_1$ in $Cl_{0,3}$ is $\log \mathbf{e}_1 = \pi \mathbf{e}_1/2$. It is easy to check that after multiplication of logarithm by $1/3$ and exponentiation, we obtain the cubic root $\sqrt[3]{\mathbf{e}_1} = (\sqrt{3} + \mathbf{e}_1)/2$ that coincides with *Mathematica* result $\sqrt[3]{\mathrm{i}} = \sqrt[6]{-1} = (\sqrt{3} + \mathrm{i})/3$.

Example 10 [$Cl_{3,0}$. *Example* 2 *from* [1]]. Logarithm of MV $\mathsf{A} = -1 + \mathbf{e}_3 - \mathbf{e}_{12} + I/2$ in $Cl_{3,0}$ is

$$\log \mathsf{A} = \log \frac{\sqrt{5}}{2} - \frac{\log 5}{2}\mathbf{e}_3 + \frac{1}{2}\left(\pi - \arctan \frac{4}{3}\right)\mathbf{e}_{12} + \left(-\pi + \arctan \frac{1}{2}\right) I.$$

Multiplication of logarithm by $1/2$ and exponentiation gives the root $\sqrt{\mathsf{A}} = (\mathbf{e}_3 + \mathbf{e}_{12})/2 - I$, which coincides with root 4 of Example 2 in [1].

Like the trigonometric and hyperbolic functions may be expressed by exponentials (Euler and de Moivre formulas), the inverse hyperbolic functions may be defined by logarithms. Therefore, we can use the following definitions in GA to compute the inverse hyperbolic and trigonometric functions of MV argument A.

For *inverse hyperbolic functions*:

$$\begin{aligned}
\operatorname{artanh} \mathsf{A} &= \frac{1}{2}\big(\log(1 + \mathsf{A}) - \log(1 - \mathsf{A})\big),\\
\operatorname{arcoth} \mathsf{A} &= \begin{cases} \frac{1}{2}(\log(1 + \mathsf{A}^{-1}) - \log(1 - \mathsf{A}^{-1})), & \mathsf{A} \neq 0,\\ \frac{\pi}{2} I, & \mathsf{A} = 0, \end{cases}\\
\operatorname{arcosh} \mathsf{A} &= \log\big(\mathsf{A} + \sqrt{\mathsf{A} - 1}\,\sqrt{\mathsf{A} + 1}\big),\\
\operatorname{arsinh} \mathsf{A} &= \log\big(\mathsf{A} + \sqrt{\mathsf{A}^2 + 1}\big).
\end{aligned}$$

For *inverse trigonometric functions*:

$$\begin{aligned}
\arcsin \mathsf{A} &= -I \log\big(\mathsf{A}I + \sqrt{1 - \mathsf{A}^2}\big),\\
\arccos \mathsf{A} &= \frac{\pi}{2} + I \log\big(\mathsf{A}I + \sqrt{1 - \mathsf{A}^2}\big),\\
\arctan \mathsf{A} &= \frac{I}{2}\big(\log(1 - I\mathsf{A}) - \log(1 + I\mathsf{A})\big),\\
\operatorname{arccot} \mathsf{A} &= \begin{cases} \frac{1}{2} I(\log(1 - I\mathsf{A}^{-1}) - \log(1 + I\mathsf{A}^{-1})), & \mathsf{A} \neq 0,\\ \frac{\pi}{2}, & \mathsf{A} = 0. \end{cases}
\end{aligned}$$

These formulas are similar to those in the theory of real and complex functions except that instead of the imaginary unit the pseudoscalar appears in trigonometric functions. However, earlier we have found [1] that in GA the functions with the square root, in general, are multivalued. Thus, at a first sight, it may appear that the listed above equations with square root are not valid in all circumstances. Nonetheless, our numerical experiments show that they, in fact, are satisfied for all possible *individual* plus/minus pairs of square roots[6] (see more examples in the preprint [6]).

[6]This property does not allow us to write the equality sign between GA general expression $\log \sqrt{\mathsf{B}}$ and $\log(\mathsf{B})/2$.

7 Discussion and conclusions

The logarithm together with the exponential [2, 8] and square root [1] are the most important functions in Clifford geometric algebra (GA). Starting from the respective exponential functions, we presented here, as far as we know, for the first time the basis-free formulas for logarithms in all 3D GAs. The formulas for both the generic and special cases may be directly applied in GA programming. They were cross-checked using the basis-free GA exponential functions found in [2]. The derived formulas were implemented in *Mathematica* and tested with thousands of randomly generated multivectors [5]. In all cases the exponentiation of the logarithm was found to simplify to the initial MV.

Using numerical experiments [5], we observed that, in accord with the suggestion in [7], the principal value of the logarithm can be defined as a GA logarithm having the smallest determinant norm. In almost all cases the principal MV logarithm is attained by setting arbitrary integer parameters c_i in generic logarithms in Theorems 1–3 to zero. Exceptions from this rule, however, may occur in the case of simple and very specific MVs, when the commuting MVs may exist (Sections 3.3 and 4.3) and therefore not restricted by additional free MVs (Eqs. (12) and (24)). Apart from discrete parameters c_i, we have also found that continuous parameters represented by free unit vectors $\hat{\mathbf{u}}$ or bivectors $\hat{\mathcal{U}}$ may be included in the logarithm in special cases as well. The parameters vanish after exponentiation of the logarithm and do not contribute to the MV norm. Recently, we have found that such free parameters may be also introduced into lower-dimensional, quaternionic-type Clifford algebras [9]. However, more investigations are needed in this direction.

Also, relations between the GA logarithm and square root of MV were investigated. The well-known formula $\sqrt{\mathsf{A}} = \exp(\log(\mathsf{A})/2)$ served as an additional check of correctness of GA logarithm. Unfortunately, this formula allows to compute only a single square root from many possible roots that may exist in GA [1]. Nevertheless, such a comparison was found to be very useful for testing purposes. Indeed, a test of square root of a MV is an algebraic problem since it reduces to a solution of system of algebraic equations. On the other hand, inversion of exponential used in finding the GA logarithm in the present paper requires solving a system of complicated transcendental equations [9], a problem which is much more difficult (but at the same time more general) task. The mentioned exp-log relation also allows to check the condition whether the MV logarithm exists at all. Indeed, since we know that exponential can be computed for all MVs and multiplication by factor $1/2$ cannot impose any restriction, it follows that it is the $\log \mathsf{A}$ function, which determines the existence condition for $\sqrt{\mathsf{A}}$ and vice versa. As a test, we have checked using our algorithm [1] that for each MV, indeed, there exists a single square root that is in agreement with the identity $\exp(\log(\mathsf{A})/2) = \sqrt{\mathsf{A}}$.

In conclusion, in the present paper the basis-free expressions have been found for GA logarithms in all 3D real algebras. The logarithm exists for all MVs in case of real $Cl_{0,3}$ algebra. In Clifford algebra $Cl_{3,0}$ (and $Cl_{1,2}$) the logarithm exists for almost all MVs except very small class that satisfies the condition $(a_+^2 + a_-^2 = 0) \wedge (a_0^2 + a_{123}^2 = 0)$. For example, the logarithm of MV $\mathbf{e}_1 \pm \mathbf{e}_{12}$ cannot be computed in Euclidean $Cl_{3,0}$ algebra. On the other hand, in $Cl_{2,1}$ algebra the GA logarithm (as well as square root) is absent in large sectors of a real coefficient space.

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