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Joint Discrete Universality in the Selberg–Steuding Class

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Abstract: In the paper, we consider the approximation of analytic functions by shifts from the wide class $\widetilde{S}$ of L-functions. This class was introduced by A. Selberg, supplemented by J. Steuding, and is defined axiomatically. We prove the so-called joint discrete universality theorem for the function $L(s) \in \widetilde{S}$. Using the linear independence over $\mathbb{Q}$ of the multiset $\{(h_j \log p : p \in \mathbb{P}), j = 1, \dots, r; 2\pi\}$ for positive h_j, we obtain that there are many infinite shifts $(L(s+ikh_1), \dots, L(s+ikh_r))$, $k = 0, 1, \dots$, approximating every collection $(f_1(s), \dots, f_r(s))$ of analytic non-vanishing functions defined in the strip $\{s \in \mathbb{C} : \sigma_L < \sigma < 1\}$, where σ_L is a degree of the function $L(s)$. For the proof, the probabilistic approach based on weak convergence of probability measures in the space of analytic functions is applied.

Keywords: analytic functions; discrete shifts; limit theorem; simultaneous approximation; Selberg–Steuding class; weak convergence

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1. Introduction

One of the most important branches of the function theory is the approximation of analytic functions, and is widely used not only in mathematics but also in other natural sciences. In the 1980s, it was discovered that there exist analytic objects that approximate large classes of analytic functions. S.M. Voronin found [1] that the first such object as the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, given by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1,$$

where $\mathbb{P}$ is the set of all prime numbers. As is well-known, $\zeta(s)$ has the meromorphic continuation of the whole complex plane with $\mathrm{Res}_{s=1}\,\zeta(s) = 1$. Voronin proved [1] (see also [2]) that if $0 < c < \frac{1}{4}$, the function $f(s)$ is continuous and non-vanishing on the disc $|s| \le c$, and analytic in the interior of that disc, then there exists a real number $\tau = \tau(\epsilon, f)$ such that

$$\max_{|s| \le c} \left| \zeta\left(s + \frac{3}{4} + it\right) - f(s) \right| < \epsilon$$

for any $\epsilon > 0$.

Thus, Voronin reported that all non-vanishing analytic functions on the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, and uniformly on discs can be approximated by shifts $\zeta(s + i\tau)$ of one and the same function $\zeta(s)$. The Bohr–Courant theorem [3] claims that the set

$$\{\zeta(\sigma + it) : t \in \mathbb{R}\}$$

is dense everywhere on a complex plane for every fixed $\frac{1}{2} < \sigma \leq 1$. From here, it follows that the set of values of the function $\zeta(s)$ is very rich. Thus, in terms of approximation, the function $\zeta(s)$ is universal, and this might be natural in view of the remark above.

We denote by $\mathcal{H}(D)$ the space of the analytic on D functions equipped with the topology of uniform convergence on the compacta. Since the space $\mathcal{H}(D)$ has an infinite-dimension, the Voronin theorem is a infinite-dimensional extension of the Bohr–Courant denseness theorem.

The above-mentioned Voronin universality theorem has a more general statement which follows the Mergelyan theorem on the approximation of analytic functions by polynomials [4]. We denote by $\mathcal{K}(D)$ the set of compact subsets of the strip D with connected complements, and by $\mathcal{H}_0(K, D)$ the class of continuous non-vanishing functions on $K \in \mathcal{K}(D)$ that are analytic in the interior of K. Moreover, we let mes$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the following statement on the $\zeta(s)$'s universality is known, see, for example, [5–9].

Theorem 1. *Suppose that $K \in \mathcal{K}(D)$ and $f(s) \in \mathcal{H}_0(K, D)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{mes}\left\{\tau \in [0, T] : \sup_{s\in K} |f(s) - \zeta(s + i\tau)| < \epsilon\right\} > 0.$$

The inequality of the theorem shows the infinitude of shifts of $\zeta(s + i\tau)$ approximating a given function $f(s) \in H_0(K, D)$.

The statement of Theorem 1 was influenced by a probabilistic method proposed in [6]. The initial Voronin method based on the Riemann-type rearrangement theorem in the Hilbert space was developed in [7,8].

Since τ in the shifts $\zeta(s + i\tau)$ of Theorem 1 is an arbitrary real number, Theorem 1 is called a continuous universality theorem. Parallel to continuous universality theorems for zeta-functions, there are discrete universality theorems when τ takes values from a certain discrete set. These were proposed by A. Reich [10] for Dedekind zeta-functions of algebraic number fields $\mathbb{K}$. If $\mathbb{K} = \mathbb{Q}$, we deal with a discrete universality for the Riemann zeta-function. As an example, we now state a classical result in the following (see [6]).

Theorem 2. *Suppose that $K \in \mathcal{K}(D)$, $f(s) \in \mathcal{H}_0(K, D)$ and $h > 0$. Then, for every $\epsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leq k \leq N : \sup_{s\in K} |f(s) - \zeta(s + ikh)| < \epsilon\right\} > 0.$$

Here $\#A$ denotes the number of elements of the set $A \subset \mathbb{R}$, and N runs over the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Note that discrete universality theorems were also investigated in [6–8].

Some other functions given by a Dirichlet series also fulfil the property of universality in the Voronin sense. For example, Dirichlet L-functions $L(s, \chi)$ with arbitrary Dirichlet character χ,

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1,$$

are universal, as was mentioned by Voronin in [2]. Let $\mathfrak{A} = \{a_m : m \in \mathbb{N}\} \subset \mathbb{C}$ be a periodic sequence. Then the periodic zeta-function

$$\zeta(s; \mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \quad \sigma > 1,$$

also has the universal approximation property [11]. For values of the parameters α and λ, the Hurwitz zeta-function $\zeta(s, \alpha)$ and Lerch zeta-function $L(\lambda, \alpha, s)$, for $\sigma > 1$, respectively given by

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s} \quad \text{and} \quad L(\lambda,\alpha,s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s},$$

are universal (see [12]). In other words, they approximate analytic functions from the class $\mathcal{H}(K,D)$ considered continuous on K and analytic in the interior of K functions. This observation leads to certain conjectures. For example, by the Linnik–Ibragimov conjecture (or programme), see [8], all functions in a certain half-plane defined by a Dirichlet series, with analytic continuation left of the absolute convergence abscissa and satisfying some natural growth hypotheses are universal in the Voronin sense. However, currently there are Dirichlet series which their universality is not known, for example, the function $L(\lambda,\alpha,s)$ with an algebraic irrational parameter. Results in this direction for the Hurwitz zeta-function $\zeta(s,\alpha)$, as in [13], are presented.

To obtain more general results, the universality of separate functions and some classes of functions are considered. One such class was introduced by A. Selberg (see [14,15]), known as the Selberg class $\mathcal{S}$. The structure of the class $\mathcal{S}$ was studied by various authors, see [8,16–20], but until now its structure was not completely known. However, the class includes all main zeta- and L-functions, for example, $\zeta(s)$, $L(s,\chi)$, the zeta-functions of certain cusp forms, etc. The Selberg class $\mathcal{S}$ is defined axiomatically, with its functions

$$L(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}, \quad a(m) \in \mathbb{C},$$

satisfying four axioms. Recall that the notation $a \ll_\theta b$, $b > 0$, means that there is a positive constant $c = c(\theta)$ such that $|a| \leq cb$, and that $\Gamma(s)$ denotes the Euler gamma-function. The axioms of the class $\mathcal{S}$ have the names:

(1) (Ramanujan conjecture). The estimate $a(m) \ll_\epsilon m^\epsilon$ is valid with any $\epsilon > 0$.

(2) (Analytic continuation). For some $l \in \mathbb{N}_0$, $(s-1)^l L(s)$ in an entire function of finite order.

(3) (Functional equation). Let

$$\Lambda_L(s) = L(s) q^s \prod_{j=1}^{j_0} \Gamma(\lambda_j s + \alpha_j),$$

where q, $\lambda_j \in \mathbb{R}^+$, and $\alpha_j \in \mathbb{C}$ such that $\Re\alpha_j \geq 0$. Then the functional equation of the form

$$\Lambda_L(s) = w\Lambda_L(1-\bar{s})$$

is valid. Here, $|w| = 1$, and, as usual, by $\bar{s}$ we denote the conjugate of s.

(4) (Euler product). Let

$$\log L_p(s) = \sum_{l=1}^{\infty} \frac{b(p^l)}{p^s}$$

with coefficients $b(p^l)$ such that $b(p^l) \ll p^{\alpha l}$, $\alpha < \frac{1}{2}$. Then the representation

$$L(s) = \prod_{p \in \mathbb{P}} L_p(s)$$

holds.

Axioms (1)–(4) of the class $\mathcal{S}$ are insufficient to prove universality as they do not include the analogue of the prime number theorem. Therefore, J. Steuding, who was first to study the class $\mathcal{S}$ with an emphasis on universality [8], introduced the following axioms.

(5) There exists $\kappa > 0$ such that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p \leq x} |a(p)|^2 = \kappa,$$

where function $\pi(x)$ counts the number of primes up to x. Moreover, in [8] the Euler product of the type

(6)

$$L(s) = \prod_{p \in \mathbb{P}} \prod_{j=1}^{l} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}$$

was required with some complex $\alpha_j(p)$.

For the universality for the above functions, we need one important ingredient of the class $\mathcal{S}$. For $L \in \mathcal{S}$, the quantity

$$d_L = 2 \sum_{j=1}^{j_0} \lambda_j$$

is called the degree of the function L. The degree is an deep characteristic of the class $\mathcal{S}$. If $d_L = 1$, then $L(s)$ coincides with $\zeta(s)$ or $L(s + ia, \chi)$ with some $a \in \mathbb{R}$. For $L \in \mathcal{S}$, let

$$\sigma_L = \max\left(\frac{1}{2}, 1 - \frac{1}{d_L}\right).$$

We denote by $D_{\sigma_L} = \{s \in \mathbb{C} : \sigma_L < \sigma < 1\}$, $K(D_{\sigma_L})$ the class of compact subsets of the strip D_{σ_L} with connected complements, and $\mathcal{H}_0(K, D_{\sigma_L})$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then, in [8], the following universality theorem has been proved.

Theorem 3. *Suppose that $L(s)$ satisfies Axioms (2), (3), (5) and (6). Let $K \in K(D_{\sigma_L})$ and $f(s) \in \mathcal{H}_0(K, D_{\sigma_L})$. Then, for every $\epsilon > 0$, the inequality*

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{mes}\left\{\tau \in [0, T] : \sup_{s \in K} |f(s) - L(s + i\tau)| < \epsilon\right\} > 0$$

holds.

In [21], Axiom (6) was removed. Thus, Theorem 3 holds for the so-called Selberg–Steuding class $\widetilde{S}$; more precisely, for the functions belonging to the Selberg class and satisfying Axiom (5).

The discrete version of Theorem 3 has been obtained in [22].

Theorem 4. *Suppose that $L(s)$, K and $f(s)$ are the same as in Theorem 3. Then, for every $h > 0$ and $\epsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N+1} \#\left\{0 \le k \le N : \sup_{s \in K} |f(s) - L(s + ikh)| < \epsilon\right\} > 0.$$

We can consider a simultaneous approximation of a tuple of analytic functions by a tuple of shifts of zeta- or L-functions. This type of universality is called joint universality. This phenomenon of a Dirichlet series was also introduced by Voronin. In [23], he studied the joint functional independence of Dirichlet L-functions using the joint universality. Of course, the joint universality is more complicated, but, on the other hand, it is more interesting. Obviously, in the case of joint universality, the approximating shifts require some independence conditions. For example, Voronin used Dirichlet L-functions with pairwise non-equivalent Dirichlet characters. Later, the joint universality theorems were proven for zeta-functions defined by a Dirichlet series with periodic coefficients, Matsumoto zeta-functions, and automorphic L-functions. For these proofs, see the very informative paper [9].

This paper deals with the discrete joint universality property for L-functions for the class $\widetilde{S}$. Let

$$L(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s},$$

$h_1, ..., h_r$ be fixed positive numbers, and $\underline{h} = (h_1, \ldots, h_r)$. We define the multiset

$$A(\mathbb{P}, \underline{h}, 2\pi) = \{(h_j \log p : p \in \mathbb{P}),\ j = 1, \ldots, r;\ 2\pi\},$$

and then we prove the following theorem.

Theorem 5. *Suppose that $L(s) \in \widetilde{S}$, and the set $A(\mathbb{P}, \underline{h}, 2\pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}(D_L)$ and $f_j(s) \in \mathcal{H}_0(K_j, D_L)$. Then, for every $\underline{h} \in (\mathbb{R}^+)^r$ and $\epsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{1 \le j \le r} \sup_{s \in K_j} |f_j(s) - L(s + ikh_j)| < \epsilon \right\} > 0.$$

Moreover, for all but at most countably many $\epsilon > 0$, the limit

$$\lim_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{1 \le j \le r} \sup_{s \in K_j} |f_j(s) - L(s + ikh_j)| < \epsilon \right\}$$

exists and is positive.

In [24], a joint continuous universality theorem for a function $L(s) \in \widetilde{S}$ on the approximation of analytic functions by shifts $\big(L(s + ia_1\tau), \ldots, L(s + ia_r\tau)\big)$ with linear independence over $\mathbb{Q}$ real algebraic numbers $a_1, \ldots, a_r$ was obtained.

For example, for $r = 3$, we can take $h_1 = 1$, $h_2 = \sqrt{2}$, and $h_3 = \sqrt{3}$ in Theorem 5.

We denote by $\mathcal{B}(\mathcal{X})$ the Borel σ-field of the space $\mathcal{X}$, and let P and P_n, where $n \in \mathbb{N}$, be probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. We report that P_n converges weakly to P as $n \to \infty$, and write $P \xrightarrow[n \to \infty]{w} P$, if, for all bounded continuous functions $g(x)$ on $\mathcal{X}$,

$$\lim_{n \to \infty} \int_{\mathcal{X}} g(x) dP_n = \int_{\mathcal{X}} g(x) dP.$$

We derive Theorem 5 from a probabilistic joint discrete limit theorem on weakly convergent probability measures in the space of analytic functions. For proof of the latter theorem, we consider the weak convergence of probability measures on the infinite-dimensional torus, and in the space of analytic functions for certain absolutely convergent Dirichlet series. After this, we show a comparison in the mean between the initial L-function and functions defined by an absolutely convergent Dirichlet series. This will give the desired joint discrete limit theorem for the tuple of functions we are interested in.

2. Case of the Torus

We define the infinite-dimensional torus as

$$\mathbb{T} = \prod_{p \in \mathbb{P}} \{s \in \mathbb{C} : |s| = 1\},$$

where $\mathbb{T}$ is the infinite Cartesian product over prime numbers of unit circles. Since each circle is a compact set, by the Tikhonov theorem, $\mathbb{T}$ with the product topology and operation of pairwise multiplication is a compact topological abelian group. Now, we construct the set

$$\mathbb{T}^r = \mathbb{T}_1 \times \ldots \times \mathbb{T}_r,$$

where $\mathbb{T}_j = \mathbb{T}$, $j = 1, \ldots, r$. Then, the Tikhonov theorem again shows that $\mathbb{T}^r$ is a compact topological group. We denote by $\mathbf{t} = (\mathbf{t}_1, .., \mathbf{t}_r)$, $\mathbf{t}_j \in \mathbb{T}_j$, $\mathbf{t}_j = (\mathbf{t}_j(p) : p \in \mathbb{P})$, $j = 1, \ldots, r$, the elements of $\mathbb{T}^r$.

For $A \in \mathcal{B}(\mathbb{T}^r)$, we set

$$Q_{N,\mathbb{T}^r,\underline{h}}(A) = \frac{1}{N+1}\#\Big\{0 \le k \le N : \Big((p^{-ikh_1} : p \in \mathbb{P}), \dots, (p^{-ikh_r} : p \in \mathbb{P}) \Big) \in A \Big\}.$$

In this section, we consider the weak convergence for $Q_{N,\mathbb{T}^r,\underline{h}}$ as $N \to \infty$.

Proposition 1. *Suppose that the set $A(\mathbb{P}, \underline{h}, 2\pi)$ is linearly independent over $\mathbb{Q}$. Then, $Q_{N,\mathbb{T}^r,\underline{h}} \xrightarrow[n\to\infty]{w} m^H$, where m^H is the probability Haar measure on $(\mathbb{T}^r, \mathcal{B}(\mathbb{T}^r))$.*

Proof. The characters of the $\mathbb{T}^r$ are of the form

$$\prod_{j=1}^{r} \prod_{p\in\mathbb{P}}{}^{*} \mathbf{t}_j^{l_{jp}}(p)$$

with integers l_{jp}, where the star indicates that only a finite number of l_{jp} are not zeroes. Therefore, the Fourier transform $\mathcal{F}_{N,\mathbb{T}^r,\underline{h}}(\underline{l}_1, .., \underline{l}_r)$, $\underline{l}_j = (l_{jp} : l_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \dots, r$, can be represented by

$$\begin{aligned} \mathcal{F}_{N,\mathbb{T}^r,\underline{h}}(\underline{l}_1, .., \underline{l}_r) &= \int_{\mathbb{T}^r} \prod_{j=1}^{r} \prod_{p\in\mathbb{P}}{}^{*} \mathbf{t}_j^{l_{jp}}(p)\, dQ_{N,\mathbb{T}^r,\underline{h}} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \prod_{j=1}^{r} \prod_{p\in\mathbb{P}}{}^{*} p^{-ikl_{jp}h_j} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \exp\Big\{ -ik \sum_{j=1}^{r} h_j \sum_{p\in\mathbb{P}}{}^{*} l_{jp} \log p \Big\}. \end{aligned} \tag{1}$$

By a continuity theorem on the compact groups, for the proof of Proposition 1, it is sufficient to show that the Fourier transform $\mathcal{F}_{N,\mathbb{T}^r,\underline{h}}(\underline{l}_1, .., \underline{l}_r)$ converges, as $N \to \infty$, to the Fourier transform

$$\mathcal{F}_{m^H}(\underline{l}_1, \dots, \underline{l}_r) = \begin{cases} 1 & \text{if} \quad (\underline{l}_1, \dots, \underline{l}_r) = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{otherwise} \end{cases}$$

of the Haar measure m^H. Here, $\underline{0} = (0, 0, \dots)$.

Equality (1), obviously, gives

$$\mathcal{F}_{N,\mathbb{T}^r,\underline{h}}(\underline{0}, \dots, \underline{0}) = 1. \tag{2}$$

Thus, it remains to consider only the case $(\underline{l}_1, \dots, \underline{l}_r) \ne (\underline{0}, \dots, \underline{0})$. Since the set $A(\mathbb{P}, \underline{h}, 2\pi)$ is linearly independent over $\mathbb{Q}$, we have, in this case,

$$\exp\Big\{ -i \sum_{j=1}^{r} h_j \sum_{p\in\mathbb{P}}{}^{*} l_{jp} \log p \Big\} \ne 1. \tag{3}$$

Actually, if (3) is false, then

$$\sum_{j=1}^{r} h_j \sum_{p\in\mathbb{P}}{}^{*} l_{jp} \log p = 2\pi m$$

for some $m \in \mathbb{Z}$ and the integers $l_{jp} \ne 0$. However, this contradicts the assumption that the set $A(\mathbb{P}, \underline{h}, 2\pi)$ is linearly independent. Now, using (3) and the formula for the sum of geometric progressions, we deduce from (1) that, for $(\underline{l}_1, \dots, \underline{l}_r) \ne (\underline{0}, \dots, \underline{0})$,

$$\mathcal{F}_{N,\mathbb{T}^r,\underline{h}}(\underline{l}_1, \dots, \underline{l}_r) = \frac{1 - \exp\Big\{ -i(N+1) \sum_{j=1}^{r} h_j \sum^{*}_{p\in\mathbb{P}} l_{jp} \log p \Big\}}{(N+1)\Big(1 - \exp\Big\{ -i \sum_{j=1}^{r} h_j \sum^{*}_{p\in\mathbb{P}} l_{jp} \log p \Big\}\Big)}.$$

Hence,

$$\lim_{N\to\infty} \mathcal{F}_{N,\mathbb{T}^r,\underline{h}}(\underline{l}_1,\ldots,\underline{l}_r) = 0$$

for $(\underline{l}_1,\ldots,\underline{l}_r) \neq (\underline{0},\ldots,\underline{0})$. This, together with (2), shows that

$$\lim_{N\to\infty} \mathcal{F}_{N,\mathbb{T}^r,\underline{h}}(\underline{l}_1,\ldots,\underline{l}_r) = \mathcal{F}_{m^H}(\underline{l}_1,\ldots,\underline{l}_r),$$

thus proving the Proposition 1.

□

We apply Proposition 1 for the proof of weak convergence for the measures defined by means of certain absolutely convergent Dirichlet series connected to the function $L(s)$. We fix a number $\beta > \frac{1}{2}$, and

$$v_n(m;\beta) = \exp\left\{ -\left(\frac{m}{n}\right)^{\beta} \right\}, \quad m,n \in \mathbb{N}.$$

We define the functions

$$L_n(s) = \sum_{m=1}^{\infty} \frac{a(m)v_n(m;\beta)}{m^s}$$

and

$$L_n(s,\mathbf{t}_j) = \sum_{m=1}^{\infty} \frac{a(m)\mathbf{t}_j(m)v_n(m;\beta)}{m^s}, \quad j = 1,\ldots,r,$$

where, for $m \in \mathbb{N}$,

$$\mathbf{t}_j(m) = \prod_{p^l \| m} \mathbf{t}_j^l(p).$$

If $L(s) \in \widetilde{S}$, then $a(m) \ll m_\epsilon^\epsilon$ with arbitrary $\epsilon > 0$. Obviously, $v_n(m;\beta)$ decreases exponentially with respect to m. Therefore, the series for $L_n(s)$ and $L_n(s,\mathbf{t}_j)$ are absolutely convergent for $\sigma > \sigma_a$ with arbitrary finite σ_a and fixed $n \in \mathbb{N}$. Let

$$\underline{L}_n(s+ik\underline{h}) = \big(L_n(s+ikh_1),\ldots,L_n(s+ikh_r)\big)$$

and

$$\underline{L}_n(s,\mathbf{t}) = \big(L_n(s,\mathbf{t}_1),\ldots,L_n(s,\mathbf{t}_r)\big).$$

Moreover, let $\mathcal{H}(D_L)$ stand for the space of analytic on D_L functions endowed with the topology of uniform convergence on compact sets, and let

$$\mathcal{H}^r(D_L) = \prod_{j=1}^{r} \mathcal{H}(D_L).$$

For $A \in \mathcal{B}(\mathcal{H}^r(D_L))$, we set

$$P_{N,n,\underline{h}}(A) = \frac{1}{N+1}\#\big\{0 \le k \le N : \underline{L}_n(s+ik\underline{h}) \in A\big\}.$$

Proposition 2. *On $(\mathcal{H}^r(D_L),\mathcal{B}(\mathcal{H}^r(D_L)))$, a probability measure P_n exists such that $P_{N,n,\underline{h}} \xrightarrow[N\to\infty]{w} P_n$.*

Proof. Let the mapping $u_n : \mathbb{T}^r \to \mathcal{H}^r(D_L)$ be given by $u_n(\mathbf{t}) = \underline{L}_n(s,\mathbf{t})$. The absolute convergence of the series for $L_n(s,\mathbf{t}_j)$, $j = 1,\ldots,r$, implies the continuity of u_n. Hence, u_n is

$(\mathbb{T}^r, \mathcal{H}^r(D_L))$-measurable. Therefore, every probability measure P on $(\mathbb{T}^r, \mathcal{B}(\mathbb{T}^r))$ induces the unique probability measure Pu_n^{-1} on $(\mathcal{H}^r(D_L), \mathcal{B}(\mathcal{H}^r(D_L)))$ given by

$$Pu_n^{-1}(A) = P(u_n^{-1}A), \quad A \in \mathcal{B}(\mathcal{H}^r(D_L)).$$

Let $Q_{N,\mathbb{T}^r,\underline{h}}$ be from Proposition 1. Then, for every $A \in \mathcal{B}(\mathcal{H}^r(D_L))$,

$$\begin{aligned} P_{N,n,\underline{h}}(A) &= \frac{1}{N+1}\#\left\{0 \le k \le N : \left((p^{-ikh_j} : p \in \mathbb{P}), j = 1, \dots, r\right) \in u_n^{-1}A\right\} \\ &= Q_{N,\mathbb{T}^r,\underline{h}}(u_n^{-1}A) = Q_{N,\mathbb{T}^r,\underline{h}}u_n^{-1}(A). \end{aligned}$$

Hence, we have $P_{N,n,\underline{h}} = Q_{N,\mathbb{T}^r,\underline{h}}u_n^{-1}$. Therefore, Proposition 1, the continuity of u_n and Theorem 5.1 in [25] show that $P_{N,n,\underline{h}} \xrightarrow[N\to\infty]{w} P_n$, where $P_n = m^H u_n^{-1}$. $\square$

We see that the measure P_n is independent of $\underline{h}$. This allows us to obtain the weak convergence of P_n as $n \to \infty$, and identify the limit measure. Let

$$L(s, \mathbf{t}_j) = \sum_{m=1}^{\infty} \frac{a(m)\mathbf{t}_j(m)}{m^s}, \quad j = 1, \dots, r.$$

It is known [8] that the Dirichlet series for $L(s, \mathbf{t}_j)$, for almost all $\mathbf{t}_j$, is uniformly convergent on compact subsets of the strip D_L. Thus, $L(s, \mathbf{t}_j)$, for $j = 1, \dots, r$, is a $\mathcal{H}(D_L)$-valued random element. The probability Haar measure m^H on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ is the product of the Haar measure m_j^H on $(\mathbb{T}_j, \mathcal{B}(\mathbb{T}_j))$, i.e., for $A = A_1 \times \ldots \times A_r \in \mathcal{B}(\mathbb{T}^r)$,

$$m^H(A) = m_1^H(A_1) \cdot \ldots \cdot m_r^H(A_r).$$

The above remarks show that

$$\underline{L}(s, \mathbf{t}) = \left(L(s, \mathbf{t}_1), \dots, L(s, \mathbf{t}_r)\right)$$

is a $\mathcal{H}^r(D_L)$-valued random element defined on the probability space $(\mathbb{T}^r, \mathcal{B}(\mathbb{T}^r))$. We denote by $P_{\underline{L}}$ the distribution of $\underline{L}(s, \mathbf{t})$.

The measure P_n coincides with that studied in the continuous case in [24]. Therefore, we have the following proposition.

Lemma 1. *The relation $P_n \xrightarrow[n\to\infty]{w} P_{\underline{L}}$ holds. Moreover, the support of the measure $P_{\underline{L}}$ is set as*

$$\left(\left\{g \in \mathcal{H}(D_L) : \text{either } g(s) \neq 0 \ \ or \ \ g(s) \equiv 0\right\}\right)^r.$$

Proof. The first assertion of the lemma is contained in Lemma 7 in [24], while the second one is in Lemma 9 in [24]. $\square$

3. Limit Theorem

We start this section with a mean value estimate for the collection of L-functions we are interested in.

Let

$$\underline{L}(s + ik\underline{h}) = \left(L(s + ikh_1), \dots, L(s + ikh_r)\right).$$

In this section, we estimate the distance between $\underline{L}(s + ik\underline{h})$ and $L_n(s + ik\underline{h})$ in the mean. Let $\underline{d}$ be the metric on the space $\mathcal{H}^r(D_L)$, i.e., for $\underline{g}_l = (g_{l1}, \dots, g_{lr})$, $l = 1, 2$,

$$\underline{d}(\underline{g}_1, \underline{g}_2) = \max_{1 \le m \le r} d(g_{1m}, g_{2m}),$$

and d is the metric in $\mathcal{H}(D_L)$ which induces its uniform convergence topology on compact sets.

Lemma 2. *For arbitrary positive fixed numbers $h_1, \dots, h_r$,*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \underline{d}\big(\underline{L}(s+ik\underline{h}), \underline{L}_n(s+ik\underline{h})\big) = 0.$$

Proof. Since

$$d(g_1, g_2) = \sum_{j=1}^{\infty} 2^{-j} \frac{\sup_{s\in K_j} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_j} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in \mathcal{H}(D_L),$$

where $\{K_j : j \in \mathbb{N}\} \subset D_L$ is a certain sequence of compact sets, it suffices to show that, for every compact set $K \subset D_L$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |L(s+ikh_j) - L_n(s+ikh_j)| = 0, \quad j = 1, \dots, r. \tag{4}$$

We fix a compact set K, a positive number h, and $L(s) \in \widetilde{S}$. We use the integral representation [24]

$$L_n(s) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} L(s+z) l_n(z;\beta)\, dz, \tag{5}$$

where

$$l_n(s;\beta) = \frac{1}{\beta} \Gamma\left(\frac{s}{\beta}\right) n^s,$$

and the fixed number $\beta > \frac{1}{2}$ is the same as in the definition of $v_n(m;\beta)$. There exists $\delta = \delta(K)$ such that $\sigma_L + 2\delta \le \sigma \le 1 - \delta$ for $\sigma + it \in K$. Thus, $\beta_1 \overset{def}{=} \sigma - \sigma_L - \delta > 0$. Let $\beta = \sigma_L + \delta$. The integrand in (5) has a simple pole at the point $z = 0$, and a possible simple pole at the point $z = 1 - s$. Therefore, by the residue theorem and (1),

$$L_n(s) - L(s) = \frac{1}{2\pi i} \int_{-\beta_1-i\infty}^{-\beta_1+i\infty} L(s+z) l_n(z;\beta)\, dz + r(s),$$

where

$$r(s) = \operatorname*{Res}_{z=1-s} L(s+z) l_n(z;\beta) = \gamma l_n(1-s;\beta),$$

and $\gamma = \operatorname{Res}_{s=1} L(s)$. If $\alpha = 0$ in Axiom (2), then $r(s) = 0$. Hence, for $s = \sigma + it \in K$,

$$\begin{aligned}
& L(s+ikh) - L_n(s+ikh) \\
& = \frac{1}{2\pi i} \int_{-\infty}^{\infty} L(s+ikh+\sigma_L-\sigma+\delta+i\tau) l_n(\sigma_L-\sigma+\delta+i\tau;\beta)\, d\tau + r(s+ikh) \\
& = \frac{1}{2\pi i} \int_{-\infty}^{\infty} L(\sigma_L+\delta+ikh+i\tau) l_n(\sigma_L+\delta-s+i\tau)\, d\tau + r(s+ikh) \\
& \ll \int_{-\infty}^{\infty} \big|L(\sigma_L+\delta+ikh+i\tau)\big| \sup_{s\in K} |l_n(\sigma_L+\delta-s+i\tau)|\, d\tau + \sup_{s\in K} |r(s+ikh)|.
\end{aligned}$$

From this, we have

$$\begin{aligned}
& \frac{1}{N+1} \sum_{k=2}^{N} \sup_{s\in K} |L(s+ikh) - L_n(s+ikh)| \\
& \qquad \ll \int_{-\infty}^{\infty} \left(\frac{1}{N+1} \sum_{k=2}^{N} |L(\sigma_L+\delta+ikh+i\tau)| \right) \sup_{s\in K} |l_n(\sigma_L+\delta-s+i\tau)|\, d\tau \\
& \qquad\quad + \frac{1}{N+1} \sum_{k=2}^{N} \sup_{s\in K} |r(s+ikh)|.
\end{aligned} \tag{6}$$

By the Cauchy–Schwarz inequality,

$$\frac{1}{N+1}\sum_{k=2}^{N}|L(\sigma_L+\delta+ikh+i\tau)| \ll \left(\frac{1}{N}\sum_{k=2}^{N}|L(\sigma_L+\delta+ikh+i\tau)|^2\right)^{\frac{1}{2}}. \tag{7}$$

To estimate the last mean square, we apply the Gallagher lemma, see Lemma 1.4 in [26], and the known estimate [8]

$$\int_{-T}^{T}|L(\sigma+it)|^2dt \ll_\sigma T \tag{8}$$

which is valid for fixed σ, $\sigma_L < \sigma < 1$. Application of the Gallagher lemma gives

$$\begin{aligned}&\sum_{k=2}^{N}|L(\sigma_L+\delta+ikh+i\tau)|^2\\ &\quad\ll_h \int_{\frac{3}{2}h}^{Nh}|L(\sigma_L+\delta+iv+i\tau)|^2dv+\\ &\qquad+\left(\int_{\frac{3}{2}h}^{Nh}|L(\sigma_L+\delta+iv+i\tau)|^2dv\int_{\frac{3}{2}h}^{Nh}|L'(\sigma_L+\delta+iv+i\tau)|^2dv\right)^{\frac{1}{2}}.\end{aligned} \tag{9}$$

The Cauchy integral formula together with (8) gives, for $\sigma_L < \sigma < 1$, the bound

$$\int_{-T}^{T}|L'(\sigma+it)|^2dt \ll_\sigma T.$$

This, and (8) and (9) lead to the estimate

$$\sum_{k=2}^{N}|L(\sigma_L+\delta+ikh+i\tau)|^2 \ll_{h,\delta} N(1+|\tau|). \tag{10}$$

To estimate $l_n(\sigma_L+\delta-s+i\tau)$ for $s \in K$, we use the well-known estimate

$$\Gamma(\sigma+it) \ll e^{-c|t|}, \quad c>0,$$

which is valid for large $|t|$ uniformly in any fixed strip. Thus, for $s \in K$, we find

$$l_n(\sigma_L+\delta-s+i\tau) \ll_\beta n^{\sigma_L+\delta-\sigma}e^{-\frac{c}{\beta}|\tau-t|} \ll_{\beta,K} n^{-\delta}c^{-c_1|\tau|}$$

with $c_1 > 0$. Now, the latter estimate, and (7) and (10) show that

$$\begin{aligned}&\int_{-\infty}^{\infty}\left(\frac{1}{N+1}\sum_{k=2}^{N}|L(\sigma_L+\delta+ikh+i\tau)|\right)\sup_{s\in K}|l_n(\sigma_L+\delta-s+i\tau)|d\tau\\ &\quad\ll_{\beta,K,h,\delta} n^{-\delta}\int_{-\infty}^{\infty}e^{-c_1|\tau|}(1+|\tau|)^{\frac{1}{2}}d\tau \ll_{\beta,K,h,\delta} n^{-\delta}.\end{aligned} \tag{11}$$

Similarly, the definition of $r(s)$ yields that, for $s \in K$,

$$r(s+ikh) \ll_\beta n^{1-\sigma}e^{-\frac{c}{\beta}|kh+t|} \ll_{\beta,K} n^{1-\sigma_L-2\delta}e^{-c_2kh}$$

with $c_2 > 0$. Hence,

$$\begin{aligned}\frac{1}{N+1}\sum_{k=2}^{N}\sup_{s\in K}|r(s+ikh)| &\ll_{\beta,K} n^{1-\sigma_L-2\delta}\,\frac{1}{N}\sum_{k=2}^{N}e^{-c_2kh}\\ &\ll_{\beta,K,h} n^{1-\sigma_L-2\delta}\left(\frac{\log N}{N}+\frac{1}{N}\sum_{k\geq\log N}^{\infty}e^{-c_2kh}\right)\\ &\ll_{\beta,K,h} n^{1-\sigma_L-2\delta}\frac{\log N}{N}.\end{aligned}$$

This, and (6) and (11) lead to the estimate

$$\frac{1}{N+1}\sum_{k=2}^{N}\sup_{s\in K}|L(s+ikh)-L_n(s+ikh)| \ll_{\beta,K,h,\delta} \left(n^{-\delta}+n^{1-\sigma_L-2\delta}\frac{\log N}{N}\right).$$

Therefore, taking $N\to\infty$ and then $n\to\infty$, we obtain

$$\lim_{n\to\infty}\liminf_{N\to\infty}\frac{1}{N+1}\sum_{k=2}^{N}\sup_{s\in K}|L(s+ikh)-L_n(s+ikh)|=0.$$

Since, obviously,

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{1}\sup_{s\in K}|L(s+ikh)-L_n(s+ikh)|=0,$$

thus proving (4). □

Now we are ready to prove the desired joint discrete limit theorem for the collection of L-functions belonging to the class $\widetilde{S}$. For $A\in\mathcal{B}(\mathcal{H}^r(D_L))$, we set

$$P_{N,\underline{h}}(A)=\frac{1}{N+1}\#\big\{0\le k\le N:\underline{L}(s+ik\underline{h})\in A\big\}.$$

Let P_n and $P_{\underline{L}}$ be the same as in Lemma 1.

Theorem 6. *Suppose that $L(s)\in\widetilde{S}$, and the set $A(\mathbb{P},\underline{h},2\pi)$ is linearly independent over $\mathbb{Q}$. Then $P_{N,\underline{h}}\xrightarrow[N\to\infty]{w}P_{\underline{L}}$.*

Proof. In view of Lemma 1, it suffices to show that P_n and $P_{N,\underline{h}}$ have the same limit measure as $n\to\infty$ and $N\to\infty$, respectively. We denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution.

On some probability space $(\Omega,\mathcal{A},P)$, we define the random variable ξ_N by

$$P\{\xi_N=k\}=\frac{1}{N+1},\quad k=0,1,\dots,N.$$

Let the $\mathcal{H}^r(D_L)$-valued random elements $X_{N,n,\underline{h}}$ and $X_{N,\underline{h}}$ be defined by

$$X_{N,n,\underline{h}}=X_{N,n,\underline{h}}(s)=\underline{L}_n(s+i\underline{h}\xi_N)$$

and

$$X_{N,\underline{h}}=X_{N,\underline{h}}(s)=\underline{L}(s+i\underline{h}\xi_N).$$

Then the assertion of Proposition 2 can be written in the form

$$X_{N,n,\underline{h}}\xrightarrow[N\to\infty]{\mathcal{D}}P_n. \tag{12}$$

Moreover, by Lemma 1,

$$X_n\xrightarrow[n\to\infty]{\mathcal{D}}P_{\underline{L}}, \tag{13}$$

where X_n is the $\mathcal{H}^r(D_L)$-valued random element with distribution P_n. Application of Lemma 2 and defining the above random elements show that, for $\epsilon>0$,

$$\begin{aligned}
&\lim_{n\to\infty}\limsup_{N\to\infty}P\{\underline{d}(X_{N,\underline{h}},X_{N,n,\underline{h}})\ge\epsilon\}\\
&=\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\#\Big\{0\le k\le N:\underline{d}\big(\underline{L}(s+ik\underline{h}),\underline{L}_n(s+ik\underline{h})\big)\ge\epsilon\Big\}\\
&\le\frac{1}{\epsilon(N+1)}\sum_{k=0}^{N}\underline{d}\big(\underline{L}(s+ik\underline{h}),\underline{L}_n(s+ik\underline{h})\big)=0.
\end{aligned}$$

Taking into account the separability of the space $(\mathcal{H}^r(D_L), \underline{d})$, the latter equality, and (12) and (13), we deduce that the hypotheses of Theorem 4.2 in [25] are satisfied. Therefore, we have

$$X_{N,\underline{h}} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\underline{L}}.$$

From last relation we obtain the assertion of the theorem. □

4. Proof of Theorem 5

The proof of Theorem 5 we derive from Theorem 6, Lemma 1 and the Mergelyan theorem mentioned in Section 1 (see [4]).

Proof of Theorem 5. Since $f_j(s) \neq 0$ on K_j, application of the Megelyan theorem for $\log f_j(s)$ implies the existence of polynomials $q_1(s), \dots, q_r(s)$ such that

$$\sup_{1\le j\le r} \sup_{s\in K_j} \left| f_j(s) - e^{q_j(s)} \right| < \frac{\epsilon}{2}. \tag{14}$$

In view of the second part of Lemma 1, the tuple $\left(e^{q_1(s)}, \dots, e^{q_r(s)}\right)$ is an element of the support of the measure $P_{\underline{L}}$. Therefore, the set

$$\mathcal{G}(\epsilon) = \left\{ (g_1, \dots, g_r) \in \mathcal{H}^r(D_L) : \sup_{1\le j\le r} \sup_{s\in K_j} |g_j(s) - e^{q_j(s)}| < \frac{\epsilon}{2} \right\}$$

is an open neighbourhood of the support element, and thus by a property of supports,

$$P_{\underline{L}}(\mathcal{G}(\epsilon)) > 0. \tag{15}$$

Now, Theorem 6 and Theorem 2.1 in [25] give

$$\liminf_{N\to\infty} P_{N,n,\underline{h}}(\mathcal{G}(\epsilon)) \ge P_{\underline{L}}(\mathcal{G}(\epsilon)) > 0. \tag{16}$$

Inequality (14) shows the inclusion of $\mathcal{G}(\epsilon) \subset \mathcal{G}_1(\epsilon)$, where

$$\mathcal{G}_1(\epsilon) = \left\{ (g_1, \dots, g_r) \in \mathcal{H}^r(D_L) : \sup_{1\le j\le r} \sup_{s\in K_j} |g_j(s) - f_j(s)| < \epsilon \right\}.$$

Therefore, by (16),

$$\liminf_{N\to\infty} P_{N,n,\underline{h}}(\mathcal{G}_1(\epsilon)) > 0,$$

and we have the first assertion of the theorem.

For the proof of second inequality of the theorem, we observe that, for different values of ϵ, the boundaries of $\mathcal{G}_1(\epsilon)$ do not intersect. This remark implies that the set $\mathcal{G}_1(\epsilon)$ is a continuity set of the measure $P_{\underline{L}}$ for all but at most countably many $\epsilon > 0$. This result, Theorem 6 and Theorem 2.1 in [25], in virtue of (15), imply

$$\liminf_{N\to\infty} P_{N,n,\underline{h}}(\mathcal{G}_1(\epsilon)) = P_{\underline{L}}(\mathcal{G}_1(\epsilon)) \ge P_{\underline{L}}(\mathcal{G}(\epsilon)) > 0$$

for all but at most countably many $\epsilon > 0$.

Theorem 5 is therefore proven. □

5. Concluding Remarks

In this paper we have obtained that every tuple $(f_1(s), \dots, f_r(s))$ of analytic non-vanishing functions in the strip D_L can be approximated simultaneously by discrete shifts $\big(L(s + ikh_1), \dots, L(s + ikh_r)\big)$, where $L(s)$ is a Dirichlet series from the Selberg–Steuding class, and the multiset $\{(h_j \log p : p \in \mathbb{P}), j = 1, \dots, r; 2\pi\}$ with positive $h_1, \dots, h_r$ is linearly independent over a field of rational numbers. For proof of the above theorem, results of a continuous universality theorem from [24] were applied.

We conjecture that Theorem 5 can be extended to include approximations by shifts $\big(L_1(s+ikh_1),\ldots,L_r(s+ikh_r)\big)$, where $L_1(s),\ldots,L_r(s)$ are functions from the Selberg–Steuding class. For this, a modification to Lemma 1 is needed.

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References

1. Voronin, S.M. Theorem on the "universality" of the Riemann zeta-function. *Math. USSR-Izv.* **1975**, *9*, 443–453. [CrossRef]
2. Karatsuba, A.A.; Voronin, S.M. *The Riemann Zeta-Function*; De Gruyter Expositions in Mathematics 5; W. de Gruyter: Berlin, Germany, 1992.
3. Bohr, H.; Courant, R. Neue Anwendungen der Theorie der diophantischen Approximationen auf die Riemannsche Zetafunktion. *J. Math.* **1914**, *144*, 249–274. [CrossRef]
4. Mergelyan, S.N. Uniform approximations to functions of a complex variable. *Am. Math. Soc. Transl.* **1952**, *101*, 99.
5. Laurinčikas, A. *Limit Theorems for the Riemann Zeta-Function*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 1996.
6. Bagchi, B. The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and Other Allied Dirichlet Series. Ph.D. Thesis, Indian Statistical Institute, Calcutta, India, 1981.
7. Gonek, S.M. Analytic Properties of Zeta and *L*-Functions. Ph.D. Thesis, University of Michigan, Ann Arbor, MI, USA, 1979.
8. Steuding, J. *Value Distribution of L-Functions*; Lecture Notes Math 1877; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2007.
9. Matsumoto, K. A survey on the theory of universality for zeta and *L*-functions. In *Number Theory. Plowing and Starring through High Wave Forms, Proceedings of 7th China-Japan Seminar, Fukuoka, Japan, 28 October–1 November 2013*; Kaneko, M., Kanemitsu, S., Liu, J., Eds.; Series on Number Theory and Its Applications 11; World Scientific: Hackensack, NJ, USA, 2015; pp. 95–144.
10. Reich, A. Werteverteilung von Zetafunktionen. *Arch. Math.* **1980**, *34*, 440–451. [CrossRef]
11. Laurinčikas, A.; Šiaučiūnas, D. Remarks on the universality of the periodic zeta function. *Math. Notes* **2006**, *80*, 532–538. [CrossRef]
12. Laurinčikas, A.; Garunkštis, R. *The Lerch Zeta-Function*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 2002.
13. Sourmelidis, A.; Steuding, J. On the value-distribution of Hurwitz zeta-functions with algebraic parameter. *Constr. Approx.* **2022**, *55*, 829–860. [CrossRef]
14. Selberg, A. Old and new conjectures and results about a class of Dirichlet series. In Proceedings of the Amalfi Conference on Analytic Number Theory, Maiori, Amalfi, Italy, 25–29 September 1989; Bombieri, E., Ed.; Universitá di Salerno: Salerno, Italy, 1992; pp. 367–385.
15. Selberg, A. *Collected Papers. Volume I. Reprint of the 1989 Original*; Springer Collected Works in Mathematics; Springer: Berlin/Heidelberg, Germany, 2014.
16. Kaczorowski, J.; Perelli, A. On the structure of the Selberg class. II. Invariants and conjectures. *J. Reine Angew. Math.* **2000**, *524*, 73–96. [CrossRef]
17. Kaczorowski, J.; Perelli, A. On the structure of the Selberg class. IV. Basic invariants. *Acta Arith.* **2002**, *104*, 97–116. [CrossRef]
18. Kaczorowski, J.; Perelli, A. On the prime number theorem for the Selberg class. *Arch. Math.* **2003**, *80*, 255–263.
19. Perelli, A. Non-linear twists of *L*-functions: A survey. *Milan J. Math.* **2010**, *78*, 117–134. [CrossRef]
20. Kaczorowski, J.; Perelli, A. Structural invariants of *L*-functions and applications: A survey. *Riv. Mat. Univ. Parma (N.S.)* **2022**, *13*, 137–159.
21. Nagoshi, H.; Steuding, J. Universality for *L*-functions in the Selberg class. *Lith. Math. J.* **2010**, *50*, 293–311. [CrossRef]
22. Laurinčikas, A.; Macaitienė, R. Discrete universality in the Selberg class. *Proc. Steklov Inst. Math.* **2017**, *299*, 143–156. [CrossRef]
23. Voronin, S.M. On the functional independence of Dirichlet *L*-functions. *Acta Arith.* **1975**, *27*, 493–503. (In Russian)

24. Kačinskaitė, R.; Laurinčikas, A.; Žemaitienė, B. On joint universality in the Selberg–Steuding class. *Mathematics* **2023**, *11*, 737. [CrossRef]
25. Billingsley, P. *Convergence of Probability Measures*, 2nd ed.; Willey: Chichester, UK, 1999.
26. Mongomery, H.L. *Topics in Multiplicative Number Theory*; Lecture Notes in Mathematics 227; Springer: Berlin/Heidelberg, Germany, 1971.