mathematics

Article

On Joint Universality in the Selberg–Steuding Class

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Abstract: The famous Selberg class is defined axiomatically and consists of Dirichlet series satisfying four axioms (Ramanujan hypothesis, analytic continuation, functional equation, multiplicativity). The Selberg–Steuding class $\mathcal{S}$ is a complemented Selberg class by an arithmetic hypothesis related to the distribution of prime numbers. In this paper, a joint universality theorem for the functions L from the class $\mathcal{S}$ on the approximation of a collection of analytic functions by shifts $\big(L(s+ia_1\tau),\dots,L(s+ia_r\tau)\big)$, where $a_1,\dots,a_r$ are real algebraic numbers linearly independent over the field of rational numbers, is obtained. It is proved that the set of the above approximating shifts is infinite, its lower density and, with some exception, density are positive. For the proof, a probabilistic method based on weak convergence of probability measures in the space of analytic functions is applied together with the Backer theorem on linear forms of logarithms and the Mergelyan theorem on approximation of analytic functions by polynomials.

Keywords: limit theorem; Selberg–Steuding class; universality; weak convergence

MSC: 11M06; 11M41

Citation: Kačinskaitė, R.; Laurinčikas, A.; Žemaitienė, B. On Joint Universality in the Selberg–Steuding Class. *Mathematics* **2023**, *11*, 737. https://doi.org/10.3390/math11030737

Academic Editor: Sitnik Sergey

Received: 28 December 2022
Revised: 30 January 2023
Accepted: 30 January 2023
Published: 1 February 2023

1. Introduction

Let $\{a_m : m \in \mathbb{N}\}$ be a sequence of complex numbers, and $s=\sigma+it$ be a complex variable. In analytic number theory, Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s}, \quad \sigma > \sigma_0,$$

are very important analytic objects. The latter series are of ordinary type, general Dirichlet series

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s}$$

where $\{\lambda_m\}$ is an increasing to $+\infty$ sequence of real numbers, are also studied. The majority of the so-called zeta- and L-functions, including the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

and Dirichlet L-functions

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1,$$

where $\chi(m)$ is a Dirichlet character, whose analytic properties play the crucial role for investigation of prime numbers in the set $\mathbb{N}$ and arithmetic progressions, respectively, are defined by Dirichlet series.

Without a class of Dirichlet L-functions, there are several classes of Dirichlet series cultivated in analytic number theory. Among them, the classes of Hurwitz-type zeta-functions, Lerch zeta-functions, Matsumoto zeta-functions, Epstein zeta-functions, and others. The famous number theorist A. Selberg in [1] introduced the class $\mathcal{S}$ of Dirichlet series including some classical number theoretical zeta- and L-functions, and stated hypotheses on that class. The Selberg class became an object of numerous studies. We recall the hypotheses which satisfy the functions

$$L(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}$$

of the class $\mathcal{S}$. As usual, we denote by $\Gamma(s)$ the Euler gamma-function.

1. For every $\varepsilon > 0$, the estimate $a(m) \ll m^{\varepsilon}$ is valid.
2. There exists an integer α such that $(s-1)^{\alpha} L(s)$ is an entire function of finite order.
3. The function $L(s)$ satisfies the functional equation

$$\Lambda_L(s) = w\Lambda_L(1-\bar{s}),$$

where

$$\Lambda_L(s) = L(s)q^s \prod_{j=1}^{J} \Gamma(\lambda_j s + \mu_j)$$

with positive numbers q and λ_j, and complex numbers w and μ_j such that $\Re \mu_j \geq 0$ and $|w| = 1$, and $\bar{s}$ denotes the complex conjugate of s.
4. The function $L(s)$ has the representation

$$L(s) = \prod_p L_p(s)$$

over the prime numbers with

$$\log L_p(s) = \sum_{l=1}^{\infty} \frac{b(p^{\alpha})}{p^s},$$

and $b(p^l) \ll p^{\theta l}$, $\theta < \frac{1}{2}$.

Much attention is devoted to the structure of the class $\mathcal{S}$. For results, see Section 6.1 of [2]. In the theory of the class $\mathcal{S}$, the degree of the function $L \in \mathcal{S}$ defined as

$$d_L = 2\sum_{j=1}^{J} \lambda_j$$

occupies an important place. For example, it is known that if $0 \leq d_L < 1$, then $L(s) \equiv 1$, while if $d_L = 1$, then $L(s)$ are the Riemann zeta-function, or shifted Dirichlet L-function $L(s + i\theta, \chi)$, $\theta \in \mathbb{R}$. There exists a conjecture that the class $\mathcal{S}$ consists of all automorphic L-functions. For example, L-functions of normalized holomorphic new forms have a degree $d_L = 2$.

In this paper, we are interested in the universality of functions of the class $\mathcal{S}$, i.e., on the approximation of a whole class of analytic functions by shifts $L(s + i\tau)$, $\tau \in \mathbb{R}$, $L(s) \in \mathcal{S}$. Recall that the universality property for $\zeta(s)$ which is a member of $\mathcal{S}$ was obtained by S.M. Voronin in [3]. For an improved version of the Voronin theorem, we use the following notation. Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$, and let $\text{meas}\{A\}$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Suppose that $K \subset D$ is a compact set with connected complement, and $f(s)$ is a continuous non-vanishing function on K and analytic in the interior of K. Then, for every $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \epsilon\right\} > 0$$

(see [2,4–6]). A similar assertion also is true for all Dirichlet L-functions.

The first universality result related to the class $\mathcal{S}$ was obtained by J. Steuding in [2]. Let, for $L \in \mathcal{S}$,

$$\sigma_L = \max\left(\frac{1}{2}, 1 - \frac{1}{d_L}\right)$$

and $D_L = \{s \in \mathbb{C} : \sigma_L < \sigma < 1\}$. In addition to the hypothesis 4 of the class $\mathcal{S}$, it was required the existence of a polynomial Euler product

$$L(s) = \prod_p \prod_{j=1}^{m} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}, \quad \alpha_j(p) \in \mathbb{C}. \tag{1}$$

Moreover , one more arithmetic condition

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} |a_j(p)|^2 = \kappa \tag{2}$$

with a certain positive κ and $\pi(x) = \sum_{p\le x} 1$ was used. Denote by $\mathcal{K}_L$ the class of compact subset of the strip D_L with connected complements, and by $H_{0L}(K)$, $K \in \mathcal{K}_L$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. Denote by $\widehat{S}$ the class satisfying hypotheses of the class $\mathcal{S}$, and (1) and (2). Then, the following theorem is true [2].

Theorem 1. *Suppose that $L \in \mathcal{S} \cap \widehat{S}$. Let $K \in \mathcal{K}_L$ and $f(s) \in H_{0L}$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to 0} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |L(s+i\tau) - f(s)| < \epsilon\right\} > 0.$$

We note that the class $\mathcal{S} \cap \widehat{S}$ consists of all functions satisfying axioms 2 and 3 of class $\mathcal{S}$, and (1) and (2).

In [7], Theorem 1 was improved, namely, the condition (1) was not used. More precisely, Theorem 1 is valid for $L \in \mathcal{S}$ satisfying (2).

For zeta- and L-functions, the joint universality also is considered. In this case, a collection of analytic functions is simultaneously approximated by a collection of shifts of zeta- or L-functions. The first result in this direction also belongs to S.M. Voronin. In [8], he obtained the joint universality for Dirichlet L-functions with nonequivalent characters and applied it for a theorem on joint functional independence of L-functions. More general results on joint universality were obtained for the periodic and periodic Hurwitz zeta-functions as well as for Matsumoto zeta-function (see, for example, [9–12]). Joint universality theorems also can be proved using only one zeta- or L-function with different shifts. Our aim is to obtain a joint universality theorem for functions from the Selberg–Steuding class $\mathcal{S}_1$ of functions belonging to the class $\mathcal{S}$ and satisfying the condition (2). The main result of the paper is the following theorem.

Theorem 2. *Suppose that $L \in \mathcal{S}_1$, and real algebraic numbers $a_1, \dots, a_r$ are linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}_L$ and $f_j(s) \in H_{0L}(K_j)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{1\le j\le r}\sup_{s\in K_j} |L(s+ia_j\tau) - f_j(s)| < \epsilon\right\} > 0.$$

Moreover, "liminf" can be replaced by "lim" for all but at most countably many $\epsilon > 0$.

The proof of Theorem 2 is based on weak convergence of probability measures in the space of analytic functions.

2. Limit Lemmas on a Group

We start to consider the weak convergence of probability measures with a case of one compact group. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of a topological space $\mathbb{X}$, and define the set

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p,$$

where $\mathbb{P}$ denotes the set of all prime numbers, and $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all $p \in \mathbb{P}$. By the classical Tikhonov theorem, the infinite-dimensional torus Ω, with the product topology and operation of pairwise multiplication, is a compact topological Abelian group. Define one more set

$$\Omega^r = \Omega_1 \times \ldots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$. Then, again, by the Tikhonov theorem, Ω^r is a compact topological Abelian group. Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H can be defined. This gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$. For $p \in \mathbb{P}$, denote by $\omega_j(p)$ the pth component of an element $\omega_j \in \Omega$, $j = 1, \dots, r$, and by $\omega = (\omega_1, \dots, \omega_r)$ the elements of Ω^r. Let, for brevity, $\underline{a} = (a_1, \dots, a_r)$.

Now, we will prove a limit lemma on weak convergence for

$$Q_{T,\underline{a}}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \left((p^{-a_1\tau} : p \in \mathbb{P}), \dots, (p^{-a_r\tau} : p \in \mathbb{P})\right) \in A\right\},$$

$A \in \mathcal{B}(\Omega^r)$, as $T \to \infty$. For its proof, we apply the following result of A. Baker (see [13]).

Lemma 1. *Suppose that the logarithm $\log\lambda_1, \dots, \log\lambda_r$ of algebraic numbers $\lambda_1, \dots, \lambda_r$ are linearly independent over $\mathbb{Q}$. Then, for any algebraic numbers $\beta_0, \beta_1, \dots, \beta_r$ not all simultaneously zero, the inequality*

$$|\beta_0 + \beta_1 \log\lambda_1 + \ldots + \beta_r \log\lambda_r| > h^{-C},$$

where h is the maximum of the heights of the numbers $\beta_0, \beta_1, \dots, \beta_r$, and C is an effective constant depending on r, $\lambda_1, \dots \lambda_r$ and the maximum of the powers of the numbers $\beta_0, \beta_1, \dots, \beta_r$, is valid.

Lemma 2. *Suppose that $a_1, \dots, a_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$. Then, $Q_{T,\underline{a}}$ converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. For the proofs of weak convergence of probability measures on groups, it is convenient to use the method of Fourier transforms. Thus, denote by $F_{T,\underline{a}}(\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \dots, r$, the Fourier transform of $Q_{T,\underline{a}}$, i.e.,

$$F_{T,\underline{a}}(\underline{k}_1, \dots, \underline{k}_r) = \int_\Omega \left(\prod_{j=1}^r \prod_{p\in\mathbb{P}}{}^* \omega_j^{k_{jp}}(p) dQ_{T,\underline{a}}\right),$$

where the star $*$ shows that only a finite number of integers k_{jp} is distinct from zero. By the definition of $Q_{T,\underline{a}}$, we have

$$\begin{aligned} F_{T,\underline{a}}(\underline{k}_1, \ldots, \underline{k}_r) &= \frac{1}{T} \int_0^T \left(\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} p^{-a_j k_{jp} \tau} \right) d\tau \\ &= \frac{1}{T} \int_0^T \exp\left\{ -i\tau \sum_{j=1}^{r} a_j \sum_{p \in \mathbb{P}}{}^{*} k_{jp} \log p \right\} d\tau. \end{aligned} \tag{3}$$

Obviously ,

$$F_{T,\underline{a}}(\underline{0}, \ldots, \underline{0}) = 1, \tag{4}$$

where $\underline{0}$ is a collection consisting from zeros. Now, suppose that $(\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0})$. Let, for brevity, $\underline{\underline{k}} = (\underline{k}_1, \ldots, \underline{k}_r)$,

$$A_{\underline{a},\underline{\underline{k}}} \stackrel{def}{=} \sum_{j=1}^{r} a_j \sum_{p \in \mathbb{P}}{}^{*} k_{jp} \log p = \sum_{p \in \mathbb{P}}{}^{*} c_p \log p,$$

where

$$c_p = \sum_{j=1}^{r} a_j k_{jp}.$$

In this case, there exists j such that $\underline{k}_j \neq \underline{0}$. Therefore, k_{jp} are not all zero. Since the numbers a_j are linearly independent over $\mathbb{Q}$, the algebraic numbers c_p are not all simultaneously zero. It is well known that the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, for $A_{\underline{a},\underline{\underline{k}}}$, Lemma 1 is applicable, and we obtain that $A_{\underline{a},\underline{\underline{k}}} \neq 0$. Hence, integrating in (3), we find

$$F_{T,\underline{a}}(\underline{k}_1, \ldots, \underline{k}_r) = \frac{1 - \exp\{ -iTA_{\underline{a},\underline{\underline{k}}} \}}{iTA_{\underline{a},\underline{\underline{k}}}}.$$

This together with (4) shows that

$$\lim_{T \to \infty} F_{T,\underline{a}}(\underline{k}_1, \ldots, \underline{k}_r) = \begin{cases} 1 & \text{if} \quad (\underline{k}_1, \ldots, \underline{k}_r) = (\underline{0}, \ldots, \underline{0}), \\ 0 & \text{if} \quad (\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0}), \end{cases}$$

and the lemma is proved because the right-hand side of the last equality is the Fourier transform of the Haar measure m_H. □

We will apply Lemma 2 to obtain a joint limit lemma in the space of analytic functions for absolutely convergent Dirichlet series. Denote by $H(D_L)$ the space of analytic on D_L functions equipped with topology of uniform convergence on compacta, and set

$$H^r(D_L) = \underbrace{H(D_L) \times \ldots \times H(D_L)}_{r}.$$

Let $\theta > 0$ be a fixed number,

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\theta} \right\}, \quad m, n \in \mathbb{N},$$

and

$$L_n(s) = \sum_{m=1}^{\infty} \frac{a_L(m) v_n(m)}{m^s}.$$

Since $a_L(m) \ll m^{\epsilon}$ and $v_n(m)$ is decreasing exponentially with respect to m, the latter series is absolutely convergent in any half-plane $\sigma > \sigma_0$. Extend the functions $\omega_j(p)$, $p \in \mathbb{P}$, $j = 1, \dots, r$, to the set $\mathbb{N}$ of all positive integers by

$$\omega_j(p) = \prod_{p^l \| m} \omega_j^l(p), \quad m \in \mathbb{N},$$

where $p^l \| m$ means that $p^l | m$ but $p^{l+1} \nmid m$, and define

$$L_n(s, \omega_j) = \sum_{m=1}^{\infty} \frac{a_L(m)\omega_j(m)v_n(m)}{m^s}, \tag{5}$$

the series also being absolutely convergent for $\sigma > \sigma_0$. Define

$$\underline{L}_n(s, \omega) = \big(L_n(s, \omega_1), \dots, L_n(s, \omega_r)\big),$$

and $h_n : \Omega^r \to H^r(D_L)$ by $h_n(\omega) = \underline{L}_n(s, \omega)$. Since the series $L_n(s, \omega_j)$, $j = 1, \dots, r$, are absolutely convergent in any half-plane, the mapping h_n is continuous. Therefore, every probability measure P on $(\Omega^r, \mathcal{B}(\Omega^r))$ defines the unique probability measure Ph_n^{-1} on $(H^r(D_L), \mathcal{B}(H^r(D_L)))$, where

$$Ph_n^{-1}(A) = P(h_n^{-1}A), \quad A \in \mathcal{B}(H^r(D_L)).$$

For $A \in \mathcal{B}(H^r(D_L))$, define

$$P_{T,n,\underline{a}}(A) = \frac{1}{T}\text{meas}\big\{\tau \in [0, T] : \underline{L}_n(s + i\underline{a}\tau) \in A\big\},$$

where

$$\underline{L}_n(s + i\underline{a}\tau) = \big(\underline{L}_n(s + ia_1\tau), \dots, \underline{L}_n(s + ia_r\tau)\big).$$

Moreover , a property of preservation of weak convergence under continuous mappings (see, for example, Theorem 5.1 of [14]), leads to the following lemma.

Lemma 3. *Suppose that $a_1, \dots, a_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$. Then, $P_{T,n,\underline{a}}$ converges weakly to the measure $V_n \overset{def}{=} m_H h_n^{-1}$ as $T \to \infty$.*

Proof. By the definitions of $P_{T,n,\underline{a}}$ and $Q_{T,\underline{a}}$, and the mapping h_n, for every $A \in \mathcal{B}(H^r(D_L))$, we have

$$\begin{aligned} P_{T,n,\underline{a}}(A) &= \frac{1}{T}\text{meas}\big\{\tau \in [0, T] : \big(\big(p^{-ia_1\tau} : p \in \mathbb{P}\big), \dots, \big(p^{-ia_r\tau} : p \in \mathbb{P}\big)\big) \in h_n^{-1}A\big\} \\ &= Q_{T,\underline{a}}(h_n^{-1}A) = Q_{T,\underline{a}}h_n^{-1}(A). \end{aligned}$$

Thus , $P_{T,n,\underline{a}} = Q_{T,\underline{a}}h_n^{-1}$. Thus, the continuity of h_n, Lemma 2 and Theorem 5.1 of [14] prove the lemma. □

Consider one more measure

$$\widehat{P}_{T,n,\underline{a}}(A) = \frac{1}{T}\text{meas}\big\{\tau \in [0, T] : \underline{L}_n(s + i\underline{a}\tau, \widehat{\omega}) \in A\big\}, \quad A \in \mathcal{B}(H^r(D_L)).$$

Lemma 4. *Suppose that $a_1, \dots, a_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$. Then, $\widehat{P}_{T,n,\underline{a}}$ with every $\widehat{\omega} \in \Omega^r$ also converges weakly to the measure V_n as $T \to \infty$.*

Proof. Define the mapping $\widehat{h}_n : \Omega^r \to H^r(D_L)$ by $\widehat{h}_n(\omega) = \underline{L}_n(s, \omega\widehat{\omega})$. Then, the mapping $\widehat{h}_n$ remains continuous, and repeating the arguments of the proof of Lemma 3, we obtain that $\widehat{P}_{T,n,\underline{a}}$ converges weakly to the measure $\widehat{V}_n \overset{def}{=} m_H \widehat{h}_n^{-1}$ as $T \to \infty$. By the definitions of $\widehat{h}_n$ and h_n, we have $\widehat{h}_n(\omega) = h_n(h(\omega))$ with $h(\omega) = \omega\widehat{\omega}$. At this moment, we use the invariance of the Haar measure m_H, i.e., that

$$m_H(\omega A) = m_H(A\omega) = m_H(A)$$

for all $A \in \mathcal{B}(\Omega^r)$ and $\omega \in \Omega^r$. Thus, we find

$$\widehat{V}_n = m_H(h_n h)^{-1} = (m_H h^{-1})h_n^{-1} = m_H h_n^{-1} = V_n.$$

□

3. Limit Theorems

In this section, we will prove a joint limit theorem for the function $L(s)$ from class $\mathcal{S}_1$. More precisely, we will consider the weak convergence for

$$P_{T,\underline{a}}(A) \stackrel{def}{=} \frac{1}{T}\text{meas}\{\tau \in [0,T] : \underline{L}(s+i\underline{a}\tau) \in A\}, \quad A \in \mathcal{B}(H^r(D_L)),$$

where

$$\underline{L}(s+i\underline{a}\tau) = \big(L(s+ia_1\tau), \dots, L(s+ia_r\tau)\big),$$

as $T \to \infty$. For the proof, we will apply Lemmas 3 and 4, some ergodicity results, and estimates for difference $|L(s+i\underline{a}\tau) - \underline{L}_n(s+i\underline{a}\tau)|$. We start with the latter problem.

Recall the metric in the space $H^r(D_L)$. For $g_1, g_2 \in H(D_L)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|}.$$

Here, $\{K_l : l \in \mathbb{N}\} \subset D_L$ is a sequence of compact embedded sets such that

$$\bigcup_{l=1}^{\infty} K_l = D_L,$$

and each compact set $K \subset D_L$ lies in K_l for some l. Then, ρ is a metric in $H(D_L)$ inducing the topology of uniform convergence on compacta. For $\underline{g}_1 = (g_{11}, \dots, g_{1r})$, $\underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D_L)$, taking

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1\le j\le r} \rho(g_{1j}, g_{2j}),$$

we have a metric in $H^r(D_L)$ inducing the product topology.

Lemma 5. *Suppose that $a_1, \dots, a_r$ are arbitrary real numbers. Then,*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\rho}(\underline{L}(s+i\underline{a}\tau), \underline{L}_n(s+i\underline{a}\tau))d\tau = 0.$$

Proof. Let the number θ come from the definition of $v_n(m)$, and

$$l_n(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)n^s.$$

Then, the Mellin formula

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \Gamma(s)c^{-s}ds = e^{-c}, \quad b, c > 0,$$

implies the representation (see, for example, [2])

$$L_n(s) = \frac{1}{2\pi i}\int_{\theta_1-i\infty}^{\theta_1+i\infty} L(s+z)l_n(z)\frac{dz}{z},$$

where $\theta_1 > \frac{1}{2}$. Hence, by the residue theorem,

$$L_n(s) - L(s) = \frac{1}{2\pi i}\int_{-\theta_2-i\infty}^{-\theta_2+i\infty} L(s+z)l_n(z)\frac{dz}{z} + R(s), \tag{6}$$

where $\theta_2 > 0$ and

$$R(s) = \operatorname*{Res}_{z=1-s} L(s+z)\frac{l_n(z)}{z} = \widehat{a}\cdot\frac{l_n(1-s)}{1-s}, \quad \widehat{a} = \operatorname*{Res}_{s=1} L(s).$$

Let $K \subset D_L$ be an arbitrary compact set. We fix $\epsilon > 0$ such that $\sigma_L + 2\epsilon \leq \sigma \leq 1-\epsilon$ for all $s = \sigma + it \in K$, and put $\theta_2 = \sigma - \sigma_L - \epsilon$. Then, $\theta_2 > 0$ for $\sigma + it \in K$. This and equality (2), for $s = \sigma + it \in K$ and $a \in \mathbb{R}$, gives

$$\begin{aligned} &L_n(s+ia\tau) - L(s+ia\tau) \\ &\ll \int_{-\infty}^{\infty} |L(\sigma_L+\epsilon-\sigma+\sigma+it+ia\tau+iv)|\cdot\left|\frac{l_n(\sigma_L+\epsilon-\sigma+iv)}{\sigma_L+\epsilon-\sigma+iv}\right|dv + |\widehat{a}|\left|\frac{l_n(1-s-ia\tau)}{1-s-ia\tau}\right|. \end{aligned}$$

Taking v in place of $t+v$, we have, for $s \in K$,

$$\begin{aligned} &L_n(s+ia\tau) - L(s+ia\tau) \\ &\ll \int_{-\infty}^{\infty} |L(\sigma_L+\epsilon+ia\tau+iv)|\cdot\sup_{s\in K}\left|\frac{l_n(\sigma_L+\epsilon-s+iv)}{\sigma_L+\epsilon-s+iv}\right|dv + |\widehat{a}|\sup_{s\in K}\left|\frac{l_n(1-s-ia\tau)}{1-s-ia\tau}\right|. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{1}{T}\int_0^T \sup_{s\in K}|L(s+ia\tau) - L_n(s+ia\tau)|d\tau \\ &\ll \int_{-\infty}^{\infty}\left(\frac{1}{T}\int_0^T |L(\sigma_L+\epsilon+ia\tau+iv)|d\tau\right)\sup_{s\in K}\left|\frac{l_n(\sigma_L+\epsilon-s+iv)}{\sigma_L+\epsilon-s+iv}\right|dv \\ &\quad +|\widehat{a}|\cdot\frac{1}{T}\int_0^T \sup_{s\in K}\left|\frac{l_n(1-s-ia\tau)}{1-s-ia\tau}\right|d\tau \\ &\stackrel{def}{=} I_T^{(1)} + I_T^{(2)}. \end{aligned} \tag{7}$$

It is known [2] that, for fixed $\sigma_L < \sigma < 1$,

$$\int_{-T}^{T} |L(\sigma+it)|^2 dt \ll_{\sigma,L} T.$$

This, for the same σ and $v \in \mathbb{R}$, gives

$$\int_0^T |L(\sigma+ia\tau+iv)|^2 d\tau = \frac{1}{a}\int_v^{aT+v} |L(\sigma+it)|^2 dt \ll_{\sigma,a} T(1+|v|). \tag{8}$$

Using the well-known estimate

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad c > 0, \tag{9}$$

we find that, for all $s \in K$,

$$\begin{aligned} \frac{l_n(\sigma_L+\epsilon-s+iv)}{\sigma_L+\epsilon-s+iv} &\ll_\theta n^{\sigma_L+\epsilon-\sigma}\left|\Gamma\left(\frac{1}{\theta}(\sigma_L+\epsilon-\sigma+it+iv)\right)\right| \\ &\ll_\theta n^{-\epsilon}\exp\{-c_1|v-t|\} \end{aligned}$$

$$\ll_{\theta,K} \exp\{-c_2|v|\}, \quad c_1, c_2 > 0.$$

This and (8) show that

$$I_T^{(1)} \ll_{\epsilon,L,\theta,a,K} \ll n^{-\epsilon} \int_{-\infty}^{\infty} (1+|v|)^{\frac{1}{2}} \exp\{-c_2|v|\} dv \ll_{\epsilon,L,\theta,a,K} n^{-\epsilon}. \tag{10}$$

Similarly, by (9), for $s \in K$,

$$\frac{l_n(1-s-ia\tau)}{1-s-ia\tau} \ll_\theta n^{1-\sigma} \exp\{-c_3|t+a\tau|\} \ll_{\theta,K,a} n^{1-\sigma_L-2\epsilon} \exp\{-c_4|\tau|\}, \quad c_4 > 0.$$

Thus,

$$I_T^{(2)} \ll_{\theta,K,a} n^{-\epsilon} \frac{1}{T} \int_0^T \exp\{-c_4|\tau|\} d\tau \ll_{\theta,K,a} \frac{\log T}{T}.$$

The latter estimate, (10) and (7) prove that, for every compact set $K \subset D_L$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} |L(s+ia\tau) - L_n(s+ia\tau)| d\tau = 0.$$

Therefore, the lemma follows from the definitions of the metrics ρ and $\underline{\rho}$. □

Now, for $\omega \in \Omega^r$, let

$$\underline{L}(s,\omega) = \big(L(s,\omega_1), \ldots, L(s,\omega_r)\big),$$

where

$$L(s,\omega_j) = \sum_{m=1}^{\infty} \frac{a_L(m)\omega_j(m)}{m^s}, \quad j = 1, \ldots, r.$$

Then, it is known [2] that the latter series, for almost all ω_j, are uniformly convergent on the compact subset of the half-plane $\sigma > \sigma_L$. Since the Haar measure m_H is the product of the Haar measures m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j))$, we have that $\underline{L}(s,\omega)$ is the $H^r(D_L)$-valued random element. Moreover, an analogue of Lemma 5 is valid.

Lemma 6. *Suppose that $a_1, \ldots, a_r$ are arbitrary real numbers. Then, for almost all $\omega \in \Omega^r$,*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\rho}\big(\underline{L}(s+i\underline{a}\tau,\omega), \underline{L}_n(s+i\underline{a}\tau,\omega)\big) d\tau = 0.$$

Proof. It is known [2] that, for almost all $\omega \in \Omega$,

$$\int_{-T}^{T} |L(\sigma+it,\omega)|^2 dt \ll_{\sigma,L} T.$$

Therefore, repeating the proof of Lemma 5, we obtain that, for a compact set $K \in D_L$ and real number a,

$$\frac{1}{T} \int_0^T \sup_{s\in K} |L(s+ia\tau,\omega) - L_n(s+ia\tau,\omega)| d\tau \ll_{\epsilon,L,\theta,a,K} n^{-\epsilon} \tag{11}$$

with certain $\epsilon > 0$. In this case, in the analogous of estimate (7), we have not the second term on the right-hand side. Since $m_H = m_{1H} \times \ldots \times m_{rH}$, estimate (11) and the definitions of the metrics ρ and $\underline{\rho}$ prove the lemma. □

Now, we are ready to consider the measure $P_{T,\underline{a}}$.

Theorem 3. *Suppose that real algebraic numbers $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$. Then, on $(H^r(D_L), \mathcal{B}(H^r(D_L)))$, there exists a probability measure P such that $P_{T,\underline{a}}$ converges weakly to P as $T \to \infty$.*

Proof. Recall that a family of probability measures $\{Q\}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is called tight if, for every $\epsilon > 0$, there exists a compact set $K = K(r) \subset \mathbb{X}$ such that

$$Q(K) > 1 - \epsilon$$

for all Q.

Denote by V_{nj} marginal measures of the measure V_n, $j = 1, \dots, r$. Since the series for $L_n(s)$ is absolutely convergent, we obtain by a standard way that the sequence $\{V_{nj} : n \in \mathbb{N}\}$ is tight, $j = 1, \dots, r$. Then, for every $\epsilon > 0$, there exists a compact set $K_j \subset H(D_L)$ such that, for all $n \in \mathbb{N}$,

$$V_{nj}(K_j) > 1 - \frac{\epsilon}{r}, \quad j = 1, \dots, r. \tag{12}$$

Let $K = K_1 \times \dots \times K_r$. Then, K is a compact set in $H^r(D_L)$. Moreover, by (12), for all $n \in \mathbb{N}$,

$$V_n(H^r(D_L) \setminus K) \le \sum_{j=1}^{r} V_{nj}(H(D_L) \setminus K_j) < \epsilon.$$

Thus,

$$V_n(K) > 1 - \epsilon$$

for all $n \in \mathbb{N}$. Hence, the sequence $\{V_n\}$ is tight. Therefore, by the Prokhorov theorem, see [14], the sequence $\{V_n\}$ is relatively compact. This means that every sequence of $\{V_n\}$ contains a subsequence $\{V_{n_k}\}$ weakly convergent to a certain probability measure P on $(H^r(D_L), \mathcal{B}(H^r(D_L)))$ as $k \to \infty$.

Denote by X_n the $H^r(D_L)$-valued random element having the distribution V_n, and by $\xrightarrow{\mathcal{D}}$ the convergence in distribution. Then, we have

$$X_{n_k} \xrightarrow[k \to \infty]{\mathcal{D}} P. \tag{13}$$

On the certain probability space with measure μ, define the random variable ξ_T which is uniformly distributed on $[0, T]$. Moreover, let

$$X_{T,n,\underline{a}} = X_{T,n,\underline{a}}(s) = \underline{L}_n(s + i\underline{a}\xi_T)$$

and

$$Y_{T,\underline{a}} = Y_{T,\underline{a}}(s) = \underline{L}(s + i\underline{a}\xi_T).$$

By Lemma 3,

$$X_{T,n,\underline{a}} \xrightarrow[T \to \infty]{\mathcal{D}} X_n, \tag{14}$$

and Lemma 5 implies, for every $\epsilon > 0$,

$$\begin{aligned} &\lim_{n \to \infty} \limsup_{T \to \infty} \mu\{\underline{\rho}(Y_{T,\underline{a}}, X_{T,n,\underline{a}}) \ge \epsilon\} \\ &\quad \le \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{\epsilon T} \int_0^T \underline{\rho}\big(\underline{L}(s + i\underline{a}\tau), \underline{L}_n(s + i\underline{a}\tau)\big) d\tau = 0. \end{aligned}$$

This together with relations (13) and (14) shows that all hypotheses of Theorem 4.2 from [2] are satisfied. Therefore,

$$Y_{T,\underline{a}} \xrightarrow[T\to\infty]{\mathcal{D}} P, \tag{15}$$

and this proves the theorem. □

By (15), the measure P is independent on the sequence $\{X_{n_k}\}$. Since the sequence $\{X_n\}$ is relatively compact, it follows that

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{16}$$

On $(H^r(D_L), \mathcal{B}(H^r(D_L)))$, define one more measure

$$\widehat{P}_{T,\underline{a}}(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \underline{L}(s + i\underline{a}\tau, \omega) \in A\}$$

for almost all $\omega \in \Omega$. Then, by (15), Lemmas 4 and 6, similarly as above, we obtain the analogue of Theorem 3.

Theorem 4. *Suppose that real algebraic numbers $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Then, $\widehat{P}_{T,\underline{a}}$ also converges weakly to the measure P as $T \to \infty$.*

4. Identification of the Measure *P*

For the proof of Theorem 2, the explicit form of the limit measure in Theorems 3 and 4 is needed. For this, some elements of ergodic theory can be applied.

For brevity, we set

$$\underline{a}(\tau) = \Big((p^{-ia_1\tau} : p \in \mathbb{P}), \ldots, (p^{-ia_r\tau} : p \in \mathbb{P}) \Big), \quad \tau \in \mathbb{R},$$

and define

$$E(\tau, \omega) = \underline{a}(\tau)\omega, \quad \omega \in \Omega^r.$$

Then, $E(\tau, \omega)$ is a measurable measure preserving transformation of the group Ω^r, and $\{E(\tau, \omega) : \tau \in \mathbb{R}\}$ form a group of these transformations. For $A \in \mathcal{B}(\Omega^r)$, let $A(\tau) = E(\tau, A)$. If the sets A and $A(\tau)$ differ one from another at most by a set of m_H-measure zero, then the set A is called invariant. All invariant sets form a σ-field. If this field consists only of sets of m_H-measure 1 or 0, then the group $\{E(\tau, \omega)\}$ is called ergodic.

Lemma 7. *Suppose that real algebraic numbers $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Then, the group $\{E(\tau, \omega) : \tau \in \mathbb{R}\}$ is ergodic.*

Proof. The characters χ of the group Ω^r are of the form

$$\chi(\omega) = \prod_{j=1}^{r} \prod_{p\in\mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p), \tag{17}$$

where the sign $*$ means that only a finite number of integers k_{jp} is not zero. This already was used in the proof of Lemma 2 for the definition of the Fourier transform of the measure $Q_{T,\underline{a}}$. Suppose that A is an invariant set with respect to $\{E(\tau, \omega)\}$, and χ is a nontrivial character of Ω^r, i.e., $\chi(m) \not\equiv 1$. Then, by (17), $(\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0})$, and thus $A_{\underline{a},\underline{k}} \neq 0$ in the notation used in the proof of Lemma 2. Therefore, there exists a real number τ_0 such that

$$\chi(\underline{a}(\tau_0)) = \exp\{-i\tau_0 A_{\underline{a},\underline{k}}\} \neq 1. \tag{18}$$

Take the indicator function $\mathbb{I}_A$ of the set A. In virtue of invariance of the set A, we have

$$\mathbb{I}_A\big(\underline{a}(\tau_0)\omega\big) = \mathbb{I}_A(\omega)$$

for almost all $\omega \in \Omega^r$. Hence, denoting by $\widehat{g}$ the Fourier transform of a function g, we find

$$\begin{aligned}\widehat{\mathbb{I}}_A(\chi) &= \chi\big(\underline{a}(\tau_0)\big)\int_{\Omega^r}\mathbb{I}_A\big(\underline{a}(\tau_0)\omega\big)\chi(\omega)dm_H \\ &= \chi\big(\underline{a}(\tau_0)\big)\int_{\Omega^r}\mathbb{I}_A(\omega)\chi(\omega)dm_H = \chi\big(\underline{a}(\tau_0)\big)\widehat{\mathbb{I}}_A(\chi).\end{aligned}$$

Therefore, in view of (18),

$$\widehat{\mathbb{I}}_A(\chi) = 0. \tag{19}$$

Now, suppose that χ_0 denotes the trivial character of Ω^r, and $\widehat{\mathbb{I}}_A(\chi_0) = c$. Then, taking into account (19), we have

$$\widehat{\mathbb{I}}_A(\chi) = c\int_{\Omega^r}\chi(m)dm_h = \widehat{c}(\chi)$$

for an arbitrary character χ of Ω^r. This shows that $\mathbb{I}_A(\omega) = c$ for almost all $\omega \in \Omega^r$. However, $\mathbb{I}_A$ is the indicator function of the A, thus, $\mathbb{I}_A(\omega) = 1$ or $\mathbb{I}_A(\omega) = 0$ for almost all $\omega \in \Omega^r$. In other words, $m_H(A) = 1$ or $m_H(A) = 0$, and the lemma is proved. □

Denote by $P_{\underline{L}}$ the distribution of the $H^r(D_L)$-valued random element $\underline{L}(s,\omega)$, i.e.,

$$P_{\underline{L}}(A) = m_H\{\omega \in \Omega^r : \underline{L}(s,\omega) \in A\}, \quad A \in \mathcal{B}(H^r(D_L)).$$

Lemma 8. *The measure P in Theorems 3 and 4 coincide with $P_{\underline{L}}$.*

Proof. Suppose that A is a continuity set of the measure P, i.e., $P(\partial A) = 0$, where ∂A denotes the boundary of A. Then, Theorem 4 together with the equivalent of weak convergence of probability measure in terms of continuity sets (see, for example, Theorem 2.1 of [14]) yields

$$\lim_{T\to\infty}\widehat{P}_{T,\underline{a}}(A) = P(A). \tag{20}$$

On $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$, define the random variable

$$\theta(\omega) = \begin{cases}1 & \text{if} \quad \underline{L}(s,\omega) \in A, \\ 0 & \text{otherwise.}\end{cases}$$

Lemma 7 implies the ergodicity of the random process $\theta\big(E(\tau,\omega)\big)$. Therefore, by the Birkhoff–Khintchine ergodic theorem (see, for example, [15]), we obtain

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\theta\big(E(\tau,\omega)\big)d\tau = \mathbb{E}\theta = P_{\underline{L}}(A), \tag{21}$$

where $\mathbb{E}\theta$ is the expectation of θ. However, by the definitions of $E(\tau,\omega)$ and θ,

$$\frac{1}{T}\int_0^T\theta\big(E(\tau,\omega)\big)d\tau = \frac{1}{T}\text{meas}\big\{\tau \in [0,T] : \underline{L}(s+i\underline{a}\tau,\omega) \in A\big\} = \widehat{P}_{T,\underline{a}}(A).$$

This and (21) show that

$$\lim_{T\to\infty}\widehat{P}_{T,\underline{a}}(A) = P_{\underline{L}}(A).$$

Thus, in view of (20), $P(A) = P_{\underline{L}}(A)$ for all continuity sets A of P. It is well known that continuity sets constitute the defining class. Thus, $P = P_{\underline{L}}$. □

It remains to find a support of the measure $P_{\underline{L}}$. We recall that the support of $P_{\underline{L}}$ is a minimal closed set S_L such that $P_{\underline{L}}(S_L) = 1$.

Let

$$S_L = \{g \in H(D_L) : g(s) = 0 \text{ or } g(s) \equiv 0\}.$$

Lemma 9. *The support of the measure $P_{\underline{L}}$ is the set S_L^r.*

Proof. It is known that the support of the measure

$$P_L(A) \overset{def}{=} m_{jH}\{\omega_j \in \Omega_j : L(s, \omega_j) \in A\}, \quad A \in \mathcal{B}(H(D_L)), \quad j = 1, \dots, r,$$

is the set S_L (see [2] or [7]). Since the space $H(D_L)$ is separable, we have

$$\mathcal{B}(H^r(D_L)) = \underbrace{\mathcal{B}(H(D_L)) \times \dots \times \mathcal{B}(H(D_L))}_{r}.$$

Therefore, it suffices to consider the measure $P_{\underline{L}}$ on rectangular sets

$$A = A_1 \times \dots \times A_r, \quad A_j \in \mathcal{B}(H(D_L)), \quad j = 1, \dots, r.$$

Moreover, $m_H = m_{1H} \times \dots \times m_{rH}$. These remarks show that

$$m_H\{\omega \in \Omega^r : \underline{L}(s, \omega) \in A\} = \prod_{j=1}^{r} m_{jH}\{\omega_j \in \Omega_j : L(s, \omega_j) \in A_j\}.$$

This and the minimality of the support prove the lemma. □

5. Proof of Theorem 2

Theorem 2 follows from Theorem 3, Lemmas 8 and 9, and the Mergelyan's theorem on the approximation of analytic functions by polynomials (see [16]).

Proof of Theorem 2. *Case of lower density:*

By the Mergelyan theorem, there exist polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1 \le j \le r} \sup_{s \in K_j} |f_j(s) - e^{p_j(s)}| < \frac{\epsilon}{2}. \tag{22}$$

Let

$$G_\epsilon = \left\{(g_1, \dots, g_r) \in H^r(D_L) : \sup_{1 \le j \le r} \sup_{s \in K_j} |g_j(s) - e^{p_j(s)}| < \epsilon\right\}.$$

Then, in view of Lemma 9, G_ϵ is an open neighborhood of an element $\left(e^{p_1(s)}, \dots, e^{p_r(s)}\right)$ of the support S_L^r of the measure $P_{\underline{L}}$. Therefore, by a property of the support,

$$P_{\underline{L}}(G_\epsilon) > 0. \tag{23}$$

Hence, by Theorem 3, Lemma 8, and the equivalent of weak convergence in terms of open sets (see, for example, Theorem 2.1 of [14]),

$$\liminf_{T \to \infty} P_{T,\underline{a}}(G_\epsilon) \ge P_{\underline{L}}(G_\epsilon) > 0.$$

Thus, by the definitions of $P_{T,\underline{\epsilon}}$ and G_ϵ,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{1 \le j \le r} \sup_{s \in K_j} |L(s + ia_j\tau) - e^{p_j(s)}| < \frac{\epsilon}{2}\right\} > 0. \tag{24}$$

Define one more set

$$A_\epsilon = \left\{ (g_1, \dots, g_r) \in H^r(D_L) : \sup_{1 \le j \le r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \epsilon \right\}.$$

Then, in view of (22), we have $G_\epsilon \subset A_\epsilon$. This and (24) show that

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{1 \le j \le r} \sup_{s \in K_j} |L(s + ia_j\tau) - f_j(s)| < \epsilon \right\} > 0.$$

Case of density: The boundaries ∂A_ϵ of the set A_ϵ do not intersect for different values of ϵ. Therefore, $P_{\underline{L}}(\partial A_\epsilon) > 0$ at most for countable many $\epsilon > 0$, i.e., A_ϵ is a continuity set of $P_{\underline{L}}$ for all but at most countable many $\epsilon > 0$. Moreover, since $G_\epsilon \subset A_\epsilon$, we have $P_{\underline{L}}(A_\epsilon) > 0$ by (23). Therefore, Theorem 3, Lemma 8, and the equivalent of weak convergence in terms of continuity sets (see Theorem 2.1 of [14]) yield

$$\lim_{T \to \infty} P_{T,\underline{a}}(A_\epsilon) = P_{\underline{L}}(A_\epsilon) > 0$$

for all but at most countably many $\epsilon > 0$. This and the definitions of $P_{T,\underline{a}}$ and A_ϵ complete the proof of the theorem. □

6. Concluding Remarks

Let $L(s)$ be a function from the Selberg–Steuding class. Combining algebraic, analytic, and probabilistic methods, we obtain a theorem on simultaneous approximations of a collection of analytic functions $(f_1(s), \dots, f_r(s))$ in the strip $D_L = \{s \in \mathbb{C} : \sigma_L < \sigma < 1\}$ by a collection of shifts $(L(s + ia_1\tau), \dots, L(s + ia_r\tau))$, where $a_1, \dots, a_r$ are real algebraic numbers linearly independent over the field of rational numbers, and $\sigma_L > \frac{1}{2}$ is a certain number depending on L. More precisely, we proved that the set of the above shifts has a positive lower density, or even positive lower density for all but not most countable many accuracies of approximation. Thus, the set of approximating shifts is infinite.

It is important that the measure V_n is independent of the used shift and converges weakly to the measure $P_{\underline{L}}$. This result can be used for the proof of other joint theorems on joint approximation of analytic functions by more complicated shifts including discrete shifts.

Author Contributions: Conceptualization, R.K., A.L. and B.Ž.; methodology, R.K., A.L. and B.Ž.; investigation, R.K., A.L. and B.Ž.; writing—original draft preparation, R.K., A.L. and B.Ž.; writing—review and editing, R.K., A.L. and B.Ž. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the referees for useful remarks and comments.

Conflicts of Interest: The authors declare no conflict of interest.

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