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On Joint Discrete Universality of the Riemann Zeta-Function in Short Intervals

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Abstract. In the paper, we prove that the set of discrete shifts of the Riemann zeta-function $(\zeta(s+2\pi i a_1 k), \dots, \zeta(s+2\pi i a_r k))$, $k \in \mathbb{N}$, approximating analytic non-vanishing functions $f_1(s), \dots, f_r(s)$ defined on $\{s \in \mathbb{C} : 1/2 < \mathrm{Re}s < 1\}$ has a positive density in the interval $[N, N+M]$ with $M = o(N)$, $N \to \infty$, with real algebraic numbers $a_1, \dots, a_r$ linearly independent over $\mathbb{Q}$. A similar result is obtained for shifts of certain absolutely convergent Dirichlet series.

Keywords: Riemann zeta-function, universality, weak convergence.

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1 Introduction

Let, as usual, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{P}$ and $\mathbb{C}$ denote the sets of all real, rational, integer, non-negative integer, positive integer, prime and complex numbers, respectively, and $s = \sigma + it$, $\sigma, t \in \mathbb{R}$, be a complex variable. The Riemann zeta-function $\zeta(s)$ is defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

and has the analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. The function $\zeta(s)$ was already known to Euler, and its importance was demonstrated by Riemann in 1859, however, till our days remains one of the most interesting and keeping many problems analytic object. In 1975, Voronin proved [18] that $\zeta(s)$ has a very good approximation property called universality. This means that a wide class of analytic functions is approximated by shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$. More precisely, Voronin obtained that if $0 < r < 1/4$, and the function $f(s)$ is continuous and non-vanishing in the disc $|s| \leqslant r$, and analytic in the interior of that disc, then, for every $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that

$$\max_{|s| \leqslant r} |\zeta(s + 3/4 + i\tau) - f(s)| < \varepsilon.$$

The Voronin theorem by various authors was improved and extended, see the monograph [17] and the informative paper [11]. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then a modern form of the Voronin theorem is the following, see, for example, [6, 17]: suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\Big\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\Big\} > 0.$$

Here meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The above inequality shows that there exist infinitely many shifts approximating a given function $f(s) \in H_0(K)$.

Joint approximation of analytic functions by shifts of the function $\zeta(s)$ was considered in [14]. Let $a_1 = 1, a_2, \dots, a_r$ be algebraic numbers linearly independent over $\mathbb{Q}$, and, for $j = 1, \dots, r$, $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every real $a \neq 0$ and $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\Big\{\tau \in [0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s + i a_j a \tau) - f_j(s)| < \varepsilon\Big\} > 0.$$

The above approximation theorems are of continuous type because τ in approximating shifts can take arbitrary real values. Reich in [16] proposed a discrete version of approximation theorems when τ takes values from a certain discrete set, for example, an arithmetic progression. Denote by $\#A$ the cardinality of the set A, and suppose that N runs over the set $\mathbb{N}_0$. Then a special case of Reich's theorem says that if $K \in \mathcal{K}$, $f(s) \in H_0(K)$, then, for every real $h \neq 0$ and $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s \in K} |\zeta(s + ikh) - f(s)| < \varepsilon\Big\} > 0.$$

The joint discrete approximation by generalized shifts $\zeta(s+i\varphi_j(k))$ with $\varphi_j(k) = k^{\alpha_j}(\log k)^{\beta_j}$ for some reals α_j and β_j was considered in [15]. The discrete universality for more general zeta-functions was studied, for example, in [2] and [5].

All above mentioned universality theorems are not effective in the sense that any concrete approximating shifts are not known. Since, for their proofs, the measure theory is applied, founding of concrete shifts is not possible. Therefore, effectivization of universality theorems is reduced to indication of intervals containing approximating values of τ or k. This leads to the type of universality theorems in short intervals. The first result in this direction was obtained in [7]. Suppose that $T^{1/3}(\log T)^{26/15} \leqslant H \leqslant T$, $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{H}\text{meas}\Big\{\tau \in [T, T+H] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\Big\} > 0.$$

Thus, there exists the interval $[T, T+H]$ of length $H = o(T)$, $T \to \infty$, containing infinitely many values τ such that $\zeta(s+i\tau)$ approximate the function $f(s)$. The joint version of [7] was obtained in [9] and [10], and the discrete version in [8].

The aim of this paper is to prove a joint discrete universality theorem for the Riemann zeta-function in short intervals. For statements of results, we need some definitions and notations. Denote by $H(D)$ the space of analytic functions on D endoved with the topology of uniform convergence on compacta. Let $\mathcal{B}(\mathbb{X})$ be the Borel σ-field of the space $\mathbb{X}$, and $\gamma = \{s \in \mathbb{C} : |s| = 1\}$.

Define the set $\Omega = \prod_{p\in\mathbb{P}} \gamma_p$, where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group.

Let $\Omega^r = \Omega_1 \times \cdots \times \Omega_r$, where $\Omega_j = \Omega$ for $j = 1, \dots, r$. Then again Ω^r is a topological Abelian group. Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H can be defined, and we have the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$. Denote by $\omega_j(p)$ the pth component of an element $\omega_j \in \Omega$, $j = 1, \dots, r$, $p \in \mathbb{P}$, and by $\omega = (\omega_1, \dots, \omega_r)$ the elements of Ω^r, and on the probability space $(\Omega^r,\mathcal{B}(\Omega^r),m_H)$, define the $H^r(D)$,$H^r(D) = \underbrace{H(D) \times \cdots \times H(D)}_{r}$,

-valued random element $\underline{\zeta}(s, \omega) = (\zeta(s, \omega_1), \dots, \zeta(s, \omega_r))$, where

$$\zeta(s, \omega_j) = \prod_{p\in\mathbb{P}} \left(1 - \omega_j(p)/p^s\right)^{-1}, \quad j = 1, \dots, r.$$

For real numbers $a_1, \dots, a_r$, let $\widehat{a} = \max_{1\leqslant j\leqslant r} |a_j|$ and $\widehat{\widehat{a}} = \max_{1\leqslant j\leqslant r} |a_j|^{-1}$.

Theorem 1. *Suppose that $a_1, \dots, a_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$ and $\widehat{\widehat{a}}(\widehat{a}N)^{1/3}(\log \widehat{a}N)^{26/15} \leqslant M \leqslant N-3$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{M+1}\Big\{N \leqslant k \leqslant N+M : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + 2\pi i a_j k) - f_j(s)| < \varepsilon\Big\}$$
$$\geqslant m_H\Big\{\omega \in \Omega^r : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s, \omega_j) - f_j(s)| < \varepsilon\Big\} > 0.$$

Moreover, the limit

$$\lim_{N\to\infty} \frac{1}{M+1} \Big\{ N \leqslant k \leqslant N+M : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+2\pi i a_j k) - f_j(s)| < \varepsilon \Big\}$$
$$= m_H \Big\{ \omega \in \Omega^r : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s,\omega_j) - f_j(s)| < \varepsilon \Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

It turns out that a certain function generated by $\zeta(s)$ also has the same approximation properties as $\zeta(s)$ but is given by absolutely convergent series. Let $\theta > 1/2$ be a fixed number, and $v_u(m) = \exp\{-(m/u)^\theta\}$ for $u > 0$ and $m \in \mathbb{N}$. Define

$$\zeta_u(s) = \sum_{m=1}^{\infty} \frac{v_u(m)}{m^s}.$$

Since $v_u(m)$ decreases exponentially with respect to m, the latter series converges absolutely for $\sigma > \sigma_0$ with every finite σ_0. For $u \to \infty$, $v_u(m)$ tends to 1, however, is not possible to take $u \to \infty$ in the definition of $\zeta_u(s)$. Nevertheless, it turns out that $\zeta_u(s)$ is close to $\zeta(s)$ in the mean, and this is sufficient for the proof of the following statement.

Theorem 2. *Suppose that* $a_1, \ldots, a_r$ *are real algebraic numbers linearly independent over* $\mathbb{Q}$, $u_N \to \infty$ *and* $u_N \ll \exp\{o(N)\}$, *and* $\widehat{\widehat{a}}(\widehat{a}N)^{1/3}(\log \widehat{a}N)^{26/15} \leqslant M \leqslant N-3$. *For* $j = 1, \ldots, r$, *let* $K_j \in \mathcal{K}$ *and* $f_j(s) \in H_0(K_j)$. *Then the limit*

$$\lim_{N\to\infty} \frac{1}{M+1} \Big\{ N \leqslant k \leqslant N+M : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta_{u_N}(s+2\pi i a_j k) - f_j(s)| < \varepsilon \Big\}$$
$$= m_H \Big\{ \omega \in \Omega^r : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s,\omega_j) - f_j(s)| < \varepsilon \Big\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Note that the power $1/3$ in Theorems 1 and 2 is not final, and depends on mean square estimates in short intervals.

For the proof of Theorems 1 and 2, we will apply arguments of weak convergence of probability measures.

2 Mean square estimates

Mean square estimates play an important role in the proofs of universality for zeta-functions, and this role is crucial in the case of short intervals. Recall that the notation $a \ll_\theta b$, $b > 0$, means that there exists a constant $C(\theta) > 0$ such that $|a| \leqslant C(\theta) b$.

Lemma 1. *Suppose that* $1/2 < \sigma \leqslant 13/22$ *is fixed, and* $T^{1/3}(\log T)^{26/15} \leqslant H \leqslant T$. *Then uniformly in* H

$$\int_{T-H}^{T+H} |\zeta(\sigma+it)|^2 \, \mathrm{d}t \ll_\sigma H.$$

Proof. The lemma is a special case of Theorem 7.1 from [4] for the exponential pair $(4/11, 6/11)$. □

Lemma 2. *Suppose that $1/2 < \sigma \leqslant 13/22$ is fixed, and $a \in \mathbb{R} \setminus \{0\}$. Then uniformly in H, $|a|^{-1}(|a|T)^{1/3}(\log|a|T)^{26/15} \leqslant H \leqslant T$,*

$$\int_T^{T+H} |\zeta(\sigma + iat + i\tau)|^2 \, \mathrm{d}t \ll_{\sigma,a} H\,(1 + |\tau|)\,.$$

Proof. Lemma 1 implies that uniformly in H, $|a|^{-1}(|a|T)^{1/3}(\log|a|T)^{26/15} \leqslant H \leqslant T$,

$$\begin{aligned}\int_{T-H}^{T+H} |\zeta(\sigma + iat)|^2 \, \mathrm{d}t &= \int_{T-H}^{T+H} |\zeta(\sigma + i|a|t)|^2 \, \mathrm{d}t \\ &= \frac{1}{|a|} \int_{|a|(T-H)}^{|a|(T+H)} |\zeta(\sigma + it)|^2 \, \mathrm{d}t \ll_\sigma H. \end{aligned} \tag{2.1}$$

We have

$$\int_T^{T+H} |\zeta(\sigma + iat + i\tau)|^2 \, \mathrm{d}t = \int_{T+\tau/a}^{T+H+\tau/a} |\zeta(\sigma + iat)|^2 \, \mathrm{d}t \ll \int_{T-H-|\tau|/|a|}^{T+H+|\tau|/|a|} |\zeta(\sigma + iat|^2 \, \mathrm{d}t. \tag{2.2}$$

Therefore, if $H + |\tau|/|a| \leqslant T$, then, by (2.1),

$$\int_T^{T+H} |\zeta(\sigma + iat + i\tau)|^2 \, \mathrm{d}t \ll_\sigma H + |\tau|/|a| \ll_{\sigma,a} H\,(1 + |\tau|)\,.$$

It is well known that, for fixed $1/2 < \sigma < 1$,

$$\int_{-T}^{T} |\zeta(\sigma + it)|^2 \, \mathrm{d}t \ll_\sigma T. \tag{2.3}$$

If $H + |\tau|/|a| > T$, then $T + H + |\tau|/|a| \leqslant 2(H + |\tau|/|a|)$ and $T - H - |\tau|/|a| > -2(H + |\tau|/|a|)$. Therefore, in view of (2.2) and (2.3),

$$\begin{aligned}\int_T^{T+H} |\zeta(\sigma + iat + i\tau)|^2 \, \mathrm{d}t &\ll \int_{T-H-|\tau|/|a|}^{T+H+|\tau|/|a|} |\zeta(\sigma + iat)|^2 \, \mathrm{d}t \\ \ll_a \int_{-2|a|(H+|\tau|/|a|)}^{2|a|(H+|\tau|/|a|)} |\zeta(\sigma + it)|^2 \, \mathrm{d}t &\ll_{\sigma,a} H + |\tau| \ll_{\sigma,a} H\,(1 + |\tau|)\,. \end{aligned}$$

□

Since we consider the discrete universality, we need mean square estimates of discrete type in short intervals. For this, the Gallagher lemma connecting discrete and continuous mean squares of certain function is useful.

Lemma 3. *Suppose that T_0, $T \geqslant \delta > 0$, $\mathcal{T}$ is a finite non-empty set in the interval $[T_0 + \delta/2, T_0 + T - \delta/2]$, and*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x|<\delta}} 1.$$

Let the complex valued function $S(t)$ be continuous on $[T_0, T_0+T]$ and have a continuous derivative on (T_0, T_0+T). Then,

$$\sum_{t\in\mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leqslant \frac{1}{\delta}\int_{T_0}^{T_0+T} |S(x)|^2\,\mathrm{d}x + \left(\int_{T_0}^{T_0+T} |S(x)|^2\,\mathrm{d}x \int_{T_0}^{T_0+T} |S'(x)|^2\,\mathrm{d}x\right)^{1/2}.$$

A proof of the lemma can be found, for example, in [13], Lemma 1.4.

Now, Lemmas 2 and 3 lead to the following statement.

Lemma 4. *Suppose that $1/2 < \sigma \leqslant 13/22$ is fixed, and $a \in \mathbb{R}\setminus\{0\}$. Then uniformly in M, $|a|^{-1}(|a|N)^{1/3}(\log|a|N)^{26/15} \leqslant M \leqslant N-3$,*

$$\sum_{k=N}^{N+M} |\zeta(\sigma+iak+i\tau)|^2 \ll_{\sigma,a} M\,(1+|\tau|)\,.$$

Proof. We take in Lemma 3 $\delta = 1$, $T_0 = N-1$, $T = M+2$, and $\mathcal{T} = \{N, N+1, \dots, N+M\}$. Clearly, $N_\delta(x) = 1$. Then an application of Lemma 3 with $S(t) = \zeta(\sigma+iat+i\tau)$ gives

$$\sum_{k=N}^{N+M} |\zeta(\sigma+iak+i\tau)|^2 \ll \int_{N-1}^{N+M+1} |\zeta(\sigma+iat+i\tau)|^2\,\mathrm{d}t + \left(\int_{N-1}^{N+M+1} |\zeta(\sigma+iat+i\tau)|^2\,\mathrm{d}t \int_{N-1}^{N+M+1} |\zeta'(\sigma+iat+i\tau)|^2\,\mathrm{d}t\right)^{1/2}. \quad (2.4)$$

In view of Lemma 2,

$$\int_{N-1}^{N+M+1} |\zeta(\sigma+iat+i\tau)|^2\,\mathrm{d}t \ll_{\sigma,a} M\,(1+|\tau|)\,.$$

Hence, by the Cauchy integral formula,

$$\int_{N-1}^{N+M+1} |\zeta'(\sigma+iat+i\tau)|^2\,\mathrm{d}t \ll_{\sigma,a} M\,(1+|\tau|)\,.$$

The latter two estimates together with (2.4) prove the lemma. □

Next we apply Lemma 4 for estimation of the mean of the difference between $\zeta(s)$ and $\zeta_u(s)$.

Lemma 5. *Suppose that $a \in \mathbb{R}\setminus\{0\}$, $K \subset D$ is a compact set, and $|a|^{-1}(|a|N)^{1/3}(\log|a|N)^{26/15} \leqslant M \leqslant N-3$. Then,*

$$\frac{1}{M+1}\sum_{m=N}^{N+M} \sup_{s\in K} |\zeta(s+iak) - \zeta_u(s+iak)| \ll_{\varepsilon,\theta,K} u^{-\varepsilon} + u^{1/2-2\varepsilon}\exp\{-c_2|a|N\}, \quad c_2 > 0.$$

Proof. Denote, as usual, by $\Gamma(s)$ the Euler gamma-function, and put

$$l_u(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)u^s,$$

where θ is from definition of $v_u(m)$. Then the Mellin formula

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\Gamma(s)d^{-s}\,\mathrm{d}s = \mathrm{e}^{-d}, \quad b,d>0,$$

imply the representation

$$\zeta_u(s) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z)\frac{l_u(z)}{z}\,\mathrm{d}z. \tag{2.5}$$

Fix $0<\varepsilon<1/11$ such that $1/2+2\varepsilon \leqslant \sigma \leqslant 1-\varepsilon$ for $\sigma+it\in K$. For such σ, we see that $\theta_1 \stackrel{\text{def}}{=} 1/2+\varepsilon-\sigma<0$. Then, putting $\theta=1/2+\varepsilon$ and applying residue theorem, we deduce from (2.5), for $s\in K$,

$$\zeta_u(s)-\zeta(s) = \frac{1}{2\pi i}\int_{\theta_1-i\infty}^{\theta_1+i\infty}\zeta(s+z)\frac{l_u(z)}{z}\,\mathrm{d}z + \frac{l_u(1-s)}{1-s}.$$

Therefore, for $s\in K$,

$$\begin{aligned}
&\zeta_u(s+iak)-\zeta(s+iak)\\
&=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\zeta\left(\frac{1}{2}+\varepsilon+it+iak+i\tau\right)\frac{l_u(1/2+\varepsilon-\sigma+i\tau)}{1/2+\varepsilon-\sigma+i\tau}\,\mathrm{d}\tau+\frac{l_u(1-s-iak)}{1-s-iak}\\
&=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\zeta\left(\frac{1}{2}+\varepsilon+iak+i\tau\right)\frac{l_u(1/2+\varepsilon-s+i\tau)}{1/2+\varepsilon-s+i\tau}\,\mathrm{d}\tau+\frac{l_u(1-s-iak)}{1-s-iak}\\
&\ll\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+\varepsilon+iak+i\tau\right)\right|\sup_{s\in K}\left|\frac{l_u(1/2+\varepsilon-s+i\tau)}{1/2+\varepsilon-s+i\tau}\right|\,\mathrm{d}\tau\\
&\quad+\sup_{s\in K}\left|\frac{l_u(1-s-iak)}{1-s-iak}\right|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{M+1}\sum_{k=N}^{N+M}\sup_{s\in K}|\zeta(s+iak)-\zeta_u(s+iak)|\\
&\quad\ll\int_{-\infty}^{\infty}\left(\frac{1}{M+1}\sum_{k=N}^{N+M}\left|\zeta\left(\frac{1}{2}+\varepsilon+iak+i\tau\right)\right|\right)\sup_{s\in K}\left|\frac{l_u(1/2+\varepsilon-s+i\tau)}{1/2+\varepsilon-s+i\tau}\right|\,\mathrm{d}\tau\\
&\quad+\frac{1}{M+1}\sum_{k=N}^{N+M}\sup_{s\in K}\left|\frac{l_u(1-s-iak)}{1-s-iak}\right| \stackrel{\text{def}}{=} I+S.
\end{aligned}\tag{2.6}$$

Lemma 4 implies the estimate

$$\begin{aligned}
&\frac{1}{M+1}\sum_{k=N}^{N+M}\left|\zeta\left(\frac{1}{2}+\varepsilon+iak+i\tau\right)\right|\\
&\quad\ll\left(\frac{1}{M+1}\sum_{k=N}^{N+M}\left|\zeta\left(\frac{1}{2}+\varepsilon+iak+i\tau\right)\right|^2\right)^{1/2}\ll_{\varepsilon,a}(1+|\tau|)^{1/2}.
\end{aligned}\tag{2.7}$$

For large $|t|$, there exists a constant $c > 0$ such that

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\} \tag{2.8}$$

uniformly in $\sigma \in [\sigma_1, \sigma_2]$ with every $\sigma_1 < \sigma_2$. Therefore, for

$$\frac{l_u(1/2 + \varepsilon - s + i\tau)}{1/2 + \varepsilon - s + i\tau} \ll_\theta u^{1/2+\varepsilon-\sigma} \exp\left\{-\frac{c}{\theta}|\tau - k|\right\} \ll_{\theta,K} u^{-\varepsilon} \exp\{-c_1|\tau|\}, \quad c_1 > 0,$$

because t is bounded for $s \in K$. This together with (2.7) shows that

$$I \ll_{\varepsilon,\theta,K} u^{-\varepsilon} \int_{-\infty}^{\infty} (1 + |\tau|)^{1/2} \exp\{-c_1|\tau|\} \, \mathrm{d}\tau \ll_{\varepsilon,\theta,K} u^{-\varepsilon}. \tag{2.9}$$

Similarly as above, using (2.8), we find for all $s \in K$

$$\frac{l_u(1 - s - iak)}{1 - s - iak} \ll_\theta u^{1-\sigma} \exp\left\{-\frac{c}{\theta}|t + ak|\right\} \ll_{\theta,K} u^{1/2-2\varepsilon} \exp\{-c_2|a|k\}, \quad c_2 > 0.$$

Hence,

$$S \ll_{\theta,K} u^{1/2-2\varepsilon} \frac{1}{M+1} \sum_{k=N}^{N+M} \exp\{-c_2|a|k\} \ll_{\theta,K} u^{1/2-2\varepsilon} \exp\{-c_2|a|N\}.$$

This, (2.9) and (2.6) prove the lemma. □

3 Limit theorems

In this section, we will discuss the weak convergence for some measures defined on $(H^r(D), \mathcal{B}(H^r(D)))$. We start with a limit lemma on the space Ω^r. For its proof the following result on linear forms in logarithms of algebraic numbers will be useful, see, for example, [1].

Lemma 6. *Suppose that the system of logarithms* $\log \lambda_1, \dots, \log \lambda_r$ *of algebraic numbers* $\lambda_1, \dots, \lambda_r$ *is linearly independent over* $\mathbb{Q}$. *Then, for any algebraic numbers* $\beta_0, \dots, \beta_r$ *not all simultaneously zero, the inequality*

$$|\beta_0 + \beta_1 \log \lambda_1 + \cdots + \beta_r \log \lambda_r| > h^{-c},$$

where h *is the maximum of the heights of the numbers* $\beta_0, \dots, \beta_r$, *and* c *is an effective constant depending on* r, $\lambda_1, \dots, \lambda_r$ *and the maximum of the powers of the numbers* $\beta_0, \dots, \beta_r$.

For $A \in \mathcal{B}(\Omega^r)$, define

$$Q_{N,M}(A) = \frac{1}{M+1} \#\big\{N \leqslant k \leqslant N + M : \big((p^{-2\pi i a_1 k} : p \in \mathbb{P}), \dots, (p^{-2\pi i a_r k} : p \in \mathbb{P})\big) \in A\big\}.$$

Lemma 7. *Suppose that $a_1, \dots, a_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$, and $\widehat{\widehat{a}}(N\widehat{a})^{1/3}(\log N\widehat{a})^{26/15} \leqslant M \leqslant N-3$. Then $Q_{N,M}$ converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We use the Fourier transform method. The Fourier transform $g_{N,M}(\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \dots, r$, is defined by

$$g_{N,M}(\underline{k}_1, \dots, \underline{k}_r) = \int_{\Omega^r} \left(\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p) \right) \mathrm{d}Q_{N,M},$$

where the star "$*$" indicates that only a finite number of integers k_{jp} are distinct from zero. Hence, the definition of $Q_{N,M}$ implies

$$\begin{aligned} g_{N,M}(\underline{k}_1, \dots, \underline{k}_r) &= \frac{1}{M+1} \sum_{k=N}^{N+M} \left(\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} p^{-2\pi i a_j k k_{jp}} \right) \\ &= \frac{1}{M+1} \sum_{k=N}^{N+M} \exp\left\{ -2\pi i k \sum_{j=1}^{r} a_j \sum_{p \in \mathbb{P}}{}^{*} k_{jp} \log p \right\}. \end{aligned} \tag{3.1}$$

Obviously,

$$g_{M,N}(\underline{0}, \dots, \underline{0}) = 1. \tag{3.2}$$

Now suppose that $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$. Then there exists $j \in \{1, \dots, r\}$ such that $\underline{k}_j \neq \underline{0}$. Hence, $k_{j\widehat{p}} \neq 0$ for some prime $\widehat{p}$. Define

$$\kappa_{\widehat{p}} = \sum_{j=1}^{r} a_j k_{j\widehat{p}}.$$

Since the numbers $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$, $\kappa_{\widehat{p}} \neq 0$. The set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, an application of Lemma 6 gives

$$A_{\underline{k}_1, \dots, \underline{k}_r} \stackrel{\text{def}}{=} \sum_{j=1}^{r} \sum_{p \in \mathbb{P}}{}^{*} a_j k_{jp} \log p = \sum_{p \in \mathbb{P}}{}^{*} \kappa_{\widehat{p}} \log p \neq l, \quad l \in \mathbb{Z}.$$

Thus, in (3.1) we have the geometric progression, and find

$$g_{N,M}(\underline{k}_1, \dots, \underline{k}_r) = \frac{\exp\{-iNA_{\underline{k}_1, \dots, \underline{k}_r}\} - \exp\{-2\pi i(N+M+1\}A_{\underline{k}_1, \dots, \underline{k}_r}\}}{(M+1)(1 - \exp\{-2\pi i A_{\underline{k}_1, \dots, \underline{k}_r}\})}.$$

This and (3.2) show that

$$\lim_{N \to \infty} g_{N,M}(\underline{k}_1, \dots, \underline{k}_r) = \begin{cases} 1, & \text{if } (\underline{k}_1, \dots, \underline{k}_r) = (\underline{0}, \dots, \underline{0}), \\ 0, & \text{if } (\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H, we obtain by a continuity theorem for probability measures on compact groups that $Q_{N,M}$ converges weakly to m_H as $N \to \infty$. □

Let, for brevity, $\underline{a} = (a_1, \dots, a_r)$, and

$$\underline{\zeta}_u(s + 2\pi i\underline{a}k) = (\zeta_u(s + 2\pi i a_1 k), \dots, \zeta_u(s + 2\pi i a_r k)),$$
$$\underline{\zeta}(s + 2\pi i\underline{a}k) = (\zeta(s + 2\pi i a_1 k), \dots, \zeta(s + 2\pi i a_r k)).$$

For $A \in \mathcal{B}(H^r(D))$ and $n \in \mathbb{N}$, define

$$P_{N,M,n}(A) = \frac{1}{M+1} \# \left\{ N \leqslant k \leqslant N + M : \underline{\zeta}_n(s + 2\pi i\underline{a}k) \in A \right\}.$$

Let $w_u : \Omega^r \to H^r(D)$ be given by $w_u(\omega) = \underline{\zeta}_u(s, \omega)$, where

$$\underline{\zeta}_u(s, \omega) = (\zeta_u(s, \omega_1), \dots, \zeta_u(s, \omega_r)),$$
$$\zeta_u(s, \omega_j) = \sum_{m=1}^{\infty} \frac{\omega_j(m) v_u(m)}{m^s}, \quad \omega_j(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad j = 1, \dots, r.$$

By the definition of w_u, we have

$$w_u \left((p^{-2\pi i a_1 k} : p \in \mathbb{P}), \dots, (p^{-2\pi i a_r k} : p \in \mathbb{P}) \right) = \underline{\zeta}_u(s + 2\pi i\underline{a}k).$$

Therefore,

$$P_{N,M,n}(A) = \frac{1}{M+1} \# \{ N \leqslant k \leqslant N + M : \left((p^{-2\pi i a_1 k} : p \in \mathbb{P}), \dots, (p^{-2\pi i a_r k} : p \in \mathbb{P}) \right) \in w_n^{-1} A \}$$

for every $A \in \mathcal{B}(H^r(D))$. Thus, $P_{N,M,n} = Q_{N,M} w_n^{-1}$. Since the series for $\zeta_n(s, \omega_j)$, $j = 1, \dots, r$, are absolutely convergent, the mapping w_n is continuous. Therefore, Lemma 7, Theorem 5.1 of [3], and the notation $V_n = m_H w_n^{-1}$ lead to the following limit lemma.

Lemma 8. *Under hypotheses of Lemma 7, $P_{N,M,n}$ converges weakly to the measure V_n as $n \to \infty$.*

The measure V_n is independent on any hypotheses, and is important for the future. From Lemma 8 and proof of Theorem 3 in [9], the following limit lemma follows. Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(s, \omega)$, and put

$$S = \{ g \in H(D) : g(s) \neq 0 \text{ on } D \text{ or } g(s) \equiv 0 \}.$$

Lemma 9. *The measure V_n converges weakly to $P_{\underline{\zeta}}$ as $n \to \infty$. Moreover, the support of $P_{\underline{\zeta}}$ is the set S^r.*

Now we are ready to prove a limit theorem for

$$P_{N,M}(A) = \frac{1}{M+1} \# \left\{ N \leqslant N + M : \underline{\zeta}(s + 2\pi i\underline{a}k) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)).$$

For this, it suffices to show that $P_{N,M}$ as $N \to \infty$ and V_n as $n \to \infty$ converge weakly to the same limit measure. We will use the convergence in distribution ($\xrightarrow{\mathcal{D}}$).

Theorem 3. *Under hypotheses of Theorem 1, $P_{N,M}$ converges weakly to the measure $P_{\underline{\zeta}}$ as $N \to \infty$.*

Proof. On a certain probability space with measure μ, define a random variable $\theta_{N,M}$ having the distribution

$$\mu\{\theta_{N,M} = k\} = 1/(M+1), \quad k = N, \dots, N+M.$$

Denote by $X_n = X_n(s)$ the $H^r(D)$-valued random element with distribution V_n. Moreover, define the $H^r(D)$-valued random element

$$X_{N,M,n} = X_{N,M,n}(s) = \underline{\zeta}_n(s + 2\pi i \underline{a} \theta_{N,M}).$$

Then, in view of Lemma 8,

$$X_{N,M,n} \xrightarrow[N\to\infty]{\mathcal{D}} X_n, \tag{3.3}$$

and, by Lemma 9,

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P_{\underline{\zeta}}. \tag{3.4}$$

Define one more $H^r(D)$-valued random element

$$X_{N,M} = X_{N,M}(s) = \underline{\zeta}(s + 2\pi i \underline{a} \theta_{N,M}).$$

Let, for $g_1, g_2 \in H(D)$,

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence embedded compact sets such that $D = \bigcup_{l=1}^{\infty} K_l$, and every compact set $K \subset D$ lies in a certain K_l. For example, we can take embedded closed rectangles. Then ρ is a metric in $H(D)$ inducing the topology of uniform convergence on compacta. Let $\underline{g}_1 = (g_{11}, \dots, g_{1r}), \underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D)$. Then,

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1\leqslant j\leqslant r} \rho(g_{1j}, g_{2j})$$

is a metric in $H^r(D)$ inducing the product topology.

Now, return to Lemma 5. Using the definition of the metric $\underline{\rho}$, we obtain from Lemma 5 that

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{M+1} \sum_{k=N}^{N+M} \underline{\rho}\left(\underline{\zeta}(s + 2\pi i \underline{a} k), \underline{\zeta}_n(s + 2\pi i \underline{a} k)\right) = 0.$$

Therefore, the definitions of $X_{N,M}$ and $X_{N,M,n}$ show that, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} \limsup_{N\to\infty} \mu\left\{\underline{\rho}(X_{N,M}, X_{N,M,n}) \geqslant \varepsilon\right\} \\ &\quad \leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(M+1)} \sum_{k=N}^{N+M} \underline{\rho}\left(\underline{\zeta}(s + 2\pi i \underline{a} k), \underline{\zeta}_n(s + 2\pi i \underline{a} k)\right) = 0. \end{aligned}$$

This, (3.3), (3.4) and Theorem 4.2 of [3] imply the relation

$$X_{N,M} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\underline{\zeta}},$$

which is equivalent to the assertion of the theorem. □

For the proof of Theorem 2, we need the analogue of Theorem 3 for

$$P_{N,M,u_N}(A) = \frac{1}{M+1}\#\left\{N \leqslant k \leqslant N+M : \underline{\zeta}_{u_N}(s+2\pi i\underline{a}k) \in A\right\}, \quad A \in \mathcal{B}(H^r(D)).$$

Theorem 4. *Under hypotheses of Theorem 2, P_{N,M,u_N} converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$.*

Proof. Let $\theta_{N,M}$ and $X_{N,M}$ be the same random objects as in proof of Theorem 3. Define one more $H^r(D)$-valued random element

$$X_{M,N,u_N} = X_{M,N,u_N}(s) = \underline{\zeta}_{u_N}(s+2\pi i\underline{a}k\theta_{M,N}).$$

Let F be a fixed closed set of the space $H^r(D)$, $\varepsilon > 0$ and $F_\varepsilon = \{\underline{g} \in H^r(D) : \underline{\rho}(\underline{g}, F) \leqslant \varepsilon\}$, where $\underline{\rho}(\underline{g}, F) = \inf_{\underline{g}_1 \in F} \rho(\underline{g}, \underline{g}_1)$. Then again the set F_ε is closed.

By Theorem 3 and the equivalent of weak convergence in terms of closed sets, see Theorem 2.1 of [3],

$$\limsup_{N\to\infty} \mu\left\{X_{N,M} \in F_\varepsilon\right\} \leqslant P_{\underline{\zeta}}(F_\varepsilon). \tag{3.5}$$

Since

$$\{X_{N,M,u_N} \in F\} \subset \{X_{N,M} \in F_\varepsilon\} \cup \left\{\underline{\rho}(X_{M,N}, X_{M,N,u_N}) \geqslant \varepsilon\right\},$$

we have

$$\mu\left\{X_{N,M,u_N} \in F\right\} \leqslant \mu\left\{X_{N,M} \in F_\varepsilon\right\} + \mu\left\{\underline{\rho}(X_{M,N}, X_{M,N,u_N}) \geqslant \varepsilon\right\}. \tag{3.6}$$

In virtue of Lemma 5,

$$\lim_{N\to\infty} \mu\left\{\underline{\rho}(X_{M,N}, X_{M,N,u_N}) \geqslant \varepsilon\right\} \leqslant \frac{1}{\varepsilon(M+1)} \sum_{k=M}^{M+N} \underline{\rho}\left(\underline{\zeta}(s+2\pi i\underline{a}k), \underline{\zeta}_{u_N}(s+2\pi i\underline{a}k)\right) = 0.$$

This, (3.5) and (3.6) show that

$$\limsup_{N\to\infty} \mu\left\{X_{N,M,u_N} \in F\right\} \leqslant P_{\underline{\zeta}}(F_\varepsilon).$$

Taking $\varepsilon \to 0+$, we obtain that

$$\limsup_{N\to\infty} \mu\left\{X_{N,M,u_N} \in F\right\} \leqslant P_{\underline{\zeta}}(F),$$

thus, P_{N,M,u_N} converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$. □

4 Proof of the main theorems

We continue with recalling the famous Mergelyan theorem on approximation of analytic functions by polynomials [12].

Lemma 10. *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complement, and $g(s)$ is a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |g(s) - p(s)| < \varepsilon.$$

Proof. (Proof of Theorem 1). Since $f_j(s) \neq 0$ on K_j, an application of Lemma 10 for $\log f_j(s)$, $j = 1, \dots, r$, shows that there exist polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| f_j(s) - \mathrm{e}^{p_j(s)} \right| < \varepsilon/2. \tag{4.1}$$

Define the set

$$G_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \varepsilon/2 \right\}.$$

Then, in view of Lemma 9, the set G_ε is an open neighbourhood of the element $(\mathrm{e}^{p_1(s)}, \dots, \mathrm{e}^{p_r(s)})$ which lies in the support S^r of the measure $P_{\underline{\zeta}}$. Thus,

$$P_{\underline{\zeta}}(G_\varepsilon) > 0. \tag{4.2}$$

Define one more set

$$\widehat{G}_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$

Then, inequality (4.1) implies the inclusion $G_\varepsilon \subset \widehat{G}_\varepsilon$. Thus $P_{N,M}(\widehat{G}_\varepsilon) > 0$ in virtue of (4.2), and, by Theorem 3 and the equivalent of weak convergence in terms of open sets, see Theorem 2.1 of [3],

$$\liminf_{N\to\infty} P_{N,M}(\widehat{G}_\varepsilon) \geqslant P_{\underline{\zeta}}(\widehat{G}_\varepsilon) > 0. \tag{4.3}$$

Therefore, the definitions of $P_{N,M}$ and $\widehat{G}_\varepsilon$ prove the first assertion of the theorem.

To prove the second assertion of the theorem, we observe that the boundary $\partial\widehat{G}_\varepsilon$ lies in the set

$$\left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K} |g_j(s) - f_j(s)| = \varepsilon \right\}.$$

Therefore, the boundaries $\partial\widehat{G}_{\varepsilon_1}$ and $\partial\widehat{G}_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. Hence, $P_{\underline{\zeta}}(\partial\widehat{G}_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$, thus, the set $\widehat{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$ for all but at most countably many

$\varepsilon > 0$. Therefore, by Theorem 3 and the equivalent of weak convergence in terms of continuity sets, see Theorem 2.1 of [3], the limit

$$\lim_{N\to\infty} P_{N,M}(\widehat{G}_\varepsilon) = P_{\underline{\zeta}}(\widehat{G}_\varepsilon)$$

exists for all but at most countably many $\varepsilon > 0$, and is positive in view of (4.3). Thus, the definitions of $P_{N,M}$ and $\widehat{G}_\varepsilon$ give the second assertion of the theorem. □

Proof. (Proof of Theorem 2). The mapping $w : H^r(D) \to \mathbb{R}$ given by

$$w(g_1,\dots,g_r) = \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |g_j(s) - f_j(s)|, \quad (g_1,\dots,g_r) \in H^r(D),$$

is continuous. Therefore, Theorem 4 and Theorem 5.1 of [3] show that

$$\frac{1}{M+1}\#\Big\{N \leqslant k \leqslant N+M : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |\zeta_{\underline{u}_N}(s+2\pi i\underline{a}k)-f_j(s)|\in A\Big\}, A\in\mathcal{B}(\mathbb{R}),$$

converges weakly to

$$m_H\Big\{\omega \in \Omega^r : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |\zeta(s,\omega_j) - f_j(s)| \in A\Big\}, \quad A \in \mathcal{B}(\mathbb{R}),$$

as $N \to \infty$. From this, it follows that the corresponding distribution function

$$\frac{1}{M+1}\#\Big\{N \leqslant k \leqslant N+M : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |\zeta_{\underline{u}_N}(s+2\pi i\underline{a}k) - f_j(s)| < \varepsilon\Big\}$$

converges weakly to the distribution function

$$m_H\Big\{\omega \in \Omega^r : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |\zeta(s,\omega_j) - f_j(s)| < \varepsilon\Big\}$$

as $N \to \infty$, i.e.,

$$\lim_{N\to\infty}\frac{1}{M+1}\#\Big\{N \leqslant k \leqslant N+M : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |\zeta_{\underline{u}_N}(s+2\pi i\underline{a}k) - f_j(s)| < \varepsilon\Big\}$$
$$= m_H\Big\{\omega \in \Omega^r : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |\zeta(s,\omega_j) - f_j(s)| < \varepsilon\Big\} \qquad (4.4)$$

for all continuity points ε of the right-hand side. Since the set of discontinuity points of a distribution function is at most countable, we find that (4.4) is true for all but at most countably many $\varepsilon > 0$. The positivity of the right-hand side of (4.4) was obtained in the proof of Theorem 1. □

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