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ON THE STABILITY OF FINITE DIFFERENCE SCHEMES FOR HYPERBOLIC EQUATION WITH NONLOCAL INTEGRAL BOUNDARY CONDITIONS

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Nomenclature

Abbreviations

CF	Characteristic function, page 43
FDM	Finite difference method, page 17
FDS	Finite difference scheme, page 15
NBC	Nonlocal boundary condition, page 1
PDE	Partial differential equation, page 3
PDE	Partial differential equation, page 24
SLP	Sturm–Liouville Problem, page 59

Notation

$\mathcal{O}(\cdot)$	Asymptotic notation, page 21
ω	Discrete grid, page 19
U_i^j	Grid function, page 19
δ_i^j	Kronecker delta, page 63
$\lfloor x \rfloor$	Largest integer less than or equal to x, page 45
$\overline{\partial}_t$	Operator of first order derivative in time, page 19
δ_x^2	Operator of second order derivative in space, page 19
$\overline{\partial}_t^2$	Operator of second order derivative in time, page 19
$\rho(\cdot)$	Spectral radius, page 25

Introduction

0.1 Relevance of the research

Nonlocal conditions are known for scientists for at least 150 years. For example, in 1896 V.A. Steklov [93] investigated mathematical model of metal core cooling, where nonlocal conditions were considered as a linear combination of the values of unknown function and its derivatives in various boundary points (see [94, pp. 63–75, Стеклов 1983]). Steklov considered heat equation

$$g\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial U}{\partial x}\right) - mU, \quad g = c\rho, \qquad x \in [0, l],$$

with initial condition $U(0, x) = f(x)$, and boundary conditions of the general form

$$\begin{aligned} L(U) \equiv \quad & a_1 U(t,0) + a_2 \frac{\partial U(t,0)}{\partial x} + a_3 U(t,l) + a_4 \frac{\partial U(t,l)}{\partial x} = 0, \\ L_1(U) \equiv \quad & b_1 U(t,0) + b_2 \frac{\partial U(t,0)}{\partial x} + b_3 U(t,l) + b_4 \frac{\partial U(t,l)}{\partial x} = 0, \end{aligned} \tag{0.1}$$

where a_k and b_k $(k = 1, 2, 3, 4)$ are constant coefficients. We call nonlocal conditions (0.1) as *classical nonlocal boundary conditions* because they link together values of unknown function and its derivatives only on the boundary. Problems with the same type classical NBCs were also investigated in 1933 by T. Carleman [14], in 1964 by R.W. Beals [8] and F.E. Browder [12].

In 1963 J.R. Cannon published an article [13], where the nonlocal integral boundary condition

$$\int_0^1 u(x,t)dx = \phi(t) \tag{0.2}$$

was considered. Nonlocal condition (0.2) links together values of the unknown function on the boundary and in the inner domain. That is exactly the *nonlocal condition*, which led to the new field of nonlocal problems and their numerical modelling. Conditions are called nonlocal, when together (or instead of) with boundary conditions, another conditions, which connect solution (or/and its derivatives) on boundary with inner domain, are formulated. Nonlocal conditions arise mainly when the data on the boundary cannot be measured directly. It is sometimes better to impose nonlocal conditions since the measurements needed by a nonlocal condition may be more precise than the measurement given by a local condition. Investigation of such problems is of special interest in the point of general partial differential equations theory as well as in the point of mathematical modelling applications.

Bitsadze–Samarskii nonlocal conditions

In 1969 A.A. Samarskii and A.V. Bitsazde made a report [10] about existence and uniqueness of solutions of a Laplace equation

$$\mathit{\Delta} u(x,y) = 0, \qquad -l < x < l, \ 0 < y < 1,$$

with boundary conditions

$$\begin{aligned} u(x,0) = \phi_1(x), \ u(x,1) = \phi_2(x), &\qquad -l \leqslant x \leqslant l, \\ u(-l,y) = \phi_3(y), \ u(0,y) = u(l,y) &\qquad 0 \leqslant y \leqslant 1, \end{aligned}$$

where ϕ_1, ϕ_2, and ϕ_3 are known continuous functions. Starting from this paper, conditions of the type

$$u|_{\text{boundary}} = au(\xi) + b, \quad \xi \in \text{inner domain}, \ a, b \in \mathbb{R}$$

are called Bitsadze–Samarskii nonlocal conditions. In ten years (1977–1987) there were published articles by N.I. Ionkin and coauthors [36, Ионкин 1977], [37, Ionkin and Moiseev 1980], Samarskii [75, 1980] and others [91, Soldatov and Shkhanukov 1987] and [41, Капанадзе 1987]. Nowadays, problems with Bitsadze–Samarskii conditions are investigated by a worldwide group of scientists. For example, in 2008 A. Ashyralyev published a paper [2], where he considered an elliptic problem

$$-u''(t) + Au(t) = f(t) \quad (0 \leqslant t \leqslant 1), \quad u(0) = \phi, \quad u(1) = u(\lambda) + \psi, \ 0 \leqslant \lambda < 1,$$

where A is a positive operator in an arbitrary Banach space. He proved the coercive inequalities in Banach space for the solutions of the formulated problem, and investigated solvability in different Banach spaces.

The eigenspectrum analysis of Sturm–Liouville and elliptic finite difference operators with two-point Bitsadze–Samarskii NBCs is made in 2015 by Elsaid et al. [27]. The authors considered elliptic PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \qquad 0 < x < 1, \quad 0 < y < 1,$$

with the boundary conditions

$$\begin{aligned} u(x,0) &= u_1(x), \\ u(x,1) &= u_2(x), \\ u(0,y) &= \gamma_1 u(1,y), \\ u(\xi,y) &= \gamma_2 u(1-\xi,y), \end{aligned}$$

where ξ, γ_1, and γ_2 are given constants such that $0 < \xi < 1-\xi < 1$. Firstly, the authors consider the eigenvalue problem for Sturm–Liouville finite difference operator with given nonlocal boundary conditions and then the results obtained from this problem are utilized to study the two-dimensional difference eigenvalue problem. The eigenvalue analysis is made very similar, as in the articles of lithuanian mathematicians R. Čiegis, M. Sapagovas, A. Štikonas (see e.g. [17–19, 59]). The authors, using the separation of variables technique, combined together the properties and relations of one-dimensional problems to obtain the corresponding ones of the two-dimensional case.

Multipoint nonlocal conditions

We call a nonlocal condition multipoint if it links values of the unknown function and its derivatives on at least three points of the inner domain and boundary.

A general notion of multipoint condition is provided by B. Pelloni and D. Smith [60, November 2015]. The authors consider initial-multipoint value problem (we use authors' notation)

$$[\partial_t + a(-\imath\partial_x)^n)]q(x,t) = 0 \qquad (x,t) \in (0,1)\times(0,T), \tag{0.3}$$

$$q(x,0) = q_0(x) \qquad x \in [0,1], \tag{0.4}$$

$$\sum_{k=0}^{n-1}\sum_{r=0}^{m} b^r_{kj}\partial^k_x q(\eta_r,t) = g_j(t), \qquad t \in [0,T],\ j = \overline{0,n-1}, \tag{0.5}$$

where $m, n \in \mathbb{N}$, and $0 = \eta_0 < \eta_1 < \eta_2 < \cdots < \eta_m = 1$, $b_{kj}^r \in \mathbb{C}$ for $k, j = \overline{0, n-1}$, $r = \overline{0, m}$. The authors assume $g_j(t) \in C^\infty[0, T]$ with $T > 0$ a fixed constant, and that the initial datum is compatible with the boundary data in the sense that

$$q_0 \in C^n(0,1) \quad \text{and} \quad \sum_{k=0}^{n-1} \sum_{r=0}^{m} b_{kj}^r \partial_x^k q_0(\eta_r, 0) = g_j(0).$$

Coefficient a satisfies

$$\begin{cases} \{e^{\imath\Phi} : \Phi \in [0, \pi]\} & \text{if } n \text{ even}, \\ \{\imath, -\imath\} & \text{if } n \text{ odd}. \end{cases}$$

Condition (0.5) is called multipoint nonlocal condition.

An example of multipoint nonlocal initial boundary problem is provided by D.G. Gordeziani *et al.* [29, 2010]. The authors try to find a regular solution of a problem

$$Lu(\bar{x}) = F(\bar{x}), \quad \bar{x} = (x_0, \ldots, x_n) \in \Omega \subset \mathbb{R}^{n+1},$$

where

$$Lu = -\sum_{i=0}^{n} \frac{\partial}{\partial x_i} \left[K_i(\bar{x}) \frac{\partial u}{\partial x_i} \right] + K(\bar{x}) u,$$

$K_i(\cdot) \geqslant \alpha_i = \text{const} > 0$, $i = \overline{0, n}$. $K(\cdot)$ satisfy boundary condition $u(\bar{x}) = \phi(\bar{x})$, $\bar{x} \in \bar{S}_\Gamma$, where $\bar{S}_\Gamma = \{\bar{x} : x_0 \in [0,1], x_1, \ldots, x_n \in \Gamma\}$, Γ is a boundary and general nonlocal boundary conditions

$$\begin{aligned} \alpha_1 \left.\frac{\partial u(x_0, x)}{\partial x_0}\right|_{x_0=0} + \beta_1 u(0, x) &= \gamma_1 u(\eta_1, x) + \delta_1 \frac{1}{\xi_1} \int_0^{\xi_1} u(x_0, x) dx_0 = \phi_1(x), \\ \alpha_2 \left.\frac{\partial u(x_0, x)}{\partial x_0}\right|_{x_0=1} + \beta_2 u(1, x) &= \gamma_2 u(\eta_2, x) + \delta_2 \frac{1}{1-\xi_2} \int_{\xi_2}^{1} u(x_0, x) dx_0 = \phi_2(x), \end{aligned}$$

where $x = (x_1, \ldots, x_n)$, $0 < \xi_1 \leqslant \xi_2 < 1$; ϕ_1, ϕ_2, ϕ, and F are smooth functions; $0 < \eta_1 \leqslant \eta_2 < 1$, α_i, β_i, γ_i, δ_i $(i = 1, 2)$ are known parameters. The authors proved the existence and uniqueness of the solutions of differential problem and formulated difference analog of the formulated problem.

Multipoint boundary conditions are also investigated with nonlinear PDEs. In the paper [21, Das *et al.* 2010] authors present an algorithm for the numerical solution of the second order multi-point boundary value problem

$$u''(x) + g(u, u') = f(x), \quad 0 \leqslant x \leqslant 1,$$

with classical initial condition and multipoint boundary condition

$$u(0) = \alpha, \quad u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i) + \gamma_i,$$

where $\eta_i \in (0,1)$, $i = \overline{0,m}$, α_i and γ_i are known constants. The algorithm is based on the homotopy perturbation approach and the solutions are calculated in the form of a rapid convergent series. The described method yields more realistic series solutions that converge rapidly to the exact solutions.

Integral nonlocal conditions

Integral nonlocal conditions, among all the nonlocal problems, are worth of investigation as a natural generalization of discrete nonlocal conditions. Integral conditions describes the relationship between boundary and inner domain as a sort of some average. Conditions of such type often occur in problems related to fluid mechanics [50, Нахушев 1982] and [85, Shelukhin 1993], hydrodynamics [84, Шелухин 1995] and [16, Чудновский 1976], linear thermoelasticity [22, 23, Day 1983, 1985], vibrations [99, Volkodavov and Zhukov 1998], biology [51, Нахушев 1995], plasma theory [25, Diaz and Rakotoson 1996], particle diffusion [49, Mu *et al.* 2010], heat conduction [13, Cannon 1963], etc.

One of the main authors investigating nonlocal integral conditions is L.S. Pulkina [43, 61–68]. In the article [64, 2011] Pulkina considers an equation

$$u_{tt} - (a_{ij}(x,t)u_{x_i})_{x_j} + c(x,t) = f(x,t) \tag{0.6}$$

in a bounded domain $\Omega \in \mathbb{R}^n$ with smooth boundary $\partial\Omega$, Q is the cylinder $\Omega \times (0,T)$, $T < \infty$, $S = \partial\Omega \times (0,T)$ is the lateral boundary of Q. Author sets a problem: find a function $u(x,t)$ that is a solution of (0.6) in Q, satisfies initial condition

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$$

and the following nonlocal integral condition for $n > 1$:

$$\alpha \left.\frac{\partial u}{\partial \nu}\right|_S + \int_\Omega K(x,t)u(x,t)dx = 0. \tag{0.7}$$

Here $\partial u/\partial\nu \equiv a_{ij}(x,t)u_{x_i}(x,t)\nu_i|_S$, $\nu(x) = (\nu_1,\dots,\nu_n)$ is an outward normal to $\partial\Omega$ at the current point, $K(x,t)$ is given weight.

In a special case $n = 1$ the lateral boundary of $Q = (0,l)\times(0,T)$ separates into two parts: $x = 0$ and $x = l$. As a consequence nonlocal condition (0.7) separates into two nonlocal conditions:

$$\gamma_1 u_x(0,t) + \rho_1 \int_0^l K_1(x,t)u(x,t)dx = 0,$$
$$\gamma_2 u_x(1,t) + \rho_2 \int_0^l K_2(x,t)u(x,t)dx = 0,$$

where $\rho_1^2 + \rho_2^2 > 0$. Pulkina proved the existence and uniqueness of the generalized solution in the Sobolev space $\hat{W}_2^1(Q) = \{v(x,t)\colon v \in W_2^1(Q),\ v(x,T) = 0\}$.

The existence and uniqueness of a strong solutions of the singular problem with integral conditions for parabolic equation was proved by A.L. Marhoune and A. Memou in [48, 2015]. The authors in the rectangle $\Omega = [0,1]\times[0,T]$ considered the equation

$$\mathcal{L}u \equiv \frac{\partial u}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) = f(x,t),$$

with the initial condition

$$u(x,0) = \phi(x), \quad x \in [0,1],$$

the Dirichlet condition

$$u(1,t) = 0, \quad t \in [0,T],$$

and the nonlocal condition

$$\int_0^\alpha u(x,t)dx + \int_\beta^1 u(x,t)dx = 0, \quad 0 \leqslant \alpha \leqslant \beta \leqslant 1,\ t \in [0,T],$$

with given functions $\phi(x)$ and $f(x,t)$ and the matching conditions

$$\phi(1) = 0, \qquad \int_0^\alpha \phi(x)dx + \int_\beta^1 \phi(x)dx = 0.$$

Existence and uniqueness of solutions is proved using the energy inequalities.

Stable nonlocal hyperbolic difference problems

The main research object of this thesis is stability of FDS for hyperbolic problems, so we mark up some articles dedicated to this thematic. The first one is written by A. Ashyralyev and E. Ozturk in 2014 [4]. The authors investigate stability of FDS for Bitsadze–Samarskii type nonlocal boundary value problem involving integral condition. They consider elliptic differential equation in a Hilbert space

$$-\frac{d^2u(t)}{dt^2} + Au(t) = f(t), \quad 0 < t < 1, u(0) = \phi,\ u(1) = \int_0^1 \rho(\lambda)u(\lambda)d\lambda + \psi \quad (0.8)$$

with the self-adjoint positive definite operator A with a closed domain $D(A) \subset H$. Here, let $f(t)$ be a given abstract continuous function defined on $[0,1]$ with values in H; ϕ, and ψ are elements of $D(A)$ and $\rho(t)$ is a scalar function. The authors consider fourth order of the accuracy difference scheme

$$
\begin{aligned}
&-\frac{u_{k+1}-2u_k-u_{k-1}}{\tau^2}+Ay_k+\frac{\tau^2}{12}A^2u_k=\phi_k,\\
&\phi_k=f(t_k)+\frac{\tau^2}{12}\left(\frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{\tau^2}+Af(t_k)\right),\ t_k=k\tau,\ k=\overline{1,N-1},\\
&u_0=\phi,\\
&u_N=\frac{\tau}{3}\left(\rho(t_0)u_0+\rho(t_N)u_N\right)+\frac{\tau}{3}\left(4\sum_{k=1}^{N/2}\rho(t_{2k-1})u_{2k-1}+2\sum_{k=1}^{N/2-1}\rho(t_{2k})u_{2k}\right)+\phi
\end{aligned}
$$

for the approximate solution of differential equation (0.8). The stability estimates for the solution of this difference scheme are established. Since A is a self-adjoint positive definite operator the authors can use some techniques of A.A. Samarskii and A.V. Gulin [70, 1973]. The authors proved theorems on the stability estimates, almost coercive stability estimates for the solution of difference scheme for elliptic equations.

Quite exotic third order stable FDS for the hyperbolic multipoint nonlocal boundary value problem is presented by O. Yildirim and M. Uzun [100, 2015]. The authors consider hyperbolic problem

$$
\begin{aligned}
\frac{d^2u(t)}{dt^2}+Au(t) &= f(t),\quad 0\leqslant t\leqslant 1,\\
u(0) &= \sum_{r=1}^{n}\alpha_r u(\lambda_r)+\phi,\\
u_t(0) &= \sum_{r=1}^{n}\beta_r u_t(\lambda_r)+\psi,
\end{aligned}
$$

where $0<\lambda_1<\lambda_2<\ldots<\lambda_n\leqslant 1$, A is a self-adjoint positive definite operator with domain $D(A)$ in a Hilbert space H. Authors associate above defined problem with the corresponding third order of accuracy difference scheme

$$
\begin{aligned}
&\tau^{-2}(u_{k+1}-2u_k+u_{k+1})+\frac{2}{3}Au_k+\frac{1}{6}A(u_{k+1}+u_{k-1})+\frac{1}{12}\tau^2A^2u_{k+1}=f_k,\\
&f_k=\frac{2}{3}f(t_k)+\frac{1}{6}\left(f(t_{k+1})+f(t_{k-1})\right)-\frac{1}{12}\tau^2\left(-Af(t_{k+1})+f''(t_{k+1})\right),\\
&\qquad t_k=k\tau,\ 1\leqslant k\leqslant N-1,\ N\tau=1,
\end{aligned}
$$

$$\left(I-\imath\tau A^2\right)u_0=\sum_{k=1}^{n}\left\{u_{[\lambda_k/\tau]}+\tau^{-1}\left(u_{[\lambda_k/\tau]}-u_{[\lambda_k/\tau]-1}\right)\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\right)\right.$$
$$+\frac{3}{2}\left(f_{[\lambda_k/\tau]}-Au_{[\lambda_k/\tau]}\right)\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^2+\frac{7}{6}\left(f'_{[\lambda_k/\tau]}-\tau^{-1}A\left(u_{[\lambda_k/\tau]}-u_{[\lambda_k/\tau]-1}\right)\right)$$
$$\left.\cdot\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^3\right\}+\phi,$$
$$\left(I+\imath\tau A^2\right)\tau^{-1}(u_1-u_0)=\sum_{k=1}^{n}\beta_k\left\{\tau^{-1}\left(u_{[\lambda_k/\tau]}-u_{[\lambda_k/\tau]-1}\right)+\left(f_{[\lambda_k/\tau]}-Au_{[\lambda_k/\tau]}\right)\right.$$
$$\cdot\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)+\frac{1}{2!}\left(f'_{[\lambda_k/\tau]}-\tau^{-1}A\left(u_{[\lambda_k/\tau]}-u_{[\lambda_k/\tau]-1}\right)\right)\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^2$$
$$\left.+\frac{1}{3!}\left(f''_{[\lambda_k/\tau]}-Af_{[\lambda_k/\tau]}+A^2u_{[\lambda_k/\tau]}\right)\left(\lambda_k-\left[\frac{\lambda_k}{\tau}\right]\tau\right)^3\right\}+\psi+f_{1,1}\tau,$$

where $f_{1,1}=\{f(0)+\left(-f(0)+\tau f'(0)\right)/2-f'(0)\tau/3\}$.

This scheme is a third order of accuracy unconditionally stable difference scheme for the approximate solution of hyperbolic multipoint nonlocal boundary value problem in a Hilbert space with self-adjoint positive definite operator. The stability is established without any assumptions in respect of grid steps h and τ.

More detailed literature review on a particular thematics is presented in the introduction of each chapter.

0.2 Aims and problems

In this section we provide the directions of the research presented in this thesis.

- **Stability conditions for the explicit FDS.** In Chapter 1 we investigate the explicit FDS for the hyperbolic problem with two integral boundary conditions. We approximate integrals by the trapezoid formula and formulate the two-layered FDS with the block transition matrix $\mathbf{S}$. We investigate the eigenstructure of this matrix, formulate and prove the sufficient stability condition.

- **Stability conditions for the weighted FDS.** In Chapter 2 we investigate a class of weighted FDS with one weight parameter. We use the generalized characteristic functions to investigate eigenspectrum (complex and real) of discrete problem. We obtain the structure of eigenspectrum, formulate and prove stability conditions according to boundary parameters and weights of FDS.

- **Stability regions for the FDS with two weights.** In Chapter 3 we consider a class of weighted FDS with two weights. Numerically modelling characteristic functions we obtain stability regions and restrictions on weights σ_1 and σ_2.

- **Stability of the weighted FDS with partial integrals.** In Chapter 4 we generalize the hyperbolic problem. We investigate integral boundary conditions of the general type. We obtain equivalence conditions for the Sturm–Liouville problem (which can be generalized to the evolution equations) to the algebraic eigenvalue problem. These conditions obtained assuming a general type of integral conditions (containing weight functions in the integral NBCs). Moreover, we investigate hyperbolic problem with partial integrals in the boundaries.

0.3 Methods

The research methodology of this thesis is based of the approximation of hyperbolic equations by finite difference scheme. We use functional and complex analysis to investigate the spectrum of the difference operators. We use numerical integration to approximate integrals in the boundary conditions. We use Java and Maple for mathematical modelling and simulation of the experiment.

0.4 Defended Statements

- The sufficient stability condition of the explicit FDS for hyperbolic equation with integral NBCs is $\gamma_0 + \gamma_1 < 2$ under the condition $\tau \leqslant h$.

- The sufficient stability condition of the weighted FDS (with one weight σ) for hyperbolic equation with integral NBCs is $\gamma_0+\gamma_1 < 2$ and $\sigma > \frac{1}{4} - \frac{1}{\tau^2\lambda_{\max}}$.

- The FDS for hyperbolic equation with integral NBCs (with one weight σ) is unstable if the spectrum has complex eigenvalues.

- The weighted FDS for hyperbolic equation with integral NBCs (with two weights σ_1 and σ_2) has a stability region if $\sigma_1 \geqslant \sigma_2$. If the spectrum is real, then the second stability condition is $\sigma_1 + \sigma_2 \geqslant 1/2$.

- The stability region of weighted FDS for hyperbolic equation with integral NBCs (with two weights σ_1 and σ_2) is bounded if $\sigma_1 - \sigma_2 < 1/2$, elsewise stability region unbounded.

0.5 Originality

The research object of this doctoral thesis is the stability of finite difference approximation of the hyperbolic one-dimensional problem with nonlocal integral boundary conditions. Most of the results presented in this doctoral thesis are new for the formulated problem and have not appeared before in the scientific literature. Some methods used in the investigation of the referred problem have been recently used by scientists to investigate other fields of mathematical physics. However, application of these methods to the hyperbolic problems is completely new. We hope, that the present thesis is a step in generalization the theory of approximation of nonlocal initial boundary value problems.

0.6 Applications

Modern problems of natural sciences lead to necessity of generalization of classical mathematical physics problems and to formulate qualitatively new problems, including nonlocal problems for differential equations. In the past several decades, many physical phenomena have been formulated into nonlocal mathematical models: electrolytic refining of non-ferrous metals [47, Lyubanova 2014], deformation of metals under high strain rates [1, Ahada *et al.* 2014; and references therein], the phenomena of Ohmic heating (see [28, Fan *et al.* 2014] and [56, Olmstead *et al.* 1994] and references therein), thermal electricity [26, Du and Fan 2013], superconductivity [11, Van Bockstal and Slodička 2015], flow of fluids through fissured rocks [92, Soltanalizadeh *et al.* 2014], etc.

Stability analysis is an essential part of the concepts in the theory of numerical schemes. In the numerical modelling of linear PDEs stability notion, along with consistency notion, are two main characteristics of FDS, which lead to the convergence of the scheme. Obtained stability conditions are useful numerically modelling any processes described by hyperbolic equations (dynamics of ground waters with NBCs of type [9, Beilin 2001] and [24, Dehghan 2005], oscillations [30, Gordeziani and Avalishvili 2000], irrigation models [83, Сербина 2007], electromagnetic field [101, Золина 1966], etc). Eigenspectrum analysis results can be used for constructing new difference schemes and for the investigations of the stability regions of certain finite difference schemes.

0.7 Dissemination of results

Publications

The results of the doctoral research are published in five research papers. Two papers are published in journals indexed by ISI Web of Science

1. J. Novickij, A. Štikonas, On the stability of a weighted finite difference scheme for wave equation with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **19**(3):460–475, 2014.

2. F.F. Ivanauskas, Yu.A. Novitski, and M.P. Sapagovas. On the stability of an explicit difference scheme for hyperbolic equations with nonlocal boundary conditions, *Differ. Equ.*, **49**(7):849–856, 2013 (transl. from: Ф.Ф. Иванаускас, Ю.А. Новицкий, М.П. Сапаговас, Об устойчивости явной разностной схемы для гиперболических уравнений с нелокальными краевыми условиями, *Дифференц. Уравнения*, **49**(7), с. 877–884, 2013).

Three papers are published in the proceeding of the conferences

3. J. Novickij, A. Skučaitė, and A. Štikonas, On the Stability of a Weighted Finite Difference Scheme for Hyperbolic Equation with Integral Boundary Conditions, In *Proc. Numerical Mathematics and Advanced Applications – ENUMATH 2015*, *Lect. Notes Comput. Sci. Eng.*, Vol. 112, Springer International Publishing, 2016 (Accepted, in press).

4. J. Novickij, A. Štikonas, On the equivalence of discrete Sturm–Liouville problem with nonlocal boundary conditions to the algebraic eigenvalue problem, *Liet. matem. rink. Proc. LMS, Ser. A*, **56**:66–71, 2015.

5. J. Novickij, A. Štikonas, On the stability of a finite difference scheme with two weights for wave equation with nonlocal conditions, *Liet. matem. rink. Proc. LMS, Ser. A*, **55**:22–27, 2014.

Conferences

The results of this thesis were presented in the following international conferences:

- *Hyp2016*, Aachen, Germany, August 1–5, 2016.
- *Actual Problems in Theory of Partial Differential Equations, dedicated to the centenary of Andrey V. Bitsadze*, Moscow, Russia, June 15–18, 2016.
- *MMA2016*, Tartu, Estonia, June 1–4, 2016.
- *ENUMATH2015*, Ankara, Turkey, September 14–18, 2015.
- *MMA2015*, Sigulda, Latvia, May 26–29, 2015.
- *MMA2014*, Druskininkai, Lithuania May 26–29, 2014.
- *MMA2013 & AMOE2013*, Tartu, Estonia, May 27–30, 2013.
- *MMA2012*, Tallinn, Estonia, June 6–9, 2012.

The results of the thesis were also presented in the local LMD53–LMD56 conferences in 2012–2015 years and in the mathematical seminar of Department of Differential Equations and Numerical Mathematics on May 24, 2016.

Conference abstracts

- J. Novickij and A. Štikonas, On the Stability of Discrete Nonlocal Hyperbolic Boundary Problem, *Hyp2016 abstracts*, Aachen, Germany, 2016.
- J. Novickij and A. Štikonas, On the stability of discrete hyperbolic equation with nonlocal integral boundary conditions, *Тезисы докладов*, Москва, 2016.

- J. Novickij and A. Štikonas, On the stability of discrete nonlocal hyperbolic boundary problem, *Abstracts of MMA2016*, Tartu, Estonia, 2016.
- J. Novickij, A. Skučaitė, and A. Štikonas, Spectrum analysis of the weighted finite difference scheme for the wave equation with the nonlocal integral boundary conditions, *Book of abstracts, European conference on numerical mathematics and advanced applications*, Ankara, Turkey, 2015.
- J. Novickij and A. Štikonas, Spectrum analysis of the weighted finite difference scheme for the wave equation with integral boundary conditions, *Abstracts of MMA2015*, Sigulda, Latvia, 2015.
- J. Novickij and A. Štikonas, Stability of the weighted finite-difference scheme for hyperbolic equation with two nonlocal integral conditions, *Abstracts of MMA2014*, Druskininkai, Lithuania, 2014.
- J. Novickij and A. Štikonas, An analysis of properties of weighted difference schemes for nonlocal hyperbolic problems, *Abstracts of MMA2012 & AMOE2013*, Tartu, Estonia, 2013.
- J. Novickij, F. Ivanauskas, and M. Sapagovas, On the stability of an explicit difference scheme for hyperbolic equations with nonlocal boundary conditions, *Abstracts of MMA2012*, Tallinn, Estonia, 2012.

Training courses

During the doctoral studies were made several research visits:

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Chapter 1

Stability of an explicit finite difference scheme

1.1 Introduction

In the last decades, the demand to study processes described by equations of mathematical physics with rather complicated nonclassical additional conditions has increased. Throughout this thesis we are interested in the stability of the finite difference schemes for nonclassical hyperbolic problems. The main principles of the general theory of stability for difference schemes were started to investigate by A.A. Samarskii in the mid 1960's [72–74]. In the article [72, Самарский 1967] the author considers two sets of difference schemes, the two-layer

$$B(t)\frac{y(t+\tau)-y(t)}{\tau}+A(t)y(t)=\phi(t), \quad y(0)=y_0, y_0\in H_N, 0\leqslant t=t_j<t_0,$$

and the three-layer

$$B(t)\frac{y(t+\tau)-y(t-\tau)}{2\tau}+R(t)\left(y(t+\tau)-2y(t)+y(t-\tau)\right)+A(t)y(t)=\phi(t),$$
$$y(0)=y_0, y(\tau)=y_1, \quad y_0, y_1\in H_N.$$

A difference scheme is interpreted as an operator equation in the Euclidean space. Necessary and sufficient stability conditions are formulated in the form of energy inequalities. The stability is formulated as the lack of increase with time of the solution energy norm, which is defined by a self-adjoint positive operator.

In the cases, when boundary conditions are nonlocal, transition operator is not self-adjoint and even not positive. One of the articles, investigating stability of the difference schemes with nonlocal boundary condition is a work of Gulin *et al.* [33, 2001]. Authors consider parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = u_0(x), \qquad 0 < x < 1, \ t > 0,$$

with nonlocal boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t), \quad u(0,t) = 0.$$

The authors obtained the necessary and sufficient stability condition

$$0 < \tau \leqslant \frac{1}{\lambda_k}\left(2 - \frac{|p_k|}{\lambda_k\sqrt{\varDelta_k}}\right), \quad p_k = -\frac{2}{h}\sin 2\pi kh, k = \overline{1,m},$$

where $\varDelta_k = r_k\beta_k - \alpha_k^2 > 0$, for the difference scheme in a special norm in Euclidean space H_D, where

$$D = \begin{bmatrix} 1 & \alpha_k \\ r_k^{-1}\alpha_k & \beta_k \end{bmatrix}$$

is the self-adjoint positive operator $D\colon H \to H$, ($r_k = 1$ for $k = \overline{1, m-1}$ and $r_m = 0.5$, $\alpha_k, \beta_k > 0$, $k = \overline{1,m}$) which defines the energy norm $\|y\|_D = \sqrt{(Dy, Dy)}$).

Hyperbolic problems with nonlocal integral boundary conditions have not been studied so broadly as, say, parabolic or elliptic problems. A. Ashyralyev and N. Aggez in the paper [3, 2011] dealt with the stability of a difference scheme for the multidimensional hyperbolic equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{m}(a_r(x)u_{x_r})_{x_r} = f(t,x), \quad x = (x_1,\dots,x_m) \in \varOmega, \quad 0 < t < 1,$$

with the initial conditions

$$u(0,x) = \int_0^1 \alpha(\rho)u(\rho,x)\,d\rho + \phi(x), \quad u_t(0,x) = \psi(x), \quad x \in \overline{\varOmega},$$

one of which is an integral condition, and the classical boundary condition

$$u(t,x) = 0, \quad 0 < t < 1, \quad x \in S,$$

where $\varOmega$ is the open unit cube in m-dimensional Euclidean space

$$\varOmega = \{x = (x_1,\dots,x_m) : 0 < x_j < 1, 1 \leqslant j \leqslant m\} \subset \mathbb{R}^m$$

with boundary S, $\overline{\varOmega} = \varOmega \cup S$, $\alpha_r(x) \geqslant \alpha > 0$ $(x \in \varOmega)$, $\phi(x)$, $\psi(x)$ $(x \in \overline{\varOmega})$ and $f(t,x)$ $(t \in (0,1), x \in \varOmega)$ are given smooth functions. Stability conditions in the class L_2 were obtained.

The stability of difference schemes for the nonlocal hyperbolic problems was studied by A. Ashyralyev and O. Yildirim in [5, 2011]. Multidimensional hyperbolic equation with Dirichlet condition is considered

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{m} (a_r(x) u_{x_r})_{x_r} = f(t,x), \\ x = (x_1, \dots, x_m) \in \Omega, \quad 0 < t < 1, \\ u(0,x) = \sum\limits_{j=1}^{n} \alpha_j u(\lambda_j, x) + \phi(x), \quad x \in \overline{\Omega}, \\ u_t(0,x) = \sum\limits_{k=1}^{n} \beta_k u_t(\lambda_k, x) + \psi(x), \quad x \in \overline{\Omega}, \\ u(t,x) = 0, \quad x \in S, \end{cases}$$

here $\Omega = \{x = (x_1, \dots, x_m) : 0 < x_j < 1, 1 \leqslant j \leqslant m\}$ is the open unit cube in the m-dimensional Euclidean space $\mathbb{R}^m$ with boundary S, $\overline{\Omega} = \Omega \cup S$. Stability conditions in a special norm $\|\cdot\|_{L_{2h}}$ were obtained and numerical analysis was done.

The existence and uniqueness of solutions of differential equations were studied in [44, Kozhanov and Pul'kina 2006] and [63, Пулькина 2004].

The hyperbolic equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = F(x,t), \qquad 0 < x < 1, \quad 0 < t \leqslant T,$$

with the initial conditions

$$u(x,0) = r(x), \quad u_t(x,0) = s(x), \qquad 0 \leqslant x \leqslant 1,$$

the Dirichlet boundary condition

$$u(0,t) = p(t)$$

and the nonlocal integral boundary condition

$$\int_0^1 u(x,t)\, dx = q(t), \quad 0 < t \leqslant T.$$

was considered by M. Ramezani *et al.* in [69, 2008]. Numerical methods combining the finite difference method and the spectral method were suggested for solving such equations.

In this chapter, we give a sufficient condition for the stability of an explicit difference scheme for a hyperbolic equation with nonlocal integral boundary conditions. By using a method applied earlier to parabolic equations with nonlocal boundary conditions [79, Sapagovas 2012], we rewrite a three-layer difference scheme in the form of an equivalent two-layer scheme. By analyzing the spectrum of the transition matrix of the two-layer scheme, we obtain sufficient conditions for the stability of the three-layer scheme depending on the parameters occurring in the integral boundary conditions. To analyze the stability of a difference nonlocal hyperbolic problem, we use an explicit three-layer difference scheme and approximate the nonlocal boundary conditions by the trapezoid quadrature formula. By representing this scheme in the form of a second-order operator-difference equation and by using some transformations, one can obtain a two-layer scheme equivalent to this three-layer scheme [71, p. 364, Самарский и Гулин 1989]. To study the spectrum of the transition matrix of the two-layer scheme, we define the norms of matrices and vectors. The analysis of the structure of the spectrum of the transition matrix (see [81, Sapagovas *et al.* 2012] and [78, Сапаговас и Штиконас 2005]) and the use of a generalized nonlinear eigenvalue problem permit one to state the main result of the present paper, a sufficient condition for the stability of an explicit difference scheme for hyperbolic equations with integral boundary conditions.

This chapter is based on an article, published together with Profs. Feliksas Ivanauskas and Mifodijus Sapagovas [39, 2013].

1.2 Notation

In this thesis we consider the one-dimensional in space hyperbolic equation with corresponding classical and nonclassical boundary and initial conditions. We are interested in the numerical solutions of the formulated problem. Obviously, the numerical solution of the mathematical problem does not correspond to the differential solution for all values of the unknown function in a certain domain. The classical way (e.g. [77, Samarskii 2001]) to overcome this is to define a finite set of points in the problem's domain and look for the numerical solutions only on the points of this set. Any such set of simulation elements is called a grid (in some literature a term mesh is used. Grids are typically a set of simulation elements that have a well defined structure to their alignment, while meshes are often more general, they may be unstructured and use various shapes of elements). The isolated points are called grid nodes. A function U, defined on the grid nodes, is called a grid function. We consider $x \in [0, L]$ and $t \in [0, T]$. Throughout this thesis we use the following grids.

$$\begin{aligned}
\overline{\omega}^h &:= \left\{x_i \colon x_i = ih, i = \overline{0, N}\right\}; \ h = L/N, \\
\overline{\omega}^h_{1/2} &:= \left\{x_{i-1/2} = (x_{i-1} + x_i)/2, i = \overline{1, N}, x_{-1/2} = x_0, x_{N+1/2} = x_N\right\}; \\
&\qquad h_{i+1/2} = x_{i+1/2} - x_{i-1/2}, i = \overline{0, N}, \\
\overline{\omega}^\tau &:= \left\{t^j : t^j = j\tau, j = \overline{0, M}\right\}; \ \tau = T/M,
\end{aligned}$$

$$\omega^h := \{x_1, \ldots, x_{N-1}\}, \quad \widetilde{\omega}^\tau := \left\{t^1, \ldots, t^M\right\}, \quad \omega^\tau := \left\{t^1, \ldots, t^{M-1}\right\},$$

where $N + 1$ and $M + 1$ are the numbers of grid points for x and t directions, accordingly, and N, $M \geqslant 2$.

We use notation $U_i^j := U(x_i, t_j)$ for the grid function, defined on the grid (or parts of it) $\overline{\omega}^h \times \overline{\omega}^\tau$. Instead of writing indices we denote $\check{U}^j := U^{j-1}$ and $\hat{U}^j := U^{j+1}$ on grids $\widetilde{\omega}^\tau$ and $\omega^\tau \cup \{t_0\}$, respectively. We define a space grid operator

$$\delta_x^2 : \overline{\omega}^h \to \omega^h, \quad \left(\delta_x^2 U\right)_i := \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2},$$

and the time grid operators

$$\begin{aligned}
\overline{\partial}_t : \overline{\omega}^\tau \to \widetilde{\omega}^\tau, \quad \overline{\partial}_t U &:= \frac{U - \check{U}}{\tau}, \\
\overline{\partial}_t^2 : \overline{\omega}^\tau \to \omega^\tau, \quad \overline{\partial}_t^2 U &:= \frac{\check{U} - 2U + \hat{U}}{\tau^2}.
\end{aligned}$$

1.3 Statement of the problem

Consider the hyperbolic equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad x \in (0,1), \ t \in (0,T], \tag{1.1}$$

with the classical initial conditions

$$u(x,0) = \phi(x), \quad x \in [0,1], \tag{1.2}$$

$$\frac{\partial u(x,0)}{\partial t} = \psi(x), \quad x \in [0,1], \tag{1.3}$$

and the nonlocal integral boundary conditions

$$u(0,t) = \gamma_0 \int_0^1 u(\xi,t)\,d\xi + \mu_1(t), \quad t \in [0,T], \tag{1.4}$$

$$u(1,t) = \gamma_1 \int_0^1 u(\xi,t)\,d\xi + \mu_2(t), \quad t \in [0,T], \tag{1.5}$$

where γ_0 and γ_1 are given numerical parameters and $f(x,t)$, $\phi(x)$, $\psi(x)$, $\mu_1(t)$, and $\mu_2(t)$ are given functions.

Remark 1.1. The initial data of the hyperbolic problem, stated in Chapters 1–4 is compatible with the boundary data up to the required smoothness.

Now we state a discrete analog of the differential problem (1.1)–(1.5).

1.4 Finite difference schemes

The principle of finite difference schemes consists in approximating the differential operator by replacing the derivatives in the equation using differential quotients. The domain is partitioned in space and in time and approximations of the solution are computed at the space or time points. The approximation error is an error between the numerical and exact solutions. In this section we consider the wave equation (1.1) with the initial conditions (1.2)–(1.3). Nonclassical boundary conditions (1.4)–(1.5) are considered separately.

The main concept of any finite difference scheme is based of the definition of the derivative of a continuously differentiable in the interval $x \in [0, L]$ function u

$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}.$$

Below we state some FDS concepts about approximation of derivatives (e.g. [77, Samarskii 2001], [7, Бахвалов 2011]).

Approximation of the first order derivatives. When h tends to 0, the quotient in the right-hand side provides an approximation of the derivative. There are some approximations, according to the chosen point in the neighbourhood of point x

$$u'(x) \approx \frac{u(x+h)-u(x)}{h}, \qquad \text{(right difference derivative)}$$

$$u'(x) \approx \frac{u(x)-u(x-h)}{h}. \qquad \text{(left difference derivative)}$$

If the function is smooth enough in the neighbourhood of x, it is possible to expand it in a Taylor series. This allows one to quantify the approximation error. Suppose $u \in C^3$ in the neighbourhood of $x \in [0, L]$, then for any $h > 0$ we have

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x^+), \tag{1.6}$$

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u'''(x^-), \tag{1.7}$$

where, $x^+ \in [x, x+h]$ and $x^- \in [x-h, x]$.

Big $\mathcal{O}$ notation. Suppose $f(x)$ and $g(x)$ are two functions defined on some subset $X \subset \mathbb{R}$. We write

$$f(x) = \mathcal{O}(g(x))$$

if and only if there exists constant C such that

$$|f(x)| \leqslant C|g(x)|, \quad x \in X.$$

For example, $X = (-\varepsilon, +\varepsilon)$, $\varepsilon > 0$, is zero neighborhood.

Using the definition of the difference derivative we have

$$\begin{aligned} \frac{u(x+h)-u(x)}{h} &= u'(x) + \frac{h}{2}u''(x) + \mathcal{O}(h^2), \\ \frac{u(x)-u(x-h)}{h} &= u'(x) - \frac{h}{2}u''(x) + \mathcal{O}(h^2). \end{aligned}$$

By subtracting Taylor series expansions (1.6) and (1.7), thanks to the intermediate value theorem, we have

$$\frac{u(x+h)-u(x-h)}{2h} = u'(x) + \frac{h^2}{6}u'''(x^\pm), \qquad \text{(central difference derivative)}$$

where $x^\pm \in [x-h, x+h]$. So, the approximation errors are

$$\begin{aligned} \frac{u(x+h)-u(x)}{h} - u'(x) &= \mathcal{O}(h), \\ \frac{u(x)-u(x-h)}{h} - u'(x) &= \mathcal{O}(h), \\ \frac{u(x+h)-u(x-h)}{2h} - u'(x) &= \mathcal{O}(h^2). \end{aligned}$$

Approximation of the second order derivatives. Suppose $u \in C^4[0, L]$. Like previously, we use Taylor expansions up to fourth order

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^{(4)}(x^+),$$
$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^{(4)}(x^-),$$

Using the intermediate value theorem, we write

$$\frac{u(x-h) - 2u(x) + u(x+h)}{h^2} = u'' + \frac{h^2}{12}u^{(4)}(x^{\pm}).$$

So, the second order difference derivative approximates u'' of the order 2, this means $\left(u(x-h) - 2u(x) + u(x+h)\right)/h^2 - u'' = \mathcal{O}(h^2)$.

1.5 Explicit finite difference scheme

We consider a rectangular domain $\overline{\omega}^h \times \overline{\omega}^\tau$. We write the original differential equation (1.1) at the point $(x_i, t_j) \in \omega^h \times \omega^\tau$

$$\frac{\partial^2 u(x_i, t_j)}{\partial t^2} - \frac{\partial^2 u(x_i, t_j)}{\partial x^2} = f(x_i, t_j)$$

Using Taylor expansion, we obtain

$$\frac{u(x_i, t_{j-1}) - 2u(x_i, t_j) + u(x_i, t_{j+1})}{\tau^2} - \frac{u(x_{i-1}, t_j) - 2u(x_i, t_j) + u(x_{i+1}, t_j)}{h^2}$$
$$= f(x_i, t_j) + E(x_i, t_j),$$

where

$$E(x_i, t_j) = \frac{h^2}{12}\frac{\partial^4 u(x_i^{\pm}, t_j)}{\partial x^4} - \frac{\tau^2}{12}\frac{\partial^4 u(x_i, t_j^{\pm})}{\partial t^4} = \mathcal{O}(\tau^2 + h^2)$$

is a truncation error of the approximation of wave equation (1.1). We consider finite difference scheme

$$\overline{\partial}_t^2 U - \delta_x^2 U = F, \qquad (x_i, t_j) \in \overline{\omega}^h \times \overline{\omega}^\tau, \tag{1.8}$$

where $F := F_i^j = f(x_i, t_j)$.

The initial conditions (1.2)–(1.3) are approximated as follows with the accuracy $\mathcal{O}(h^2)$

$$U^0 = \Phi, \qquad x_i \in \overline{\omega}^h, \tag{1.9}$$
$$\overline{\partial}_t U^1 = \Psi, \qquad x_i \in \overline{\omega}^h, \tag{1.10}$$

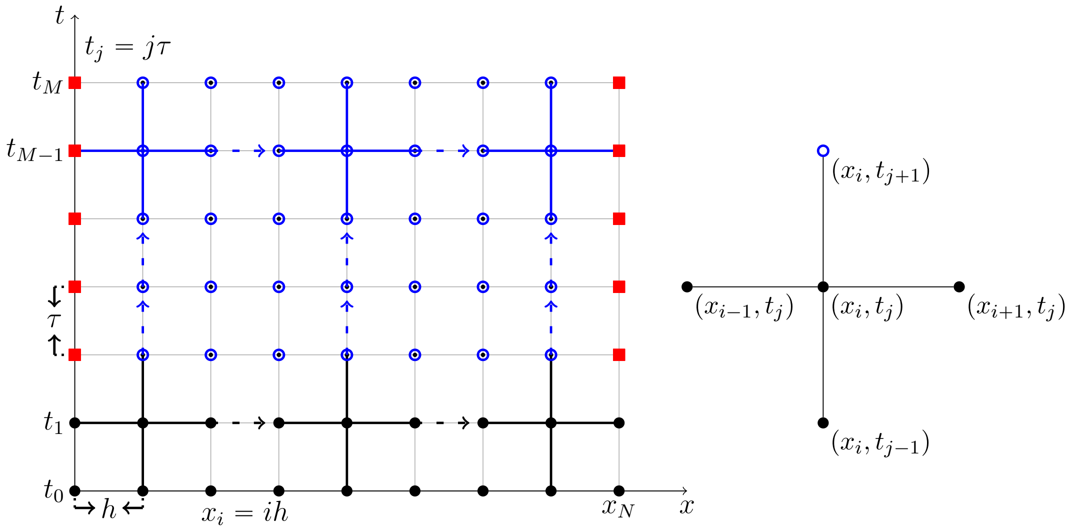


Fig. 1.1: Grid stencil and computation algorithm of the explicit three-layer FDS: • — known values, ∘ — computed stencil values, ■ — computed boundary values.

where $\Phi := \Phi_i = \phi(x_i)$ and $\Psi := \Psi_i = \psi(x_i) + \frac{\tau}{2}\left(\delta_x^2 U^0 + f(x_i, t_0)\right)$. The order of the approximation of Eq. (1.10) is $\mathcal{O}(\tau^2)$.

We replace the boundary conditions by the trapezoid quadrature formula

$$U_0^{j+1} = \gamma_0 h \left(\frac{U_0^{j+1} + U_N^{j+1}}{2} + \sum_{i=1}^{N-1} U_i^{j+1} \right) + {\mu_1}^{j+1}, \tag{1.11}$$

$$U_N^{j+1} = \gamma_1 h \left(\frac{U_0^{j+1} + U_N^{j+1}}{2} + \sum_{i=1}^{N-1} U_i^{j+1} \right) + {\mu_2}^{j+1}. \tag{1.12}$$

The error of approximation of trapezoid formula is $u''(\xi, t)h^2/2$, where $\xi \in [0, 1]$, and is of the order $\mathcal{O}(h^2)$.

Finally, we have formulated difference problem (1.8)–(1.12) with an order of approximation of hyperbolic boundary-value problem (1.1)–(1.5) equals to $\mathcal{O}(\tau^2 + h^2)$.

The five-point stencil of the explicit FDS is shown on Fig. 1.1. The unknown value on the $j+1$ layer is computed using the known values on j and $j-1$ layers. The general algorithm of solving hyperbolic boundary-value problem is shown on Fig. 1.1. The values of the grid $\overline{\omega}^h$ on first and second ($j = 0$ and $j = 1$) time layers are known from two classical initial conditions. Then, using explicit FDS stencil, values of the grid ω^h on the third layer are found. Then, using the boundary conditions values U_0^3 and U_N^3 are calculated. Repeating this algorithm for the remaining time layers, one can solve the problem numerically.

1.6 von Neumann stability

There are two basic concepts in the theory of numerical schemes. These are the notions of consistency and stability. For the numerical scheme to be useful it is important, that both of these properties are fulfilled. For a linear PDE Lax–Richtmyer theorem is valid [46, Lax and Richtmyer 1956]

$$\text{consistency} + \text{stability} \Longleftrightarrow \text{convergence}.$$

A finite difference approximation is considered **consistent** if by reducing the grid and time step size, the truncation error terms could be made to approach zero. In that case the solution to the difference equation would approach the true solution to the PDE.

A finite difference approximation is **stable** if the errors decay as the computation proceeds from one layer to the next. Stability of a finite difference approximation is assessed using Von-Neumann stability analysis (see e.g. [38, Isaacson and Keller 1994] and [90, Smith 1985]).

For linear constant coefficient finite difference schemes such as (1.8), a complete stability analysis is possible, because the numerical algorithm equations can be solved exactly by separation of variables. This means then that any solution of the scheme can be written as a Fourier expansion. Assume we have a Fourier expansion in space of desired function

$$u(x,t) = \sum_{\omega} \hat{u}(t) e^{\imath\omega x}.$$

Now, we take just one term and evaluate it at point (x_i, t_j)

$$u(x_i, t_j) \approx \hat{u}(t_j) e^{\imath\omega x_i}$$

These expressions can be plugged directly into any finite difference scheme to check for stability. The **growth rate** (in some literature **amplification factor**) G is defined as

$$G = \left| \frac{\hat{u}(t_{j+1})}{\hat{u}(t_j)} \right|,$$

where $\hat{u}(t_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x, t_j) e^{-\imath\omega x} dx$. For stability we need

$$G \leqslant 1 \text{ for all } \omega. \tag{1.13}$$

Conditional stability means the stability on a certain condition.

Since the operator of the FDS (1.8)–(1.12) is not symmetric and positively defined (due to nonlocal boundary conditions), we cannot apply the energy norm stability analysis methods. So, in this thesis we use the spectral stability analysis.

1.7 Definition of matrix and vector norm

We define the norm of any $m \times m$ matrix $\mathbf{M}$ as follows

$$\|\mathbf{M}\|_2 = \sqrt{\max_{1\leqslant i\leqslant m} \lambda_i(\mathbf{M}^*\mathbf{M})} = \sigma_{\max}(\mathbf{M}),$$

where $\mathbf{M}^*$ is the conjugate transpose of $\mathbf{M}$ and $\sigma_{\max}$ is the largest singular value of $\mathbf{M}$. If $\mathbf{M}$ is normal matrix (commutes with its conjugate transpose $\mathbf{M}\mathbf{M}^* = \mathbf{M}^*\mathbf{M}$) then

$$\sigma_{\max}(\mathbf{M}) = \rho(\mathbf{M}) := \max_{1\leqslant i\leqslant m} |\lambda_i(\mathbf{M})|.$$

We define the associated vector norm by the formula

$$\|\mathbf{V}\|_2 = \sqrt{\sum_{i=1}^{m} |V_i|^2}. \tag{1.14}$$

According to [71, p. 353, Самарский и Гулин 1989], a sufficient stability condition of the two-layer scheme is

$$\|\mathbf{M}\| \leqslant 1,$$

where $\|\cdot\|$ is an arbitrary norm, $\mathbf{M}$ is a transition matrix of two-layer scheme.

1.8 Reduction to a two-layer scheme

From conditions (1.11) and (1.12), treated as a system of two linear equations with two unknowns U_0^{j+1} and U_N^{j+1}, we express these unknowns via U_i^{j+1}, $i = \overline{1, N-1}$, and obtain

$$U_0^{j+1} = \frac{\gamma_0 h \left({\mu_2}^{j+1} + 2\sum\limits_{i=1}^{N-1} U_i^{j+1} \right) - {\mu_1}^{j+1}(2 - \gamma_1 h)}{2 - h(\gamma_0 + \gamma_1)}, \tag{1.15}$$

$$U_N^{j+1} = \frac{\gamma_1 h \left({\mu_1}^{j+1} + 2\sum\limits_{i=1}^{N-1} U_i^{j+1} \right) - {\mu_2}^{j+1}(2 - \gamma_0 h)}{2 - h(\gamma_0 + \gamma_1)}. \tag{1.16}$$

By substituting expressions (1.15) and (1.16) into Eq. (1.8) for $i = 1$ and $i = N-1$, we rewrite system (1.8)–(1.12) in the form

$$\mathbf{I}U^{j+1} + \mathbf{B}U^j + \mathbf{I}U^{j-1} = \tau^2\mathbf{F}, \tag{1.17}$$

$$\mathbf{B} = -\left(2\mathbf{I} - \tau^2\boldsymbol{\Lambda}\right), \tag{1.18}$$

where $\mathbf{I}$ is the identity matrix, U^j is $(N-1)$-vector, $\mathbf{F} = \left(\widetilde{F}_1, \ldots, \widetilde{F}_{N-1}\right)^\top$, where $\widetilde{F}_i = F_i$, $i = \overline{2, N-2}$ and $\widetilde{F}_i = \widetilde{F}_i\left(F_i, \mu_1, \mu_2\right)$, $i = 1, N-1$, and

$$\boldsymbol{\Lambda} = \frac{1}{h^2}\begin{pmatrix} 2-a & -1-a & -a & -a & \ldots & -a & -a & -a & -a \\ -1 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2 & -1 \\ -b & -b & -b & -b & \ldots & -b & -b & -1-b & 2-b \end{pmatrix}, \tag{1.19}$$

$$a = \frac{h\gamma_0}{1 - \frac{h}{2}\left(\gamma_0 + \gamma_1\right)}, \quad b = \frac{h\gamma_1}{1 - \frac{h}{2}\left(\gamma_0 + \gamma_1\right)},$$

is $(N-1) \times (N-1)$ matrix.

Remark 1.2. A conversion of expressions (1.15) and (1.16) to algebraic problem (1.17) is possible under the condition $(\gamma_0 + \gamma_1) \neq 2/h$. A general case of equivalence of the boundary value problem to the algebraic one is investigated in the Section 4.4.

We represent the three-layer scheme (1.17) as an equivalent two-layer scheme [71, p. 364, Самарский и Гулин 1989]. To this end, we supplement the scheme (1.17) with the trivial condition $U^j = U^j$

$$\begin{pmatrix} U^{j+1} \\ U^j \end{pmatrix} = \begin{pmatrix} 2\mathbf{I} - \tau^2\boldsymbol{\Lambda} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}\begin{pmatrix} U^j \\ U^{j-1} \end{pmatrix} + \begin{pmatrix} \tau^2 f^j \\ \mathbf{0} \end{pmatrix}. \tag{1.20}$$

We define the $2(N-1)$-vector

$$V^{j+1} = \begin{pmatrix} U^{j+1} \\ U^j \end{pmatrix}.$$

This vector combines the solution of the difference problem at points of two time layers. Now system (1.20) can be represented as

$$V^{j+1} = \mathbf{S}V^j + \mathbf{G}, \tag{1.21}$$

where

$$\mathbf{S} = \begin{pmatrix} 2\mathbf{I} - \tau^2\mathbf{\Lambda} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} -\mathbf{B} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \tau^2\mathbf{F} \\ \mathbf{0} \end{pmatrix}.$$

According to [80, Sapagovas 2008] and [31, Гулин *и др.* 2006], one can study the stability conditions for the two-layer difference scheme (1.21) by analyzing the spectrum of the matrix $\mathbf{S}$. Note that the matrix $\mathbf{S}$, as well as the matrix $\mathbf{\Lambda}$ is nonsymmetric.

1.9 Matrix S spectrum structure

Let μ be an eigenvalue of the matrix $\mathbf{S}$; i.e., consider the eigenvalue problem

$$\det(\mathbf{S} - \mu\mathbf{I}) = 0. \tag{1.22}$$

Lemma 1.3. *The following equality for block matrix is valid*

$$\det \begin{pmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{M} & \mathbf{0} \end{pmatrix} = \det \mathbf{M} \det(-\mathbf{L}),$$

where $\mathbf{K}$, $\mathbf{L}$, *and* $\mathbf{M}$ *are arbitrary* $n \times n$ *matrices.*

Proof. The proof follows from a decomposition like

$$\det \begin{pmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{M} & \mathbf{0} \end{pmatrix} = (-1)^{n \cdot n + n} \det \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ -\mathbf{K} & -\mathbf{L} \end{pmatrix} = \det \left(\begin{pmatrix} \mathbf{M} & \mathbf{0} \\ -\mathbf{K} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{L} \end{pmatrix} \right)$$
$$= \det \mathbf{M} \det(-\mathbf{L}). \qquad \square$$

It follows from this lemma, that

$$\det \begin{pmatrix} -\mathbf{B} - \mu\mathbf{I} & -\mathbf{I} \\ \mathbf{I} & -\mu\mathbf{I} \end{pmatrix} = \det \begin{pmatrix} -\mathbf{B} - \mu\mathbf{I} & -\mathbf{I} - \mu\mathbf{B} - \mu^2\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}$$
$$= \det\left(\mu^2\mathbf{I} + \mu\mathbf{B} + \mathbf{I}\right).$$

So, we have a nonlinear eigenvalue problem

$$\left(\mu^2\mathbf{I} + \mu\mathbf{B} + \mathbf{I}\right)\mathbf{V} = 0, \tag{1.23}$$

which is rather well studied for the case of symmetric matrix $\mathbf{B}$ (e.g., see [45, p. 23, Lancaster 1966]). We have thereby proved the following assertion.

Lemma 1.4. *The eigenvalues of the matrix* **S** *coincide with the eigenvalues of the generalized nonlinear eigenvalue problem (1.23).*

Note that the number of eigenvalues of problem (1.23) is $2(N-1)$, where $N-1$ is the order of the matrix **B**.

Let us clarify the relationship between the eigenvalues μ of the matrix **S** of order $2(N-1)$ and the eigenvalues λ of the matrix $\mathbf{\Lambda}$ of order $(N-1)$. To this end, we present the assertion proved in [79, Сапаговас 2012].

Consider the eigenvalue problem

$$\begin{cases} \dfrac{U_{i-1}-2U_i+U_{i+1}}{h^2}+\lambda U_i=0, \quad i=1,2,\dots,N-1, \\ U_0=\gamma_0 h\left(\dfrac{U_0+U_N}{2}+\sum_{i=1}^{N-1}U_i\right)+\mu_1, \\ U_N=\gamma_1 h\left(\dfrac{U_0+U_N}{2}+\sum_{i=1}^{N-1}U_i\right)+\mu_2. \end{cases}$$

If $h<2/(\gamma_0+\gamma_1)$, then, for arbitrary values of the parameters $\gamma_0,\gamma_1\in\mathbb{R}$, all eigenvalues λ of the matrix $\mathbf{\Lambda}$ are real and distinct; in addition, the following assertions hold.

1. If $\gamma_0+\gamma_1<2$, then all eigenvalues are positive.

2. If $\gamma_0+\gamma_1=2$, then $\lambda=0$ is an eigenvalue, and the remaining $N-2$ eigenvalues are positive.

3. If $\gamma_0+\gamma_1>2$, then one eigenvalue is negative, and the remaining ones are positive.

Remark 1.5. The extended version on the formulated assertion is presented in the Chapter 2.

In all three cases, the positive eigenvalues λ_k can be found from the relation

$$\lambda_k=\frac{4}{h^2}\sin^2\frac{\alpha_k h}{2},$$

where the α_k are the solutions of the equation

$$\tan\frac{\alpha_k}{2}=\frac{2}{h(\gamma_0+\gamma_1)}\tan\frac{\alpha_k h}{2}$$

in the interval $(0,2\pi N)$.

By substituting an eigenvector $\mathbf{V}_k$, $k = \overline{1, N-1}$ of matrix $\mathbf{\Lambda}$ into equation (1.23) we obtain

$$\mu^2 \mathbf{I}\mathbf{V}_k + \mu \mathbf{B}\mathbf{V}_k + \mathbf{I}\mathbf{V}_k = \left[\mu^2 \lambda_k(\mathbf{I}) + \mu \lambda_k(\mathbf{B}) + \lambda_k(\mathbf{I})\right] \mathbf{V}_k = 0.$$

Hence it follows that

$$\mu^2 \lambda_k(\mathbf{I}) + \mu \lambda_k(\mathbf{B}) + \lambda_k(\mathbf{I}) = 0, \qquad k = \overline{1, N-1}. \tag{1.24}$$

Lemma 1.6. *For the three-layer scheme (1.8)–(1.12), to each eigenvalue λ_k ($k = \overline{1, N-1}$) of the matrix $\mathbf{\Lambda}$ there correspond two eigenvalues μ_k^1 and μ_k^2 of the matrix $\mathbf{S}$:*

$$\mu_k^{1,2} = \left(1 - \frac{\tau^2 \lambda_k}{2}\right) \pm \sqrt{\left(1 - \frac{\tau^2 \lambda_k}{2}\right)^2 - 1}. \tag{1.25}$$

Proof. From relation (1.18) we obtain

$$\lambda_k(\mathbf{I}) = 1, \quad \lambda_k(\mathbf{B}) = -(2 - \tau^2 \lambda_k), \quad \lambda_k(\mathbf{I}) = 1.$$

By substituting these values into (1.24) and solving the resulting equation, and by making simple transformations, we get relations (1.25). □

Lemma 1.7. *Let λ_k and $\mathbf{V}_k$, be an eigenvalue and an eigenvector, respectively, of the matrix $\mathbf{\Lambda}$, and let μ_k^m, $m = 1, 2$, be the eigenvalues of the matrix $\mathbf{S}$, corresponding to λ_k, $\mu_k^1 \neq \mu_k^2$. Then*

$$\mathbf{W}_k^m = \begin{pmatrix} \mathbf{V}_k \\ \dfrac{1}{\mu_k^m} \mathbf{V}_k \end{pmatrix}, \qquad m = 1, 2, \tag{1.26}$$

are linearly independent eigenvectors of the matrix $\mathbf{S}$.

Proof. Consider the expression $\mathbf{S}\mathbf{W}_k^m$. By using formulas (1.22), (1.26), and

$$\mu_k^m \mathbf{B}\mathbf{V}_k + \mathbf{I}\mathbf{V}_k = -(\mu_k^m)^2 \mathbf{I}\mathbf{V}_k \iff -\mathbf{B}\mathbf{V}_k - \frac{1}{\mu_k^m} \mathbf{I}\mathbf{V}_k = \mu_k^m \mathbf{I}\mathbf{V}_k,$$

we obtain the equation

$$\begin{pmatrix} -\mathbf{B} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{V}_k \\ \frac{1}{\mu_k^m} \mathbf{V}_k \end{pmatrix} = \begin{pmatrix} -\mathbf{B}\mathbf{V}_k - \frac{1}{\mu_k^m} \mathbf{I}\mathbf{V}_k \\ \mathbf{I}\mathbf{V}_k \end{pmatrix} = \mu_k^m \begin{pmatrix} \mathbf{V}_k \\ \frac{1}{\mu_k^m} \mathbf{V}_k \end{pmatrix}, \tag{1.27}$$

where $m = 1, 2$. Equation (1.27) coincides with the definition of an eigenvalue of the problem $\mathbf{S}\mathbf{W}_k^m = \mu_k^m \mathbf{W}_k^m$, $m = 1, 2$. The inequality $\mu_k^1 \neq \mu_k^2$ provides the linear independence of the eigenvectors $\mathbf{V}_k^1$ and $\mathbf{V}_k^2$. □

Theorem 1.8. *For the three-layer scheme (1.8)–(1.12), the relation* $\rho(\mathbf{S}) = 1$ *holds for any* $h > 0$ *and* $\tau \leqslant h$ *if and only if all eigenvalues* λ_k *of the matrix* $\mathbf{\Lambda}$ *are nonnegative.*

Proof. We use relations (1.25) and estimate $|\mu_k^m|$ ($k = \overline{1, N-1}$, $m = 1, 2$) depending on λ_k. Then, on the basis of the assumptions of the theorem on the sign of eigenvalues of the matrix $\mathbf{\Lambda}$ we obtain the inequality $\lambda_k \geqslant 0$, which is equivalent to the condition $\gamma_0 + \gamma_1 \leqslant 2$; therefore,

$$\lambda_k = \frac{4}{h^2} \sin^2 \frac{\alpha_k h}{2} \leqslant \frac{4}{h^2}.$$

Consider the following cases separately.

1. $\lambda_k > 0$ and $\mu_k^m \in \mathbb{R}$, $m = 1, 2$. Consequently, $|1 - \tau^2\lambda_k/2| > 1$, which is equivalent to the inequalities $\tau^2\lambda_k/2 < 0$ and $\tau^2\lambda_k/2 > 2$. However, for this problem, we have $0 < \tau^2\lambda_k/2 < 2$ (since $\tau > 0$, $\lambda_k > 0$, $\tau^2\lambda_k/2 \leqslant 2\tau^2/h^2 \leqslant 2$); therefore, this case is impossible.

2. The case of $\lambda_k > 0$ and $\mu_k^m \in \mathbb{C}$, $m = 1, 2$, is possible if $|1 - \tau^2\lambda_k/2| < 1$ or, which is the same $0 < \tau^2\lambda_k/2 < 2$. Then

$$\begin{aligned} \mu_k^m &= \left(1 - \frac{\tau^2\lambda_k}{2}\right) \pm \imath\sqrt{1 - \left(1 - \frac{\tau^2\lambda_k}{2}\right)^2}, \\ |\mu_k^m|^2 &= \left(1 - \frac{\tau^2\lambda_k}{2}\right)^2 + 1 - \left(1 - \frac{\tau^2\lambda_k}{2}\right)^2 = 1, \end{aligned}$$

   where $m = 1, 2$.

3. If $\lambda_k < 0$, then at least one of the eigenvalues does not satisfy the stability condition, because

$$|\mu_k^m| = \left| \left(1 + \frac{\tau^2|\lambda_k|}{2}\right) \pm \sqrt{\left(1 + \frac{\tau^2|\lambda_k|}{2}\right)^2 - 1} \right| > 1,$$

   where $m = 1, 2$.

4. If $\lambda_k = 0$, then $|\mu_k^m| = 1$, $m = 1, 2$.

It follows from the second and the fourth cases that the assertion of the theorem holds. □

Therefore, a sufficient condition for the stability of the difference scheme (1.8)–(1.12) is given by the inequality $\gamma_0 + \gamma_1 < 2$ under the condition $\tau \leqslant h$.

Remark 1.9. If $\gamma_0 + \gamma_1 = 2$, then, by Theorem 1.8, $\rho(\mathbf{S}) = 1$, but one of the eigenvalue of the matrix $\mathbf{\Lambda}$ is zero. The corresponding values μ_k^1 and μ_k^2 defined by relation (1.25) coincide and are equal to unity. Consequently, the vectors $v_k^{1,2}$ are not linearly independent in this case.

1.10 Numerical experiment

We take a model problem such that the function $u(x,t) = x^3 + t^3$ is an analytic solution of problem (1.1)–(1.5). Then we obtain the problem

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = 6t - 6x, \qquad x \in (0,1), \quad t \in (0,T], \tag{1.28}$$

$$u(x,0) = x^3, \quad \frac{\partial u(x,0)}{\partial t} = 0, \tag{1.29}$$

$$u(0,t) = \gamma_0 \int_0^1 u(x,t)\,dx + t^3 - \gamma_0\left(\frac{1}{4} + t^3\right), \tag{1.30}$$

$$u(1,t) = \gamma_1 \int_0^1 u(x,t)\,dx + 1 + t^3 - \gamma_1\left(\frac{1}{4} + t^3\right). \tag{1.31}$$

We apply the difference scheme (1.8)–(1.12) (with known functions f, ϕ, ψ, μ_1, and μ_2) to the differential problem (1.28)–(1.31). We find the maximum relative error by the formula

$$\mathit{\Delta} U = \max_{0<i<N} \left| \frac{u(x_i,t_M) - U_i^M}{u(x_i,t_M)} \right|,$$

where $u(x_i,t_M)$ is the analytic solution of the differential problem (1.28)–(1.31) at the point $x = x_i$, $t = t_M$, and U_i^j is the solution of the corresponding difference problem.

As follows from the Table 1.1, the maximum relative error of the model problem for sufficiently small T ($T = 1$) remains a quantity of the same order under small changes of the parameter $\gamma_0 + \gamma_1$ ($1.8 \leqslant \gamma_0 + \gamma_1 \leqslant 2.2$), occurring in the boundary conditions. For larger T ($T = 50$), there is a sharp jump (followed by further growth) in the maximum relative error as the solution of the difference problem crosses the line $\gamma_0 + \gamma_1 = 2$, which shows that the difference scheme

Table 1.1: Maximum relative error in the numerical experiment.

$\gamma_0+\gamma_1$	ΔU		
	$T=1$	$T=10$	$T=50$
0	$3.1\cdot10^{-10}$	$6.8\cdot10^{-13}$	$1.4\cdot10^{-13}$
0.2	$6.1\cdot10^{-8}$	$1.2\cdot10^{-10}$	$4.5\cdot10^{-13}$
0.4	$1.3\cdot10^{-7}$	$1.8\cdot10^{-10}$	$1.4\cdot10^{-12}$
0.6	$2.2\cdot10^{-7}$	$9.2\cdot10^{-11}$	$2.4\cdot10^{-12}$
⋮	⋮	⋮	⋮
1.4	$8.3\cdot10^{-7}$	$3.2\cdot10^{-10}$	$9.0\cdot10^{-12}$
1.6	$1.1\cdot10^{-6}$	$1.8\cdot10^{-9}$	$2.9\cdot10^{-12}$
1.8	$\mathbf{1.4\cdot10^{-6}}$	$\mathbf{2.9\cdot10^{-9}}$	$\mathbf{3.3\cdot10^{-11}}$
2.0	$\mathbf{1.9\cdot10^{-6}}$	$\mathbf{1.5\cdot10^{-7}}$	$\mathbf{3.0\cdot10^{-8}}$
2.2	$\mathbf{2.4\cdot10^{-6}}$	$\mathbf{9.0\cdot10^{-5}}$	$\mathbf{1.2\cdot10^{13}}$
2.4	$3.1\cdot10^{-6}$	$6.1\cdot10^{-3}$	$1.4\cdot10^{23}$
⋮	⋮	⋮	⋮
3.8	$1.8\cdot10^{-5}$	$2.2\cdot10^{6}$	$5.3\cdot10^{66}$
4	$2.4\cdot10^{-5}$	$2.2\cdot10^{7}$	$6.1\cdot10^{71}$

$h=0.001$ and $\tau=0.0005$.

is unstable in the domain $\gamma_0+\gamma_1>2$. In turn, this shows that the condition $\gamma_0+\gamma_1\leqslant 2$ is sufficient for the stability of our difference scheme for sufficiently large values of the parameter T.

1.11 Conclusions and final remarks

- For the explicit three-layer scheme, the relation $\rho(\mathbf{S})=1$ holds for any $h>0$ and $\tau\leqslant h$ if and only if all eigenvalues λ_k of the matrix $\mathbf{\Lambda}$ are nonnegative.
- A sufficient condition for the stability of the explicit finite difference scheme is given by the inequality $\gamma_0+\gamma_1<2$ under the condition $\tau\leqslant h$.

Remark 1.10. When studying the spectrum of the matrix $\mathbf{S}$, one encounters the problem on the equivalence of the norm $\|\cdot\|_2$ and the classical norms used in numerical methods. The equivalence for a similarly defined norm was proved by A.V. Gulin in [31, 32] under different boundary conditions.

Remark 1.11. Note one more important fact if there exists a $\lambda(\mathbf{\Lambda})<0$, then $|\mu(\mathbf{S})|>1$. It follows from this inequality (since $|\lambda(\mathbf{S})|\leqslant\|\mathbf{S}\|$), that any norm of the matrix $\mathbf{S}$ is strictly larger than unity, $\|\mathbf{S}\|>1$. It follows that the instability of a difference scheme in the norm $\|\cdot\|_2$ leads to the instability of that scheme in an arbitrary norm.

Chapter 2

Stability of a weighted finite difference scheme

2.1 Introduction

In this chapter, we investigate a wide class of finite difference schemes — weighted schemes. Approximating differential problem, we consider weight $\sigma \in \mathbb{R}$ in the finite difference scheme:

$$U^{(\sigma)} := \sigma U^{j+1} + (1 - 2\sigma)U^{j} + \sigma U^{j-1}. \tag{2.1}$$

This allows us to investigate the full class of difference schemes. When $\sigma = 0$ we have explicit scheme, at $\sigma = 1/2$ the scheme is Crank–Nicolson, and at other σ values the scheme is implicit.

In this chapter we use Characteristic Function method introduced in [98, Štikonas and Štikonienė 2009]. The spectrum and characteristic functions for eigenvalues of Sturm–Liouville problem are widely investigated in 2005–2007 by S. Pečiulytė, O. Štikonienė, A. Štikonas, and M. Sapagovas in [58, 59, 82, 96]. For example, the following problems

$$-u'' = \lambda u, \quad t \in (0, 1),$$

with one classical boundary condition $u(0) = 0$ and other nonlocal boundary condition

$$u(1) = \gamma u(\xi) \quad \text{or} \quad u(1) = \gamma \int_0^{\xi} u(t)\,dt \quad \text{or} \quad u(1) = \gamma \int_{\xi}^{1} u(t)\,dt,$$

where $\gamma \in \mathbb{R}$, $0 \leqslant \xi \leqslant 1$, were investigated. There exist eigenvalues of two types: the first type eigenvalues do not depending on γ, and the second type eigenvalues which do not depend on γ [82, Sapagovas and Štikonas 2005]. The complex eigenvalues exist for these problems. Complex eigenvalues of these Sturm–Liouville problems (with integral boundary condition) were investigated in [86–88, Skučaitė *et al.* 2010, 2013, 2015].

We obtain a sufficient condition for the stability of a weighted difference scheme for a hyperbolic equation with nonlocal integral boundary conditions. By using a method applied earlier to explicit difference scheme for hyperbolic equations with nonlocal boundary conditions [39, Ivanauskas *et al.* 2013], we rewrite a three-layer difference scheme in the form of an equivalent two-layer scheme. By analyzing the spectrum of the transition matrix of the two-layer scheme, we obtain sufficient conditions for the stability of the three-layer scheme depending on the parameters occurring in the integral boundary conditions and not depending on the weight parameter used in scheme.

To obtain the stability estimates of a difference nonlocal hyperbolic problem, we use a weighted three-layer difference scheme and approximate the nonlocal integral conditions by the trapezoid quadrature formula. By representing this scheme in a form of the second-order operator-difference equation and by using some transformations, one can obtain a two-layer scheme equivalent to this three-layer scheme [77, p. 364, Samarskii 2001]. To study the spectrum of the transition matrix of the two-layer scheme, we define special norms of matrices and vectors. The analysis of the structure of the spectrum of the transition matrix [81, Sapagovas *et al.* 2012] and [82, Sapagovas and Štikonas 2005] and the use of a generalized nonlinear eigenvalue problem permit one to state the main result of the present paper, a sufficient condition for the stability of a weighted difference scheme for hyperbolic equations with integral boundary conditions.

This chapter is based on an article, published in 2014 [54, Novickij and Štikonas].

2.2 Notation

In this chapter notations defined in Chapter 1 are valid. Let $\overline{H}$ and H be spaces of real grid functions on $\overline{\omega}^h$ and ω^h, respectively. If U and V are grid functions, then the following notation is valid

$$[U,V] := \sum_{i=0}^{N} U_i V_i h_{i+1/2} = \frac{U_0 V_0 h}{2} + (U,V) + \frac{U_N V_N h}{2}, \quad U,V \in \overline{H}, \tag{2.2}$$

$$(U,V) := \sum_{i=1}^{N-1} U_i V_i h, \quad U,V \in H. \tag{2.3}$$

Remark 2.1. Notation (2.2) is used for the approximation of an integral by a trapezoid quadrature formula

$$[1,V] = V_0 \frac{h}{2} + \sum_{i=1}^{N-1} V_i h + V_N \frac{h}{2}.$$

Lemma 2.2. *The following relation is valid*

$$[1, e^{\pm \imath z x}] = [1, e^{\pm \imath z (1-x)}] = h \sin(z/2) \tan^{-1}(zh/2) e^{\pm \imath z/2}, \quad z \in \mathbb{C}, \; x \in \overline{\omega}_h. \tag{2.4}$$

Proof. First, we calculate $[1, y^i]$, where y^i, $i = \overline{0,N}$, $y \in \mathbb{C}$, is a power function. If $y \neq 1$

$$[1,y^i] = h\left(\frac{1+y^N}{2} + \sum_{i=1}^{N-1} y^i\right) = h\left(\frac{1+y^N}{2} + \frac{\left(y^N - y\right)}{(y-1)}\right) = h\frac{\left(y^N-1\right)(y+1)}{2(y-1)},$$

$$[1, y^{N-i}] = \frac{h}{2} y^N + \sum_{i=1}^{N-1} y^{N-i} h + \frac{h}{2} y^0 = \frac{h}{2} y^N + \sum_{i=1}^{N-1} y^i h + \frac{h}{2} y^0 = [1, y^i].$$

Now, we substitute exponential function $y = e^{\imath z h}$, $z \neq 0$, instead of power function

$$\begin{aligned} [1, e^{\pm \imath z x}] &= [1, e^{\pm \imath z(1-x)}] = h e^{\pm \imath z/2} \left(e^{\imath z/2} - e^{-\imath z/2}\right) \frac{\imath}{2} \tan^{-1}(zh/2) \\ &= h e^{\pm \imath z/2} \sin(z/2) \tan^{-1}(zh/2). \end{aligned}$$

If $y = 1$, then

$$[1,1] = \frac{h}{2} + (N-1)h + \frac{h}{2} = Nh = 1. \qquad \square$$

Using Euler's formula, we obtain

$$[1, \sin(zx)] = [1, \sin(z(1-x))] = h \sin^2(z/2) \tan^{-1}(zh/2), \tag{2.5}$$

$$[1, \cos(zx)] = h \sin(z/2) \cos(z/2) \tan^{-1}(zh/2), \tag{2.6}$$

and, using the fact that trapezoid formula is exact for linear polynomials, we also have

$$[1,1] = 1, \quad [1,x] = 1/2, \tag{2.7}$$

and

$$[1, (-1)^i] = 0, \quad [1, (-1)^i x_i] = \frac{1}{4} h^2 \left((-1)^N - 1\right) = \begin{cases} 0, & \text{for even } N, \\ \frac{h^2}{2}, & \text{for odd } N. \end{cases} \tag{2.8}$$

2.3 Problem formulation

Consider the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad (x,t) \in (0,L) \times (0,T], \tag{2.9}$$

with the classical initial conditions

$$u|_{t=0} = \phi(x), \quad x \in [0,L], \tag{2.10}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x), \quad x \in [0,L] \tag{2.11}$$

and the additional nonlocal integral boundary conditions

$$u(0,t) = \gamma_0 \int_0^L u(x,t)\,dx + v_l(t), \quad t \in [0,T], \tag{2.12}$$

$$u(1,t) = \gamma_1 \int_0^L u(x,t)\,dx + v_r(t), \quad t \in [0,T], \tag{2.13}$$

where $f(x,t)$, $\phi(x)$, $\psi(x)$, $v_l(t)$, and $v_r(t)$ are given functions, and γ_0 and γ_1 are given real parameters. We are interested in sufficiently smooth solutions of the nonlocal problem (2.9)–(2.13).

For the sum of integral parameters we use notation $\gamma = \gamma_0 + \gamma_1$. Without loss of generality, we use transformation $x = Lx'$ to investigate the problem in the interval $[0,1]$ instead of $[0,L]$. Then new $c' = c/L$. Further we consider $c' = 1$ for simplicity.

Now we state a difference analog of the differential problem (2.9)–(2.13). We define a weighted FDS approximating the original differential equation (2.9)

$$\overline{\partial}_t^2 U - \delta_x^2 U^{(\sigma)} = F, \quad (x_i, t_j) \in \omega^h \times \omega^\tau, \tag{2.14}$$

where σ is a weight parameter. The initial conditions are approximated as follows

$$U^0 = \varPhi, \quad x_i \in \overline{\omega}^h, \tag{2.15}$$

$$\overline{\partial}_t U^1 = \varPsi, \quad x_i \in \overline{\omega}^h. \tag{2.16}$$

Boundary conditions are approximated by trapezoid formula

$$U_0 = \gamma_0 [1, U] + V_l, \quad t^j \in \widetilde{\omega}^\tau \backslash \{t^1\}, \tag{2.17}$$

$$U_N = \gamma_1[1, U] + V_r, \quad t^j \in \widetilde{\omega}^\tau \backslash \{t^1\}. \tag{2.18}$$

In the problem (2.14)–(2.18) we approximate functions f, ϕ, ψ, v_l and v_r by grid functions F, Φ, Ψ, V_l, and V_r.

Remark 2.3. Properly choosing right-hand side functions in (2.14)–(2.18) one can obtain required approximation accuracy. For example, if $\Psi = \psi + 0.5\tau(\delta_x^2 U^0 + f^0)$ the differential problem (2.9)–(2.13) is approximated by (2.14)–(2.18) with accuracy $\mathcal{O}(\tau^2 + h^2)$.

2.4 Equivalence of the three-layer scheme to a two-layer scheme

Equations (2.17)–(2.18) is a system of two linear equations for unknowns U_0 and U_N. We express these unknowns via inner points U_i, $i = \overline{1, N-1}$, and obtain

$$U_0 = \widetilde{\gamma}_0(1, U) + \widetilde{V}_l, \tag{2.19}$$

$$U_N = \widetilde{\gamma}_1(1, U) + \widetilde{V}_r, \tag{2.20}$$

where $\widetilde{\gamma}_0 = \gamma_0 d^{-1}$, $\widetilde{\gamma}_1 = \gamma_1 d^{-1}$, $d = 1 - h\gamma/2 \neq 0$; $\widetilde{V}_l = (V_l + hc)d^{-1}$, $\widetilde{V}_r = (V_r - hc)d^{-1}$, $c = (\gamma_0 V_r - \gamma_1 V_l)/2$.

Remark 2.4. The restriction on coefficient $d = 1 - h\gamma/2 \neq 0$ is set in order to the equivalence of the boundary value problem (2.14)–(2.18) to the algebraic problem (2.21). More detailed equivalence is investigated in the Section 4.4.

By substituting expressions (2.19) and (2.20) into Eq. (2.14) for $i = 1$ and $i = N - 1$ we rewrite it in the form

$$\mathbf{A}\hat{\mathbf{U}} + \mathbf{B}\mathbf{U} + \mathbf{C}\check{\mathbf{U}} = \tau^2\mathbf{F}, \tag{2.21}$$

$$\mathbf{A} = \mathbf{C} = \mathbf{I} + \tau^2\sigma\mathbf{\Lambda}, \ \mathbf{B} = -2\mathbf{I} + \tau^2(1 - 2\sigma)\mathbf{\Lambda}, \tag{2.22}$$

where $\hat{\mathbf{U}} = U^{j+1}$, $\check{\mathbf{U}} = U^{j-1}$, $j = \overline{1, N-1}$; $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and

$$\mathbf{\Lambda} = \frac{1}{h^2} \begin{pmatrix} 2-\widetilde{\gamma}_0 h & -1-\widetilde{\gamma}_0 h & -\widetilde{\gamma}_0 h & \dots & -\widetilde{\gamma}_0 h & -\widetilde{\gamma}_0 h & -\widetilde{\gamma}_0 h \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -\widetilde{\gamma}_1 h & -\widetilde{\gamma}_1 h & -\widetilde{\gamma}_1 h & \dots & -\widetilde{\gamma}_1 h & -1-\widetilde{\gamma}_1 h & 2-\widetilde{\gamma}_1 h \end{pmatrix} \tag{2.23}$$

are $(N-1)\times(N-1)$ matrices, $\mathbf{I}$ is the identity matrix, and $\mathbf{F} = \left(\widetilde{F}_1, \dots, \widetilde{F}_{N-1}\right)^\top$, where $\widetilde{F}_i = F_i$, $i = \overline{2, N-2}$ and $\widetilde{F}_i = \widetilde{F}_i\left(F_i, V_l, V_r\right)$, $i = 1, N-1$.

Remark 2.5. Let suppose that all eigenvalues of matrix $\mathbf{\Lambda}$ are real. In this case $\det \mathbf{A} > 0$ if the following condition is satisfied (follows from matrix $\mathbf{A}$ form (2.22))

$$-\frac{1}{\tau^2 \max(0, \lambda_{\max})} < \sigma < -\frac{1}{\tau^2 \min(0, \lambda_{\min})}. \tag{2.24}$$

Matrix $\mathbf{A}^{-1}$ exists for such σ.

We represent the three-layer scheme (2.21) as an equivalent two-layer scheme (analogously as in Chapter 1).

$$\widehat{\mathbf{W}} = \mathbf{S}\mathbf{W} + \mathbf{G}, \tag{2.25}$$

where

$$\widehat{\mathbf{W}} = \begin{pmatrix} \hat{\mathbf{U}} \\ \mathbf{U} \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} \mathbf{U} \\ \check{\mathbf{U}} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \tau^2 \mathbf{A}^{-1}\mathbf{F} \\ \mathbf{0} \end{pmatrix}. \tag{2.26}$$

Remark 2.6. The structure of two-layer scheme is the same that in Chapter 1. The main difference is the structure of matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

2.5 Structure of the spectrum of the matrix $\mathbf{\Lambda}$

Eigenvalue problem

$$\mathbf{\Lambda}\mathbf{U} = \lambda\mathbf{U},$$

for $(N-1)\times(N-1)$ matrix $\mathbf{\Lambda}$ is equivalent to the eigenvalue problem for the difference operator with nonlocal boundary conditions

$$-\delta_x^2 U = \lambda U, \quad U \in \omega^h, \tag{2.27}$$

$$U_0 = \gamma_0[1, U], \quad U_N = \gamma_1[1, U]. \tag{2.28}$$

First, note one important property of the three-layer scheme (2.21).

Lemma 2.7. *The matrices* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *of the three-layer scheme (2.21) defined by relations (2.22) have a common system of eigenvectors.*

Remark 2.8. The lemma can be stated in a different form: the eigenvectors of the matrix $\mathbf{\Lambda}$ are eigenvectors of the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

Lemma 2.9 (See Sapagovas 2012 [79])**.** *For arbitrary values of the parameters* $\gamma_0, \gamma_1 \in \mathbb{R}$, *all eigenvalues* λ *of the matrix* $\mathbf{\Lambda}$ *are real and simple, moreover, the following assertions hold*

1) if $\gamma = \gamma_0 + \gamma_1 < 2$, *then all eigenvalues are positive;*

2) if $\gamma = 2$, *then there exists one zero eigenvalue, other eigenvalues are positive;*

3) if $2 < \gamma < 2/h$, *then there exists one negative eigenvalue, all other are positive.*

Now we specify few additional properties of eigenvalues λ, which are not stated in Lemma 2.9.

Remark 2.10. First, we enumerate all the eigenvalues $\lambda_1 < \ldots < \lambda_{N-1}$ of the problem (2.27)–(2.28) in the ascending order using the classical case $\gamma_0 = 0$, $\gamma_1 = 0$ (in this case $\gamma = 0$).

Lemma 2.11. *Additional properties of the eigenvalues are valid*

1) if $\gamma < 2$, *then* $\lambda \in (0, 4/h^2)$;

2) if $\gamma \nearrow 2/h$, *then* $\lambda_1 \to -\infty$;

3) if $\gamma = 2/h$, *then boundary conditions* (2.17)–(2.18) *are not equivalent to conditions* (2.19)–(2.20);

4) if $\gamma \searrow 2/h$, *then* $\lambda_1 \to +\infty$;

5) if $\gamma > 2/h$, then all the eigenvalues λ are positive.

Proof. We further prove all the properties in the following eigenspectrum investigation.

Instead of investigating eigenvalues $\lambda \in \mathbb{C}$ we use a bijection $\lambda = \lambda(q)$ from domain $\mathbb{C}_q$ to $\mathbb{C}$

$$\lambda = \frac{4}{h^2}\sin^2\frac{qh}{2}, \quad q := \alpha + \imath\beta, \tag{2.29}$$

where $\mathbb{C}_q = \{q = \alpha\colon 0 < \alpha < \pi/h\} \cup \{q = \imath\beta\colon \beta \geqslant 0\} \cup \{q = \pi/h + \imath\beta\colon \beta \geqslant 0\}$. The points $q = 0$ and $q = \pi/h$ are the branch points of the map (2.29). Therefore every eigenvalue $\lambda_i = \lambda(q_i)$ conforms to q_i, $i = \overline{1, N-1}$ and vice versa. A numeration of $\{q_1, \ldots, q_{N-1}\}$ coincides with the numeration of $\{\lambda_1, \ldots, \lambda_{N-1}\}$ ($\{\lambda_2, \ldots, \lambda_{N-1}\}$ and $\{q_2, \ldots, q_{N-1}\}$, for $\gamma = 2/h$).

Now we investigate the spectrum of matrix $\mathbf{\Lambda}$ in detail.

a) the case of $q \neq 0$, $q \neq \pi/h$. The general solution of (2.27) is of the form

$$U = C_0 \cos(qx) + C_1 \sin(qx), \quad x \in \overline{\omega}^h.$$

By substituting it into (2.28) we have

$$\begin{aligned} \big(\gamma_0[1, \cos(qx)] - 1\big)C_0 + \gamma_0[1, \sin(qx)]C_1 &= 0, \\ \big(\gamma_1[1, \cos(qx)] - \cos q\big)C_0 + \big(\gamma_1[1, \sin(qx)] - \sin q\big)C_1 &= 0. \end{aligned} \tag{2.30}$$

A nontrivial solutions of system (2.30) exist if its determinant is equal to zero

$$\gamma_0[1, \sin(qx)\cos q - \cos(qx)\sin q] - \gamma_1[1, \sin(qx)] + \sin q = 0,$$

or simplifying this formula we have

$$-\gamma_0[1, \sin(q(1-x))] - \gamma_1[1, \sin(qx)] + \sin q = 0.$$

Using expression (2.4) we get an equation for q

$$\gamma h \cdot \frac{\sin^2(q/2)\cos(qh/2)}{\sin(qh/2)} = \sin q. \tag{2.31}$$

In this formula functions $\sin(qh/2)$ and $\cos(qh/2)$ are never equal to zero in $\mathbb{C}_q \setminus \{0, \pi/h\}$ (since a sine function has only real zero points in a complex plane and function $\sin(qh/2)$ has no zero points in the interval $(0, \pi/h)$). We rewrite Eq. (2.31) in the form

$$\sin(q/2) \cdot \big(\gamma h \sin(q/2) - 2\cos(q/2)\tan(qh/2)\big) = 0. \tag{2.32}$$

The roots of Eq. (2.32) can be found from two equations

$$\sin(q/2) = 0, \tag{2.33a}$$

$$\gamma h \sin(q/2) - 2\cos(q/2)\tan(qh/2) = 0. \tag{2.33b}$$

The roots of Eq. (2.33a) are called Constant Eigenvalue points (see [98, Štikonas and Štikonienė 2009]) because they do not depend on γ, and

$$q_{2k} = 2k\pi, \quad k = \overline{0, N_0}, \quad N_0 := \lfloor (N-1)/2 \rfloor. \tag{2.34}$$

The roots of Eq. (2.33b) depend on γ. Such type of roots is called Nonconstant Eigenvalue points.

Now we divide this equation by $\sin(q/2)$ and get expression for γ

$$\gamma = 2h^{-1}\tan^{-1}(q/2)\tan(qh/2). \tag{2.35}$$

A function $\gamma = \gamma(q)$ is called *Complex-Real Characteristic Function (CF)* [98]. The roots q_{2k+1}, $k = \overline{0, N_1}$, $N_1 := \lfloor N/2 \rfloor$ can be found as γ-points of the CF (2.35).

b) the case of $\lambda = q = 0$. In this case the general solution of (2.27) is

$$U_i = C_0 + C_1 ih.$$

By substituting it into (2.28) we have

$$\begin{aligned} C_0 &= \gamma_0 (C_0 + C_1/2), \\ C_0 + C_1 &= \gamma_1 (C_0 + C_1/2). \end{aligned}$$

A nontrivial solution exists if

$$0 = \begin{vmatrix} \gamma_0 - 1 & \gamma_0/2 \\ \gamma_1 - 1 & \gamma_1/2 - 1 \end{vmatrix} = \begin{vmatrix} -1 & \gamma_0/2 \\ 1 & \gamma_1/2 - 1 \end{vmatrix} = \begin{vmatrix} -1 & \gamma_0/2 \\ 0 & \gamma/2 - 1 \end{vmatrix} = 1 - \gamma/2.$$

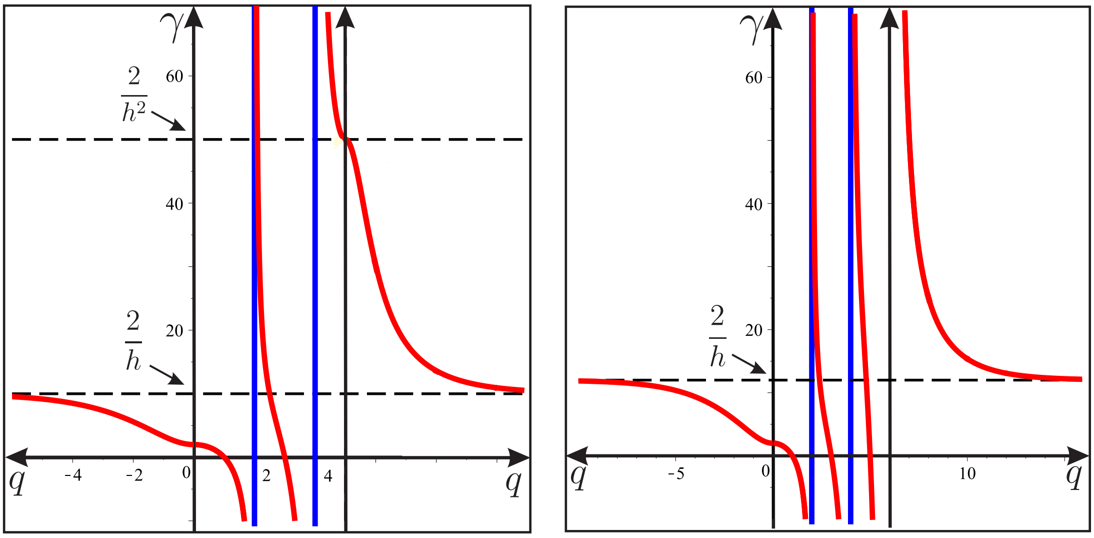
So, we have zero eigenvalue, when $\gamma = 2$.

c) the case of $q = \pi/h$ ($\lambda = 4/h^2$). Now the general solution of (2.27) is

$$U_i = (-1)^i (C_0 + C_1 ih).$$

For even N we have (see (2.8))

$$\begin{aligned} C_0 &= 0, \\ C_0 + C_1 &= 0, \end{aligned}$$



(a) The case of odd number of grid points, $N = 5$ $(h = 1/5)$.

(b) The case of even number of grid points, $N = 6$ $(h = 1/6)$.

Fig. 2.1: Generalized characteristic function $\gamma(\pi q)$.

and for this case only trivial solution exists. For odd N we obtain

$$C_0 = -\gamma_0 h^2 C_1/2,$$

$$C_0 + C_1 = \gamma_1 h^2 C_1/2.$$

A nontrivial solution exists if

$$0 = \begin{vmatrix} 1 & \gamma_0 h^2/2 \\ 1 & 1 - \gamma_1 h^2/2 \end{vmatrix} = \begin{vmatrix} 1 & \gamma_0 h^2/2 \\ 0 & 1 - \gamma h^2/2 \end{vmatrix} = 1 - \gamma h^2/2.$$

So, eigenvalue $\lambda = 4/h^2$ exists for odd N if $\gamma = 2/h^2$.

Remark 2.12. *The generalized CF*, is plotted on Fig. 2.1. All the $N-1$ roots of Eq. (2.32) belong to a union of three intervals $\{q = \alpha \in [0, \pi/h]\} \cup \{q = \imath\beta\colon \beta \geqslant 0\} \cup \{q = \pi/h + \imath\beta\colon \beta \geqslant 0\}$ (if $\gamma = 2/h$ then we have $N-2$ roots). We plot a graph of function (2.35) in each interval: $\gamma = \gamma(\alpha)$, $\alpha \in [0, \pi/h]$; $\gamma := \gamma_-(\beta) = \gamma(\imath\beta)$, $\beta \geqslant 0$; $\gamma := \gamma_+(\beta) = \gamma(\pi/h + \imath\beta)$, $\beta \geqslant 0$. We combine them on one graph of *Real CF*. Finally, we add vertical lines $q = \alpha = 2\pi k$, $k = \overline{0, N_1}$, which correspond to CEs, and get generalized CF. As one can see Real CF asymptotes coincide with CE points. We plot CF $\gamma(\pi\alpha)$ graph in Fig. 2.1, so that in the classical case ($\gamma_0 = 0$, $\gamma_1 = 0$) graph intersects α axis in the integer values (the index of eigenvalues). We note that CF is smooth at the point $q = 0$ for all N; CF is smooth at the point $q = \pi/h$ for odd N only and has a pole at this point for even N.

Remark 2.13. The generalized CF on Fig. 2.1 describes an eigenspectrum for all $\gamma \in \mathbb{R}$. According to [98, Štikonas and Štikonienė 2009] at the critical points (the points, where $\gamma'(q) = 0$), two real eigenvalues merge and two conjugate complex eigenvalues appear.

All CE points are poles of CF. For even N we have additional pole at $q = \pi/h$. So, we have poles at points:

$$p_k = 2\pi k,\ \ k = 1, \ldots, \lfloor N/2 \rfloor.$$

Zeroes of CF are at the points

$$z_l = 2\pi l - \pi,\ \ l = 1, \ldots, \lfloor N/2 \rfloor.$$

Lemma 2.14. *Real CF is decreasing function in the intervals* $(+\infty\imath; 0)$, $(0; p_1)$, $(p_1; p_2)$, $\ldots$, $(p_{M-1}; p_M)$, $(p_M; \pi/h)$, *and* $(\pi/h; +\infty\imath)$, *where* $M = \lfloor N/2 \rfloor$.

Proof. Functions $y_1 = x \tanh^{-1} x$, $x > 0$ and $y_2 = x \tan^{-1} x$, $0 < x < \pi$ are decreasing and $y_1, y_2 < 1$ (see [57, Pečiulytė 2007]).

Consider following functions for $h \in (0, 1)$:

$$g_1(h) = \frac{\sinh(hx)}{hx},\ \ x > 0; \quad g_2(h) = \frac{\sin(hx)}{hx},\ \ 0 < x < \frac{\pi}{h}.$$

The derivatives of these functions are

$$\begin{aligned} g_1'(h) &= \frac{\sinh(hx)}{h^2 x}\big(1 - xh \tanh^{-1}(xh)\big) < 0; \\ g_2'(h) &= \frac{\sin(hx)}{h^2 x}\big(1 - xh \tan^{-1}(xh)\big) < 0. \end{aligned}$$

So, the inequalities

$$\frac{\sinh(hx)}{hx} > \frac{\sinh x}{x},\ \ x > 0; \quad \frac{\sin(hx)}{hx} > \frac{\sin x}{x},\ \ 0 < x < \frac{\pi}{h},$$

are valid for $h \in (0, 1)$, or

$$\sinh(hx) - h \sinh x > 0,\ \ x > 0; \quad \sin(hx) - h \sin x > 0,\ \ 0 < x < \frac{\pi}{h}.$$

Real CF in for $\beta \in (+\infty\imath; 0)$ is equal to

$$\gamma_-(\beta) = \frac{2 \cosh\left(\frac{\beta}{2}\right)}{h \sinh\left(\frac{\beta}{2}\right)} \cdot \frac{\sinh\left(\frac{\beta h}{2}\right)}{\cosh\left(\frac{\beta h}{2}\right)},$$

and the derivative is equal to

$$\gamma'_-(\beta) = -\frac{\sinh(h\beta) - h\sinh\beta}{2h\sinh^2\left(\frac{\beta}{2}\right)\cosh^2\left(\frac{\beta h}{2}\right)} < 0.$$

Analogously, for $\alpha \in (0, \pi/h)$,

$$\gamma_-(\alpha) = \frac{2}{h}\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \cdot \frac{\sin\left(\frac{\alpha h}{2}\right)}{\cos\left(\frac{\alpha h}{2}\right)},$$

$$\gamma'_-(\alpha) = -\frac{\sin(h\alpha) - h\sin\alpha}{2h\sin^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha h}{2}\right)} < 0.$$

We proved that Real CF is decreasing function in the intervals $(+\infty\imath; 0)$, $(0; p_1)$, $\ldots$, $(p_{M-1}; p_M)$. So, in each interval (p_k, p_{k+1}), $k = 1, \ldots, M-1$ we have exactly one eigenvalue point.

Since $\lim_{\beta\to+\infty}\gamma_-(\beta) = 2/h$, we have one negative eigenvalue for $2 < \gamma < 2/h$, and one positive eigenvalue $0 < q_1 < 2$ for $\gamma < 2$. For $\gamma \leqslant 2$ all eigenvalues are found.

If $\gamma > 2$ situation depends on N: if N is even, then negative, zero eigenvalues and positive $\lambda_1 < 2$ eigenvalue do not exist, and $q_1 \in (\pi/h; \pi/h + \infty\imath)$; if N is odd, then $q_1 \in (p_M; \pi/h)$ for $\gamma > 2/h^2$ and $q_1 \in (\pi/h; \pi/h + \infty\imath)$ for $2/h < \gamma < 2/h^2$. CF is decreasing in $(\pi/h; \pi/h + \infty\imath)$, because in this interval we have exactly one eigenvalue point.

We note, that $\gamma'(\pi/h) = 0$ for odd N, but the point $q = \pi/h$ is a first order branch point for $\lambda = \lambda(q)$ and complex eigenvalues at this point do not appear. The same situation is for the branch point $q = 0$. □

Conclusion 2.15. *Matrix* $\mathbf{\Lambda}$ *has only real eigenvalues.*

In general (except the case of $\gamma = 2/h$) the eigenvectors are real and form the complete eigenvector system $\{\mathbf{V}_1, \ldots, \mathbf{V}_{N-1}\}$ (we have $N-2$ eigenvectors $\{\mathbf{V}_2, \ldots, \mathbf{V}_{N-1}\}$, when $\gamma = 2/h$). We call two eigenvectors equal if they are linearly dependent. These eigenvectors can be expressed by general formula

$$V_{ki} = \sin(q_k x_i) - \gamma_0[1, \sin(q_k(x_i - x))], \quad k \in \overline{1, N-1}. \tag{2.36}$$

Note that $q_k = q_k(\gamma)$. So, V_{ki} also depends on γ. Eq. (2.36) can be rewritten as

$$V_{ki} = \sin(q_k(1 - x_i)) - \gamma_1[1, \sin(q_k(x_i - x))], \quad k \in \overline{1, N-1}.$$

Remark 2.16. We can rewrite Eq. (2.36) in such forms

a) if $q_k = \alpha_k \in (0, \pi/h)$ then

$$V_i = \sin(\alpha_k x_i) - \gamma_0[1, \sin(\alpha_k(x_i - x))]; \tag{2.37}$$

b) if $q_1 = \imath\beta$ $(\gamma \in (2, 2/h))$ then

$$V_{1i} = \sinh(\beta x_i) - \gamma_0[1, \sinh(\beta(x_i - x))]; \tag{2.38}$$

c) if $q_1 = \pi/h + \imath\beta$ $(\gamma \in (2/h, 2/h^2)$ if N is odd; $\gamma \in (2/h, \infty)$ if N is even) then

$$V_{1i} = (-1)^i\big(\sinh(\beta x_i) - \gamma_0[1, \sinh(\beta(x_i - x))]\big); \tag{2.39}$$

d) if $q_1 = 0$ $(\gamma = 2)$ then

$$V_{1i} = x_i - \gamma_0[1, x_i - x] = x_i - \gamma_0(x_i - 1/2); \tag{2.40}$$

e) if $q_{N-1} = \pi/h$ $(\gamma = 2/h^2$, N is odd) then

$$V_{1i} = (-1)^i(-\gamma_0 h^2 + 2x_i). \tag{2.41}$$

Expressions (2.40)–(2.41) are the limit versions of the formula (2.36) at the points $q = 0$ and $q = \pi/h$ (as well as Eqs. (2.37)–(2.39)).

2.6 Stability of finite difference scheme

First, we note one important property of the three-layer scheme (2.21) with $(N-1) \times (N-1)$ matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ defined by Eqs. (2.22)–(2.23). We use notation $\lambda_k(\mathbf{A})$, $\lambda_k(\mathbf{B})$, $\lambda_k(\mathbf{C})$ for the k-th eigenvalue of matrix $\mathbf{A}$, $\mathbf{B}$ or $\mathbf{C}$ accordingly. We investigate the case of the complete $N-1$ eigenvector system $\{\mathbf{V}_1, \dots, \mathbf{V}_{N-1}\}$ (in the case $\gamma \neq 2/h$).

Lemma 2.17. *The matrices* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *have a common system of eigenvectors. More precisely, the eigenvectors of the matrix* $\mathbf{\Lambda}$ *are the eigenvectors of the matrices* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$.

Proof. The eigenvectors of the matrix $\mathbf{\Lambda}$ are also the eigenvectors of the unit matrix $\mathbf{I}$. So, since $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are the linear combination of matrices $\mathbf{I}$ and $\mathbf{\Lambda}$, the formulated lemma is valid. □

Let μ be the eigenvalue of the $2(N-1)\times 2(N-1)$ matrix $\mathbf{S}$ (see Eq. (2.26)). We have

$$\det(\mathbf{S}-\mu\mathbf{I}) = \det\begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B}-\mu\mathbf{I} & -\mathbf{I} \\ \mathbf{I} & -\mu\mathbf{I} \end{pmatrix}$$
$$= \det\begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B}-\mu\mathbf{I} & -\mu^2\mathbf{I}-\mathbf{A}^{-1}\mathbf{B}\mu-\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \quad (2.42)$$
$$= \det\big(\mathbf{A}\mu^2+\mathbf{B}\mu+\mathbf{C}\big)\det\big(\mathbf{A}^{-1}\big).$$

We get a characteristic equation for the eigenvalues of the generalized nonlinear eigenvalue problem

$$\big(\mu^2\mathbf{A}+\mu\mathbf{B}+\mathbf{C}\big)\mathbf{U}=0, \quad \mathbf{U}\neq\mathbf{0}. \quad (2.43)$$

Problem (2.43) is rather well studied for the case of symmetric matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ (e.g., see [45, p. 23, Lancaster 1966]). We note that the eigenvalues μ of the matrix $\mathbf{S}$ coincide with the eigenvalues of the generalized nonlinear eigenvalue problem (2.43). The number of eigenvalues of problem (2.43) is $2(N-1)$. Let us clarify the relationship between the eigenvalues μ of the matrix $\mathbf{S}$ and the eigenvalues λ of the matrix $\mathbf{\Lambda}$.

By substituting an eigenvector $\mathbf{V}_k$ of matrix $\mathbf{\Lambda}$ (see Eq. (2.36)) into Eq. (2.43), we obtain

$$\big(\mu^2\mathbf{A}+\mu\mathbf{B}+\mathbf{C}\big)\mathbf{V}_k=\big(\mu^2\lambda_k(\mathbf{A})+\mu\lambda_k(\mathbf{B})+\lambda_k(\mathbf{C})\big)\mathbf{V}_k=0. \quad (2.44)$$

So, eigenvalues of the matrix $\mathbf{S}$ satisfy the quadratic equation

$$\mu^2\lambda_k(\mathbf{A})+\mu\lambda_k(\mathbf{B})+\lambda_k(\mathbf{C})=0, \quad k=\overline{1,N-1}. \quad (2.45)$$

Lemma 2.18. *Each eigenvalue $\lambda_k\big(\mathbf{\Lambda}\big)$, $k=\overline{1,N-1}$ corresponds to two eigenvalues μ_k^1 and μ_k^2 of the matrix $\mathbf{S}$:*

$$\mu_k^m=-b_k\pm\sqrt{b_k^2-1}, \quad m=1,2, \quad (2.46)$$

where $b_k=\left(-1+\tau^2(1/2-\sigma)\lambda_k\right)/\left(1+\tau^2\sigma\lambda_k\right), \quad k=\overline{1,N-1}.$

Proof. Using relations (2.22), we calculate $\lambda_k(\mathbf{A})=\lambda_k(\mathbf{C})=1+\tau^2\sigma\lambda_k$, $\lambda_k(\mathbf{B})=-2+\tau^2(1-2\sigma)\lambda_k$. By substituting these values into (2.45) and solving the resulting equation, we obtain relations (2.46) for eigenvalues of matrix $\mathbf{S}$. □

Remark 2.19. Equation (2.46) determines the relation between eigenvalues μ_k^m and λ_k. The value of μ_k^m can be complex as well as real, depending on the parameters σ, τ and eigenvalues λ_k.

Lemma 2.20. *Let λ_k and $\mathbf{V}_k$ be an eigenvalue and an eigenvector of the matrix $\mathbf{\Lambda}$, respectively. Let μ_k^1 and μ_k^2 be the eigenvalues of matrix $\mathbf{S}$ corresponding to λ_k, $\mu_k^1 \neq \mu_k^2$. Then*

$$\mathbf{W}_k^m = \begin{pmatrix} \mathbf{V}_k \\ (\mu_k^m)^{-1}\mathbf{V}_k \end{pmatrix}, \quad m = 1, 2, \;\; k = \overline{1, N-1}, \tag{2.47}$$

are linearly independent eigenvectors of the matrix $\mathbf{S}$.

Proof. Consider the eigenvalue problem $\mathbf{SW} = \mu_k^m \mathbf{W}$, $m = 1$ or $m = 2$. Using definition of matrix $\mathbf{S}$ (see Eq. (2.26)), we have

$$\begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix} = \mu_k^m \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix}, \quad m = 1, 2, \;\; k = \overline{1, N-1}, \tag{2.48}$$

where $\mathbf{W} = \left(\mathbf{W}_1, \mathbf{W}_2\right)^\top$ is an eigenvector. So, two equalities are valid

$$-\mathbf{A}^{-1}\mathbf{B}\mathbf{W}_1 - \mathbf{W}_2 = \mu_k^m \mathbf{W}_1, \tag{2.49}$$

$$\mathbf{W}_1 = \mu_k^m \mathbf{W}_2. \tag{2.50}$$

Substituting Eq. (2.50) into Eq. (2.49) and multiplying it by $\mu_k^m \mathbf{A}$, we get an analogue of formula (2.44): $((\mu_k^m)^2\mathbf{A} + \mu_k^m\mathbf{B} + \mathbf{A})\mathbf{W}_1 = 0$. Every $\mathbf{V}_k$, $k = \overline{1, N-1}$, satisfies Eq. (2.44) with $\mu = \mu_k^m$. So, we can take $\mathbf{W}_1 = \mathbf{V}_k$, $k = \overline{1, N-1}$. Then from Eq. (2.50) it follows that $\mathbf{W}_2 = (\mu_k^m)^{-1}\mathbf{V}_k$. □

Remark 2.21. If $\mu_k^1 \neq \mu_k^2$, $k = \overline{1, N-1}$, then we have $2(N-1)$ linear independent eigenvectors $\mathbf{W}_k^m$, $m = 1, 2$, $k = \overline{1, N-1}$, which form a complete eigenvector system. If eigenvalues μ_k^m, $m = 1, 2$ are complex, then eigenvectors $\mathbf{W}_k^m$ are also complex.

A polynomial satisfies the *root condition* if all the roots of this polynomial are in the closed unit disc of complex plane and roots of magnitude 1 are simple [34, Hairer *et al.* 1987] and [71, Samarskii and Gulin 1989]. For polynomial of the second order

$$A\mu^2 + B\mu + C, \quad A \neq 0, \;\; B, C \in \mathbb{C}, \tag{2.51}$$

the following theorem is valid.

Theorem 2.22. (*See* [95, Štikonas 1998]) *The roots of the second order polynomial are in the closed unit disc of complex plane and those roots of magnitude* 1 *are simple if*

$$|C|^2 + |\bar{A}B - \bar{B}C| \leqslant |A|^2, \tag{2.52a}$$

$$|B| < 2|A|. \tag{2.52b}$$

We rewrite the quadratic equation (2.45) in a form

$$p(\mu) := a\mu^2 - 2(a-\eta)\mu + a = 0, \tag{2.53}$$

where $a = 1 + \tau^2\sigma\lambda \in \mathbb{R}$, $\eta = \tau^2\lambda/2 \in \mathbb{R}$. For this real polynomial $p(\mu)$, inequality (2.52a) is trivial, and $p(\mu)$ has two complex roots of magnitude 1. The strong inequality (2.52b) ensures that these roots are simple [95, Štikonas 1998]. So, polynomial $p(\mu)$ satisfies the root condition if and only if

$$|a - \eta| < |a| \tag{2.54}$$

(see (2.52b)).

Remark 2.23. If polynomial (2.45) satisfies the root condition, then $\rho(\mathbf{S}) = 1$.

Theorem 2.24. *If* $\gamma < 2$ *and*

$$\sigma > \frac{1}{4} - \frac{1}{\tau^2\lambda_{\max}}, \tag{2.55}$$

then the weighted FDS (2.14)–(2.18) *is stable.*

Proof. Let us analyze condition (2.54). If $a \leqslant 0$, then $a < a - \eta < -a$. In this case, we have $\eta < 0$ or $\lambda < 0$, which contradicts the assumption $\gamma < 2$. If $a > 0$, then $-a < a - \eta < a$. If $\gamma < 2$, then $\lambda_k > 0$, $k = \overline{1, N-1}$, and inequality $\eta/2 > 0$ is valid. So, we have $a > \eta/2 > 0$. We rewrite the inequality $a > \eta/2$ as

$$\sigma > \frac{1}{4} - \frac{1}{\tau^2\lambda}. \tag{2.56}$$

If $\sigma > 1/4 - 1/(\tau^2\lambda_{\max})$, then (2.56) is valid for all λ_k, $k = \overline{1, N-1}$. □

Remark 2.25. The obtained inequality (2.55) is an analogue of the stability inequality for three-layered difference schemes with classical Dirichlet boundary conditions (see [77, Samarskii 2001]).

Remark 2.26. While $\gamma < 2$, the eigenvalues λ_k, $k = \overline{1, N-1}$, are in the interval $(0, 4/h^2)$. So, we can use inequality

$$\sigma \geqslant \frac{1}{4} - \frac{h^2}{4\tau^2}$$

instead of the condition (2.55). If $\sigma \geqslant 1/4$, then the weighted FDS in unconditionally stable. If $\sigma = 0$, then the FDS is stable under the condition $\tau \leqslant h$.

2.7 Conclusions and final remarks

- The sufficient stability condition ($\gamma < 2$ and $\sigma > 1/4 - 1/(\tau^2\lambda_{\max})$) for the three-layered weighted finite difference scheme is obtained.
- The weighted FDS in unconditionally stable under the condition $\sigma \geqslant 1/4$ ($\gamma < 2$).
- The stability condition (2.55) for the weight σ is the same as in the classical case $\gamma_0 = \gamma_1 = 0$.
- The spectrum of the matrix $\mathbf{\Lambda}$ is investigated. Eigenvalues are real, and eigenvectors form a complete system (except the case of $\gamma = 2/h$).
- The spectrum of $\mathbf{\Lambda}$ is qualitatively different for the cases of odd and even number of grid points N.
- If $\gamma > 2/h^2$ and the number of grid points N is odd, then the spectrum of matrix $\mathbf{\Lambda}$ is in the interval $(0, 4/h^2)$ (as well as in the case of $\gamma < 2$).
- If $\gamma > 2/h$, then all the eigenvalues λ_k, $k = \overline{1, N-1}$, are positive, but eigenvalue $\lambda_{\max}$ could be greater than $4/h^2$. This affects the condition on σ.

Chapter 3

Stability of finite difference scheme with two weights

3.1 Introduction

Two-weight finite difference schemes for evolution equations are investigated by authors infrequently. One can find an investigation of two-weight scheme for a time-dependent advection-diffusion problem in the article of N.M. Chadha and N. Madden [15, 2011]. The authors consider the numerical solution of a one-dimensional advection-diffusion problem

$$\frac{\partial \Phi}{\partial t} + L\Phi = 0, \quad L := a\frac{\partial \Phi}{\partial x} - \epsilon\frac{\partial^2 \Phi}{\partial x^2}, \text{ for } (x,t) \in (0,l) \times (0,T],$$

subject to the boundary and initial conditions

$$\begin{aligned} &\Phi(0,t) = g_0(t), \quad \Phi(l,t) = g_l(t), && t \in [0,T], \\ &\Phi(x,0) = f(x), && x \in [0,l]. \end{aligned}$$

The authors consider following difference operators:

$$L_\phi^N u_j := (-\epsilon\delta_{xx} + a\delta_x)u_j, \qquad \text{where } \delta_x := \phi D^- + (1-\phi)D^0,$$

where D^0 is the standard discrete centered approximation, and D^- is the left approximation. The authors then introduce the parameter θ that weights the scheme between being implicit and explicit in nature:

$$\delta_t\Phi_j^n + L_\phi^N(\theta\Phi_j^{n+1} + (1-\theta)\Phi_j^n) = 0, \qquad j = \overline{1, N-1}, n = \overline{1, M}.$$

$\theta = 0$ and $\phi = 0$ correspond to forward Euler with central differencing; $\theta = 0$ and $\phi = a\delta t/\delta x$ give the standard Lax-Wendroff scheme; $\theta = 1$ and $\phi = 0$ give the backward Euler method with central differencing; $\theta = 1/2$ gives Crank-Nicolson type methods.

The authors investigate monotonicity, stability regions and optimal values of the parameters, illustrating results with the numerical experiments.

In this chapter we investigate the stability region of the FDS with two parameters (see [77, Samarskii 2001]) for the hyperbolic equation with two integral NBCs. By using the root criterion (see [95, Štikonas 1998] and [40, Jachimavičienė *et al.* 2014]) we obtain regions on a complex plane, where FDS is stable. A.A. Samarskii in book [77, 2001], using the energy inequality technique, obtained the stability conditions for the classical hyperbolic problem. We have generalized the results presented in [54, Novickij and Štikonas 2014], by using more general scheme. We note, that FDS with more general boundary conditions may have complex eigenvalues.

This chapter is based on an article, published in 2014 [53, Novickij and Štikonas].

3.2 Finite difference scheme

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad (x,t) \in (0,L) \times (0,T], \tag{3.1}$$

with the classical initial conditions

$$u|_{t=0} = \phi(x), \quad x \in [0,L], \tag{3.2}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x), \quad x \in [0,L] \tag{3.3}$$

and integral NBC

$$u(0,t) = \gamma_0 \int_0^L u(x,t)\,dx + v_l(t), \quad t \in [0,T], \tag{3.4}$$

$$u(1,t) = \gamma_1 \int_0^L u(x,t)\,dx + v_r(t), \quad t \in [0,T], \tag{3.5}$$

where $f(x,t)$, $\phi(x)$, $\psi(x)$, $v_l(t)$, and $v_r(t)$ are given functions, and γ_0 and γ_1 are given real parameters. We are interested in sufficiently smooth solutions of the nonlocal problem (3.1)–(3.5). We can investigate problem (3.1)–(3.5) in the interval $[0,1]$ instead of $[0,L]$ using transformation $x = Lx'$. Then new $c' = c/L$. Further we consider $c' = 1$, without losing of generality, for simplicity.

Now we state a difference analogue of the differential problem (3.1)–(3.5). We denote $U^{(\sigma)} = \sigma_1\check{U} + (1-\sigma_1-\sigma_2)U + \sigma_2\hat{U}$, $\sigma_1,\sigma_2 \in \mathbb{R}$. We define a FDS approximating the original differential equation (3.1) (see [77, Samarskii 2001]):

$$\overline{\partial}_t^2 U - \delta_x^2 U^{(\sigma)} = F, \quad (x_i,t_j) \in \omega^h \times \omega^\tau. \tag{3.6}$$

The initial conditions are approximated as follows:

$$U^0 = \varPhi, \quad x_i \in \overline{\omega}^h, \tag{3.7}$$

$$\overline{\partial}_t U^1 = \varPsi, \quad x_i \in \overline{\omega}^h. \tag{3.8}$$

We rewrite the boundary conditions:

$$U_0 = \gamma_0[1,U] + V_l, \quad t^j \in \tilde{\omega}^\tau \backslash \{t^1\}, \tag{3.9}$$

$$U_N = \gamma_1[1,U] + V_r, \quad t^j \in \tilde{\omega}^\tau \backslash \{t^1\}. \tag{3.10}$$

In the problem (3.6)–(3.10) we approximate functions f, ϕ, ψ, v_l and v_r by grid functions F, $\varPhi$, $\varPsi$, V_l, and V_r. In the case $\sigma_1 = \sigma_2 = \sigma$ stability of FDS (3.6)–(3.10) is equal to the one, investigated in [54, Novickij and Štikonas 2014] and Chapter 2.

Equations (3.9)–(3.10) is a system of two linear equations for unknowns U_0 and U_N. We express these unknowns via inner points U_i, $i = \overline{1, N-1}$, and obtain

$$U_0 = \tilde{\gamma}_0(1,U) + \tilde{V}_l, \tag{3.11}$$

$$U_N = \tilde{\gamma}_1(1,U) + \tilde{V}_r, \tag{3.12}$$

where $\tilde{\gamma}_0 = \gamma_0 d^{-1}$, $\tilde{\gamma}_1 = \gamma_1 d^{-1}$, $d = 1 - h\gamma/2 > 0$; $\tilde{V}_l = (V_l + hc)d^{-1}$, $\tilde{V}_r = (V_r - hc)d^{-1}$, $c = (\gamma_0 V_r - \gamma_1 V_l)/2$. By substituting expressions (3.11) and (3.12) into Eq. (3.6) for $i = 1$ and $i = N-1$ we rewrite it in the form

$$\mathbf{A}\hat{\mathbf{U}} + \mathbf{B}\mathbf{U} + \mathbf{C}\check{\mathbf{U}} = \tau^2\mathbf{F}, \tag{3.13}$$

$$\mathbf{A} = \mathbf{I} + \tau^2\sigma_1\mathbf{\Lambda},\ \mathbf{B} = -2\mathbf{I} + \tau^2(1-\sigma_1-\sigma_2)\mathbf{\Lambda},\ \mathbf{C} = \mathbf{I} + \tau^2\sigma_2\mathbf{\Lambda}, \tag{3.14}$$

where **A**, **B**, **C**, and

$$\mathbf{\Lambda} = \frac{1}{h^2}\begin{pmatrix} 2-\widetilde{\gamma}_0 h & -1-\widetilde{\gamma}_0 h & -\widetilde{\gamma}_0 h & \dots & -\widetilde{\gamma}_0 h & -\widetilde{\gamma}_0 h & -\widetilde{\gamma}_0 h \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -\widetilde{\gamma}_1 h & -\widetilde{\gamma}_1 h & -\widetilde{\gamma}_1 h & \dots & -\widetilde{\gamma}_1 h & -1-\widetilde{\gamma}_1 h & 2-\widetilde{\gamma}_1 h \end{pmatrix} \tag{3.15}$$

are $(N-1)\times(N-1)$ matrices, **I** is the identity matrix. Finally, $\mathbf{F} = \big(\widetilde{F}_1,\dots,\widetilde{F}_{N-1}\big)^\intercal$, where $\widetilde{F}_i = F_i$, $i = \overline{2, N-2}$ and $\widetilde{F}_i = \widetilde{F}_i\big(F_i, V_l, V_r\big)$, $i = 1, N-1$. The spectrum of matrix $\mathbf{\Lambda}$ is fully investigated in §3 of paper [54, Novickij and Štikonas 2014] and Chapter 2 of this dissertation. According to that paper's Lemma 1 and Remark 2 under certain conditions ($\gamma < 2$) spectrum is real and is in the interval $(0, 4/h^2)$.

We represent the three-layer scheme (3.13) as an equivalent two-layer scheme

$$\widehat{\mathbf{W}} = \mathbf{S}\mathbf{W} + \mathbf{G}, \tag{3.16}$$

using notations

$$\widehat{\mathbf{W}} = \begin{pmatrix} \widehat{\mathbf{U}} \\ \mathbf{U} \end{pmatrix},\ \mathbf{W} = \begin{pmatrix} \mathbf{U} \\ \check{\mathbf{U}} \end{pmatrix},\ \mathbf{S} = \begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix},\ \mathbf{G} = \begin{pmatrix} \tau^2\mathbf{A}^{-1}\mathbf{F} \\ \mathbf{0} \end{pmatrix}. \tag{3.17}$$

According to [54, Novickij and Štikonas 2014] eigenvalues μ of the matrix **S** could be found as the roots of the quadratic equation

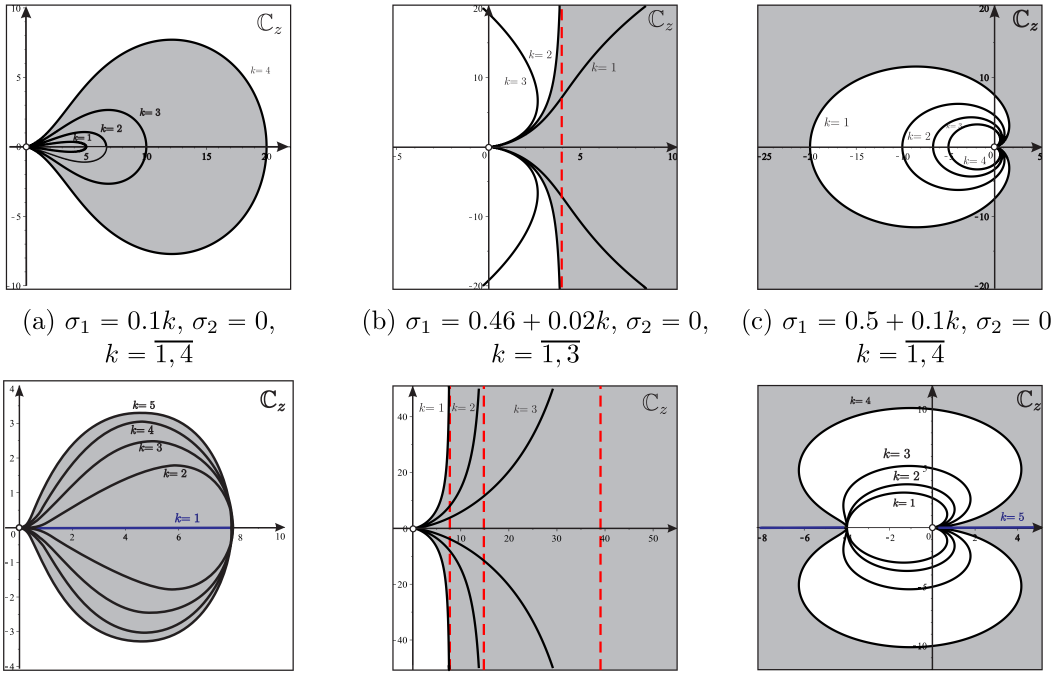
$$\mu^2\lambda_k(\mathbf{A}) + \mu\lambda_k(\mathbf{B}) + \lambda_k(\mathbf{C}) = 0, \quad k = \overline{1, N-1}, \tag{3.18}$$

where λ_k are the eigenvalues of the matrix $\mathbf{\Lambda}$.

The aim of the following section is to investigate the spectrum of the weighted FDS independently of boundary conditions.

3.3 Stability regions

In general, under various boundary conditions, eigenvalues of operator $\mathbf{\Lambda}$ could be complex numbers. A polynomial $p(\mu,\lambda) := a(\lambda)\mu^2 + b(\lambda)\mu + c(\lambda)$ satisfies



(a) $\sigma_1 = 0.1k$, $\sigma_2 = 0$, $k = \overline{1,4}$

(b) $\sigma_1 = 0.46 + 0.02k$, $\sigma_2 = 0$, $k = \overline{1,3}$

(c) $\sigma_1 = 0.5 + 0.1k$, $\sigma_2 = 0$, $k = \overline{1,4}$

(d) $\sigma_1 + \sigma_2 = 0.25$, $\sigma_1 = 0.07k$, $k = \overline{1,5}$

(e) $\sigma_1 + \sigma_2 = 0.5$, $\sigma_1 = 0.48 - 0.05k$, $k = \overline{1,3}$

(f) $\sigma_1 + \sigma_2 = 1$, $\sigma_1 = 0.1k$, $k = \overline{1,5}$

Fig. 3.1: Stability regions for different values of weights σ_1 and σ_2.

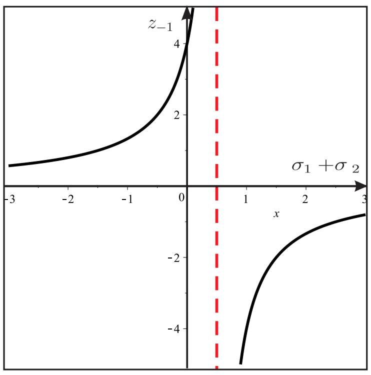


Fig. 3.2: Function $z_{-1} = z_{-1}(\sigma_1 + \sigma_2)$.

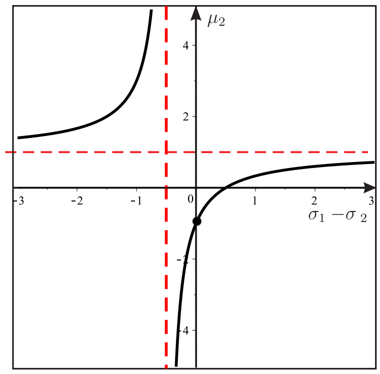


Fig. 3.3: Function $\mu_2 = \mu_2(\sigma_1 - \sigma_2)$.

the root condition if all the roots of that polynomial are in the closed unit disc of complex plane and roots of magnitude 1 are simple (see [95, Štikonas 1998] and [40, Jachimavičienė *et al.* 2014]). If polynomial $p(\mu, \lambda) := a(\lambda)\mu^2 + b(\lambda)\mu + c(\lambda)$ satisfies the root condition, then we say that λ is in stability region defined by equation $p(\mu, \lambda) = 0$. Denoting $z := \tau\lambda$ and substituting it into (3.18) we have:

$$P(\mu, z) := (1 + z\sigma_1)\mu^2 - 2\left(1 - \frac{1}{2}(1 - \sigma_1 - \sigma_2)z\right)\mu + (1 + z\sigma_2) = 0, \tag{3.19}$$

or expressing z:

$$z(\mu) = -\frac{(\mu - 1)^2}{\sigma_1\mu^2 + (1 - \sigma_1 - \sigma_2)\mu + \sigma_2}. \tag{3.20}$$

Substituting $\mu = e^{\imath\varphi}$, $\varphi \in (-\pi, +\pi]$ into Eq. (3.19) we obtain the formula for the boundary of the stability region:

$$z(\varphi) = \frac{2(1-\cos\varphi)\,(1-(\sigma_1+\sigma_2)(1-\cos\varphi)-(\sigma_1-\sigma_2)\imath\sin\varphi)}{(1-(\sigma_1+\sigma_2)(1-\cos\varphi))^2+(\sigma_1-\sigma_2)\sin^2\varphi}. \tag{3.21}$$

One can see that $\operatorname{Re} z(\varphi)$ is even function and $\operatorname{Im} z(\varphi)$ is odd function, so the stability region is symmetric to the real axis (see Fig. 3.1), and boundary intersects it in two points (except of the confluent region when $\sigma_1 = \sigma_2$). If $\mu_1 = 1$, then the first intersection point $z_0 = 0$. By substituting $\mu_1 = -1$ to (3.19), we find the second $z(\varphi)$ intersection point with the real axis (see Fig. 3.2):

$$z_{-1} = \frac{4}{1-2(\sigma_1+\sigma_2)}. \tag{3.22}$$

To find the second root μ_2 of the Eq. (3.19), while the first $\mu_1 = -1$ we use Viète formula $\mu_1\mu_2 = -\mu_2 = (1+z\sigma_2)/(1+z\sigma_1)$ and relation (3.22) for z_{-1}:

$$\mu_2 = \frac{2(\sigma_1-\sigma_2)-1}{2(\sigma_1-\sigma_2)+1}. \tag{3.23}$$

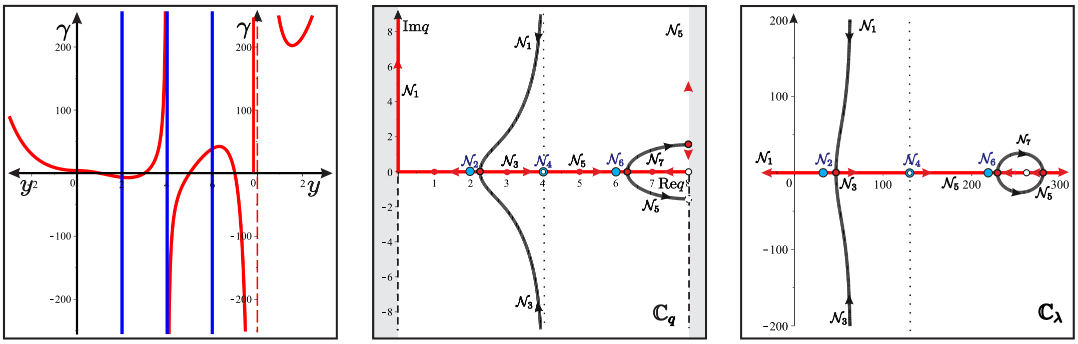
If $\sigma_1 < \sigma_2$, then $|\mu_2| > 1$ (see Fig. 3.3) and the root condition is not satisfied. For the case $\sigma_1 = \sigma_2 = \sigma$ we investigate the discriminant of $P(\mu, z)$ of (3.19):

$$D(P(\mu,z)) = \left((\sigma_1-\sigma_2)^2 - 2(\sigma_1+\sigma_2)+1\right) z^2 - 4z = 0. \tag{3.24}$$

One root of Eq. (3.24) is $z_0 = 0$, and the second root is:

$$z = \frac{4}{1-2(\sigma_1+\sigma_2)+(\sigma_1-\sigma_2)^2}. \tag{3.25}$$

Using relation (3.22) we see, that z is on the boundary in the case of $\sigma_1 = \sigma_2 = \sigma$ and the contour of stability region is on the real axis. The contour consists of two parts: $[0, z_{-1}]$ and $(z_{-1}, 0)$. If $\sigma = 1/4$, then $z_{-1} = \infty$, and if $\sigma > 1/4$, then $z_{-1} < 0$ and the contour gets over the infinity to the negative values (see Figs. 3.2 and 3.3). In this case the roots of (3.19) are $\mu = e^{\pm\imath\varphi}$. For $\sigma_1 > \sigma_2$ we investigate a mapping $z = z(\mu) : \mathbb{C}_\mu \to \mathbb{C}_z$, which is conformal mapping at the point $\mu = -1$. We investigate the monotonicity of the mapping z at the point $\mu = -1$: $z'(-1) = -4(\sigma_1-\sigma_2)/(2(\sigma_1+\sigma_2)-1)^2$. So, if $\sigma_1 > \sigma_2$, then z is decreasing at the point $\mu = -1$, and z defines a boundary of stability region. If $\sigma_1 > \sigma_2$, then $|\mu_2| < 1$ and the root condition is satisfied.



(a) Characteristic function (b) Spectrum view on $\mathbb{C}_q$ (c) Spectrum view on $\mathbb{C}_\lambda$

Fig. 3.4: Eigenspectrum of the numerical problem with one integral NBC.

Example 3.1. Let us take boundary conditions of the form $u(0,t) = 0$ and $u(1,t) = \gamma_1 \int_{1/4}^{3/4} u(x,t)\,dx$ (differential SLP was studied in [19, Sapagovas *et al.* 2004]). The spectrum of formulated discrete problem (integrals approximated with trapezoid formula) was investigated in article [88, Skučaitė and Štikonas 2015]. The study is based on the investigation of characteristic curves on the part of a complex plane C_q, where $\lambda = 4/h^2 \sin^2(\pi q h/2)$ (Fig. 3.4). The points of the spectrum belongs to a spectrum curves. These curves $\mathcal{N}_j$, $j = \overline{1,7}$ are shown in Figs. 3.4(b) and 3.4(c). Every spectrum point moves along the spectrum curve while $\gamma \in (-\infty, +\infty)$. One can compare Fig. 3.4(c) with the stability regions shown in Fig. 3.1, keeping in mind relation $z = \tau^2\lambda$.

The same situation is general for NBCs with not full integrals (see [88, Skučaitė and Štikonas 2015]). Except some special cases there exist complex eigenvalues.

Corollary 3.2. *FDS is unstable for sufficiently small* $\tau \leqslant \tau^*$ *if the corresponding SLP has complex eigenvalues.*

Remark 3.3. If the corresponding SLP has complex eigenvalues then FDS can be stable for some intervals of $\tau > 0$ only if we select special σ_1 and σ_2 values in the case $\sigma_1 > \sigma_2$, $\sigma_1 + \sigma_2 > 0.5$, $\gamma_0 = 0$, and $\gamma_1 < \gamma_*$. In the case $\gamma_0 \neq 0$ and $\gamma_1 \neq 0$ situation is more complex and is under investigation.

3.4 Conclusions and final remarks

- FDS with two weight parameters has a stability region if $\sigma_1 \geqslant \sigma_2$. If the spectrum is in the interval $(0, \infty)$, then the second stability condition is $\sigma_1 + \sigma_2 \geqslant 1/2$ (the same stability condition was obtained in [77, Samarskii 2001] for problem with classical boundary conditions and symmetrical and positive matrix $\mathbf{\Lambda}$).

- The stability region depends on the $\sigma_1 - \sigma_2$ value. While $\sigma_1 - \sigma_2 < 1/2$ the stability region is bounded, elsewise — unbounded.

- FDS is unstable for sufficiently small $\tau \leqslant \tau^*$ if the corresponding Sturm–Liouville problem has complex eigenvalues.

- For the case $\gamma_0 = 0$ and bounded γ_1, if $\sigma_1 > \sigma_2$ and $\sigma_1 + \sigma_2 > 0.5$, then FDS has stability regions.

Chapter 4

Stability of a weighted difference scheme with generalized integral conditions

4.1 Introduction

As a result of technological progress during the last couple decades, there has been an interest investigating problems with rather complicated nonclassical conditions modeling natural, physical, chemical and other processes. There often arise problems described by equations of mathematical physics. In connection with this fact it is natural to investigate whether the problem is well-posed. To understand the behaviour of real processes it is natural to investigate solvability condition on the stationary problems. The solvability results for various type differential problems with nonlocal conditions can be found in [20, Čiupaila *et al.* 2013].

The solvability of nonlocal problems for second-order ordinary differential equations is investigated in [42, Kiguradze and Kiguradze 2011]. The authors consider boundary value problem

$$\begin{aligned} &u'' = f(t,u),\\ &\int_a^b u^{(i-1)}(s)d\phi_s(s) = c_i \quad (i = 1,2), \end{aligned}$$

where $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is a function satisfying local Caratheodory conditions, $c_i \in \mathbb{R}$ $(i = 1,2)$, and $\phi_i\colon [a,b]\to\mathbb{R}$ $(i = 1,2)$ are functions of bounded variation such that

$$\phi_i(a) = 0, \quad \phi_i(b) = 1, \ (i = 1, 2),$$

with one of the following four conditions

$$\begin{aligned}
\phi_i(s) &> 1 \text{ for } a < s < b \ (i = 1, 2),\\
\phi_i(s) &< 0 \text{ for } a < s < b \ (i = 1, 2),\\
\phi_1(s) &> 1, \quad \phi_2(s) < 0 \text{ for } a < s < b \ (i = 1, 2),\\
\phi_1(s) &< 0, \quad \phi_2(s) > 1 \text{ for } a < s < b \ (i = 1, 2).
\end{aligned}$$

The authors presented sufficient conditions which guarantee: solvability, unique solvability, and the existence of at least three distinct solutions of formulated problem.

The solvability of nonlocal multipoint boundary value problems for quasi-linear systems of hyperbolic equations is presented in [6, Assanova and Imanchiev 2015]. The authors consider the following nonlocal multi-point boundary value problem on $\overline{\Omega} = [0, T] \times [0, \omega]$ for a second-order system of quasilinear hyperbolic equations

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t \partial x} = A(t, x)\frac{\partial u}{\partial x} + f\left(t, x, u, \frac{\partial u}{\partial t}\right), \quad u \in R^n,\\
&\sum_{i=0}^{m} \left\{P_i(x)\frac{\partial u(t_i, x)}{\partial x} + S_i(x)\frac{\partial u(t_i, x)}{\partial t} + U_i(x)u(t_i, x)\right\} = \phi(x), \quad x \in [0, \omega],\\
&u(t, 0) = \psi(t), \quad t \in [0, T],
\end{aligned}$$

where $u(t, x) = col(u_1(t, x), u_2(t, x), \ldots, u_n(t, x))$ is the unknown function, the $n \times n$ matrices $A(t, x)$, $P_i(x)$, $S_i(x)$, $U_i(x)$, $i = \overline{0, m}$, and the n-vector function $f(t, x, u, \frac{\partial u}{\partial t}$ are continuous on $\bar{\Omega} \times R^n \times R^n$, the n-vector function ϕ is continuous on $[0, \omega]$, and the n-vector function ψ is continuously differentiable on $[0, T]$, $0 = t_0 < \ldots < t_{m-1} < t_m = T$. The authors establish sufficient coefficient conditions of the unique solvability of formulated problem by introducing some additional functions and applying related results for families of multi-point boundary value problems for systems of ordinary differential equations.

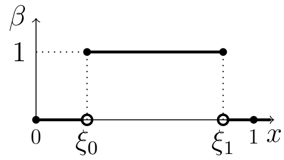
In this chapter we investigate the solvability of the discrete Sturm–Liouville problem with two nonlocal boundary conditions of the general form. We investigate the condition when the discrete Sturm–Liouville problem can be transformed to an algebraic eigenvalue problem. We also provide the examples of the solvability conditions for the most popular nonlocal boundary conditions.

This chapter is based on an article, published in 2015 [55, Novickij and Štikonas] and partly on [52, Novickij *et al.* 2015].

4.2 Notation

In this chapter notations defined in Chapter 1–3 are valid. We define the following piecewise constant function

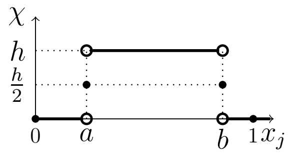
$$\beta(x;\xi_0,\xi_1) := \begin{cases} 0 & x < \xi_0, \\ 1 & \xi_0 \leqslant x \leqslant \xi_1, \\ 0 & \xi_1 < x. \end{cases}$$



,

where $0 \leqslant \xi_0 < \xi_1 \leqslant 1$; and its difference analog

$$\chi_{[a,b]}(x_j) = \begin{cases} 0 & x_j < a \text{ or } x_j > b, \\ \frac{h}{2} & x_j = a \text{ or } x_j = b, \\ h & a < x_j < b. \end{cases}$$



.

We use the following notation, to define discrete function in the inner domain

$$\mathring{U} = \begin{cases} 0 & i = 0,\ i = N, \\ U & \text{otherwise.} \end{cases}$$

We denote δ_i^j as the Kronecker delta

$$\delta_i^j = \begin{cases} 0 & \text{if } j \neq i, \\ 1 & \text{if } j = i. \end{cases}$$

We also use the following sum notation

$$[U,V] = \sum_{i=0}^{N} U_i V_i, \quad U,V \in H.$$

4.3 Problem formulation

Consider the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad (x,t) \in (0,1) \times (0,T], \tag{4.1}$$

with the classical initial conditions

$$u|_{t=0} = \phi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x), \quad x \in [0,1], \tag{4.2}$$

and the additional nonlocal integral boundary conditions

$$\begin{aligned} u(0,t) &= \gamma_0 \int_0^1 \beta^0(x) u(x,t)\, dx + v_l(t), \quad t \in [0,T], \\ u(1,t) &= \gamma_1 \int_0^1 \beta^1(x) u(x,t)\, dx + v_r(t), \quad t \in [0,T], \end{aligned} \tag{4.3}$$

where $f(x,t)$, $\phi(x)$, $\psi(x)$, $v_l(t)$, and $v_r(t)$ are given functions, γ_0 and γ_1 are given parameters, $\beta^0(x)$ and $\beta^1(x)$ are weight functions. Further we consider $c' = 1$ for simplicity. We are interested in sufficiently smooth solutions of the nonlocal problem (4.1)–(4.3) (all the coefficients in (4.1)–(4.3) are smooth enough that the solution $U \in C^{4,4}$). We consider piecewise constant weight functions $\beta^0(x) := \beta(x; \xi_0^0, \xi_1^0)$ and $\beta^1(x) = \beta(x; \xi_0^1, \xi_1^1)$.

4.4 Equivalence of discrete Sturm–Liouville problem to the algebraic eigenvalue problem

Sturm–Liouville problem

We consider discrete Sturm–Liouville operator

$$\mathcal{L}U := -\delta(P\delta U) + QU = \lambda U, \quad x_i \in \omega^h, \tag{4.4}$$

where P, Q are real functions and

$$\left(\delta(P\delta U)\right)_i := \frac{P_{i+1/2}\left(U_{i+1} - U_i\right) - P_{i-1/2}\left(U_i - U_{i-1}\right)}{h^2},$$

with two nonlocal boundary conditions of general form

$$\langle k_0, U\rangle = \gamma_0 \langle n_0, U\rangle, \quad \langle k_1, U\rangle = \gamma_1 \langle n_1, U\rangle, \tag{4.5}$$

where $\langle\cdot,\cdot\rangle$ is a linear functional, $\langle k_i, U\rangle$ is the classical part and $\langle n_i, U\rangle$ is a nonlocal part of boundary conditions, $i = 0, 1$. For example $\langle \delta^N, U\rangle = U_N, \quad \langle \delta^0, U\rangle = U_0, \quad \langle \delta_i, \delta^j\rangle = \delta_i^j$.

Now we investigate the condition when problem (4.4)–(4.5) can be transformed to the algebraic eigenvalue problem. The algebraic problem is degenerate if its determinant equals to zero. We rewrite boundary conditions (4.5) in the following form

$$\left\langle k_0-\gamma_0 n_0,\delta^0\right\rangle U_0+\left\langle k_0-\gamma_0 n_0,\delta^N\right\rangle U_N=\langle\gamma_0 n_0-k_0,\mathring{U}\rangle, \tag{4.6}$$

$$\left\langle k_1-\gamma_1 n_1,\delta^0\right\rangle U_0+\left\langle k_1-\gamma_1 n_1,\delta^N\right\rangle U_N=\langle\gamma_1 n_1-k_1,\mathring{U}\rangle. \tag{4.7}$$

Equations (4.6)–(4.7) form a system of linear equations respect to boundary values of the function U

$$\begin{pmatrix}\left\langle k_0-\gamma_0 n_0,\delta^0\right\rangle & \left\langle k_0-\gamma_0 n_0,\delta^N\right\rangle\\ \left\langle k_1-\gamma_1 n_1,\delta^0\right\rangle & \left\langle k_1-\gamma_1 n_1,\delta^N\right\rangle\end{pmatrix}\begin{pmatrix}U_0\\ U_N\end{pmatrix}=\begin{pmatrix}\langle\gamma_0 n_0-k_0,\mathring{U}\rangle\\ \langle\gamma_1 n_1-k_1,\mathring{U}\rangle\end{pmatrix}. \tag{4.8}$$

System (4.8) degenerates if

$$\begin{vmatrix}\left\langle k_0-\gamma_0 n_0,\delta^0\right\rangle & \left\langle k_0-\gamma_0 n_0,\delta^N\right\rangle\\ \left\langle k_1-\gamma_1 n_1,\delta^0\right\rangle & \left\langle k_1-\gamma_1 n_1,\delta^N\right\rangle\end{vmatrix}=0,$$

or in the expanded form

$$\gamma_0\gamma_1\mathrm{D}(n_0,n_1)+\gamma_0\mathrm{D}(n_0,k_1)+\gamma_1\mathrm{D}(n_1,k_0)+\mathrm{D}(k_0,k_1)=0, \tag{4.9}$$

where

$$\begin{aligned}\mathrm{D}(n_0,n_1)&=\begin{vmatrix}\langle n_0,\delta^0\rangle & \langle n_0,\delta^N\rangle\\ \langle n_1,\delta^0\rangle & \langle n_1,\delta^N\rangle\end{vmatrix},\quad \mathrm{D}(k_1,n_0)=\begin{vmatrix}\langle k_1,\delta^0\rangle & \langle k_1,\delta^N\rangle\\ \langle n_0,\delta^0\rangle & \langle n_0,\delta^N\rangle\end{vmatrix},\\ \mathrm{D}(n_1,k_0)&=\begin{vmatrix}\langle n_1,\delta^0\rangle & \langle n_1,\delta^N\rangle\\ \langle k_0,\delta^0\rangle & \langle k_0,\delta^N\rangle\end{vmatrix},\quad \mathrm{D}(k_0,k_1)=\begin{vmatrix}\langle k_0,\delta^0\rangle & \langle k_0,\delta^N\rangle\\ \langle k_1,\delta^0\rangle & \langle k_1,\delta^N\rangle\end{vmatrix}.\end{aligned} \tag{4.10}$$

In general case Eq. (4.9) describe a second degree algebraic curve on the plane (γ_0,γ_1). The classification of the curves of such type is given in [97, Štikonas 2011]. We call a set of points (γ_0,γ_1), satisfying Eq. (4.9), the *Degeneration Curve* for the problem (4.4)–(4.5).

We denote matrix

$$\mathrm{A}=\begin{pmatrix}a_{00} & a_{01}\\ a_{10} & a_{11}\end{pmatrix}=\begin{pmatrix}D(n_0,n_1) & D(n_0,k_1)\\ D(k_0,n_1) & D(k_0,k_1)\end{pmatrix}. \tag{4.11}$$

Each matrix A corresponds to one of the five types of Degeneration Curves. More detailed classification is shown in Table 4.1. We have 16 types of matrices overall

Table 4.1: Classification of the Degeneration Curves.

Case	Curve in plane	Matrix A				
		a_{00}	a_{01}	a_{10}	a_{11}	
1	whole plane	0	0	0	0	
2	empty set	0	0	0	a_{11}	
3a	line	0	a_{01}	0	0	
3b		0	0	a_{10}	0	
3c		0	a_{01}	0	a_{11}	
3d		0	0	a_{10}	a_{11}	
3e		0	a_{01}	a_{10}	0	
3f		0	a_{01}	a_{10}	a_{11}	
4a	two lines	a_{00}	0	0	0	
4b		a_{00}	a_{01}	0	0	
4c		a_{00}	0	a_{10}	0	
4d		a_{00}	a_{01}	a_{10}	a_{11}	$\det A = 0$
5a	hyperbola	a_{00}	a_{01}	a_{10}	a_{11}	$\det A \neq 0$
5b		a_{00}	0	0	a_{11}	
5c		a_{00}	a_{01}	0	a_{11}	
5d		a_{00}	0	a_{10}	a_{11}	
5e		a_{00}	a_{01}	a_{10}	0	

and one type is split into two cases ($\det A = 0$ and $\det A \neq 0$). So, the next lemma is valid for the Degeneration Curve (as well as for the Characteristic Curve in [97, Štikonas 2011]).

Lemma 4.1. *A Degeneration Curve for problem* (4.4)–(4.5) *in the plane* $\mathbb{R}^2$ *can be one of the following five types:*

1. *If* $D(n_0, n_1) = D(k_0, k_1) = D(n_0, k_1) = D(k_0, n_1) = 0$ *then the curve is whole plane;*
2. *If* $D(n_0, n_1) = D(n_0, k_1) = D(k_0, n_1) = 0$, $D(k_0, k_1) \neq 0$ *then the curve is empty set;*
3. *If* $D(n_0, n_1) = 0$, $D(n_0, k_1) \neq 0$ *or* $D(n_0, n_1) = 0$, $D(k_0, n_1) \neq 0$ *then the curve is line;*
4. *If* $D(n_0, n_1) \neq 0$ *and* $\det A = 0$ *then the curve is union of vertical and horizontal lines;*
5. *If* $D(n_0, n_1) \neq 0$ *and* $\det A \neq 0$ *then the curve is hyperbola.*

Remark 4.2. We see, that Degeneration Curve in the plane $\mathbb{R}^2$ cannot be algebraic curve such as ellipse, parabola, point, parallel lines, double line.

Remark 4.3. If det A $\neq 0$ then the line (Case 3) is neither vertical nor horizontal (see Cases 3e,f in Table 4.1), otherwise we have single vertical or single horizontal line (see Cases 3a-d in Table 4.1).

Remark 4.4. Investigated problem can be easily extended from plane $\mathbb{R}^2$ to the cone $\mathbb{T}^2$ analogously as it was done in [97, Štikonas 2011].

Applications

Example 4.5 (Nonlocal integral boundary conditions). We consider Dirichlet integral boundary conditions with weights. The boundary conditions (4.5) are of the form

$$\langle \delta^0, U \rangle := U_0 = \gamma_0 [\chi^0, U], \quad \langle \delta^N, U \rangle := U_N = \gamma_1 [\chi^1, U], \tag{4.12}$$

where χ^0 and χ^1 are the weight functions. In the general case we have the following nondegeneracy condition:

$$\begin{vmatrix} 1 - \gamma_1 \chi^1_N/2 & \gamma_1 \chi^0_N/2 \\ \gamma_0 \chi^1_0/2 & 1 - \gamma_0 \chi^0_0/2 \end{vmatrix} \neq 0.$$

The degeneration curve is of the following form

$$\frac{1}{4} \begin{vmatrix} \chi^0_0 & \chi^0_N \\ \chi^1_0 & \chi^1_N \end{vmatrix} \gamma_0 \gamma_1 - \frac{1}{2} \chi^0_0 \gamma_0 - \frac{1}{2} \chi^1_N \gamma_1 + 1 = 0;$$

If $\chi^0 \equiv h$ and $\chi^1 \equiv h$, then the full integral which was investigated in [54, Novickij and Štikonas 2014]. The degeneration curve is of the following form

$$-\frac{h}{2}(\gamma_0 + \gamma_1) + 1 = 0;$$

In the case of classical boundary conditions $\chi^0 \equiv 0$ and $\chi^1 \equiv 0$,the degeneration curve is a whole plane (see Table 4.1 case 1).

Example 4.6 (Bitsadze–Samarskii NBC). We consider boundary conditions of the Bitsadze–Samarskii form

$$\langle \delta^0, U \rangle = \gamma_0 \langle \delta^{s_0}, U \rangle := \gamma_0 U_{s_0}, \quad \langle \delta^N, U \rangle = \gamma_1 \langle \delta^{s_1}, U \rangle := \gamma_1 U_{s_1}. \tag{4.13}$$

In this case the degeneration curve is of the form

$$\begin{vmatrix} \delta^{s_0}_0 & \delta^{s_0}_N \\ \delta^{s_1}_0 & \delta^{s_1}_N \end{vmatrix} \gamma_0 \gamma_1 - \delta^{s_0}_0 \gamma_0 - \delta^{s_1}_N \gamma_1 + 1 = 0 \tag{4.14}$$

As one can see from Eq. (4.14) the classification for the degeneration curves in the case of Bitsadze–Samarskii nonlocal boundary condition is the same as for the integral conditions, except the coefficients. For the investigated case the classifications depends on whether the nonlocal point is inner or boundary.

Example 4.7 (Multipoint NBC). We consider boundary conditions of the following form

$$U(0) = \gamma_0 \sum_{i=0}^{N} \alpha_i^0 U(\xi_i), \quad U(1) = \gamma_1 \sum_{i=0}^{N} \alpha_i^1 U(\xi_i),$$

where $\alpha_k = \sum_{j=0}^{N} \alpha_k^j \delta_j$, $k = 0, 1$; $0 \leqslant \xi_0 < \ldots < \xi_N \leqslant 1$. We rewrite NBCs in the following form

$$U_0 = \gamma_0 \langle \alpha^0, U \rangle, \quad U_N = \gamma_1 \langle \alpha^1, U \rangle. \tag{4.15}$$

The method of investigating multipoint case is similar to the method in Example 4.6. The form of the degeneration curve is equivalent to the Eq. (4.14)

$$\begin{vmatrix} \alpha_0^{s_0} \delta_0^{s_0} & \alpha_N^{s_0} \delta_N^{s_0} \\ \alpha_0^{s_1} \delta_0^{s_1} & \alpha_N^{s_1} \delta_N^{s_1} \end{vmatrix} \gamma_0 \gamma_1 - \alpha_0^{s_0} \delta_0^{s_0} \gamma_0 - \alpha_N^{s_1} \delta_N^{s_1} \gamma_1 + 1 = 0 \tag{4.16}$$

Example 4.8 (Left and right rectangle rules for integral NBC). We consider boundary conditions (4.15) with the following notation

$$[U, V]_l := \sum_{i=0}^{N-1} U_i V_i h, \quad [U, V]_r := \sum_{i=1}^{N} U_i V_i h$$

corresponding to the left and right rectangle rules respectively. So the degeneration curves are of the following forms:

$$hB_0^0 \gamma_0 - 1 = 0 \text{ for the left rectangle rule,}$$
$$hB_N^1 \gamma_1 - 1 = 0 \text{ for the right rectangle rule.}$$

Remark 4.9. Examples 1–4 describe all the cases mentioned in the Table 4.1, except of an empty set (case 2). This situation is valid when the boundary conditions are of the following form

$$\int_{\xi_0}^{\xi_1} \beta_0(x) U(x,t) dx = a_0, \quad \int_{\xi_2}^{\xi_3} \beta_1(x) U(x,t) dx = a_1,$$

where $0 < \xi_0 \leqslant \xi_1 < 1$, $0 < \xi_2 \leqslant \xi_3 < 1$, $a_0, a_1 \in \mathbb{R}$, β_0 and β_1 are weight functions.

Remark 4.10. The technique investigated in this section is suitable for defining the solvability conditions for different stationary and non-stationary problems with nonlocal boundary conditions. As one can see only the boundary conditions are needed to define the solvability. It is enough to define the operators, corresponding to the classical and nonlocal parts of the boundary conditions. Obtained solvability condition mostly depends only on the values of the operators of the nonlocal parts on the boundaries.

4.5 Difference problem

Now we state a difference analogue of the differential problem (4.1)–(4.3). We define a weighted FDS approximating original differential equation (4.1):

$$\overline{\partial}_t^2 U - \delta_x^2 U^{(\sigma)} = F, \quad \left(x_i, t^j\right) \in \omega^h \times \omega^\tau, \tag{4.17}$$

where σ is a weight parameter. The initial conditions are approximated as follows:

$$U^0 = \Phi, \ \overline{\partial}_t U^1 = \Psi \quad x_i \in \overline{\omega}^h, \tag{4.18}$$

We rewrite boundary conditions using the defined inner products:

$$\begin{aligned} U_0 &= \ \gamma_0[\chi^0, U] + V_l, \quad t^j \in \overline{\omega}^\tau, \\ U_n &= \ \gamma_1[\chi^1, U] + V_r, \quad t^j \in \overline{\omega}^\tau. \end{aligned} \tag{4.19}$$

The functions χ^0 and χ^1 in the Eq (4.19) correspond to the weight functions in Eq. (4.3). In the problem (4.17)–(4.19) we approximate functions f, ϕ, ψ, v_l and v_r by grid functions $F \in H \times H_\tau$; $\Phi, \Psi \in \overline{H}$; and $V_l, V_r \in \overline{H}_\tau$.

Remark 4.11. We consider without loss of generality that functions χ^0 and χ^1 are defined on the uniform grid $\overline{\omega}^h$.

Remark 4.12. Both boundary conditions (4.19) and the initial conditions (4.18) are defined at the points t^0 and t^1. At these points conditions are consistent. Properly choosing right hand side functions in (4.17)–(4.19) one can obtain required approximation accuracy. For example, if $\Psi = \psi + 0.5\tau(\delta_x^2 U^0 + f^0)$ the differential problem (4.1)–(4.3) is approximated by (4.17)–(4.19) with accuracy $\mathcal{O}(\tau^2 + h^2)$.

Finite difference scheme

Conditions (4.19) form a system of two linear equations for unknowns U_0 and U_n. We express these unknowns via inner points U_i, $i = \overline{1, n-1}$, and obtain

$$U_0 = \tilde{\gamma}_0(\tilde{\chi}^0, U) + \widetilde{V}_0, \quad U_n = \tilde{\gamma}_1(\tilde{\chi}^1, U) + \widetilde{V}_1, \tag{4.20}$$

where $\tilde{\gamma}_0 = \gamma_0 d^{-1}$, $\tilde{\gamma}_1 = \gamma_1 d^{-1}$ and

$$\begin{aligned} \tilde{\chi}^0 &= \Big(1 - \frac{h\gamma_1\chi_n^1}{2}\Big)\chi^0 + \frac{h\gamma_1\chi_n^0}{2}\chi^1, \quad \widetilde{V}_0 &= d^{-1}\Big(\frac{h\gamma_0}{2}\chi_n^0 V_r + \Big(1 - \frac{h\gamma_1}{2}\chi_n^1\Big)V_l\Big), \\ \tilde{\chi}^1 &= \frac{h\gamma_0\chi_0^1}{2}\chi^0 + \Big(1 - \frac{h\gamma_0\chi_0^0}{2}\Big)\chi^1, \quad \widetilde{V}_1 &= d^{-1}\Big(\frac{h\gamma_1}{2}\chi_0^1 V_l + \Big(1 - \frac{h\gamma_0}{2}\chi_0^0\Big)V_r\Big), \end{aligned}$$

$$d = \frac{\gamma_0\gamma_1}{4}\begin{vmatrix} \chi_0^0 & \chi_n^0 \\ \chi_0^1 & \chi_n^1 \end{vmatrix} - \frac{1}{2}(\gamma_0\chi_0^0 + \gamma_1\chi_n^1) + 1.$$

Problem (4.17), (4.20), according to section 4.4, can be transformed to the algebraic problem if $d \neq 0$.

By substituting expressions (4.20) into Eq. (4.17) for $i = 1$ and $i = n-1$ we rewrite it in the canonical three-layer form

$$\mathbf{A}\hat{\mathbf{U}} + \mathbf{B}\mathbf{U} + \mathbf{C}\check{\mathbf{U}} = \tau^2\mathbf{F}. \tag{4.21}$$

Then, analogously as in Chapter 2, we represent the three-layer scheme (4.21) as an equivalent two-layer scheme

$$\widehat{\mathbf{W}} = \mathbf{S}\mathbf{W} + \mathbf{G}, \tag{4.22}$$

where

$$\mathbf{W} = \begin{pmatrix} \mathbf{U} \\ \check{\mathbf{U}} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B} & -\mathbf{A}^{-1}\mathbf{C} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \tau^2\mathbf{A}^{-1}\mathbf{F} \\ \mathbf{0} \end{pmatrix}. \tag{4.23}$$

Remark 4.13. The structure of two-layer scheme is the same that in Chapters 1 and 2. The main difference is the structure of matrices **A**, **B**, and **C**.

Spectrum Analysis

We investigate an eigenvalue problem

$$\mathbf{\Lambda}\mathbf{U} = \lambda\mathbf{U},$$

for $(n-1)\times(n-1)$ matrix $\mathbf{\Lambda}$ which is in general equivalent to SLP for the difference operator with nonlocal boundary conditions

$$-\delta_x^2 U = \lambda U, \quad U \in \omega^h, \tag{4.24}$$

$$\begin{aligned} U_0 &= \gamma_0[\chi^0, U], \\ U_n &= \gamma_1[\chi^1, U]. \end{aligned} \tag{4.25}$$

Instead of investigating eigenvalues $\lambda \in \mathbb{C}_\lambda := \mathbb{C}$ we use a bijection $\lambda = \lambda(q)$ from complex plane $\mathbb{C}_q$ to $\mathbb{C}_\lambda$:

$$\lambda = \frac{4}{h^2}\sin^2\frac{qh}{2}, \quad q := \alpha + \imath\beta \tag{4.26}$$

where $\mathbb{C}_q = \{q = \alpha\colon 0 < \alpha < \pi/h\} \cup \{q = \imath\beta\colon \beta \geqslant 0\} \cup \{q = \pi/h + \imath\beta\colon \beta \geqslant 0\}$. The points $q = 0$ and $q = \pi/h$ are the branch points of the map (4.26). So, every eigenvalue $\lambda_i = \lambda(q_i)$ conforms to q_i, $i = \overline{1, n-1}$ and vice versa.

Now we investigate the spectrum of matrix $\mathbf{\Lambda}$ in detail. The general solution of (4.24) in the case of $q \neq 0$, $q \neq \pi/h$ is $U = C_0\cos(qx) + C_1\sin(qx)$, $x \in \overline{\omega}^h$. By substituting it into (4.25) we have

$$\begin{aligned} \bigl(\gamma_0[\chi^0, \cos(qx)] - 1\bigr)C_0 + \gamma_0[\chi^0, \sin(qx)]C_1 &= 0, \\ \bigl(\gamma_1[\chi^1, \cos(qx)] - \cos q\bigr)C_0 + \bigl(\gamma_1[\chi^1, \sin(qx)] - \sin q\bigr)C_1 &= 0. \end{aligned} \tag{4.27}$$

A nontrivial solutions of system (4.27) exist if its determinant is equal to zero

$$\gamma_0\gamma_1 \begin{vmatrix} [\chi^0, \cos(qx)] & [\chi^0, \sin(qx)] \\ [\chi^1, \cos(qx)] & [\chi^1, \sin(qx)] \end{vmatrix} - \gamma_0[\chi^0, \sin q(1-x)] - \gamma_1[\chi^1, \sin(qx)] + \sin q = 0.$$

Example 4.14. If $\chi^{0,1}_{[0,1]} \equiv 1$, we have the following characteristic function (see [54, Novickij and Štikonas 2014])

$$(\gamma_0 + \gamma_1)h\frac{\sin^2(q/2)\cos(qh/2)}{\sin(qh/2)} - \sin q = 0.$$

If $\chi^0 \equiv 0$ and $\chi^1 = \chi_{[\xi_1,\xi_2]}$ the characteristic function is the following (see [89, Skučaitė 2016])

$$\frac{\gamma_1 h}{2} - \frac{\sin(\pi q)}{\cos(\pi\xi_1 q) - \cos(\pi\xi_2 q)} \cdot \tan(\pi q h/2) = 0.$$

4.6 Conclusions and final remarks

- If boundary conditions satisfy the relation $\gamma_0\gamma_1\mathrm{D}(n_0, n_1) + \gamma_0\mathrm{D}(n_0, k_1) + \gamma_1\mathrm{D}(n_1, k_0) + \mathrm{D}(k_0, k_1) = 0$, then the three-layer finite difference scheme is not defined.

- The Degeneration Curve in the plane $\mathbb{R}^2$ could only be algebraic curve such as hyperbola, line and two lines. Two trivial cases (whole plane and empty set) are also possible.

- The characteristic function for the discrete hyperbolic problem with partial integral nonlocal boundary conditions is found.

Conclusions

- The sufficient stability condition of the explicit FDS for hyperbolic equation with integral NBCs is $\gamma_0 + \gamma_1 < 2$ under the condition $\tau \leqslant h$.
- The weighted FDS (with one weight σ) in unconditionally stable under the condition $\sigma \geqslant 1/4$ for $\gamma_0 + \gamma_1 < 2$. This means that there are no restrictions on τ and h.
- $\mathbf{\Lambda}$ eigenvalues of the weighted FDS (with one weight σ) are real, and eigenvectors form a complete system (except the case of $\gamma_0 + \gamma_1 = 2/h$).
- The $\mathbf{\Lambda}$ spectrum of the weighted FDS (with one weight σ) is qualitatively different (in some sense) for the cases of odd and even number of grid points N.
- If $2 > \gamma_0 + \gamma_1 > 2/h^2$ and the number of grid points N is odd, then the $\mathbf{\Lambda}$ spectrum is in the interval $(0, 4/h^2)$.
- The sufficient stability condition of the weighted FDS (with one weight σ) for hyperbolic equation with integral NBCs is $\gamma_0 + \gamma_1 < 2$ and $\sigma > \frac{1}{4} - \frac{1}{\tau^2 \lambda_{\max}}$.
- The FDS for hyperbolic equation with integral NBCs (with one weight σ) is unstable if the spectrum has complex eigenvalues.
- The weighted FDS for hyperbolic equation with integral NBCs (with two weights σ_1 and σ_2) has a stability region if $\sigma_1 \geqslant \sigma_2$. If the spectrum is real, then the second stability condition is $\sigma_1 + \sigma_2 \geqslant 1/2$.
- The stability region of weighted FDS for hyperbolic equation with integral NBCs (with two weights σ_1 and σ_2) is bounded if $\sigma_1 - \sigma_2 < 1/2$, elsewise — undounded.

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