VILNIUS UNIVERSITY

AGNĖ SKUČAITĖ

INVESTIGATION OF THE SPECTRUM FOR STURM–LIOUVILLE PROBLEM WITH A NONLOCAL INTEGRAL CONDITION

Doctoral dissertation

Physical sciences, mathematics (01P)

Vilnius, 2016

Doctoral dissertation was written in 2011–2015 at Vilnius University

Scientific supervisor:

prof. dr. Artūras Štikonas

Vilnius University, Physical sciences, Mathematics – 01P

VILNIAUS UNIVERSITETAS

AGNĖ SKUČAITĖ

ŠTURMO IR LIUVILIO UŽDAVINIO SU INTEGRALINE NELOKALIĄJA SĄLYGA SPEKTRO TYRIMAS

Daktaro disertacija

Fiziniai mokslai, matematika (01P)

Vilnius, 2016

Disertacija rengta 2011–2015 metais Vilniaus universitete.

Mokslinis vadovas:

prof. dr. Artūras Štikonas

Vilniaus universitetas, fiziniai mokslai, matematika – 01P

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Notation

$x \in X$	–	x is an element of X
$X \cup Y$	–	is the union of sets X and Y
$X \cap Y$	–	is the intersection of sets X and Y
$\varnothing$	–	empty set
$X \times Y$	–	Cartesian product of sets X and Y
$\mathbb{N}$	–	1, 2, 3,... the set of positive integers
$\mathbb{N}_0$	–	0, 1, 2, 3,... the set of natural numbers
$\mathbb{N}_e$	–	2, 4, 6,... the set of an even integers
$\mathbb{N}_o$	–	1, 3, 5,... the set of an odd integers
$\mathbb{N}_k$	–	$k\mathbb{N} := \{n \in \mathbb{N} \colon n = km, m \in \mathbb{N}\}, k \in \mathbb{N}$
$\mathbb{Z}$	–	the set of integers
$\mathbb{Q}$	–	the set of rational numbers
$\mathbb{I}$	–	the set of irrational numbers
$\mathbb{R}$	–	the set of real numbers
$\overline{\mathbb{R}}$	–	extended set of real numbers $\mathbb{R} \cup \{-\infty, +\infty\}$
$\mathbb{C}$	–	the set of complex numbers
$\overline{\mathbb{C}}$	–	extended set of complex numbers $\mathbb{C} \cup \{\infty\}$
$\gcd(n, m)$	–	the greatest common divisor of two integers n and m

Introduction

1 Problem formulation

In the theory of differential equations, the basic concepts have been formulated studying the problems of classical mathematical physics. However, the modern problems motivate to formulate and investigate the new ones, for example, a class of nonlocal problems. Differential problems with nonclassical Boundary Conditions (BC) is quite a widely investigated area of differential equations theory, which is very often used in other sciences. During the physical experiment, in many cases it is impossible to measure data on the boundaries, but a relations of data in the inner domain are known. If value of a solution on the boundary is related with some expression of value(s) inside of the given domain, where we solve problem, then BC of such type are called *Nonlocal Boundary Conditions* (NBC).

The main object of the dissertation is Sturm–Liouville problem with nonlocal integral BC.

2 Topicality of the problem

Differential problems with Nonlocal Conditions (NC) are quite a widely investigated area of mathematics. Differential problems with NCs are not yet completely and properly investigated, as it is a wide research area. A. Bitsadze and A. Samarskii in [2, 1969] formulated a separate class of nonlocal elliptic problems. Later, the generalizations of this problem was investigated in [53, Samarskii 1980], [82, Skubachevskii 1986], [39, Kiškis 1988]. The first paper, dedicated to the second order partial differential equation with nonlocal integral BCs is

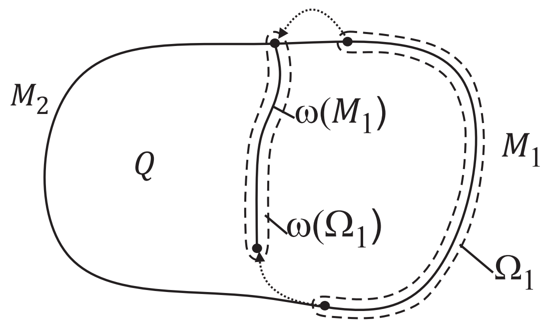


Fig. 1. Domain $\overline{Q}$.

written by Cannon, where the heat equation is investigated [4, 1963]. Nonlocal problems appear in various physics: [11, 12, Day 1982,1985], [16, Gordeziani and Avalishvili 2000], [18, Gordeziani 2000], [28, Ionkin and Morozova 2000], biological and ecological [44, Nakhushev 1995], [13, Eloe and Henderson 1997], chemistry [75, Schugerl 1987], [7, Choi and Chan 1992] and other science's problems [29, Ionkin], [15, Gordeziani and Davitashvili 1999], [1, Avalishvili and Gordeziani 2003]. The problems with NBCs were investigated for parabolic equations [29, Ionkin 1977], [37, Kamynin 1964], for elliptic equations [2, Bitsadze and Samarskii 1969], [19, Gushchin and Mikhaĭlov 1994], [88, Wang 2002], and for hyperbolic equations [17, Gordeziani and Avalishvili 2001], [91, Zikirov 2007], etc. Integral BCs are the special case of a more general nonlocal BC for stationary BVP [50, 51, Roman and Štikonas 2009, 2010], [85, Štikonas and Roman 2009].

In Lithuania, the problems with NCs had been started to investigate, in 1977. Two scientists of the Institute of Mathematics and Cybernetics: M. Sapagovas and T. Veidaitė published an article about differential problem with NC [87, Veidaitė *et al.* 1977]. Professor Sapagovas found the scientific school and main area of investigations at this school are investigation of the problem with NBC. Later, with his doctoral student R. Čiegis, professor was investigating elliptic and parabolic problems with Bitsadze–Samarskii boundary conditions [6, Čiegis 1984], [63, Sapagovas 1984].

Sturm–Liouville Problem (SLP) is important investigating the existence and uniqueness of the classical stationary problems. The problems of such type are not self-adjoint, their spectrum can be negative or complex, so the investigation of such type problems is very complicated.

As it was mentioned before, in 1969 Bitsadze and Samarskii formulated new

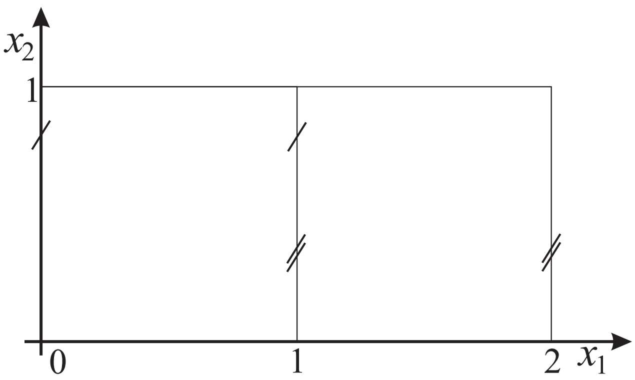


Fig. 2. Domain Q.

nonlocal Boundary Value Problem (BVP) for the elliptic equation [2, Bitsadze and Samarskii], which appears in the plasma theory:

$$\begin{aligned}(Aw)(x) &= -\sum_{i,j=1}^{n} a_{ij}(x)w_{x_i,x_j}(x) + \sum_{i=1}^{n} \alpha_i(x)w_{x_i}(x) \\ &\quad +\alpha_0(x)w(x) = f_0(x), \quad x \in \Omega, \end{aligned} \tag{1}$$

$$w(x)|_{M_1} = b(x)w(\omega(x))|_{M_1} + f_1(x), \quad x \in M_1, \tag{2}$$

$$w(x)|_{M_2} = f_2(x), \qquad x \in M_2, \tag{3}$$

where

$$\sum_{i,j=1}^{n} \alpha_{i,j}(x)\xi_i\xi_j > 0, \quad 0 \neq \xi \in \mathbb{R}^n, \quad x \in \overline{Q},$$

$Q \subset \mathbb{R}^n$ — bounded region with boundary ∂Q, $M_1 \subset \partial Q$ — $(n-1)$ dimension open subset, $M_2 = \partial Q \setminus M_1$ — subset; $\omega(x)$ — C^∞ such diffeomorphism, that $\omega\colon \Omega_1 \longrightarrow \omega(\Omega_1)$, where $\Omega_1 \subset M_1$, $\omega(\Omega_1) \subset Q$; $a_{ij}, a_i, a_0, b \in C^\infty(\mathbb{R}^n)$ (see Figure 1).

Unlike the formulated problem, authors solved [2, Bitsadze and Samarskii] the particular case (1)–(3):

$$\begin{aligned} -\triangle\omega(x) &= f_0(x), && x \in Q = (0,2) \times (0,1), \\ \omega(x_1, 0) &= \omega(x_1, 1) = 0, && 0 \leqslant x_1 \leqslant 2, \\ \omega(0, x_2) &= \gamma_1\omega(1, x_2), \quad \omega(2, x_2) = \gamma_2\omega(1, x_2), && 0 \leqslant x_2 \leqslant 1, \end{aligned}$$

where $\gamma_1 = 0$, $\gamma_2 = 1$, $\triangle$ — Laplace operator; $x = (x_1, x_2)$ (see Figure 2). As can be seen $(\{2\} \times [0,1]) \cap (\{1\} \times [0,1]) = \varnothing$, so, the problem was reduced to the second order integral Fredholm equation. Uniqueness and existence were proved using induction and the maximum principle.

Problems with two-points and multi-points NBCs were analyzed by Il'in, Moiseev and Ionkin in [21–23, Il'in and Moiseev 1976, 1987, 1987], [27, Ionkin and Moiseev 1980]. The investigation of the spectrum and other similar problems for differential equations with nonlocal Bitsadze–Samarskii or multipoint BCs are also analyzed in papers [66, Sapagovas 2000], [45, Pečiulytė and Štikonas 2006], [68, Sapagovas and Štikonas 2005], [86, Štikonas and Štikonienė 2009]; and integral conditions in [46, Pečiulytė and Štikonas 2007], [77, Skučaitė *et al.* 2010], [84, Štikonas 2014] etc.

B. Chanane in his paper [5, 2009], use the regularized sampling method introduced recently to compute the eigenvalues of SLPs with NCs:

$$\begin{cases} -y'' + q(x)y = \lambda y, & x \in [0,1] \\ x_0(y) = 0, & x_1(y) = 0, \end{cases}$$

where $q \in L^1$ and x_0 and x_1 are continuous linear functionals defined by:

$$x_0(y) = \int_0^1 [y(t)\mathrm{d}\psi_1(t) + y'(t)\mathrm{d}\psi_2(t)], \quad x_1(y) = \int_0^1 [y(t)\mathrm{d}\phi_1(t) + y'(t)\mathrm{d}\phi_2(t)],$$

where x_0 and x_1 are independent, and ψ_1, ψ_2, ϕ_1 and ϕ_2 are functions of bounded variations, integration is in the sense of Riemann–Stieltjes. The author has used the regularized sampling method and has obtained much higher estimates of the eigenvalues without computing multiple integrals or taking a high number of terms in the cardinal series involved. Also, two numerical examples have been presented to illustrate the effectiveness of the method.

The n-th order BVP is widely analyzed by X. Hao *et. al.* [20, 2015]. The nonlinear nth-order singular nonlocal BVP:

$$\begin{cases} u^{(n)}(t) + \lambda a(t) f(t, u(t)) = 0, & t \in (0,1) \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & u(1) = \int_0^1 u(s)\mathrm{d}A(s), \end{cases}$$

under some conditions is considered to be the first eigenvalue corresponding to the relevant linear operator, where $\int_0^1 u(s)\mathrm{d}A(s)$ is given by a Riemann–Stieltjes integral with a signed measure, a may be singular at $t = 0$ and/or $t = 1$, $f(t,x)$ may also have singularity at $x = 0$. The existence of positive solutions is obtained by means of the fixed point index theory in cones, and two explicit examples are given to illustrate the results.

The inverse spectral problems for Sturm–Liouville operators with NBCs are studied in [90, Yurko and Yang 2014]. The authors considered the differential equation:

$$-y'' + q(x)y = \lambda y, \quad x \in (0, T),$$

and linear forms:

$$U_j(y) := \int_0^T y(t) d\sigma_j(t), \quad j = 1, 2,$$

where $q(x) \in L(0, T)$ is a complex-valued function, $\sigma_j(t)$ are complex-valued bounded variation functions, continuous from the right for $t > 0$. Such boundary conditions can be rewritten in the following nonlocal form:

$$U_j(y) := H_j y(0) + \int_0^T y(t) d\sigma_{j0}(t), \quad j = 1, 2,$$

where, H_j is finite limit $H_j := \sigma_j(+0) - \sigma_j(0)$, σ_{j0} are complex-valued bounded variations functions continuous from the right.

The authors study the spectra of the inversed problems introducing the Weyl-type function and two spectra, which are generalizations of the Weyl function and Borg's inverse problem for the classical Sturm-Liouville operator. The uniqueness theorems are proved and some counterexamples are given for the two formulated problems.

The question about the existence of the solutions of the nonlinear SLP with integral BC is analyzed in the article [89, Yang 2006]. The author consider SLP with integral BCs:

$$\begin{aligned}
&-(au')' + bu = g(t)f(t, u), \qquad t \in (0, 1),\\
&(\cos\gamma_0)u(0) - (\sin\gamma_0)u'(0) = \textstyle\int_0^1 u(\tau) d\alpha(\tau),\\
&(\cos\gamma_1)u(1) - (\sin\gamma_1)u'(1) = \textstyle\int_0^1 u(\tau) d\beta(\tau),
\end{aligned}$$

where $a \in C^1([0, 1], (0, \infty))$ and $b \in C([0, 1], [0, \infty))$, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ and $g \in C((0, 1) \times [0, \infty)) \cap L(0, 1)$, $\int_0^1 g(t)dt > 0$, α and β right continuous on $[0, 1))$ and left continuous at $t = 1$. The author proved the existence of nontrivial solutions for the formulated problem using the topological degree arguments and cone theory.

The results of Lithuanian mathematicians in the field of differential and numerical problems with NBCs are very important. Prof. M. Sapagovas was not only a pioneer in the study of such problems, but also the founder of the scientific school in Vilnius. The first problem with NBCs came from the applications, and it was the investigation of a mercury droplet in electric contact, given the droplet volume [54, 55, 62, 63, 69, 70, Sapagovas 1978-1984]. Difference scheme for two-dimensional elliptic problem with an integral condition was constructed in [55,64, Sapagovas 1983, 1984]. Scientific supervisor of Sapagovas (Kiev, 1963–1965) Prof. V. Makarov also began investigating problems with NBCs [40–42, 1984-1985]. Sapagovas and his doctoral student (Vilnius, 1982–1985) Čiegis investigated elliptic and parabolic problems with integral and Bitsadze–Samarskii type NBCs and Finite-Difference Schemes (FDS) for them [6,71]. They published some new results about numerical solutions for problems with NBCs in [73, 74, Sapagovas and Čiegis 1987], [72, Sapagovas 1988], [8, Čiegis 1988].

Sapagovas with co-authors began to study eigenvalues for Bitsadze–Samarskii type

$$u(0) = 0, \quad u(1) = \gamma u(\xi), \quad 0 < \xi < 1, \tag{4}$$

and integral type NBCs

$$u(0) = \gamma_0 \int_0^1 \alpha_0(x)u(x)\,\mathrm{d}x, \qquad u(1) = \gamma_1 \int_0^1 \alpha_1(x)u(x)\,\mathrm{d}x, \tag{5}$$

[65, 66, Sapagovas 2000, 2002], [9, Sapagovas *et al.* 2004], [68, Sapagovas and Štikonas 2005]. They showed that there exists eigenvalues, which do not depend on parameters γ_0 or γ_1 in boundary conditions and complex eigenvalues may exist. The eigenvalue problems, investigation of the spectra, analysis of nonnegative solutions and similar problems for the operators with NBCs of Bitsadze–Samarskii or of integral-type are given in the papers [9, Čiupaila *et al.* 2004], [25, Infante 2003], [26, Infante 2005]. Complex eigenvalues for differential operators with NBCs are less investigated than the real case. Some results of these eigenvalues for a problem with one Samarskii–Bitsadze NBC are published [68, Sapagovas and Štikonas 2005], [86, Štikonas and Štikonienė 2009]. Sapagovas with co-authors analyzed the spectrum of discrete SLP, too. These results can be applied to prove the stability of FDS for nonstationary problems and the convergence of iterative methods. Numerical methods were proposed for parabolic and iterative methods for solving two-dimensional elliptic equation with Bitsadze–Samarskii or integral type NBCs: Alternating Direction Method (ADM) for a two-dimensional parabolic equation with NBC [58, Sapagovas *et al.* 2007], FDS of increased order of accuracy for the Poisson equation with NCs [67, Sapagovas 2008], FDS for two-dimensional elliptic equation with NC [34, Jakubėlienė *et al.* 2009], the fourth-order ADM for FDS with NC [61, Sapagovas and Štikonienė 2009], ADM for the Poisson equation with variable weight coefficients in an integral condition [60, Sapagova *et al.*]; ADM for a mildly nonlinear elliptic equation with integral type NCs [60, Sapagovas *et al.* 2011], FDS for nonlinear elliptic equation with NC [10, Čiupaila *et al.* 2013]. Spectral analysis was applied for two- and three-layer FDS for parabolic equations with NBCs: FDS for one-dimensional differential operator with integral type NCs [52, Sajavičius and Sapagovas 2009], [59, Sapagovas *et al.* 2012], [57, Sapagovas 2012]. Stability analysis was done for FDS in the case of one- and two-dimensional parabolic equation with NBCs [30, Ivanauskas *et al.* 2009], [36, Jesevičiūtė and Sapagovas 2008], [56, Sapagovas 2008].

3 Aims and problems

The main aim of the dissertation is the analysis of the differential or the discrete Sturm–Liouville Problem with integral NBC. To investigate the spectrum of SLP we study the following problems:

- To investigate the spectrum for SLP with integral NBC depending on three parameters.
  - Location of the zeroes, poles and Constant Eigenvalue points of the Characteristic Function.
  - Qualitative analysis of Spectrum Curves.
  - Trajectories of the Critical Points.
  - Bifurcations of Spectrum Curves.
- To investigate the spectrum of SLP with integral NBC depending on two parameters.
  - Dependence on parameters γ and ξ in the case BCs: $u(1) = \gamma \int_{\xi}^{1} u(t)\,\mathrm{d}t$, $u(1) = \gamma \int_{\xi}^{1} u(t)\,\mathrm{d}t$, $u(1) = \gamma \int_{\xi}^{1-\xi} u(t)\,\mathrm{d}t$.

  - Bifurcation points of Spectrum Curves.

- To investigate the spectrum of discrete Sturm–Liouville Problem (dSLP) with NBC depending on three parameters. Integral is approximated by trapezoid formula.

  - Characteristic Function, it's zeroes, poles and Constant Eigenvalues points.
  - Properties of Spectrum Curves.
  - Dependence on number of grid points and parameters in NBC.
  - Spectrum Curves near Special points.

- To investigate the special cases of dSLP with two parameters in NBC.

  - Characteristic Function, it's zeroes, poles and Constant Eigenvalues points.
  - Properties of Spectrum Curves.
  - Dependence on number of grid points and parameters in NBC.
  - Spectrum Curves near Special points.
  - Influence of approximation type (trapezoid formula or Simpson's rule) of NBC.

4 Methods

Characteristic Function (CF) analysis is using for investigation of the spectrum for differential and discrete SLP with NBC [86, Štikonas and Štikonienė 2009]. The properties of the spectrum for such type problems depend on CF zeros, poles, Constant Eigenvalue (CE) points and critical points of CF. Investigations of real and complex parts of the spectrum are provided with the results of numerical experiments. Some results are given as graphs of CF, trajectories in Phase Space (ξ_1, ξ_2) and bifurcation diagrams.

5 Actuality and novelty

Most of the results presented in this work are completely new and have not appeared before in the scientific literature. Although the results do not embrace all the possible variants of the spectrum this thesis contributes to a better understanding of the spectrum for SLP with NBCs.

6 Structure of the dissertation and main results

Dissertation consists of introduction, four chapters, conclusions and bibliography. In the first chapter we investigate SLP with integral NBC depending on three parameters. The qualitative study of the Spectral Curves was done. We found zeroes, poles, CE points and critical points. Classification of such points is done. We numerically investigated and found the trajectories of different type critical points in the Phase Space.

In the second chapter we investigate special cases of SLP with one nonlocal integral boundary condition ($u(1) = \gamma \int_\xi^1 u(t)\,\mathrm{d}t,\ u(1) = \gamma \int_\xi^1 u(t)\,\mathrm{d}t,\ u(1) = \gamma \int_\xi^{1-\xi} u(t)\,\mathrm{d}t$). Some new properties of CF were found. We investigate how the spectrum depends on NBCs parameters.

In the third chapter we analyzed dSLP corresponding to the problem in the first chapter. NBC was approximated by trapezoidal rule. We investigate how the spectrum depends on the number of grid points. The behavior of Spectrum Curves in the neighbourhood of special points ($q = 0$, $q = n$ and $q = \infty$) was analyzed.

In the fourth chapter we investigate special cases of dSLP with one integral boundary condition. The nonlocal boundary condition was approximated by trapezoidal or Simpson's rule. We investigate how the spectrum depends on the number of grid points. Some properties, depending on approximation, were obtained.

7 Dissemination of results

The results of this thesis were presented in the following international conferences:

- *MMA2016*, Tartu, Estonia, June 1–4, 2016;
  "Spectrum curves of discrete Sturm—Liouville problem with integral condition";
- *ENUMATH2015*, Ankara, Turkey, September 14–18, 2015;
  "Eigenspectrum analysis of the Sturm–Liouville problem with nonlocal integral boundary condition";
- *MMA2015*, Sigulda, Latvia, May 26–29, 2015;
  "Investigation of the spectrum for Sturm–Liouville problem with partial integral condition";
- *MMA2014*, Druskininkai, Lithuania, May 26–29, 2014;
  "Investigation of critical and bifurcation points for Sturm–Liouville problem with integral boundary condition";
- *MMA2013*, Tartu, Estonia, May 27–30, 2013;
  "Investigation of spectrum for finite difference scheme with integral boundary condition";
- *MMA2012*, Tallinn, Estonia, June 6–9, 2012;
  "Investigation Sturm–Liouville problems with integral boundary condition";
- *MMA2011*, Sigulda, Latvia, May 25–28, 2011;
  "Investigation Sturm–Liouville problems with integral boundary condition";
- *MMA2010*, Druskininkai, Lithuania, May 24–27, 2010;
  "Investigation Discrete Sturm–Liouville Problems with Nonlocal Boundary Conditions";

and other conferences

- *LMD*, Kaunas, Lithuania, June 16-17, 2015;
  "Zeroes and poles of a characteristic function for Sturm—Liouville problem with nonlocal integral condition";
- *LMD*, Vilnius, Lithuania, June 26–27, 2014;
  "The dynamics of Sturm–Liouville problem's with integral BCs bifurcation points";

- *LMD*, Vilnius, Lithuania, June 19–20, 2013;
  "Investigation of the spectrum of the Sturm—Liouville problem with a nonlocal integral condition";

- *LMD*, Vilnius, Lithuania, June 16–17, 2011;
  "Investigation Sturm–Liouville problems with integral boundary condition";

- *MLD*, Šiauliai, Lithuania, June 17–18, 2010;
  "Investigation of complex eigenvalues for finite difference scheme with integral type nonlocal boundary condition";

- *MMM2010*, Kaunas, Lithuania, April 8–9, 2010;
  "Investigation Sturm–Liouville problems with integral boundary condition";

- *LMD*, Vilnius, Lithuania, June 18–19, 2009;
  "Investigation of complex eigenvalues for stationary problems with nonlocal integral boundary condition";

- *MMM2009*, Kaunas, Lithuania, April 2–3, 2009;
  "Investigation of complex eigenvalues for Sturm–Liouville problems with nonlocal integral boundary condition";

8 Publications

Results of the research were published in the following scientific papers:

Main publications:

- A. Skučaitė, A. Štikonas. Spectrum Curves for Sturm–Liouville Problem with Integral Boundary Condition, Math. Model. Anal., **20**(6):802–818, 2015.

- A. Skučaitė, K. Skučaitė-Bingelė, S. Pečiulytė, A. Štikonas. Investigation of the Spectrum for the Sturm–Liouville Problem with One Integral Boundary Condition. Nonlinear Anal. Model. Control, **15**(4):501–512, 2010.

Other publications:

- A. Skučaitė, A. Štikonas. Zeroes and Poles of a Characteristic Function for Sturm–Liouville Problem with Nonlocal Integral Condition. Liet. matem. rink. Proc. LMS, Ser. A, **56**:95–100, 2015.

- A. Skučaitė, A. Štikonas. Investigation of the Spectrum of the Sturm–Liouville Problem with a Nonlocal Integral Condition. Liet. matem. rink. Proc. LMS, Ser. A, **54**:67-72, 2013.

- A. Skučaitė, A. Štikonas. Investigation of the Sturm–Liouville Problems with Integral Boundary Condition. Liet. matem. rink. Proc. LMS, Ser. A, **52**:297-302, 2011.

- A. Skučaitė, S. Pečiulytė, A. Štikonas. Investigation of Complex Eigenvalues for Sturm–Liuville Problem with Nonlocal Integral Boundary Condition. Mathematics and Mathematical modelling, 5:24-32, 2009.

9 Traineeships

During the doctoral studies were made several research visits:

- Two week visit to Trento, Italy, Trento Winter School on Numerical Methods 2015, February 2–13, 2015.

- One week visit to Barcelona, Spain, JISD2014, Universitat Politècnica de Catalunya, June 16–20, 2014.

10 Scientific projects

The author of the dissertation took part in the scientific project "Investigation of stationary problems with nonlocal boundary conditions, numerical analysis and applications". Funded by Research Council of Lithuania (No. MIP-047/2014).

11 Acknowledgements

I would like to express the deepest appreciation to my scientific supervisor Prof. Dr. Artūras Štikonas for his invaluable support, understanding, and patience during the doctoral studies. His guidance helped me in all the time of research and writing of this thesis.

Besides my advisor, I would like to thank to the staff of Department of Human and Medical Genetics of Vilnius University, for for their support.

I would especially like to thank Kancevičienė Alina and Žėkienė Ona for making me interested in mathematics at the time I was studying at "Šeštokai Secondary School". Without Your confidence in my math skills and motivation I would not be studying mathematics.

I wish to express my sincere thanks to UAB "VTEX" management for the opportunity to combine work and scientific researches.

Last but not the least, I would like to thank to my parents for believing in me, their support and help during my studies.

Chapter 1

Sturm–Liouville problem with a nonlocal integral condition

In this chapter Sturm–Liouville Problem (SLP) is analyzed:

$$-u'' = \lambda u, \quad t \in (0,1),$$

with one classical and another integral NBC:

$$u(0) = 0, \qquad u(1) = \gamma \int_{\xi_1}^{\xi_2} u(t)\,\mathrm{d}t,$$

with parameters $\gamma \in \mathbb{R}$ and $\boldsymbol{\xi} \in S_{\boldsymbol{\xi}} := \{(\xi_1, \xi_2) \in (0,1)^2, \xi_1 < \xi_2\}$. The cases $\boldsymbol{\xi} = (0,1)$ and $\boldsymbol{\xi} = (1/4, 3/4)$ were analyzed in [9, Čiupaila *et al.* 2004]. Such problem has been investigated in [47, Pečiulytė and Štikonas 2009], [43, Mikalauskaitė 2011] and some new results were obtained. It should be noted that in [43, Mikalauskaitė 2011] complex eigenvalues are analyzed only for special cases of $\boldsymbol{\xi}$ with rational components. These studies are extended in this thesis. The main aim of this thesis is to investigate the influence of parameters γ, ξ_1, ξ_2 for the spectrum of SLP and the behavior of the Critical points of Characteristic Function (CF). CF method was described in [86, Štikonas and Štikonienė 2009] for problem with one Bitsadze–Samarskii type NBC. Critical points of the CF are important for the study of multiple eigenvalues. These points are connected with bifurcations points in Phase Space $S_{\boldsymbol{\xi}}$ of parameter $\boldsymbol{\xi} = (\xi_1, \xi_2)$. The limit cases ($\boldsymbol{\xi} = (0, \xi)$ and $\boldsymbol{\xi} = (\xi, 1)$, $\xi \in [0,1]$), were analyzed in these papers [48, Pečiulytė *et al.* 2005], [77, Skučaitė *et al.* 2010] and also in Chapter 2. The special case $\boldsymbol{\xi} = (\xi, 1-\xi)$, $\xi \in [0, 1/2]$), is presented in [79, Skučaitė and Štikonas 2013]. Real CF and Real Critical points were studied for problems with one two-points NBC [45, Pečiulytė and Štikonas 2006]. Negative Critical points for problems with two-point or integral NBCs with one parameter ξ were investigated in paper [49, Pečiulytė *et al.* 2008], too.

In Section 1 the problem is formulated and some useful notations are introduced. The classification of the special points (zeroes, poles and CE points) is done in Section 2. The main results about the CF Critical points are presented in Section 3. In Section 4 some remarks and conclusions are given. Certain parts of this chapter are published in [80, 81].

1 Formulation of the problem

Let us analyze the SLP:

$$-u'' = \lambda u, \quad t \in (0,1), \tag{1.1}$$

$\lambda \in \mathbb{C}_\lambda := \mathbb{C}$, with one classical BC:

$$u(0) = 0, \tag{1.2}$$

and another integral type NBC:

$$u(1) = \gamma \int_{\xi_1}^{\xi_2} u(t)\,\mathrm{d}t \tag{1.3}$$

with parameters $\gamma \in \mathbb{R}$, $\boldsymbol{\xi} = (\xi_1, \xi_2) \in S_{\boldsymbol{\xi}}$.

For the case $\gamma = 0$ (classical one) eigenvalues are well known:

$$\lambda_k = (k\pi)^2, \quad v_k(t) = \sin(k\pi t), \quad k \in \mathbb{N}.$$

Note that the same classical problem is obtained in the limit case $\xi_1 = \xi_2$.

If $\lambda = 0$, then all the functions $u(t) = Cu_0(t)$, where $u_0(t) := t$, satisfy the equation (1.1)–(1.2). By substituting this solution into NBC (1.3) we derive, that the nontrivial solution ($C \neq 0$) exists if $1 = \gamma(\xi_2^2 - \xi_1^2)/2$. So, eigenvalue $\lambda = 0$ exists if and only if $\gamma = 2/(\xi_2^2 - \xi_1^2)$.

In the case $\lambda \neq 0$ entire function $u_q(t) := \sin(\pi q t)/(\pi q)$ is defined. Functions $u(t) = Cu_q(t)$ satisfy equation (1.1) with $\lambda = (\pi q)^2$, $q \neq 0$, and BC (1.2). If $q \in \mathbb{C}_q := \{q = x + \imath y \in \mathbb{C} : x = 0, y \geqslant 0 \text{ or } x > 0\}$, then a map $\lambda = (\pi q)^2$ is a bijection between $\mathbb{C}_q$ and $\mathbb{C}_\lambda$ (see Figure 1.1 and [86, Štikonas and Štikonienė

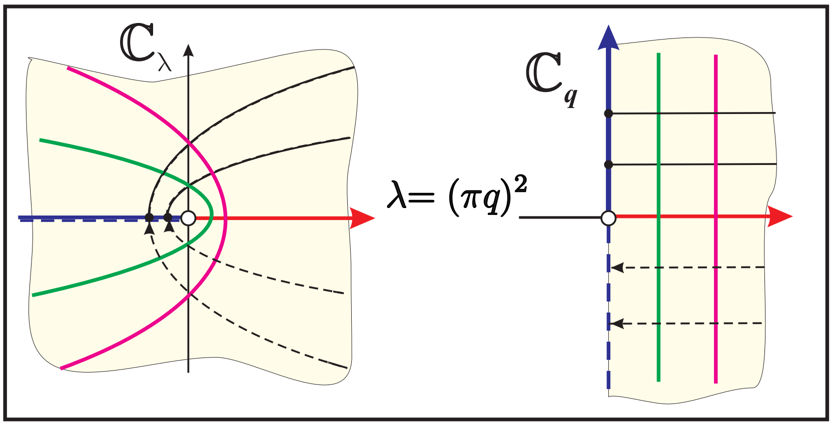


Fig. 1.1. Bijection between $\mathbb{C}_\lambda$ and $\mathbb{C}_q$.

2009]). Note, that $q = 0$ corresponds to $\lambda = 0$ in this bijection and $u_0 = \lim_{q\to 0} u_q(q)$. Bijection $\lambda = (\pi q)^2$ is a conformal mapping, except the point $q = 0$. The point $\lambda = 0$ is the first order branch point of the function $\lambda = \lambda(q) = (\pi q)^2$. Real eigenvalues correspond to $\mathbb{R}_q = \mathbb{R}_q^- \cup \mathbb{R}_q^+ \cup \mathbb{R}_q^0 \subset \mathbb{C}_q$, and $\mathbb{R}_q^- := \{q \in \mathbb{C}_q : x = 0, y > 0\}$, $\mathbb{R}_q^+ := \{q \in \mathbb{C}_q : x > 0, y = 0\}$, $\mathbb{R}_q^0 := \{q = 0\}$ correspond to negative, positive, zero eigenvalues, respectively. If λ is eigenvalue for SLP, then $q \in \mathbb{C}_q$ which corresponds to this λ we call *Eigenvalue point*.

Remark 1.1. If bijection $\lambda = (\pi q)^2$ is used instead of $\lambda = q^2$, then the spectrum points coincide with $\mathbb{N}$ in the classical case $\gamma = 0$, i.e. $q_k = k \in \mathbb{N}$.

A nontrivial solution of the problem (1.1)–(1.3) exists if q is a root of the equation:

$$u_q(1) = \gamma \int_{\xi_1}^{\xi_2} u_q(t)dt. \tag{1.4}$$

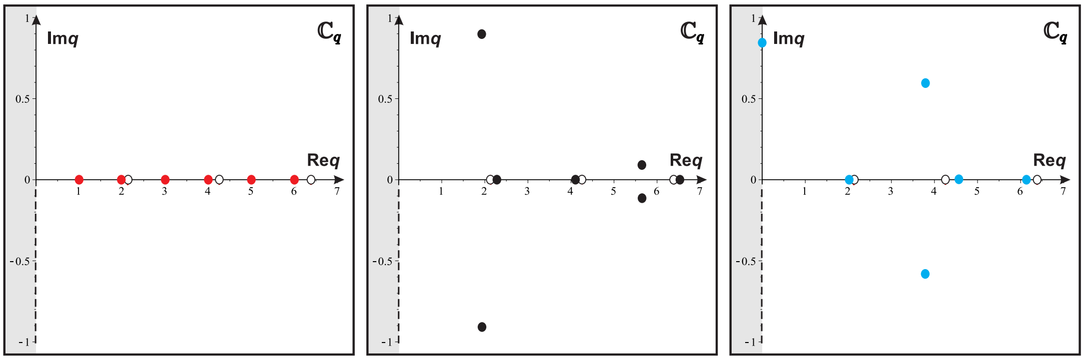
For NBC (1.3) two entire functions are introduced:

$$Z(z) := \frac{\sin(\pi z)}{\pi z}; \quad P_{\boldsymbol{\xi}}(z) := 2\frac{\sin(\pi z(\xi_1 + \xi_2)/2)}{\pi z} \cdot \frac{\sin(\pi z(\xi_2 - \xi_1)/2)}{\pi z}. \tag{1.5}$$

Zeroes of these functions are important for the description of the spectrum. Zeroes of the function $Z(q)$, $q \in \mathbb{C}_q$, coincide with Eigenvalue points in the classical case $\gamma = 0$. The equality (1.4) can be rewritten in the form:

$$Z(q) = \gamma P_{\boldsymbol{\xi}}(q), \quad q \in \mathbb{C}_q. \tag{1.6}$$

In Figure 1.2, the roots (not all) of this equation for $\gamma = -17, 0, +17$ in the case $\boldsymbol{\xi} = (0.32, 0.61)$ can be seen. Complex roots exist for $\gamma = -17, +17$.



(a) $\bullet$ – zero ($\gamma = 0$); $\circ$ – pole (b) $\bullet$ $\gamma = -17$ (c) $\bullet$ $\gamma = +17$

Fig. 1.2. A part of the spectrum for SLP (1.1)–(1.3), $\boldsymbol{\xi} = (0.32, 0.61)$.

We define the *Constant Eigenvalue* (CE) as the eigenvalue that does not depend on parameter γ. For any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue point* (CE point) $q \in \mathbb{C}_q$ and $\lambda = (\pi q)^2$ [86, Štikonas and Štikonienė 2009]. For NBC (1.3) we can find CE points as solutions of the following system:

$$Z(q) = 0, \quad P_{\boldsymbol{\xi}}(q) = 0,$$

i.e., CE point $c \in \mathbb{N}$ and $P_{\boldsymbol{\xi}}(c) = 0$. The notation $\mathcal{C}$ or $\mathcal{C}_{\boldsymbol{\xi}}$ is used for the set of all CE points. For a CE point, the set of γ-values in $\mathbb{C}_q \times \mathbb{R}$ is a vertical line.

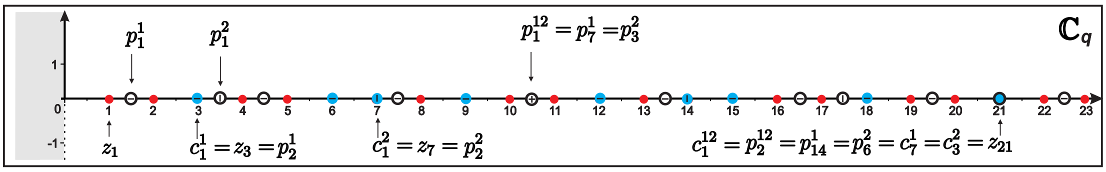
If $q \notin \mathbb{N}$, i.e. $Z(q) \neq 0$, and q satisfies equation $P_{\boldsymbol{\xi}}(q) = 0$, then the equality (1.5) is not valid for all γ and such point q is *a pole point*. Notation of the pole point is connected with meromorphic function:

$$\gamma_c(z) = \frac{Z(z)}{P_{\boldsymbol{\xi}}(z)}, \quad z \in \mathbb{C}. \tag{1.7}$$

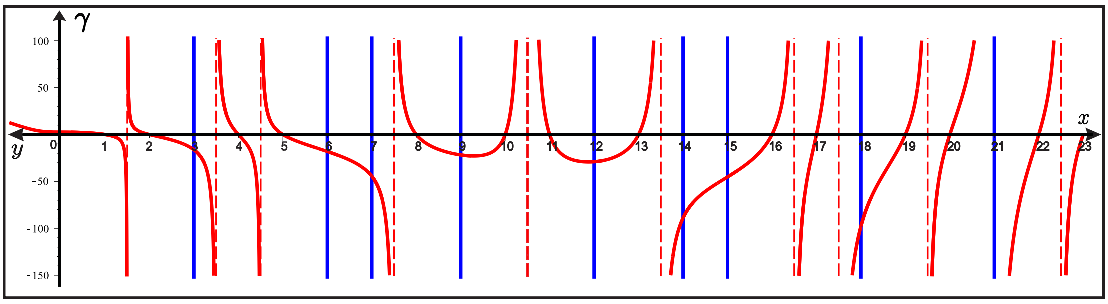
This function is obtained by expressing γ from the equation (1.5). If the denominator has a zero at $z = p$ and the numerator does not, then the value of the function will be infinite and we have a pole. If both parts have a zero at $z = p$, then the multiplicities of these zeroes must be compared. For our problem all zeroes $z_k = k \in \mathbb{N}$, of function $Z(z)$ are simple and positive if $z \in \mathbb{C}_q$. It follows that function $P_{\boldsymbol{\xi}}(z) = 2P^1_{\boldsymbol{\xi}}(z)P^2_{\boldsymbol{\xi}}(z)$, where:

$$P^1_{\boldsymbol{\xi}}(z) := \sin(\pi z(\xi_1 + \xi_2)/2)/(\pi z), \quad P^2_{\boldsymbol{\xi}}(z) := \sin(\pi z(\xi_2 - \xi_1)/2)/(\pi z). \tag{1.8}$$

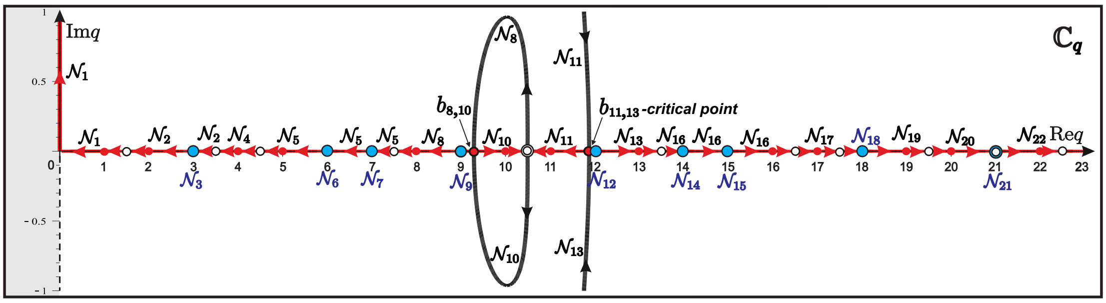
Zeroes of the functions $P^1_{\boldsymbol{\xi}}$, $P^2_{\boldsymbol{\xi}}$ in the domain $\mathbb{C}_q$ are simple and positive, too. So, zeroes of function $P_{\boldsymbol{\xi}}$ can be simple or the second order. The restriction of the meromorphic function γ_c on $\mathbb{C}_q$ can be called *Complex Characteristic Function*



(a) ● – zero; ○ – pole; ● – CE point; ● – Critical point.



(b) Real CF

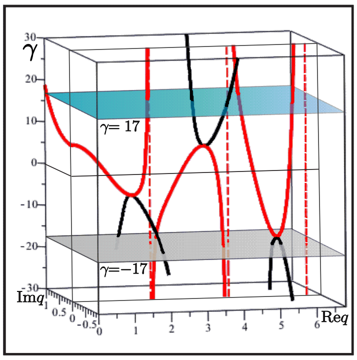


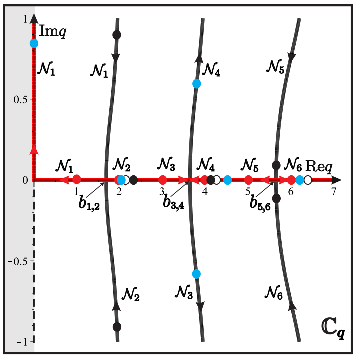
(c) Projection Complex-Real CF into $\mathbb{C}_q$.

Fig. 1.3. Zeroes, poles, CE points for SLP (1.1)–(1.3), $\boldsymbol{\xi} = (8/21, 20/21)$.

(Complex CF) [86, Štikonas and Štikonienė 2009]. We define the value of this function at point p, $P_{\boldsymbol{\xi}}(p) = 0$ as a limit $\gamma_c(p) := \lim_{q\to p} Z(q)/P_{\boldsymbol{\xi}}(q)$. This limit is finite $\gamma_c(p) = \frac{Z'(p)}{P'_{\boldsymbol{\xi}}(p)} \neq 0$ (removable singularity) if $p \in \mathbb{N}$ is the first order zero of function $P_{\boldsymbol{\xi}}$ and the limit is infinite (function γ_c has the first order pole) if $p \in \mathbb{N}$ is the second order zero of the function $P_{\boldsymbol{\xi}}$ or $p \notin \mathbb{N}$. For example, in Figure 1.3(a) such points can be seen in the case $\boldsymbol{\xi} = (8/21, 20/21)$.

All *Nonconstant Eigenvalues* (which depend on the parameter γ) are γ-points of *Complex-Real Characteristic Function* ($\mathbb{C}$-$\mathbb{R}$ CF or CF) [86, Štikonas and Štikonienė 2009]. In Figure 1.4(a) CF graph in the case $\boldsymbol{\xi} = (0.32, 0.61)$ can be seen. Complex-Real CF $\gamma(q)$ is the restriction of the function $\gamma_c(q)$ on a set $\mathcal{N}^\gamma := \{q \in \mathbb{C}_q \colon \mathrm{Im}\gamma_c(q) = 0\}$. *Real* CF is the restriction of the Complex-Real CF $\gamma(q)$ on a set $\mathbb{R}_q := \{q \in \mathbb{C}_q \colon \lambda = (\pi q)^2 \in \mathbb{R}\}$ and describes only real eigenvalues. One can see the Real CF graph in Figure 1.3(b) for $\boldsymbol{\xi} = (8/21, 20/21)$ and in Figure 1.4(c) for $\boldsymbol{\xi} = (0.32, 0.61)$. The vertical solid lines correspond to the CE


(a) Complex-Real CF


(b) Spectrum Domain

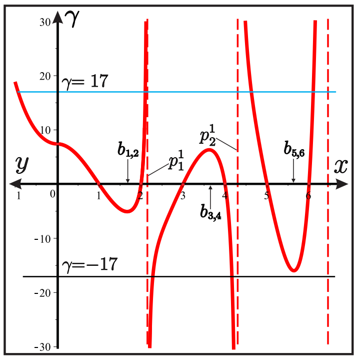
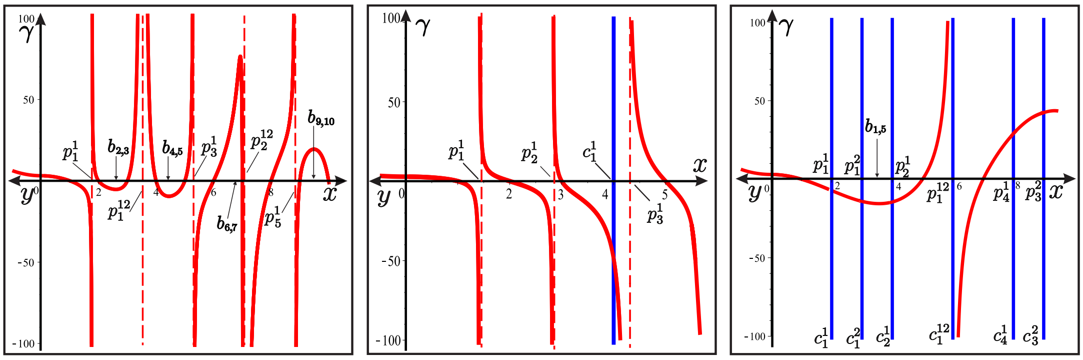

(c) Real CF

Fig. 1.4. CF for $\boldsymbol{\xi} = (0.32, 0.61)$ and its projections.

points, vertical dashed lines cross the x-axis at the points of poles. For some cases, the vertical line of the CE point is coincident with the vertical asymptotic line at the point of a pole.

Spectrum Domain is the set $\mathcal{N} = \mathcal{N}^\gamma \cup \mathcal{C}$. The example of the Spectrum Domain we can see in Figure 1.4(b). We also add the eigenvalue points ($\gamma = -17, 0, +17$) from Figure 1.2 and pole points ($\gamma = \infty$). Eigenvalue points for $\gamma \in \mathbb{R}$ exist only in this domain. Spectrum Domain is symmetric with respect to the real axis for $\operatorname{Re} q > 0$. CF $\gamma(q)$ describes the value of the parameter γ at the point $q \in \mathcal{N}^\gamma$ (see Figure 1.4(a)) such that there exists the eigenvalue $\lambda = (\pi q)^2$. For each $\gamma_0 \in \mathbb{R}$ the set $\mathcal{N}(\gamma_0) := \gamma^{-1}(\gamma_0)$ is the set of all Nonconstant Eigenvalue points. So, Spectrum Domain $\mathcal{N} = \cup_{\gamma\in\mathbb{R}}\mathcal{N}(\gamma) \cup \mathcal{C}$. For example, $\mathcal{N}(0) \cup \mathcal{C}$ corresponds to a spectrum of the classical case. If $q \in \mathcal{N}^\gamma$ and $\gamma_c'(q) \neq 0$ (q is not a Critical point of CF) then $\mathcal{N}(\gamma)$ is smooth parametric curve $\mathcal{N} \colon \mathbb{R} \to \mathbb{C}_q$ and arrow can be added on this curve. Arrows indicate the direction in which γ is increasing. So, eigenvalue point is moving along this curve when parameter γ is increasing. If $\gamma = 0$ then the eigenvalue points are $q = z_k = k \in \mathbb{N}$. So, the part of $\mathcal{N}(\gamma)$ for this point can be enumerated by the classical case $\mathcal{N}_k(0) = z_k$, $k \in \mathbb{N}$. For every CE point $c_j = j$ we define $\mathcal{N}_j = \{c_j\}$, i.e. every such $\mathcal{N}_j$ has one point only (see Figure 1.3(c), Figure 1.4(b)). We call every $\mathcal{N}_k$, $k \in \mathbb{N}$, a *Spectrum Curve*. Spectrum Domain $\mathcal{N}$ is a countable union of Spectrum Curves $\mathcal{N}_k$. Different Spectrum Curves may have a common point. For example, CE point may be on other $\mathcal{N}_k$ or few Spectrum Curves that intersect at the Critical point b, where $\gamma_c'(b) = 0$. For the $\gamma \to \pm\infty$ Spectrum Curve $\mathcal{N}_k(\gamma)$ approaches a



(a) $\boldsymbol{\xi} = (\sqrt{2}/5, 3\sqrt{2}/5)$, $p_1^{12} = p_2^1 = p_1^2$

(b) $\boldsymbol{\xi} = (\frac{2\sqrt{2}-1}{4}, \frac{2\sqrt{2}+1}{4})$, $p_1^2 = c_1^1$

(c) $\boldsymbol{\xi} = (1/6, 5/6)$, $c_3^1 = c_2^2 = c_1^{12}$

Fig. 1.5. Real CF and CE for different $\boldsymbol{\xi}$ values.

pole point or a point $q = \infty$. For the analysis of the Spectrum Curves we must know zeroes, poles and CE point of CF.

2 Zeroes, poles and Constant Eigenvalues points of the Characteristic Function

We use notation: $\xi = \xi_1/\xi_2$, $\xi_+ = \xi_1 + \xi_2$, $\xi_- = \xi_2 - \xi_1$. If $\xi_i \in \mathbb{Q}$, $i = 1, 2$, then we use a rule: $\xi_i = m_i/n_i$, $m_i, n_i \in \mathbb{N}$. The such rule we use for $\xi = \xi_1/\xi_2 \in \mathbb{Q}$: $\xi = m/n$, $m, n \in \mathbb{N}$. If $\xi_1, \xi_2 \in \mathbb{Q}$, then $m = m_1 n_2$, $n = m_2 n_1$, $n_+ = n_- = n_1 n_2$, $m_+ = m_2 n_1 + m_1 n_2$, $m_- = m_2 n_1 - m_1 n_2$.

All zeroes of the functions Z, $P^1_{\boldsymbol{\xi}}$, $P^2_{\boldsymbol{\xi}}$ (see (1.5) and (1.8)) in $\mathbb{C}_q$ are simple (of the first order), real and positive:

$$z_k = k \in \mathbb{N}, \quad p^1_k = \tfrac{2}{\xi_+} k, \ k \in \mathbb{N}, \quad p^2_k = \tfrac{2}{\xi_-} k, \ k \in \mathbb{N}. \tag{2.1}$$

We denote the corresponding sets of points as $\overline{\mathcal{Z}}$, $\overline{\mathcal{Z}}^1_{\boldsymbol{\xi}}$, $\overline{\mathcal{Z}}^2_{\boldsymbol{\xi}}$. Then a set $\mathcal{Z}_{\boldsymbol{\xi}} = \mathcal{Z}^1_{\boldsymbol{\xi}} + \mathcal{Z}^2_{\boldsymbol{\xi}} + \mathcal{Z}^{12}_{\boldsymbol{\xi}}$ describes all zeroes of the function $P_{\boldsymbol{\xi}}$, where $\mathcal{Z}^1_{\boldsymbol{\xi}} := \overline{\mathcal{Z}}^1_{\boldsymbol{\xi}} \setminus \mathcal{Z}^{12}_{\boldsymbol{\xi}}$ and $\mathcal{Z}^2_{\boldsymbol{\xi}} := \overline{\mathcal{Z}}^2_{\boldsymbol{\xi}} \setminus \mathcal{Z}^{12}_{\boldsymbol{\xi}}$ are two families of the first order zeroes, $\mathcal{Z}^{12}_{\boldsymbol{\xi}} := \overline{\mathcal{Z}}^1_{\boldsymbol{\xi}} \cap \overline{\mathcal{Z}}^2_{\boldsymbol{\xi}}$ is family of the second order zeroes.

Remark 1.2. If $\xi \in \mathbb{Q}$ then $\xi_1, \xi_2 \in \mathbb{Q}$ or $\xi_1, \xi_2 \notin \mathbb{Q}$. If $\xi \notin \mathbb{Q}$ then $\xi_1 \notin \mathbb{Q}$ or $\xi_2 \notin \mathbb{Q}$, or both $\xi_1, \xi_2 \notin \mathbb{Q}$.

For (real) CF we consider the following sets: a set of poles $\mathcal{P}_{\boldsymbol{\xi}} := \mathcal{P}^1_{\boldsymbol{\xi}} + \mathcal{P}^2_{\boldsymbol{\xi}} + \mathcal{P}^{12}_{\boldsymbol{\xi}}$, where $\mathcal{P}^1_{\boldsymbol{\xi}} := \mathcal{Z}^1_{\boldsymbol{\xi}} \setminus \overline{\mathcal{Z}}$ and $\mathcal{P}^2_{\boldsymbol{\xi}} := \mathcal{Z}^2_{\boldsymbol{\xi}} \setminus \overline{\mathcal{Z}}$ are two families of the poles of the first

Table 1.1. Zeroes, poles and CE points of CF (special cases), $m, l, m_1, m_2, n_1, n_2 \in \mathbb{N}$. "+" means that the set above is nonempty, "−" means that the set above is empty.

Case subcase	Example $\boldsymbol{\xi} = (\xi_1, \xi_2)$	Poles $\mathcal{P}^1_{\boldsymbol{\xi}}$ $\mathcal{P}^2_{\boldsymbol{\xi}}$ $\mathcal{P}^{12}_{\boldsymbol{\xi}}$	CE points $\mathcal{C}^1_{\boldsymbol{\xi}}$ $\mathcal{C}^2_{\boldsymbol{\xi}}$ $\mathcal{C}^{12}_{\boldsymbol{\xi}}$	$\mathcal{Z}_{\boldsymbol{\xi}}$	Remarks
$\xi \in \mathbb{Q}$, $\xi_1, \xi_2 \notin \mathbb{Q}$, $l > 1$:					
$\xi \neq \frac{l-1}{l+1}$	$\left(\frac{\sqrt{2}}{20}, \frac{\sqrt{2}}{5}\right)$	$+\ +\ +$	$-\ -\ -$	$+$	$p^2_1 < p^{12}_1$
$\xi = \frac{l-1}{l+1}$	$\left(\frac{\sqrt{2}}{5}, \frac{3\sqrt{2}}{5}\right)$	$+\ -\ +$	$-\ -\ -$	$+$	$p^2_1 = p^{12}_1$
$\xi \notin \mathbb{Q}$, $l > 1$, $m > 2$:					
$\xi_+, \xi_- \notin \mathbb{Q}$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right); \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right)$	$+\ +\ -$	$-\ -\ -$	$+$	$p^1_1 < p^2_1$
$\xi_+ \in \mathbb{Q}$, $\xi_+ \neq \frac{2}{l}$, $\xi_- \notin \mathbb{Q}$	$\left(\frac{3-\sqrt{2}}{8}, \frac{3+\sqrt{2}}{8}\right)$	$+\ +\ -$	$+\ -\ -$	$+$	$p^1_1 < c^1_1$
$\xi_- \in \mathbb{Q}$, $\xi_- \neq \frac{2}{m}$, $\xi_+ \notin \mathbb{Q}$	$\left(\frac{5\sqrt{2}-7}{24}, \frac{5\sqrt{2}+7}{24}\right)$	$+\ +\ -$	$-\ +\ -$	$+$	$p^2_1 < c^2_1$
$\xi_+ = \frac{2}{l}$, $\xi_- \notin \mathbb{Q}$	$\left(\frac{2-\sqrt{2}}{4}, \frac{2+\sqrt{2}}{4}\right)$	$-\ +\ -$	$+\ -\ -$	$+$	$p^1_1 = c^1_1$
$\xi_- = \frac{2}{m}$, $\xi_+ \notin \mathbb{Q}$	$\left(\frac{2\sqrt{2}-1}{4}, \frac{2\sqrt{2}+1}{4}\right)$	$+\ -\ -$	$-\ +\ -$	$+$	$p^2_1 = c^2_1$
$\xi_1 = m_1/n_1, \xi_2 = m_2/n_2 \in \mathbb{Q}$ (a)–(d) $p^k_1 < c^k_1, k = 1, 2, 12$, $p^k_1 < p^{12}_1, k = 1, 2$, (m)–(q) $n = n_1 = n_2 = m_1 + m_2$:					
(a)	$\left(\frac{8}{21}, \frac{20}{21}\right)$	$+\ +\ +$	$+\ +\ +$	$+$	$c^1_1, c^2_1 < c^{12}_1$
(b)	$\left(\frac{10}{27}, \frac{25}{27}\right)$	$+\ +\ +$	$-\ +\ +$	$+$	$c^2_1 < c^1_1 = c^{12}_1$
(c)	$\left(\frac{2}{25}, \frac{18}{25}\right)$	$+\ +\ +$	$+\ -\ +$	$+$	$c^1_1 < c^2_1 = c^{12}_1$
(d)	$\left(\frac{6}{17}, \frac{15}{17}\right)$	$+\ +\ +$	$-\ -\ +$	$+$	$c^1_1 = c^2_1 = c^{12}_1$
(e)	$\left(\frac{4}{11}, \frac{10}{11}\right)$	$+\ +\ -$	$-\ -\ +$	$+$	$p^{12}_1 = c^{12}_1$, $p^1_1 < c^1_1 = c^2_1$
(f)	$\left(\frac{4}{9}, \frac{8}{9}\right)$	$+\ -\ +$	$+\ -\ +$	$+$	$p^2_1 = p^{12}_1$, $p^1_1 < c^1_1 < c^{12}_1$
(g)	$\left(\frac{2}{7}, \frac{6}{7}\right)$	$+\ -\ +$	$-\ -\ +$	$+$	$p^2_1 = p^{12}_1$, $p^1_1 < c^1_1 = c^{12}_1$
(h)	$\left(\frac{5}{12}, \frac{11}{12}\right)$	$+\ -\ -$	$+\ +\ +$	$+$	$p^2_1 = c^2_1 < c^{12}_1$, $p^1_1 < c^1_1$
(i)	$\left(\frac{1}{2}, \frac{5}{6}\right)$	$+\ -\ -$	$+\ -\ +$	$+$	$p^2_1 = c^2_1 = c^{12}_1$, $p^1_1 < c^1_1 < c^{12}_1$
(k)	$\left(\frac{1}{5}, \frac{13}{15}\right)$	$+\ -\ -$	$-\ +\ +$	$+$	$p^2_1 = c^2_1 < c^{12}_1$, $c^1_1 = c^{12}_1$
(l)	$\left(\frac{1}{5}, \frac{3}{5}\right)$	$+\ -\ -$	$-\ -\ +$	$+$	$p^2_1 = c^{12}_1 = c^1_1$
(m)	$\left(\frac{1}{10}, \frac{9}{10}\right)$	$-\ +\ -$	$+\ +\ +$	$+$	$p^1_1 = c^1_1$,
(n)	$\left(\frac{1}{7}, \frac{6}{7}\right)$	$-\ +\ -$	$+\ -\ +$	$+$	$p^1_1 = c^1_1$, $p^2_1 < c^2_1 = c^{12}_1$
(p)	$\left(\frac{1}{6}, \frac{5}{6}\right)$	$-\ -\ -$	$+\ +\ +$	$+$	$p^1_1 = c^1_1$, $c^1_1 < c^2_1 = p^2_1$
(q)	$\left(\frac{1}{4}, \frac{3}{4}\right)$	$-\ -\ -$	$+\ -\ +$	$+$	$p^1_1 = c^1_1$, $p^2_1 = c^{12}_1$

order, a set of the second order poles $\mathcal{P}^{12}_{\boldsymbol{\xi}} := \mathcal{Z}^{12}_{\boldsymbol{\xi}} \setminus \overline{\mathcal{Z}}$; a set of the CE points $\mathcal{C}_{\boldsymbol{\xi}} := \mathcal{C}^1_{\boldsymbol{\xi}} + \mathcal{C}^2_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, where $\mathcal{C}^1_{\boldsymbol{\xi}} := \mathcal{Z}^1_{\boldsymbol{\xi}} \cap \overline{\mathcal{Z}}$ and $\mathcal{C}^2_{\boldsymbol{\xi}} := \mathcal{Z}^2_{\boldsymbol{\xi}} \cap \overline{\mathcal{Z}}$ are sets of the points with removable singularity, $\mathcal{C}^{12}_{\boldsymbol{\xi}} := \mathcal{Z}^{12}_{\boldsymbol{\xi}} \cap \overline{\mathcal{Z}}$ is the set of the points with the first order pole, too; a set of zeroes $\mathcal{Z}_{\boldsymbol{\xi}} := \overline{\mathcal{Z}} \setminus \mathcal{C}_{\boldsymbol{\xi}}$. To describe the points of these sets we will use the following results.

Remark 1.3. The sets $\mathcal{P}^1_{\boldsymbol{\xi}}$, $\mathcal{P}^2_{\boldsymbol{\xi}}$, $\mathcal{P}^{12}_{\boldsymbol{\xi}}$, $\mathcal{C}^1_{\boldsymbol{\xi}}$, $\mathcal{C}^2_{\boldsymbol{\xi}}$, $\mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{Z}_{\boldsymbol{\xi}}$ points have form $q_k = \alpha k$, $k \in \mathbb{N}$, $\alpha \geqslant 1$, or can be empty. So, nonempty sets are described by the first point ($k=1$). Since $1 < p^1_1 < p^2_1 \leqslant p^{12}_1$, the set $\mathcal{Z}_{\boldsymbol{\xi}} \neq \varnothing$. We note, that $2 < p^2_1$, too.

Let a, b, $c \in \mathbb{Z}$.

Theorem 1.4 (Гельфонд 1978, [14])**.** *If* $\gcd(a,b)=1$ *and* $[x_0, y_0]$ *is any solution of equation:*

$$ax + by + c = 0, \tag{2.2}$$

then all solutions of this equation have a form:

$$x = x_0 - bt \qquad y = y_0 + at, \qquad t \in \mathbb{Z}.$$

Remark 1.5 (Гельфонд 1978, [14]). Any solution $[x_0, y_0]$ of (2.2), $\gcd(a,b)=1$, can be found using Euclidean algorithm. We take ratio $\frac{a}{b}$. Let q_1 be the quotient and r_2 be a residual of a division a of b. Then $a = q_1 b + r_2$, where $r_2 < b$. The coefficient b can be written in the same form: $b = q_2 r_2 + r_3$, $r_3 < r_2$, where q_2-quotient, r_3-residual of a division b of r_2. Then we get the sequence:

$$\begin{cases} a &= q_1 b + r_2, & r_2 < b, \\ b &= q_2 r_2 + r_3, & r_3 < r_2, \\ r_2 &= q_3 r_3 + r_4, & r_4 < r_3, \\ & \dots & \\ r_{n-2} &= q_{n-1} r_{n-1} + r_n, & r_n < r_{n-1}, \\ r_{n-1} &= q_n r_n, & \end{cases}$$

where $r_{n+1} = 0$. Using coefficients $q_1, ..., q_n$ we construct two new sequences of numbers P_n and Q_n:

$$\begin{array}{ll} P_0 = 1, & Q_0 = 0, \\ P_1 = q_1, & Q_1 = 1, \\ P_2 = P_1 q_2 + P_0, & Q_2 = Q_1 q_2 + Q_0, \\ P_3 = P_2 q_3 + P_1, & Q_3 = Q_2 q_3 + Q_1, \\ \dots & \dots \\ P_n = P_{n-1} q_n + P_{n-2}, & Q_n = Q_{n-1} q_n + Q_{n-2}. \end{array}$$

Then the pair of numbers (x_0, y_0) which are solutions of the equation $ax+by+c=0$, is:

$$x_0 = (-1)^{n-1} c Q_{n-1}, \qquad y_0 = (-1)^n c P_{n-1}. \tag{2.3}$$

Lemma 1.6. *If $\alpha, \beta > 0$, then the equation:*

$$\alpha k = \beta l, \quad k, l \in \mathbb{N}, \tag{2.4}$$

has solutions if and only if $\alpha/\beta \in \mathbb{Q}$ and all solutions have a form:

$$k = \frac{bt}{\gcd(a,b)}, \qquad l = \frac{at}{\gcd(a,b)}, \quad t \in \mathbb{N}, \tag{2.5}$$

where $\alpha/\beta = a/b$, $a, b \in \mathbb{N}$.

Proof. If $r = \frac{\alpha}{\beta} \notin \mathbb{Q}$, then $rk = l$. So, there is no such $k, l \in \mathbb{N}$, that this equality will be valid.

If $\frac{\alpha}{\beta} = \frac{a}{b} \in \mathbb{Q}$, then from equation (2.4) we have:

$$ak = bl, \quad k, l \in \mathbb{N}. \tag{2.6}$$

Then equation (2.6) can be rewritten in the following form:

$$\frac{ak}{\gcd(a,b)} = \frac{bl}{\gcd(a,b)}, \quad k, l \in \mathbb{N}, \tag{2.7}$$

where $\frac{a}{\gcd(a,b)}$ and $\frac{b}{\gcd(a,b)}$ are coprime numbers. One solution for this equation is $(0,0)$. So, from the Theorem 1.4 (see, [14, Гельфонд 1978]) follows, that all solutions of the equality (2.6) have a form:

$$k = \frac{bt}{\gcd(a,b)}, \qquad l = \frac{at}{\gcd(a,b)}, \quad t \in \mathbb{N}.$$

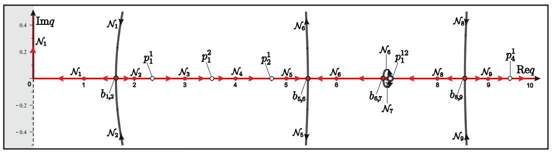
□

Theorem 1.7. *If $\xi \notin \mathbb{Q}$, then the second order zeroes for the function $P_{\boldsymbol{\xi}}(z)$ do not exist, i.e. $\mathcal{Z}^{12}_{\boldsymbol{\xi}} = \varnothing$. If $\xi \in \mathbb{Q}$, then a set $\mathcal{Z}^{12}_{\boldsymbol{\xi}}$ describes the second order zeroes:*

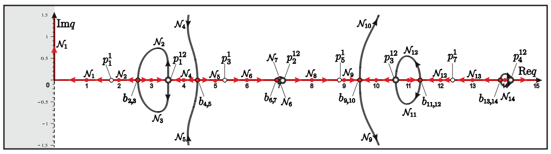
$$p^{12}_k = 2n/(\xi_2 d_p)k = 2m/(\xi_1 d_p)k, \ k \in \mathbb{N}, \quad d_p = \gcd(n-m, n+m). \tag{2.8}$$

Proof. The second order pole appears if zeros from the first family $p^1_k = \frac{2}{\xi_+}k$ consider with zeros from the second family $p^2_k = \frac{2}{\xi_-}k$, $k \in \mathbb{N}$:

$$\frac{2}{\xi_+} l = \frac{2}{\xi_-} k, \quad l, k \in \mathbb{N}. \tag{2.9}$$



(a) $\boldsymbol{\xi} = (\sqrt{2}/10, \sqrt{2}/2)$



(b) $\boldsymbol{\xi} = (\sqrt{2}/5, (3\sqrt{2})/5)$

Fig. 1.6. Spectrum Curves for $\xi_1/\xi_2 = m/n \in \mathbb{Q}$, $\xi_1, \xi_2 \notin \mathbb{Q}$.

Equation (2.9) has solution if

$$\frac{\xi_-}{\xi_+} = \frac{\xi_2 - \xi_1}{\xi_2 + \xi_1} = \frac{1 - \xi_1/\xi_2}{1 + \xi_1/\xi_2} = \frac{1 - \xi}{1 + \xi} = \frac{n - m}{n + m} \in \mathbb{Q} \Leftrightarrow \frac{m}{n} \in \mathbb{Q}.$$

So, using Lemma 1.6 we conclude, that all solutions of equality (2.9) have form

$$l = \frac{n + m}{d} t, \qquad k = \frac{n - m}{d} t, \quad t \in \mathbb{N},$$

where $d = \gcd(n + m, n - m)$. Substituting solution to the equation (2.9) we conclude, that the second order zeros of the function $P_{\boldsymbol{\xi}}(z)$ are defined by the formula

$$p_t^{12} = \frac{2(m + n)t}{d(\xi_2 + \xi_1)} = \frac{2n}{\xi_2 d} t = \frac{2m}{\xi_1 d} t, \quad t \in \mathbb{N}. \tag{2.10}$$

□

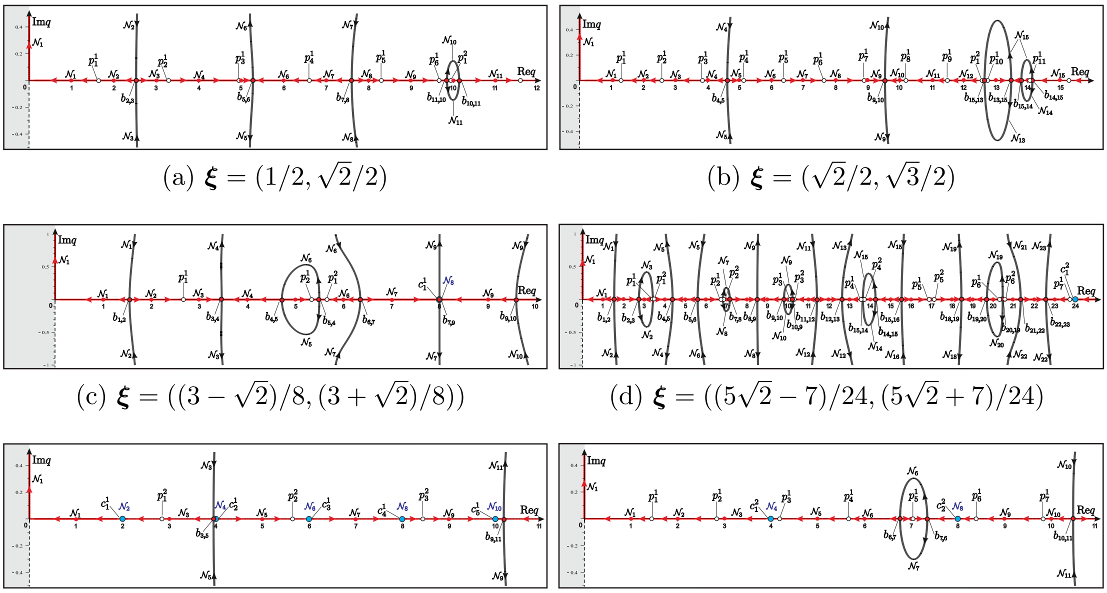
The case $\xi = m/n \in \mathbb{Q}$, $\xi_1, \xi_2 \notin \mathbb{Q}$. In this case ξ_+, $\xi_- \notin \mathbb{Q}$. Consequently, there exist no constant eigenvalues ($\mathcal{C}_{\boldsymbol{\xi}} = \varnothing$, (see Figure 1.6)). So, CF has two families of the first order poles in $\mathcal{P}_{\boldsymbol{\xi}}^1$ and $\mathcal{P}_{\boldsymbol{\xi}}^2$, respectively, and the second order poles in $\mathcal{P}_{\boldsymbol{\xi}}^{12}$ (see formulae (2.1)–(2.4) for calculation p_k^1, p_k^2, p_k^{12}, $k \in \mathbb{N}$). Note, that $\mathcal{P}_{\boldsymbol{\xi}}^2 = \varnothing$ for $\xi = (l-1)/(l+1)$, $1 < l \in \mathbb{N}$. In this special case $p_1^2 = p_1^{12}$ (see Figure 1.5(a) and Table 1.1).

The case $\xi \notin \mathbb{Q}$. In this case at least one number ξ_+ or ξ_- is irrational (and at least one number ξ_1 or ξ_2 is irrational). If $\xi_+ \notin \mathbb{Q}$ and $\xi_- \notin \mathbb{Q}$ then CF has two families of the first order poles $\mathcal{P}_{\boldsymbol{\xi}}^1$ and $\mathcal{P}_{\boldsymbol{\xi}}^2$, respectively, and $\mathcal{P}_{\boldsymbol{\xi}}^{12} = \mathcal{C}_{\boldsymbol{\xi}} = \varnothing$.

Theorem 1.8. *If* $\xi_+ \in \mathbb{Q}$, *then* $\mathcal{C}_{\boldsymbol{\xi}}^1 \neq \varnothing$ *and CE points exist:*

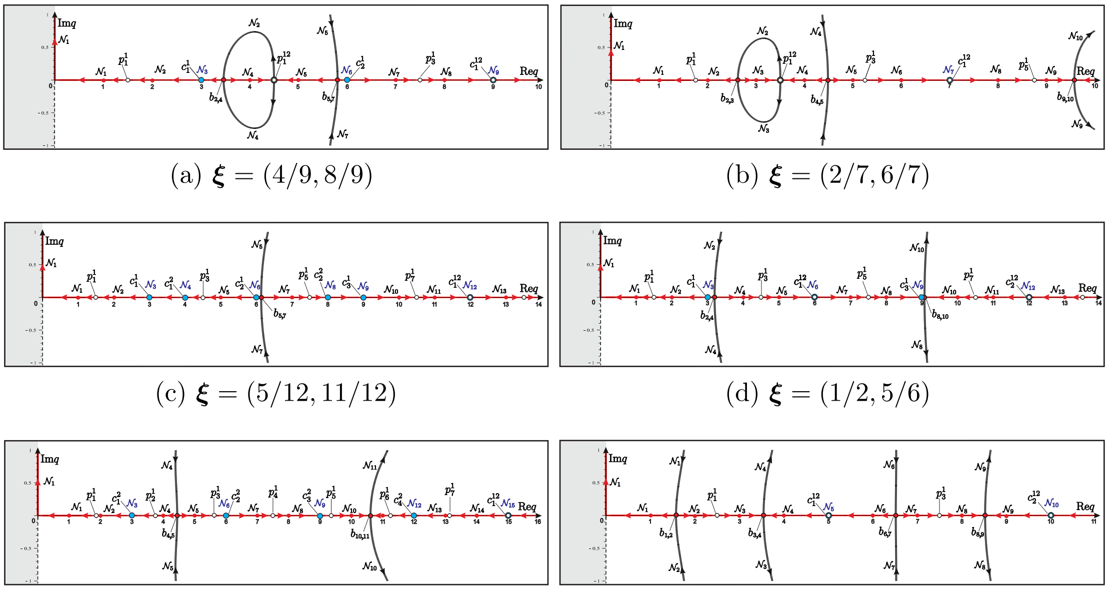
$$c_k^1 = p_{m_+/d_1 k}^1 = z_{2n_+/d_1 k} = 2n_+/d_1 k, \;\; k \in \mathbb{N}, \quad d_1 = \gcd(2n_+, m_+). \tag{2.11}$$

Proof. For the problem (1.1)–(1.3) all zeros are real and positive $z_l = l$, $l \in \mathbb{N}$ (see Figure 1.7(c),(e)). CE points from the set $\mathcal{C}^1$ appear if poles from the first



(a) $\boldsymbol{\xi} = (1/2, \sqrt{2}/2)$
(b) $\boldsymbol{\xi} = (\sqrt{2}/2, \sqrt{3}/2)$
(c) $\boldsymbol{\xi} = ((3-\sqrt{2})/8, (3+\sqrt{2})/8))$
(d) $\boldsymbol{\xi} = ((5\sqrt{2}-7)/24, (5\sqrt{2}+7)/24)$
(e) $\boldsymbol{\xi} = ((2-\sqrt{2})/4, (2+\sqrt{2})/4)$
(f) $\boldsymbol{\xi} = ((2\sqrt{2}-1)/4, (2\sqrt{2}+1)/4)$

Fig. 1.7. Spectrum Curves for $\xi_1/\xi_2 \notin \mathbb{Q}$.



(a) $\boldsymbol{\xi} = (4/9, 8/9)$
(b) $\boldsymbol{\xi} = (2/7, 6/7)$
(c) $\boldsymbol{\xi} = (5/12, 11/12)$
(d) $\boldsymbol{\xi} = (1/2, 5/6)$
(e) $\boldsymbol{\xi} = (1/5, 13/15)$
(f) $\boldsymbol{\xi} = (1/5, 3/5)$

Fig. 1.8. Spectrum Curves for $\xi_1 = m_1/n_1, \xi_2 = m_2/n_2 \in \mathbb{Q}$.

family $p_k^1 = \frac{2}{\xi_+}k$, $k \in \mathbb{N}$, coincide with zeroes:

$$\frac{2}{\xi_+}k = l, \;\; k, l \in \mathbb{N}. \tag{2.12}$$

From proof of Lemma 1.6 we have, that if $\xi_+ \notin \mathbb{Q}$ then (2.12) has no solutions. If $\xi_+ = \xi_2 + \xi_1 = \frac{m_1}{n_1} + \frac{m_2}{n_2} = \frac{m_+}{n_+} \in \mathbb{Q}$ then formula (2.12) can be rewritten in the following form:

$$\frac{2n_+}{d_1}k = \frac{m_+}{d_1}l, \quad k, l \in \mathbb{N}. \tag{2.13}$$

Then the solutions of the equation (2.13) are:

$$l = \frac{2n_+}{d_1}t, \qquad k = \frac{m_+}{d_1}t, \quad t \in \mathbb{N}. \tag{2.14}$$

From equations (2.12) and (2.14) we conclude that first family CE points are:

$$c_t^1 = \frac{2m_+}{\xi_+ d_1}t = \frac{2n_1n_2}{m_2n_1 + m_1n_2} \cdot \frac{m_2n_1 + m_1n_2}{d_1}t = \frac{2n_+}{d_1}t = \frac{2n_1n_2}{d_1}t, \quad t \in \mathbb{N}. \tag{2.15}$$

□

Remark 1.9. If $\xi_1 + \xi_2 \neq 2/l$, $1 < l \in \mathbb{N}$ then $\mathcal{P}_{\boldsymbol{\xi}}^{12} = \varnothing$, $\mathcal{C}_{\boldsymbol{\xi}}^2 = \varnothing$ and $\mathcal{C}_{\boldsymbol{\xi}}^{12} = \varnothing$, but $\mathcal{P}_{\boldsymbol{\xi}}^1 \neq \varnothing$ and $\mathcal{C}_{\boldsymbol{\xi}}^1 \neq \varnothing$. In addition, if $\xi_1 + \xi_2 = 2/l$ is not satisfied, then the set $\mathcal{P}_{\boldsymbol{\xi}}^1$ is empty, because $p_1^1 = c_1^1$.

Theorem 1.10. *If $\xi_- \in \mathbb{Q}$, then $\mathcal{C}_{\boldsymbol{\xi}}^2 \neq \varnothing$:*

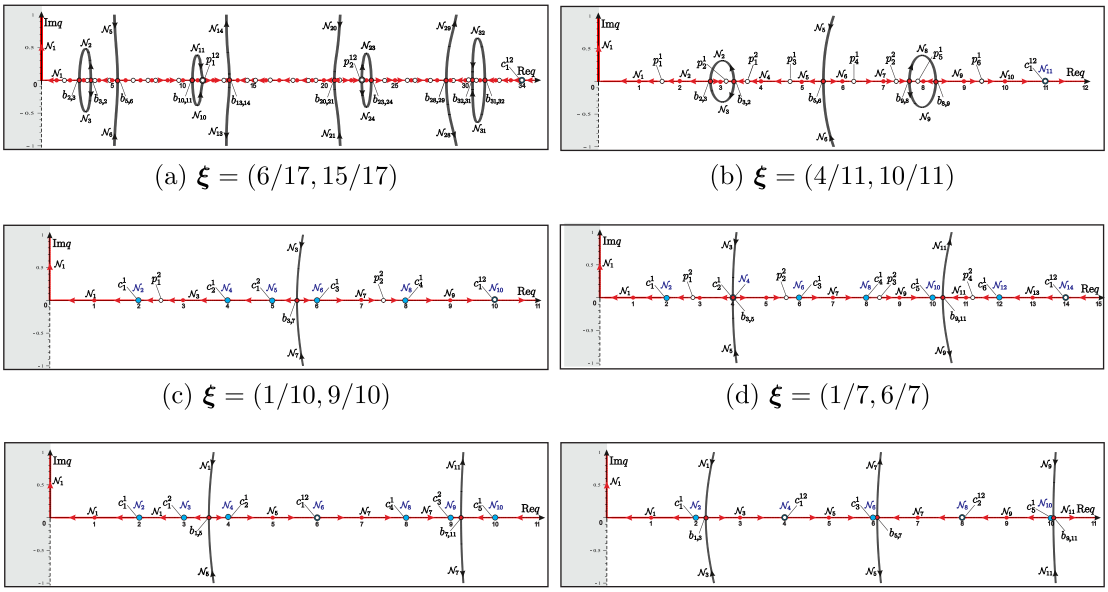
$$c_k^2 = p_{m_-/d_2k}^2 = z_{2n_-/d_2k} = 2n_-/d_2k, \ k \in \mathbb{N}, \quad d_2 = \gcd(2n_-, m_-). \tag{2.16}$$

Proof. The proof follows from the proof of Theorem 1.8. □

CF has removable singularities in these CE points (there is one family of such points) and the first order poles in the set $\mathcal{P}_{\boldsymbol{\xi}}^1 + \mathcal{P}_{\boldsymbol{\xi}}^2$.

Remark 1.11. The set $\mathcal{P}_{\boldsymbol{\xi}}^2 = \varnothing$ for $\xi_- = 2/m$, $2 < m \in \mathbb{N}$, because $p_1^2 = c_1^2$ (see Figure 1.5(b) and Table 1.1). The other sets of constant eigenvalue points ($\mathcal{C}_{\boldsymbol{\xi}}^1$, $\mathcal{C}_{\boldsymbol{\xi}}^{12}$) and poles ($\mathcal{P}_{\boldsymbol{\xi}}^{12}$) are empty if $\xi_2 - \xi_1 \neq 2/m$ (Figure 1.7(d)). If this condition is not satisfied then, additionally, $\mathcal{P}_{\boldsymbol{\xi}}^2 = \varnothing$ (see Figure 1.7(f)).

The case $\xi_1, \xi_2 \in \mathbb{Q}$. In this case the set $\mathcal{C}_{\boldsymbol{\xi}}^{12} \neq \varnothing$ and there exist a few special cases for other sets of poles and CE points. For example, if $\boldsymbol{\xi} = (8/21, 20/21)$, then all sets $\mathcal{P}_{\boldsymbol{\xi}}^1$, $\mathcal{P}_{\boldsymbol{\xi}}^2$, $\mathcal{P}_{\boldsymbol{\xi}}^{12}$, $\mathcal{C}_{\boldsymbol{\xi}}^1$, $\mathcal{C}_{\boldsymbol{\xi}}^2$, $\mathcal{C}_{\boldsymbol{\xi}}^{12}$ and $\mathcal{Z}_{\boldsymbol{\xi}}$ are not empty (see Figure 1.3); if $\boldsymbol{\xi} = (6/17, 15/17)$, then all sets are not empty, except $\mathcal{C}_{\boldsymbol{\xi}}^1$, $\mathcal{C}_{\boldsymbol{\xi}}^2$ ($\mathcal{C}_1^1 = \mathcal{C}_1^2 = \mathcal{C}_1^{12}$ (see Figure 1.9(a)). Further, if $\boldsymbol{\xi} = (4/11, 10/11)$, then the sets $\mathcal{P}_{\boldsymbol{\xi}}^{12} = \mathcal{C}_{\boldsymbol{\xi}}^1 = \mathcal{C}_{\boldsymbol{\xi}}^2 = \varnothing$ and $\mathcal{P}_1^{12} = \mathcal{C}_1^{12}$ (Figure 1.9(b)). In the cases Figure 1.8 the set $\mathcal{P}_{\boldsymbol{\xi}}^1 \neq \varnothing$ however the set $\mathcal{P}_{\boldsymbol{\xi}}^2 = \varnothing$. In the cases of Figure 1.8(a),(b) exist second order pole (there Figure 1.8 $\mathcal{P}_{\boldsymbol{\xi}}^{12} = \varnothing$). The set $\mathcal{C}_{\boldsymbol{\xi}}^1 \neq \varnothing$ for the $\boldsymbol{\xi} = (4/9, 8/9)$, $\boldsymbol{\xi} = (5/12, 11/12)$, $\boldsymbol{\xi} = (1/2, 5/6)$ and the set $\mathcal{C}_{\boldsymbol{\xi}}^2 \neq \varnothing$ for $\boldsymbol{\xi} = (5/12, 11/12)$. The first family CE do



(a) $\boldsymbol{\xi} = (6/17, 15/17)$ (b) $\boldsymbol{\xi} = (4/11, 10/11)$

(c) $\boldsymbol{\xi} = (1/10, 9/10)$ (d) $\boldsymbol{\xi} = (1/7, 6/7)$

(e) $\boldsymbol{\xi} = (1/6, 5/6)$ (f) $\boldsymbol{\xi} = (1/4, 3/4)$

Fig. 1.9. Spectrum Curves for $\xi_1 = m_1/n_1, \xi_2 = m_2/n_2 \in \mathbb{Q}$.

not exist in the case Figure 1.8(a) and both sets $\mathcal{C}^1_{\boldsymbol{\xi}}$ and $\mathcal{C}^2_{\boldsymbol{\xi}}$ are empty for the case Figure 1.8(b)–(f). For the fixed $\boldsymbol{\xi}$ values, when $n_1 = n_2 = n$ and $m_1 + m_2 = n$ (see Figure 1.9(c)–(f): $\mathcal{P}^1_{\boldsymbol{\xi}} = \varnothing$ and $\mathcal{P}^{12}_{\boldsymbol{\xi}} = \varnothing$. In contrast to the case shown in Figure 1.9(e)–(f), when $\boldsymbol{\xi} = (1/10, 9/10)$ and $\boldsymbol{\xi} = (1/7, 6/7)$, then the set $\mathcal{P}^2_{\boldsymbol{\xi}}$ is not empty (see Figure 1.9(c)–(f)). For all examples in Figure 1.9(c)–(f), the set $\mathcal{C}^1_{\boldsymbol{\xi}} \neq \varnothing$, but CE points depending on the second order pole family are obtained only for examples in Figure 1.9(c),(e). For this instance, we can use expressions (2.1)–(2.6) and get formulae for poles and CE points ($k \in \mathbb{N}$).

Theorem 1.12. *If $\xi_1 \in \mathbb{Q}$ and $\xi_2 \in \mathbb{Q}$, then the pole points of the sets $\mathcal{P}^1_{\boldsymbol{\xi}} + \mathcal{P}^{12}_{\boldsymbol{\xi}} + \mathcal{C}^1_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{P}^2_{\boldsymbol{\xi}} + \mathcal{P}^{12}_{\boldsymbol{\xi}} + \mathcal{C}^2_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{P}^{12}_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$ for the problem (1.1)–(1.3) are*

$$p^1_k = \frac{2n_1n_2k}{m_+}, \qquad p^2_k = \frac{2n_1n_2k}{m_-}, \qquad p^{12}_k = \frac{2n_1n_2k}{\gcd(m_-, m_+)}, \qquad k \in \mathbb{N}.$$

Theorem 1.13. *If $\xi_1 \in \mathbb{Q}$ and $\xi_2 \in \mathbb{Q}$ then the CE points c^1_k and c^2_k for the problem (1.1)–(1.3) are equal to:*

$$c^1_k = \frac{2n_1n_2k}{\gcd(2n_\pm, m_+)}, \qquad c^2_k = \frac{2n_1n_2k}{\gcd(2n_\pm, m_-)}, \qquad k \in \mathbb{N}.$$

CE points of the first family and the second family are points of the sets $\mathcal{C}^1_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$ and $\mathcal{C}^2_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$.

Theorem 1.14. *If $\xi_1 \in \mathbb{Q}$ and $\xi_2 \in \mathbb{Q}$, then CE points of the set $\mathcal{C}^{12}_{\boldsymbol{\xi}}$ for the problem (1.1)–(1.3) are equal to:*

$$c^{12}_k = \frac{2n_1n_2k}{\gcd(2n_\pm, m_+, m_-)}, \qquad k \in \mathbb{N}.$$

Proof. If the second order pole points $p^{12}_k = \frac{2n_1n_2k}{\gcd(m_-,m_+)}$, $k \in \mathbb{N}$, coincide with zero $z_l = l \in \mathbb{N}$, then we get the point of the set $\mathcal{C}^{12}_{\boldsymbol{\xi}}$:

$$\frac{2n_1n_2k}{\gcd(m_+, m_-)} = l, \quad k, l \in \mathbb{N}. \tag{2.17}$$

This equation can be rewritten in the other form:

$$2n_\pm k = \gcd(m_+, m_-)l, \quad k, l \in \mathbb{N}, \tag{2.18}$$

For numbers $2n_\pm$ and $\gcd(m_+, m_-)$ we can find the greatest common divisor $d_3 = \gcd(2n_\pm, \gcd(m_+, m_-)) = \gcd(2n_\pm, m_+, m_-)$ and then the equation (2.18) we can rewrite in the other form:

$$\frac{2n_\pm}{d_3}k = \frac{\gcd(m_+, m_-)}{d_3}l, \quad k, l \in \mathbb{N}. \tag{2.19}$$

Then by the Lemma 1.6 all solutions of the equation (2.17) have a form:

$$l = \frac{2n_\pm t}{d_3}, \qquad k = \frac{\gcd(m_+, m_-)t}{d_3}, \quad t \in \mathbb{N}. \tag{2.20}$$

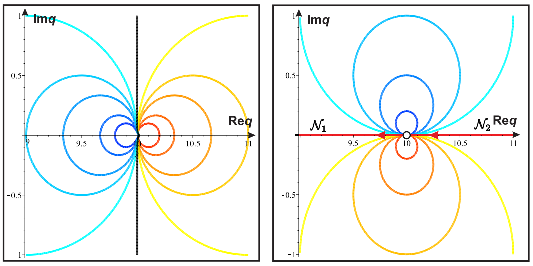
From equations (2.17) and (2.20) we obtain that CE points of the set $\mathcal{C}^{12}_{\boldsymbol{\xi}}$ are

$$c^{12}_t = \frac{2n_\pm t}{\gcd(2n_\pm, m_+, m_-)}, \quad t \in \mathbb{N}.$$

□

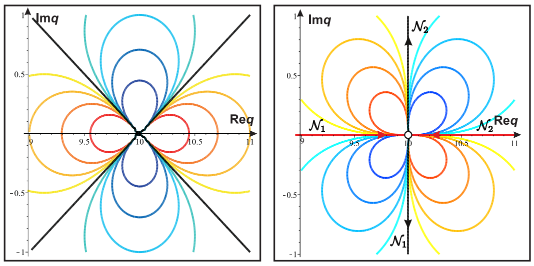
The set $\mathcal{C}^{12}_{\boldsymbol{\xi}} \neq \varnothing$ for all $\boldsymbol{\xi}$. $\mathcal{P}^1_{\boldsymbol{\xi}} = \varnothing$ for $p^1_1 = c^1_1$; $\mathcal{P}^2_{\boldsymbol{\xi}} = \varnothing$ for $p^2_1 = p^{12}_1$ or $p^2_1 = c^2_1$ or $p^2_1 = c^{12}_1$; $\mathcal{P}^{12}_{\boldsymbol{\xi}} = \varnothing$ for $p^{12}_1 = c^{12}_1$ or $p^1_1 = c^1_1$ or $p^2_1 = c^2_1$; $\mathcal{C}^1_{\boldsymbol{\xi}} = \varnothing$ for $c^1_1 = c^{12}_1$; $\mathcal{C}^2_{\boldsymbol{\xi}} = \varnothing$ for $c^2_1 = c^{12}_1$; $\mathcal{C}^{12}_{\boldsymbol{\xi}} = \varnothing$ for $c^2_1 = c^1_1$ (see Table 1.1).

Some information on the first or the second order poles can be presented as contour lines of the functions $(z-10)^{-1}$ and $(z-10)^{-2}$. Real CF in neighbourhood of the first order pole are shown in Figure 1.10 and Figure 1.11. In this case there are two Spectrum Curves $\mathcal{N}_1$ and $\mathcal{N}_2$ on the real axis (see Figure 1.10(b)). In the neighbourhood of the first order pole there exist only real eigenvalues (see Figure 1.12). The Spectrum Curves $\mathcal{N}_1$, $\mathcal{N}_2$ and the CF in neighbourhood of the second order poles are presented in Figure 1.11(b) and Figure 1.13.



(a) real part (b) image part

Fig. 1.10. The first order pole.



(a) real part (b) image part

Fig. 1.11. The second order pole.

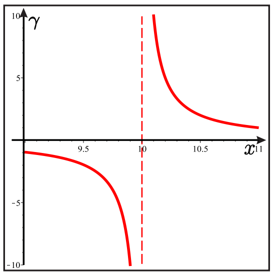
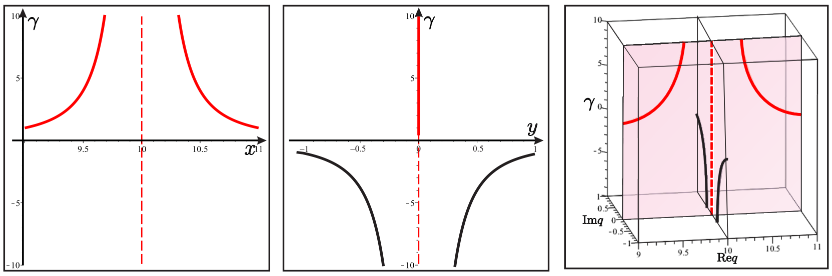


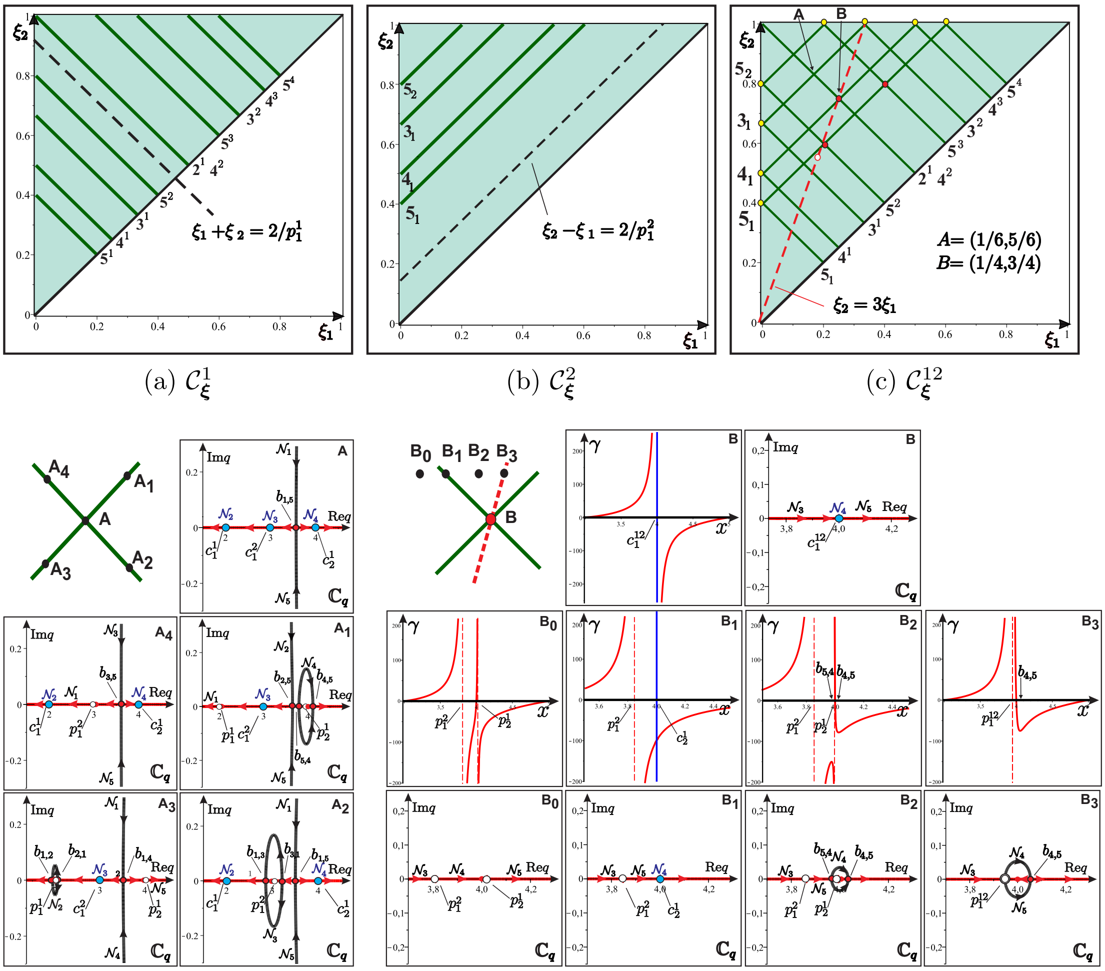
Fig. 1.12. Real CF. A neighbourhood of the first order pole.



(a) Real CF (b) complex part (c) 3D view

Fig. 1.13. A neighbourhood of the second order pole.

We have two families $\mathcal{C}^1_{\boldsymbol{\xi}}$ and $\mathcal{C}^2_{\boldsymbol{\xi}}$ of CE points (these eigenvalues do not exist if $\xi_1, \xi_2 \notin \mathbb{Q}$, but $\xi \in \mathbb{Q}$). The dependence of CE points on NBC parameters ξ_1 and ξ_2 are presented in Phase Space $S_{\boldsymbol{\xi}}$ (see Figure 1.14). The CE points of the first family $\mathcal{C}^1_{\boldsymbol{\xi}}$ are on the lines $\xi_1 + \xi_2 = 2k/l$, $l \in \mathbb{N} \setminus \{1\}$, which are perpendicular to the line $\xi_2 = \xi_1$ (see Figure 1.14(a)). The CE points of the second family $\mathcal{C}^2_{\boldsymbol{\xi}}$ are on the lines $\xi_2 - \xi_1 = 2k/l$, $l \in \mathbb{N} \setminus \{1, 2\}$, which are parallel to the line $\xi_2 = \xi_1$ (see Figure 1.14(b)). Notation l^k or l_k near the line shows that the CE point is $c^1_k = l$ or $c^2_k = l$, accordingly. The intersection points of the CE lines from the different families with the same number l give the set $\mathcal{C}^{12}_{\boldsymbol{\xi}}$ (see Figure 1.14(c)). We have the first order pole p^1_1 or p^2_1 in the lines $\xi_1 + \xi_2 = 2\pi/p^1_1$ or $\xi_2 - \xi_1 = 2\pi/p^2_1$, too. The double pole is in the line $\xi_2 = n/m \cdot \xi_1$ (see Figure 1.14(c), $m = 1$, $n = 3$). We analyze two points in Phase Space $S_{\boldsymbol{\xi}}$: $A = (1/6, 5/6)$ and $B = (1/4, 3/4)$. The point A corresponds to the situation without poles ($p^1_1 = c^1_1$, $p^2_1 = c^2_1$, see Table 1.1 and Figure 1.5(c)), point B corresponds to the situation with first order pole in CE point. If $\boldsymbol{\xi}$ is moving across line (A_2, A_4) or (A_1, A_3) then at the CE point the complex part of Spectrum Curve is arising or disappearing in $\mathbb{C}_q$ (see Figure 1.14(d)). In this case the complex part of the Spectrum Curve is between



(a) $\mathcal{C}_\xi^1$ (b) $\mathcal{C}_\xi^2$ (c) $\mathcal{C}_\xi^{12}$

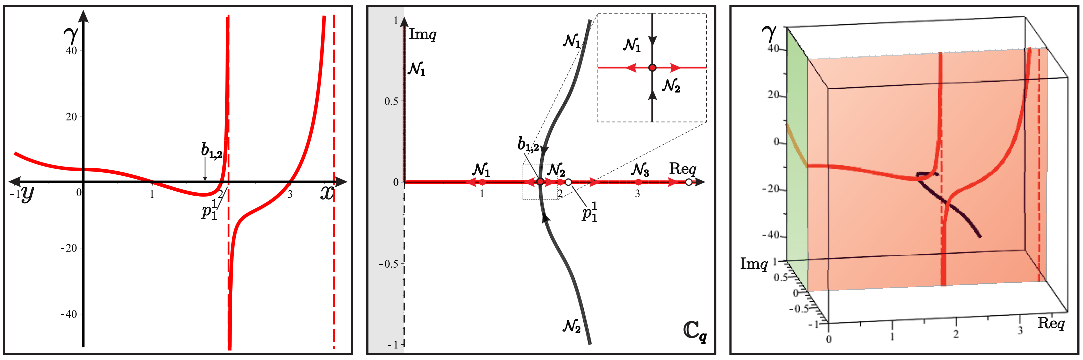
(d) point A (e) point B

Fig. 1.14. CE points ($c_k = \pi k$, $k = 1, \ldots, 5$) in Phase Space S_ξ and Spectrum Curves in the points A and B and in their neighbourhood.

two Critical points. We have the same situation near point B (see Figure 1.14(e), $B_0 \to B_1 \to B_2$). At the point B_3 two first order poles create the second order pole and the complex part of the Spectrum Curve is between the Critical point and this pole. All complex parts of the Spectrum Curve are disappearing in the point B.

3 Critical points

If $\gamma_c'(b) = 0$, $b \in \mathbb{C}$, then we have a *Critical point* b of the Complex CF, and value $\gamma_c(b)$ is a *critical value* of the Complex CF [46, Pečiulytė and Štikonas 2007], [49, Pečiulytė *et al.* 2008]. Critical points of the Complex CF are saddle points of this function. For Real CF Critical points can be a half-saddle points



(a) Real CF (b) Spectrum Curves (c) 3D view

Fig. 1.15. The first order Real Critical point, $\boldsymbol{\xi} = (0.2, 0.75)$.

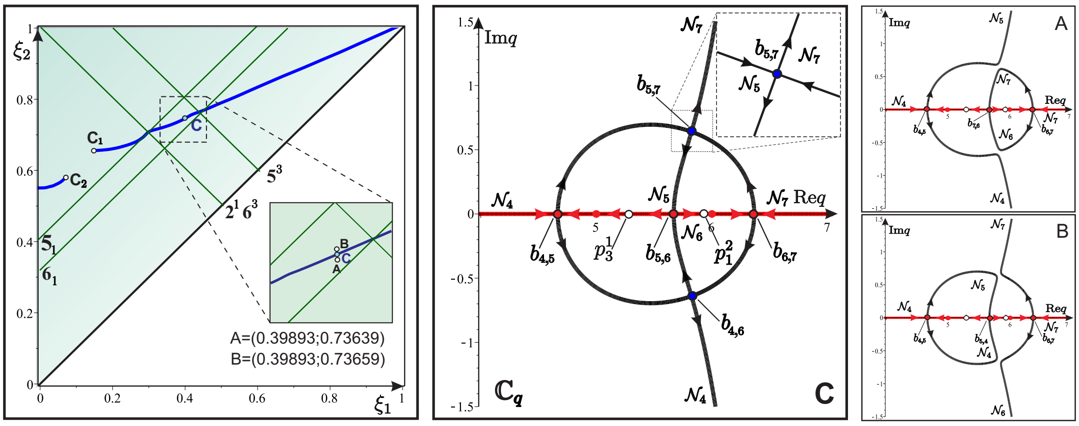
(for example, such point is branch point $q = 0$), or maximum, minimum points or inflection (saddle) points. If the function γ_c at the Critical point $b \in \mathbb{C}_q$ satisfies $\gamma_c'(b) = 0, \ldots, \gamma_c^k(b) = 0$, $\gamma_c^{(k+1)}(b) \neq 0$, then b is called the *k-order Critical point*. At this point Spectrum Curves change direction and the angle between the old and the new direction is $\frac{\pi}{k+1}$. We use "right hand rule". So, the Spectrum Curve turns to the right.

The index of Critical point is formed of the indices of the Spectrum Curves, which is not a constant eigenvalue point, intersecting in this Critical point. The index of Complex (the first order) Critical point $b \in \mathbb{C}_q \setminus \mathbb{R}_q$ is formed by indexes of two Spectral Curves intersecting in this Critical point and the first index is always smaller. If the Critical point is real, then the left index coincide with the index of Spectral Curve, which is defined by the smaller real λ values, and the right index is defined by greater λ values. We put the indexes of other Spectrum Curves in the accending order between left and right indices (see Figure 1.16(b)–Figure 1.19(b)).

Remark 1.15. A point $q = 0$ is the first order Critical point in the domain $\mathbb{C}_q$, but $\lambda = 0$ is not Critical point in $\mathbb{C}_\lambda$, because q is a branch point of $\lambda = \lambda(q)$. The order of a Critical point at branching point is not invariant. Therefore, we investigate these points separately.

3.1 The first order Critical points

For SLP (1.1)–(1.3) there exist two types of the first order Critical points. The first type Critical point appears for $\overset{\circ}{\mathbb{R}}_q = \mathbb{R}_q^- \cup \mathbb{R}_q^+$. In this case, multiple eigenvalue



(a) (b) (c)

Fig. 1.16. a) The trajectory of the first order Complex Critical point in Phase Space. $C = (0.39893, 0.73649\ldots)$; b)-c) Spectrum Curves ir the neighbourhood of the first order Complex Critical point C.

is real (usually double or triple, where the Critical point coincide with CE point). *The first order Real Critical point* $b \in \overset{\circ}{\mathbb{R}}_q$, can be find from the following equation (for fixed ξ_1 and ξ_2):

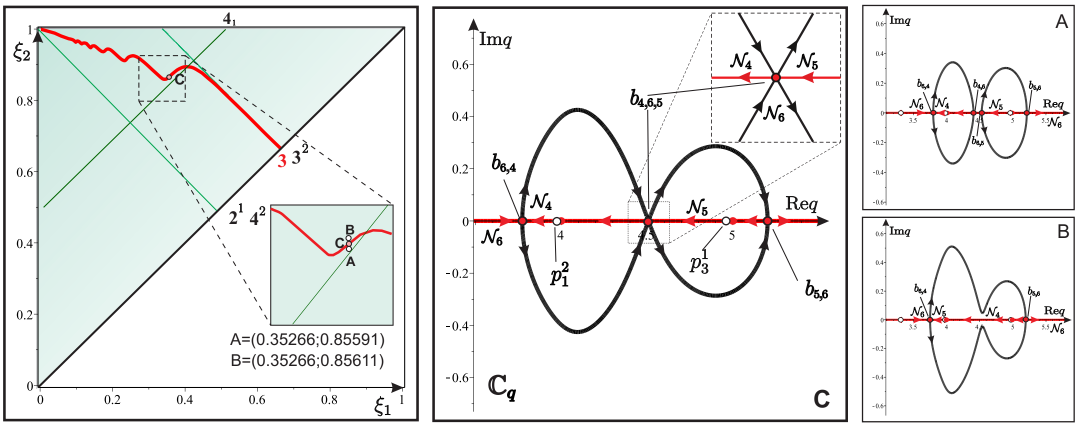
$$\gamma'(b; \boldsymbol{\xi}) = 0. \tag{3.1}$$

CF in the neighbourhood of the first order Real Critical point is presented in Figure 1.15. Real Critical point of CF exists in the typical situations : 1) between two zeroes do not exist pole, 2) between two poles do not exist zero. For SLP (1.1)–(1.3) all the first order Real Critical points are positive.

The first order Complex Critical point $b = x + \imath y \in \mathbb{C}_q \setminus \mathbb{R}_q$ can be calculated solving the system of equations:

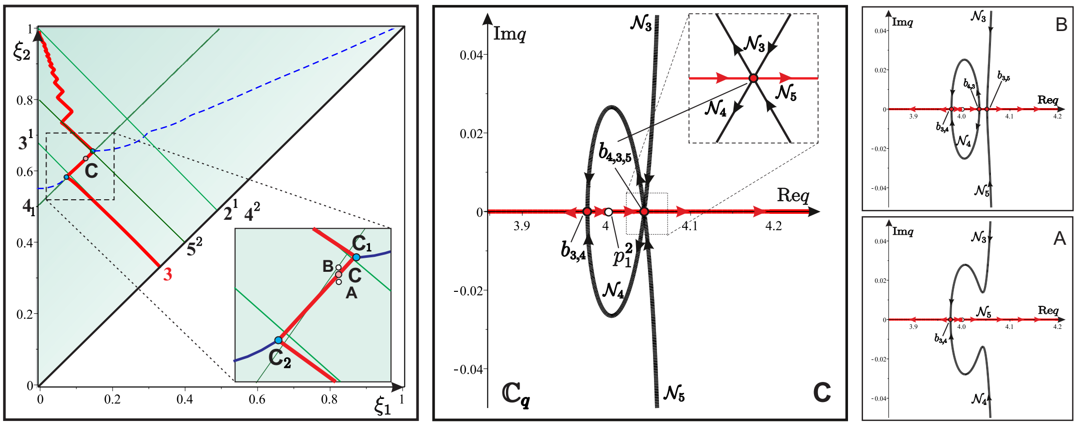
$$\operatorname{Im} \gamma(b; \boldsymbol{\xi}) = 0, \qquad \operatorname{Re} \gamma'(b; \boldsymbol{\xi}) = 0, \qquad \operatorname{Im} \gamma'(b; \boldsymbol{\xi}) = 0. \tag{3.2}$$

Spectrum Curves in $\mathbb{C}_q \setminus \mathbb{R}_q$ are symmetrical with respect to x-axis and we have pair conjugate Complex Critical points always. The solution of system (3.2) is a trajectory in Phase Space $S_{\boldsymbol{\xi}}$. Two trajectories of the such Complex Critical points are presented in Figure 1.16(a). The Spectrum Curves for $\boldsymbol{\xi} = (0.39893, 0.73649...)$ with Complex Critical point is presented in Figure 1.16(b). Every point of the trajectory in $S_{\boldsymbol{\xi}}$ has similar Spectrum Curves in the neighbourhood of a Critical point. If Phase Point moves across this trajectory, then the view of the Spectrum Curves are qualitative different (see Figure 1.16(a),(c), points A and B). In this example (see Figure 1.16(b), point C) we have both cases of the first order Critical



(a) (b) (c)

Fig. 1.17. a) The trajectory of the second order Critical point in Phase Space, $C = (0.35266, 0.85601\dots)$; b)-c) Spectrum Curves ir the neighbourhood of second order Critical point C.



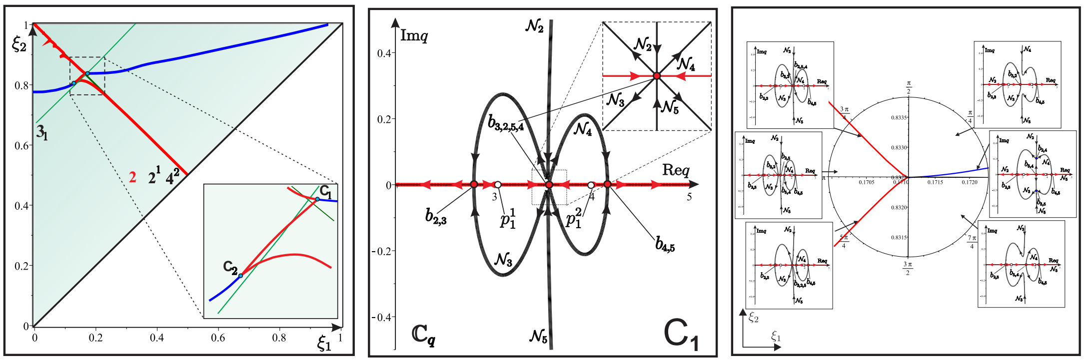
(a) (b) (c)

Fig. 1.18. a) The trajectory of the second order Critical point $C = (0.11625, 0.616239\dots)$, $A = (0.11625, 0.616238\dots)$, $B = (0.11625, 0.616240\dots)$, $C_1 = (0.15454\dots, 0.64970\dots)$, $C_2 = (0.07331\dots, 0.57495\dots)$; b)-c) Spectrum Curves ir the neighbourhood of the second order Critical point C.

points: $b_{4,5}$, $b_{5,6}$, $b_{6,7}$ are Real Critical points and $b_{4,6}$, $b_{5,7}$ are pair Complex Critical points. The gap between two trajectories in $S_{\boldsymbol{\xi}}$ (points C_1 and C_2) will be explained further (see Figure 1.16(a)).

3.2 The second order and the third order Critical points

The *second order Critical point* appears when two the first order Real Critical points coincide in the same point b. Second order Critical point $b \in \overset{\circ}{\mathbb{R}}_q$ can be



(a) (b) (c)

Fig. 1.19. a) Intersection of trajectories of the third order Critical points; b) The third order Critical point $C_1 = (0.17122\ldots, 0.83250\ldots)$; c) Bifurcation diagram in the neighbourhood of the third order Critical point C_1.

found from the following equation:

$$\gamma'(b;\boldsymbol{\xi}) = 0, \qquad \gamma''(b;\boldsymbol{\xi}) = 0.$$

For SLP (1.1)–(1.3) all the second order Real Critical points are positive. Two trajectories of such the second order Critical points in Phase Space are shown in Figure 1.17(a) and Figure 1.18(a). In the Figure 1.17(b) and Figure 1.18(b) Spectrum Curves are presented in the point C which is on corresponding trajectory of the second order Critical point and in Figure 1.17(c) and Figure 1.18(c) Spectrum Curves in the Phase Points A and B near this trajectory can be seen. Points $b_{4,6,5}$ and $b_{4,3,5}$ in Figure 1.17(b) and Figure 1.18(b) are the second order Critical points. Points C_1 and C_2 in Figure 1.18(a) are the same as in Figure 1.16(a). So, the gap between Phase Points C_1 and C_2 is the part of the second order Critical point trajectory (see Figure 1.18(a)).

Numerical calculations show that such gaps exist for $\xi_1 + \xi_2 \lessapprox 1$ (see Figure 1.16(a), Figure 1.18(a), Figure 1.19(a)). The gap boundary points C_1 and C_2 are the third order Critical points $b \in \overset{\circ}{\mathbb{R}}_q$ and they can be found from the system:

$$\gamma'(b;\boldsymbol{\xi}) = 0, \qquad \gamma''(b;\boldsymbol{\xi}) = 0, \qquad \gamma'''(b;\boldsymbol{\xi}) = 0.$$

The views of Spectrum Curves in point C_1 and in the neighbourhood of this third order Critical point $b_{3,2,5,4}$ are presented in Figure 1.19(b)–(c). At this point the trajectory of the second order Critical point changes direction and the pair of the first order Complex Critical points become real ($y = 0$ and positive).

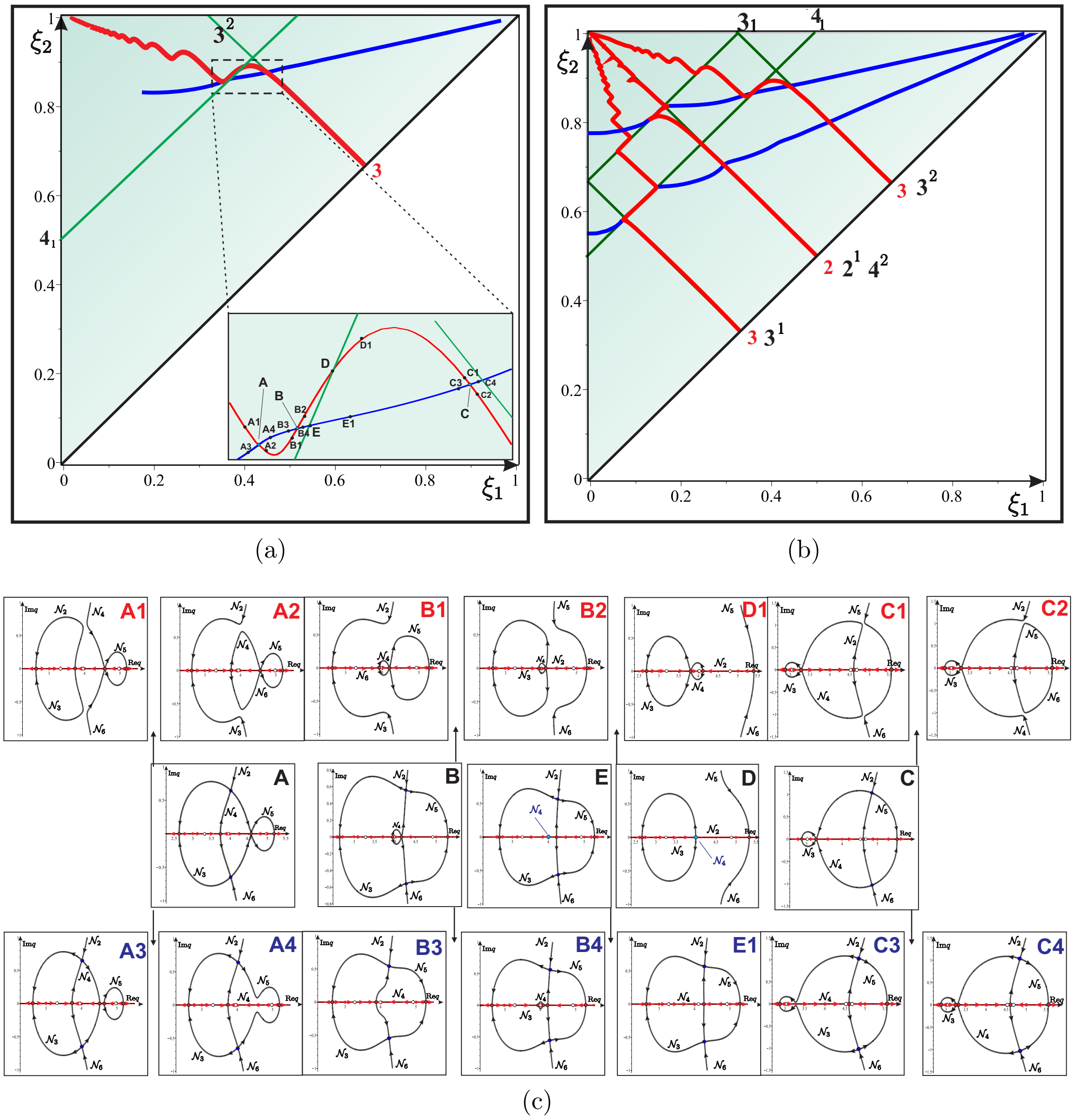


Fig. 1.20. The trajectories of the third order and the second order Critical points and Spectrum Curves, $A = (0.3526\ldots, 0.8560\ldots)$, $B = (0,3600\ldots, 0.8601\ldots)$, $C = (0.4491\ldots, 0.8771\ldots)$, $D = (0.3660\ldots, 0.8660\ldots)$, $E = (0.3603\ldots, 0.8603\ldots)$.

If $\xi_1 + \xi_2 \gtrapprox 1$ (see Figure 1.17(a), Figure 1.20(a)) then the trajectory of the second order Critical point is "smooth" curve. This trajectory intersects with the first order Complex Critical point trajectory without the third order Critical points, i.e. the pair of the Complex Critical points do not reach the real axis. Typical Spectrum Curves are presented in Figure 1.20(c).

The general behaviour of the second and the third order trajectories of Phase Space is more complicated. For small x three trajectories are shown in Figure 1.20(b). The second order trajectories leave points $\boldsymbol{\xi} = (1/3, 1/3)$, $(1/2, 1/2)$, $(2/3, 2/3)$ for $x = 3, 2, 3$, accordingly. All these trajectories approach Phase Point

$\boldsymbol{\xi} = (0, 1)$. There is not the second order Critical point for integral NBC with $\xi_2 = 1$ or $\xi_1 = 0$. Trajectories of the first order Complex Critical points start at points $\boldsymbol{\xi} = (0, b)$ and move towards a point which corresponds to the third order Critical point and after "gap" these trajectories approach Phase Point $\boldsymbol{\xi} = (1, 1)$.

Spectrum Curves at the point $q = 0$. Taylor series for $\gamma(q)$ at the point $q = 0$ is

$$\gamma(q) = \frac{2}{\xi_2^2 - \xi_1^2} - \frac{2 - \xi_1^2 - \xi_2^2}{\xi_2^2 - \xi_1^2} q^2 + \mathcal{O}(q^4). \tag{3.3}$$

Multiplier of q^2 is always negative. At this point Spectrum Curve $\mathcal{N}_1$ turn orthogonal to the right, so, the point $q = 0$ is the first order Critical point in $\mathbb{C}_q$. A point $\lambda = 0$ is not Critical one in the complex plane $\mathbb{C}_\lambda$ because $\lambda = 0$ is a branch point of a mapping $\lambda = \lambda(q)$.

4 Conclusions

In this chapter the spectrum for SLP with integral NBC depending on three parameters was analyzed.

Qualitative view of the Spectrum Curves with respect to parameters ξ_1 and ξ_2 in integral BC, the location of the zeroes, poles of the CF and CE points were investigated. We found all such points in the case SLP (1.1)–(1.3). One of our results is the classification of poles and zero points. The dependence of zeros and poles on the integral BC parameters ξ_1 and ξ_2 was analyzed. CE non-existence condition (sets $\mathcal{C}_{\boldsymbol{\xi}}^1$, $\mathcal{C}_{\boldsymbol{\xi}}^2$ and $\mathcal{C}_{\boldsymbol{\xi}}^{12}$ are empty) is $\xi_1/\xi_2 \in \mathbb{Q}$, $\xi_1, \xi_2 \notin \mathbb{Q}$. If the following condition $\xi_1/\xi_2 \notin \mathbb{Q}$ is satisfied, then $\mathcal{P}_{\boldsymbol{\xi}}^{12} = \varnothing$ and $\mathcal{C}_{\boldsymbol{\xi}}^{12} = \varnothing$. For all ξ_1 and ξ_2 satisfying condition $\xi_1, \xi_2 \in \mathbb{Q}$, the set $\mathcal{C}_{\boldsymbol{\xi}}^{12}$ is not empty.

Critical points of CF are important for numerically analysis of complex eigenvalues and Spectrum Curves in the complex plane. We found trajectories of the first order Complex Critical points and the second order (real) Critical points. In this chapter we described how Spectrum Curves depends on parameters ξ_1 and ξ_2. We analyzed the first order Real and Complex Critical points, trajectories of the first order Complex Critical points and the second order Critical points in the Phase Space $S_{\boldsymbol{\xi}}$, find location of the third order Critical points.

Investigation of the Spectrum Curves gives useful information about the spectrum for problems with NBC.

Chapter 2

Sturm–Liouville problem with integral NBC (special cases)

In this chapter, SLP for equation $-u'' = \lambda u$ with two cases of integral NBC is analyzed. We investigate how the eigenvalues of these problems depend on two parameters γ and ξ in the integral NBC. As the theoretical investigation of the complex spectrum is a difficult problem, we present the results of modelling and computational analysis and illustrate the existing situation in graphs. In Section 1, we formulate the SLP with clasical BC ($u(0) = 0$ or $u'(0) = 0$) and integral NBC:

$$u(1) = \gamma \int\limits_{\xi}^{1} u(t)\ \mathrm{d}t \quad \text{or} \quad u(1) = \gamma \int\limits_{0}^{\xi} u(t)\ \mathrm{d}t.$$

Also, in Section 1, we present the earlier gathered results on CF, zeroes, poles, CE and Critical points in all cases [48, 49, Pečiulytė *et al.* 2005, 2008]. In Section 2 a short review of real eigenvalues properties of the analyzed problems is given. These results are discussed in the previous papers [48, 49, Pečiulytė *et al.* 2005, 2008], [83, Štikonas 2007] and they are useful for investigating complex eigenvalues. The behaviour of Spectrum Curves in the complex part of the spectrum is presented in Section 3.

The Section 4 presents some new results on the spectrum for the second order differential problem with symmetric interval in the integral

$$u(1) = \gamma \int_{\xi}^{1-\xi} u(t)\,\mathrm{d}t.$$

We investigate how the spectrum of this problem depends on the integral NBC parameters γ, ξ in this symmetric case. Some special conclusions are given on the complex spectra of this problem in Section 4.1. Some results are presented as graphs of Real and Complex CF. Similar problem for $\boldsymbol{\xi} = (0, 1)$ and $\boldsymbol{\xi} = (1/4, 3/4)$ were discussed in [9, Čiupaila *et al.* 2004]. Certain parts of this chapter are published in [76, 77, 79].

Remark 2.1. In Chapter 1: $\xi = \xi_1/\xi_2$. In this chapter, we use notation ξ instead of ξ_1 and ξ_2: $\boldsymbol{\xi} = (0, \xi)$, $\boldsymbol{\xi} = (\xi, 1)$, $\boldsymbol{\xi} = (\xi, 1 - \xi)$. If $\xi \in \mathbb{Q}$, then we use rule: $\xi = m/n$, $m, n \in \mathbb{N}$, and m, n are coprime numbers, i.e., $\gcd(n, m) = 1$.

1 Sturm–Liouville problem with integral type NBC

Let us consider a SLP with one classical BC:

$$-u'' = \lambda u, \quad t \in (0, 1), \tag{1.1}$$

$$u(0) = 0, \tag{1.2}$$

and another integral NBC (two cases):

$$u(1) = \gamma \int_{\xi}^{1} u(t)\ \mathrm{d}t \quad \text{(Case 1)}, \tag{1.3_1}$$

$$u(1) = \gamma \int_{0}^{\xi} u(t)\ \mathrm{d}t \quad \text{(Case 2)}, \tag{1.3_2}$$

with parameters $\gamma \in \mathbb{R}$ and $\xi \in [0, 1]$. Also the SLP (1.1) with the BC is analyzed:

$$u'(0) = 0 \tag{1.4}$$

on the left side, and with integral NBC (1.3) on the right side of the interval. We enumerate these cases from Case 1′ to Case 2′.

In Case 1, 1′ for $\xi = 0$ and Case 2, 2′ for $\xi = 1$ we have the same integral NBC. In the general case, the eigenvalues $\lambda \in \mathbb{C}$ and eigenfunctions $u(t)$ are the complex functions. For $\gamma = \infty$, we get NBC:

$$\int_{\xi}^{1} u(t)\ \mathrm{d}t = 0, \quad 0 \leqslant \xi < 1, \qquad \int_{0}^{\xi} u(t)\ \mathrm{d}t = 0, \quad 0 < \xi \leqslant 1. \tag{$1.5_{1,2}$}$$

Note that the index in the number of a formula (for example in formula (1.3)) denotes the case. If there is no index, then the rule (or results) holds on in all the cases of NBCs. If we write two indexes in the number of formulae, as in (1.5), then the first part of this formula is related to Case 1 and the second part is related to Case 2. If we write one index, then the formula is related to one case.

Remark 2.2 (**classical case**). If $\gamma = 0$ or $\xi = 1$ in problem (1.1), (1.2), (1.3_1) or problem (1.1), (1.4), (1.3_1) and $\gamma = 0$ or $\xi = 0$ in problem (1.1),(1.2), (1.3_2) or problem (1.1), (1.4), (1.3_2), we have the SLP with the classical BCs and in this case eigenvalues and eigenfunctions are well known:

$$\lambda_k = (k\pi)^2, \quad u_k(t) = \sin(k\pi t), \quad k \in \mathbb{N}, \tag{$1.6_{1,2}$}$$

$$\lambda_k = \left(k - \frac{1}{2}\right)^2 \pi^2, \quad u_k(t) = \cos\left(\left(k - \frac{1}{2}\right)\pi t\right), \quad k \in \mathbb{N}. \tag{$1.6_{1',2'}$}$$

If $\lambda = 0$, then the function $u(t) = ct$ satisfies problem (1.1)–(1.2) and the function $u(t) = c$ satisfies problem (1.1), (1.4). By substituting these solutions into NBCs, we derive that there exists a nontrivial solution ($c \neq 0$) if:

$$1 - \gamma\frac{1-\xi^2}{2} = 0, \qquad 1 - \gamma\frac{\xi^2}{2} = 0, \tag{$1.7_{1,2}$}$$

$$1 - \gamma(1-\xi) = 0, \qquad 1 - \gamma\xi = 0. \tag{$1.7_{1',2'}$}$$

Lemma 2.3. *The eigenvalue $\lambda = 0$ exists if, and only if*

$$\gamma = \frac{2}{1-\xi^2}, \qquad \gamma = \frac{2}{\xi^2}, \tag{$1.8_{1,2}$}$$

$$\gamma = \frac{1}{1-\xi}, \qquad \gamma = \frac{1}{\xi}. \tag{$1.8_{1',2'}$}$$

In general, if $\lambda \neq 0$ and eigenvalues $\lambda = (\pi q)^2$, then the solution of problem (1.1)–(1.2) is $u = c\sin(\pi q t)$ and the solution of problem (1.1), (1.4) is $u(t) = \cos(\pi q t)$. In both cases ($q = 0$ and $q \neq 0$), we can write one formula for the nontrivial solutions $u = c\sin(\pi q t)/(\pi q) = c\sinh(-\imath\pi q t)/(\pi q)$ of BC (1.2) and $u = c\cos(\pi q t) = c\cosh(-\imath\pi q t)$ of BC (1.4), where $q \in \mathbb{C}_q$.

Let us return to the problems (1.1)–(1.3) and (1.1), (1.3), (1.4). If $\lambda \neq 0$, the NBC is satisfied and there exists a nontrivial solution (eigenfunction) if $q \in \mathring{\mathbb{C}}_q :=$

$\mathbb{C}_q \setminus \{0\}$ is the root of the equation:

$$f(q) := 2\gamma \frac{\sin((1+\xi)\pi q/2)\sin((1-\xi)\pi q/2)}{(\pi q)^2} - \frac{\sin(\pi q)}{\pi q} = 0, \tag{1.9$_1$}$$

$$f(q) := 2\gamma \frac{\sin^2(\xi\pi q/2)}{(\pi q)^2} - \frac{\sin(\pi q)}{\pi q} = 0, \tag{1.9$_2$}$$

$$f(q) := 2\gamma \frac{\cos((1+\xi)\pi q/2)\sin((1-\xi)\pi q/2)}{\pi q} - \cos(\pi q) = 0, \tag{1.9$_{1'}$}$$

$$f(q) := \gamma \frac{\sin(\xi\pi q)}{\pi q} - \cos(\pi q) = 0. \tag{1.9$_{2'}$}$$

CE were analyzed in papers [48, 49, Pečiulytė *et al.* 2005, 2008]. CE points, we get as a roots of the system:

$$\begin{cases} Z(q) := \sin(\pi q) = 0, \\ P_\xi(q) := \cos(\xi\pi q) - \cos(\pi q) = 0, \end{cases} \qquad \begin{cases} Z(q) := \sin(\pi q) = 0, \\ P_\xi(q) := 1 - \cos(\xi\pi q) = 0, \end{cases} \tag{1.10$_{1,2}$}$$

$$\begin{cases} Z(q) := \cos(\pi q) = 0, \\ P_\xi(q) := \sin(\pi q) - \sin(\xi\pi q) = 0, \end{cases} \qquad \begin{cases} Z(q) := \cos(\pi q) = 0, \\ P_\xi(q) := \sin(\xi\pi q) = 0. \end{cases} \tag{1.10$_{1',2'}$}$$

CEs exist only for rational $\xi = m/n \in (0,1)$, $m, n \in \mathbb{N}$, and those eigenvalues are equal to $\lambda_k = (\pi c_k)^2$, $k \in \mathbb{N}$, where CE points c_k are given by formulae shown in Table 2.1. $\mathcal{C}$ is the set of all CE points (as in Chapter 1).

Table 2.1. CE points c_k, $k \in \mathbb{N}$.

Case	$n-m \in \mathbb{N}_e$	$n-m \in \mathbb{N}_o$	$m \in \mathbb{N}_e$	$m \in \mathbb{N}_o$
Case 1	nk	$2nk$		
Case 1′	$n(k-1/2)$	$2n(k-1/2)$		
Case 2			nk	$2nk$
Case 2′			$n(k-1/2)$	$2n(k-1/2)$

All nonconstant eigenvalues are γ-points of the meromorphic Complex CF:

$$\gamma(q) := \frac{\pi q \sin(\pi q)}{\cos(\xi\pi q) - \cos(\pi q)} = \frac{\pi q \sin(\pi q)}{2\sin((1+\xi)\pi q/2)\sin((1-\xi)\pi q/2)}, \tag{1.11$_1$}$$

$$\gamma(q) := \frac{\pi q \sin(\pi q)}{1 - \cos(\xi\pi q)} = \frac{\pi q \sin(\pi q)}{2\sin^2(\xi\pi q/2)}, \tag{1.11$_2$}$$

$$\gamma(q) := \frac{\pi q \cos(\pi q)}{\sin(\pi q) - \sin(\xi\pi q)} = \frac{\pi q \cos(\pi q)}{2\cos((1+\xi)\pi q/2)\sin((1-\xi)\pi q/2)}, \tag{1.11$_{1'}$}$$

$$\gamma(q) := \frac{\pi q \cos(\pi q)}{\sin(\xi\pi q)}. \tag{1.11$_{2'}$}$$

So, we can find the eigenvalues $\lambda = (\pi q)^2$ in two ways: as CEs from (1.10) (only for rational ξ); as nonconstant eigenvalues, using CF (1.11).

For the investigation of CEs as well as for the analysis of complex eigenvalues, zero and pole points of the CF are important.

Lemma 2.4. *Zero points of the function Z are of the first order. These positive zeroes are equal to:*

$$z_k := k, \qquad k \in \mathbb{N}, \tag{$1.12_{1,2}$}$$

$$z_k := (k - 1/2), \quad k \in \mathbb{N}. \tag{$1.12_{1',2'}$}$$

Lemma 2.5. *Points $p_k^{12} = 2k/\xi, k \in \mathbb{N}$, are the second order zeroes of function P_ξ in Case 2 and there are not the first order zeroes in this case. In Cases 1, 1′, 2′ all zeroes of function P_ξ belong to one of the families of the first order zeroes:*

$$p_k^1 := \frac{2k}{1+\xi}, \quad k \in \mathbb{N}, \qquad \text{and} \qquad p_l^2 := \frac{2l}{1-\xi}, \quad l \in \mathbb{N}, \tag{1.13_1}$$

$$p_k^1 := \frac{2(k-1/2)}{1+\xi}, \quad k \in \mathbb{N}, \qquad \text{and} \qquad p_l^2 := \frac{2l}{1-\xi}, \quad l \in \mathbb{N}, \tag{$1.13_{1'}$}$$

$$p_k^1 := \frac{k}{\xi}, \quad k \in \mathbb{N}. \tag{$1.13_{2'}$}$$

If $\xi \notin \mathbb{Q}$, then CE points do not exist, and point $p_k^{12}, k \in \mathbb{N}$, in Case 2 are the second order pole points, $p_k^1, k \in l$, $p_l^2, l \in \mathbb{N}$, are the first order poles of CF.

Proof. Formulae for zeroes of P_ξ points we get as zeroes of denominators in fractions (1.11). It is obvious, that zeros of P_ξ in Case 2 are of the second order only. If $\xi \notin \mathbb{Q}$, then $(1+\xi)/(1-\xi) \notin \mathbb{Q}$. So, the second order zeroes of P_ξ do not exist in the Case 1 and Case 1′. If $\xi \notin \mathbb{Q}$, then all points $p \notin \mathbb{Q}$, where $p = p_k^1, p_l^2$, $k, l \in \mathbb{N}$ and CE points do not exist. So, all zeros of P_ξ are poles of CF. □

In remaining part of this section we present result for $\xi \in \mathbb{Q}$, $\xi = m/n$ and $\gcd(n, m) = 1$, $n, m \in \mathbb{N}$. If $\xi = 0$, then $n = 1$, $m = 0$. For sets of zeros, poles, CE points we use the same notation as in Chapter 1, with agreement $\overline{\mathcal{Z}}_\xi^2 = \varnothing$ in the Case 2′ and ξ instead of $\boldsymbol{\xi}$.

Case 1. Theorems 1.12, 1.13, 1.14 (Chapter 1) are valid with $\xi_1 = \xi = m/n$, $m < n$, $\xi_2 = 1 = 1/1$, $n_+ = n_- = n$, $m_+ = n + m$, $m_- = n - m$.

Lemma 2.6. *If $\xi \in \mathbb{Q}$, then in Case 1 points $q_j = j \in \mathbb{N}_n$, $n - m \in \mathbb{N}_e$ or $q_j = j \in \mathbb{N}_{2n}$, $n - m \in \mathbb{N}_o$, are the first order poles of CF and CE points. Other P_ξ zeroes are the second order poles of CF.*

Proof. If $n+m \in \mathbb{N}_e$, then $n-m \in \mathbb{N}_e$, too, and $d = \gcd(n+m, n-m) = 2$. From Theorem 1.12 we get

$$p_t^{12} = nt, \quad t \in \mathbb{N}. \tag{1.14}$$

If $n+m \in \mathbb{N}_o$, then $n-m \in \mathbb{N}_o$, too, and $\gcd(n+m, n-m) = 1$. So, we have:

$$p_t^{12} = 2nt, \quad t \in \mathbb{N}. \tag{1.15}$$

These points are zeroes of the function $Z(q)$ as well, therefore they are CE point, too. Thus, in Case 1 all the points p_k^1, $k \in \mathbb{N}_{2n}$, or p_l^2, $l \in \mathbb{N}_{2n}$, are poles of the first order. □

Case 2. Theorems 1.12, 1.13, 1.14 (Chapter 1) are valid with $\xi_2 = \xi = m/n$, $m > 0$, $\xi_1 = 0 = 0/1$, $n_+ = n_- = n$, $m_+ = m$, $m_- = m$.

Points $p_k^{12} = k/m$, $k \in \mathbb{N}_{2n}$, are poles of the second order, except the case $k \in \mathbb{N}_{mn}$, $m \in \mathbb{N}_e$, and $k \in \mathbb{N}_{2mn}$, $m \in \mathbb{N}_o$. In this case, we have the first order pole at CE point. There are no poles of the second order for $m = 1$ and $m = 2$.

Case 1′.

Lemma 2.7. *If $\xi \in \mathbb{Q}$, $(n-m)/\gcd(n+m, n-m) \in \mathbb{N}_e$, $n, m \in \mathbb{N}_o$, then in Case 1′ points $p_k^1 = 2n(k-1/2)/(n+m)$, $k \in \mathbb{N}$, coincide with points from the second family $p_l^2 = 2nl/(n-m)$, $l \in \mathbb{N}$, at the points:*

$$p_t^{12} = 2n\left(\frac{n-m}{\gcd(n+m, n-m)}t + l_0\right)/(n-m), \qquad t \in \mathbb{N}, \tag{1.16}$$

where (k_0, l_0) are any solution of the equation:

$$\frac{n-m}{\gcd(n+m, n-m)}k - \frac{n+m}{\gcd(n+m, n-m)}l - \frac{n-m}{2\gcd(n+m, n-m)} = 0, \quad k, l \in \mathbb{N},$$

such that $-\frac{n-m}{\gcd(n+m, n-m)} < l_0 \leqslant 0$.

Proof. If a point from the first family $p_k^1 = 2n(k-1/2)/(n+m)$, $k \in \mathbb{N}$, coincides with a point from the second family $p_l^2 = 2nl/(n-m)$, $l \in \mathbb{N}$, then

$$\frac{n(2k-1)}{n+m} = \frac{2nl}{n-m}, \quad k, l \in \mathbb{N},$$

or

$$(2k-1)(n-m) = 2l(n+m), \quad k, l \in \mathbb{N}. \tag{1.17}$$

From equality (1.17) we obtain that $n-m \in \mathbb{N}_e$, i.e., $n, m \in \mathbb{N}_o$. The equation (1.17) can be rewritten in another form:

$$(n-m)k-(n+m)l-\frac{n-m}{2}=0. \tag{1.18}$$

For numbers $n-m \in \mathbb{N}_e$ and $n+m \in \mathbb{N}_e$ we find $d=\gcd(n-m, n+m) \geqslant 2$. Then $n-m=da,\ n+m=db$, where $\gcd(a,b)=1$, $0<a,b \in \mathbb{N}$. So, we have equation

$$ak-bl-\frac{a}{2}=0, \quad k,l \in \mathbb{N}. \tag{1.19}$$

If $a \notin \mathbb{N}_e$ then the equation (1.19) has no solution for $k,l \in \mathbb{N}$. If $a \in \mathbb{N}_e$, then $b \in \mathbb{N}_o$ and from Theorem 1.4 in Chapter 1 we obtain that for equation (1.19) exist solutions and all solutions have a form:

$$k=k_0+bt, \qquad l=l_0+at, \quad t \in \mathbb{N}, \tag{1.20}$$

where (k_0,l_0) is any solution of the equation (1.19). For fixed ξ value we always can find (k_0,l_0), but we select only such (k_0,l_0) that $-a<l_0 \leqslant 0$. Then the second order zeroes point are defined by the formula:

$$p_t^{12}=\frac{2n}{d}t+\frac{2nl_0}{n-m}, \quad t \in \mathbb{N}. \tag{1.21}$$

□

Example 2.8. $\xi=1/3$. *Then* $m,n \in \mathbb{N}_o$, $d=\gcd(n-m,n+m)=\gcd(2,4)=2$, $(n-m)/d=2/2=1 \notin \mathbb{N}_e$ *and the equation* (1.19) *is*

$$k-2l-\frac{1}{2}=0, \qquad k,l \in \mathbb{N}. \tag{1.22}$$

This equation has no solution for $k,l \in \mathbb{N}$.

$\xi=1/5$, $m,n \in \mathbb{N}_o$. *Then* $d=\gcd(4,6)=2$ *and* $(n-m)/d=4/2=2 \in \mathbb{N}_e$. *The equation*

$$ak-bl-\frac{a}{2}=2k-3l-1=0, \qquad k,l \in \mathbb{N} \tag{1.23}$$

has solutions and one of the solution is the point $[-1,-1]$. *So, another solution has a form* $k=-1+3t$ *and* $l=-1+2t$, $t \in \mathbb{N}$. *Finally, for* $\xi=m/n=1/5$

$$p_t^{12}=\frac{5(2t-1)}{2}, \qquad t \in \mathbb{N}.$$

Lemma 2.9. *If $\xi = m/n \in \mathbb{Q}$, and $(n-m)/\gcd(2n, n+m) \in \mathbb{N}_e$, $n, m \in \mathbb{N}_o$, then in Case 1′ the points $p_k^1 = 2n(k-1/2)/(n+m)$, $k \in \mathbb{N}$, coincide with zeroes points $z_l = l - 1/2$, $l \in \mathbb{N}$, and we have CE points*

$$c_t^1 = l_0 + \frac{2nt}{\gcd(2n, n+m)} - \frac{1}{2}, \qquad t \in \mathbb{N}, \tag{1.24}$$

where (k_0, l_0) is any solution of the equation

$$\frac{2n}{\gcd(2n, n+m)} l - \frac{n+m}{\gcd(2n, n+m)} k - \frac{n-m}{2\gcd(2n, n+m)} = 0, \quad k, l \in \mathbb{N},$$

such that $-\frac{2n}{\gcd(2n,n+m)} < l_0 - 1/2 \leqslant 0$.

Proof. CE points of the first family appear if point of the first family $p_k^1 = 2(k - 1/2)/(1+\xi)$, $k \in \mathbb{N}$, coincides with zero $z_l = l - 1/2$, $l \in \mathbb{N}$, i.e.:

$$\frac{2n(k-1/2)}{n+m} = l - 1/2, \quad k, l \in \mathbb{N},$$

or

$$2n(2k-1) = (n+m)(2l-1), \quad k, l \in \mathbb{N}. \tag{1.25}$$

From the equation above we have that this equality is valid if $n+m \in \mathbb{N}_e$, i.e., $n, m \in \mathbb{N}_o$. This equation can be rewritten in the other form:

$$2nk - (n+m)l - \frac{n-m}{2} = 0, \quad k, l \in \mathbb{N},$$

or

$$ak - bl - \frac{a-b}{2} = 0, \quad k, l \in \mathbb{N}, \tag{1.26}$$

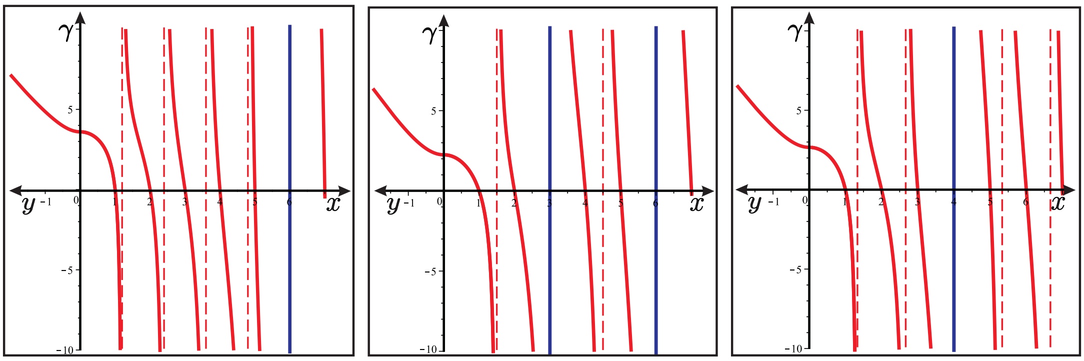
where $a = 2n/\gcd(2n, n+m)$, $b = (n+m)/\gcd(2n, n+m)$. If $a - b \notin \mathbb{N}_e$, then the equation (1.26) has no solution for $k, l \in \mathbb{N}$. Otherwise, all solutions of the equation (1.26) have a form:

$$k = k_0 + bt, \qquad l = l_0 + at, \quad t \in \mathbb{N}, \tag{1.27}$$

where (k_0, l_0) is any solution of the equation (1.19) (see Theorem 1.4 in Chapter 1). For SLP (1.1), (1.4), (1.3_1) all poles and zeros are positive, so we choose (k_0, l_0) only such that $1/2 - a < l_0 \leqslant 1/2$. Then the formula:

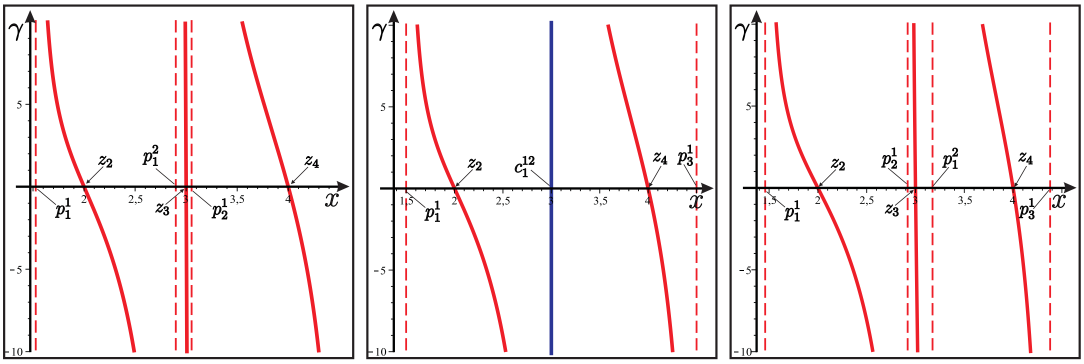
$$c_t^1 = l_0 + \frac{2n}{\gcd(2n, n+m)} t - \frac{1}{2}, \quad t \in \mathbb{N},$$

describe the CE points (the first family). □



(a) $\xi = 2/3$ (b) $\xi = 1/3$ (c) $\xi = 1/2$

Fig. 2.1. Real CF $\gamma(x)$ for various ξ in Case 1.



(a) $\xi = 0.31$ (b) $\xi_c = 1/3$ (c) $\xi = 0.37$

Fig. 2.2. Real CF $\gamma(x)$ in the neighborhood of the CE point in Case 1.

Lemma 2.10. *If $\xi = m/n \in \mathbb{Q}$, and $(n-m)/\gcd(2n, n-m) \in \mathbb{N}_e$, $n, m \in \mathbb{N}_o$, then in Case 1′ the points $p_k^2 = 2k/(n-m)$, $k \in \mathbb{N}$, coincide with points $z_l = l - 1/2$, $l \in \mathbb{N}$, at the CE points points*

$$c_t^2 = l_0 + \frac{2nt}{\gcd(2n, n-m)} - \frac{1}{2}, \quad t \in \mathbb{N}, \tag{1.28}$$

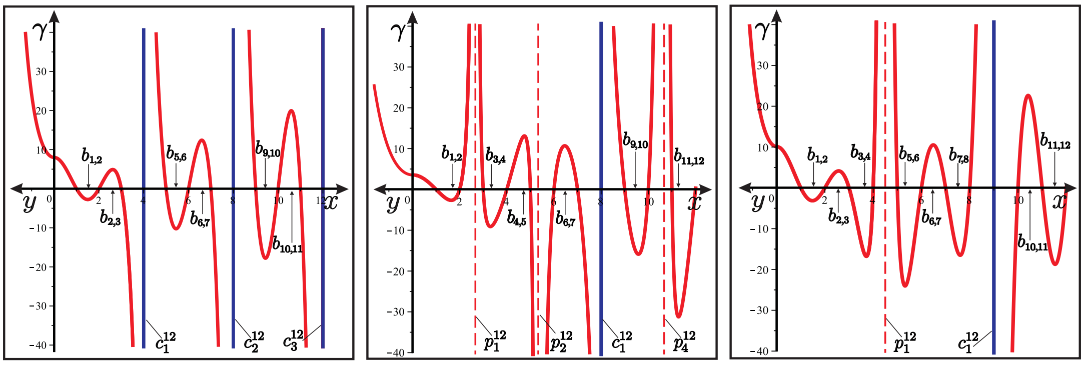
where (k_0, l_0) is any solution of the equation

$$\frac{2n}{\gcd(2n, n-m)} l - \frac{n-m}{\gcd(2n, n-m)} k - \frac{n-m}{2\gcd(2n, n-m)} = 0, \quad k, l \in \mathbb{N},$$

such that $-\frac{2n}{\gcd(2n, n-m)} < l_0 - 1/2 \leqslant 0$.

Proof. The proof is analogous as proof of Lemma 2.9. □

Case 2′. Points $p_k^1 = k/m$, $k \in \mathbb{N}_n$, are CE points if $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$, otherwise points p_k^1 are the first order poles.



(a) $\xi = 1/2$ (b) $\xi = 3/4$ (c) $\xi = 4/9$

Fig. 2.3. Real CF $\gamma(x)$ for various ξ in Case 2.

2 Real eigenvalues of the Sturm–Liouville problem

We have Real CF if $q \in \mathbb{R}_q := \mathbb{R}_q^- \cup \mathbb{R}_q^0 \cup \mathbb{R}_q^+$. $\mathbb{R}_q^- := \{q = \imath y, y > 0\} = \{q = -\imath x, x < 0\}$ and we use argument $x \in \mathbb{R}$ for Real CF. So, in this case, for CF (1.11) the Real CF are:

$$\gamma(x) := \begin{cases} \dfrac{\pi x \sinh(\pi x)}{2\sinh((1+\xi)\pi x/2)\sinh((1-\xi)\pi x/2)}, & x \leqslant 0, \\ \dfrac{\pi x \sin(\pi x)}{2\sin((1+\xi)\pi x/2)\sin((1-\xi)\pi x/2)}, & x \geqslant 0, \end{cases} \tag{2.1$_1$}$$

$$\gamma(x) := \begin{cases} \dfrac{\pi x \sinh(\pi x)}{2\sinh^2(\xi\pi x/2)}, & x \leqslant 0, \\ \dfrac{\pi x \sin(\pi x)}{2\sin^2(\xi\pi x/2)}, & x \geqslant 0, \end{cases} \tag{2.1$_2$}$$

$$\gamma(x) := \begin{cases} \dfrac{\pi x \cosh(\pi x)}{2\cosh((1+\xi)\pi x/2)\sinh((1-\xi)\pi x/2)}, & x \leqslant 0, \\ \dfrac{\pi x \cos(\pi x)}{2\cos((1+\xi)\pi x/2)\sin((1-\xi)\pi x/2)}, & x \geqslant 0, \end{cases} \tag{2.1$_{1'}$}$$

$$\gamma(x) := \begin{cases} \dfrac{\pi x \cosh(\pi x)}{\sinh(\xi\pi x)}, & x \leqslant 0, \\ \dfrac{\pi x \cos(\pi x)}{\sin(\xi\pi x)}, & x \geqslant 0. \end{cases} \tag{2.1$_{2'}$}$$

Those functions are useful for the investigation of real negative, zero, and positive eigenvalues. The graphs of these Real CF for some parameter ξ values are presented in Figure 2.1, Figure 2.2, Figure 2.3 and Figure 2.7. More properties of the Real CF and real spectrum for same cases are investigated in [48, 49, Pečiulytė *et al.* 2005, 2008].

2.1 The spectra in Cases 1, $1'$

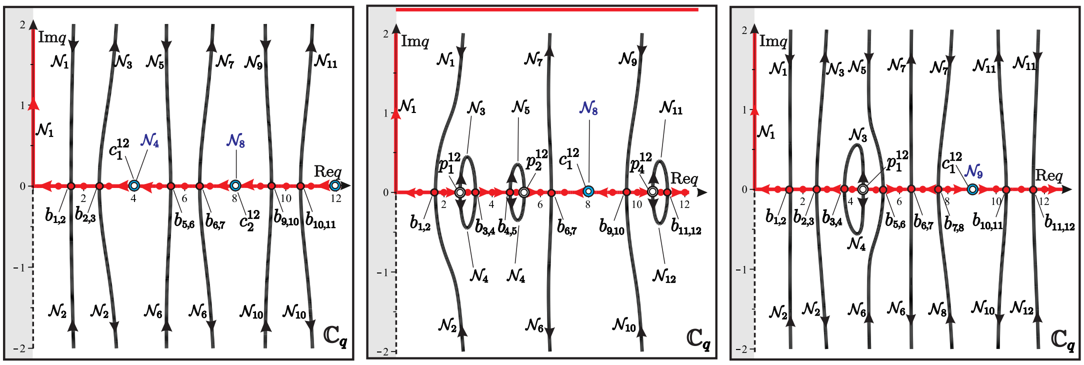
The spectra for problems (1.1), (1.2), (1.3_1) and (1.1), (1.4), (1.3_1) lie on the real axis as shown in papers [48, 49, Pečiulytė *et al.* 2005, 2008].

The function $\gamma(x)$ is a monotone decreasing function in each interval (α, β), where α and β are the points of the first order poles. Real CF for $\xi = 2/3$, $1/3$, $1/2$ we can see in Figure 2.1. For example, if $\xi = \xi_c = 1/3$, then $x = 3$ is the first order pole for CF and CE point. If we take $\xi < \xi_c$ or $\xi > \xi_c$, then we have two first order poles near zero point $x = 3$. Such a situation is shown in Figure 2.2. We have the same situation with the spectrum in Case $1'$. So, if the poles p_2^1 and p_1^2 move toward the zero point z_3, then a part of the graph of the CF, that was in (p_1^2, p_2^1), becomes a vertical line, i.e., we have a CE point $c_1^{12} = p_2^1 = p_1^2 = z_3$ for $\xi = \xi_c$. For $\xi > \xi_c$ we have the interval (p_2^1, p_1^2), i.e., the poles change places with each other.

3 Complex eigenvalues of the Sturm–Liouville problem

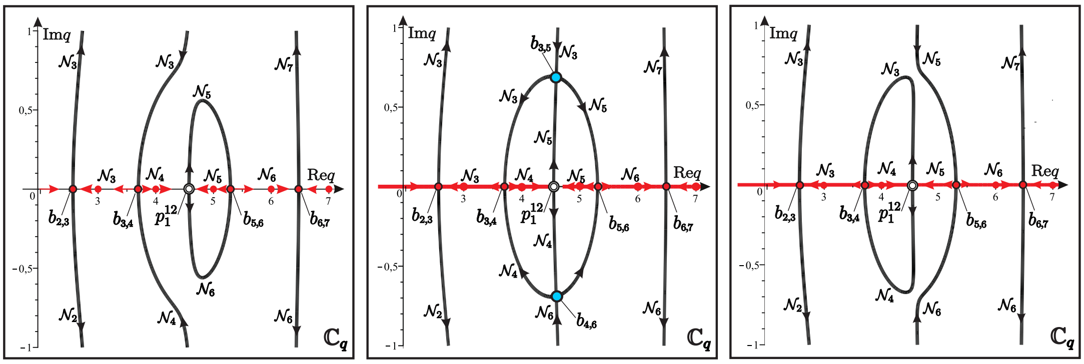
In the recent scientific literature there are many papers, in which real eigenvalues of the SLP are analyzed. However, a complex spectrum of this problem is considerably less investigated [68, Sapagovas and Štikonas 2005], [86, Štikonas and Štikonienė 2009].

It is important to investigate complex eigenvalues of the SLPs (1.1)–(1.3) and (1.1), (1.4), (1.3) with $\gamma \in \mathbb{R}$. The poles of the function $\gamma(q)$ are eigenvalues of the problems (1.1)–(1.3) and (1.1), (1.4), (1.3) in the case $\gamma = \infty$. All zeros and poles of the meromorphic function $\gamma(q)$ are on the positive part of the real axis. From (1.11) and from the properties of sine and cosine functions we obtain that all zeros of this function are real numbers $q = k \in \mathbb{N}$ in Cases 1, 2 and $q = (k - 1/2), k \in \mathbb{N}$ in Cases $1'$,$2'$. So, only positive zeroes and poles exist in $\mathbb{R}_q^+ \subset \mathbb{C}_q$.

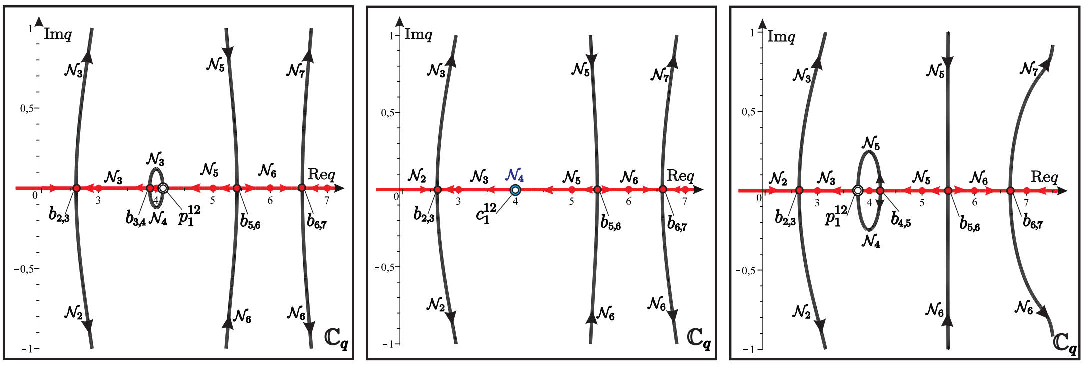


(a) $\xi = 1/2$ (b) $\xi = 3/4$ (c) $\xi = 4/9$

Fig. 2.4. Spectrum Curves for various ξ in Case 2.



(a) $\xi = 0.437$ (b) $\xi = \xi_k = 0.43963\ldots$ (c) $\xi = 0.44$



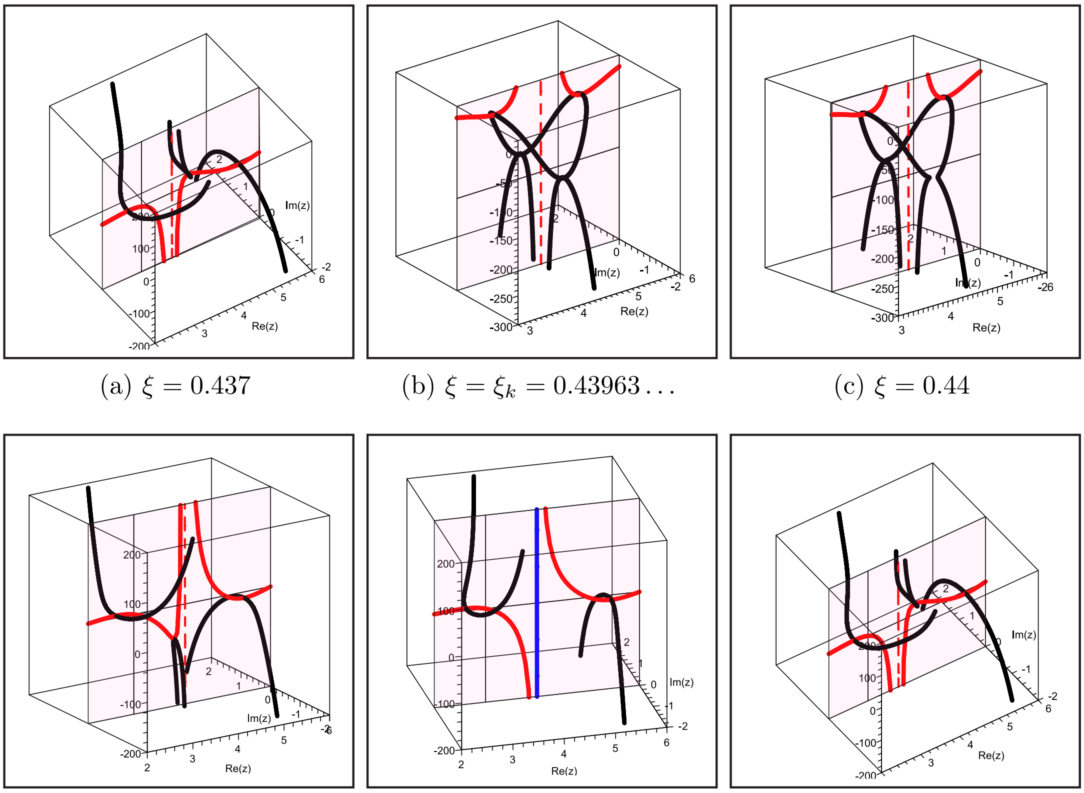
(d) $\xi = 0.485$ (e) $\xi = \xi_c = 0.5$ (f) $\xi = 0.53$

Fig. 2.5. Spectrum Curves for various ξ in Case 2.

3.1 Dynamics of Spectrum Curves in Case 2

In this case, the spectrum of complex eigenvalues is more complicated. By changing the value of the parameter ξ we get various types of the Spectrum Domain $\mathcal{N}$ and Spectrum Curves.

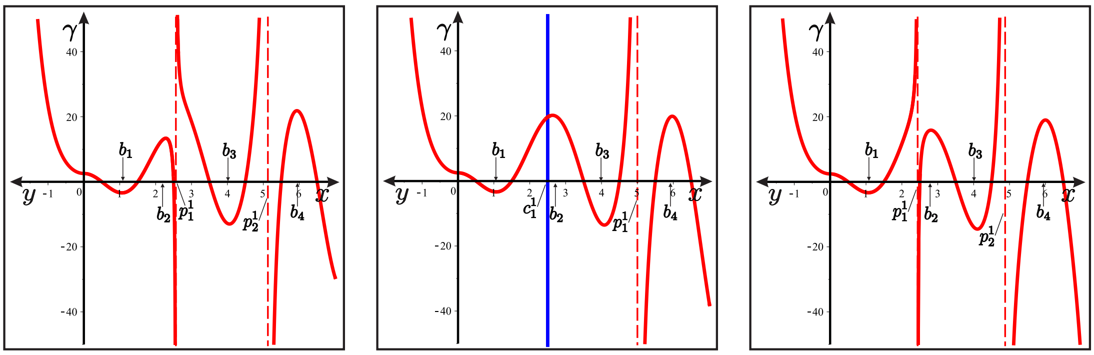
A qualitative view of dependence of Spectrum Curves on the parameter ξ can



(a) $\xi = 0.437$ (b) $\xi = \xi_k = 0.43963\ldots$ (c) $\xi = 0.44$

(d) $\xi = 0.485$ (e) $\xi = \xi_c = 0.5$ (f) $\xi = 0.53$

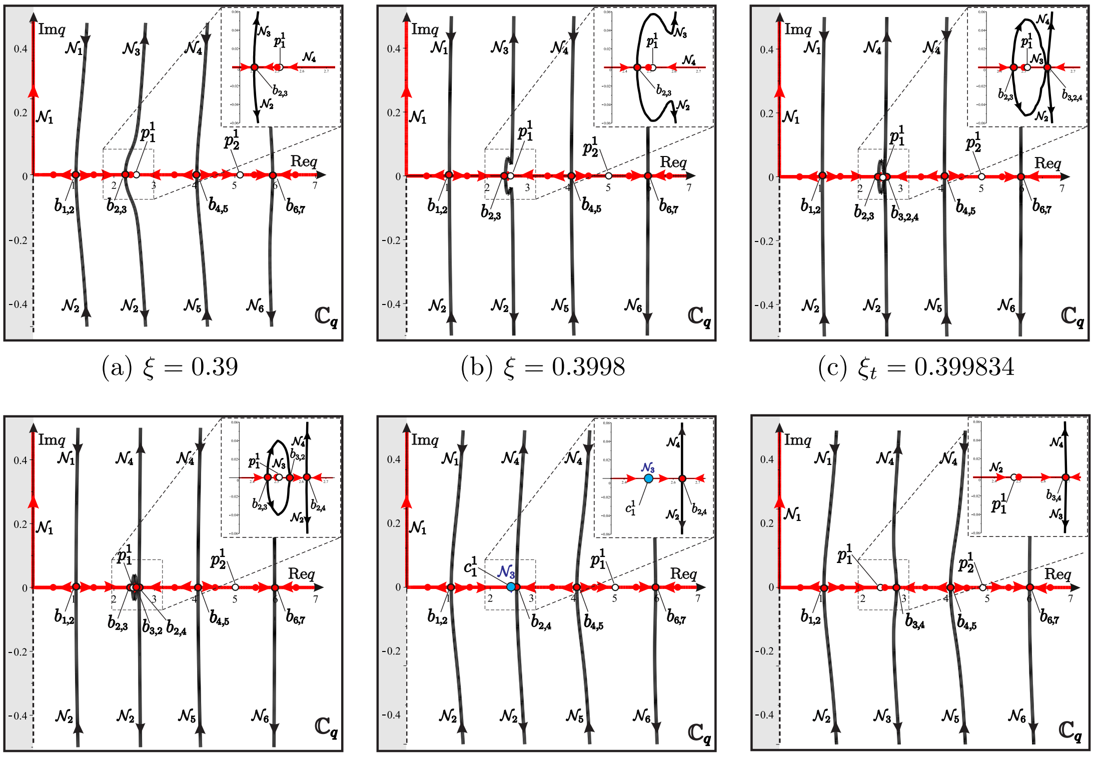
Fig. 2.6. Complex-Real CF $\gamma(q)$ for various ξ in Case 2, 3D-view.



(a) $\xi = 0.39$ (b) $\xi = 0.4$ (c) $\xi = 0.41$

Fig. 2.7. Real CF $\gamma(x)$ for various ξ in Case $2'$.

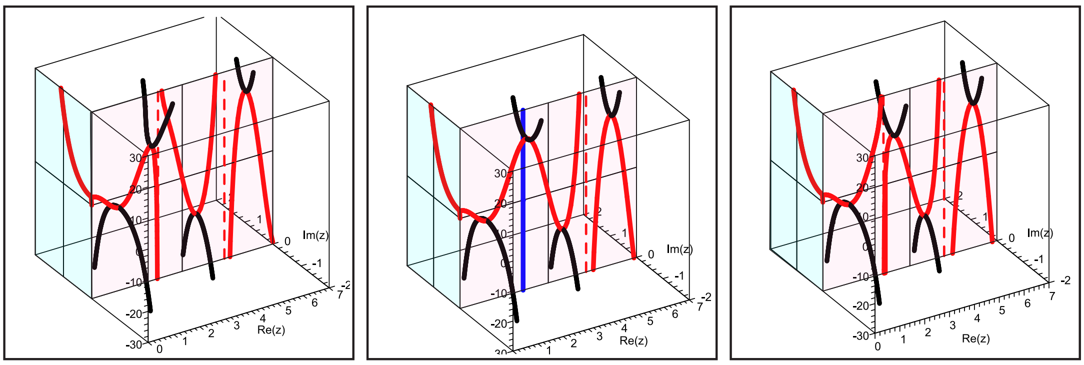
be seen in Figure 2.4. In Case 2, there are two types of bifurcation. The first type consists of bifurcation where two different complex curves join at the first order Complex Critical point. We get the second type by changing the value of the parameter ξ, so that zero and second order pole points of the CF become coincident with the Critical points (in which CEs exist) and the loop type curves disappear.



(a) $\xi = 0.39$ (b) $\xi = 0.3998$ (c) $\xi_t = 0.399834$

(d) $\xi = 0.3999$ (e) $\xi = \xi_c = 0.4$ (f) $\xi = 0.41$

Fig. 2.8. Spectrum Curves for various ξ in Case 2′.



(a) $\xi = 0.39$ (b) $\xi = \xi_c = 0.4$ (c) $\xi = 0.41$

Fig. 2.9. Complex-Real CF $\gamma(q)$ for various ξ in Case 2′.

Figure 2.5 shows how the Spectrum Curves are changing depending on the parameter ξ value near to $\xi_k = 0.43963\ldots$ (we call it the first order Complex Critical point in the complex part of $\mathbb{C}_q$) and ξ_c (CE point) points. There the Spectrum Curves make a loop. In this example, the value of ξ is increasing from 0.437 to 0.53. When $\xi \lesssim \xi_k$, Spectrum Curves $\mathcal{N}_3$ and $\mathcal{N}_5$ (or $\mathcal{N}_4$ and $\mathcal{N}_6$) become close, and when $\xi = \xi_k$, those different Spectrum Curves join each other at the

Critical point $b_{3,5}$ (and $b_{4,6}$) (see Figure 2.5(a)–(c)). Next, when $\xi \gtrsim \xi_k$, the loop type curve is on the left side. The order of the poles does not change in this bifurcation. Zero is inside the loop. As $\xi \in (\xi_k; \xi_c)$, the loop tightens and intersects the real axis at the second order pole and Critical points. When zero and pole consist with the Critical point, we have CE point. At $\xi = \xi_c$ bifurcation is "symmetrical". We can see 3D view of Complex-Real CF in Figure 2.6.

3.2 Dynamics of Spectrum Curves in Case 2′

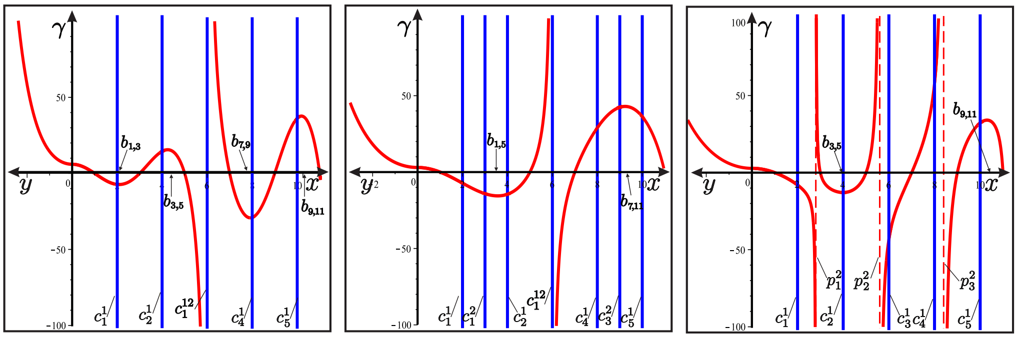
The complex spectrum in Case 2′ is also quite complicated for same parameters ξ values. In this case, index of Spectrum Curve $\mathcal{N}_k$ corresponds to classical case, i.e., $q = k - 1/2 \in \mathcal{N}_k$.

In Figure 2.8 it is shown how the spectrum of complex eigenvalues is approaching to the constant eigenvalue point ξ_c. If $\xi < \xi_c = 2/5$, the pole moves toward zero from right side and the first order Critical point moves toward zero from the left. When the pole is approaching to zero, the Spectrum Curves $\mathcal{N}_2$ and $\mathcal{N}_3$ bend (make a loop) and intersect in the second order Critical point $b_{3,2,4}$ when $\xi = \xi_t$. If ξ is growing, then the pole moves to the left and the second order Critical point $b_{3,2,4}$ is divided into two first order Critical points $b_{3,2}$ and $b_{2,4}$ (loop type complex curve retire from the other complex curve). Loop type complex curve consist of two Spectrum Curves ($\mathcal{N}_2$ and $\mathcal{N}_3$) and two first order Critical points ($b_{2,3}$ and $b_{3,2}$). Inside this loop exists first order pole and zero. If the ξ value is increasing from 0.3999 to 0.4, the loop is shrinking and when $\xi = \xi_c = 0.4$, pole, zero and two first order Critical point meet, we have a CE point. If ξ is growing, the pole moves to the left from zero point. We can see 3D view of Complex-Real CF in Figure 2.9, too.

4 Sturm–Liouville problem with symmetric interval in the integral

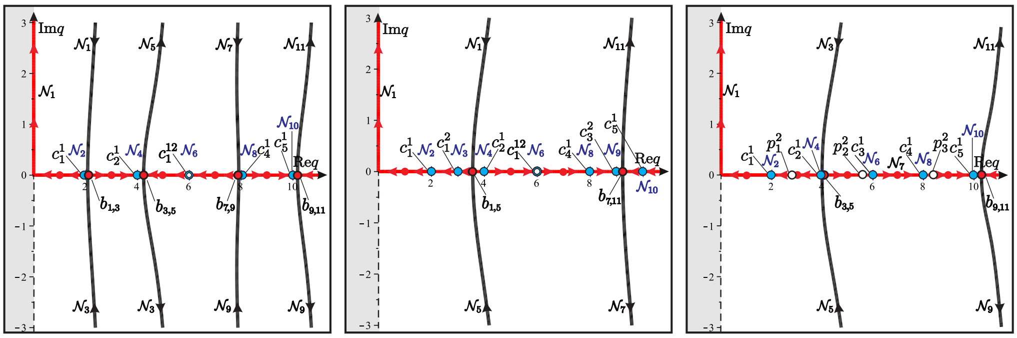
Let us consider the SLP with one classical BC:

$$-u'' = \lambda u, \qquad u(0) = 0 \quad t \in (0, 1), \tag{4.1}$$



(a) $\xi = 1/3$ (b) $\xi = 1/6$ (c) $\xi = 1/7$

Fig. 2.10. Real CF for problem (4.1)–(4.2).



(a) $\xi = 1/3$ (b) $\xi = 1/6$ (c) $\xi = 1/7$

Fig. 2.11. Spectrum Curves for problem (4.1)–(4.2).

and other integral NBC:

$$u(1) = \gamma \int_{\xi}^{1-\xi} u(t)dt, \tag{4.2}$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in (0, 1/2)$. If $\gamma = 0$ or $\xi = 1/2$, then we obtain a problem with the classical BC. The case $\xi = 0$ we analyse in the previous section.

We get the eigenvalue $\lambda = 0$ of problem (4.1)–(4.2) if and only if $\gamma = \frac{2}{1-2\xi}$. Solutions of problem (4.1) are $u = c\sin(\pi qt)/(\pi q)$, $q \in \mathbb{C}_q$. Substituting this solution into the second BC we derive the condition for existence of nontrivial solution:

$$\frac{\sin(\pi q)}{\pi q} = \gamma \frac{2\sin(\pi q/2)\sin(\pi q(1-\xi)/2)}{(\pi q)^2}$$

CE points q of the problem (4.1)–(4.2) can be defined as a roots of the equation and the system:

$$\sin(\frac{\pi q}{2}) = 0; \qquad \begin{cases} \cos(\frac{\pi q}{2}) = 0, \\ \sin(\frac{\pi q(1-2\xi)}{2}) = 0. \end{cases} \tag{4.3}$$

CE, existing for all $\xi \in (0, 1/2)$, are called the *first type* CEs. All other CEs are called the *second type* CE.

Lemma 2.11. *The first type CE are $\lambda_k = (\pi c_k^1)^2$, $c_k^1 = 2k$, $k \in \mathbb{N}$. The second type CE exist only for $\xi = \frac{m}{n} \in \mathbb{Q}$, $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$, $n/2 \in \mathbb{N}_o$, $\gcd(n/2, m) = 1$, and these eigenvalue are equal to $\lambda_k = (\pi c_k^2)^2$, $c_k^2 = \frac{n}{2}(2k+1)$, $k \in \mathbb{N}$.*

Proof. The first part of this lemma is obvious. For the second type CE we solve equation

$$\frac{2}{1-2\xi} l = 2k-1, \quad l, k \in \mathbb{N}. \tag{4.4}$$

If $\xi \notin \mathbb{Q}$, then CEs do not exist. If $\xi = \frac{m}{n} \in \mathbb{Q}$, then we have equation

$$\frac{nl}{\gcd(n, n-2m)} - \frac{n-2m}{\gcd(n, n-2m)} k + \frac{n-2m}{2\gcd(n, n-2m)} = 0, \quad l, k \in \mathbb{N}. \tag{4.5}$$

For $n \in \mathbb{N}_o$, $\gcd(n, n-2m) = 1$ and equation (4.5) has not solution. For $n \in \mathbb{N}_e$, $\gcd(n, n-2m) = 2$ $(\gcd(n/2, m) = 1)$. Equation

$$\frac{n}{2} l - \frac{n-2m}{2} k + \frac{\frac{n}{2} - m}{2} = 0, \quad l, k \in \mathbb{N}, \tag{4.6}$$

has solution if $n/2 \in \mathbb{N}_o$, $m \in \mathbb{N}_o$. For such n and m we solve equation

$$\frac{n}{2} l = \frac{n-2m}{4} \tilde{k}, \quad l \in \mathbb{N}, \tilde{k} \in \mathbb{N}_o, \tag{4.7}$$

and get $\tilde{k} = \frac{n}{2} t$, $t \in \mathbb{N}$. But $\tilde{k} \in \mathbb{N}_o$. So, the second type CE point are

$$c_k^2 = \frac{n}{2}(2k+1), \quad k \in \mathbb{N}. \tag{4.8}$$

□

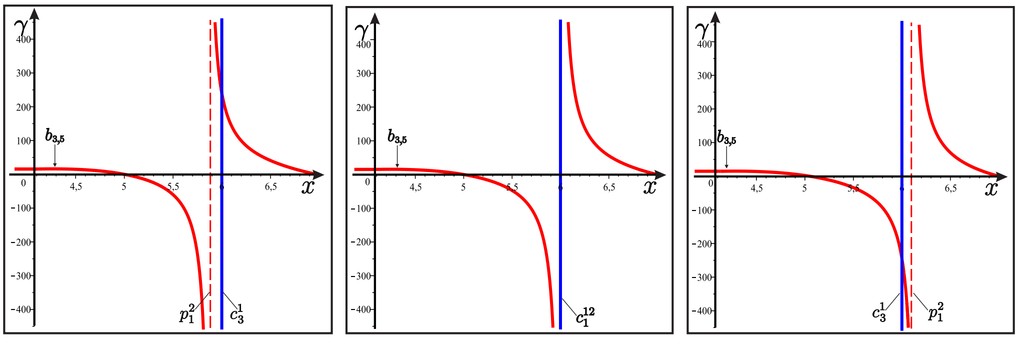
All Nonconstant Eigenvalues points are γ-points of the meromorphic function (Complex-Real CF):

$$\gamma(q) := \frac{Z(q)}{P_\xi(q)}. \tag{4.9}$$

In this case, $Z(q) := \cos(\pi q/2)$, $P_\xi(q) := \sin\left(\pi q(1-2\xi)/2\right)/(\pi q)$. Real CF is:

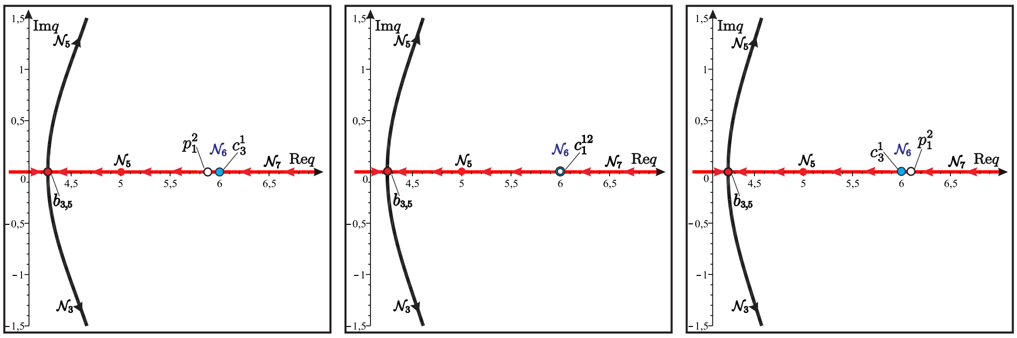
$$\gamma(x) := \begin{cases} \frac{\pi x \cosh(\pi x/2)}{\sinh\left(\frac{\pi x(1-2\xi)}{2}\right)}, & x < 0; \\ \frac{\pi x \cos(\pi x/2)}{\sin\left(\frac{\pi x(1-2\xi)}{2}\right)}, & x \geqslant 0. \end{cases} \tag{4.10}$$

This function is useful for the investigation of zero, real negative, real positive, and complex eigenvalues. The graphs of this Real CF for some values of the



(a) $\xi = 0.33$ (b) $\xi = 1/3$ (c) $\xi = 0.336$

Fig. 2.12. Real CF in the neighborhood of the first type CE point.



(a) $\xi = 0.33$ (b) $\xi = 1/3$ (c) $\xi = 0.336$

Fig. 2.13. Spectrum Curves in the neighborhood of the first type CE point.

parameter ξ are presented in Figure 2.10. In this case, there exists only one negative eigenvalue as $\gamma > \frac{2}{1-2\xi}$.

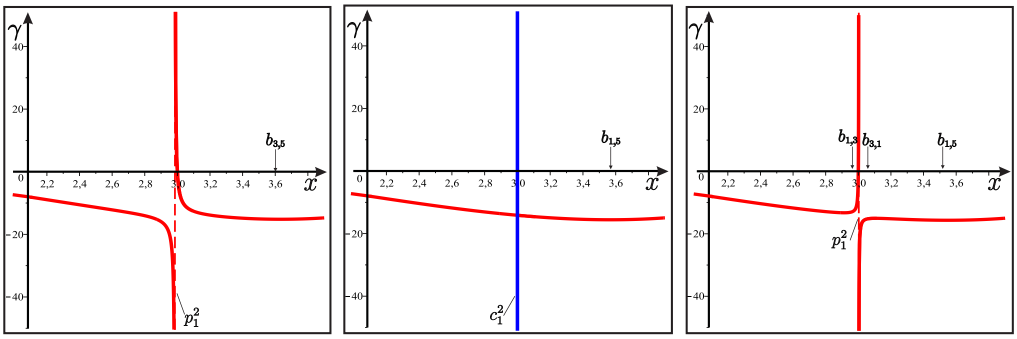
Zero points of the function $Z(q)$ are of the first order and they are equal to $z_l := 2l - 1, l \in \mathbb{N}$. Points

$$p_k^1 = \frac{2k}{1-2\xi}, \quad k \in \mathbb{N}, \tag{4.11}$$

are the first order zeroes of function $P_\xi(q)$. If $\xi \notin \mathbb{Q}$, then all p_k^1, $k \in \mathbb{N}$, are the first order poles.

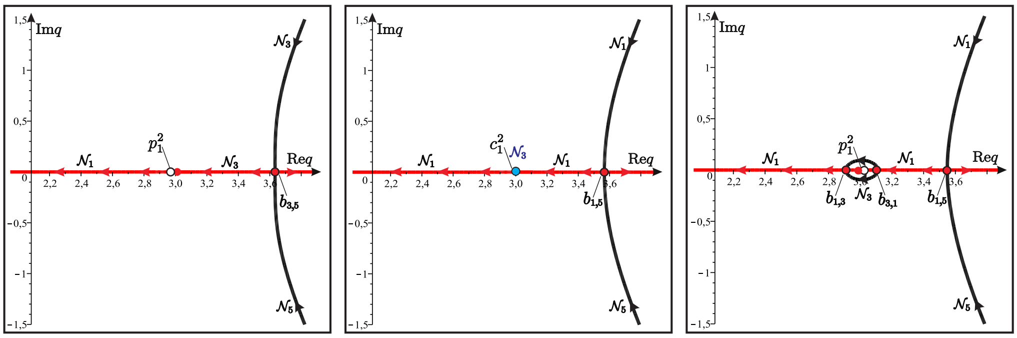
Lemma 2.12. *If $\xi = m/n \in \mathbb{Q}$, then points p_l^1, $l = \frac{(n-2m)t}{\gcd(n,n-2m)}$, $t \in \mathbb{N}$ are the first order poles at the first type CE points. If $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$, $n/2 \in \mathbb{N}_o$, $\gcd(n/2, m) = 1$, then points p_l^1, $l = \frac{n-2m}{4}(2t+1)$, $t \in \mathbb{N}$, are the second type CE points. Otherwise, p_l^1 are the first order poles.*

Proof. Proof follows from the Proof of Lemma 2.11. □



(a) $\xi = 0.165$ (b) $\xi = 0.1(6)$ (c) $\xi = 0.167$

Fig. 2.14. Real CF in the neighborhood of the second type CE point.



(a) $\xi = 0.165$ (b) $\xi = 0.1(6)$ (c) $\xi = 0.167$

Fig. 2.15. Spectrum Curves in the neighborhood of the second type CE point.

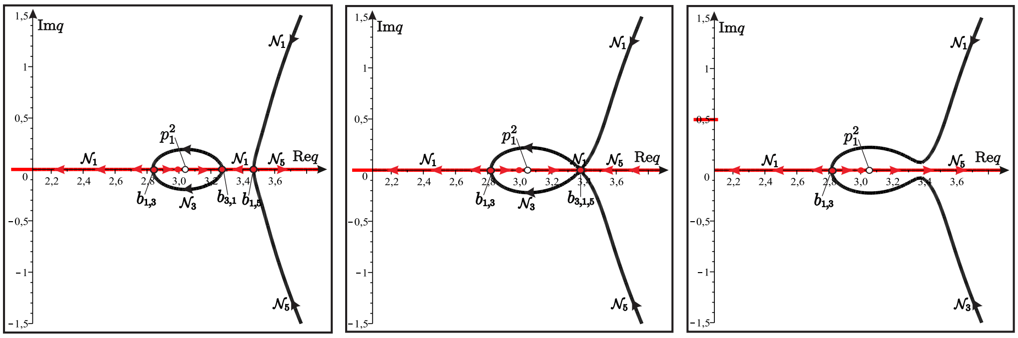
If we change the value of ξ from 0 to 1/2, then the poles (4.10) are moving from the left to the right side of CE point. Figure 2.12 and Figure 2.13 shows a qualitative view of the Real CF $\gamma(q)$ in the neighborhood of the first type CE point.

In Figure 2.14 and Figure 2.15, we see the view of a real part of the spectrum when the pole point moves towards the zero point. When the pole point is coincident with the zero point, we obtain a CE point.

4.1 Complex eigenvalues

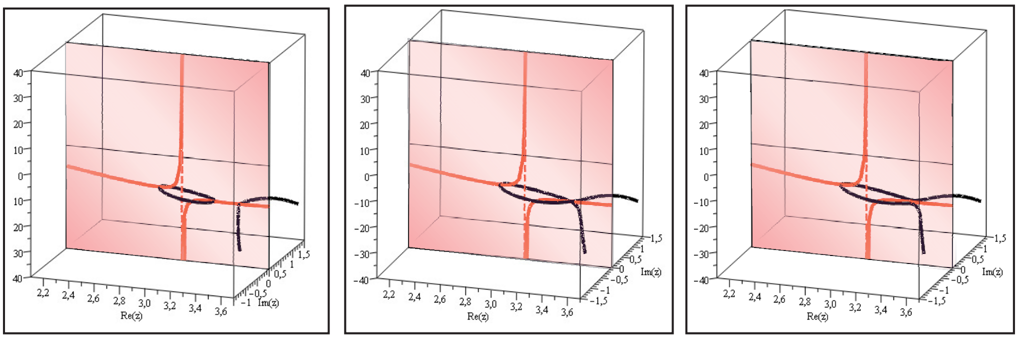
A complex spectrum of similar problems is investigated in [48, Pečiulytė *et al.*], [77, Skučaitė *et al.*]. We can see a qualitative view of dependence of a complex part of the spectrum on some values of the parameter ξ in Figure 2.11.

The view of the complex part of spectrum, where the pole moves towards and over the CE point is shown in Figure 2.13.



(a) $\xi = 0.1682$ (b) $\xi = 0.168(3)$ (c) $\xi = 0.1684$

Fig. 2.16. Spectrum Curves in the neighborhood of the Critical point of the second order.



(a) $\xi = 0.1682$ (b) $\xi = 0.168(3)$ (c) $\xi = 0.1684$

Fig. 2.17. Complex-Real CF in the neighborhood of Critical point of the second order.

Figure 2.15 and Figure 2.16 shows how the Spectrum Curves dependent on the values of the parameter ξ. If the value of ξ is increasing, the pole point moves towards and over the zero point and then there appears a loop type curve, which consists of Spectrum Curves ($\mathcal{N}_1$ and $\mathcal{N}_3$) and two first order Critical points ($b_{1,3}$ and $b_{3,1}$). While ξ value is increasing, the *loop* grows too and when $\xi = 0.168(3)$ this loop type curve joins the other complex curve in the second order Critical point $b_{3,1,5}$. This loop type curve disappears, as $\xi > 0.168(3)$ (see, Figure 2.16). The 3D view of this situation is shown in Figure 2.17.

5 Conclusions

In this chapter the complex spectrum of the SLP with the classical or first type BC on the left side of the interval and integral NBC of two types on the right side of the interval was analyzed.

- All eigenvalues of the SLP in Case 1 and Case $1'$ are real. Complex eigenvalues do not exist for all ξ.

- In Case 2 there are two types of bifurcation: two different Complex Spectrum Curves join at the Complex Critical point; a loop type curve disappears, when the zero and pole points of the CF become coincident with the Critical point, i.e., points, at which CE exists.

- In Case $2'$ exists the second order (Real) Critical point. This point appears when two Spectrum Curves intersect. Later this point splits up into two the first order Critical points. Also, in this case two types of bifurcation exist: two different Spectrum Curves intersect in the second order Critical point; two first order Critical point coincide with zero and pole at the CE point.

Also SLP with the symmetric interval in the integral was investigated. Complex eigenvalues for differential problem (4.1)–(4.2) always exist and for some values of ξ the complex part of the spectrum is quite complicated:

- If the pole moves towards and over the CE point, complex eigenvalues do not appear.

- If the pole moves towards and over the zero point, then the loop type complex curve appears at the second order Critical point. This loop type complex curve disappears when it joins other complex curve at the second order Critical point.

Chapter 3

Discrete Sturm–Liouville problem

SLP is very important for investigation of the existence and the uniqueness of the solutions for classical stationary problems. Such problems are complicated, not self-adjoint and spectrum for such problems may be not positive (or real). Using the CF we analyze spectrum of the nonlocal SLP. In [86, Štikonas and Štikonienė 2009], the CF method for investigation of the spectrum for such problems was used.

1 Introduction

In this chapter some new results on a spectrum in a complex plane for the discrete Sturm–Liouville problem (dSLP) are presented. The definition of CE points and the CF is introduced for the discrete SLP. The method of the CF is used for the analysis of complex eigenvalues and qualitative behaviour (dynamics) of Spectrum Curves.

Discrete SLP:

$$
\begin{gathered}
-\frac{U_{i-1}-2U_i+U_{i+1}}{h^2}=\lambda U_i,\quad i=1\dots N-1,\\
\frac{U_0+U_N}{2}+\sum_{i=1}^{N-1}U_i=0,\qquad \frac{U_N}{2}+\sum_{i=1}^{N-1}U_i ih=0,
\end{gathered}
$$

corresponding to differential SLP with NBCs:

$$
\int_0^1 u(t)\,\mathrm{d}t=0,\quad \int_0^1 tu(t)\,\mathrm{d}t=0
$$

was analysed by Jachimavičienė, Jesevičiūtė and Sapagovas [32, 2009]. These results are part of doctoral dissertation of Jesevičiūtė (Jokšienė) [35, 2010]. Jachimavičienė gathered some new results [31,33, 2009] about the spectrum for NBCs:

$$U_0 = \gamma_0 h\left(\frac{U_0+U_N}{2}+\sum_{i=1}^{N-1}U_i\right), \quad U_N = \gamma_1 h\left(\frac{U_0+U_N}{2}+\sum_{i=1}^{N-1}U_i\right).$$

In this chapter we investigate Finite-Difference Scheme for differential SLP:

$$-u'' = \lambda u, \quad t \in (0,1), \quad \lambda \in \mathbb{C}_\lambda := \mathbb{C}, \tag{1.1}$$

with one classical BC and another integral type NBC:

$$u(0) = 0, \qquad u(1) = \gamma \int_{\xi_1}^{\xi_2} u(t)\,\mathrm{d}t, \tag{1.2}$$

where NBC's parameter $\gamma \in \mathbb{R}$ and $\boldsymbol{\xi} \in S_{\boldsymbol{\xi}} := \{(\xi_1,\xi_2) \in [0,1]^2 \colon 0 \leqslant \xi_1 < \xi_2 \leqslant 1\}$.

The remaining part of this chapter is organized as follows: in Section 2 the discrete problem is stated. In Section 3 CE, poles and zeros of CF are analyzed. The structure of the spectrum (Spectrum Curves in the domain and at the special points $q = 0$, $q = n$ and $q = \infty$) of Discrete SLP is investigated in Sections 4. Section 5 contains some brief conclusions and comments.

2 Discrete SLP

We introduce a uniform grids and we use notation $\overline{\omega}^h = \{t_j = jh, j = \overline{0,n}; nh = 1\}$ for $2 \leqslant n \in \mathbb{N}$. $\mathbb{N}^h := (0,n) \cap \mathbb{N}$, $\overline{\mathbb{N}}^h := \mathbb{N}^h \cup \{0,n\}$, $\mathbb{N}^h_o := (0,n) \cap \mathbb{N}_o$, $\mathbb{N}^h_e := (0,n) \cap \mathbb{N}_e$. Also, we make an assumption, that ξ_1 and ξ_2 are coincident with grid points, i.e., $\xi_1 = m_1 h = m_1/n$, $\xi_2 = m_2 h = m_2/n$, $\boldsymbol{m} \in S^h_{\boldsymbol{\xi}} := \{(m_1,m_2) \colon 0 \leqslant m_1 < m_2 \leqslant n, m_1, m_2 \in \overline{\mathbb{N}}^h\}$. So, $\boldsymbol{\xi} = \boldsymbol{m}/n = (m_1/n, m_2/n)$, $\xi = \xi_1/\xi_2 = m_1/m_2$, $\xi_+ = \xi_1 + \xi_2 = m_+/n$, $\xi_- = \xi_2 - \xi_1 = m_-/n$, where $m_+ := m_1 + m_2$, $m_- := m_2 - m_1$.

We will introduce a space H of real grid function on $\overline{\omega}^h$. We use notation:

$$[U,V] := \sum_{j=0}^{n} U_j V_j. \tag{2.1}$$

to describe the approximation of the integral in the BC. For real function this notation corresponds to the inner product in the space H. We will use this definition for complex function, too. In this chapter NBC we approximate by trapezoidal formula:

$$\int_a^b u\mathrm{d}t \approx u_{m_1}\frac{h}{2} + \sum_{i=m_1+1}^{m_2-1} u_i h + u_{m_2}\frac{h}{2} = [\chi_{[a,b],j}, u],$$

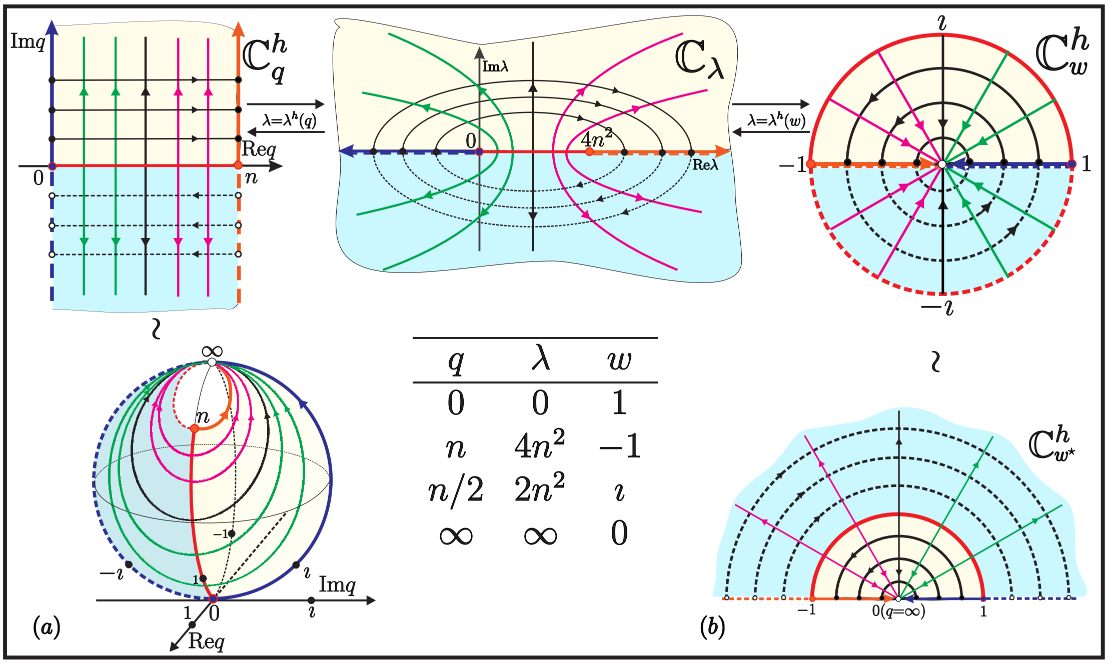


Fig. 3.1. Bijective mappings: $\lambda = \frac{4}{h^2}\sin^2(\frac{\pi q h}{2})$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_q^h$; $\lambda = \frac{2}{h^2}(1-\frac{w-w^{-1}}{2})$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_w^h$; (a) $\overline{\mathbb{C}}_q^h$ on *Riemann sphere*; (b) Domain $\mathbb{C}_{w^\star}^h$ on the upper half-plane.

where:

$$\chi_{[a,b],j} = \chi_{[a,b]}(t_j) = \begin{cases} 0 & \text{for } t_j < a, \text{ or } t_j > b, \\ \frac{h}{2} & \text{for } t_j = a, \text{ or } t_j = b, \\ h & \text{for } a < t_j < b, \end{cases} \qquad t_j \in \overline{\omega}^h, \tag{2.2}$$

$a, b \in \overline{\omega}^h$ and $a < b$, i.e., $a = t_\alpha = \alpha h$, $b = t_\beta = \beta h$, $\alpha, \beta \in \overline{\mathbb{N}}^h$.

We approximate differential SLP (1.1)–(1.2) by the *Finite-Difference Scheme* (FDS):

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j \in \mathbb{N}^h, \tag{2.3}$$

$$U_0 = 0, \qquad U_n = \gamma[\chi_{[\xi_1,\xi_2]}, U] = \gamma h\Big(\frac{U_{m_1} + U_{m_2}}{2} + \sum_{k=m_1+1}^{m_2-1} U_k\Big), \tag{2.4}$$

where right-hand side of (2.4) corresponds to trapezoidal formula for the integral BC.

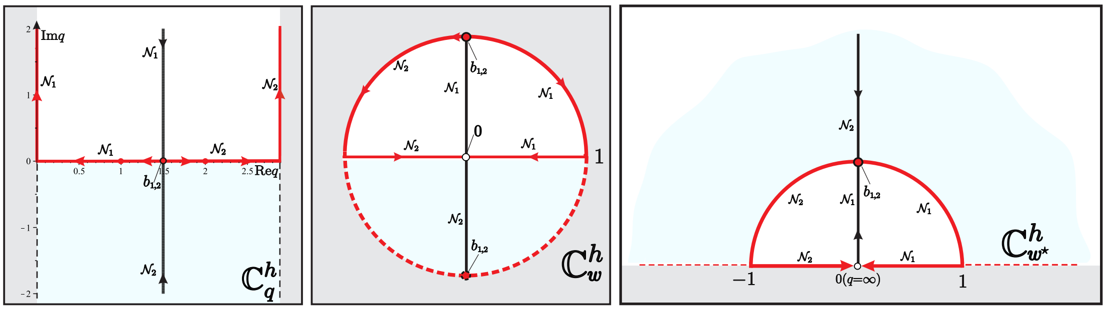


Fig. 3.2. Spectrum Curves in $\mathbb{C}_q^h$, $\mathbb{C}_w^h$ and $\mathbb{C}_{w^\star}^h$ ($n = 3$, $\boldsymbol{m} = (0, 1)$).

Lemma 3.1. *Let $t \in \overline{\omega}^h$, then the following equalities are hold:*

$$[\chi_{[a,b]}, e^{\imath zt}] = h e^{\imath z(a+b)/2} \sin \frac{z(b-a)}{2} \tan^{-1} \frac{zh}{2}, \tag{2.5}$$

$$[\chi_{[a,b]}, \cos(zt)] = h \sin \frac{z(b-a)}{2} \cos \frac{z(b+a)}{2} \tan^{-1} \frac{zh}{2}, \tag{2.6}$$

$$[\chi_{[a,b]}, \sin(zt)] = h \sin \frac{z(b-a)}{2} \sin \frac{z(b+a)}{2} \tan^{-1} \frac{zh}{2}. \tag{2.7}$$

Remark 3.2. In the case $z = 0$ we understand the equalities (2.5)–(2.6) as $[\chi_{[a,b]}, 1] = b - a$ and the equality (2.7) as $[\chi_{[a,b]}, 0] = 0$.

Proof of Lemma 3.1. We can consider the grid function $Y = Y_j = y^j$, for $y \in \mathbb{C}$, i.e., $Y_0 = 1$, $Y_1 = y, \ldots, Y_n = y^n$. If $y \neq 1$, then we have:

$$\begin{aligned}
[\chi_{[a,b]}, Y] &= h\Big(\frac{y^\alpha + y^\beta}{2} + \sum_{j=\alpha+1}^{\beta-1} y^j\Big) \\
&= h\Big(\frac{y^\alpha + y^\beta}{2} + \sum_{j=0}^{\beta-1} y^j - \sum_{j=0}^{\alpha} y^j\Big) = h\,\frac{\left(y^\beta - y^\alpha\right)(y+1)}{2(y-1)} \\
&= h\,\frac{y^{\frac{\beta+\alpha}{2}}\left(y^{\frac{\beta-\alpha}{2}} - y^{-\frac{\beta-\alpha}{2}}\right)}{2\imath} \cdot \imath\frac{(y+1)}{(y-1)}.
\end{aligned}$$

In the case $y^j = e^{\imath z t_j}$, $z \neq 0$, $t_j \in \overline{w}^h$ i.e., we have:

$$\begin{aligned}
[\chi_{[a,b]}, e^{\imath zt}] &= h e^{\frac{\imath z(a+b)}{2}} \frac{e^{\frac{\imath z(b-a)}{2}} - e^{-\frac{\imath z(b-a)}{2}}}{2\imath} \tan^{-1}\frac{zh}{2} \\
&= h e^{\frac{\imath z(a+b)}{2}} \sin\frac{z(b-a)}{2} \tan^{-1}\frac{zh}{2}.
\end{aligned} \tag{2.8}$$

Using the formula above, as a result we get formulas:

$$\begin{aligned}
[\chi_{[a,b]}, \cos(zt)] &= \frac{1}{2}[\chi_{[a,b]}, e^{\imath zt}] + \frac{1}{2}[\chi_{[a,b]}, e^{-\imath zt}] \\
&= h \cos\frac{z(b+a)}{2} \sin\frac{z(b-a)}{2} \tan^{-1}\frac{zh}{2}; \\
[\chi_{[a,b]}, \sin(zt)] &= \frac{1}{2\imath}[\chi_{[a,b]}, e^{\imath zt}] - \frac{1}{2\imath}[\chi_{[a,b]}, e^{-\imath zt}] \\
&= h \sin\frac{z(b-a)}{2} \sin\frac{z(b+a)}{2} \tan^{-1}\frac{zh}{2}.
\end{aligned}$$

If $z = 0$, then the trapezoidal formula is exact, so $[\chi_{[a,b]}, 1] = b - a$. □

The function

$$\lambda = \lambda^h(q) := \tfrac{4}{h^2}\sin^2(\pi q h/2), \tag{2.9}$$

is bijection between $\mathbb{C}_\lambda := \mathbb{C}$ and $\mathbb{C}_q^h$, where $\mathbb{C}_q^h := \mathbb{R}_q^{h-} \cup \{0\} \cup \mathbb{R}_q^h \cup \{n\} \cup \mathbb{R}_q^{h+} \cup \mathbb{C}_q^{h+} \cup \mathbb{C}_q^{h-}$ where $\mathbb{R}_q^h := \{q = x \colon 0 < x < n\}$, $\mathbb{R}_q^{h-} := \{q = \imath y \colon y > 0\}$, $\mathbb{R}_q^{h+} := \{q = n + \imath y \colon y > 0\}$, $\mathbb{C}_q^{h+} := \{q = x + \imath y \colon 0 < x < n, y > 0\}$, $\mathbb{C}_q^{h-} := \{q = x + \imath y \colon 0 < x < n, y < 0\}$ [86, Štikonas and Štikonienė 2009]. Then for any eigenvalue $\lambda \in \mathbb{C}_\lambda$ there exists the *eigenvalue point* $q \in \mathbb{C}_q^h$. We use notation $\overline{\mathbb{C}}_q^h = \mathbb{C}_q^h \cup \{\infty\}$ for *Riemann sphere* (see Figure 3.1). It follows, that $\lambda < 0$ for $q \in \mathbb{R}_q^{h-}$, $0 < \lambda < 4/h^2$ for $q \in \mathbb{R}_q^h$, $\lambda > 4/h^2$ for $q \in \mathbb{R}_q^{h+}$. Points $\lambda = 0$, $\lambda = 4/h^2$ are the first order branch points of the function $\lambda = \lambda^h(q)$. We note that for differential problem (1.1)–(1.2) eigenvalues are defined by the formula $\lambda = (\pi q)^2$, $q \in \mathbb{C}_q$ [86, Štikonas and Štikonienė 2009]. We also use bijection $\lambda_w = \lambda^h(w) := \frac{2}{h^2}(1 - (w - w^{-1})/2)$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_w^h := \{w \in \mathbb{C} : |w| \leqslant 1, w \neq 0\}$ (see Figure 3.1). The $\mathbb{C}_w^h$ will be used for investigation of eigenvalues in the neighborhood of $\lambda = \infty$ ($w = 0$). This bijection maps $\lambda < 0$ to the interval $w \in (0, 1)$, $0 < \lambda < 4/h^2$ to the upper unit semidisk, $\lambda > 4/h^2$ to the interval $(-1, 0)$, and complex λ points correspond to the points w, $\operatorname{Im} w \neq 0$, inside the unit circle (see Figure 3.1). In the domain $\mathbb{C}_w^h$ the points $w = \pm 1$ correspond to a branch points of the function λ_w^h. The function $w = \mathrm{e}^{\imath\pi h q}$ maps $\mathbb{C}_q^{h+}$ to the unit semidisk and $\mathbb{C}_q^{h-}$ to outer part of unit semidisk in $\mathbb{C}_{w^\star}^h$. The corresponding points in the different domains are shown in the table (see Figure 3.1).

Remark 3.3. Point $w = 0$ in Figure 3.1 and Figure 3.2 (see domain $\mathbb{C}_w^h$) corresponds to the point $\lambda = \infty$ in domain $\mathbb{C}_\lambda$ and $q = \infty$ in domain $\mathbb{C}_q^h$.

Using formula (2.9), the equation (2.3) can be rewritten in form:

$$U_{j+1} - 2\cos(\pi q h)U_j + U_{j-1} = 0, \qquad j \in \mathbb{N}^h, \tag{2.10}$$

where $q = x + \imath y \in \mathbb{C}_q^h$. The general solution of the difference equation (2.10) is:

$$\begin{aligned} &U = C_1 \sin(\pi q t_j) + C_2 \cos(\pi q t_j) \text{ for } q \neq 0, n;\\ &U = C_1 t_j + C_2 \text{ for } q = 0;\\ &U = C_1(-1)^j t_j + C_2(-1)^j \text{ for } q = n. \end{aligned} \tag{2.11}$$

From classical BC $U_0 = 0$ we get $C_2 = 0$. So, nontrivial eigenfunctions are of the form: $V = t_j$ for $q = 0$; $V = (-1)^j t_j$ for $q = n$; $V = \sin(\pi q t_j)$ for $q \neq 0, n$.

If $\lambda = 0$, then all functions $u(t) = Ct$ satisfy the equation and the first BC (2.3)–(2.4)(left side). By substituting this solution into NBC (2.4)(right side) we have that the eigenvalue $\lambda = 0$ ($q = 0$) for SLP (2.3)–(2.4), if and only if $\gamma = \frac{2n^2}{m_2^2 - m_1^2}$. Note, that for the differential case $\lambda = 0$ if $\gamma = \frac{2}{\xi_2^2 - \xi_1^2}$. The eigenvalue $\lambda = 4/h^2$ exist for $q = n$, if and only if:

$$\gamma = \frac{2}{h^2} N_0, \qquad N_0 := \frac{2(-1)^{n-m_2}}{1-(-1)^{m_2-m_1}} = \begin{cases} \infty, & \text{for } m_2 - m_1 \in \mathbb{N}_e;\\ +1, & \text{for } m_2 - m_1 \in \mathbb{N}_o,\ n - m_2 \in \mathbb{N}_e;\\ -1, & \text{for } m_2 - m_1 \in \mathbb{N}_o,\ n - m_2 \in \mathbb{N}_o. \end{cases}$$

If we substitute $V = \sin(\pi q t_j)$ into (2.4) then by the Lemma 3.1 we get equation for $q \in \mathbb{C}_q^h$:

$$\begin{aligned} \sin(\pi q) &= \gamma h \frac{\cos(\pi \xi_1 q) - \cos(\pi \xi_2 q)}{2} \tan^{-1} \frac{\pi q h}{2}\\ &= \gamma h \sin \frac{\pi q(\xi_2 - \xi_1)}{2} \sin \frac{\pi q(\xi_2 + \xi_1)}{2} \tan^{-1} \frac{\pi q h}{2}. \end{aligned} \tag{2.12}$$

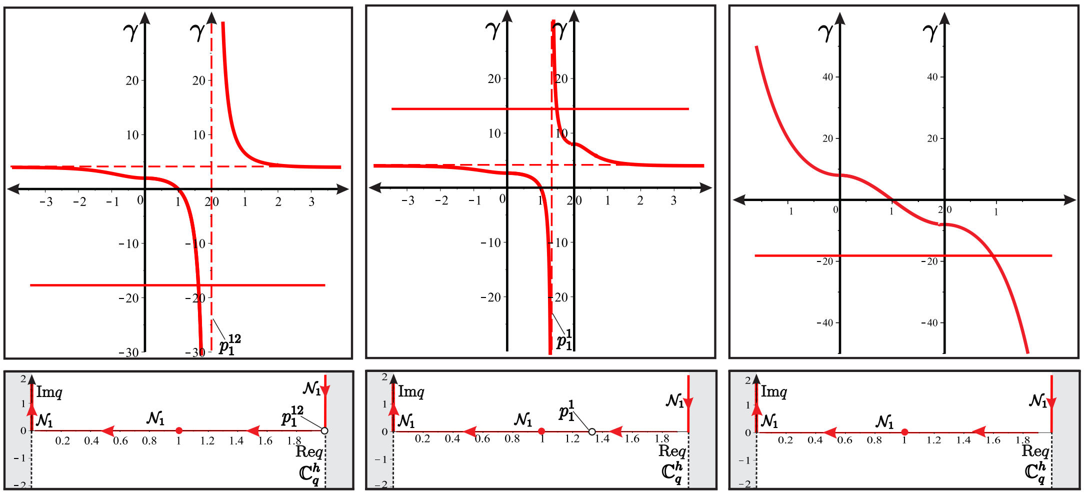
The equation (2.12) can be rewritten in a more convenient form:

$$\frac{1}{h} \cdot \frac{\sin(\pi q)}{\pi q} \cdot \frac{\tan(\pi q h/2)}{\pi q} = \gamma \frac{\sin \frac{\pi q(\xi_2 - \xi_1)}{2} \sin \frac{\pi q(\xi_2 + \xi_1)}{2}}{\pi^2 q^2}. \tag{2.13}$$

This equation is valid (as limit cases) for $q = 0, n$, too. If hq is sufficiently small, then $\tan \frac{\pi q h}{2} \approx \frac{\pi q h}{2}$. So, in limit case, the equation (2.13) is the same as for differential problem [80, Skučaitė and Štikonas 2015].

If $\gamma = 0$, we have the classical BCs and all the $n-1$ eigenvalues for the classical FDS are positive and algebraically simple and do not depend on the parameters ξ_1 and ξ_2:

$$\lambda_k(0) = \lambda^h(q_k(0)),\ U_j^k(0) = \sin(\pi q_k(0) t_j),\ q_k(0) = k \in \mathbb{N}^h. \tag{2.14}$$



(a) $\boldsymbol{m} = (0, 2)$ (b) $\boldsymbol{m} = (1, 2)$ (c) $\boldsymbol{m} = (0, 1)$

Fig. 3.3. CF for $n = 2$.

3 Characteristic Function, Constant Eigenvalues Points, Poles

We introduce the entire functions:

$$Z^h(z) := Z(z) \cdot \frac{1}{h} \tan(\pi z h/2); \quad Z(z) := \frac{\sin(\pi z)}{\pi z}; \quad P_{\boldsymbol{\xi}}(z) := P^1_{\boldsymbol{\xi}}(z) P^2_{\boldsymbol{\xi}}(z);$$
$$P^1_{\boldsymbol{\xi}}(z) := \frac{\sin(\pi z(\xi_1 + \xi_2)/2)}{\pi z}; \quad P^2_{\boldsymbol{\xi}}(z) := \frac{\sin(\pi z(\xi_2 - \xi_1)/2)}{\pi z}.$$

Zeroes of the functions $Z(q)$, $Z^h(q)$, $q \in \mathbb{C}^h_q$, coincide with the eigenvalue points in the classical case $\gamma = 0$. Zeroes of the functions $P^1_{\boldsymbol{\xi}}$, $P^2_{\boldsymbol{\xi}}$ in the domain $\mathbb{C}^h_q$ are simple and positive. We can rewrite equality (2.13) in the form:

$$Z^h(q) = \gamma P_{\boldsymbol{\xi}}(q), \quad q \in \mathbb{C}^h_q. \tag{3.1}$$

We define the *constant eigenvalue* (CE) as the eigenvalue that does not depend on the parameter γ. For any CE $\lambda_c \in \mathbb{C}$ there exists a *Constant Eigenvalue point* $q_c \in \mathbb{C}^h_q$ (CE point) [86, Štikonas and Štikonienė 2009] and $\lambda_c = \lambda^h(q_c)$. For SLP (2.1)–(2.2) all CE points are real and we can find them as solutions of the following system:

$$Z(q) = 0, \quad P_{\boldsymbol{\xi}}(q) = 0, \quad q \in (0, n).$$

The notation $\mathcal{C}$ is used for the set of all CE points.

If $q \notin \mathbb{N}^h$, i.e. $Z^h(q) \neq 0$, and q satisfies equation $P_{\boldsymbol{\xi}}(q) = 0$, then the equality (3.1) is not valid for all γ and such point q is *a Pole Point*. Notation of the pole point is connected with meromorphic function:

$$\gamma_c(z) = \frac{Z^h(z)}{P_{\boldsymbol{\xi}}(z)}, \quad z \in \mathbb{C}. \tag{3.2}$$

This function is obtained by expressing γ from equation (3.1). We call the restriction of meromorphic function γ_c on $\mathbb{C}_q^h$ as *Complex-Real Characteristic Function* ($\mathbb{C}$-$\mathbb{R}$ CF or CF) [86, Štikonas and Štikonienė 2009], and denote this function as:

$$\begin{aligned}\gamma(q) &= \frac{Z^h(q)}{P_{\boldsymbol{\xi}}(q)} = \frac{Z(q)}{P_{\boldsymbol{\xi}}(q)} \cdot \frac{1}{h}\tan(\pi q h/2) = \\ &= \frac{\sin(\pi q)}{\cos(\pi\xi_1 q) - \cos(\pi\xi_2 q)} \cdot \frac{1}{h}\tan(\pi q h/2), \quad q \in \mathbb{C}_q^h.\end{aligned} \tag{3.3}$$

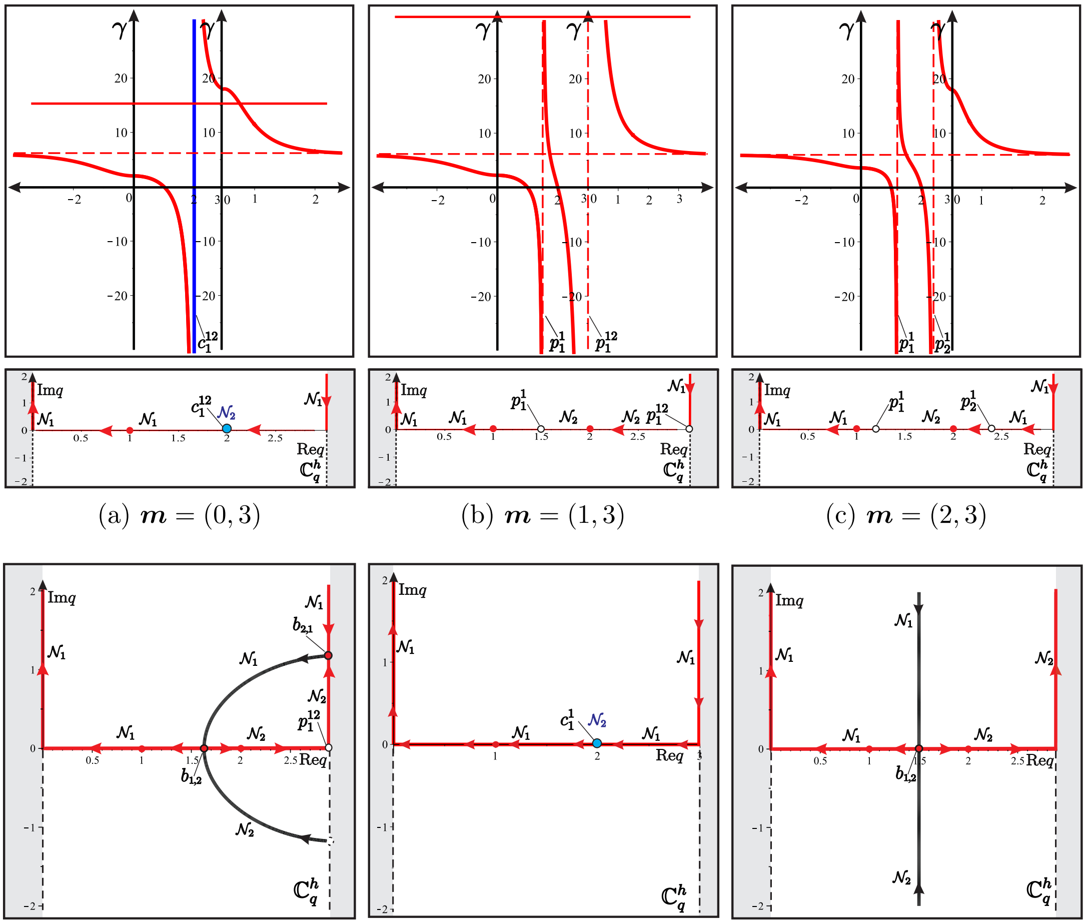
All *nonconstant eigenvalues* (which depend on the parameter γ) are γ-points of *Complex-Real Characteristic Function* (Complex-Real CF) [86, Štikonas and Štikonienė 2009]. Complex-Real CF $\gamma(q)$ is the restriction of function $\gamma_c(q)$ on a set $\mathcal{N}^\gamma := \{q \in \mathbb{C}_q^h \colon \operatorname{Im}\gamma_c(q) = 0\}$. *Real CF* $\gamma(q)$ is defined on the domain $\{q \in \mathbb{C}_q^h \colon \lambda \in \mathbb{R}\}$ and describes only real eigenvalues. We plot the graph of Real CF for eigenvalue points $0 < x < n$ in the middle graph; $x = 0$, $y > 0$ in the left half plane and $x = n$, $y > 0$ in the right half plane. Two γ-axes correspond to points $q = 0, n$. The Real CF graph can be seen in the Figure 3.3(a)–(c) and Figure 3.4(a)–(c) (top pictures) for $n = 2, 3$. We note, that there are no complex eigenvalues in the case $n = 2$ for any m_1 and m_2 values and in case $n = 3$ if $m_2 = n$ and $m_1 = 0, ..., n-1$ or $m_2 = 2$ and $m_1 = 1$ (see Figure 3.3 and Figure 3.4). Vertical blue and red dash lines are added at the CE and poles points.

There exists the horizontal asymptote in the case $m_2 = n$:

$$\gamma(\infty) = \lim_{q\to\infty} \gamma(q) = \frac{2}{h} N_1, \quad N_1 = \begin{cases} \infty & m_2 \neq n; \\ 1 & m_2 = n, \end{cases}$$

(see Figure 3.3(a)–(b) and Figure 3.4(a)–(c), a horizontal dashed red line).

Equations for zeroes and poles of CF are the same as in differential case in Chapter 1 and Chapter 2 [80], but in the discrete case zeroes and poles are located in $(0, n)$. Function $\tan(\frac{\pi q h}{2})$ has pole in the point $q = n$. So, additional pole can be at this point. We have this pole if $N_0 = \infty$, i.e., if $m_2 - m_1 \in \mathbb{N}_e$. We include this pole into the family of the second order poles, but it will be the first order pole by definition, because $\lambda = 4/h^2$ is a branch point.



(a) $\boldsymbol{m} = (0, 3)$ (b) $\boldsymbol{m} = (1, 3)$ (c) $\boldsymbol{m} = (2, 3)$

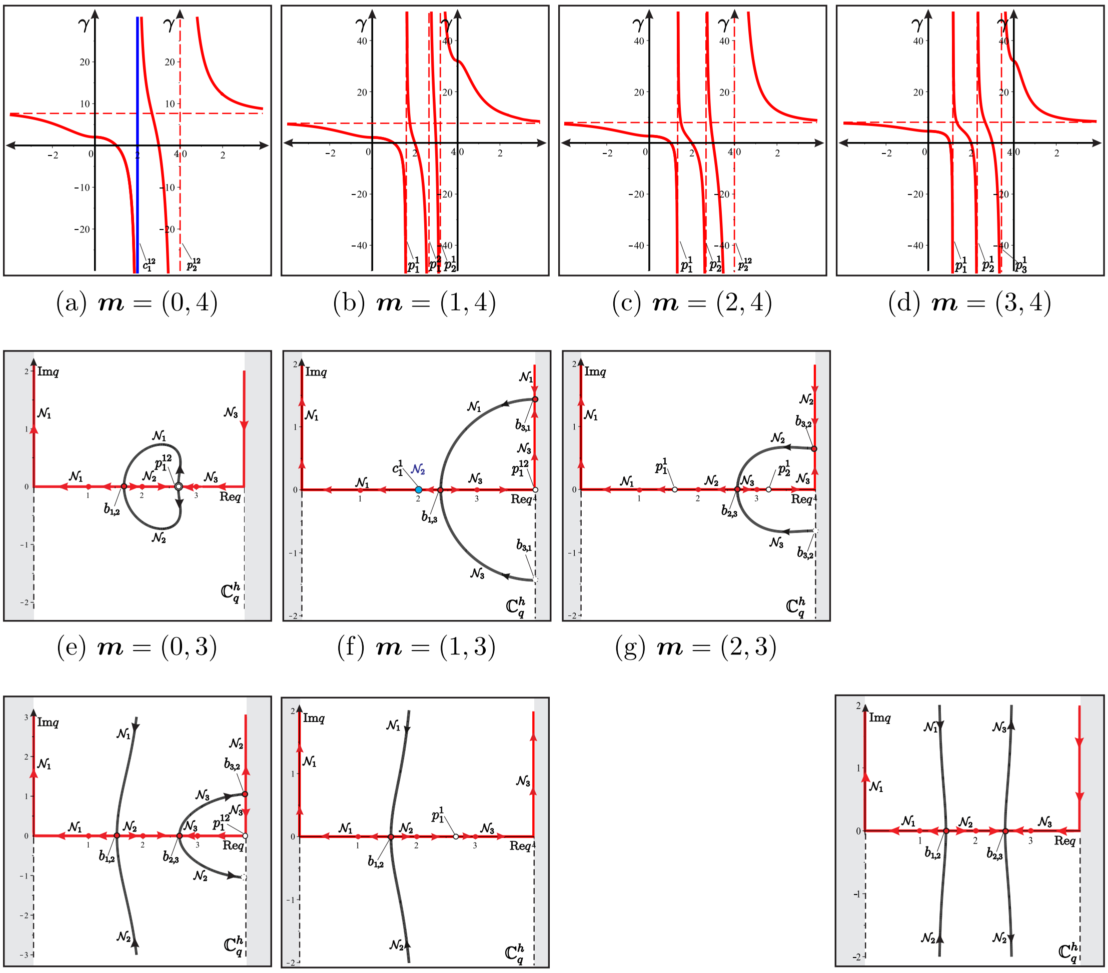
(d) $\boldsymbol{m} = (0, 2)$ (e) $\boldsymbol{m} = (1, 2)$ (f) $\boldsymbol{m} = (0, 1)$

Fig. 3.4. Complex–Real CF for $n = 3$.

All zeroes of the functions Z, P_ξ^1, P_ξ^2 in $(0, +\infty)$ are simple (of the first order), real and positive:

$$z_k = k \in \mathbb{N}, \quad p_k^1 = \tfrac{2}{\xi_+} k, \ k \in \mathbb{N}, \quad p_k^2 = \tfrac{2}{\xi_-} k, \ k \in \mathbb{N}. \tag{3.4}$$

We denote these sets Z, P^1, P^2 and denote $\overline{\mathcal{Z}} := Z \cap (0, n)$, $\overline{\mathcal{Z}}_\xi^1 := P^1 \cap (0, n]$, $\overline{\mathcal{Z}}_\xi^2 := P^2 \cap (0, n]$. Then a set $\mathcal{Z}_\xi = \mathcal{Z}_\xi^1 + \mathcal{Z}_\xi^2 + \mathcal{Z}_\xi^{12}$ describes all zeroes of the function P_ξ, where $\mathcal{Z}_\xi^1 := \overline{\mathcal{Z}}_\xi^1 \setminus \mathcal{Z}_\xi^{12}$ and $\mathcal{Z}_\xi^2 := \overline{\mathcal{Z}}_\xi^2 \setminus \mathcal{Z}_\xi^{12}$ are two families of the first order zeroes, $\mathcal{Z}_\xi^{12} := \overline{\mathcal{Z}}_\xi^1 \cap \overline{\mathcal{Z}}_\xi^2$ is family of the second order zeroes. For CF we consider the following sets: a set of poles $\mathcal{P}_\xi := \mathcal{P}_\xi^1 + \mathcal{P}_\xi^2 + \mathcal{P}_\xi^{12}$, where $\mathcal{P}_\xi^1 := \mathcal{Z}_\xi^1 \setminus \overline{\mathcal{Z}}$ and $\mathcal{P}_\xi^2 := \mathcal{Z}_\xi^2 \setminus \overline{\mathcal{Z}}$ are two families of the poles of the first order, a set of the second order poles $\mathcal{P}_\xi^{12} := \mathcal{Z}_\xi^{12} \setminus \overline{\mathcal{Z}}$; a set of the CE points $\mathcal{C}_\xi := \mathcal{C}_\xi^1 + \mathcal{C}_\xi^2 + \mathcal{C}_\xi^{12}$, where $\mathcal{C}_\xi^1 := \mathcal{Z}_\xi^1 \cap \overline{\mathcal{Z}}$ and $\mathcal{C}_\xi^2 := \mathcal{Z}_\xi^2 \cap \overline{\mathcal{Z}}$ are sets of the points with removable singularity, $\mathcal{C}_\xi^{12} := \mathcal{Z}_\xi^{12} \cap \overline{\mathcal{Z}}$ is the set of the points with the first order pole, too;

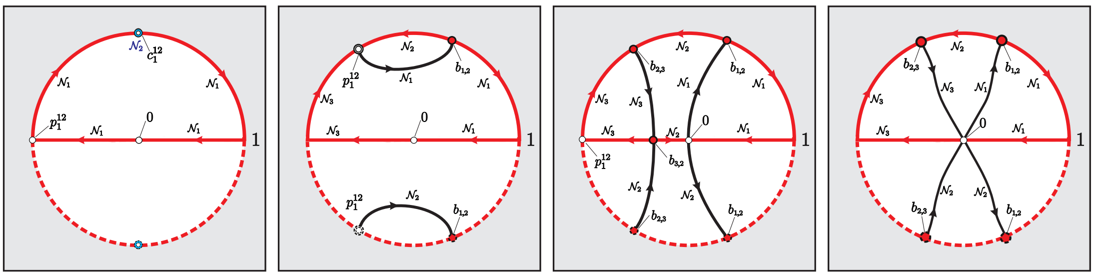


(a) $\boldsymbol{m} = (0, 4)$ (b) $\boldsymbol{m} = (1, 4)$ (c) $\boldsymbol{m} = (2, 4)$ (d) $\boldsymbol{m} = (3, 4)$

(e) $\boldsymbol{m} = (0, 3)$ (f) $\boldsymbol{m} = (1, 3)$ (g) $\boldsymbol{m} = (2, 3)$

(h) $\boldsymbol{m} = (0, 2)$ (i) $\boldsymbol{m} = (1, 2)$ (j) $\boldsymbol{m} = (0, 1)$

Fig. 3.5. Complex–Real CF for $n = 4$.



(a) $\boldsymbol{m} = (0, 4)$ (b) $\boldsymbol{m} = (0, 3)$ (c) $\boldsymbol{m} = (0, 2)$ (d) $\boldsymbol{m} = (0, 1)$

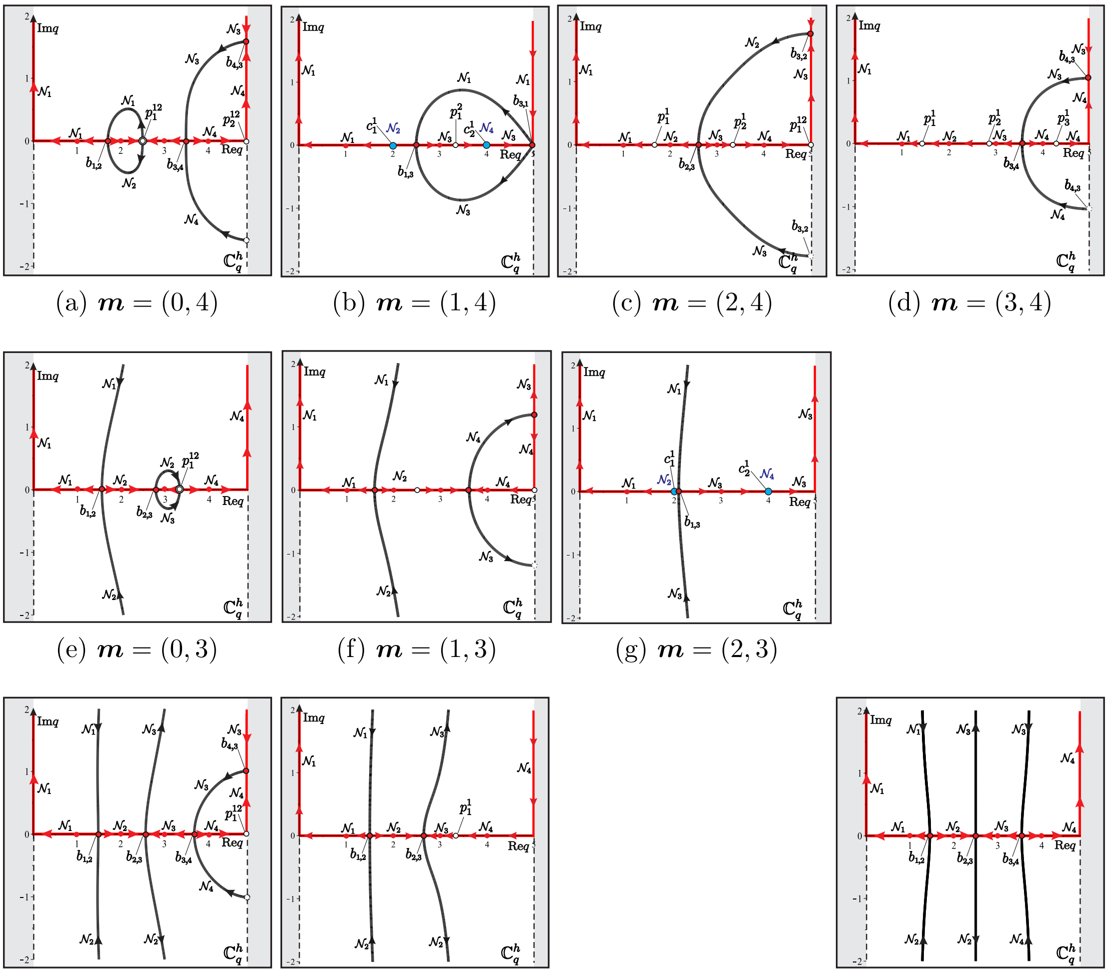
Fig. 3.6. Bijective mappings in domain $\mathbb{C}_w^h$ for different $\boldsymbol{m}$ values ($n = 4$).

a set of zeroes $\mathcal{Z}_{\boldsymbol{\xi}} := \overline{\mathcal{Z}} \setminus \mathcal{C}_{\boldsymbol{\xi}}$.

So, formulae for poles and CE points are:

$$p_k^1 = 2nk/m_+, \quad p_k^2 = 2nk/m_-, \quad p_k^{12} = 2nk/\gcd(m_+, m_-),$$

$$c_k^1 = 2nk/\gcd(2n, m_+), \; c_k^2 = 2nk/\gcd(2n, m_-), \; c_k^{12} = 2n/\gcd(2n, m_+, m_-)$$



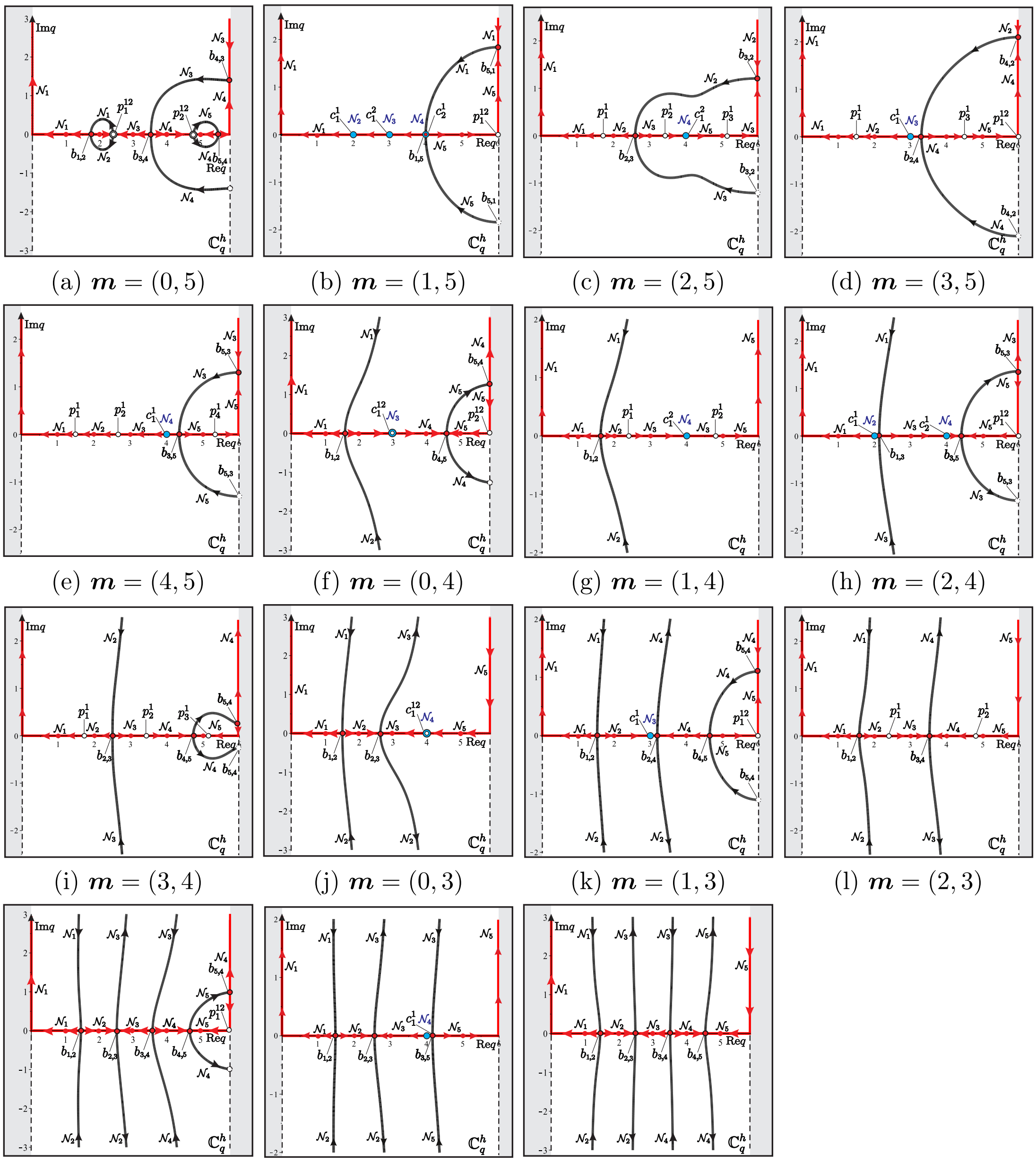
(a) $\boldsymbol{m} = (0, 4)$ (b) $\boldsymbol{m} = (1, 4)$ (c) $\boldsymbol{m} = (2, 4)$ (d) $\boldsymbol{m} = (3, 4)$

(e) $\boldsymbol{m} = (0, 3)$ (f) $\boldsymbol{m} = (1, 3)$ (g) $\boldsymbol{m} = (2, 3)$

(h) $\boldsymbol{m} = (0, 2)$ (i) $\boldsymbol{m} = (1, 2)$ (j) $\boldsymbol{m} = (0, 1)$

Fig. 3.7. Spectrum Curves for $n = 5$.

for the sets $\mathcal{P}^1_{\boldsymbol{\xi}} + \mathcal{P}^{12}_{\boldsymbol{\xi}} + \mathcal{C}^1_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{P}^2_{\boldsymbol{\xi}} + \mathcal{P}^{12}_{\boldsymbol{\xi}} + \mathcal{C}^2_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{P}^{12}_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{C}^1_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{C}^2_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{C}^{12}_{\boldsymbol{\xi}}$, respectively. The points of these sets have form $q_k = \alpha k$, $k \in \mathbb{N}^\star := \{k \in \mathbb{N}, k = 1 \ldots k_{\max}\}$, where $k_{\max} = \lfloor n/q_1 \rfloor$ for poles and $k_{\max} = \lfloor (n-1)/q_1 \rfloor$ for CE points, $\alpha \geqslant 1$, or can be empty. So, nonempty sets are described by the first point q_1 ($q_1 = p^1_1$, p^2_1, p^{12}_1, c^1_1, c^2_1, c^{12}_1). The second order pole exists for $n = 4$, $m_1 = 0$, $m_2 = 3$ at the point $q = 2.(6)$ (see Figure 3.5(e)), $n = 5$, $m_1 = 0$, $m_2 = 4$ at the point $q = 2.5$ and $m_1 = 0$, $m_2 = 3$ at the point $q = 3.(3)$ (see Figure 3.7(a) and Figure 3.7(e)) and $n = 6$, $m_1 = 0$, $m_2 = 4$ at the point $q = 3$ (see Figure 3.8(f)). In these figure we can see additional the second order pole at $q = n$ for $m_2 - m_1 \in \mathbb{N}_e$. The Figure 3.6 shows the schemes of bijection mapping for different m_1 and m_2 in $\mathbb{C}^h_w$ ($n = 4$). We have Critical point and CE point in the case $n = 6$, $m_1 = 1$, $m_2 = 5$ at the point $q = 4$ (see Figure 3.8(b)). If $m_2 + m_1 = n$, then exist two types CE Points. First type CE Points do not



(a) $\boldsymbol{m} = (0,5)$ (b) $\boldsymbol{m} = (1,5)$ (c) $\boldsymbol{m} = (2,5)$ (d) $\boldsymbol{m} = (3,5)$

(e) $\boldsymbol{m} = (4,5)$ (f) $\boldsymbol{m} = (0,4)$ (g) $\boldsymbol{m} = (1,4)$ (h) $\boldsymbol{m} = (2,4)$

(i) $\boldsymbol{m} = (3,4)$ (j) $\boldsymbol{m} = (0,3)$ (k) $\boldsymbol{m} = (1,3)$ (l) $\boldsymbol{m} = (2,3)$

(m) $\boldsymbol{m} = (0,2)$ (n) $\boldsymbol{m} = (1,2)$ (o) $\boldsymbol{m} = (0,1)$

Fig. 3.8. Spectrum Curves for $n = 6$.

depend on NBC parameters ($c_1^1 = 2$). The second type CE Points can be defined by the formula $c_1^2 = \frac{3n}{2} = 3(2n_1 + 1)$, $k \in \mathbb{N}$, where m_1 and $n = 2(2n_1 + 1)$ are coprime numbers (see Figure 3.4(a), (e), Figure 3.5(a), (f), Figure 3.7(b), (g) and others). More results about Poles and CE Points are presented in [80, Skučaitė and Štikonas 2015] (the cases when ξ_1 and ξ_2 are rational).

4 Spectrum Curves

Spectrum Domain for dSLP is the set $\mathcal{N} = \mathcal{N}^\gamma \cup \mathcal{C} \subset \mathbb{C}_q^h$. Function γ_c has real values on $\mathcal{N}$ except for the pole points. For each $\gamma_0 \in \mathbb{R}$ a set $\mathcal{N}(\gamma_0) := \gamma^{-1}(\gamma_0)$ is the set of all eigenvalue points for nonconstant eigenvalues. So, Spectrum Domain $\mathcal{N} = \cup_{\gamma\in\mathbb{R}}\mathcal{N}(\gamma)\cup\mathcal{C}$. Though zeroes, poles and CE points are practically the same for SLP and dSLP, but multiplier $\tan\frac{\pi q h}{2}$ little change the Spectrum Curves in the other points.

If $\gamma = 0$ then the eigenvalue points are $q = z_k = k \in \mathbb{N}$. So, we can numerate the part of $\mathcal{N}(\gamma)$ for this point by the classical case $\mathcal{N}_k(0) = z_k \in \mathbb{N}$. For every CE point $c_j = j$ we define $\mathcal{N}_j = \{c_j\}$, i.e. every such $\mathcal{N}_j$ has one point only. We call every $\mathcal{N}_k$, $k \in \mathbb{N}$, a *Spectrum Curve* [80]. For $\gamma \to \pm\infty$ Spectrum Curve $\mathcal{N}_k(\gamma)$, which is not CE point, approaches a pole point or the point $q = \infty$. One can see the Spectrum Curves in Figure 3.4(d)–(f) and Figure 3.5(e)–(j) for $n = 3, 4$.

4.1 Spectrum Curves at the points $q = 0$, $q = n$ and $q = \infty$

$\boldsymbol{q = 0}$: Taylor series for CF at the point $q = 0$ is

$$\gamma(q) = \frac{2n^2}{m_2^2 - m_1^2} + \frac{1}{6(m_2^2 - m_1^2)}(1 - 2n^2 + m_2^2 + m_1^2)q^2 + \mathcal{O}(q^4). \tag{4.1}$$

For $n > 1$ we estimate $m_1^2 + m_2^2 \leqslant (n-1)^2 + n^2 = 2n^2 - 2n + 1 < 2n^2 - 1$. So, the second term is negative, and we have $\gamma'(0) = 0$, $\gamma''(0) \neq 0$ at the first order branch point 0. At this point Spectrum Curve $\mathcal{N}_1$ turn orthogonal to the right, i.e. the first positive eigenvalue point reaches $q = 0$ and then this point moves along the imaginary axis. So, point $q = 0$ has properties of the first order Critical point in domain $\mathbb{C}_q^h$, but the point $\lambda = 0$ is not a Critical point in domain $\mathbb{C}_\lambda$ for CF.

$\boldsymbol{q = n}$: If $m_- = m_2 - m_1 \in \mathbb{N}_o$ then $|N| = 1$ and the Taylor series for $\gamma(q)$ at the point $q = n$ is:

$$\begin{aligned}\gamma(q) := {} & 2Nn^2 - \frac{N}{2}\left(\frac{1+2n^2}{3} - (m_2^2 + m_1^2)\right)(q-n)^2 + \\ & - \frac{N}{24n^2}\left(\frac{6n^4 + 10n^2 - 1}{15} - (m_2^4 + m_1^4) + \right. \\ & \left. + (3(m_2^2 + m_1^2) - 2n^2 - 1)(m_2^2 + m_1^2)\right)(q-n)^4 + \mathcal{O}((q-n)^6).\end{aligned} \tag{4.2}$$

So, in this case, $\gamma'(n) = 0$. If condition:

$$m_2^2 + m_1^2 = (1 + 2n^2)/3 \tag{4.3}$$

is valid (for example, $n = 5$, $m_1 = 1$, $m_2 = 4$) then $\gamma''(n) = 0$, too. This point has properties of the second order Critical point in domain $\mathbb{C}_q^h$ and the first order Critical point in domain $\mathbb{C}_\lambda$ (see Figure 3.5(c)). So, by definition $q = n$ is the first order Critical point. The next term is vanished if

$$m_2^4 + m_1^4 = (6n^2 + 10n^2 - 1)/15 + (3(m_2^2 + m_1^2) - 2n^2 - 1)(m_2^2 + m_1^2). \tag{4.4}$$

Both terms (4.3) and (4.4) are vanished if there exists a solution of the system:

$$m_2^2 + m_1^2 = (1 + 2n^2)/3, \quad m_2^2 m_1^2 = (n^2 - 1)(n^2 - 4)/45$$

where $n, m_1, m_2 \in \mathbb{N}$. By Vieta's formulas m_1^2 and m_2^2 are roots of the second order equation and m_1^2 and m_2^2 will be integers, if the expression $80s^2 + 24s + 1$ is square of an integer number, where $n^2 = 15s + 1$. However, there is no such s. So, in domain $\mathbb{C}_q^h$ Critical point at $q = n$ can be first order or second order only. If condition (4.3) is not valid, then the point $q = n$ is a pole or correspond to the first order branch point.

$\boldsymbol{q = \infty}$**:** CF can be rewritten in the following form by using Euler formula:

$$\gamma(q) = \frac{e^{\imath\pi q} - e^{-\imath\pi q}}{e^{\imath\pi q\xi_1} + e^{-\imath\pi q\xi_1} - e^{\imath\pi q\xi_2} - e^{-\imath\pi q\xi_2}} \cdot \frac{1 - e^{\pi q h}}{1 + e^{\pi q h}} \cdot \frac{2}{h}. \tag{4.5}$$

Then CF could be expressed in terms of argument w:

$$\begin{aligned}\gamma(w) &= \frac{w^n - w^{-n}}{w^{m_1} + w^{-m_1} - w^{m_2} - w^{-m_2}} \cdot \frac{1 - w}{1 + w} \cdot \frac{2}{h} \\ &= \frac{w^{-n}}{w^{-m_2}} \cdot \frac{(1 - w^{2n})(1 - w)}{(1 - w^{m_2+m_1} - w^{m_2-m_1} + w^{2m_2})(1 + w)} \cdot \frac{2}{h}, \quad w \in \mathbb{C}_w^h.\end{aligned} \tag{4.6}$$

Lemma 3.4. *At the point $w = 0$ a formula is valid:*

$$\gamma(w) = \frac{1}{w^{n-m_2}} \cdot \frac{2}{h} \cdot (1 + \mathcal{O}(w)). \tag{4.7}$$

Proof. From (4.5) we have

$$\gamma(w) = \frac{1}{w^{n-m_2}} \cdot \frac{(1 + \mathcal{O}(w^{2n}))(1 + \mathcal{O}(w))(1 + \mathcal{O}(w))}{(1 + \mathcal{O}(w^{2m_2}))} \cdot \frac{2}{h}. \tag{4.8}$$

□

So, $\frac{1}{w^{n-m_2}}$ describes the properties of CF at the point $w = 0$. Point $w = 0 \notin \mathbb{C}^h_w$ is an isolated singularity point. If $m_2 = n$, then $\lim_{w\to 0}\gamma(w) = 2/h$ and we have *removable singularity* point. In this case the same Spectrum Curve enters and leaves the point $w = 0$ (see Figure 3.6(a)). Additionally, there are no complex Spectrum Curves in the neighbourhood of the point $w = 0$. If $m_2 \leqslant n-1$, then the point $w = 0$ is a pole point. The difference $n - m_2$ show the order of the pole (see equation (4.6) and Figure 3.6). In the case $n - m_2 = 1$ two real Spectrum Curves enter or leave the point $w = 0$ (see Figure 3.6(b)). On the right side of the first order pole point $w = 0$ one spectrum curve ($\mathcal{N}_1 \subset (0, 1)$) enters and on the left side of this point one spectrum curve ($\mathcal{N}_{n-1} \subset (-1, 0)$) leaves this point. If $n - m_2 \geqslant 2$ then there exist additional Spectrum Curves that enter and leave the point $w = 0$.

The point $q = \infty \in \overline{\mathbb{C}}^h_q$ is a pole or removable singularity point. The order of this point is $n_\infty = n - m_2$. So, the the order of $q = \infty$ does not depend on m_1. Then n_∞ Spectrum Curves enters on this point and the same number of Spectrum Curves leaves this point, if $\gamma \to \pm\infty$. Note, that incoming Spectral Curves alternates with outgoing (see all Figures). If $n = m_2$, then the point $q = \infty$ is a removable singularity point. Finally, we formulate few obvious lemmas.

Lemma 3.5. *The formula $n_p + n_{ce} + 1 = m_2$ is valid, where n_p is number of poles (including the order), n_{ce} is number of CE Points.*

Proof. Each Spectrum Curves has limit points in poles or in CE point. So, $n_\infty + n_p + n_{ce} = n - 1$. But $n_\infty = n - m_2$. So, $n_p + n_{ce} = n - 1$. □

Lemma 3.6. *There are $n_{cr} = n_\infty + n_c + n_{2p} - 1$ Critical points (including the order) on $\mathbb{C}^h_q$, where n_{2p} is the number of the second order poles, n_c is the number of the parts of Spectrum Curves in the complex part of $\mathbb{C}^h_q$ between two Critical points.*

Lemma 3.7. *If $m_2 = n$ (does not depend on m_1 value) complex eigenvalues do not exist. In this case the point $q = \infty$ is a removable singularity point (the same Spectrum Curve enters and leaves this point). There exists a horizontal asymptote $\gamma(\infty) = \lim_{q\to\infty}\gamma(q) = 2/h$ as well.*

5 Remarks and conclusions

- Zeroes, poles of CF, CE point for dSLP in the interval $(0, n)$ are only real and the same as in the case of differential SLP. In the case of dSLP at $q = n$ there is additional pole if $m_2 - m_1 \in \mathbb{N}_e$. There are not poles for $\lambda \leqslant 0$ and $\lambda > 4/h^2$.

- Critical points may exist for $q \in \mathbb{R}_q^{h+}$, i.e., for $\lambda > 4/h^2$. There are not Critical point for negative λ.

- The point $q = n$ is the critical point of the second order for $m_2^2 + m_1^2 = (1 + 2n^2)/3$.

- The number of Spectrum Curves parts in the neighbourhood of ∞ is equal to $2(n - m_2)$ if $n > m_2$, and there is one Spectrum Curve if $n = m_2$.

Chapter 4

Discrete Sturm–Liouville problem with integral nonlocal boundary condition (special cases)

1 Problem formulation

Let us consider a SLP with one classical BC:

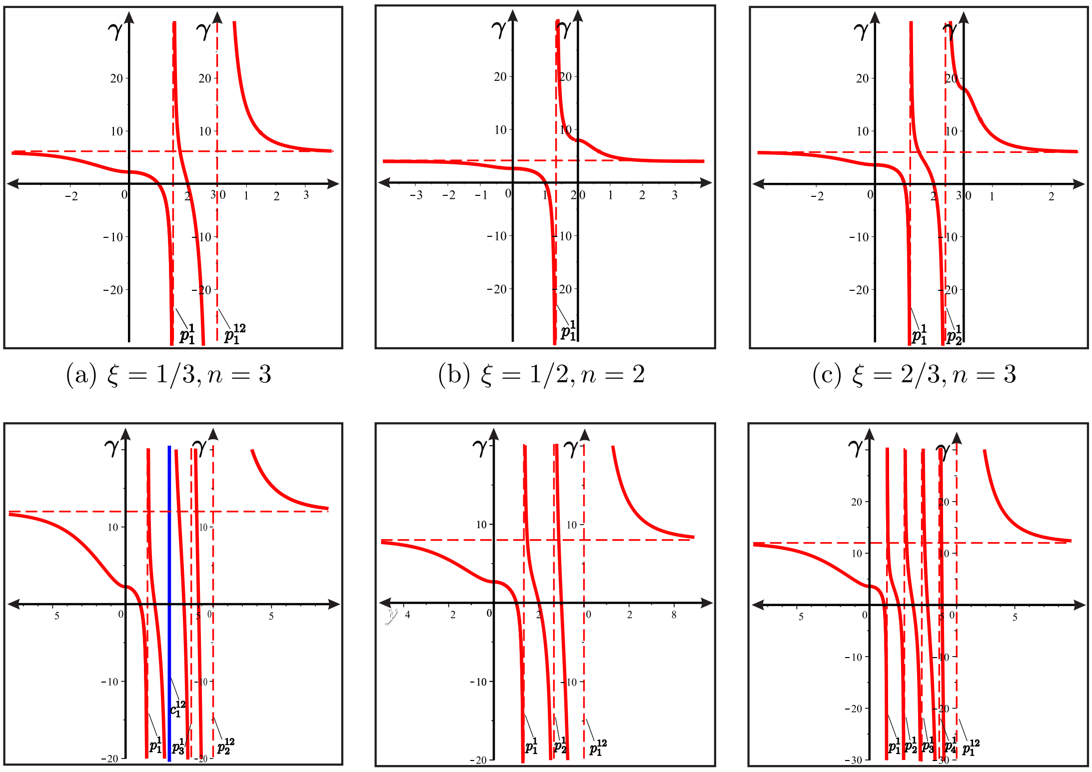
$$-u'' = \lambda u, \quad t \in (0,1), \quad u(0) = 0, \tag{1.1}$$

and an integral NBC:

$$u(1) = \gamma \int_{\xi}^{1} u(t)\,\mathrm{d}t, \qquad \text{(Case 1)} \qquad (1.2_1)$$

$$u(1) = \gamma \int_{0}^{\xi} u(t)\,\mathrm{d}t, \qquad \text{(Case 2)} \qquad (1.2_2)$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in [0,1]$. Same results on the spectrum view in complex part of the spectrum for differential problem were presented in Chapter 2. CF and its Spectrum Domain $\mathcal{N}$ for these problems are described in [86, Štikonas and Štikonienė 2009]. This chapter is based on [78].



(a) $\xi = 1/3, n = 3$ (b) $\xi = 1/2, n = 2$ (c) $\xi = 2/3, n = 3$

(d) $\xi = 1/3, n = 6$ (e) $\xi = 1/2, n = 4$ (f) $\xi = 2/3, n = 6$

Fig. 4.1. Real CF for various ξ of discrete problems in Case 1 (approximated by trapezoid).

2 The case of an approximation by the trapezoidal rule

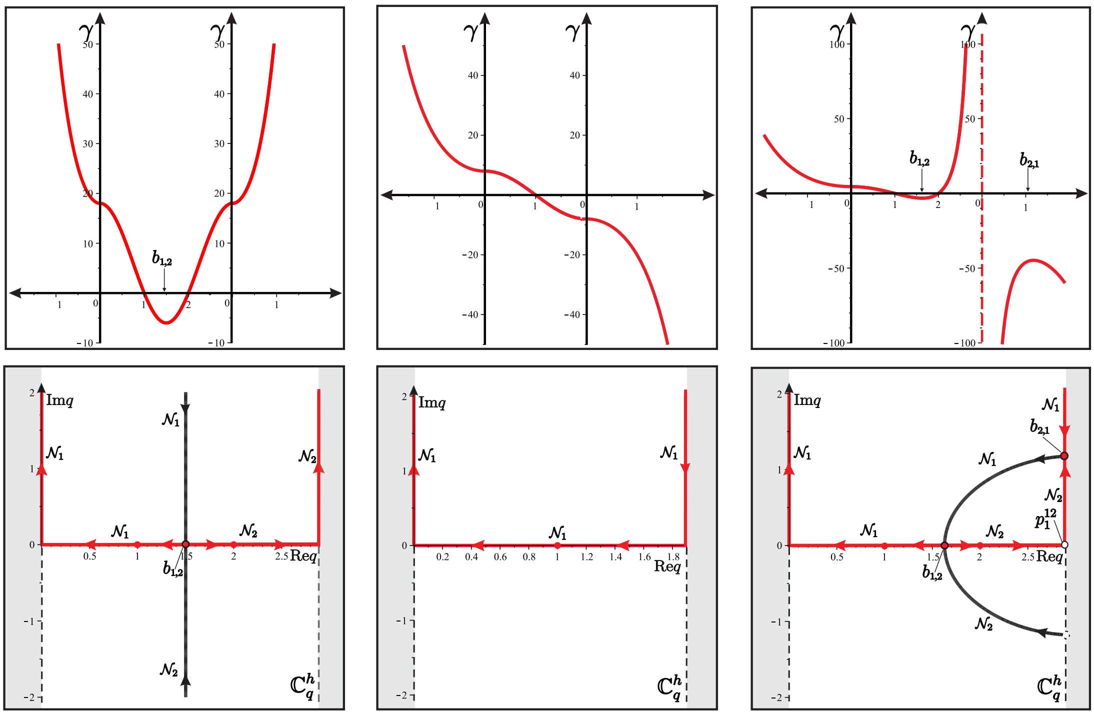
In the interval $[0, 1]$, a uniform grid $\omega^h = \{t_j = jh, j = \overline{0, n};\ n \in \mathbb{N}, nh = 1\}$ is introduced. Also, we make an assumption that ξ is coincident with a grid point, i.e., $\xi = mh = m/n$, $m = \overline{0, n}$. Let us denote the greatest common divisor by $K := \gcd(n, m)$ and $N := n/K$, $M := m/K$. Then $\xi = M/N$, too. We approximate differential problem (1.1)–(1.2) by the FDS:

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, n-1}, \tag{2.1}$$

$$U_0 = 0, \tag{2.2}$$

$$U_n = \gamma h \Big(\frac{U_m + U_n}{2} + \sum_{k=m+1}^{n-1} U_k \Big), \tag{2.3$_1$}$$

$$U_n = \gamma h \Big(\frac{U_0 + U_m}{2} + \sum_{k=1}^{m-1} U_k \Big). \tag{2.3$_2$}$$



(a) $\xi = 1/3$, $n = 3$ (b) $\xi = 1/2$, $n = 2$ (c) $\xi = 2/3$, $n = 3$

Fig. 4.2. Real CF and Spectrum Curves for problem (2.1),(2.2),(2.3_2) (Case 2).

We investigate eigenvalues for the FDS. Equations (2.1)–(2.2) in another form:

$$U_{j+1} - 2\cos(\pi q h)U_j + U_{j-1} = 0, \quad \lambda = \tfrac{4}{h^2}\sin^2(\pi q h/2), \quad U_0 = 0, \qquad (2.4)$$

where $q = x + \imath y \in \mathbb{C}_q^h$. More about domain $\mathbb{C}_q^h$ and bijection between $\mathbb{C}_\lambda$ and $\mathbb{C}_q^h$ see in Chapter 3.

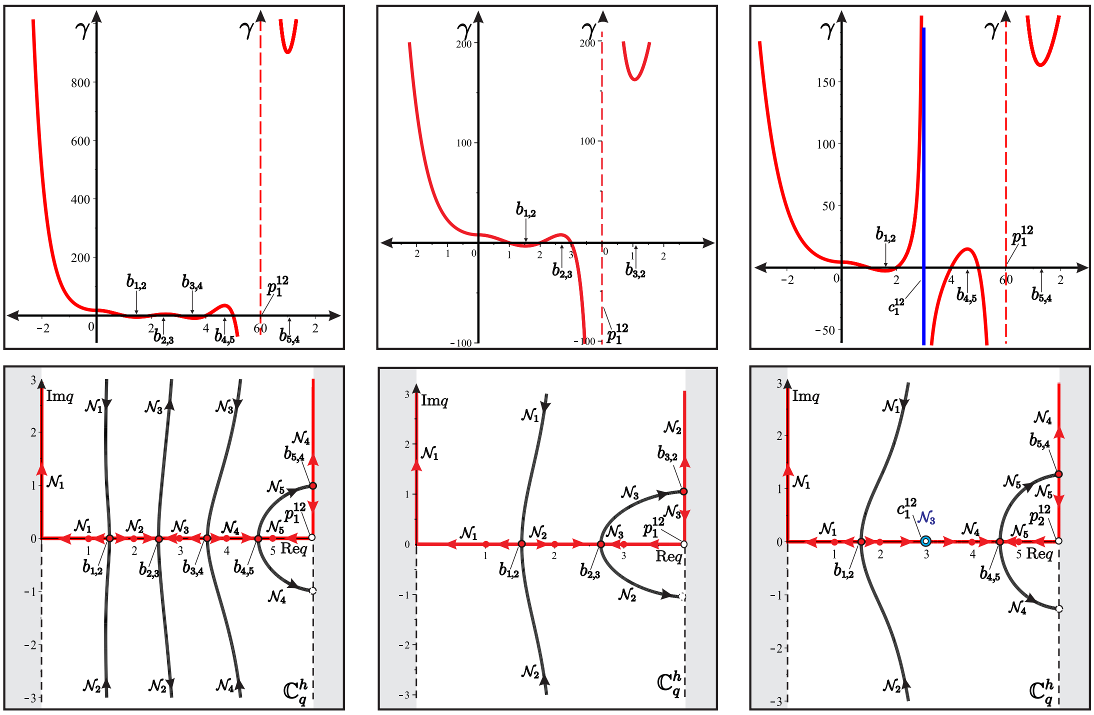
The general solution of difference equation (2.3) could be expressed by the formulae (2.11) in Chapter 3.

It can be seen from the BC (2.2) that $C_2 = 0$. After substituting this solution to NBC (2.3_1) in Case 1 (or (2.3_2) in Case 2) we get that eigenvalues $q \neq 0, n$ if $q = q(\gamma, \xi)$ are roots of the equations:

$$\sin(\pi q) - \gamma \frac{h\big(\cos(\xi\pi x) - \cos(\pi x)\big)}{2\tan(\pi q h/2)} = 0; \qquad (2.5_1)$$

$$\sin(\pi q) - \gamma \frac{h\sin^2(\xi\pi x/2)}{\tan(\pi q h/2)} = 0. \qquad (2.5_2)$$

We have the eigenvalue $\lambda = 0$ for problem (2.1)–(2.3), if and only if $\gamma = \frac{2}{1-\xi^2}$ in Case 1 and $\gamma = \frac{2}{\xi^2}$ in Case 2 (the same conditions are for the differential case). The eigenvalue $\lambda = 4/h^2$ can be found, if and only if $\gamma = \frac{2}{h^2} \cdot \frac{2}{1-(-1)^{n-m}}$ in Case 1 ($n - m \in \mathbb{N}_o$), and $\gamma = \frac{2}{h^2} \cdot \frac{2(-1)^n}{(-1)^m - 1}$ in Case 2 ($m \in \mathbb{N}_o$).



(a) $\xi = 1/3$, $n = 6$ (b) $\xi = 1/2$, $n = 4$ (c) $\xi = 2/3$, $n = 6$

Fig. 4.3. Real CF and Spectrum Curves for problem (2.1),(2.2),(2.3_2) (Case 2).

If $\gamma = 0$, we have the classical BCs and all the $n-1$ eigenvalues for the classical FDS are positive and algebraically simple and do not depend on the parameter ξ (see (2.14) in Chapter 3). If $q = x \in (0, 1/h)$, then $\lambda \in (0, 4/h^2)$ and the eigenvalues of problem (2.1)–(2.3) are calculated by the formula $\lambda_k = \frac{4}{h^2}\sin^2(\frac{\pi x_k h}{2})$, where x_k are roots of the equation:

$$\sin(\pi x) - \gamma h \tan^{-1}(\pi x h/2)\big(\cos(\xi\pi x) - \cos(\pi x)\big)/2 = 0; \qquad (2.6_1)$$

$$\sin(\pi x) - \gamma h \tan^{-1}(\pi x h/2)\sin^2(\xi\pi x/2) = 0. \qquad (2.6_2)$$

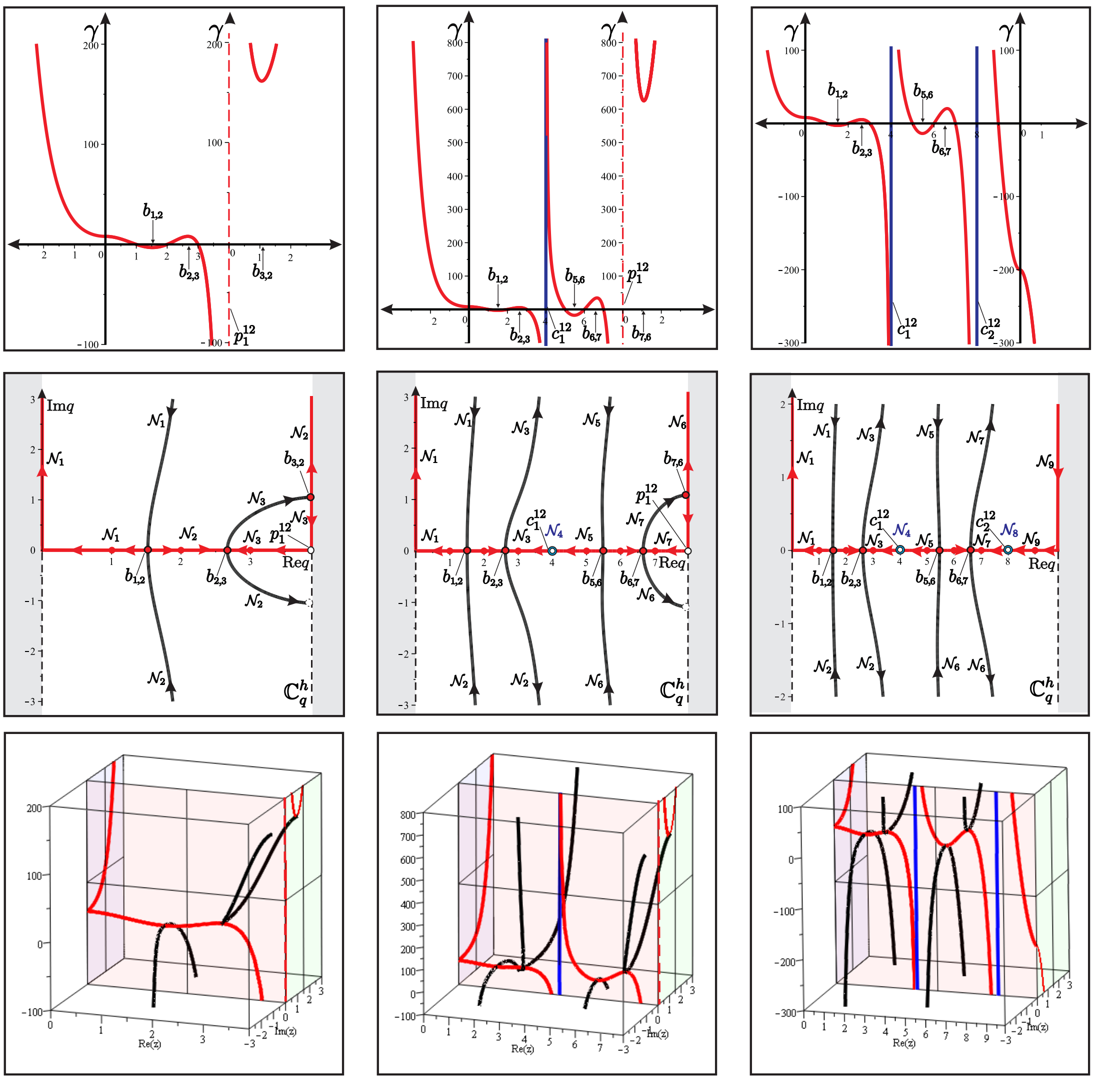
CE points are equal to:

$$c_k = 2Nk,\ N - M \in \mathbb{N}_o, \qquad c_k = Nk,\ N - M \in \mathbb{N}_e. \qquad (2.7_1)$$

$$c_k = 2Nk,\ M \in \mathbb{N}_o, \qquad c_k = Nk,\ M \in \mathbb{N}_e, \qquad (2.7_2)$$

$k \in \mathbb{N}$ such that $c_k \in (0, n)$.

Other (nonconstant) eigenvalues (which depend on the parameter γ) as γ-points are defined on the set $\mathbb{C}_q^h$ (see Figure 4.1(a)–(c) and Figure 4.2).



(a) $\xi = 1/2$, $n = 4$ (b) $\xi = 1/2$, $n = 8$ (c) $\xi = 1/2$, $n = 10$

Fig. 4.4. Real CF, Spectrum Curves and Complex-Real CF.

Lemma 4.1. *If $0 \leqslant m < l \leqslant 1$, then CF of the FDS:*

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, n-1}, \tag{2.8}$$

$$U_0 = 0, \tag{2.9}$$

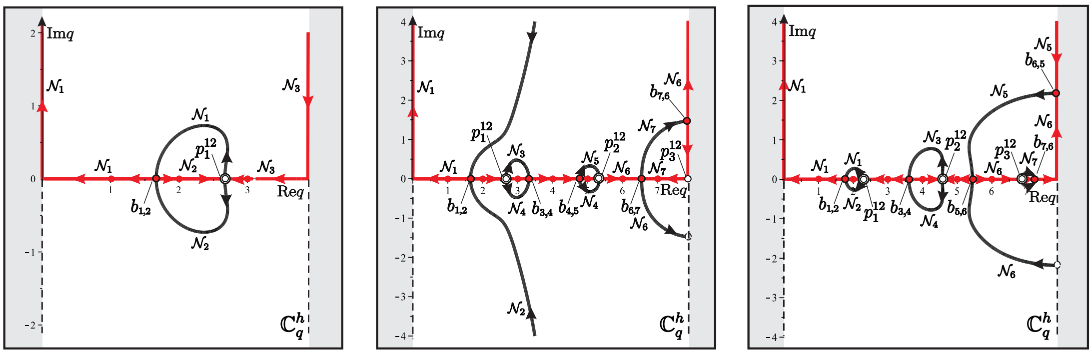
$$U_n = \gamma h\Big(\frac{U_m + U_l}{2} + \sum_{k=m+1}^{l-1} U_k\Big) \tag{2.10}$$

is

$$\gamma = \frac{\sin(\pi q h n)}{\cos(\pi q h m) - \cos(\pi q h l)} \cdot \tan\frac{\pi q h}{2} \cdot \frac{2}{h}. \tag{2.11}$$

Proof. Proof follows from Lemma 3.1. □

On the case $l = n$, $m = m$ (Case 1) and $l = m$, $m = 0$ (Case 2) we get that



(a) $\xi = 3/4$, $n = 4$ (b) $\xi = 3/4$, $n = 8$ (c) $\xi = 7/8$, $n = 8$

Fig. 4.5. Spectrum Curves in Case 2.

CF of the discrete problem (2.1)–(2.3):

$$\gamma_{t1} = \frac{\sin(\pi q)}{\cos(\xi\pi q) - \cos(\pi q)} \cdot \tan\frac{\pi q h}{2} \cdot \frac{2}{h}, \tag{2.12$_1$}$$

$$\gamma_{t2} = \frac{\sin(\pi q)}{2\sin^2(\xi\pi q/2)} \cdot \tan\frac{\pi q h}{2} \cdot \frac{2}{h}. \tag{2.12$_2$}$$

Lemma 4.2. *For the problem (2.1)–(2.3$_1$) exist two families of the first order zeroes:* $p_k^1 = \frac{2n\pi k}{n+m}$, $k \in \mathbb{N}^\star$ *and* $p_k^2 = \frac{2n\pi k}{n-m}$, $k \in \mathbb{N}^\star$. *For the problem (2.1)–(2.2), (2.3$_2$) there exist second order zeroes:* $p_k^{12} = \frac{2nk}{m}$, $k \in \mathbb{N}^\star$.

Proof. This Lemma is a part of Chapter 3 results. □

In Case 1 zero point from the first family coincides with zero point from the second family at CE points (see (2.7$_1$) and Figure 2.2 in Chapter 2).

If hq is a sufficiently small number, then $\tan\frac{\pi q h}{2} \cdot \frac{2}{\pi q h} \approx 1$. It follows that, in this case, the discrete CF is similar to the CF of the differential problem [48, Pečiulytė *et al.* 2005].

If $q = \imath y$, $y > 0$, then $\lambda_k = -\frac{4}{h^2}\sinh^2(\frac{y_k h}{2}) < 0$ and Real CF is:

$$\gamma_{t1-} = \frac{\sinh(\pi q)}{\cosh(\xi\pi q) - \cosh(\pi q)} \cdot \tanh\frac{\pi q h}{2} \cdot \frac{2}{h}, \tag{2.13$_1$}$$

$$\gamma_{t2-} = \frac{\sinh(\pi q)}{2\sinh^2(\xi\pi q/2)} \cdot \tanh\frac{\pi q h}{2} \cdot \frac{2}{h}. \tag{2.13$_2$}$$

If $q = n + \imath y$, $y > 0$, then $\lambda_k = -\frac{4}{h^2}\cosh^2(\frac{y_k h}{2}) > 4/h^2$ and:

$$\gamma_{t1+} = \frac{\sinh(\pi q)}{\cosh(\xi\pi q) - \cosh(\pi q)} \cdot \tanh\frac{\pi q h}{2} \cdot \frac{2}{h}, \tag{2.14$_1$}$$

$$\gamma_{t2+} = \frac{\sinh(\pi q)}{2\sinh^2(\xi\pi q/2)} \cdot \tanh\frac{\pi q h}{2} \cdot \frac{2}{h}. \tag{2.14$_2$}$$

The selection of the parameter n influences the spectrum structure of the discrete problem.

3 Investigation of the spectrum structure

In the case of the problem (2.1)–(2.2), (2.3_1) only real eigenvalues exist. The grid point $q = n$ is a pole point for even $n - m$ in Case 1. Otherwise, Spectrum Curve moves through the point $q = n$ (from $\mathbb{R}_q^h$ to $\mathbb{R}_q^{h+}$ or from $\mathbb{R}_q^{h+}$ to $\mathbb{R}_q^h$). Real CF for the FDS (2.1)–(2.2), (2.3_1) is presented in Figure 4.1. In Case 1 there exists horizontal asymptote $\gamma = \frac{2}{h}$ for Real CF. If $\gamma = \frac{2}{h}$, then $n - 2$ eigenvalues exist, only. If $n \to \infty$, then the view of the spectrum becomes similar to the spectrum for differential SLP in Chapter 2.

In Case 2 the complex part of the spectrum is more complicated then in Case 1 (see Figure 4.2–Figure 4.5). The grid point $q = n$ is a pole of the second order in domain $\mathbb{C}_q^h$ and the first order pole in domain $\mathbb{C}_\lambda$ for even m in Case 2. Real CF for FDS (2.1)–(2.2), (2.3_2) is shown in the Figure 4.2(a)–(c) and Figure 4.3(a)–(c)(top pictures).

Complex part of the spectrum for FDS (2.1)–(2.2), (2.3_2) are presented in Figure 4.2(a)–(c) and Figure 4.3(a)–(c)(bottom pictures). For some parameter ξ values only real eigenvalues (see Figure 4.2(b)) exist. As shown in the Figure 4.5, for some parameter values Spectrum Curves make loops.

Figure 4.2, Figure 4.3, Figure 4.4 show real and complex parts of the spectrum for different number of grid point n for (2.1)–(2.2), (2.3_2). As shown in the Figure 4.4, the number of grid points influences the spectrum structure. If grid point is increasing, then the spectrum view becomes more similar to differential SLP in Chapter 2.

4 The case of an approximation by Simpson's rule

In the interval $[0, 1]$, a uniform grid $\omega^h = \{t_j = jh, j = \overline{0, 2n};\ 2nh = 1\}$ is introduced. Also, we make the an assumption, that ξ is coincident with the grid point, i.e., $\xi = 2mh = m/n$, $m = \overline{0, n}$. We approximate differential problem (1.1), (1.2) by FDS:

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, 2n-1}, \tag{4.1}$$

$$U_0 = 0, \tag{4.2}$$

$$U_{2n} = \frac{\gamma h}{3}\Big(U_{2m} + U_{2n} + 4\sum_{k=m+1}^{n} U_{2k-1} + 2\sum_{k=m+1}^{n-1} U_{2k}\Big), \tag{4.3$_1$}$$

$$U_{2n} = \frac{\gamma h}{3}\Big(U_0 + U_{2m} + 4\sum_{k=1}^{m} U_{2k-1} + 2\sum_{k=1}^{m-1} U_{2k}\Big). \tag{4.3$_2$}$$

From the general solution (see (2.11) in Chapter 3) and the BC $U_0 = 0$ yields that $C_2 = 0$. By substituting such a solution function to NBC (4.3$_1$) or (4.3$_2$), we derive that the eigenvalue exists for $q \neq 0, n$ if $q = q(\gamma, \xi)$ is the root of the equation:

$$\sin(\pi q) - \frac{\gamma h}{3}\big(\cos(\xi\pi q) - \cos(\pi q)\big) \cdot \frac{(2 + \cos(\pi q h))}{\sin(\pi q h)} = 0; \tag{4.4$_1$}$$

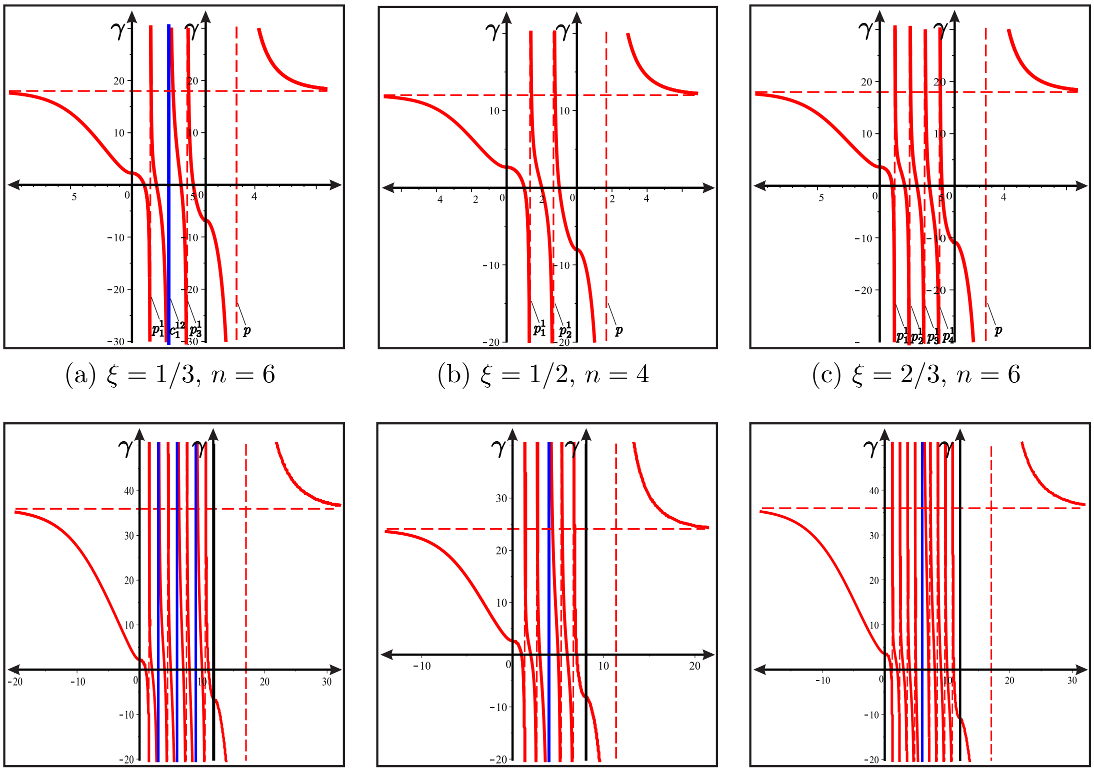
$$\sin(\pi q) - \frac{2\gamma h}{3}\sin^2(\xi\pi q/2) \cdot \frac{(2 + \cos(\pi q h))}{\sin(\pi q h)} = 0. \tag{4.4$_2$}$$

We get the eigenvalue $\lambda = 0$ for problem (4.1)–(4.3), if and only if $\gamma = \frac{2}{1-\xi^2}$ in Case 1 and $\gamma = \frac{2}{\xi^2}$ in Case 2 (the same conditions are for differential (1.1)–(1.2) and for FDS (2.1)–(2.3)). If $\gamma = 0$, then all the $n-1$ eigenvalues could be defined by equation (2.14) in Chapter 3. All eigenvalues $\lambda_k(\gamma, \xi)$ and eigenfunctions $U^k(\gamma, \xi)$ can be enumerated as: $\lambda_k(0, \xi) = \lambda_k^0$, $k = 1 \ldots n-1$. If $q = x \in (0, 1/h)$, then $\lambda \in (0, 4/h^2)$ and we calculate the eigenvalues of problem (4.1)–(4.3) by the formula $\lambda_k = \frac{4}{h^2}\sin^2(\frac{\pi x_k h}{2})$, where x_k are roots of the equation:

$$\sin(\pi x) - \frac{\gamma h}{3}\big(\cos(\xi\pi x) - \cos(\pi x)\big) \cdot \frac{2 + \cos(\pi x h)}{\sin(\pi x h)} = 0; \tag{4.5$_1$}$$

$$\sin(\pi x) - \frac{2\gamma h}{3}\sin^2\left(\frac{\xi\pi x}{2}\right) \cdot \frac{2 + \cos(\pi x)h}{\sin(\pi x h)} = 0. \tag{4.5$_2$}$$

Let us denote the greatest common divisor $K := \gcd(2n, 2m)$ and $N := 2n/K$, $M := 2m/K$. Then $\xi = M/N$, too. Then CE points c_k are described by the formula (2.7) (the same formula is for (2.1)–(2.3)). Other (nonconstant) eigenvalues (which depend on the parameter γ) as γ-point of the CF (see Figure 4.6–Figure 4.9) are defined on the set $\mathbb{C}_q^h$.



(a) $\xi = 1/3$, $n = 6$ (b) $\xi = 1/2$, $n = 4$ (c) $\xi = 2/3$, $n = 6$

(d) $\xi = 1/3$, $n = 12$ (e) $\xi = 1/2$, $n = 8$ (f) $\xi = 2/3$, $n = 12$

Fig. 4.6. Discrete problem (4.1)–(4.2), (4.3_1), Real CF for various ξ values.

Lemma 4.3. *If $0 \leqslant m < l \leqslant 1$, then CF for FDS:*

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, 2n-1}, \tag{4.6}$$

$$U_0 = 0, \tag{4.7}$$

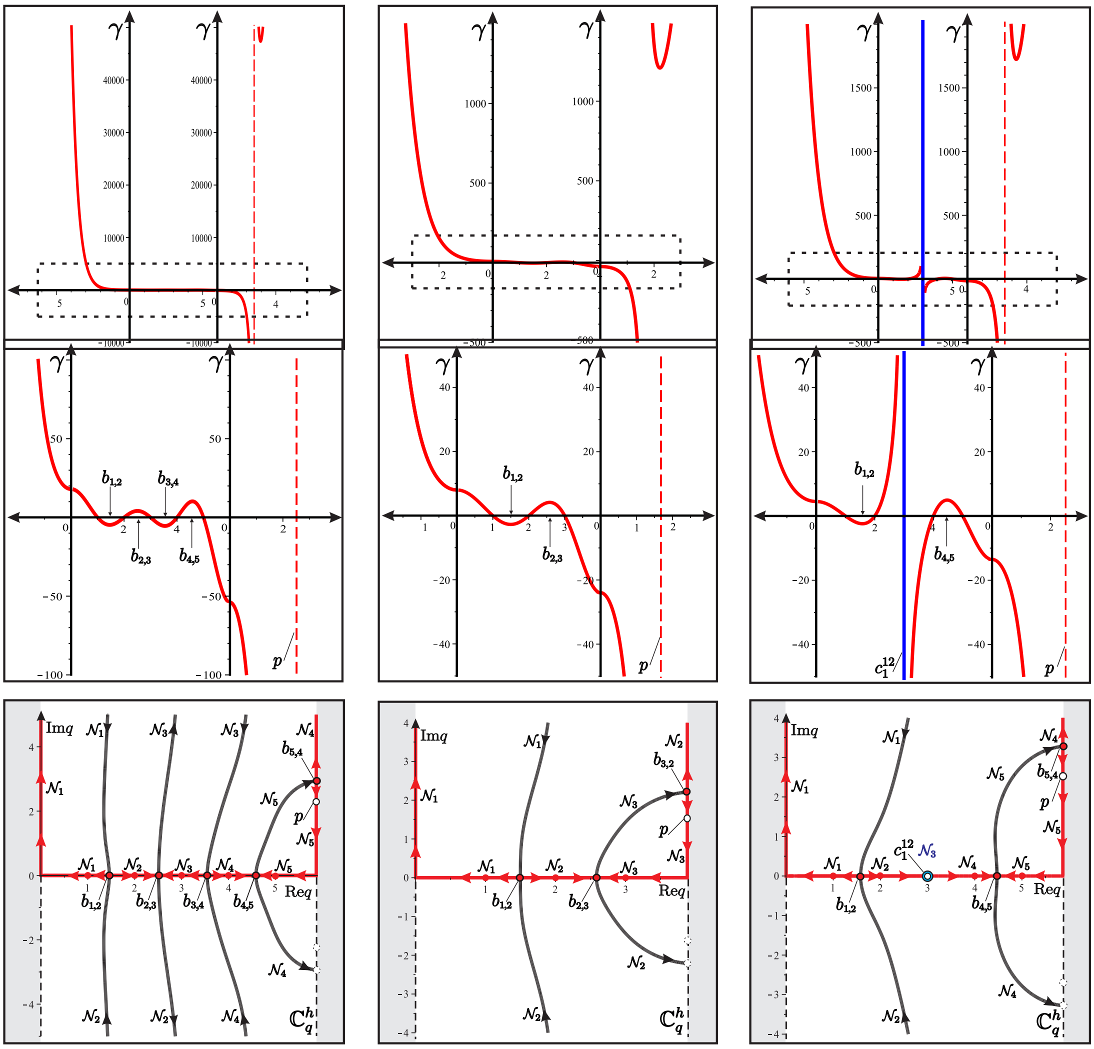
$$U_{2n} = \frac{\gamma h}{3}\Big(U_{2m} + U_{2l} + 4\sum_{k=m+1}^{l} U_{2k-1} + 2\sum_{k=m+1}^{l-1} U_{2k}\Big), \tag{4.8}$$

is:

$$\gamma = \frac{\sin(\pi q h n)}{\cos(\pi q h m) - \cos(\pi q h l)} \cdot \frac{3\sin(\pi q h)}{h(2 + \cos(\pi q h))}. \tag{4.9}$$

Proof. First of all, it should be mentioned that the parameters of FDS (4.6)–(4.8) are $m = 0, 1, \ldots, n$, $l = 0, 1, \ldots, n$, $l > m$, where $2n$ is a number of grid points. After the assumption that $U_j = y^j$, the NBC (4.8) can be rewritten in another form:

$$y^{2n} = \frac{\gamma h}{3}\Big(y^{2m} + y^{2l} + 4\sum_{k=m+1}^{l} y^{2k-1} + 2\sum_{k=m+1}^{l-1} y^{2k}\Big). \tag{4.10}$$



(a) $\xi = 1/3$, $n = 6$ (b) $\xi = 1/2$, $n = 4$ (c) $\xi = 2/3$, $n = 6$

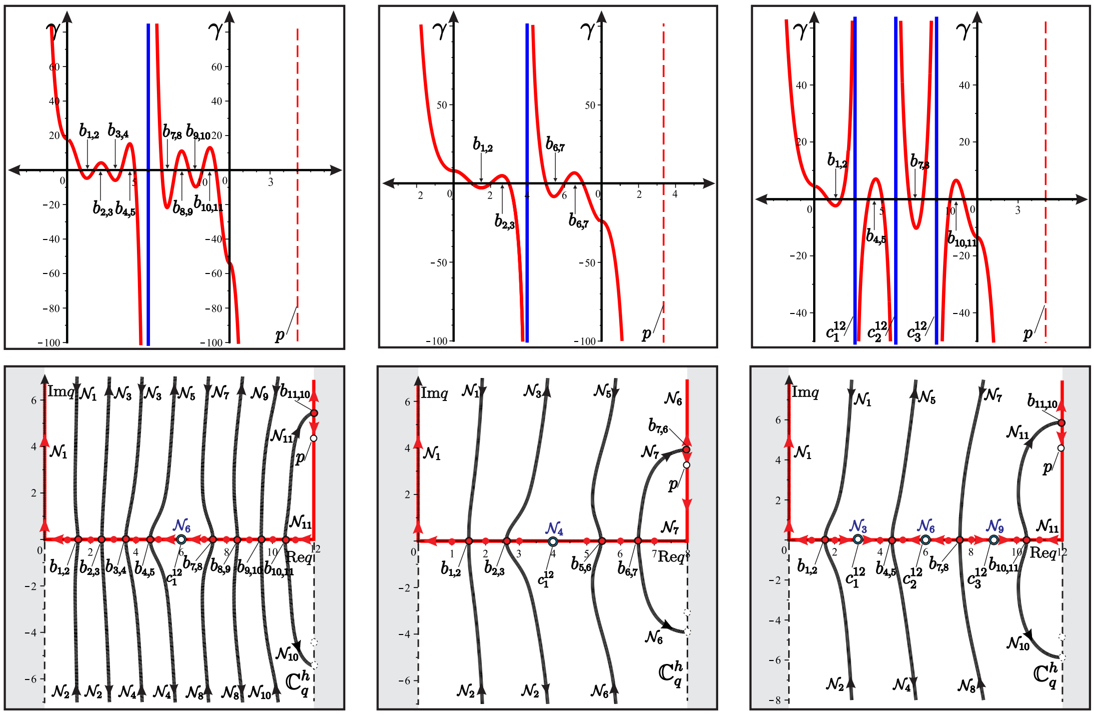
Fig. 4.7. Discrete problem (4.1)–(4.2), (4.3$_2$), Real CF and Spectrum Curves for various ξ values.

Applying the formula of geometric series in the case $y \neq \pm 1$, we obtain:

$$\sum_{k=m+1}^{l} y^{2k-1} = y^{2m+1} + y^{2m+3} + \cdots + y^{2l-1} = y^{2m+1}(1 + y^2 + \cdots + y^{2l-2m-2})$$
$$= y^{2m+1} \cdot \frac{1 - y^{2l-2m}}{1 - y^2}, \qquad (4.11_1)$$
$$\sum_{k=m+1}^{l-1} y^{2k} = y^{2m+2} + y^{2m+4} + \cdots + y^{2l-2} = y^{2m+2}(1 + y^2 + \cdots + y^{2l-2m-4})$$
$$= y^{2m+2} \cdot \frac{1 - y^{2l-2m-2}}{1 - y^2}. \qquad (4.11_2)$$

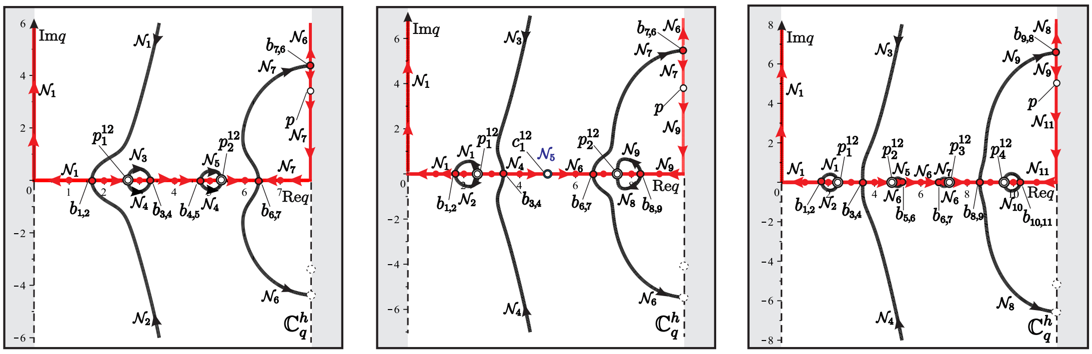
Applying these expressions the equation (4.10) can be rewritten in the another form:

$$y^{2n} = \frac{\gamma h}{3} \left(y^{2m} + y^{2l} + 4y^{2m+1} \cdot \frac{1 - y^{2l-2m}}{1 - y^2} + 2y^{2m+2} \cdot \frac{1 - y^{2l-2m-2}}{1 - y^2} \right). \qquad (4.12)$$



(a) $\xi = 1/3,\ n = 12$ (b) $\xi = 1/2,\ n = 8$ (c) $\xi = 2/3,\ n = 12$

Fig. 4.8. Discrete problem (4.1)–(4.2), (4.3$_2$), Real CF and Spectrum Curves for various ξ values.



(a) $\xi = 3/4,\ n = 8$ (b) $\xi = 4/5,\ n = 10$ (c) $\xi = 5/6,\ n = 12$

Fig. 4.9. Spectrum curves for discrete problem (4.1),(4.2), (4.3$_2$).

After some operations we get:

$$y^{2n} = \frac{\gamma h}{3} \cdot \frac{(y^{2l} - y^{2m})(y^2 + 4y + 1)}{y^2 - 1}. \tag{4.13}$$

If $y = \mathrm{e}^{\imath\pi q h}$, $q \neq 0$, $q \neq n$, then the equation (4.13) can be rewritten in another form:

$$\mathrm{e}^{\imath 2\pi q h n} = \frac{\gamma h}{3} \cdot \frac{(\mathrm{e}^{\imath 2\pi q h l} - \mathrm{e}^{\imath 2\pi q h m})(\mathrm{e}^{\imath 2\pi q h} + 4\mathrm{e}^{\imath\pi q h} + 1)}{\mathrm{e}^{\imath 2\pi q h} - 1}. \tag{4.14}$$

By the Euler's formula we have $y^k = \mathrm{e}^{\imath\pi q h k} = \cos(\pi q h k) + \imath \sin(\pi q h k)$. This expression we apply to equation (4.11), and we get:

$$\begin{aligned}
&\cos(2\pi qhn) + \imath\sin(2\pi qhn) = \\
&\quad = \frac{\gamma h}{3}\cdot(\cos(2\pi qhl) + \imath\sin(2\pi qhl) - \cos(2\pi qhm) + \imath\sin(2\pi qhm))\cdot \\
&\quad \cdot\frac{(\cos(\pi qh)+\imath\sin(\pi qh))^2 + 4(\cos(\pi qh)+\imath\sin(\pi qh)) + 1}{(\cos(\pi qh)+\imath\sin(\pi qh))^2 - 1}.
\end{aligned} \tag{4.15}$$

After some operations with real and image part of the equation (4.15), we get:

$$\begin{aligned}
&\cos(2\pi qhn) + \imath\sin(2\pi qhn) = \\
&\quad = \imath\frac{\gamma h}{3}\frac{(\cos(2\pi qhl) - \cos(2\pi qhm))(\cos(\pi qh)+2)}{\sin(\pi qh)} + \\
&\quad + \frac{\gamma h}{3}\cdot\frac{(\sin(2\pi qhl) - \sin(2\pi qhm))(\cos(\pi qh)+2)}{\sin(\pi qh)}.
\end{aligned} \tag{4.16}$$

We take only imaginary part of this expression:

$$\sin(2\pi qhn) = \frac{\gamma h}{3}\frac{(\cos(2\pi qhl) - \cos(2\pi qhm))(\cos(\pi qh)+2)}{\sin(\pi qh)}. \tag{4.17}$$

Generally, the characteristic function of FDS (4.6)–(4.6) is expressed in the formula (4.17). Cases $q = 0$, $q = n$ are valid as limit cases. □

On the case $l = n$, $m = m$ (Case 1) and $l = m$, $m = 0$ (Case 2) we get that CF of the discrete problem (4.1)–(4.3):

$$\gamma_{S1} = \frac{\sin(2\pi qhn)}{\cos(2\pi qhm) - \cos(2\pi qhn)}\cdot\frac{3\sin(\pi qh)}{h(2+\cos(\pi qh))}, \tag{4.18$_1$}$$

$$\gamma_{S2} = \frac{\sin(2\pi qhn)}{2\sin^2(2\pi qhm/2)}\cdot\frac{3\sin(\pi qh)}{h(2+\cos(\pi qh))}. \tag{4.18$_2$}$$

Zeros of the CFs (γ_{S1} and γ_{S2}) are positive and first order $z_k = k \in \mathbb{N}$ such that $z_k \in (0; 2n)$. Pole points could be defined by the same formula as for the approximation by the trapezoidal formula (see Lemma 4.2). In the case of approximation by Simpson's rule there exists a pole point in $\mathbb{R}_q^{h+} := \{q = 2n + \imath y, y > 0\}$. This pole we can find from the analysis of the multiplier $\frac{3\sin(\pi qh)}{qh(2+\cos(\pi qh))}$ (see (4.18)). The numerator of this fraction is equal to zero only for real q. So, in the domain $\mathbb{R}_q^{h+}$ CE point does not exist, but there exists a pole.

Lemma 4.4. *For the FDS (4.1)–(4.3) with integral NBC approximated by Simpson's rule in $\mathbb{R}_q^{h+}$ there exists the first order pole $p = 2n + \imath(2n\ln(2+\sqrt{3}))/\pi$.*

Proof. If $q \in \mathbb{R}_q^{h+}$ then the equation $-\cos(\pi q h) = 2$ can be rewritten in another form:

$$
\begin{aligned}
-\cos(\pi h(2n + \imath y)) &= -\cos\pi\cos(\imath\pi h y) \\
&= \cosh(\pi h y) = 2. \qquad (4.19)
\end{aligned}
$$

So, the equation above is equal to $e^{\pi h y} + e^{-\pi h y} = 4$. Denote $e^{\pi h y} = \lambda$. Then we have quadratic equation for λ:

$$\lambda^2 - 4\lambda + 1 = 0. \qquad (4.20)$$

The roots of this equation are

$$\lambda = 2 \pm \sqrt{3}. \qquad (4.21)$$

The formulas mentioned above allow us to get this formula:

$$y = \frac{2n}{\pi}\ln(2 + \sqrt{3}), \qquad (4.22)$$

because we look for $y > 0$. □

This pole point for different grid point number is shown in Figure 4.6–Figure 4.9. For example, if $2n = 6$, then $p \approx 6 \pm \imath\ 2.51520430$ (see Figure 4.6(a), (c), Figure 4.7(a), (c)).

For $2hn = 1$ and $2hm = \xi$, from the formula (4.18) we get:

$$
\begin{aligned}
\gamma_{S1} &= \frac{q\sin(\pi q)}{\cos(\xi\pi q) - \cos(\pi q)} \cdot \frac{3\sin(\pi q h)}{qh(2 + \cos(\pi q h))}, \qquad (4.23_1) \\
\gamma_{S2} &= \frac{q\sin(\pi q)}{2\sin^2(\xi\pi q/2)} \cdot \frac{3\sin(\pi q h)}{qh(2 + \cos(\pi q h))}. \qquad (4.23_2)
\end{aligned}
$$

If hq is a sufficiently small number, then $\frac{3}{\pi q h} \cdot \frac{\sin(\pi q h)}{2+\cos(\pi q h)} \approx 1$. It follows that, in this case, the discrete CF is similar to the CF of the differential problem [48, Pečiulytė *et al.* 2005].

If $q = \imath y$, $y > 0$, then $\lambda_k = -\frac{4}{h^2}\sinh^2(\frac{y_k h}{2}) < 0$ and Real CF is:

$$
\begin{aligned}
\gamma_{S1-} &= \frac{x\sinh(\pi x)}{\cosh(\xi\pi x) - \cosh(\pi x)} \cdot \frac{3\sinh(\pi x h)}{xh(2 + \cosh(\pi x h))}, \qquad (4.24_1) \\
\gamma_{S2-} &= \frac{x\sinh(\pi x)}{2\sinh^2(\xi\pi x/2)} \cdot \frac{3\sinh(\pi x h)}{h(2 + \cosh(\pi x h))}. \qquad (4.24_2)
\end{aligned}
$$

If $q = 2n + \imath y$, $y > 0$, then $\lambda_k = -\frac{4}{h^2}\cosh^2(\frac{y_k h}{2}) < 4/h^2$ and:

$$\gamma_{S1+} = \frac{x\sinh(\pi x)}{\cosh(\xi\pi x) - \cosh(\pi x)} \cdot \frac{3\sinh(\pi x h)}{xh(2+\cosh(\pi x h))}, \tag{4.25$_1$}$$

$$\gamma_{S2+} = \frac{x\sinh(\pi x)}{2\sinh^2(\xi\pi x/2)} \cdot \frac{3\sinh(\pi x h)}{h(2+\cosh(\pi x h))}. \tag{4.25$_2$}$$

In addition, the selection of the parameter n influences the spectrum structure of discrete problem. If the number of grid points grows, then the structure of spectrum becomes more similar to the differential problem.

5 Spectrum Curves at the points $q = 0$, $q = 2n$ and $q = \infty$

$q = 0$: Taylor series for $\gamma(q)$ at the point $q = 0$ is:

$$\gamma_{S1}(q) = \frac{2(2n)^2}{(2n)^2 - (2m)^2} - \frac{1}{6}q^2 + \mathcal{O}(q^4), \tag{5.1$_1$}$$

$$\gamma_{S2}(q) = \frac{2(2n)^2}{(2m)^2} - \frac{1}{6} \cdot \frac{2(2n)^2 - (2m)^2}{(2m)^2} q^2 + \mathcal{O}(q^4). \tag{5.1$_2$}$$

The second term is negative and not equals to zero ($2(2n)^2 - (2m)^2 > 0$ for all n and m, $m \leqslant n$ in Case 2), so the point $q = 0$ is the first order Critical point in domain $\mathbb{C}_q^h$, but $\lambda = 0$ is not a Critical point in domain $\mathbb{C}_\lambda$, because it is a first order branch point. At this point Spectrum Curve $\mathcal{N}_1$ turns orthogonal to the right, i.e. the first positive eigenvalue point reaches $q = 0$ and then this point moves along imaginary axis, as well as in the case of approximation integral condition by trapezoidal formula (see Chapter 3).

$q = 2n$: Taylor series for $\gamma(q)$ at the point $q = 2n$ is

$$\gamma_{S1}(q) = -\frac{24(2n)^2}{(2n)^2 - 4(2m)^2} + \frac{1}{2}\frac{7(2n)^2 - 4(2m)^2 + 8}{(2n)^2 - 4(2m)^2}(q - 2n)^2 + \mathcal{O}((q-2n)^4), \tag{5.2$_1$}$$

$$\gamma_{S2}(q) = -\frac{6(2n)^2}{(2m)^2} + \frac{1}{2} \cdot \frac{2(2n)^2 - (2m)^2 + 8}{(2m)^2}(q - 2n)^2 + \mathcal{O}((q-2n)^4). \tag{5.2$_2$}$$

It can be seen in Case 1, that $q = 2n$ is a first order Critical point in domain $\mathbb{C}_q^h$, because $n \geqslant m$ and $7(2n)^2 - 4(2m)^2 + 8 \neq 0$ for all n and m. In domain $\mathbb{C}_\lambda$, the point $\lambda = 4/h^2$ is not a Critical point. In Case 1, the second term negative if

$m < n/2$. In Case 2 the inequality $2(2n)^2 - (2m)^2 + 8 > 0$, is valid for $n \geqslant m$, too. In this case the second term is positive for all n and m. So, point $q = 2n$ is the first order Critical point in domain $\mathbb{C}_q^h$, but not a Critical point in domain $\mathbb{C}_\lambda$.

$\boldsymbol{q = \infty}$**:** Using Euler formula the equation (4.23) we can rewrite in the following form:

$$\gamma_{S1} = \frac{-(e^{\imath\pi q} - e^{-\imath\pi q})(e^{\imath\pi qh} - e^{-\imath\pi qh})}{(e^{\imath\pi q\xi} + e^{-\imath\pi q\xi} - e^{\imath\pi q} - e^{-\imath\pi q})(4 + e^{\imath\pi qh} + e^{-\imath\pi qh})} \cdot \frac{3}{h}, \tag{5.3$_1$}$$

$$\gamma_{S2} = \frac{(e^{\imath\pi q} - e^{-\imath\pi q})(e^{\imath\pi qh} - e^{-\imath\pi qh})}{(e^{\imath\pi q\xi} - 2 + e^{-\imath\pi q\xi})(4 + e^{\imath\pi qh} + e^{-\imath\pi qh})} \cdot \frac{3}{h}. \tag{5.3$_2$}$$

Then CF could be expressed in terms of $w = e^{\imath qh}$

$$\begin{aligned}\gamma_{S1} &= \frac{-(w^{2n} - w^{-2n})(w - w^{-1})}{(w^{2m} + w^{-2m} - w^{2n} - w^{-2n})(4 + w + w^{-1})} \cdot \frac{3}{h} \\ &= \frac{-w^{-2n-1}(1 - w^{4n})(1 - w^2)}{-w^{-2n-1}(1 - w^{2n+2m} - w^{2n-2m} + w^{4n})(1 + 4w + w^2)} \cdot \frac{3}{h},\end{aligned} \tag{5.4$_1$}$$

$$\begin{aligned}\gamma_{S2} &= \frac{(w^{2n} - w^{-2n})(w - w^{-1})}{(w^{2m} - 2 + w^{-2m})(4 + w + w^{-1})} \cdot \frac{3}{h} \\ &= \frac{w^{-2n-1}(1 - w^{4n})(1 - w^2)}{w^{-2m-1}(1 + w^{4m} - 2w^{2m})(1 + 4w + w^2)} \cdot \frac{3}{h}\end{aligned} \tag{5.4$_2$}$$

where $w \in \mathbb{C}_w^h$ (see Chapter 3). We can rewrite (5.4) as

$$\gamma_{S1} = \frac{1}{w^0} \frac{(1 + \mathcal{O}(w^{4n}))(1 + \mathcal{O}(w^2))}{(1 + \mathcal{O}(w^{4n}))(1 + 4\mathcal{O}(w))} \cdot \frac{3}{h} = \frac{3}{h}\mathcal{O}(1), \tag{5.5$_1$}$$

$$\gamma_{S2} = \frac{1}{w^{2n-2m}} \frac{(1 + \mathcal{O}(w^{4n}))(1 + \mathcal{O}(w^2))}{(1 + \mathcal{O}(w^{4m}))(1 + 4\mathcal{O}(w))} \cdot \frac{3}{h} = \frac{1}{w^{2n-2m}} \cdot \frac{3}{h}\mathcal{O}(1) \tag{5.5$_2$}$$

We remind, that the ratio of $\frac{1}{w^0}$ in Case 1 (or $\frac{1}{w^{2(n-m)}}$ in Case 2) describes the properties of the point $w = 0$ on $\mathbb{C}_w^h$, $w = 0 \notin \mathbb{C}_w^h$ (see Chapter 3). In the Case 1 $\lim_{w \to 0} \gamma(w) = 3/h$ and we have *removable singularity* point for all parameter ξ values. So, the same Spectrum Curve enters and leaves the point $w = 0$ and, additionally, there are no complex Spectrum Curves for all ξ values (see Figure 4.6). In Case 2, if $m = n$, then $\lim_{w \to 0} \gamma(w) = 3/h$ and we have *removable singularity* point, too. If $n - m_2 = 1$, then the point $w = 0$ is the second order pole point in $\mathbb{C}_w^h$ and one real and one complex Spectrum Curve enter to $w = 0$ and one real and one complex Spectrum Curves leaves the point (see Figure 4.7(b)–(c)). If $n - m_2 = 2$ then the point $w = 0$ is the fourth order pole point and

$2(n-2)$ complex Spectrum Curves enter and leave the point $w=0$. So, in Case 2 if $n>m$, the the point $w=0$ is pole of order even number $2(n-m)$ and $2(n-m)-2$ complex Spectrum Curves enter and leave this point in domain $\mathbb{C}_w^h$ (see Figure 4.7–Figure 4.9).

6 Investigation of the spectrum structure

For discrete problem (4.1)–(4.2),(4.3_1) complex eigenvalues do not exist as in the case of differential problem (1.1)–(1.2_1), and discrete problem (2.1)–(2.2),(2.3_1) when integral NBC was approximated using trapezoidal rule (see Figure 4.6). In this case, there exists horizontal asymptote if $\gamma = 3/h$ for Real CF. If γ satisfies the condition $\gamma = 3/h$, we have degenerate problem and there exist only $n-2$ eigenvalues. For the both cases of FDS (4.1)–(4.3) grid point $q=2n$ is not a pole for any parameter ξ value (see Figure 4.1(d)–(e), Figure 4.6(a)–(c) in Case 1 and Figure 4.3(a)–(c), Figure 4.7(a)–(c) in Case 2). So, Spectrum Curve moves through the point $q=2n$. For the FDS (4.1)–(4.3), there exists one pole in $\mathbb{R}_q^{h+}$, that does not depend on NBC's parameter ξ value, but depends on the number of grid point: $p = 2n + \imath\,\frac{2n}{\pi}\ln(2+\sqrt{3})$.

In case of FDS (4.1)–(4.2), (4.3_2) there exist real and complex eigenvalues (see Figure 4.7–Figure 4.9). In this case complex eigenvalues exist for all ξ values. For same parameter ξ values complex part of the spectrum is very complicated (see Figure 4.9)).

After comparison of the Figure 4.7 and Figure 4.7–Figure 4.8 we can see that increasing number of grid point makes the spectrum more similar to differential problem (1.1), (1.2).

7 Conclusions

The spectra of FDS's (2.1)–(2.3) and (4.1)–(4.3) in Case 1 and Case 2 are different:

- Real CF for FDS (2.1)–(2.3_1) has a horizontal asymptote if $\gamma = \frac{2}{h}$. Real CF for FDS (4.1)–(4.3_1) has a horizontal asymptote if $\gamma = \frac{3}{h}$.

- For discrete problem (2.1)–(2.3) we get pole at the point $q = n$ if $n - m$ is even in Case 1 and m is even in Case 2. In the case of problem (4.1)–(4.3) the point $q = 2n$ is not a pole for all parameter n and m, but there exists a pole $p = 2n + \imath\, \frac{2n}{\pi} \ln(2 + \sqrt{3})$ in $\mathbb{R}_q^{h+}$.

- The point $q = \infty$ for FDS (2.1)–(2.2), (2.3$_2$) is a pole (see Chapter 3). For discrete problem (4.1)–(4.3$_2$) the point $q = \infty$ is a *removable singularity* point for all $\xi \in (0, 1)$. Complex eigenvalues for the discrete problem (4.1)–(4.2), (4.3$_2$) exist for all parameters n and m, $m < n$ values, because the point $q = \infty$ is a pole of the $2(n - m)$ order. So $2(n - m)$ Spectrum Curves enter and leave this point.

- With an increase in the value n, the spectra of FDS (2.1)–(2.3) and (4.1)–(4.3) become more similar to that spectrum of the differential problem.

Conclusions

During the doctoral studies at Vilnius University we have studied the SLP with one classical and another type NBC. From the results obtained in the previous chapters we derive the following conclusions:

- In Chapter 1 we investigate the spectrum of SLP with one integral NBC depending on three parameters. One of our results is the classification of poles, zeroes and CE points. The dependence of these point on the integral BC parameters ξ_1 and ξ_2 is analyzed, too. Also, we classified Critical points and we have found trajectories (numerically) of the first order Complex Critical points and the second order (Real) Critical points in the Phase Space $S_{\boldsymbol{\xi}}$.

- In Chapter 2, the complex spectrum of the SLP with the classical or first type BC on the left side of the interval and integral NBC of two types on the right side of the interval was analyzed. In Case 1 and Case 1′ there exist only real eigenvalues. In Case 2 there are two types of bifurcation: two different Spectrum Curves intersect at the Critical point; zero and pole points coincide with the Critical point, i.e., appears CE. In Case 2′ there exists the second order Critical point when the loop type Spectrum Curve intersect with other Spectrum Curves.

  Also, SLP with the symmetric interval in the integral was analyzed. In this case, the behaviour of Spectrum Curves is quite similar to Case 2′.

- In Chapter 3 dSLP with one integral NBC depending on three parameters was analyzed. The integral condition was approximated by the trapezoidal rule. If $m_2 = n$, then there exists a horizontal asymptote $\gamma(\infty) = \lim_{q\to\infty}\gamma(q) = \frac{2}{h}$ of Real CF. The number of Spectrum Curves that enters and leaves the point $q = \infty$ depends on m_2, only. The point $q = n$ is the first order Critical point, if m_1, m_2 and n satisfy the condition $m_1^2 + m_2^2 = (1 + 2n^2)/3$.

- In Chapter 4 we have analyzed special cases of dSLP with one integral NBC. The integral condition was approximated by trapezoidal and Simpson's rule.

In case of approximation by trapezoidal rule, for some parameter ξ values the point $q = n$ is a pole. In case of approximation by Simpson's rule the point $q = 2n$ is not a pole for any parameter ξ values, but there exists the first order pole $p = 2n + \imath(2n \ln(2 + \sqrt{3}))/\pi$. If the number of grid point is increasing in both cases, the spectrum becomes similar to the differential problem.

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