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GREEN'S FUNCTIONS FOR BOUNDARY-VALUE PROBLEMS WITH NONLOCAL BOUNDARY CONDITIONS

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VILNIAUS UNIVERSITETAS

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Introduction

Problem formulation

In the dissertation Green's functions for the second-order and higher-order differential and difference equations with additional conditions, which are described by linearly independent linear functionals, are investigated. A lot of attention is given to the problems with nonlocal boundary conditions.

Topicality of the problem

In the theory of differential equations the basic concepts have been formulated studying the problems of classical mathematical physics. However, the modern problems motivate to formulate and investigate the new ones, for example, a class of nonlocal problems. Nonlocal conditions arise when we cannot measure data directly at the boundary. In this case, the problem is formulated where the value of the solution and/or a derivative is linked to a few points or the whole interval.

In 1963 J.R. Cannon [6] formulated new problem with boundary conditions which are now called *nonlocal* (N.I. Ionkin was the first who used the concept of *nonlocal boundary condition* in 1977 [24]). The similar problem was investigated by L.I. Kamynin in 1964 [27]. A.A. Samarskii and A.V. Bitsadze formulated the elliptical problem with nonlocal boundary conditions in 1969 [5]. Now some one-dimensional nonlocal conditions are called as Bitsadze–Samarskii conditions. Differential equations (for example, ordinary, elliptic, parabolic, etc.) with various types of nonlocal conditions were investigated by scientists: V.A. Il'in [16], E.I. Moiseev [17, 18], N.I. Ionkin [23, 25], A.V. Gulin [13, 12, 14, 15], J.R.L. Webb [51, 52], G. Infante [21, 22] and others. In Lithuania such problems were investigated by M. Sapagovas [10, 26, 31, 37, 38, 39, 40, 41, 42], R. Čiegis [7, 8, 9], A. Štikonas [32, 33, 43, 45] and their students.

In this dissertation Green's functions of the second- and higher-order differential and difference equations with various conditions (for example, initial, boundary or nonlocal conditions) are investigated.

In mathematics, Green's function is a type of function used to solve nonhomogeneous equations subject to boundary conditions. Green's functions play an important role in the theory of linear Ordinary and Partial Differential Equations. The term is used in physics, specifically in quantum field theory, electrodynamics and statistical field theory. It helps to investigate the existence and uniqueness of the solutions for many boundary problems.

Green's functions for problems with classical boundary conditions were analyzed by V.S. Vladimirov [48], U.M. Ascher, R.D. Russell and R.M.M. Mattheij [3], I. Stakgold [44], D.G. Duffy [11]. Besides A.A. Samarskii [34], E.S. Nikolaev [36], A.V. Gulin [35], N.S. Bahvalov, H.P. Židkov and G.M. Kobel'kov [4] studied Green's functions for problems with difference operator.

The investigation of the semi-linear problems with nonlocal boundary conditions and the existence of positive solutions are based on the investigation of Green's function for the linear problems with nonlocal boundary conditions.

Green's functions for the second- and higher-order boundary problems with various nonlocal conditions were constructed by D.R. Anderson [1, 2], J.R.L. Webb [21, 22, 49, 50], G. Infante [19, 20], R.Y. Ma [28, 29, 30], Y. Sun [46], L.X. Truong [47], B. Yang [54], Z. Zhao [55, 56] D. Xie, Y. Liu and C. Bai [53] and other scientists. In their works authors considered the existence and multiplicity of solutions applying various methods: lower and upper solution method, Leggett–Williams fixed-point theorem, Guo–Krasnoselskii fixed-point theorem, Leray–Schauder continuation principle, Avery–Peterson fixed-point theorem.

In the *first chapter* of this dissertation, the m-order linear differential equation with m additional conditions, which can be as classical as nonlocal, is considered. These conditions are formulated by means of independent linear functionals. Also in the first chapter, the formula, describing the relation between two problems with nonclassical and classical boundary conditions, is deduced.

In the *second chapter* the second-order differential equation is formulated. The results of the first chapter are applied to this problem. The examples show how Green's function for problems with varying complexity of nonlocal boundary conditions is constructed.

The differential equations, considered in the first and second chapters, can be approximated by the difference equations. In the *third chapter* the second-order difference equation with two conditions is investigated.

In the *fourth chapter* the results of the third chapter are generalized to the m-order difference equation with m additional conditions.

Research object

The main research object of the dissertation is differential and difference operators with various conditions, the solutions and Green's functions of these problems, the existence conditions of Green's functions and the application of the results for problems with nonlocal boundary conditions.

Aim and task of the work

The aim of this dissertation is to investigate the differential and discrete problems with additional conditions; to write down the expression of the solution for nonhomogeneous equation, if the fundamental solution is known; to find Green's function for these problems and to obtain a relation between two Green's functions for problems with the same equation and various conditions.

The main tasks of the work are:

1. to investigate linear space of the solutions for homogeneous equation;
2. to investigate the solutions of differential or discrete nonhomogeneous problems with the same equation, but with various conditions;
3. to find Green's functions for the differential and the difference problems and the existence conditions for them;
4. to investigate Green's functions for two problems with the same equation, but with various conditions;
5. to apply results to problems with nonlocal boundary conditions;
6. to compare results for differential or discrete problems.

Methodology of research

In the dissertation the method of variation of parameters is applied. Using this method and properties of determinants, Green's functions for differential and difference equations with variable coefficients and various boundary conditions have been obtained. Also we describe linear functionals and functional determinants, which properties have been used. By means of packages "Maple" graphs are represented.

Scientific novelty

Green's functions are often constructed to study the existence and multiplicity of the solutions for the second-order and higher-order semi-linear problems. In almost all cases Green's function is searched for problem, which is a special case of the problem of this thesis.

In this thesis Green's functions are investigated for problems with general linear conditions. The relation between two Green's functions is obtained, which helps to investigate Green's function for problem with nonlocal boundary conditions, if Green's function for problem with classical boundary conditions is known. Mostly in the various monographs and articles the formulae of this type have been obtained for the problem with classical boundary conditions, but in this work they have been deduced for problem with any (linear) conditions. So, we can write the expression of Green's function for problem with (for example, nonlocal) conditions if we know the fundamental basis of homogeneous equation and Green's function of problem with other (for example, classical) conditions. The obtained results can be applied to problems with nonlocal boundary conditions.

Practical value

In the doctoral dissertation the obtained results might be applied for solving the differential or difference equations with the conditions of various (for example, initial, boundary, nonlocal conditions), and for investigating the existence, the uniqueness and the properties of the solutions.

Defended propositions

- The necessary and sufficient condition for the existence of Green's function.
- The expressions of Green's functions for the second- and higher-order differential and difference equations with various linear conditions.

- The relation between Green's function and Green's function for the problem with initial conditions.
- The relation between two Green's functions.
- The application of the obtained results to problems with nonlocal boundary conditions.

The scope of the scientific work

The doctoral dissertation consists of the introduction, four chapters, conclusions, the list of the references and the other one of author's publications. The total scope of the doctoral dissertation is 129 pages, 4 figures and 4 tables. The results of doctoral dissertation are published in 9 publications. The results were presented at 4 national and 5 international conferences. The language of the doctoral dissertation is Lithuanian.

Chapter 1. The m-Order Differential Problem

In the first chapter we investigate m-order differential equation with additional conditions

$$\mathcal{L}u := u^{(m)} + a^{m-1}(x)u^{(m-1)} + \cdots + a^1(x)u' + a^0(x)u = f(x), \tag{1}$$

$$\langle L_i, u\rangle = f_i \in \mathbb{K}, \quad i = 1, \dots, m, \tag{2}$$

where $a^i \in C[0, L]$, $i = 0, \dots, m-1$, $f_i \in \mathbb{K} := \mathbb{R}, \mathbb{C}$, $i = 1, \dots, m$, $f \in C[0, L]$, and $L_1, \dots, L_m$ are linearly independent functionals.

Let $\boldsymbol{L} = (L_1, \dots, L_m)$ and $\boldsymbol{w} = [w^1, \dots, w^m]$ be fundamental system of homogeneous equation (1), and we denote

$$H(x) := \begin{cases} 1, & x > 0, \\ 0, & x \leqslant 0, \end{cases}, \quad G^c(x, s) := H(x-s)\frac{\widetilde{W}[\boldsymbol{w}](x, s)}{W[\boldsymbol{w}](s)}, \quad x, s \in [0, L],$$

if $W[\boldsymbol{w}](s) \neq 0$, where

$$W[\boldsymbol{w}](s) := \begin{vmatrix} w^1(s) & (w^1)'(s) & \cdots & (w^1)^{(k-2)}(s) & (w^1)^{(k-1)}(s) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ w^k(s) & (w^k)'(s) & \cdots & (w^k)^{(k-2)}(s) & (w^k)^{(k-1)}(s) \end{vmatrix}, \quad s \in [0, L],$$

$$\widetilde{W}[\boldsymbol{w}](x,s) := \begin{vmatrix} w^1(s) & (w^1)'(s) & \cdots & (w^1)^{(k-2)}(s) & w^1(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ w^k(s) & (w^k)'(s) & \cdots & (w^k)^{(k-2)}(s) & w^k(x) \end{vmatrix}, \quad x, s \in [0, L].$$

We use functional $\langle \delta_x, w \rangle = w(x)$ and determinants:

$$D(\boldsymbol{L})[\boldsymbol{w}] := \begin{vmatrix} \langle L_1, w^1 \rangle & \cdots & \langle L_k, w^1 \rangle \\ \cdots & \cdots & \cdots \\ \langle L_1, w^k \rangle & \cdots & \langle L_k, w^k \rangle \end{vmatrix},$$

$$D(\boldsymbol{L}, \delta_x)[\boldsymbol{w}, G(\cdot, s)] := \begin{vmatrix} \langle L_1, w^1 \rangle & \cdots & \langle L_k, w^1 \rangle & w^1(x) \\ \cdots & \cdots & \cdots & \cdots \\ \langle L_1, w^k \rangle & \cdots & \langle L_k, w^k \rangle & w^k(x) \\ \langle L_1(\cdot), G(\cdot, s) \rangle & \cdots & \langle L_k(\cdot), G(\cdot, s) \rangle & G(x, s) \end{vmatrix}.$$

We denote the linear space of solutions of equation (1) as $S := \{u \in C^m[0, L] \colon \mathcal{L}u = 0\}$. Let us introduce new functions

$$\hat{v}^i[\boldsymbol{u}](x) := D(L_1, \ldots, L_{i-1}, \delta_x, L_{i+1}, \ldots, L_m)[\boldsymbol{u}], \quad i = \overline{1, m}. \tag{3}$$

The next lemma is valid.

Lemma 1 [see thesis, 1.4 lema]. *Let $\{u^1, \ldots, u^m\}$ be the basis of linear space S. Then the next propositions are equivalent*:

a) *The functionals $L_1, \ldots, L_m$ are linearly independent*;

b) *The functions $\hat{v}^1, \ldots, \hat{v}^m$ are linearly independent*;

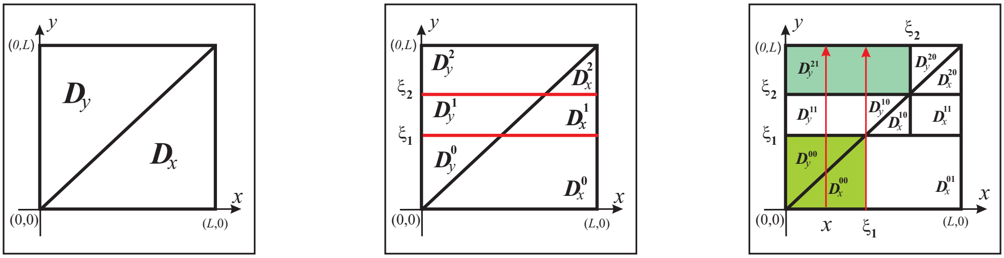
c) $D(\boldsymbol{L}) \neq 0$.

If the functionals $L_1, \ldots, L_m$ are linearly independent, and

$$v^i(x) := \frac{D(L_1, \ldots, L_{i-1}, \delta_x, L_{i+1}, \ldots, L_m)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]} = \frac{\hat{v}^i[\boldsymbol{u}](x)}{D(\boldsymbol{L})[\boldsymbol{u}]}, \quad i = \overline{1, m},$$

then two bases $\{v^1, \ldots, v^m\}$ and $\{L_1, \ldots, L_m\}$ are biorthogonal: $\langle L_i, v^j \rangle = \delta_i^j$, $i, j = \overline{1, m}$.

Let us consider the finite set $\Xi_N = \{\xi_0, \xi_1, \ldots, \xi_N\}$ where $\xi_0 = 0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = L$ ($N \in \mathbb{N}$). The lines $y = \xi_i$, $i = \overline{0, N}$ as well as $y = x$ divide the square $[0, L]^2$ into triangles and trapezoids: D_x^i, D_y^i, $i = \overline{0, N-1}$ (see Fig. 1(b)). We use this



(a) classical fundamental solution

(b) generalized fundamental solution

(c)

Figure 1. Domains of the fundamental solutions

notation for triangles and trapezoids with boundary, i.e., D_x^i, D_y^i, $i = \overline{0, N-1}$ are closed sets.

Definition 1 [see thesis, 1.2 apibrėžimas]. The function $g(x, y)$ defined on $0 \leqslant x, y \leqslant L$ is called the *Generalized Fundamental Solution* (GFS) of homogeneous differential equation (1) if it has the following properties:

1. $g(x, y)$ has m partial derivatives in x in each figure D_y^i and D_x^i, $i = \overline{0, N-1}$ and these derivatives are continuous in both x and y;
2. $g(x, y)$ (as a function of x) satisfies homogeneous equation (1) in each of those figures, i.e., $\mathcal{L}_x g(x, y) = 0$;
3. $g(x, y)$ is continuous in each rectangle $D_x^i \cup D_y^i$, $i = \overline{0, N-1}$ and has partial derivatives in x up to order $m-2$ and derivatives are continuous in both x and y in these rectangles;
4. The equality
$$\frac{\partial^{m-1} g(y+0, y)}{\partial x^{m-1}} - \frac{\partial^{m-1} g(y-0, y)}{\partial x^{m-1}} = 1$$
is valid for $y \in [0, L] \smallsetminus \varXi_N$.

Definition 2 [see thesis, 1.3 apibrėžimas]. A Generalized Fundamental Solution $G(x, y)$ of equation (1) is called *Green's function* for problem (1)–(2) if it is satisfying (as a function of x) homogeneous additional conditions (2) for $y \in [0, L] \smallsetminus \varXi_N$, i.e.,

$$\big\langle L_i(x), G(x, y) \big\rangle = 0, \quad i = 1, \dots, m, \; y \in [0, L] \smallsetminus \varXi_N.$$

If Green's function $G(x,y)$ exists for problem (1)–(2) then its solution allows the following integral representation:

$$u(x) = \big(G(x,y), f(y)\big)_X = \int_0^L G(x,y) f(y)\, dy.$$

If functionals $L_1, \dots, L_m$ are linearly independent then the following propositions about Green's functions are valid.

Lemma 2 [see thesis, 1.6 lema]. *Green's function for differential equation* (1) *with homogeneous additional conditions* $\langle L_1, u\rangle = 0$, …, $\langle L_m, u\rangle = 0$ *is equal to*:

$$\begin{aligned} G(x,s) &= \big\langle \delta_x(y) - \boldsymbol{L}(y)\boldsymbol{v}(x), G^c(y,s)\big\rangle \\ &= G^c(x,s) - \sum_{i=1}^m \big\langle L_i(y), G^c(y,s)\big\rangle \frac{D(L_1,\dots,L_{i-1},\delta_x,L_{i+1},\dots,L_m)}{D(\boldsymbol{L})} \\ &= \frac{1}{D(\boldsymbol{L})[\boldsymbol{u}]} \begin{vmatrix} \langle L_1, u^1\rangle & \cdots & \langle L_m, u^1\rangle & u^1(x) \\ \cdots & \cdots & \cdots & \cdots \\ \langle L_1, u^m\rangle & \cdots & \langle L_m, u^m\rangle & u^m(x) \\ \langle L_1(\cdot), G^c(\cdot,s)\rangle & \cdots & \langle L_m(\cdot), G^c(\cdot,s)\rangle & G^c(x,s) \end{vmatrix} \\ &= \frac{D(\boldsymbol{L},\delta_x)[\boldsymbol{u}, G^c(\cdot,s)]}{D(\boldsymbol{L})[\boldsymbol{u}]}. \end{aligned}$$

For the theoretical investigation of problems with nonlocal boundary conditions the next result about relations between Green's functions $G^u(x,s)$ and $G^v(x,s)$ of two inhomogeneous problems:

$$\begin{cases} \mathcal{L}u = f, \\ \langle l_i, u\rangle = 0, \quad i = 1,\dots,m, \end{cases} \qquad \begin{cases} \mathcal{L}v = f, \\ \langle L_i, v\rangle = 0, \quad i = 1,\dots,m, \end{cases} \tag{4}$$

is useful.

Theorem 1 [see thesis, 1.2 teorema]. *The relations between Green's functions* $G^v(x,s)$ *and* $G^u(x,s)$ *for problems* (4) *are*:

$$\begin{aligned} G^v(x,s) &= \big\langle \delta_x(y) - \boldsymbol{L}(y)\boldsymbol{v}(x), G^u(y,s)\big\rangle \\ &= G^u(x,s) - \sum_{j=1}^m \big\langle L_j(y), G^u(y,s)\big\rangle \frac{D(L_1,\dots,L_{j-1},\delta_x,L_{j+1},\dots,L_m)}{D(\boldsymbol{L})} \end{aligned}$$

$$= \frac{1}{D(\boldsymbol{L})[\boldsymbol{u}]} \begin{vmatrix} \langle L_1, u^1 \rangle & \cdots & \langle L_m, u^1 \rangle & u^1(x) \\ \cdots & \cdots & \cdots & \cdots \\ \langle L_1, u^m \rangle & \cdots & \langle L_m, u^m \rangle & u^m(x) \\ \langle L_1(\cdot), G^u(\cdot, s) \rangle & \cdots & \langle L_m(\cdot), G^u(\cdot, s) \rangle & G^u(x, s) \end{vmatrix}$$

$$= \frac{D(\boldsymbol{L}, \delta_x)[\boldsymbol{u}, G^u(\cdot, s)]}{D(\boldsymbol{L})[\boldsymbol{u}]}. \tag{5}$$

Let us investigate Green's function for the problem (1) with nonlocal boundary conditions

$$\mathcal{L}u := u^{(m)} + a^{m-1}(x)u^{(m-1)} + \cdots + a^1(x)u' + a^0(x)u = f(x), \tag{6}$$

$$\langle L_i, u \rangle := \langle \kappa_i, u \rangle - \gamma_i \langle \varkappa_i, u \rangle = 0, \quad i = \overline{1, m}. \tag{7}$$

We can write many problems with nonlocal boundary conditions (NBC) in this form, where $\langle \kappa_i, u \rangle := \langle \kappa_i(x), u(x) \rangle$ is a classical part, and $\langle \varkappa_i, u \rangle := \langle \varkappa_i(x), u(x) \rangle$, $i = \overline{1, m}$, is a nonlocal part of boundary conditions. For example, the functionals $\varkappa_i$, $i = \overline{1, m}$, can describe the multi-point ($\xi_j \in [0, 1]$, $j = \overline{1, m}$) or integral NBCs

$$\langle \varkappa, u \rangle = \sum_{j=1}^{m} \sum_{k=1}^{m} \left(\varkappa_j^k u^{(k-1)}(\xi_j) \right), \qquad \langle \varkappa, u \rangle = \int_0^1 \varkappa(t) u(t) \, dt,$$

and the functionals κ_i, $i = \overline{1, m}$, can describe the local (classical) boundary conditions.

If $\gamma_1, \ldots, \gamma_m = 0$, then problem (6)–(7) becomes classical. Suppose that there exists Green's function $G^{\mathrm{cl}}(x, s)$ for this classical case. Then, Green's function exists for problem (6)–(7) if $\vartheta := D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$. For $L_i = \kappa_i - \gamma_i \varkappa_i$, $i = \overline{1, m}$, we derive

$$\vartheta = \sum_{\sigma_1 = 0, \ldots, \sigma_m = 0}^{1} \prod_{j=1}^{m} (-\gamma_j)^{\sigma_j} D\left(\left(\varkappa_1^{\sigma_1} \kappa_1^{1-\sigma_1} \right), \ldots, \left(\varkappa_m^{\sigma_m} \kappa_m^{1-\sigma_m} \right) \right). \tag{8}$$

If we define the matrices $\mathbf{K} := (\kappa_{ij})$, $\kappa_{ij} = \langle \kappa_j, u_i \rangle$, $\mathbf{N} := (\varkappa_{ij})$, $\varkappa_{ij} = \langle \varkappa_j, u_i \rangle$, $\boldsymbol{\Gamma} := (\gamma_i \delta_{ij})$, then the condition $\vartheta \neq 0$ is equivalent to $\det(\mathbf{I} - \boldsymbol{\Gamma}\mathbf{N}\mathbf{K}^{-1}) \neq 0$. Since $\langle \kappa_i(\cdot), G^{\mathrm{cl}}(\cdot, s) \rangle = 0$, $i = \overline{1, m}$, we can rewrite formula (5) as

$$G(x, s) = G^{\mathrm{cl}}(x, s) + \sum_{j=1}^{m} \gamma_j \left\langle \varkappa_j(y), G^{\mathrm{cl}}(y, s) \right\rangle v^j(x). \tag{9}$$

The main results of this section are published in [A3,A5,A8].

Chapter 2. The Application of Green's Function to the Second-Order Differential Problems with Nonlocal Boundary Conditions

In the second chapter the results of the first chapter are applied to the second-order differential problem:

$$-(p(x)u')' + q(x)u = f(x), \tag{10}$$

$$\langle L_1, u\rangle = g_1, \qquad \langle L_2, u\rangle = g_2. \tag{11}$$

We investigate the differential equation $-u'' = f(x)$, $x \in (0,1)$ with various type of nonlocal boundary conditions:

$$
\begin{array}{llll}
u(0) = \gamma_1 u(\xi_1), & u(1) = \gamma_2 u(\xi_2); & u'(0) = \gamma_1 u(\xi_1), & u(1) = \gamma_2 u(\xi_2); \\
u(0) = \gamma_1 u(\xi_1), & u(1) = \gamma_2 u'(\xi_2); & u'(0) = \gamma_1 u(\xi_1), & u(1) = \gamma_2 u'(\xi_2); \\
u(0) = \gamma_1 u'(\xi_1), & u(1) = \gamma_2 u(\xi_2); & u'(0) = \gamma_1 u'(\xi_1), & u(1) = \gamma_2 u(\xi_2); \\
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u(0) = \gamma_1 u'(\xi_1), & u'(1) = \gamma_2 u'(\xi_2); & u'(0) = \gamma_1 u'(\xi_1), & u'(1) = \gamma_2 u'(\xi_2);
\end{array}
$$

or with integral nonlocal boundary conditions

$$u(0) = \gamma_1 \int_0^1 \alpha_1(x)u(x)\,dx, \qquad u(1) = \gamma_2 \int_0^1 \alpha_2(x)u(x)\,dx,$$

where $\alpha_1, \alpha_2 \in L^1(0,1)$. Also, the examples with the differential operators $\mathcal{L}u := -u''+u$, $\mathcal{L}u := -u'' - u$, $\mathcal{L}u := -u'' - \frac{2x}{1+x^2}u'$ are presented.

For these problems we can find the fundamental system for homogeneous equation. In all cases we find Green's function, expressing it either via Green's function for problem with initial conditions (see Lemma 2), or via Green's function for problem with classical boundary conditions, when $\gamma_1 = \gamma_2 = 0$ (see formula (9)).

The main results of this section are published in [A1,A2,B1].

Chapter 3. The Second-Order Discrete Problem

The differential equation of the second chapter can be approximated by the difference equation

$$\mathcal{L}u := a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \tag{12}$$

where $a^2, a^0 \neq 0$, which is investigated in the third chapter.

Let $X = \{0, 1, \dots, n\}$, $\tilde{X} = \{0, 1, \dots, n-2\}$. In discrete case, the notation and definition are as follows:

$$[\boldsymbol{u}]_{ij} = [u^1, u^2]_{ij} := \begin{pmatrix} u_i^1 & u_j^1 \\ u_i^2 & u_j^2 \end{pmatrix}, \qquad D[\boldsymbol{u}]_{ij} = \det[u^1, u^2]_{ij} := \begin{vmatrix} u_i^1 & u_j^1 \\ u_i^2 & u_j^2 \end{vmatrix},$$

$$W[\boldsymbol{u}]_j := \begin{vmatrix} u_{j-1}^1 & u_{j-1}^2 \\ u_j^1 & u_j^2 \end{vmatrix} = \begin{vmatrix} u_{j-1}^1 & u_j^1 \\ u_{j-1}^2 & u_j^2 \end{vmatrix} = D[\boldsymbol{u}]_{j-1,j}, \quad j = 1, \dots, n,$$

$$V[\boldsymbol{u}]_{ij} := \frac{D[\boldsymbol{u}]_{j+1,i}}{W[\boldsymbol{u}]_{j+2}} = \frac{D[\boldsymbol{u}]_{j+1,i}}{D[\boldsymbol{u}]_{j+1,j+2}}, \quad i \in X,\ j = -1, 0, 1, \dots, n-2,$$

when $W[\boldsymbol{u}]_{j+2} \neq 0$. We define $V[\boldsymbol{u}]_{i,n-1} = V[\boldsymbol{u}]_{in} = 0$, $i \in X$. Note that $V[\boldsymbol{u}]_{j+1,j} = 0$, $V[\boldsymbol{u}]_{j+2,j} = 1$ for $j \in \tilde{X}$. For the functionals $\boldsymbol{f} = (f_1, f_2)$ and the functions $\boldsymbol{w} = [w^1, w^2]$, we define the determinant:

$$D(\boldsymbol{f})[\boldsymbol{w}] :== \begin{vmatrix} \langle f_1, w^1 \rangle & \langle f_2, w^1 \rangle \\ \langle f_1, w^2 \rangle & \langle f_2, w^2 \rangle \end{vmatrix}.$$

We introduce a function $G^c \in F(X \times \tilde{X})$:

$$G_{ij}^c := H_{i-j} V_{ij} / a_j^2, \qquad \text{where } H_i := \begin{cases} 1, & i > 0, \\ 0, & i \leqslant 0. \end{cases}$$

We consider the solutions of homogeneous equation (12). Let $S \subset F(X)$ be a two-dimensional linear space of solutions. We investigate additional equations:

$$\langle L_1, u \rangle = 0, \qquad \langle L_2, u \rangle = 0, \quad u \in S, \tag{13}$$

where L_1, L_2 are linearly independent linear functionals and we use the notation $\boldsymbol{L} = (L_1, L_2)$. We introduce new functions

$$\bar{v}_i^1 := D(\delta_i, L_2)[\boldsymbol{u}], \qquad \bar{v}_i^2 := D(L_1, \delta_i)[\boldsymbol{u}]. \tag{14}$$

Lemma 3 [see thesis, 3.3 lema]. *Let $\{u^1, u^2\}$ be the basis of the linear space S. Then the following propositions are equivalent*:

a) *The functionals L_1, L_2 are linearly independent*;

b) *The functions $\bar{v}^1$, $\bar{v}^2$ are linearly independent*;

c) $D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$.

Suppose that $X_n := X = \{0, 1, \ldots, n\}$, $F(X) := \{u \mid u : X \to \mathbb{K}\}$ be a linear space of real (complex) functions. Let $A : F(X_n) \to F(X_{n-m}) = \operatorname{Im} A$ and $B : F(X_n) \to F(X_{M-n+m})$ be linear operators, $0 \leqslant m \leqslant n$. Consider the operator equation $Au = f$, where $u \in F(X_n)$ is unknown and $f \in F(X_{n-m})$ is given, with the additional operator equation $Bu = 0$.

If the solution of the problem $Au = f$, $Bu = 0$ allows the following representation:

$$u_i = \sum_{j=0}^{n-m} G_{ij} f_j, \quad i \in X_n,$$

then $G \in F(X_n \times X_{n-m})$ is called *discrete Green's function* of operator A with the additional condition $Bu = 0$. Green's function exists, if $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$. For $m = 2$, $u_i = \sum_{j=0}^{n-2} G_{ij} f_j$, $i \in X_n$.

Lemma 4 [see thesis, 3.5 lema]. *Green's function for problem* (12) *with homogeneous additional conditions $\langle L_1, u\rangle = 0$, $\langle L_2, u\rangle = 0$, where the functionals L_1 and L_2 are linearly independent, is equal to*:

$$G_{ij} = \frac{D(\boldsymbol{L}, \delta_i)[\boldsymbol{u}, G^c_{\cdot,j}]}{D(\boldsymbol{L})[\boldsymbol{u}]}, \quad i \in X, \ j \in \tilde{X}.$$

For the theoretical investigation of problems with NBCs, the next result about the relations between Green's functions G^u_{ij} and G^v_{ij} of two nonhomogeneous problems

$$\begin{cases} \mathcal{L}u = f, \\ \langle l_m, u\rangle = 0, \quad m = 1, 2, \end{cases} \qquad \begin{cases} \mathcal{L}v = f, \\ \langle L_m, v\rangle = 0, \quad m = 1, 2, \end{cases} \tag{15}$$

is useful.

Theorem 2 [see thesis, 3.2 teorema]. *If Green's function G^u exists and functionals L_1 and L_2 are linearly independent, then*

$$G^v_{ij} = \frac{D(\boldsymbol{L}, \delta_i)[\boldsymbol{u}, G^u_{\cdot,j}]}{D(\boldsymbol{L})[\boldsymbol{u}]}, \quad i \in X, \ j \in \tilde{X}. \tag{16}$$

Let us investigate Green's function for the problem with nonlocal boundary conditions

$$\mathcal{L}u := a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \quad i \in \tilde{X}, \tag{17}$$

$$\langle L_1, u\rangle := \langle \kappa_1, u\rangle - \gamma_1 \langle \varkappa_1, u\rangle = 0, \tag{18}$$

$$\langle L_2, u\rangle := \langle \kappa_2, u\rangle - \gamma_2 \langle \varkappa_2, u\rangle = 0. \tag{19}$$

We can write many problems with nonlocal boundary conditions (NBC) in this form, where $\langle \kappa_m, u\rangle := \langle \kappa_m^i, u_i\rangle$, $m = 1, 2$, is a classical part, and $\langle \varkappa_m, u\rangle := \langle \varkappa_m^i, u_i\rangle$, $m = 1, 2$, is a nonlocal part of boundary conditions.

If $\gamma_1, \gamma_2 = 0$, then problem (17)–(19) becomes classical. Suppose that there exists Green's function G_{ij}^{cl} for the classical case. Then Green's function exists for problem (17)–(19), if $\vartheta := D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$. For $L_m = \kappa_m - \gamma_m \varkappa_m$, $m = 1, 2$, we derive

$$\vartheta = D(\kappa_1 \cdot \kappa_2)[\boldsymbol{u}] - \gamma_1 D(\varkappa_1 \cdot \kappa_2)[\boldsymbol{u}] - \gamma_2 D(\kappa_1 \cdot \varkappa_2)[\boldsymbol{u}] + \gamma_1\gamma_2 D(\varkappa_1 \cdot \varkappa_2)[\boldsymbol{u}].$$

Since $\langle \kappa_m^k, G_{kj}^{\mathrm{cl}}\rangle = 0$, $m = 1, 2$, we can rewrite formula (16) as

$$\begin{aligned} G_{ij} &= G_{ij}^{\mathrm{cl}} + \gamma_1 \big\langle \varkappa_1^k, G_{kj}^{\mathrm{cl}}\big\rangle \frac{D(\delta_i, L_2)}{\vartheta} + \gamma_2 \big\langle \varkappa_2^k, G_{kj}^{\mathrm{cl}}\big\rangle \frac{D(L_1, \delta_i)}{\vartheta} \\ &= \frac{1}{\vartheta} \begin{vmatrix} \langle L_1, u^1\rangle & \langle L_2, u^1\rangle & u_i^1 \\ \langle L_1, u^2\rangle & \langle L_2, u^2\rangle & u_i^2 \\ -\gamma_1 \langle \varkappa_1^k, G_{kj}^{\mathrm{cl}}\rangle & -\gamma_2 \langle \varkappa_2^k, G_{kj}^{\mathrm{cl}}\rangle & G_{ij}^{\mathrm{cl}} \end{vmatrix}. \end{aligned} \tag{20}$$

The main results of this section are published in [A4].

Chapter 4. The m-Order Discrete Problem

In this chapter the results of the third chapter are generalized. We consider the m-order discrete problem with m additional conditions

$$a_i^m u_{i+m} + \cdots + a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \quad i \in \tilde{X} = \{0, 1, \ldots, n-m\}, \tag{21}$$

$$\langle L_1, u\rangle = g_1 \in \mathbb{K}, \quad \ldots, \quad \langle L_m, u\rangle = g_m \in \mathbb{K}. \tag{22}$$

The expression of the solution and Green's function for this problem are formulated; also the relation between two Green's functions for two problems with the same equation, but with various linear conditions, is given.

Let $\boldsymbol{L} = (L_1, \dots, L_m)$. We consider a determinant $D[\boldsymbol{u}]_{\boldsymbol{i}} : X^m \to \mathbb{K}$:

$$D[\boldsymbol{u}]_{\boldsymbol{i}} = \det[\boldsymbol{u}]_{\boldsymbol{i}} = \det\left[u^1, \dots, u^m\right]_{i_1 \dots i_m} := \begin{vmatrix} u^1_{i_1} & \cdots & u^1_{i_m} \\ \cdots & \cdots & \cdots \\ u^m_{i_1} & \cdots & u^m_{i_m} \end{vmatrix}.$$

The Wronskian determinant $W[\boldsymbol{u}]_j$ and similar determinant $\tilde{W}[\boldsymbol{u}]_{ij}$ in the theory of difference equations are denoted as follows

$$W[\boldsymbol{u}]_j := \begin{vmatrix} u^1_{j-m+1} & \cdots & u^m_{j-m+1} \\ \cdots & \cdots & \cdots \\ u^1_{j-1} & \cdots & u^m_{j-1} \\ u^1_j & \cdots & u^m_j \end{vmatrix} = D[\boldsymbol{u}]_{j-m+1,\dots,j}, \quad j = m-1, \dots, n,$$

$$\widetilde{W}[\boldsymbol{u}]_{ij} := \begin{vmatrix} u^1_{j-m+1} & \cdots & u^m_{j-m+1} \\ \cdots & \cdots & \cdots \\ u^1_{j-1} & \cdots & u^m_{j-1} \\ u^1_i & \cdots & u^m_i \end{vmatrix}, \quad i \in X,\ j = m-1, \dots, n+1.$$

Let (if $W[\boldsymbol{u}]_{j+m} \neq 0$)

$$V[\boldsymbol{u}]_{ij} := \frac{\widetilde{W}[\boldsymbol{u}]_{i,j+m}}{W[\boldsymbol{u}]_{j+m}}, \quad i \in X,\ j \in -1, \dots, n-m.$$

We define $V[\boldsymbol{u}]_{i,n-k} = 0,\ k = 0, \dots, m-1,\ i \in X$. Note that $V[\boldsymbol{u}]_{j+k,j} = 0,\ k = 1, \dots, m-1,\ V[\boldsymbol{u}]_{j+m,j} = 1$ for $j \in \tilde{X}$.

We introduce a function $G^{\mathrm{c}} \in F(X \times \tilde{X})$:

$$G^{\mathrm{c}}_{ij} := H_{i-j} V_{ij} / a^m_j.$$

Lemma 5 [see thesis, 4.2 lema]. *Green's function for problem* (21) *with the homogeneous additional conditions* $\langle L_1, u\rangle = 0,\ \dots,\ \langle L_m, u\rangle = 0$, *where functionals* $L_1, \dots, L_m$ *are linearly independent, is equal to*

$$G_{ij} = \frac{D(\boldsymbol{L}, \delta_i)[\boldsymbol{u}, G^{\mathrm{c}}_{\cdot,j}]}{D(\boldsymbol{L})[\boldsymbol{u}]}, \quad i \in X,\ j \in \tilde{X}. \tag{23}$$

For the theoretical investigation of problems with NBCs, the next result about the relations between Green's functions G^u_{ij} and G^v_{ij} of two nonhomogeneous problems

$$\begin{cases} \mathcal{L}u = f, \\ \langle l_k, u\rangle = 0, \quad k = 1, \dots, m, \end{cases} \qquad \begin{cases} \mathcal{L}v = f, \\ \langle L_k, v\rangle = 0, \quad k = 1, \dots, m, \end{cases} \tag{24}$$

is useful.

Theorem 3 [see thesis, 4.2 teorema]. *If Green's function G^u exists and the functionals $L_1, \dots, L_m$ are linearly independent, then*

$$\begin{aligned} G^v_{ij} &= \frac{D(\boldsymbol{L}, \delta_i)[\boldsymbol{u}, G^u_{\cdot,j}]}{D(\boldsymbol{L})[\boldsymbol{u}]} \\ &= \frac{1}{D(\boldsymbol{L})[\boldsymbol{u}]} \begin{vmatrix} \langle L_1, u^1\rangle & \cdots & \langle L_m, u^1\rangle & u^1_i \\ \cdots & \cdots & \cdots & \cdots \\ \langle L_1, u^m\rangle & \cdots & \langle L_m, u^m\rangle & u^m_i \\ \langle L^k_1, G^u_{kj}\rangle & \cdots & \langle L^k_m, G^u_{kj}\rangle & G^u_{ij} \end{vmatrix} \\ &= G^u_{ij} - \big\langle L^k_1, G^u_{kj}\big\rangle \frac{D(\delta_i, L_2, \dots, L_m)}{D(\boldsymbol{L})} - \cdots \\ &\quad - \big\langle L^k_m, G^u_{kj}\big\rangle \frac{D(L_1, \dots, L_{m-1}, \delta_i)}{D(\boldsymbol{L})}, \quad i \in X,\ j \in \tilde{X}. \end{aligned} \tag{25}$$

Let us investigate Green's function for the problem with nonlocal boundary conditions

$$\mathcal{L}u := a^m_i u_{i+m} + \cdots + a^1_i u_{i+1} + a^0_i u_i = f_i, \quad i \in \tilde{X}, \tag{26}$$

$$\langle L_l, u\rangle := \langle \kappa_l, u\rangle - \gamma_l \langle \varkappa_l, u\rangle = 0, \quad l = \overline{1, m}. \tag{27}$$

We can write many problems with nonlocal boundary conditions (NBC) in this form, where $\langle \kappa_l, u\rangle := \langle \kappa^i_l, u_i\rangle$, $l = \overline{1, m}$, is a classical part, and $\langle \varkappa_l, u\rangle := \langle \varkappa^i_l, u_i\rangle$, $l = \overline{1, m}$, is a nonlocal part of boundary conditions (BC).

If $\gamma_1, \dots, \gamma_m = 0$, then problem (26)–(27) becomes classical. Suppose that there exists Green's function G^{cl}_{ij} for the classical case. Then Green's function exists for problem (26)–(27), if $\vartheta := D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$. For $L_l = \kappa_l - \gamma_l \varkappa_l$, $l = \overline{1, m}$, we derive that

$$\vartheta = \sum_{j_1, \dots, j_m = 0}^{1} \gamma_1^{j_1} \cdots \gamma_m^{j_m} D\big(\big((1 - j_1)\kappa_1 - j_1 \varkappa_1\big), \dots, \big((1 - j_m)\kappa_m - j_m \varkappa_m\big)\big)[\boldsymbol{u}].$$

Since $\langle \kappa_l^k, G_{kj}^{\text{cl}} \rangle = 0$, $l = \overline{1, m}$, we can rewrite formula (25) as follows:

$$
\begin{aligned}
G_{ij} &= G_{ij}^{\text{cl}} + \gamma_1 \big\langle \varkappa_1^k, G_{kj}^{\text{cl}} \big\rangle \frac{D(\delta_i, L_2, \dots, L_m)}{\vartheta} + \cdots \\
&\quad + \gamma_m \big\langle \varkappa_m^k, G_{kj}^{\text{cl}} \big\rangle \frac{D(L_1, \dots, L_{m-1}, \delta_i)}{\vartheta} \\
&= \frac{1}{\vartheta} \begin{vmatrix} \langle L_1, u^1 \rangle & \cdots & \langle L_m, u^1 \rangle & u_i^1 \\ \cdots & \cdots & \cdots & \cdots \\ \langle L_1, u^m \rangle & \cdots & \langle L_m, u^m \rangle & u_i^m \\ -\gamma_1 \langle \varkappa_1^k, G_{kj}^{\text{cl}} \rangle & \cdots & -\gamma_m \langle \varkappa_m^k, G_{kj}^{\text{cl}} \rangle & G_{ij}^{\text{cl}} \end{vmatrix}.
\end{aligned}
\tag{28}
$$

The main results of this section are presented in [A9].

General conclusions

1. The m-order linear nonhomogeneous differential and discrete equations with various conditions are analyzed. The expressions of the solutions and Green's functions for these problems are obtained.

2. The independence of linear functionals is necessary and sufficient condition for the existence of Green's function for the problem which conditions are written with linear functionals.

3. Green's functions for problems with the same equation are related to each other. If the fundamental system of homogeneous equation and Green's function are known, then Green's function of the other problem with the same equation can be expressed via Green's function of the first problem.

4. The formulae can be applied to a very wide class of problems with nonconstant coefficients and various boundary conditions as well as nonlocal boundary conditions.

5. The expressions and relations for Green's function are the same in both differential and difference cases. The conditions of the existence of Green's function are also formulated in the same way.

List of Published Works on the Topic of the Dissertation

In the reviewed scientific periodical publications

[A1] S. Roman and A. Štikonas. Green's functions for stationary problems with nonlocal boundary conditions. *Lith. Math. J.*, **49**(2):190–202, 2009. Doi:10.1007/s10986-009-9041-0.

[A2] A. Štikonas and S. Roman. Stationary problems with two additional conditions and formulae for Green's functions. *Numer. Funct. Anal. Optim.*, **30**(9):1125–1144, 2009. Doi:10.1080/01630560903420932.

[A3] S. Roman and A. Štikonas. Third-order linear differential equation with three additional conditions and formula for Green's function. *Lith. Math. J.*, **50**(4):426–446, 2010. Doi:10.1007/s10986-010-9097-x.

[A4] S. Roman and A. Štikonas. Green's function for discrete second-order problems with nonlocal boundary conditions. *Bound. Value Probl.*, **2011**:1–23, 2011. Doi:10.1155/2011/767024.

[A5] S. Roman. Linear differential equation with additional conditions and formulae for Green's function. *Math. Model. Anal.*, **16**(3):401–417, 2011. Doi:10.3846/13926292.2011.602125.

[A6] S. Roman and A. Štikonas. Nelokaliųjų stacionariųjų kraštinių uždavinių Gryno funkcijos. *Liet. mat. rink. LMD darbai*, **48/49**:333–337, 2008.

[A7] S. Roman and A. Štikonas. The properties of Green's functions for one stationary problem with nonlocal boundary conditions. *Liet. mat. rink. LMD darbai*, **50**:340–344, 2009.

[A8] S. Roman and A. Štikonas. Linear ODE with nonlocal boundary conditions and Green's functions for such problems. *Liet. mat. rink. LMD darbai*, **51**:379–384, 2010.

[A9] S. Roman and A. Štikonas. Green's function for discrete problems with nonlocal boundary conditions. *Liet. mat. rink. LMD darbai*, **52**, 2011. (in press)

In the other editions

[B1] S. Roman and A. Štikonas. Green's functions for stationary problems with four-point nonlocal boundary conditions. In V. Kleiza, S. Rutkauskas and A. Štikonas(Eds.), *Differential Equations and Their Applications (DETA'2009)*, pp. 123–130. Kaunas University of Technology, 2009.

About the author

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GRYNO FUNKCIJOS UŽDAVINIAMS SU NELOKALIOSIOMIS KRAŠTINĖMIS SĄLYGOMIS

Problemos formulavimas

Disertacijoje tiriamos antros ir m-tosios eilės diferencialinių ir diskrečiiųjų lygčių su įvairiomis sąlygomis, kurios yra aprašytos tiesiškai nepriklausomais tiesiniais funkcionalais, *Gryno funkcijos.* Didelis dėmesys skirtas uždaviniams su nelokaliosiomis kraštinėmis sąlygomis.

Darbo aktualumas

Diferencialinių lygčių teorijoje pagrindinės sąvokos buvo suformuluotos tiriant klasikines matematinės fizikos uždavinius. Tačiau šiuolaikinės problemos skatina naujų uždavinių formulavimą ir tyrimą, pavyzdžiui, nelokaliųjų uždavinių klasė. Nelokaliosios sąlygos atsiranda tada, kai negalima tiesiogiai išmatuoti duomenų nagrinėjamo uždavinio srities krašte. Šiuo atveju formuluojami uždaviniai, kuriuose sprendinio ir / arba jo išvestinės reikšmės susijusios keliuose taškuose ar ištisame intervale.

1963 m. J. R. Kenonas nagrinėjo uždavinį, kuris dabar vadinamas *nelokaliuoju* (terminą *nelokaliosios kraštinės sąlygos*, pirmą sykį panaudojo N. I. Jonkinas 1977 m.). Panašų uždavinį tyrė L. I. Kamyninas 1964 m. Vieni iš pirmųjų nelokaliuosius kraštinius uždavinius 1969 m. nagrinėjo A. A. Samarskis ir A. V. Bitsadzė. Jų vardais pavadintos tam tikros nelokaliosios sąlygos vienamačiu atveju (t. y. Bitsadzės ir Samarskio sąlygos). Diferencialines lygtis (pavyzdžiui, paprastąją, elipsinę, parabolinę, ir kitas) su įvairaus tipo nelokaliosiomis sąlygomis nagrinėjo mokslininkai: Iljinas, Moisejevas, Jonkinas, Gulinas, Vebas, Infantė ir kt. Lietuvoje tokius uždavinius tyrė Sapagovas, Čiegis, Štikonas ir jų mokiniai.

Šioje disertacijoje tiriami antros ir aukštesnės eilės diferencialinių ir diskrečiųjų lygčių su įvairiomis sąlygomis (pavyzdžiui, su pradinėmis, kraštinėmis arba nelokaliosiomis sąlygomis) Gryno funkcijos.

Matematikoje Gryno funkcija naudojama sprendžiant nehomogenines lygtis atsižvelgiant į kraštines sąlygas. Tiesinių paprastųjų ir dalinių išvestinių diferencialinių lygčių teorijoje Gryno funkcija atlieka svarbų vaidmenį. Jos pagalba galima rasti stacionariųjų ir nestacionariųjų uždavinių sprendinius, ištirti sprendinių egzistavimą ir vienatį. Gryno funkcijos taikomos elektrostatikoje (Puasono lygties sprendimas), kondensuotųjų medžiagų teorijoje (difuzijos lygties sprendimas), kvantinėje mechanikoje (Šredingerio lygties sprendimas).

Uždavinių su klasikinėmis kraštinėmis sąlygomis Gryno funkcijas nagrinėjo V. S. Vladimirovas, U. M. Ašeris, R. M. M. Matteijus, R. D. Raselas, I. Stakgoldas, D. G. Dafis. Be to, A. A. Samarskis, E. S. Nikolajevas ir A. V. Gulinas tyrė diskrečiųjų uždavinių Gryno funkcijas.

Pusiau tiesinių uždavinių (dešinioji diferencialinės lygties pusė netiesiškai priklauso nuo ieškomosios funkcijos, o kairioji yra tiesinis diferencialinis operatorius) su nelokaliosiomis kraštinėmis sąlygomis tyrimas ir teigiamų sprendinių egzistavimas remiasi tiesinių uždavinių su nelokaliosiomis kraštinėmis sąlygomis Gryno funkcijos tyrimu.

Antros ir aukštesnės eilės kraštinių uždavinių su įvairiomis nelokaliosiomis kraštinėmis sąlygomis Gryno funkcijas konstravo D. R. Andersonas, G. Infantė, L. X. Truongas, J. R. L. Vebas, B. Jangas, Z. Zao ir kiti mokslininkai. Šiuose darbuose autoriai nagrinėjo sprendinių egzistavimą ir vienatį, naudodami įvairus metodus: apatinio ir viršutinio sprendimo metodą; Krasnoselskio nejudamojo taško teoremą; Averio ir Petersono nejudamojo taško teoremą; Lerėjaus ir Šauderio pratęsimo principą; Legeto ir Viljamso nejudamojo taško teoremą.

Šios disertacijos *pirmajame* skyriuje nagrinėjama m-tosios eilės tiesinė diferencialinė lygtis su įvairiomis sąlygomis, kurios gali būti ne tik klasikinės, bet ir nelokaliosios. Šios sąlygos formuluojamos nepriklausomų tiesinių funkcionalų pagalba. Taip pat pirmajame skyriuje išvesta formulė, apibendrinanti uždavinių su neklasikinėmis ir klasikinėmis kraštinėmis sąlygomis sąryšį.

Antrajame skyriuje suformuluota antros eilės diferencialinė lygtis. Pirmojo skyriaus rezultatai pritaikomi šiam uždaviniui. Pateikiami pavyzdžiai, kuriuose pavaizduota, kaip konstruojama uždavinių su įvairiomis nelokaliosiomis kraštinėmis sąlygomis Gryno funkcija.

Pirmojo ir antrojo skyriaus nagrinėjamas diferencialines lygtis galima aproksimuoti baigtinių skirtumų lygtimis. *Trečiajame* skyriuje tiriama antros eilės diskrečioji lygtis su dviem sąlygomis.

Ketvirtajame disertacijos skyriuje apibendrinti trečiojo skyriaus rezultatai m-tosios eilės diskrečiajai lygčiai su įvairiomis sąlygomis.

Tyrimų objektas

Disertacijos tyrimo objektas – diferencialinės ir diskrečiosios lygtys su įvairiomis sąlygomis, šių uždavinių sprendiniai ir Gryno funkcijos, jų egzistavimo sąlygos ir rezultatų pritaikymas uždaviniams su nelokaliosiomis kraštinėmis sąlygomis.

Darbo tikslas ir uždaviniai

Disertacijos tikslas – ištirti tiesinius diferencialinį ir diskretųjį uždavinius su įvairiomis sąlygomis, užrašyti nehomogeninės lygties sprendinių išraiškas, kai žinoma fundamentalioji sprendinių sistema, rasti Gryno funkcijas, taip pat nustatyti sąryšį tarp dviejų Gryno funkcijų uždaviniams su ta pačia lygtimi ir skirtingomis sąlygomis.

Siekiant numatyto tikslo buvo sprendžiami šie uždaviniai:

- ištirti homogeninės lygties sprendinių tiesinę erdvę;
- ištirti nehomoheninių diferencialinio ir diskrečiojo uždavinių sprendinius su ta pačia lygtimi, bet su skirtingomis sąlygomis;
- surasti diferencialinio ir diskrečiojo uždavinių Gryno funkcijas ir jų egzistavimo sąlygas;
- ištirti dviejų diferencialinių arba diskrečiųjų lygčių su skirtingomis sąlygomis Gryno funkcijas;
- pritaikyti gautus rezultatus uždaviniams su nelokaliosiomis kraštinėmis sąlygomis;
- palyginti diferencialinio ir diskrečiojo uždavinių rezultatus.

Tyrimų metodika

Disertacijos pagrindinių formulių išvedimui buvo taikomas konstantų varijavimo metodas. Naudojant šį metodą ir determinantų savybes, galima surasti Gryno funkcijas diferencialinėms ir diskrečiosioms lygtims su kintamais koeficientais ir įvairiomis kraštinėmis sąlygomis. Taip pat darbe aprašyti tiesiniai funkcionalai ir funkcionaliniai determinantai, kurių savybėmis buvo naudojamasi. Braižant grafikus, buvo naudojamas „Maple“ programų paketas.

Darbo mokslinis naujumas ir jo reikšmė

Tiriant antros ir aukštesnės eilės (tiek tiesinių, tiek netiesinių) uždavinių sprendinių egzistavimą ir vienatį, dažniausiai konstruojama Gryno funkcija tiesiniam uždaviniui. Beveik visais atvejais Gryno funkcija ieškoma uždaviniui, kuris yra šios disertacijos uždavinio atskiras atvejis.

Šioje disertacijoje nagrinėjamos Gryno funkcijos uždaviniams su bet kokiomis tiesinėmis sąlygomis. Rasta būtina ir pakankama Gryno funkcijos egzistavimo sąlyga. Taip pat gautas Gryno funkcijų sąryšis, kuris padeda ištirti Gryno funkciją uždaviniui su nelokaliosiomis kraštinėmis sąlygomis, žinant Gryno funkciją uždaviniui su pradinėmis arba klasikinėmis kraštinėmis sąlygomis. Dažniausiai įvairiose monografijose ir straipsniuose tokio tipo formulės buvo gautos uždaviniui su klasikinėmis kraštinėmis sąlygomis, bet šiame darbe jos yra išvestos uždaviniui su bet kokiomis (tiesinėmis) sąlygomis. Tokiu būdu, mes galime užrašyti Gryno funkcijos išraiška uždaviniui su įvairiomis (pavyzdžiui, nelokaliosiomis) sąlygomis, jei žinomos homogeninės lygties fundamentalioji bazė ir uždavinio su kitomis (pavyzdžiui, klasikinėmis) sąlygomis Gryno funkcija. Gautus rezultatus galima pritaikyti uždaviniams su nelokaliosiomis kraštinėmis sąlygomis (ir ne tik tokioms).

Darbo rezultatų praktinė reikšmė

Disertacijos gauti rezultatai gali būti panaudojami sprendžiant tiesines diferencialines arba diskrečiąsias lygtis su įvairiomis (pavyzdžiui, pradinėmis, kraštinėmis, nelokaliosiomis) tiesinėmis sąlygomis, o taip pat tiriant sprendinių egzistavimą, vienatį ir jų savybes.

Ginamieji teiginiai

- Būtina ir pakankama Gryno funkcijos egzistavimo sąlyga.
- Antros ir m-tosios eilės diferencialinio ir diskrečiojo uždavinių su įvairiomis tiesinėmis sąlygomis Gryno funkcijų išraiškos.
- Gryno funkcijos sąryšis su Gryno funkcija uždaviniui su pradinėmis sąlygomis.
- Uždavinių su ta pačia lygtimi, bet su skirtingomis sąlygomis, Gryno funkcijų sąryšis.
- Taikymai uždaviniams su nelokaliosiomis kraštinėmis sąlygomis.

Disertacijos struktūra

Disertaciją sudaro įvadas, 4 skyriai, išvados, literatūros sąrašas ir autorės publikacijų disertacijos tema sąrašas. Bendra disertacijos apimtis – 129 puslapiai, 4 grafikai, 4 lentelės. Disertacijos rezultatai skelbti 9 publikacijose. 1 publikacija priimta spaudai Lietuvos periodiniame recenzuojamame leidinyje. Šia tema skaityta 9 pranešimai mokslinėse konferencijose.

Šios disertacijos *pirmajame skyriuje* nagrinėjama m-tosios eilės diferencialinė lygtis su įvairiomis (įskaitant ir nelokaliąsias) sąlygomis. *Antrajame skyriuje* pirmojo skyriaus rezultatai pritaikyti antros eilės diferencialiniam uždaviniui. Pateikiami pavyzdžiai, kuriuose demonstruojama kaip konstruoti Gryno funkciją uždaviniams su įvairiomis nelokaliosiomis kraštinėmis sąlygomis. Antros eilės diferencialines lygtis galima aproksimuoti diskrečiosiomis lygtimis, kurios nagrinėjamos *trečiajame skyriuje*. Trečiojo skyriaus rezultatai apibendrinti *ketvirtajame skyriuje*. Šiame skyriuje nagrinėjamas m-tosios eilės diskretusis uždavinys su įvairiomis sąlygomis.

Bendrosios išvados

1. Išanalizuotos m-tosios eilės tiesinės nehomogeninės diferencialinė ir diskrečioji lygtys su įvairiomis sąlygomis. Gautos šių uždavinių sprendinių išraiškos ir Gryno funkcijos.

2. Tiesinio nehomogeninio uždavinio, kurio sąlygos užrašomos tiesiniais funkcionalais, būtina ir pakankama Gryno funkcijos egzistavimo sąlyga yra tų funkcionalų tiesinis nepriklausomumas.

3. Gryno funkcijos uždaviniams su ta pačia lygtimi yra susietos sąryšiu. Jei žinome vieną Gryno funkciją ir homogeninės lygties fundamentaliąją sistemą, tai kito uždavinio su ta pačia lygtimi, Gryno funkciją galima išreikšti per pirmojo uždavinio Gryno funkciją.

4. Formulės gali būti pritaikytos plačiai uždavinių klasei su nepastoviais koeficientais ir įvairiomis kraštinėmis sąlygomis, tame tarpe su nelokaliosiomis kraštinėmis sąlygomis.

5. Gautos Gryno funkcijų formulės yra vienodos tiek diferencialiniu, tiek diskrečiuoju atvejais. Gryno funkcijų egzistavimo sąlygos irgi formuluojamos vienodai.

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