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THE ANALYSIS OF THE GERBER-SHIU DISCOUNTED PENALTY FUNCTION

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1 Dissertation overview

1.1 The object of investigation

This thesis deals with Gerber-Shiu discounted penalty function under the classical risk model and the Erlang(2) model. Properties of the renewal counting process were also investigated.

1.2 Actualities

The ruin theory is a branch of insurance mathematics that studies theoretical aspects of insurers' solvency. This theory is based on mathematical modeling of the insurers' surplus and derivation of many ruin-related measures and quantities, such as ruin probability, insurers' surplus immediately prior to ruin, deficit at ruin, time of ruin, etc. It is also closely related with applied probability, as major part of the techniques adopted in the ruin theory are based on the application of stochastic processes. In 1903 Swedish actuary Filip Lundberg (see [5]) laid the foundations for this theory and introduced the classical compound Poisson risk model where company's surplus over time is described by the compound Poisson process. The process was further studied by H. Cramer (see [4]) and developed by other scientists. It was considered as the surplus process with dividend policies. Also interest rates were included into this process and fluctuations of the surplus were modeled by Brownian motion. The goal of all these cases was to find the distribution of the surplus prior to ruin and deficit at ruin, to derive the Laplace transform of the ruin time and to get the ruin probability. Gerber and Shiu (see [2]) in 1998 proposed to consider another quantity, which is also closely related with insurance company's work, i.e. discounted penalty function, known as the Gerber-Shiu function. This function describes the expectation of the present value of a future bankruptcy. According to the behavior of this function, insurance company can deal with risk and avoid the ruin through time. The fundamental work of Gerber and Shiu was followed by a large number of scientific publications in the ruin literature deriving the explicit expression of named function. Over past few years the asymptotics of the discounted penalty function has been studied, but only few cases were analysed and hence exploring of this subject is still very important.

The investigation of compound Poisson model and Gerber-Shiu discounted penalty function remains crucial nowadays. However, a lot of issues connected with both function and model have not been fully investigated yet. Therefore the analysis of the discounted penalty function under different risk models has still been very important and topical.

1.3 Goals and concerns

The objective of the thesis is to investigate the Gerbe-Shiu discounted penalty function in the classical risk model and to derive it's asymptotic formula in the Erlang(2) model. Special attention is devoted to the analysis of properties of the renewal counting process. The main points of the thesis are:

- To derive explicit expression of the Gerber-Shiu discounted penalty function in the classical risk model with mixed exponential claims and to examine this expression for various parameter choices;
- To obtain explicit expressions of the Gerber-Shiu discounted penalty function and discounted moments on finite time horizon and to derive Gerber-Shiu discounted penalty function on the time interval $[t_1, t_2]$;

- To obtain the asymptotic expression for the Gerber-Shiu discounted penalty function in the Erlang(2) model in case of subexponentially distributed clime sizes;
- To investigate main properties of the renewal counting process and to derive asymptotics for the tail of it's exponential moment.

1.4 Methods

General analytical methods are used in this thesis. Different techniques from probability and from functions' of complex variables theory are applied. In the proofs of the theorems' exponential transformation, Laplace transformation, renewal equation and residual methods are used.

1.5 Main results

During thesis defence the following results will be presented:

- Explicit expression of the Gerber-Shiu discounted penalty function in the classical risk model with mixed exponential claims.
- Explicit expressions of the Gerber-Shiu discounted penalty function on finite time horizon in the classical risk model with claims having exponential distribution.
- Asymptotic expression for the Gerber-Shiu discounted penalty function in the Erlang(2) model in case in case of subexponentially distributed clime sizes.
- Asymptotic property of renewal counting process.

1.6 Novelties

The conception of discounted penalty function under compound Poisson risk model, introduced by Gerber and Shiu, is very effective in solving various actuarial problems. This function is instrumental for analysis of such quantities as ruin probability, insurer's surplus prior to ruin, deficit at ruin and ruin time. Since Gerber and Shiu proposed mentioned function, the number of papers related to this methodology have been growing rapidly. In these papers the study of the discounted penalty function has been extended to more general risk models such as compound Poisson risk model perturbed by diffusion, the Cox risk model, the Markov modulated risk model, Levy model, etc. Also this function have been studied in cases where economical factors were incorporated into models, mainly, interest rate, dividends and stochastic return of investment.

In first two parts of the thesis the Gerber-Shiu discounted penalty function was deeply examined in the classical risk model. Generally, in scientific articles for this model was considered cases, when claim sizes and inter-arrival times both have exponential, Erlang and phase-type distributions. The main goal of many articles was to study properties of discounted penalty function and provide it's explicit expression. In the thesis explicit expression of named function was acquired in classical risk model under assumption that claims have mixed exponential distribution. The function was also analysed in classical risk model with exponential claims in a finite time horizon. In both cases the derived expressions of discounted penalty function were examined with various parameter choices. As a result, obtained expressions gave an opportunity to consider many analytical properties of the expected discounted penalty function.

The renewal counting process was also studied in the thesis. Since this process plays an important role in ruin theory and is widely applied with different risk models, it was interesting to investigate some asymptotic properties of this process. The asymptotic properties of renewal counting process may be used for examination of different ruin problems. Asymptotics of the tail of the exponential moment was derived in the thesis. The obtained result was applied to the asymptotics of the finite time ruin probability in a renewal risk model.

As mentioned above, only few cases of the asymptotic behavior of discounted penalty function were studied by scientist. It can be mentioned only few works. Šiaulys and Asanavičūtė [15] obtained an asymptotic formula for Gerber-Shiu discounted penalty function in compound Poisson model with subexponential claim sizes. Cheng and Tang [12] derived some asymptotic formulas for the moments of the surplus prior to ruin and deficit at ruin in the renewal risk model with convolution-equivalent clime sizes and Eralng(2) inter-arrival times. Tang and Wei [16] studied asymptotic bahavior of the Gerber-Shiu discounted penalty function in the renewal risk model with convolution-equivalent clime sizes and got some asymptotic formulas for this function.The goal of the thesis was to study the asymptotic problem in details and consider one more case. Special case of the Gerber-Shiu discounted penalty function in the Erlang(2) with subexponential claim sizes was considered. The asymptotic expression for this function was obtained. This result may be used for deeper studies of discounted penalty function.

1.7 Structure

The thesis is 63 pages long. Its' contents include introduction, four sections, conclusions and references. 61 scientific literature sources were used. The dissertation reflects investigated object, main goal and main problems of research.

1.8 Acknowledgements

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2 Dissertation's results

2.1 Main definitions and notions

The Gerber-Shiu discounted penalty function in the classical risk model was analysed in the first and second parts of the thesis. This model, proposed by E. Sparre Andersen [1] in 1957, has been applied to risk business of an insurance company. In this model the insurers' surplus process $\{U(t),\, t \geq 0\}$ is given by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \tag{2.1}$$

where u is insurer's initial surplus, c is the rate of premium income per unit time. $\{N(t),\, t \geq 0\}$ is a renewal counting process for the number of claims up to time t. As usually,

$$N(t) = \sum_{i=1}^{\infty} \mathbf{I}_{\{\theta_1+\theta_2+\ldots+\theta_i \leq t\}}, \tag{2.2}$$

where $\theta_1, \theta_2, ...$ independent identically distributed random variables, which represent the inter-arrival times. Individual claims $Y_1, Y_2, ...$ form a sequence of independent and identically distributed positive random variables with distribution function $H(y) = P(Y_1 \leq y)$ and finite mean EY_1. In addition, we suppose that claim series are independent of process $N(t)$.

In special case when θ_1 has exponential distribution, i.e.

$$P(\theta_1 \leq y) = \begin{cases} 0 & , \ y < 0, \\ 1 - e^{-\lambda y} & , \ y \geq 0, \end{cases}$$

the process $N(t)$ is homogeneous Poisson process with intensity λ and the risk model is called the classical risk model or the compound Poisson risk model.

Suppose that

$$T = \inf\{t > 0 : U(t) < 0 | U(0) = u\} \tag{2.3}$$

is the ruin time. If $U(t) \geq 0$ with all t, then $T = \infty$. The function

$$\psi(u) = P(T < \infty)$$

is called the ruin probability. Obviously, that ruin occurs with probability 1, i.e.

$$\psi(u) = P(T < \infty) = 1,$$

when $EY_1 - cE\theta_1 \geq 0$. Therefore, it assumed that safety condition should be hold and

$$c = \frac{EY_1}{E\theta_1}(1 + \widehat{\theta}),$$

with some positive security loading coefficient $\widehat{\theta} > 0$.

In 1998, Gerber and Shiu [2] proposed, instead of the probability of ruin the classical risk model, to analyze the discounted penalty function

$$\phi_\omega(u) = E(e^{-\delta T} \omega(U(T-), |U(T)|) \mathbf{I}_{\{T<\infty\}} | U(0) = u), \tag{2.4}$$

which describes the expectation of the present value of future ruin. Here $U(T-)$ denotes the surplus prior to ruin, $|U(T)|$ denotes the deficit at ruin and $\omega(x, y)$, $0 \leq x,\ y < \infty$ is some non-negative function, which can be interpreted as the penalty at the time to riun; $\delta > 0$ is a force of interest. In case when $\omega(x, y) \equiv 1$ for all x, y we have that function from (2.4) equality becomes

$$\phi(u) = E(e^{-\delta T} \mathbf{I}_{\{T<\infty\}} | U(0) = u).$$

If $\delta = 0$ and $\omega(x, y) \equiv 1$ we get

$$\psi(u) = P(T < \infty)$$

the ruin probability.

2.2 Discounted penalty function with claims having mixed exponential distribution

The discounted penalty function $\phi(u)$ in the classical risk model with mixed exponential claims was considered in the first section of the thesis. More precisely it was investigated the case, where for all $y \geq 0$ claims'

$Y_1, Y_2, ...$ distribution function is

$$P(Y_1 \leq y) = H(y) = \alpha(1 - e^{-\sigma y}) + (1 - \alpha)(1 - e^{-\nu y}), \tag{2.5}$$

where $\nu, \sigma > 0,\ \ 0 \leq \alpha \leq 1$. In this case the explicit expression of the discounted penalty function was obtained. The following statement is the main result of the first part of the dissertation.

Teorema 2.1 *Let individual claims in the classical model has d.f. H(y) defined by (2.5). Let, further, the parameter of Poisson process be $\lambda > 0$ and the relative security loading $\widehat{\theta} > 0$. Then*

$$\begin{aligned}
\phi(u) \;&=\; \sum_{n=1}^{\infty} \frac{(1-\phi)\phi^n}{(a+b)^n} \Bigg[b^n e^{-\nu u} \sum_{j=0}^{n-1} \frac{(u\nu)^j}{j!} + a^n e^{-\sigma u} \sum_{j=0}^{n-1} \frac{(u\sigma)^j}{j!} \\
&+\; \sum_{k=1}^{n-1} \binom{n}{k} (a\sigma)^k (b\nu)^{n-k} \left(\frac{(-1)^{k-1}}{(k-1)!} \mathcal{V}_1 + \frac{(-1)^{n-k-1}}{(n-k-1)!} \mathcal{V}_2 \right) \Bigg],
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{V}_1 \;&=\; \frac{e^{-\sigma u}}{(n-k-1)!} \sum_{i=0}^{k-1} \Bigg(\binom{k-1}{i} \frac{(n-k+i-1)!(\sigma-\nu)^{k-n-i}}{\sigma^{k-i}} \\
&\times \sum_{j=0}^{k-1-i} \frac{(u\sigma)^j (k-1-i)!}{j!} \Bigg),
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}_2 \;&=\; \frac{e^{-\nu u}}{(k-1)!} \sum_{i=0}^{n-k-1} \Bigg(\binom{n-k-1}{i} \frac{(i+k-1)!(\nu-\sigma)^{-i-k}}{\nu^{n-k-i}} \\
&\times \sum_{j=0}^{n-k-1-i} \frac{(u\nu)^j (n-k-1-i)!}{j!} \Bigg),
\end{aligned}$$

$$\begin{aligned}
a =& \alpha(\rho - \nu), \quad b = (1-\alpha)(\rho - \sigma), \\
\phi =& \frac{\sigma\nu(\alpha\nu + (1-\alpha)\sigma + \rho)}{(1+\widehat{\theta})(\alpha\nu + \sigma(1-\alpha))(\rho+\sigma)(\rho+\nu)}, \\
\rho =& \frac{1}{6}\sqrt[3]{E + 12\sqrt{F}} - \frac{2C - \frac{2}{3}B^2}{\sqrt[3]{E + 12\sqrt{F}}} - \frac{B}{3}, \\
E =& 36BC - 108D - 8B^3, \\
F =& 12C^2 - 3B^2C^2 - 54BCD + 81D^2 + 12B^3D, \\
B =& (\nu + \sigma) - \frac{\lambda + \delta}{c}, \\
C =& \nu\sigma - \frac{(\lambda+\delta)(\sigma+\nu)}{c} + \frac{\lambda}{c}(\alpha\sigma + (1-\alpha)\nu), \\
D =& -\frac{\delta\nu\sigma}{c}, \\
c =& \frac{\lambda(\alpha\nu + (1-\alpha)\sigma)}{\nu\sigma}(1+\widehat{\theta}).
\end{aligned}$$

The obtained formula for discounted penalty function was examined for various parameters choices. The dependency on initial capital, interest rate, security loading and intensity of the Poisson process was considered. In this case any parameter changing affects the function's behavior. Increase of claim intensity causes the increase of named function. Likewise, the large value of initial capital and safety loading coefficient have the positive impact on discounted penalty function, i.e. function's values are less when these coefficients are large. According to this behavior insurer can deal with the risk and control the work of insurance company.

2.3 The discounted penalty function on finite time horizon

The second part of the dissertation is devoted to Gerber-Shiu discounted penalty function investigation on the finite time horizon. The classical risk model with exponential claims with intensity μ was studied. i.e. when

$$\overline{H}(y) = e^{-\mu y},\ y \geq 0.$$

In this case the explicit expression of the Gerber-Shiu discounted penalty function on the finite time horizon

$$\phi(u,t) = E(e^{-\delta T}\mathbf{I}_{\{T<t\}}), \quad \delta \geq 0, \quad t > 0$$

was derived.

The expressions for discounted moments on the finite time horizon

$$\phi_m(u,t) = E(T^m e^{-\delta T}\mathbf{I}_{\{T<t\}}),\ m = 1,2,..., \quad t > 0, \quad \delta \geq 0$$

and for Gerber-Shiu discounted penalty function on time interval $[t_1, t_2]$

$$\gamma(u, t_1 t_2) = E(e^{-\delta T}\mathbf{I}_{\{t_1<T<t_2\}})$$

was obtained. Here $\delta \geq 0$ ir $t_1, t_2 > 0$.

Teorema 2.2 *Assume the claim sizes $Y_1, Y_2, ...$ in the classical risk model have an exponential distribution with parameter μ. Let, in addition, λ be the parameter of a Poisson process, $c = \lambda EY(1+\widehat{\theta})$ be a premium income rate with positive parameter $\widehat{\theta}$, let $u \geq 0$ be an initial surplus and let non-negative δ be the force of interest. Define $\nu = \delta + \lambda(2+\widehat{\theta})$ and $\kappa = \lambda\sqrt{1+\widehat{\theta}}$. Then the following statements hold:*

(a) The Gerber-Shiu discounted penalty function on the finite time horizon has the following form:

$$\begin{aligned}\phi(u,t) &= \phi e^{-(1-\phi)\mu u} - e^{-\nu t}\sum_{j=0}^{\infty}\frac{(\nu t)^j}{j!}\left(\phi e^{-(1-\phi)\mu u} - \frac{\lambda}{\nu}e^{-\mu u}\sum_{n=0}^{j-1}\frac{n+1}{n!}\left(\frac{\mu\lambda u}{\nu}\right)^n\right.\\ &\times \left.\sum_{k=0}^{[\frac{j-n+1}{2}]-1}\frac{(2k+n)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k}\right),\end{aligned}$$

here

$$\phi = \frac{\lambda(\nu - \sqrt{\nu^2 - 4\kappa^2})}{2\kappa^2}.$$

(b) The Gerber-Shiu discounted penalty function on time interval $[t_1, t_2]$ is given by

$$\begin{aligned}\gamma(u,t_1,t_2) &= \frac{\lambda}{\nu}e^{-\mu u}\sum_{n=0}^{\infty}\frac{(n+1)}{n!}\left(\frac{\mu\lambda u}{\nu}\right)^n\sum_{k=0}^{\infty}\frac{(2k+n)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k}\\ &\times \sum_{j=0}^{n+2k}\left(t_1^j e^{-\nu t_1} - t_2^j e^{-\nu t_2}\right).\end{aligned}$$

(c) Discounted moments on the finite time horizon are

$$\begin{aligned}\phi_m(u,t) &= \frac{\nu^{m+1}}{\lambda}e^{\mu u}\phi_m(u) - \frac{\lambda}{\nu^{m+1}}e^{-(\mu u+\nu t)}\sum_{j=0}^{\infty}\frac{(\nu t)^j}{j!}\\ &\times \left(\phi_m(u) - \sum_{n=0}^{j-1}\frac{n+1}{n!}\left(\frac{\mu\lambda u}{\nu}\right)^n\right.\\ &\times \left.\sum_{k=0}^{[\frac{j-n+1}{2}]-1}\frac{(2k+n+m)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k}\right),\end{aligned}$$

where

$$\phi_m(u) = \frac{e^{-\frac{\mu\theta u}{1+\theta}}}{1+\theta}\frac{(m-1)!}{\lambda^m}\sum_{i=0}^{m-1}\left(\frac{\lambda u}{c}\right)^{m-1-i}\frac{(m-i+\frac{\lambda u}{c})}{(m-1-i)!}$$
$$\times \sum_{l=0}^{i}\binom{m}{i-l}\binom{m+l-1}{l}\theta^{-m-l}.$$

Derived expressions of $\phi(u,t)$, $\phi_m(u,t)$ and $\gamma(u,t_1,t_2)$ let to consider the dependance of these functions on initial capital, claim intensity, interest rate and security loading. Hence, insurer may evaluate company's safety level by taiking into consideration the values of all mentioned parameters because they have deep impact on company's activity.

2.4 The property of the renewal counting process with application to the finite-time ruin probability

In the third section of the thesis the obeject of investigation was renewal counting process

$$N(t) = \sup\{n \geq 1 : \theta_1 + ... + \theta_n \leq t\}, \tag{2.6}$$

where $\theta_1, \theta_2, ...$ is independent identically distributed random variables. Let $\lambda(t) = EN(t) \to \infty$ as $t \to \infty$. It was considered whether the process $N(t)$ has the following property:

ASSUMPTION B^*. For every $a > \lambda$ with $\lambda = 1/E\theta < \infty$, there exists $b > 1$ such that

$$\lim_{t\to\infty}\sum_{k>at} P(N(t) \geq k)b^k = 0. \tag{2.7}$$

The proofed theorem in third section shows that any renewal counting process $N(t)$ with $EN = 1/\lambda < \infty$ satisfies Assumption B^*. In the case of an infinite mean $EN = \infty$, for large t some stronger estimates for the probabilities $P(N(t) > k)$ can be obtained and, as a result, the same relation (2.7) holds for every positive a.

Teorema 2.3 *Let the renewal counting process $N(t)$ be defined in (2.6) with $\theta, \theta_1, \theta_2, \ldots$ being a sequence of i.i.d. nonnegative r.v.s. Then:*
(a) if $E\theta = 1/\lambda < \infty$, then $N(t)$ satisfies Assumption B^*;
(b) If $E\theta = \infty$, then for every positive number $a > 0$ there exists $b > 1$ such that relation (2.7) holds.

The result of the theorem was applied to the analysis of the finite time ruin probability in the renewal risk model. Suppose that claim sizes and interarrival times in this model are satisfied the following assumptions:

ASSUMPTION $\mathcal{H}_1$ The individual claim sizes $Y_1, Y_2, \ldots$ form a sequence of i.i.d. nonnegative r.v.s with common d.f. $H(u)$ and finite mean $\beta = EY_1$.

ASSUMPTION $\mathcal{H}_2$ The interarrival times $\theta, \theta_1, \theta_2, \ldots$ are i.i.d. nonnegative r.v.s with finite mean $E\theta = 1/\lambda$. In addition, sequences $\theta, \theta_1, \theta_2, \ldots$ and $Y_1, Y_2, \ldots$ are mutually independent.

Consider a risk reserve process

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i,\ t \geq 0$$

with $N(t) = \sup\{n \geq 1 : \theta_1 + \cdots + \theta_n \leq t\}$, initial risk reserve $u = R(0) \geq 0$, premium intensity $c > 0$ and positive safety loading coefficient $\widehat{\theta} = c/\lambda - \beta$.

Denote the probability of ruin within time τ by

$$\psi(x,\tau) \quad = \quad P\Big(\inf_{0\leq s\leq\tau} U(s) < 0 | R(0) = u\Big) = P\Big(\max_{0\leq k\leq N(\tau)} \sum_{i=1}^{k}(Y_i - c\theta_i) > u\Big).$$

Before formulating the main result of the fourth part let define some distribution function class.

Definition 2.1. *D.f. $F = 1 - \overline{F}$ belongs to strongly subexponential distribution function's class (denoted by $\mathcal{S}_*$) if $\int_0^\infty \overline{F}(y)dy < \infty$ and*

$$\lim_{x\to\infty} \frac{\overline{F_u^{*2}}(x)}{\overline{F_u}(x)} = 2 \tag{2.8}$$

uniformly for $u \in [1,\infty)$, where

$$\overline{F_u}(x) = \begin{cases} \min\Big\{1, \int_x^{x+u} \overline{F}(y)dy\Big\} & \text{if } \ x \geq 0, \\ 1 & \text{if } \ x < 0. \end{cases}$$

The result of the Theorem 2.3 improved the following result from [9]:

Teorema 2.4 *Let the renewal risk model satisfy assumptions $\mathcal{H}_1$ and $\mathcal{H}_2$. If the claim size distribution $H(x) \in \mathcal{S}_*$ and interarrival times $\theta_1, \theta_2, ...$ satisfy Assumption B^*, then*

$$\psi(u,\tau) \sim \frac{1}{\mu} \int\limits_u^{u+\mu\lambda\tau} \overline{H}(y)dy \tag{2.9}$$

uniformly for $\tau \in [f(x), \infty]$ with any infinitely incresing function $f(t)$.

Therefore, according to the Theorem 2.3, Assumption B^* in Theorem 2.4 can be suppressed. Then the following statement holds:

Teorema 2.5 *Let the renewal risk model satisfy assumption $\mathcal{H}_1$ and $\mathcal{H}_2$ and the claim size distribution $H(x) \in \mathcal{S}_*$ then the asymptotic formula (2.9) holds with the same uniformity as in Theorem 2.4.*

2.5 The renewal equation for discounted penalty function

In the forth section of the thesis the Gerber-Shiu discounted penalty function $\phi(u)$ in the Erlang(2) model was investigated. In this case the insurers' surplus process $\{U(t),\, t \geq 0\}$ is from equality (2.1). The counting process $\{N(t)\}_{t\geq 0}$ describes a number of claims up to t and

$$N(t) = \sum_{i=1}^{\infty} \mathbf{I}_{\{\theta_1+\theta_2+\ldots+\theta_i \leq t\}},$$

$\theta_1, \theta_2, ...$ be a sequence of independent and identically distributed random variables, which represent the inter-arrival times, with θ_1 being the time until the first claim. In addition, we suppose that θ_1 has the Erlang(2) density function with scale parameter $\lambda > 0$:

$$k(t) = \lambda^2 t e^{-\lambda t}, \quad t \geq 0.$$

Individual claims $Y_1, Y_2, ...$ independent of $N(t)$ are non-negative, independent and identically distributed random variables with distribution $H(y) = P(Y_1 \leq y)$ and mean $EY_1 = a$. Also it is assumed that the safety loading condition holds i.e.

$$c = \frac{EY_1}{E\theta_1}(1+\widehat{\theta}) = \frac{\lambda a}{2}(1+\widehat{\theta}) \tag{2.10}$$

with some positive security loading coefficient $\widehat{\theta}$.

Sun in 2003, demonstrated that (see also Cheng and Tang [12]) in case when

$$\int_0^\infty \int_0^\infty \omega(x,y)h(x+y)dxdy < \infty \tag{2.11}$$

the discounted penalty function $\phi_\omega(u)$ from equality (2.4) satisfies the following defective renewal equation

$$\phi_\omega(u) = \frac{1}{1+\beta}\int_0^u \phi_\omega(u-y)dG(y) + \frac{1}{1+\beta}B(u), \tag{2.12}$$

here $h(y)$ is the continuous density function of individual claim sizes,

$$B(u) = \frac{\lambda^2}{c^2}(1+\beta)\int_u^\infty e^{-\rho_2(s-u)}\int_s^\infty e^{-\rho_1(x_1-s)}\int_{x_1}^\infty \omega(x_1, x_2-x_1)dH(x_2)dx_1ds,$$

$$\beta = \frac{2\lambda\delta+\delta^2}{c^2\rho_1\rho_2 - 2\lambda\delta - \delta^2}. \tag{2.13}$$

and $\rho_1 = \rho_1(\delta)$, $\rho_2 = \rho_2(\delta)$ $(0 \leq \rho_1 < (\lambda+\delta)/c < \rho_2)$ are two non-negative roots of Lundberg's fundamental equation (see for example [14])

$$\lambda^2 \int_0^\infty e^{-sx}dH(x) = (cs-\lambda-\delta)^2 \tag{2.14}$$

with $\rho_1(0) = 0$.
Furthermore, function described in equation (2.12)

$$G(y) = \frac{1}{D}\int_0^y g(x)dx$$

is called the associated "claim size" distribution, where

$$g(y) = \frac{\lambda^2}{c^2}\int_y^\infty e^{-\rho_2(s-y)}\int_s^\infty e^{-\rho_1(x-s)}dH(x)ds$$

and

$$D = \int_0^\infty g(y)dy = \frac{c^2\rho_1\rho_2 - 2\lambda\delta-\delta^2}{c^2\rho_1\rho_2} = \frac{1}{1+\beta}.$$

In the dissertation was investigated the case when $\delta > 0$ and function $\omega(x,y) \equiv 1$ for all x, y what means that the penalty at the moment T is accepted to be unit. In this case

$$\phi(u) = E(e^{-\delta T}\mathbf{I}_{\{T<\infty\}}|U(0)=u)$$

The defective renewal equation for function $\phi(u)$ without technical condition (2.11) was derived.

Teorema 2.6 *Consider the Erlang(2) risk model with safety loading condition (2.10). If $\delta > 0$ then the function $\phi(u)$ satisfies the renewal equation*

$$\phi(u) = \frac{1}{1+\beta}\int\limits_{[0,u]} \phi(u-y)dG(y) + \frac{1}{1+\beta}\overline{G}(u),$$

with distribution function G defined by equality

$$\overline{G}(u) = \frac{\lambda^2}{c^2}(1+\beta)\int\limits_u^\infty e^{-\rho_2(y-u)}\int\limits_y^\infty e^{-\rho_1(x-y)}\overline{H}(x)dxdy, \quad u \geq 0,$$

where $\rho_1, \rho_2\, (\rho_1 < \rho_2)$ are two positive roots of the equation (2.14), and the positive constant β defined in (2.13).

The result of this Theorem was applied to investigation of discounted penalty function's asymptotics in Erlang(2) model with subexponential claims sizes. It was a crucial point to refuse the condition (2.11), which requires the existence of claim size density function. In our case mentioned condition would restrict the subexponential distribution functions' class and would limit discounted penalty function's exploration.

2.6 Function's $\phi(u)$ asymptotycs in Erlang(2) model

The last dissertation section is devoted to analysis of asymptotic behavior of the Gerber-Shiu discounted penalty function in Erlang(2) model with subexponential claims. Now define the subexponential distribution functions class.

The d.f. F with support $[0,\infty)$ is called subexponential if

$$\lim_{x\to\infty}\frac{\overline{F*F}(x)}{\overline{F}(x)} = 2$$

where $F*F$ means the convolution of F with itself and $\overline{F}(x) = 1 - F(x)$ for $x \in \mathbb{R}$. This class of distribution functions is denoted by $\mathcal{S}$.

The asymptotic of discounted penalty function with subexponential claims in the classical risk model was studied by Šiaulys and Asanavičiūtė [15] . They showed, that the asymptotic formula

$$\phi(u) \sim \frac{\mu}{\delta}\overline{H}(u), \; u \to \infty,$$

holds. Here $H \in \mathcal{S}$, $\delta > 0$, and μ is the intensity of the Poison process which coincide in the case of Poison process with the parameter of inter-arrival times $\theta_1, \theta_2, \ldots$.

The case of Erlang(2) inter-arrival times and subexponential claims was investigated in the thesis. The asymptotic formula of discounted penalty function was derived and the following theorem was proved:

Teorema 2.7 *Consider the Erlang(2) model with safety loading condition, where the claim distribution function $H \in \mathcal{S}$. Then for $\delta > 0$*

$$\frac{\phi(u)}{\overline{H}(u)} \sim \frac{\lambda^2}{\delta^2 + 2\lambda\delta}, \; u \to \infty.$$

If, in addition,

$$\frac{1}{a}\int\limits_0^u \overline{H}(y)dy \in \mathcal{S}$$

then for $\delta = 0$

$$\phi(u) \sim \frac{1}{\widehat{\theta}a}\int\limits_u^\infty \overline{H}(y)dy.$$

The second part of the Theorem is well known result (see for example, [7]), in dissertation was investigated case when $\delta > 0$.

Conclusions

The Gerber-Shiu discounted penalty function was the main object of investigations in the thesis. The explicit expressions of this function was got in the classical risk model by using appropriate distributions of the claim amount. Applying these expressions the dependence of the discounted penalty function on initial capital, interest rate, safety loading and claim intensity was studied. After the analysis was completed, it was noticed, that any parameter changing has a deep impact on function's behaviour. Hence, insurer may evaluate company's safety level by taking into consideration the values of all mentioned parameters. According to this insurer may choose the profitable direction of capital investment and deal with the risk in future.

The asymptotics of the discounted penalty function in the Erlang(2) model with subexponential claims was considered in the thesis and the asymptotic formula for this function was derived. Obtained formula depends on interest rate and on parameter of inter-arrival times distribution. As a result, these parameters' changing affects the discounted penalty function's behaviour. Also asymptotic formula has a simple form what facilitates analysis of the discounted penalty function when initial capital becomes extremely large.

The renewal counting process was also studied in the thesis. Generally this process widely applied with different risk models where represents the number of claims up to some time point. Some properties of this process was investigated in the thesis and asymptotics of the tail of the exponential moment was derived. The obtained result was applied to the asymptotic's analysis of the finite time ruin probability in a renewal risk model. The asymptotic properties of renewal counting process may be also used for examination of different ruin problems.

Approbation

The main thesis' results were presented at three local Lithuanian conferences and were published in the following papers:

- J. Šiaulys, J. Kočetova, *On the Discounted Penalty Function for Claims Having Mixed Exponential Distribution*, Nonlinear Analysis: Modelling and Control, 11(4), p. 413 - 426, 2006.
- J. Kočetova, J. Šiaulys, R. Leipus, *A property of the renewal counting process with application to the finite-time ruin probability*, LMJ, 49(1), p. 55 - 61, 2008.
- J. Kočetova, J. Šiaulys, *Investigation of the Gerber-Shiu discounted penalty function on finite time horizon*, Information Technology and Control, 39(1), 18-24, 2010.
- J. Kočetova, J. Šiaulys, *Asymptotic behaviour of the Gerber-Shiu discounted penalty function in the Erlang(2) risk process with subexponential claims*, submited to the journlal "Nonlinear Analysis: Modelling and Control" in 2011.

Reziumė

Pirmoje ir antroje darbo dalyse buvo nagrinėjama Gerber-Shiu diskontuota baudos funkcija klasikiniame rizikos modelyje. Buvo tyrinėjamas atskiras šios funkcijos atvejis, esant prielaidai, kad bankroto dydis bankroto metu lygus vienam. Pirmoje darbo dalyje gauta minėtos funkcijos išraiška su žalomis, turinčiomis mišrų eksponentinį pasiskirstymą. Antroje dalyje diskontuota baudos funkcija buvo tyrinėjama baigtiniame laiko intervale su eksponentinėmis žalomis. Nagrinėjamu atveju, pritaikius žinoma bankroto laiko tankio funkcijos formulę, buvo gauti tokie dydžiai: Gerber-Shiu diskontuotos baudos funkcija baigtiniame laiko intervale, diskontuoti bankroto laiko momentai baigtiniame laiko intervale ir Gerber-Shiu diskontuotos baudos funkcija intervale $[t_1, t_2]$. Abiejuose dalyse gautos diskontuotos baudos funkcijos išraiškos buvo panaudotos tyrinėjant šios funkcijos elgesį, parenkant įvarias parametrų reikšmes.

Kadangi Gerber-Shiu funkcija paprastai nagrinėjama su įvairiais rizikos modeliais, tapo aktualu ištirti atstatymo procesą, kuris yra bet kokio modelio pagrindas. Šis procesas paprastai rodo įvykusių žalų skaičių konkrečiu laiko mementu. Trečiame disertacinio darbo skyriuje buvo gautas atstatymo proceso eksponėntinės eilės momento uodegos įvertis. Gautas rezultatas buvo pritaikytas baigtinio laiko bankroto tikimybės asimptotikos tyrimui.

Ketvirtame darbo skyriuje gauta diskontuotos baudos funkcijos asimptotinė formulė Erlang(2) modelyje su subeksponentinėmis žalomis. Ši formulė buvo gauta iš diskontuotos baudos funkcijos atstatymo lygties. Nagrinėjant atskirą baudos funkcijos atvejį, darbe buvo išvesta atstatymo lygtis atsisakant tankio egzistavimo prielaidos. Toks žingsnis buvo esminis, nes žalų tankio egzistavimas susiaurina subeksponentinių skirstinių klasę ir apriboja minėtos funkcijos tyrinėjimo galimybes. Gauta formulė leidžia išanalizuoti diskontuotos baudos funkcijos elgesį ir savybes didelėms pradinio kapitalo vertėms.

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