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LIMIT THEOREMS FOR LERCH ZETA-FUNCTIONS WITH ALGEBRAIC IRRATIONAL PARAMETER

Doctoral dissertation
Physical sciences, mathematics (01P)

Vilnius, 2009

The scientific work was carried out in 2003–2009 at Šiauliai University.
The dissertation was prepared externally.

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VILNIAUS UNIVERSITETAS

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LERCHO DZETA FUNKCIJŲ SU ALGEBRINIU IRACIONALIUOJU PARAMETRU RIBINĖS TEOREMOS

Daktaro disertacija
Fiziniai mokslai, Matematika (01P)

Vilnius, 2009

Disertacija rengta 2003–2009 metais Šiaulių universitete.
Disertacija ginama eksternu.

Mokslinis konsultantas:
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Introduction

Let $s = \sigma + it$ be a complex variable. The Lerch zeta-function $L(\lambda, \alpha, s)$ with parameters $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, $0 < \alpha \leq 1$, is defined, for $\sigma > 1$, by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}$$

and by analytic continuation elsewhere. If $\lambda \in \mathbb{Z}$, then $L(\lambda, \alpha, s)$ reduces to the

Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}, \quad \sigma > 1,$$

which is a meromorphic function with a simple pole at $s = 1$ and $\text{Res}_{s=1}\zeta(s, \alpha) = 1$. If $\lambda \notin \mathbb{Z}$, then $L(\lambda, \alpha, s)$ is an entire function. In this case, without loss of generality, we can suppose that $0 < \lambda < 1$.

The Lerch zeta-function was introduced independently in [52] and [53]. For all s, the function $L(\lambda, \alpha, s)$ with $0 < \lambda \ < 1$ satisfies the functional equation

$$L(\lambda, \alpha, 1-s) = \frac{\Gamma(s)}{(2\pi)^s}(\exp\{\frac{\pi i s}{2} - 2\pi i \alpha\lambda\}L(-\alpha, \lambda, s)+$$

$$\exp\{-\frac{\pi i s}{2} + 2\pi i \alpha(1-\lambda)\}L(\alpha, 1-\lambda, s)).$$

Several proofs of this equation are known. The first proof was given in [52]. A proof in [1] is based on a transformation formula and difference differential equation satisfied by the function $L(\lambda, \alpha, s)$. In [65], the Poisson summation formula is applied, while the paper [64] uses the Fourier series method. B.C. Berndt proposed [4] simple proofs using the contour integration as well as the Euler-Maclaurin summation formula, see also [46].

The theory of the function $L(\lambda, \alpha, s)$ is given in [46]. Chapter 5 of [46] is devoted to statistical properties of the Lerch zeta-function. There limit theorems in the sense of weak convergence of probability measures in various spaces for $L(\lambda, \alpha, s)$ can be found, see also [12], [16], [47], [49] and [50].

Actuality

The Lerch zeta-function is not so important in analytic number theory as, for example, the Riemann zeta-function or Dirichlet L-functions. On the other hand, the function $L(\lambda, \alpha, s)$ is a classical zeta-function, except for some particular cases, having no the Euler product over primes, therefore, it is interesting to compare its properties with those of zeta-functions with Euler product. Moreover, the function $L(\lambda, \alpha, s)$ depends on two parameters λ and α, and is governed by arithmetic nature of them. Thus, the Lerch zeta-function is a very attractive classical mathematical object.

An idea of application of probability methods in the theory of zeta-functions comes back to H. Bohr. He prevised that the complicated value distribution of zeta-functions can be described by probabilistic laws. H. Bohr, B. Jessen and A. Wintner [6], [7], [33] were the first who proved probabilistic limit theorems for zeta-functions. Last fifteen years is a new period of development of Bohr's approach. D. Joyner [34], B. Bagchi [2], K. Matsumoto [56]-[63], J. Steuding [66], A. Laurinčikas [40] and his students R. Kačinskaitė [35], [36], R. Šleževičienė [67], I. Belovas [3], J. Ignatavičiūtė [27]-[32], J. Genys [25], V. Garbaliauskienė [11], R. Macaitienė [55] created the modern probabilistic theory of zeta-functions having important applications in the universality theory. Therefore, this research direction has a large influence in development of mathematics.

Probabilistic limit theorems for the Lerch zeta-function with transcendental and rational parameter α were obtained by A. Laurinčikas, R. Garunkštis, K. Matsumoto, J. Steuding and others [12], [16], [19], [23], [41], [43], [44], [46], [47], [48], [50], [51]. However, the most complicated case of algebraic irrational α remained an open problem till now. In the thesis, this gap in the theory of the Lerch zeta-function is filled.

Aims and problems

The aim of the thesis is to prove probabilistic limit theorems of the Lerch zeta-function $L(\lambda, \alpha, s)$ with $\lambda \in (0, 1)$ and algebraic irrational parameter α.

The specified problems are the following:

1. To prove a limit theorem on the complex plane for $L(\lambda, \alpha, s)$ with algebraic irrational parameter α.

2. To prove a joint limit theorem on complex plane for a collection of Lerch zeta-functions with algebraic irrational parameters.

3. To prove a limit theorem in the space of analytic functions for $L(\lambda, \alpha, s)$ with algebraic irrational parameter α.

Methods

Proofs of limit theorems are based on the analytic theory of the Lerch zeta-function as well as on the theory of weak convergence of probability measures. The method of contour integration, Prokhorov's theorems and elements of ergodic theory are applied. Also, a result of Cassels on the linear independence of the system $\{\log(m + \alpha) : m \in \mathbb{C}_0\}$ with algebraic irrational α plays an important role in proofs. This is a new moment in the theory of value distribution of zeta-functions.

Novelty

All results of the thesis are new. Limit theorems for the Lerch zeta-function with algebraic irrational parameter α are obtained for the first time.

Defended results of the thesis

1. A limit theorem in the sense of weak convergence of probability measures on the complex plane for the Lerch zeta-function $L(\lambda, \alpha, s)$ with algebraic irrational parameter α.

2. A joint limit theorem in the sense of weak convergence of probability measures in the complex plane for Lerch zeta-functions with algebraic irrational parameters.

3. A limit theorem in the sense of weak convergence of probability measures in the space of analytic functions for $L(\lambda, \alpha, s)$ with algebraic irrational parameter α.

History of the problem and main results

For a long time, the Lerch zeta-function $L(\lambda, \alpha, s)$ was forgotten. Only in 1987, D. Klusch [37] obtained the asymptotic formula for the mean square of $L(\lambda, \alpha, s)$

$$\int_0^T |L(\lambda, \alpha, \sigma + it)|^2 dt \sim \begin{cases} T \log T & \text{if } \sigma = \frac{1}{2}, \\ T\zeta(2\sigma, \alpha) & \text{if } \frac{1}{2} < \sigma < 1, \end{cases}$$

as $T \to \infty$. Two years later, he gave [39] the asymptotic expansion in δ for the integral

$$\int_0^\infty |L(\lambda, \alpha, \sigma + it)|^2 e^{-\delta t} dt.$$

The above results stimulated the probabilistic investigations in the theory of the Lerch zeta-function.
The results mentioned of D. Klusch were improved in [20] by using an approximate functional equation for $L(\lambda, \alpha, s)$ obtained in [21].
Let, for $T > 0$,

$$\nu_T^t(...) = \frac{1}{T} \text{meas}\{t \in [0, T] : ...\},$$

where in place of dots a condition satisfied by t is to be written. Throughout the dissertation we suppose that $0 < \lambda < 1$. First we recall some limit theorems on the complex plane for the function $L(\lambda, \alpha, s)$. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and define

$$P_T(A) = \nu_T^t(L(\lambda, \alpha, \sigma + it) \in A), \quad A \in \mathcal{B}(\mathbb{C}).$$

In [15] the following assertion is given (see also [46]).

Theorem A. *Let $\sigma > \frac{1}{2}$ be fixed. Then, for arbitrary α, $0 < \alpha \leq 1$, there exists a probability measure P on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that the probability measure P_T converges weakly to P as $T \to \infty$.*

In Theorem A, only the existence of the limit measure for P_T is obtained. However, it is important for applications to know an explicit form of the limit measure. Such the form of the measure P in Theorem A in the case of transcendental or rational α follows from limit theorems in the space of analytic functions.

Denote by γ the unit circle on $\mathbb{C}$, i. e. $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and define

$$\Omega_1 = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for $m \in \mathbb{C}_0$. By the Tikhonov theorem, with the product topology and pointwise multiplication the infinite-dimensional torus Ω_1 is a compact topological Abelian group. Therefore, on $(\Omega_1, \mathcal{B}(\Omega_1))$, the probability Haar measure m_{1H} exists. This gives the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$. Let $\omega_1(m)$ be the projection of $\omega_1 \in \Omega_1$ to the coordinate space γ_m, $m \in \mathbb{C}_0$. For $\sigma > \frac{1}{2}$ and $\omega_1 \in \Omega_1$ define

$$L_1(\lambda, \alpha, \sigma, \omega_1) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m} \omega_1(m)}{(m + \alpha)^{\sigma}}.$$

Then $L_1(\lambda, \alpha, \sigma, \omega_1)$ is a complex-valued random variable defined on the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$. Theorem 5.2.2 of [46] implies the following result. We recall that α is transcendental if there is no polynomials $P(s) \not\equiv 0$ with rational coefficients such that $P(\alpha) = 0$.

Theorem B. *Suppose that the parameter α is transcendental. Then the probability measure P_T converges weakly to the distribution of the random variable*
$L_1(\lambda, \alpha, \sigma, \omega_1)$ as $T \to \infty$.

We recall that the distribution of $L_1(\lambda, \alpha, \sigma, \omega_1)$ is the probability measure P_{L_1} defined by

$$P_{L_1}(A) = m_{1H}(\omega_1 \in \Omega_1 : L_1(\lambda, \alpha, \sigma, \omega_1) \in A), \quad A \in \mathcal{B}(\mathbb{C}).$$

Now let

$$\Omega_2 = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. Similarly to the case of Ω_1, we obtain the probability space $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$, where m_{2H} is the probability Haar measure on $(\Omega_2, \mathcal{B}(\Omega_2))$. Denote by $\omega_2(p)$ the projection of $\omega_2 \in \Omega_2$ to the coordinate space γ_p, $p \in \mathbb{P}$. For $m \in \mathbb{C}$, we put

$$\omega_2(m) = \prod_{p^k \| m} \omega_2^k(p),$$

where $p^k \| m$ means that $p^k | m$ but $p^{k+1} \not| m$. This construction allows us to consider the case of rational α. Let $\alpha = \frac{a}{q}$, $a, q \in \mathbb{C}$, $1 \le a \le q$, and $(a, q) = 1$. For $\sigma > \frac{1}{2}$, define on $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$ the complex-valued random variable $L_2(\lambda, \alpha, s, \omega_2)$ by

$$L_2(\lambda, \alpha, s, \omega_2) = \omega_2(q) q^s \mathrm{e}^{-2\pi i \lambda \frac{a}{q}} \cdot \sum_{\substack{m=1 \\ m \equiv a (\mathrm{mod} q)}}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda \frac{m}{q}} \omega_2(m)}{m^s}, \quad \omega_2 \in \Omega_2,$$

and let P_{L_2} denote the distribution of $L_2(\lambda, \alpha, s, \omega_2)$:

$$P_{L_2}(A) = m_{2H}(\omega_2 \in \Omega_2 : L_2(\lambda, \alpha, s, \omega_2) \in A), \quad A \in \mathcal{B}(\mathbb{C}).$$

Then from Theorem 5.4.1 of [46] the following statement follows.

Theorem C. *Let $\alpha = \frac{a}{q}$, $a, q \in \mathbb{C}$, $1 \le a \le q$, and $(a, q) = 1$. Then the probability measure P_T converges weakly to the measure P_{L_2} as $T \to \infty$.*

J. Ignatavičiūtė [27], [32] obtained the discrete versions of theorems A, B, C. Theorems B and C show that it remains to consider the case of an algebraic irrational parameter α. We recall that α is algebraic number if it is a root of a polynomial with rational coefficients. Chapter 1 of this dissertation is devoted to the latter problem. So, α denotes an algebraic irrational number, $0 < \alpha < 1$.

Let

$$L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{C}_0\}.$$

In [8] J. W. S. Cassels obtained that at least 51 percent of elements of the set $L(\alpha)$ are linearly independent over the field of rational numbers $\mathbb{Q}$. Let $I(\alpha)$ be a maximal linearly independent over $\mathbb{Q}$ subset of $L(\alpha)$. If $I(\alpha) = L(\alpha)$, then we have the same situation as in the ease of Theorem B, since the set $L(\alpha)$ with transcendental α is linearly independent over $\mathbb{Q}$. Therefore, we suppose that $I(\alpha) \neq L(\alpha)$ and denote $D(\alpha) = L(\alpha) \setminus I(\alpha)$. Then, for any element $d_m \in D(\alpha)$, the set $\{d_m\} \bigcup I(\alpha)$ is linearly dependent over $\mathbb{Q}$. Therefore, there exist a finite number of elements $i_{m_1}(m), ..., i_{m_{n(m)}}(m) \in I(\alpha)$ and numbers $k_0(m), ..., k_n(m) \in \mathbb{Z} \setminus \{0\}$ such that

$$d_m = -\frac{k_1(m)}{k_0(m)} i_{m_1}(m) - ... - \frac{k_n(m)}{k_0(m)} i_{m_{n(m)}}(m).$$

Since the elements of $L(\alpha)$ are $\log(m+\alpha)$, we find that

$$m+\alpha = (m_1(m)+\alpha)^{-\frac{k_1(m)}{k_0(m)}} ... (m_{n(m)}(m)+\alpha)^{\frac{-k_n(m)}{k_0(m)}}. \tag{0.1}$$

Now define two subsets $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ of $\mathbb{C}_0$ by

$$\mathcal{M}(\alpha) = \{m \in \mathbb{C}_0 : \log(m+\alpha) \in I(\alpha)\},$$

$$\mathcal{N}(\alpha) = \{m \in \mathbb{C}_0 : \log(m+\alpha) \in D(\alpha)\},$$

and let

$$\Omega = \prod_{m \in \mathcal{M}(\alpha)} \gamma_m,$$

where $\gamma_m = \gamma$ for $m \in \mathcal{M}(\alpha)$. Then Ω is also a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and this leads to the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m, $m \in \mathcal{M}(\alpha)$. We extend the function $\omega(m)$ to the whole set $\mathbb{C}_0$ by putting, for $m \in \mathcal{N}(\alpha)$,

$$\omega(m) = \omega^{-\frac{k_1(m)}{k_0(m)}}(m_1(m))...\omega^{-\frac{k_n(m)}{k_0(m)}}(m_{n(m)}(m)) \tag{0.2}$$

if (0.1) takes place. Here the principal values of the roots are taken. So, we have that $\{\omega(m) : m \in \mathbb{C}_0\}$ is a sequence of complex-valued random variables defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

Denote by $\mathcal{A}$ a class of algebraic irrational numbers α, $0 < \alpha < 1$, for which the numbers $\frac{k_1(m)}{k_0(m)}, ..., \frac{k_{n(m)}(m)}{k_0(m)}$ in (0.2) are integer. If $\alpha \in \mathcal{A}$, then is easily seen that $\{\omega(m) : m \in \mathbb{C}_0\}$ is a sequence of pairwise orthogonal random variables on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Therefore, using the Rademacher theorem on series of pairwise orthogonal random variables, see, for example, [54]) we can obtain in a standard way that, for $\sigma > \frac{1}{2}$,

$$L(\lambda, \alpha, \sigma, \omega) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m} \omega(m)}{(m + \alpha)^{\sigma}}$$

is a complex-valued random variable defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by P_L the distribution of the random variable $L(\lambda, \alpha, \sigma, \omega)$.

The main result of Chapter 1 is the following assertion.

Theorem 1.1. *Suppose that $\lambda \in (0,1)$, $\alpha \in \mathcal{A}$, and $\sigma > \frac{1}{2}$. Then the probability measure P_T converges weakly to P_L as $T \to \infty$.*

Note that an analogue of Theorem 1.1 for Hurwitz zeta-function was obtained in [51]. However, we propose a simpler and shorter proof.

Chapter 2 is devoted to a joint limit theorem on the complex plane for Lerch zeta-functions with algebraic irrational parameters.

The first joint limit theorem in the complex plane for Lerch zeta-functions was obtained in [47], see also [46].

Theorem D. *Suppose that $r > 1$ and $\min_{1 \le j \le r} \sigma_j > \frac{1}{2}$. Then, for all real $\lambda_1, ..., \lambda_r$ and $\alpha_1, ..., \alpha_r$, $0 < \alpha_j \le 1$, $j = 1, ..., r$, there exists a probability measure P on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ such that the probability measure*

$$\nu_T^t\big((L(\lambda_1, \alpha_1, \sigma_1 + it), ..., L(\lambda_r, \alpha_r, \sigma_r + it)) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}^r),$$

converges weakly to P as $T \to \infty$.

In Theorem D, the limit measure P is not explicitly given. A joint limit theorem with explicitly given limit measure in the space of analytic functions for Lerch zeta-functions was obtained in [47], [49] and [46]. Let $D = \left\{s \in \mathbb{C}: \ \sigma > \frac{1}{2}\right\}$, and let $H(D)$ denote the space of analytic on D functions equipped with the topology of uniform on compacta. Denote by $H^r(D)$ the Cartesian product $\underbrace{H(D) \times ... \times H(D)}_{r}$. From the mentioned limit theorem in the space $H^r(D)$, a joint limit theorem on $\mathbb{C}^r$ follows.

Denote $\Omega_1^{(r)} = \prod_{j=1}^{r} \Omega_{1j}$, where $\Omega_{1j} = \Omega_1$ for $j = 1, ..., r$. Then again $\Omega_1^{(r)}$ is a compact topological Abelian group, and we have a probability space $(\Omega_1^{(r)}, \mathcal{B}(\Omega_1^{(r)}), m_{1H}^{(r)})$, where $m_{1H}^{(r)}$ is the probability Haar measure on $(\Omega_1^{(r)}, \mathcal{B}(\Omega_1^{(r)}))$. Denote by $\underline{\omega}_1 = (\omega_{11}, ..., \omega_{1r})$ the elements of $\Omega_1^{(r)}$, where $\omega_{1j} \in \Omega_{1j}$, $j = 1, ..., r$. Let, for brevity, $\underline{\lambda} = (\lambda_1, ..., \lambda_r)$, $\underline{\alpha} = (\alpha_1, ..., \alpha_r)$, $\underline{\sigma} = (\sigma_1, ..., \sigma_r)$. Define on $(\Omega_1^{(r)}, \mathcal{B}(\Omega_1^{(r)}), m_{1H}^{(r)})$ the $\mathbb{C}^r$-valued random element

$$\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}_1) = \big(L(\lambda_1, \alpha_1, \sigma_1, \omega_{11}), ..., L(\lambda_r, \alpha_r, \sigma_r, \omega_{1r})\big),$$

where, for $\sigma_j > \frac{1}{2}$,

$$L(\lambda_j, \alpha_j, \sigma_j, \omega_{1j}) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda_j m} \omega_{1j}(m)}{(m + \alpha_j)^{\sigma_j}}, \quad j = 1, ..., r.$$

Let $P_{\underline{L}}$ denote the distribution of $\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}_1)$, i. e.,

$$P_{\underline{L}}(A) = m_{1H}^{(r)}\left(\underline{\omega}_1 \in \Omega_1^{(r)} : \underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}_1) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}^r),$$

and define the probability measure

$$\underline{P}_T(A) = \nu_T^t\big((L(\lambda_1, \alpha_1, \sigma_1 + it), ..., L(\lambda_r, \alpha_r, \sigma_r + it)) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}^r).$$

The numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$ if there is no polynomial $P \not\equiv 0$ with rational coefficients, such that $P(\alpha_1, ..., \alpha_r) = 0$.

Theorem E. *Suppose that $\lambda_j \in (0, 1)$, $j = 1, ..., r$, the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over the field $\mathbb{Q}$, and $\min_{1 \le j \le r} \sigma_j > \frac{1}{2}$. Then the probability measure $\underline{P}_T$ converges weakly to the measure $P_{\underline{L}}$ as $T \to \infty$.*

If the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$, each number α_j, $j = 1, ..., r$, is transcendental. Our aim is to obtain a joint limit theorem with algebraic irrational numbers $\alpha_1, ..., \alpha_r$.

Now suppose that $\alpha_1, ..., \alpha_r$ are distinct algebraic irrational numbers, $0 < \alpha_j < 1$, $j = 1, ..., r$. Define

$$\Omega^r = \prod_{j=1}^{r} \Omega_j,$$

where

$$\Omega_j = \prod_{m \in \mathcal{M}(\alpha_j)} \gamma_m$$

with $\gamma_m = \gamma$ for $m \in \mathcal{M}(\alpha_j)$, $j = 1, ..., r$. Since each torus Ω_j is a compact topological Abelian group, Ω^r is such a group as well. Thus, we obtain a probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$, where m_H^r is the probability Haar measure on $(\Omega^r, \mathcal{B}(\Omega^r))$. Let, as above, $\underline{\omega} = (\omega_1, ..., \omega_r) \in \Omega^r$, where $\omega_j \in \Omega_j$, $j = 1, ..., r$, $\underline{\lambda} = (\lambda_1, ..., \lambda_r)$, $\underline{\alpha} = (\alpha_1, ..., \alpha_r)$ and $\underline{\sigma} = (\sigma_1, ..., \sigma_r)$. On the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$ define the $\mathbb{C}^r$-valued random element $\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega})$, for $\min\limits_{1 \le j \le r} \sigma_j > \frac{1}{2}$, by

$$\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}) = \big(L(\lambda_1, \alpha_1, \sigma_1, \omega_1), ..., L(\lambda_r, \alpha_r, \sigma_r, \omega_r)\big),$$

where

$$L(\lambda_j, \alpha_j, \sigma_j, \omega_j) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m} \omega_j(m)}{(m + \alpha_j)^{\sigma_j}}$$

and $\omega_j(m)$ is the projection of $\omega_j \in \Omega_j$ to γ_m if $m \in \mathcal{M}(\alpha_j)$ or the relation of type (0.2) if $m \notin \mathcal{M}(\alpha_j)$, $j = 1, ..., r$. Let $Q_{\underline{L}}$ denote the distribution of the random element $\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega})$. Now we can state the main result of Chapter 2.

Theorem 2.1. *Suppose that $\lambda_j \in (0, 1)$, $j = 1, \ldots, r$, and that $\alpha_1, ..., \alpha_r$ are distinct algebraic irrational numbers from the class $\mathcal{A}$ such that the set*

$$\bigcup_{j=1}^{r} I(\alpha_j)$$

is linearly independent over $\mathbb{Q}$, *and* $\min\limits_{1\le j\le r} \sigma_j > \frac{1}{2}$. *Then the probability measure* $\underline{P}_T$ *converges weakly to the measure* $Q_{\underline{L}}$ *as* $T \to \infty$.

The aim of Chapter 3 is to obtain an analogue Theorem 1.1 in the space of analytic functions.

Let $L(\lambda, \alpha, s, \omega)$ be the $H(D)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ by

$$L(\lambda, \alpha, s, \omega) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m} \omega(m)}{(m+\alpha)^s}. \tag{0.3}$$

Define

$$P_{T,H}(A) = \nu_T^{\tau}(L(\lambda, \alpha, s + i\tau) \in A), \quad A \in \mathcal{B}(H(D)).$$

The main result of Chapter 3 is the following statement.

Theorem 3.1. *Suppose that* $\lambda \in (0,1)$ *and* $\alpha \in \mathcal{A}$. *Then the probability measure* $P_{T,H}$ *converges weakly to the distribution* P_L *of the random element* $L(\lambda, \alpha, s, \omega)$ *as* $T \to \infty$.

In the case of an absolute convergence region, we can remove the condition $\alpha \in \mathcal{A}$. Let $D_1 = \{s \in \mathbb{C} : \sigma > 1\}$, and the $H(D_1)$-valued random element $L(\lambda, \alpha, s, \omega)$ is a restriction of $L(\lambda, \alpha, s, \omega)$ to $H(D_1)$.

Theorem 3.2. *Suppose that* $\lambda \in (0,1)$ *and* α *is an algebraic irrational number. Then the probability measure*

$$\nu_T^{\tau}(L(\lambda, \alpha, s + i\tau) \in A), \quad A \in \mathcal{B}(H(D_1)).$$

converges weakly to the distribution of the $H(D_1)$*-valued random element*

$L(\lambda, \alpha, s, \omega)$ *as* $T \to \infty$.

Discrete limit theorems in functional spaces for the function $L(\lambda, \alpha, s)$ are obtained in [28]-[32]. The universality and functional independence of Lerch zeta-functions are investigated in [13], [17], [19], [42], [45], [48] and [32]. The zeros distribution problems are treated in [14], [18], [19], [22] and, in connection with the Lindelöf hypothesis, in [24]. The results of the thesis are theoretical. They fill a gap in probabilistic theory of the Lerch zeta-function, and can by applied for further investigations of this function.

Approbation

The results of the thesis were presented at the Conferences of Lithuanian Mathematical Society (2007, 2008, 2009), as well as at the seminar on number theory of Vilnius University and the seminar of the faculty of Mathematics and Informatics of Šiauliai University.

Principal publications

The main results of the thesis are published in the following papers:

1. V. Garbaliauskienė, D. Genienė and A. Laurinčikas, Value-distribution of the Lerch zeta-function with algebraic irrational parameter. I, *Lith. Math. J.*, **47** (2): 163–176, 2007.

2. D. Genienė, A. Laurinčikas and R. Macaitienė, Value-distribution of the Lerch zeta-function with algebraic irrational parameter. II, *Lith. Math. J.*, **47** (4): 394–405, 2007.

3. D. Genienė, A. Laurinčikas and R. Macaitienė, Value-distribution of the Lerch zeta-function with algebraic irrational parameter. II, *Lith. Math. J.*, **48** (3): 282–293, 2008.

4. D. Genienė, The Lerch zeta-function with algebraic irrational parameter. *Liet. math. rink. LMD darbai*, **50**: 1–5, 2009.

Outline of the thesis

The thesis consists of the introduction, three chapters, conclusions, bibliography and notation. In Introduction a short review on the actuality of the research field is given, the aims and problems are stated, the methods and the novelty of results are discussed. The history of results related to the thesis is also presented. In Chapter 1, a limit theorem on the complex plane for the Lerch zeta-function with algebraic irrational parameter is proved. Chapter 2 is devoted to a joint limit theorem on the complex plane for Lerch zeta-functions with algebraic irrational parameters. Finally, in Chapter 3, the limit theorems in the space of analytic functions for the Lerch zeta-function with algebraic irrational parameter are proved.

Acknowledgment

I would like to express my gratitude to Professor A. Laurinčikas who proposed me the problems solved in the thesis and supported their solution. I also thank Šiauliai University for financial support and the member of Faculties of Mathematics and Informatics of Vilnius and Šiauliai Universities for useful suggestions.

Chapter 1

A limit theorem on the complex plane for the Lerch zeta-function with algebraic irrational parameter

Let $0 < \lambda < 1$, and α be an algebraic irrational number, $0 < \alpha \leq 1$. In this chapter, we obtain a limit theorem on the complex plane for the Lerch zeta-function defined, for $\sigma > 1$, by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m}}{(m+\alpha)^s}. \tag{1.1}$$

Since $\lambda \notin \mathbb{Z}$, the function $L(\lambda, \alpha, s)$ has analytic continuation to an entire function.

1.1. The statement of the main theorem

Denote by $\text{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$ and let, for $T > 0$,

$$\nu_T^t(...) = \frac{1}{T}\text{meas}\{t \in [0, T] : ...\},$$

where in place of dots a condition satisfied by t is to be written. Define the probability measure P_T on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ by

$$P_T(A) = \nu_T^t(L(\lambda, \alpha, \sigma + it) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

where, as usual, $\mathcal{B}(S)$ stands for the the class of Borel sets of the space S.

Let

$$L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{C}_0\},$$

and let $I(\alpha)$ be a maximal linearly independent over the field of rational numbers $\mathbb{Q}$ subset of $L(\alpha)$. We suppose, that $I(\alpha) \neq L(\alpha)$ and denote $D(\alpha) = L(\alpha) \setminus I(\alpha)$. Then, for any element $d_m \in D(\alpha)$, the set $\{d_m\} \bigcup I(\alpha)$ is already linearly dependent over $\mathbb{Q}$. Therefore, there exist a finite number of elements

$i_{m_1}(m), ..., i_{m_{n(m)}}(m) \in I(\alpha)$ and numbers $k_0(m), ..., k_n(m) \in \mathbb{Z} \setminus \{0\}$ such that

$$d_m = -\frac{k_1(m)}{k_0(m)} i_{m_1(m)}(m) - ... - \frac{k_n(m)}{k_0(m)} i_{m_{n(m)}}(m).$$

Since the elements of $L(\alpha)$ are of the form $\log(m+\alpha)$, we have that

$$\log(m+\alpha) = -\frac{k_1(m)}{k_0(m)} \log(m_1(m)+\alpha) - ... - \frac{k_n(m)}{k_0(m)} \log(m_{n(m)}(m)+\alpha).$$

Hence we find that

$$m+\alpha = (m_1(m)+\alpha)^{-\frac{k_1(m)}{k_0(m)}} ... (m_{n(m)}(m)+\alpha)^{-\frac{k_n(m)}{k_0(m)}}. \tag{1.2}$$

Now define two subsets $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ of $\mathbb{C}_0$ by

$$\mathcal{M}(\alpha) = \{m \in \mathbb{C}_0 : \log(m+\alpha) \in I(\alpha)\},$$

$$\mathcal{N}(\alpha) = \{m \in \mathbb{C}_0 : \log(m+\alpha) \in D(\alpha)\},$$

and let

$$\Omega = \prod_{m \in \mathcal{M}(\alpha)} \gamma_m,$$

where $\gamma_m = \{s \in \mathbb{C} : |s| = 1\} \stackrel{def}{=} \gamma$ for $m \in \mathcal{M}(\alpha)$. Then Ω is also a compact topological Abelian group; therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and this leads to the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m, $m \in \mathcal{M}(\alpha)$. We extend the function $\omega(m)$ to the whole set $\mathbb{C}_0$ by putting, for $m \in \mathcal{N}(\alpha)$,

$$\omega(m) = \omega^{-\frac{k_1(m)}{k_0(m)}}(m_1(m)) ... \omega^{-\frac{k_n(m)}{k_0(m)}}(m_{n(m)}(m)),$$

if relation (1.2) takes place. Here the principal values of the roots are taken. For a given algebraic irrational α, there is no any concrete information on the set $L(\alpha)$. Therefore, all hypotheses are possible. In the sequel, we suppose that the numbers $\frac{k_1(m)}{k_0(m)}, ..., \frac{k_n(m)}{k_0(m)}$ are integer for all $M \in \mathcal{N}(\alpha)$ and denote the class of such numbers α, $0 < \alpha < 1$, by $\mathcal{A}$. Then we have that $\{\omega(m) : m \in \mathbb{C}_0\}$ is a sequence of pairwise orthogonal complex-valued random variables defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. For $\sigma > \frac{1}{2}$, on $(\Omega, \mathcal{B}(\Omega), m_H)$ define the complex-valued random variable $L(\lambda, \alpha, \sigma, \omega)$ by

$$L(\lambda, \alpha, \sigma, \omega) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^{\sigma}}.$$

Denote by P_L the distribution of the random variable $L(\lambda, \alpha, \sigma, \omega)$.

The main result of this chapter is the following assertion.

Theorem 1.1. *Suppose that $\lambda \in (0,1)$, $\alpha \in \mathcal{A}$, and $\sigma > \frac{1}{2}$. Then the probability measure P_T converges weakly to P_L as $T \to \infty$.*

1.2. A limit theorem on the torus Ω

In this section, we prove the weak convergence of the probability measure

$$Q_T(A) = \nu_T^t \left(((m+\alpha)^{-it} : m \in \mathcal{M}(\alpha)) \in A \right), \quad A \in \mathcal{B}(\Omega).$$

Theorem 1.2. *Let α be algebraic irrational. Then the probability measure Q_T converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. The dual group of Ω is isomorphic to

$$\bigoplus_{m \in \mathcal{M}(\alpha)} \mathbb{Z}_m,$$

where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathcal{M}(\alpha)$. The element $\underline{k} = \{k_m : m \in \mathcal{M}(\alpha)\} \in \bigoplus_{m \in \mathcal{M}(\alpha)} \mathbb{Z}_m$, where only a finite number of integers k_m are distinct from zero, acts on Ω by

$$\omega \to \omega^{\underline{k}} = \prod_{m \in \mathcal{M}(\alpha)} \omega^{k_m}(m), \quad \omega \in \Omega.$$

Therefore, the Fourier transform $g_T(\underline{k})$ of the measure Q_T is

$$g_T(\underline{k}) = \int\limits_{\Omega} \prod_{m\in\mathcal{M}(\alpha)} \omega^{k_m}(m)\mathrm{d}Q_T =$$

$$= \frac{1}{T}\int\limits_0^T \prod_{m\in\mathcal{M}(\alpha)} (m+\alpha)^{-ik_m t}\mathrm{d}t =$$

$$= \frac{1}{T}\int\limits_0^T \exp\left\{-it\sum_{m\in\mathcal{M}(\alpha)} k_m \log(m+\alpha)\right\}\mathrm{d}t.$$

The set $I(\alpha)$ is linearly independent over $\mathbb{Q}$. Thus

$$g_T(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k}=0, \\ \frac{\exp\left\{-iT\sum\limits_{m\in\mathcal{M}(\alpha)} k_m \log(m+\alpha)\right\}-1}{-iT\sum\limits_{m\in\mathcal{M}(\alpha)} k_m \log(m+\alpha)} & \text{if} \quad \underline{k}\neq 0. \end{cases}$$

From this we have that

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k}=0, \\ 0 & \text{if} \quad \underline{k}\neq 0. \end{cases}$$

Therefore, in view of Theorem 1.4.2 of [26] we obtain that the measure Q_T converges weakly to m_H as $T\to\infty$.

Note, that Theorem 1.2 is also given in [51], Lemma 4. However, the above proof is shorter and clearer.

1.3. Limit theorems for absolutely convergent Dirichlet series

Let $\sigma_1 > \frac{1}{2}$ be fixed. For $m \in \mathbb{C}_0$, define

$$l_n(s,\alpha) = \frac{s}{\sigma_1}\Gamma\left(\frac{s}{\sigma_1}\right)(m+\alpha)^s,$$

where $\Gamma(s)$ denotes the Euler gamma-function. For $\sigma > \frac{1}{2}$, define

$$L_n(\lambda,\alpha,s)=\frac{1}{2\pi i}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} L(\lambda,\alpha,s+z)l_n(z,\alpha)\frac{dz}{z}.$$

We have $\sigma+\sigma_1>1$; therefore, for $\mathrm{Re}z=\sigma_1$, the function $L(\lambda,\alpha,s)$ is represented by the absolutely convergent Dirichlet series

$$L(\lambda,\alpha,s+z)=\sum_{m=0}^{\infty}\frac{\mathrm{e}^{2\pi i\lambda m}}{(m+\alpha)^{s+z}}.$$

Now define

$$b_n(\lambda,\alpha,m)=\frac{\mathrm{e}^{2\pi i\lambda m}}{2\pi i}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{l_n(z,\alpha)}{(m+\alpha)^z}\frac{dz}{z}.$$

Then, in view of the well-known estimates for the function $\Gamma(s)$, we find that

$$b_n(\lambda,\alpha,m)\ll_n (m+\alpha)^{-\sigma_1}\int\limits_{-\infty}^{\infty}\frac{|l_n(\sigma_1+it)|}{|\sigma_1+it|}\mathrm{d}t\ll_n (m+\alpha)^{-\sigma_1}.$$

Here $f(x)\ll_\theta g(x)$, $g(x)>0$, $x\in X$, means that there exists a constant $C=C(\theta)>0$ such that $|f(x)|\le Cg(x)$ for all $x\in X$. Therefore, the series

$$\sum_{m=0}^{\infty}\frac{b_n(\lambda,\alpha,m)}{(m+\alpha)^s}$$

converges absolutely for $\sigma>\frac{1}{2}$. Now the interchange of order of summation and integration yields

$$\sum_{m=0}^{\infty}\frac{b_n(\lambda,\alpha,m)}{(m+\alpha)^s}=\frac{1}{2\pi i}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\left(\frac{l_n(z,\alpha)}{z}\sum_{m=0}^{\infty}\frac{\mathrm{e}^{2\pi i\lambda m}}{(m+\alpha)^{s+z}}\right)\mathrm{d}z=$$

$$=L_n(\lambda,\alpha,s). \tag{1.3}$$

Now let

$$v_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\sigma_1}\right\}.$$

Then the Mellin inversion formula

$$\frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty}\Gamma(s)a^{-s}\mathrm{d}s = \mathrm{e}^{-a}, \quad a,b>0,$$

shows that

$$b_n(\lambda,\alpha,m) = \mathrm{e}^{2\pi i\lambda m}v_n(m,\alpha).$$

Thus, we have by (1.3) that

$$L_n(\lambda,\alpha,s) = \sum_{m=0}^{\infty}\frac{\mathrm{e}^{2\pi i\lambda m}v_n(m,\alpha)}{(m+\alpha)^s},$$

the series being absolutely convergent for $\sigma > \frac{1}{2}$.

For $\sigma > \frac{1}{2}$ and $\omega_0 \in \Omega$, define

$$L_n(\lambda,\alpha,s,\omega) = \sum_{m=0}^{\infty}\frac{\mathrm{e}^{2\pi i\lambda m}v_n(m,\alpha)\omega_0(m)}{(m+\alpha)^s}.$$

In this section, we consider the weak convergence of the probability measures

$$P_{T,n}(A) = \nu_T^t\left(L_n(\lambda,\alpha,\sigma+it)\in A\right), \quad A\in\mathcal{B}(\mathbb{C}),$$

and, for $\omega_0 \in \Omega$,

$$\hat{P}_{T,n}(A) = \nu_T^t\left(L_n(\lambda,\alpha,\sigma+it,\omega_0)\in A\right), \quad A\in\mathcal{B}(\mathbb{C}).$$

Let S and S_1 be two metric spaces, P be a probability measure on $(S, \mathcal{B}(S))$, and let h be S_1-valued measurable function defined on $(S, \mathcal{B}(S))$. Then P induces the unique probability measure Ph^{-1} on $(S_1, \mathcal{B}(S_1))$ defined by the equality

$$Ph^{-1}(A) = P(h^{-1}(A))$$

for $A \in \mathcal{B}(S_1)$.

Denote by D_h the set of discontinuity points of h.

Lemma 1.1. *Suppose that $P(D_h) = 0$ and P_n converges weakly to the measure P as $n \to \infty$. Then $P_n h^{-1}$ converges weakly to Ph^{-1} as $n \to \infty$.*

Proof. The lemma is Theorem 5.1 from [5].

Theorem 1.3. *Let α be algebraic irrational and $\sigma > \frac{1}{2}$. Then the probability measures $P_{T,n}$ and $\hat{P}_{T,n}$ both converge weakly to the same probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$.*

Proof. Define the function $u : \Omega \to \mathbb{C}$ by the formula

$$u(\omega) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m} v_n(m, \alpha) \omega(m)}{(m+\alpha)^{\sigma}}.$$

Since the series converges absolutely for $\sigma > \frac{1}{2}$, the function u is continuous. Moreover,

$$u\left(\{(m+\alpha)^{-it} : m \in \mathcal{M}(\alpha)\}\right) =$$

$$= \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m} v_n(m, \alpha)}{(m+\alpha)^{\sigma+it}} = L_n(\lambda, \alpha, \sigma + it).$$

Therefore, we have that $P_{T,n} = Q_T u^{-1}$, where, for $A \in \mathcal{B}(\mathbb{C})$, $Q_T u^{-1}(A) = Q_T(u^{-1}A)$, and using the continuity of u, Theorem 1.2 and Lemma 1.1, we obtain that the measure $P_{T,n}$ converges weakly to $m_H u^{-1}$ as $T \to \infty$.

Now define $u_1 : \Omega \to \Omega$ by the formula $u_1(\omega) = \omega\omega_0$. Then we obtain that

$$u\left(u_1\left(\{(m+\alpha)^{-it} : m \in \mathcal{M}(\alpha)\}\right)\right) =$$

$$= \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m} v_n(m,\alpha)\omega_0(m)}{(m+\alpha)^{\sigma+it}} = L_n(\lambda,\alpha,\sigma+it,\omega_0).$$

Therefore, similarly to the case of $P_{T,n}$, we find that the measure $\hat{P}_{T,n}$ converges weakly to $m_H(uu_1)^{-1}$ as $T \to \infty$. However, the invariance of the Haar measure m_H shows that $m_H(uu_1)^{-1} = (m_H u_1^{-1})u = m_H u^{-1}$, and the theorem is proved.

1.4. Approximation in the mean

To prove that the function $L(\lambda,\alpha,s)$ has a limit distribution, i. e., that the measure P_T converges weakly to some measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ we have to pass from the function $L_n(\lambda,\alpha,s)$ to $L(\lambda,\alpha,s)$. For this, we need an approximation of $L(\lambda,\alpha,s)$ by $L_n(\lambda,\alpha,s)$ in the mean.

Lemma 1.3. *Let $\sigma > \frac{1}{2}$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T |L(\lambda,\alpha,\sigma+it) - L_n(\lambda,\alpha,\sigma+it)| \mathrm{d}t = 0.$$

Proof. Let K be a compact subset of the half-plane $\left\{s \in \mathbb{C} : \sigma > \frac{1}{2}\right\}$. In [46], Lemma 5.2.11, it is proved that, for transcendental α,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} |L(\lambda,\alpha,\sigma+it) - L_n(\lambda,\alpha,\sigma+it)| \mathrm{d}t = 0. \tag{1.4}$$

However, it is easily seen that the proof is independent of the arithmetic of the number α. Thus, the lemma is a corollary of relation (1.4).

The case of approximation $L(\lambda,\alpha,s,\omega)$ by $L_n(\lambda,\alpha,s,\omega)$ in the mean is more complicated. First, we have to establish the boundness of the mean square for $L(\lambda,\alpha,s,\omega)$. For this, we will apply the Birkhoff-Khintchine theorem from ergodic theory.

For $t \in \mathbb{R}$, we put

$$a_t = \left\{(m+\alpha)^{-it} : m \in \mathcal{M}(\alpha)\right\},$$

and define the family $\{\varphi_t : t \in \mathbb{R}\}$ of transformations on Ω by $\varphi_t(\omega) = a_t\omega$ for $\omega \in \Omega$. Then $\{\varphi_t : t \in \mathbb{R}\}$ is an one-parameter group of measurable measure-preserving transformations on the torus Ω. We recall that a set $A \in \mathcal{B}(\Omega)$ is invariant with respect to the group $\{\varphi_t : t \in \mathbb{R}\}$ if, for each t, the sets A and $A_t = \varphi_t(A)$ differ one from another by a set of zero m_H-measure. The group $\{\varphi_t : t \in \mathbb{R}\}$ is ergodic if its σ-field of invariant sets consists only of sets having m_H-measure equal to 1 or 0.

Lemma 1.4. *Suppose that α is algebraic irrational. Then the one-parameter group $\{\varphi_t : t \in \mathbb{R}\}$ is ergodic.*

Proof. The lemma is Lemma 7 from [51].

Let Y be the space of finite real functions $y(\tau)$, $\tau \in \mathbb{R}$. If is well known that the family of finite-dimensional distributions of each random process determines a probability measure Q on $(Y, \mathcal{B}(Y))$. On the probability space $(Y, \mathcal{B}(Y), Q)$, the translation $g_u : Y \to Y$ can be defined by $g_u(y(\tau)) = g(\tau - u)$. A strongly stationary process $X(\tau, \omega)$ is called ergodic if its σ- field of invariant sets consists only of sets having Q-measure equal to 0 or 1.

The following statement is the classical Birkhoff-Khintchine theorem, see, for example, [10].

Lemma 1.5. *Suppose that a process $X(t, \omega)$ is ergodic, $\mathbb{E}|X(t, \omega)| < \infty$, and that sample paths are integrable almost surely in the Riemann sense over every finite interval. Then*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, \omega) \mathrm{d}t = \mathbb{E}X(0, \omega).$$

Lemma 1.6. *Suppose that $\alpha \in \mathcal{A}$ and $\sigma > \frac{1}{2}$. Then, for $T \to \infty$,*

$$\int_0^T |L(\lambda, \alpha, s, \sigma + it, \omega)|^2 \mathrm{d}t \ll T$$

for almost all $\omega \in \Omega$.

Proof. For $m \in \mathbb{C}_0$, we put

$$X_m(\lambda, \alpha, s, \sigma, \omega) = \frac{\mathrm{e}^{2\pi i \lambda m} \omega(m)}{(m + \alpha)^\sigma}.$$

Then, obviously,

$$\mathbb{E}|X_m(\lambda, \alpha, \sigma, \omega)|^2 = \frac{1}{(m + \alpha)^{2\sigma}}.$$

Thus, in view of the orthogonality of the random variables $\omega(m)$,

$$\mathbb{E}|L(\lambda, \alpha, \sigma, \omega)|^2 = \sum_{m=0}^{\infty} \mathbb{E}|X_m(\lambda, \alpha, \sigma, \omega)|^2 = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{2\sigma}} < \infty. \tag{1.5}$$

However,

$$|L(\lambda, \alpha, \sigma, \varphi_t(\omega))|^2 = |L(\lambda, \alpha, \sigma + it, \omega)|^2,$$

and Lemma 1.5 implies the ergodicity of the process $|L(\lambda, \alpha, \sigma+it, \omega)|^2$. Therefore, by Lemma 1.5

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |L(\lambda, \alpha, \sigma + it, \omega)|^2 \mathrm{d}t = \mathbb{E}|L(\lambda, \alpha, \sigma, \omega)|^2$$

for almost $\omega \in \Omega$. This togehther with (1.5) proves the lemma.

Theorem 1.4. *Suppose that $\alpha \in \mathcal{A}$ and $\sigma > \frac{1}{2}$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T |L(\lambda, \alpha, \sigma + it, \omega) - L_n(\lambda, \alpha, \sigma + it, \omega)| \mathrm{d}t = 0$$

for almost all $\omega \in \Omega$.

Proof. Similarly to the case of $L(\lambda, \alpha, s)$ we have that

$$L_n(\lambda, \alpha, s, \omega) = \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} L(\lambda, \alpha, s+z, \omega) l_n(z, \alpha) \frac{dz}{z}. \tag{1.6}$$

The function $L(\lambda, \alpha, s, \omega)$ is analytic in $\{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$ for almost all $\omega \in \Omega$. Let $\sigma_2 > \frac{1}{2}$, and $\sigma > \sigma_2$. Then by the residue theorem from (1.6) we deduce that

$$L_n(\lambda, \alpha, s, \omega) = \frac{1}{2\pi i} \int_{\sigma_2 - \sigma - i\infty}^{\sigma_2 - \sigma + i\infty} L(\lambda, \alpha, s+z, \omega) l_n(z, \alpha) \frac{dz}{z} + L(\lambda, \alpha, s, \omega).$$

Hence, we find that

$$L(\lambda,\alpha,\sigma+it,\omega)-L_n(\lambda,\alpha,\sigma+it,\omega)\ll\int\limits_{-\infty}^{\infty}|L(\lambda,\alpha,\sigma_2+it,\omega)|\cdot|l_n(\sigma_2-\sigma+it,\alpha)|\mathrm{d}\tau.$$

Therefore, in view of Lemma 1.6,

$$\frac{1}{T}\int\limits_0^T|L(\lambda,\alpha,\sigma+it,\omega)-L_n(\lambda,\alpha,\sigma+it,\omega)|\mathrm{d}t\ll$$

$$\ll\int\limits_{-\infty}^{\infty}\left(|l_n(\sigma_2-\sigma+i\tau,\alpha)|\frac{1}{T}\int\limits_0^T|L(\lambda,\alpha,\sigma_2+it+i\tau,\omega)|\mathrm{d}t\right)\mathrm{d}\tau\ll$$

$$\ll\int\limits_{-\infty}^{\infty}\left(|l_n(\sigma_2-\sigma+i\tau,\alpha)|\frac{1}{T}\int\limits_{-|\tau|}^{|\tau|+T}|L(\lambda,\alpha,\sigma_2+it,\omega)|\mathrm{d}t\right)\mathrm{d}\tau\ll$$

$$\ll\int\limits_{-\infty}^{\infty}|l_n(\sigma_2-\sigma+i\tau,\alpha)|(1+|\tau|)\mathrm{d}\tau \tag{1.7}$$

for almost all $\omega\in\Omega$.

Since $\sigma_2-\sigma<0$, by the definition of $l_n(s,\alpha)$ we have that

$$\lim_{n\to\infty}\int\limits_{-\infty}^{\infty}|l_n(\sigma_2-\sigma+i\tau,\alpha)|(1+|\tau|)\mathrm{d}\tau=0.$$

This and (1.7) prove the theorem.

1.5. Proof of Theorem 1.1

Let S be a metric space, P be a probability measure on $\mathcal{B}(S)$ and let ∂A denote the boundary of A. A set $A \in \mathcal{B}(S)$ is called a continuity set of the measure P if $P(\partial A) = 0$.

Lemma 1.7. *Let P and P_n, $n \in \mathbb{C}$, be probability measures on $(S, \mathcal{B}(S))$. The relations:*

1) P_n weakly converges to P as $n \to \infty$;

2) $\lim\limits_{n\to\infty} \int\limits_S f(x)\mathrm{d}P_n(x) = \int\limits_S f(x)\mathrm{d}P(x)$ for all bounded uniformly continuous real functions f;

3) $\limsup\limits_{n\to\infty} P_n(F) \leq P(F)$ for all closed sets $F \subset S$;

4) $\liminf\limits_{n\to\infty} P_n(G) \geq P(G)$ for all open sets $G \subset S$;

5) $\lim\limits_{n\to\infty} P_n(A) = P(A)$ for all continuity sets A of the measure P

are equivalent.

Proof. The lemma is Theorem 2.1 from [5].

The family $\{P\}$ of probability measure on $(S, \mathcal{B}(S))$ is called relatively compact if every sequence of elements of $\{P\}$ contains a weakly convergent subsequence. The family $\{P\}$ is called tight if for arbitrary $\varepsilon > 0$ there exists a compact set $K \in \mathcal{B}(S)$ such that $P(K) > 1 - \varepsilon$ for all P from $\{P\}$.

The following statement is the well-known Prokhorov theorem, see, for example [5] Theorems 6.1 and 6.2.

Lemma 1.8. *If the family of probability measure $\{P\}$ is tight, then it is relatively compact. If S is a separable complete metric space and the family $\{P\}$ on $(S, \mathcal{B}(S))$ is relatively compact, then it is tight.*

Now suppose that S-valued random elements $Y_n, X_{1n}, X_{2n}, ...$ are defined on the same probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), P)$ and that the space S is separable. Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution.

Lemma 1.9. *Suppose, that for every u $X_{un} \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X_n$ and $X_n \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X$. Suppose also, that, for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \limsup_{n\to\infty} P\{\varrho(X_{un}, Y_n) \geq \varepsilon\} = 0.$$

Then $Y_n \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X$.

Proof. The lemma is Theorem 4.2 from [5].

For $A \in \mathcal{B}(\mathbb{C})$ and $\omega \in \Omega$, define

$$\hat{P}_T(A) = \nu_T^t(L(\lambda, \alpha, \sigma + it, \omega) \in A).$$

Theorem 1.5. *Suppose that $\lambda \in (0,1)$, $\alpha \in \mathcal{A}$ and $\sigma > \frac{1}{2}$. Then the probability measures P_T and $\hat{P}_T$ both converge weakly to the same probability measure, say, P on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$.*

Proof. By Theorem 1.3 the probability measures $P_{T,n}$ and $\hat{P}_{T,n}$ both converge to the same probability measure, say, P_n on on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$. The first step is to show that the family of probability measures $\{P_n : n \in \mathbb{C}_0\}$ is tight.

Let θ be a uniformly distributed on $[0,1]$ random variable defined on a certain probability space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \mathbb{P})$. Define

$$X_{T,n} = X_{T,n}(\sigma) = L(\lambda, \alpha, \sigma + it\theta).$$

Then by Theorem 1.3

$$X_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} X_n, \tag{1.8}$$

where $X_n = X_n(\sigma)$ is a complex-valued random variable with distribution P_n.

Since the series for $L_n(\lambda, \alpha, s)$ converges absolutely for $\sigma > \frac{1}{2}$, we have that

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T |L_n(\lambda, \alpha, \sigma + it)|^2 \mathrm{d}t =$$

$$= \sum_{m=0}^{\infty} \frac{v_n^2(m,\alpha)}{(m+\alpha)^{2\sigma}} \leq \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{2\sigma}}.$$

Therefore, there exists a real number $0 < R < \infty$ such that

$$\sup_{n\in\mathbb{C}_0} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T |L_n(\lambda, \alpha, \sigma + it)| \mathrm{d}t \leq$$

$$\leq \sup_{n\in\mathbb{C}_0} \limsup_{T\to\infty} \left(\frac{1}{T} \int\limits_0^T |L_n(\lambda, \alpha, \sigma + it)|^2 \mathrm{d}t \right)^{\frac{1}{2}} \leq R.$$

Thus, taking $M = R\varepsilon^{-1}$ with arbitrary $\varepsilon > 0$, we find that

$$\limsup_{T\to\infty} P_{T,n}(\{s \in \mathbb{C} : |s| > M\}) = \limsup_{T\to\infty} \nu_T\left(|L_n(\lambda, \alpha, \sigma + it)| > M\right) \leq$$

$$\leq \limsup_{T\to\infty} \frac{1}{M} \int\limits_0^T |L_n(\lambda, \alpha, \sigma + it)| \mathrm{d}t \leq \varepsilon. \tag{1.9}$$

Clearly, the weak convergence of the measure $P_{T,n}$ implies that of the measure

$$\nu_T^t(|L_n(\lambda, \alpha, \sigma + it)| \in A), \quad A \in \mathcal{B}(\mathbb{R}).$$

Therefore, by Lemma 1.7 and (1.9)

$$P_n(\{s \in \mathbb{C} : |s| > M\}) \leq \liminf_{T\to\infty} P_{T,n}(\{s \in \mathbb{C} : |s| > M\}) \leq$$

$$\leq \limsup_{T\to\infty} P_{T,n}(\{s \in \mathbb{C} : |s| > M\}) \leq \varepsilon.$$

Now, taking $K_\varepsilon = \{s \in \mathbb{C} : |s| \leq M\}$, hence we obtain that

$$P_n(K_\varepsilon) \geq 1 - \varepsilon, \quad n \in \mathbb{C}_0.$$

Since K_ε is a compact set on $\mathbb{C}$, this proves the tightness of the family $\{P_n : n \in \mathbb{C}_0\}$. By the Prokhorov theorem (Lemma 1.8) now it follows that $\{P_n : n \in \mathbb{C}_0\}$ is relatively compact. Therefore, there exists $\{P_{n_1}\} \subset \{P_n\}$ such that P_{n_1} converges weakly to some probability measure P on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $n_1 \to \infty$, and we have that

$$X_{n_1} \xrightarrow[n_1\to\infty]{\mathcal{D}} P. \tag{1.10}$$

Let

$$X_T = X_T(\sigma) = L(\lambda, \alpha, \sigma + iT\theta).$$

Then using Lemma 1.3, we find that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \hat{\mathbb{P}}\left(\{|X_T(\sigma) - X_{T,n}(\sigma)| \geq \varepsilon\}\right) =$$

$$= \lim_{n\to\infty} \limsup_{T\to\infty} \nu_T^t\left(|L(\lambda, \alpha, \sigma + it) - L_n(\lambda, \alpha, \sigma + it)| \geq \varepsilon\right) \leq$$

$$\leq \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_0^T |L(\lambda, \alpha, \sigma + it) - L_n(\lambda, \alpha, \sigma + it)| \mathrm{d}t = 0.$$

Now from this, (1.10), (1.8) and from Lemma 1.9 we deduce that

$$X_T \xrightarrow[T\to\infty]{\mathcal{D}} P. \tag{1.11}$$

This shows that the measure P is independent on the sequence $\{P_{n_1}\}$. Consequently, we have that

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{1.12}$$

Now define

$$\hat{X}_{T,n} = \hat{X}_{T,n}(\sigma) = L(\lambda, \alpha, \sigma + iT\theta, \omega)$$

and

$$\hat{X}_T = \hat{X}_T(\sigma) = L(\lambda, \alpha, \sigma + iT\theta, \omega).$$

Then by the same arguments, using Theorems 1.3 and 1.4 and (1.12), we obtain that

$$\hat{X}_T \xrightarrow[T\to\infty]{\mathcal{D}} P.$$

This and (1.11) prove the theorem.

Proof of Theorem 1.1. In view of Theorem 1.5, it remains to show that $P = P_L$.

Let $A \in \mathcal{B}(\mathbb{C})$ be a fixed continuity set of the measure P. Then by Theorem 1.5 and Lemma 1.8

$$\lim_{T\to\infty} \nu_T^t \left(L(\lambda, \alpha, \sigma + it, \omega) \in A\right) = P(A). \tag{1.13}$$

On $(\Omega, \mathcal{B}(\Omega))$, define a random variable ξ by

$$\xi = \xi(\omega) = \begin{cases} 1 & \text{if} \quad L(\lambda, \alpha, \sigma, \omega) \in A, \\ 0 & \text{if} \quad L(\lambda, \alpha, \sigma, \omega) \notin A. \end{cases}$$

It is easily seen that

$$\mathbb{E}(\xi) = \int\limits_{\Omega} \xi \mathrm{d}m_H = m_H(\omega \in \Omega : L(\lambda, \alpha, \sigma, \omega) \in A) = P_L(A). \tag{1.14}$$

By Lemma 1.4 the process $\xi(\varphi_t(\omega))$ is ergodic. Therefore, by Lemma 1.5

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \xi(\varphi_t(\omega))\mathrm{d}t = \mathbb{E}(\xi) \tag{1.15}$$

for almost all $\omega \in \Omega$. However, the definitions of ξ and φ_t show that

$$\frac{1}{T} \int\limits_0^T \xi(\varphi_t(\omega))\mathrm{d}t = \nu_T^t(L(\lambda, \alpha, \sigma + it, \omega) \in A).$$

This, (1.14) and (1.15) yield

$$\lim_{T\to\infty} \nu_T^t(L(\lambda, \alpha, \sigma + it, \omega) \in A) = P_L(A)$$

for almost all $\omega \in \Omega$, and in view of (1.13), the equality $P(A) = P_L(A)$ holds for every continuity set A of P. Hence, we have that $P(A) = P_L(A)$ for all $A \in \mathcal{B}(\mathbb{C})$, and the theorem is proved.

Chapter 2

A joint limit theorem on the complex plane for Lerch zeta-function with algebraic irrational parameters

2.1. The statement of a joint limit theorem

Suppose that $\alpha_1, ..., \alpha_r$ are distinct algebraic irrational numbers from the class $\mathcal{A}$. Define

$$\Omega^r = \prod_{j=1}^{r} \Omega_j,$$

where

$$\Omega_j = \prod_{m \in \mathcal{M}(\alpha_j)} \gamma_m$$

and $\gamma_m = \gamma$ for $m \in \mathcal{M}(\alpha_j)$, $j = 1, ..., r$, $\mathcal{M}(\alpha_j)$ being defined as in section 1.1. Since each torus Ω_j is a compact topological Abelian group, Ω^r is as well. Thus, we obtain the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$, where m_H^r is the probability Haar measure on $(\Omega^r, \mathcal{B}(\Omega^r))$. Let, for brevity, $\underline{\omega} = (\omega_1, ..., \omega_r) \in \Omega^r$, where $\omega_j \in \Omega_j$, $j = 1, ..., r$, $\underline{\lambda} = (\lambda_1, ..., \lambda_r)$, $\underline{\alpha} = (\alpha_1, ..., \alpha_r)$, and $\underline{\sigma} = (\sigma_1, ..., \sigma_r)$. On the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$, define the $\mathbb{C}^r$-valued random element $\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega})$, for $\min\limits_{1 \le j \le r} \sigma_j > \frac{1}{2}$, by

$$\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}) = \big(L(\lambda_1, \alpha_1, \sigma_1, \omega_1), ..., L(\lambda_r, \alpha_r, \sigma_r, \omega_r)\big),$$

where

$$L(\lambda_j, \alpha_j, \sigma_j, \omega_j) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m} \omega_j(m)}{(m + \alpha_j)^{\sigma_j}}$$

and $\omega_j(m)$ is the projection of $\omega_j \in \Omega_j$ to the coordinate space γ_m if $m \in \mathcal{M}(\alpha_j)$, and the relation of type (0.2) if $m \in \mathcal{N}(\alpha_j)$, $j = 1, ..., r$. Let $Q_{\underline{L}}$ denote the distribution of the random element $\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega})$, i. e.

$$Q_{\underline{L}}(A) = m_H^r \left(\underline{\omega} \in \Omega^r : \underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}^r).$$

For $A \in \mathcal{B}(\mathbb{C}^r)$, define

$$\underline{P}_T(A) = \nu_T^t\big((L(\lambda_1, \alpha_1, \sigma_1 + it), ..., L(\lambda_r, \alpha_r, \sigma_r + it)) \in A\big).$$

Theorem 2.1. *Suppose that* $\lambda_j \in (0, 1)$, $j = 1, \dots, r$, *that* $\alpha_1, ..., \alpha_r$ *are distinct algebraic irrational numbers from the class* $\mathcal{A}$ *such that the set*

$$\bigcup_{j=1}^{r} I(\alpha_j)$$

is linearly independent over $\mathbb{Q}$, *and* $\min\limits_{1 \le j \le r} \sigma_j > \frac{1}{2}$. *Then the probability measure* $\underline{P}_T$ *converges weakly to the measure* $Q_{\underline{L}}$ *as* $T \to \infty$.

2.2. A limit theorem on Ω^r

In this section, we consider the weak convergence of the probability measure

$$Q_{T,\underline{\alpha}}(A) = \nu_T^t\big(\big((m + \alpha_1)^{it} : m \in \mathcal{M}(\alpha_1), ..., (m + \alpha_r)^{it} : m \in \mathcal{M}(\alpha_r)\big) \in A\big),$$

$$A \in \mathcal{B}(\Omega^r),$$

as $T \to \infty$.

Theorem 2.2. *Suppose that the numbers* $\alpha_1, ..., \alpha_r$ *satisfy the hypotheses of Theorem 2.1. Then the probability measure* Q_T *converges weakly to the measure* m_H^r *as* $T \to \infty$.

Proof. The dual group of Ω^r is isomorphic to

$$\bigoplus_{j=1}^{r} \bigoplus_{m \in \mathcal{M}(\alpha_j)} \mathbb{Z}_{mj},$$

where $\mathbb{Z}_{mj} = \mathbb{Z}$ for all $m \in \mathcal{M}(\alpha_j)$ and $j = 1, ..., r$. The element

$$\underline{k} = \big((k_{m1})_{m \in \mathcal{M}(\alpha_1)}, ..., (k_{mr})_{m \in \mathcal{M}(\alpha_r)}\big) \in \bigoplus_{j=1}^{r} \bigoplus_{m \in \mathcal{M}(\alpha_j)} \mathbb{Z}_{mj}$$

acts on Ω^r by

$$\underline{\omega} \to \underline{\omega}^{\underline{k}} = \prod_{j=1}^{r} \prod_{m \in \mathcal{M}(\alpha_j)} \omega_j^{k_{mj}}(m),$$

where only a finite number of integers k_{mj} are distinct from zero. Hence, we have that the Fourier transform $g_{T,\alpha}(\underline{k})$ of the probability measure $Q_{T,\alpha}$ is

$$g_{T,\alpha}(\underline{k}) = \int\limits_{\Omega^{(r)}} \prod_{j=1}^{r} \prod_{m \in \mathcal{M}(\alpha_j)} \omega_j^{k_{mj}} \mathrm{d}Q_T =$$

$$= \frac{1}{T} \int\limits_{0}^{T} \prod_{j=1}^{r} \prod_{m \in \mathcal{M}(\alpha_j)} (m+\alpha_j)^{ik_{mj}t} \mathrm{d}t =$$

$$= \frac{1}{T} \int\limits_{0}^{T} \exp\left\{ it \sum_{j=1}^{r} \sum_{m \in \mathcal{M}(\alpha_j)} k_{mj} \log(m+\alpha_j) \right\} \mathrm{d}t, \tag{2.1}$$

where only a finite number of integers k_{mj} are distinct from zero. Since the set

$$\bigcup_{j=1}^{r} I(\alpha_j)$$

is linearly independent over $\mathbb{Q}$, from (2.1)we obtain that

$$g_{T\alpha}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k}=\underline{0}, \\ \dfrac{\exp\left\{ iT \sum\limits_{j=1}^{r} \sum\limits_{m \in \mathcal{M}(\alpha_j)} k_{mj} \log(m+\alpha_j) \right\} - 1}{iT \sum\limits_{j=1}^{r} \sum\limits_{m \in \mathcal{M}(\alpha_j)} k_{mj} \log(m+\alpha_j)} & \text{otherwise,} \end{cases}$$

and that

$$\lim_{T\to\infty} g_{T,\alpha}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k}=\underline{0}, \\ 0 & \text{if } \underline{k}\neq\underline{0}. \end{cases}$$

This proves the theorem.

2.3. Case of absolutely convergent Dirichlet series

For a fixed $\sigma_{1j} > \frac{1}{2}$ and $m, n \in \mathbb{C}_0$, let

$$v_j(m, n, \alpha_j) = \exp\left\{-\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\sigma_{1j}}\right\},$$

and let, for $\sigma > \frac{1}{2}$,

$$L_{nj}(\lambda_j, \alpha_j, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda_j m} v_j(m, n, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, ..., r,$$

the series being absolutely convergent for $\sigma > \frac{1}{2}$, see [46]. Obviously, the series

$$L_{nj}(\lambda_j, \alpha_j, s, \omega_j) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda_j m} \omega_j(m) v_j(m, n, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, ..., r,$$

is also absolutely convergent for $\sigma > \frac{1}{2}$. For brevity, denote

$$\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}) = \big(L_{n1}(\lambda_1, \alpha_1, \sigma_1), ..., L_{nr}(\sigma_r, \alpha_r, \sigma_r)\big),$$

and, for $\underline{\omega} \in \Omega^r$,

$$\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}) = \big(L_{n1}(\lambda_1, \alpha_1, \sigma_1, \omega_1), ..., L_{nr}(\sigma_r, \alpha_r, \sigma_r, \omega_r)\big).$$

In this section, we consider the weak convergence of the probability measures

$$\underline{P}_{T,n}(A) = \nu_T^t\big(\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}^r),$$

and, for a fixed $\underline{\widehat{\omega}} \in \Omega^r$,

$$\underline{\widehat{P}}_{T,n}(A) = \nu_T^t\big(\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it, \underline{\widehat{\omega}}) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}^r),$$

where $\underline{\sigma} + it = (\sigma_1 + it, ..., \sigma_r + it)$.

Theorem 2.3. *Suppose that the numbers $\alpha_1, ..., \alpha_r$ and $\lambda_1, ..., \lambda_r$ satisfy the hypotheses of Theorem 2.1, and $\min\limits_{1 \le j \le r} \sigma_j > \frac{1}{2}$. Then on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ there exists a probability measure P_n such that both the measures $\underline{P}_{T,n}$ and $\widehat{\underline{P}}_{T,n}$ converge weakly to P_n as $T \to \infty$.*

Proof. Define a function $\underline{h}_n : \ \Omega^r \to \mathbb{C}^r$ by

$$\underline{h}_n \left((\{\omega_1(m) : \ m \in \mathcal{M}(\alpha_1)\}, \ldots, \{\omega_r(m) : \ m \in \mathcal{M}(\alpha_r)\}) \right) =$$

$$\left(\sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda_1 m} v_1(m, n, \alpha_1)}{(m+\alpha_1)^{\sigma_1} \omega_1(m)}, \ldots, \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda_r m} v_r(m, n, \alpha_r)}{(m+\alpha_r)^{\sigma_r} \omega_r(m)} \right).$$

Then the function h_n is continuous, and

$$h_n \left((\{(m+\alpha_1)^{it} : m \in \mathcal{M}(\alpha_1)\}, \ldots, \{(m+\alpha_r)^{it} : m \in \mathcal{M}(\alpha_r)\}) \right) =$$

$$= \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it).$$

This, Theorem 2.2 and Lemma 1.1 show that the probability measure $\underline{P}_{T,n}$ converges weakly to the measure $m_H^r \underline{h}_n^{-1}$ as $T \to \infty$.

Now define a new function $\widehat{\underline{h}} : \Omega^{(r)} \to \Omega^{(r)}$ by

$$\widehat{\underline{h}} \left((\{\omega_1(m) : \ m \in \mathcal{M}(\alpha_1)\}, \ldots, \{\omega_r(m) : \ m \in \mathcal{M}(\alpha_r)\}) \right)$$

$$= \left(\{\widehat{\omega}_1(m) \omega_1^{-1}(m) : \ m \in \mathcal{M}(\alpha_1)\}, \ldots, \{\omega_r(m) \omega_r^{-1}(m) : \ m \in \mathcal{M}(\alpha_r)\} \right).$$

Then one easily sees that

$$\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it, \underline{\widehat{\omega}}) =$$

$$= \underline{h}_n\big(\widehat{\underline{h}}\big((\{(m+\alpha_1)^{it} : m \in \mathcal{M}(\alpha_1)\}, \ldots, \{(m+\alpha_r)^{it} : m \in \mathcal{M}(\alpha_r)\}\big)\big)\big).$$

From this, reasoning similarly to the case of the measure $P_{T,n}$, we find that the measure $\widehat{P}_{T,n}$ converges weakly to the measure $m_H^r(\underline{h}_n\widehat{\underline{h}})^{-1}$ as $T \to \infty$. However, the Haar measure m_H^r is invariant, and, therefore, we obtained that

$$m_H^{(r)}(\underline{h}_n\widehat{\underline{h}})^{-1} = (m_H^{(r)}\widehat{\underline{h}})\underline{h}_n^{-1} = m_H^r\underline{h}_n^{-1},$$

and the theorem is proved.

2.4. Approximation in the mean

Denote

$$\varrho(\underline{z}^{(1)}, \underline{z}^{(2)}) = \left(\sum_{j=1}^{r} |z_j^{(1)} - z_j^{(2)}|^2\right)^{\frac{1}{2}},$$

$\underline{z}^{(j)} = \left(z_1^{(j)}, \ldots, z_r^{(j)}\right) \in \mathbb{C}^r$, $j = 1, 2$, the metric in $\mathbb{C}^r$ which induces the topology of $\mathbb{C}^r$.

In this section, we approximate in the mean the vectors $\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it)$ and $\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it, \underline{\omega})$ by the vectors $\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it)$ and $\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it, \underline{\omega})$, respectively.

Theorem 2.4. *Suppose that the numbers $\alpha_1, ..., \alpha_r$ and $\lambda_1, ..., \lambda_r$ satisfy the hypotheses of Theorem 2.1, and $\min\limits_{1 \le j \le r} \sigma_j > \frac{1}{2}$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \varrho\big(\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it)\big) \mathrm{d}t = 0,$$

and, for almost all $\underline{\omega} \in \Omega^r$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \varrho\big(\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it, \underline{\omega}), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it, \underline{\omega})\big) \mathrm{d}t = 0.$$

Proof. From Lemma 5.2.11 of [46], as in Lemma 1.3, it follows that, independently of the arithmetic origine of the numbers α and $\lambda \in (0,1)$, for $\sigma > \frac{1}{2}$, we have

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left|L(\lambda,\alpha,\sigma+it) - L_n(\lambda,\alpha,\sigma+it)\right| \mathrm{d}t = 0.$$

Consequently , we have that, for $\min\limits_{1\leq j\leq r} \sigma_j > \frac{1}{2}$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left|L(\lambda_j,\alpha_j,\sigma_j+it) - L_n(\lambda_j,\alpha_j,\sigma_j+it)\right| \mathrm{d}t = 0,$$

$j = 1,\dots,r$. From this, using the definition of the metric ϱ, we obtain the first assertion of the theorem.

Similarly, by Theorem 1 of [51] we have that, for $\min\limits_{1\leq j\leq r} \sigma_j > \frac{1}{2}$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left|L(\lambda_j,\alpha_j,\sigma_j+it,\omega_j) - L_n(\lambda_j,\alpha_j,\sigma_j+it,\omega_j)\right| \mathrm{d}t = 0$$

for almost all $\omega_j \in \Omega_j$, $j = 1,\dots,r$. Since the measure m_H^r is a product of the Haar measures on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1,...,r$, the statement of the theorem follows.

2.5. Proof of Theorem 2.1

We start with an analogue of Theorem 2.3. Define, for $\underline{\omega} \in \Omega^r$, the probability measure

$$\widehat{\underline{P}}_T(A) = \nu_T^t\big(\underline{L}_n(\underline{\lambda},\underline{\alpha},\underline{\sigma}+it,\underline{\omega}) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}^r).$$

Theorem 2.5. *Suppose that the numbers $\lambda_1,...,\lambda_r$ and $\alpha_1,...,\alpha_r$ satisfy the hypotheses of Theorem 2.1, and that $\min\limits_{1\leq j\leq r} \sigma_j > \frac{1}{2}$. Then on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ there exists a probability measure P such that both the measures P_T and $\widehat{\underline{P}}_T$ converge weakly to P as $T \to \infty$.*

Proof. Theorem 2.3 shows that both the measures $\underline{P}_{T,n}$ and $\widehat{\underline{P}}_{T,n}$ converge weakly to the same probability measure $\underline{P}_n$ as $T \to \infty$. It is not difficult to see that the family of probability measures $\{\underline{P}_n : n \in \mathbb{C}_0\}$ is tight. Really, the definition of $\underline{P}_{T,n}$ and the Chebyshev inequality, for $M > 0$, give

$$P_{T,n}\big(\{\underline{z} \in \mathbb{C}^r : \varrho(\underline{z}, \underline{0}) > M\}\big) = \nu_T^t\big(\varrho(\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it), \underline{0}) > M\big) \leq$$

$$\leq \frac{1}{MT} \int_0^T \varrho\big(\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}+it), \underline{0}\big) \mathrm{d}t \leq \frac{1}{M} \left(\frac{1}{T} \int_0^T \sum_{j=1}^r |L_{nj}(\lambda_j, \alpha_j, \sigma_j + it)|^2 \, \mathrm{d}t \right)^{\frac{1}{2}} =$$

$$= \frac{1}{M} \left(\sum_{j=1}^r \frac{1}{T} \int_0^T |L_{nj}(\lambda_j, \alpha_j, \sigma_j + it)|^2 \, \mathrm{d}t \right)^{\frac{1}{2}}. \tag{2.2}$$

For each $j = 1, ..., r$, the series for $L_{nj}(\lambda_j, \alpha_j, \sigma_j + it)$ converges absolutely; therefore

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T |L_{nj}(\lambda_j, \alpha_j, \sigma_j + it)|^2 \, \mathrm{d}t =$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T |L_{nj}(\lambda_j, \alpha_j, \sigma_j + it)|^2 \, \mathrm{d}t =$$

$$= \sum_{m=0}^\infty \frac{v_j^2(m, n, \alpha_j)}{(m + \alpha_j)^{2\sigma_j}} \leq \sum_{m=0}^\infty \frac{1}{(m + \alpha_j)^{2\sigma_j}} \stackrel{\text{def}}{=} R_j < \infty,$$

$j = 1, ..., r$. This together with (2.2) shows that

$$\limsup_{T \to \infty} P_{T,n}\big(\{\underline{z} \in \mathbb{C}^r : \varrho(\underline{z}, \underline{0}) > M\}\big) \leq$$

$$\sup_{n\in\mathbb{C}_0}\limsup_{T\to\infty}\frac{1}{M}\left(\sum_{j=1}^{r}\frac{1}{T}\int_0^T|L_{nj}(\lambda_j,\alpha_j,\sigma_j+it)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\leq\frac{R}{M}, \tag{2.3}$$

where

$$R=\left(\sum_{j=1}^{r}R_j\right)^{\frac{1}{2}}<\infty.$$

$$R_j=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha_j)^{2\sigma_j}}.$$

Let $\varepsilon>0$ be an arbitrary number, and let $M=R\varepsilon^{-1}$. Then, taking into account (2.3), we find that

$$\limsup_{T\to\infty}\underline{P}_{T,n}\big(\{\underline{z}\in\mathbb{C}^r:\varrho(\underline{z},\underline{0})>M\}\big)\leq\varepsilon. \tag{2.4}$$

The function $h:\mathbb{C}^r\to\mathbb{R}$ defined by $h(\underline{z})=\varrho(\underline{z},\underline{0})$, clearly, is continuous. Therefore, Theorem 2.3 and Lemma 1.1 show that the probability measure

$$\nu_T^t\big(\varrho(\underline{L}_n(\underline{\lambda},\underline{\alpha},\underline{\sigma}+it),\underline{0})\in A\big),\quad A\in\mathcal{B}(\mathbb{C}^r),$$

converges weakly to P_nh^{-1} as $T\to\infty$. The set $\{\underline{z}\in\mathbb{C}^r:\varrho(\underline{z},\underline{0})>M\}$ is open. Therefore, by Lemma 1.7 and (2.4), for all $n\in\mathbb{C}_0$, we have

$$P_n\big(\{\underline{z}\in\mathbb{C}^r:\varrho(\underline{z},\underline{0})>M\}\big)\leq$$

$$\leq\liminf_{T\to\infty}P_{T,n}\big(\{\underline{z}\in\mathbb{C}^r:\varrho(\underline{z},\underline{0})>M\}\big)\leq\varepsilon. \tag{2.5}$$

The set $K_\varepsilon=\{\underline{z}\in\mathbb{C}^r:\varrho(\underline{z},\underline{0})\leq M\}$ is compact in $\mathbb{C}^r$, and in view of (2.5)

$$P_n(K_\varepsilon)\geq1-\varepsilon$$

for all $n \in \mathbb{C}_0$. This means that the family of probability measures $\{P_n : n \in \mathbb{C}_0\}$ is tight. Hence, by the Prokhorov theorem, (see Lemma 1.8) it is relatively compact. Thus, there exists a subsequence $\{\underline{P}_{n_1}\} \subset \{\underline{P}_n\}$ such that $\underline{P}_{n_1}$ converges weakly to some probability measure $\underline{P}$ on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ as $n_1 \to \infty$. Let $\underline{X}_n(\sigma)$ be a $\mathbb{C}^r$-valued random element having the distribution $\underline{P}_n$. Then we have from above that

$$X_{n_1}(\sigma) \xrightarrow[n_1 \to \infty]{\mathcal{D}} P. \tag{2.6}$$

Now we take a uniformly distributed on $[0, 1]$ random variable θ defined on some probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \mathbb{P})$. Let

$$\underline{X}_{T,n}(\underline{\sigma}) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + i\theta T).$$

Then by Theorem 2.3 we have that

$$\underline{X}_{T,n}(\underline{\sigma}) \xrightarrow[T \to \infty]{\mathcal{D}} \underline{X}_n(\underline{\sigma}). \tag{2.7}$$

Moreover, by the first assertion of Theorem 2.4, we obtain that, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \mathbb{P}(\varrho(X_{T,n}(\underline{\sigma}), X_T(\underline{\sigma})) \geq \varepsilon) =$$

$$= \lim_{n \to \infty} \limsup_{T \to \infty} \nu_T^t \big(\varrho(\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it)) \geq \varepsilon\big) \leq$$

$$\leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{\varepsilon T} \int_0^T \varrho(\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it)) \mathrm{d}t = 0,$$

where

$$\underline{X}_T(\underline{\sigma}) = \underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + i\theta T).$$

Now this, (2.6), (2.7) and Lemma 1.9 imply the relation

$$\underline{X}_T(\underline{\sigma}) \xrightarrow[T\to\infty]{\mathcal{D}} P, \tag{2.8}$$

which is equivalent to the weak convergence of the measure $\underline{P}_T$ to $\underline{P}$ as $T \to \infty$.

To show the weak convergence of $\widehat{\underline{P}}_T$ to $\underline{P}$ as $T \to \infty$, first we observe that, in view of (2.8), the measure $\underline{P}$ is independent of the choice of the sequence $\{\underline{P}_{n_1}\}$. Thus, we have that

$$X_n(\underline{\sigma}) \xrightarrow[n\to\infty]{\mathcal{D}} \underline{P}. \tag{2.9}$$

Now define

$$\widehat{\underline{X}}_{T,n}(\underline{\sigma}) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + i\theta T, \underline{\omega})$$

and

$$\widehat{\underline{X}}_T(\underline{\sigma}) = \underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + i\theta T, \underline{\omega}).$$

Then, repeating the above arguments for $\widehat{X}_{T,n}(\underline{\sigma})$ and $\widehat{X}_T(\underline{\sigma})$, and applying Theorem 2.3 as well as the second assertion of Theorem 2.4, we obtain that the measure $\widehat{\underline{P}}_T$ also converges weakly to $\underline{P}$ as $T \to \infty$.

To complete the proof of Theorem 2.1, we need some ergodicity arguments. For $t \in \mathbb{R}$, define

$$a_{t,\underline{\alpha}} = \left(\{(m+\alpha_1)^{-it} :\ m \in \mathcal{M}(\alpha_1)\}, \ldots, \{(m+\alpha_r)^{-it} :\ m \in \mathcal{M}(\alpha_r)\}\right),$$

and let $\{\varphi_{t,\underline{\alpha}} : t \in \mathbb{R}\}$ be the one-parameter family of transformations on $\Omega^{(r)}$ defined by

$$\varphi_{t,\underline{\alpha}}(\underline{\omega}) = a_{t,\underline{\alpha}}\underline{\omega}, \quad \underline{\omega} \in \Omega^{(r)}.$$

Then we have that $\{\varphi_{t,\underline{\alpha}} : t \in \mathbb{R}\}$ is a one-parameter group of measure preserving measurable transformations on $\Omega^{(r)}$. We recall that a set $A \in \mathcal{B}(\Omega^{(r)})$ is invariant with respect to the group $\{\varphi_{t,\underline{\alpha}} : t \in \mathbb{R}\}$ if, for each $t \in \mathbb{R}$, the sets A

and $A_t = \varphi_{t,\alpha}(A)$ differ one from another by a set of m_H^r-measure zero. All invariant sets form a σ-subfield of $\mathcal{B}(\Omega^r)$. The one-parameter group $\{\varphi_{t,\underline{\alpha}} : t \in \mathbb{R}\}$ is called ergodic if its σ-field of invariant sets consists only of sets having m_H^r-measure equal to 0 or 1.

Lemma 2.1. *The one-parameter group $\{\varphi_{t,\underline{\alpha}} : t \in \mathbb{R}\}$ is ergodic.*

Proof. We already have seen in the proof of Theorem 2.2 that the dual group of $\Omega^{(r)}$ is

$$\bigoplus_{j=1}^{r} \bigoplus_{m \in \mathcal{M}(\alpha_j)} \mathbb{Z}_{mj},$$

where $\mathbb{Z}_{mj} = \mathbb{Z}$ for all $m \in \mathcal{M}(\alpha_j)$ and $j = 1, ..., r$. Therefore, if $\chi : \Omega^{(r)} \to \gamma$ is a character,

$$\chi(\underline{\omega}) = \prod_{j=1}^{r} \prod_{m \in \mathcal{M}(\alpha_j)} \omega_j^{k_{mj}}(m),$$

where only a finite number of integers k_{mj} are distinct from zero. If χ is a nonprincipal character, from this we have that

$$\chi(a_{t,\underline{\alpha}}) = \prod_{j=1}^{r} \prod_{m \in \mathcal{M}(\alpha_j)} (m + \alpha_j)^{-ik_{mj}t}.$$

Since the set

$$\bigcup_{j=1}^{r} I(\alpha_j)$$

is linearly independent over $\mathbb{Q}$,

$$\prod_{j=1}^{r} \prod_{m \in \mathcal{M}(\alpha_j)} (m + \alpha_j)^{k_{mj}} \neq 1.$$

Therefore, there exists a number $t_0 \in \mathbb{R}$ such that $\chi(a_{t_0,\underline{\alpha}}) \neq 1$.

Denote by I_A the indicator function of a set A. Let A be an invariant set of the one-parameter group $\{\varphi_{t,\underline{\alpha}} : t \in \mathbb{R}\}$. Then, for each fixed $t \in \mathbb{R}$,

$$I_A(a_{t,\underline{\alpha}}\underline{\omega}) = I_A(\underline{\omega})$$

for almost all $\omega \in \Omega^r$. Thus, the Fourier transform $\widehat{I}_A$ of I_A is

$$\widehat{I}_A(\chi) = \int\limits_{\Omega^r} \chi(\underline{\omega}) I_A(\underline{\omega}) m_H^{(r)}(\mathrm{d}\underline{\omega}) =$$

$$= \int\limits_{\Omega^{(r)}} \chi(\underline{\omega}) I_A(a_{t_0,\underline{\alpha}}\underline{\omega}) m_H^{(r)}(\mathrm{d}\underline{\omega}) =$$

$$= \chi(a_{t_0,\underline{\alpha}}) \int\limits_{\Omega^{(r)}} \chi(\underline{\omega}) I_A(\underline{\omega}) m_H^{(r)}(\mathrm{d}\underline{\omega}) = \chi(a_{t_0,\underline{\alpha}}) \widehat{I}_A(\chi).$$

Since $\chi(a_{t_0,\underline{\alpha}}) \neq 1$, this shows that $\widehat{I}_A(\chi) = 0$ for all nontrivial characters χ of $\Omega^{(r)}$.

Now let χ_0 be the principal character ($\chi_0(\underline{\omega}) = 1$ for all $\underline{\omega} \in \Omega^r$). Denote $\widehat{I}_A(\chi_0) = u$. Using

$$\int\limits_{\Omega^{(r)}} \chi(\underline{\omega}) m_H^r(\mathrm{d}\underline{\omega}) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0, \end{cases}$$

we obtain that, for any character χ of Ω^r,

$$\widehat{I}_A(\chi) = u \int\limits_{\Omega^{(r)}} \chi(\underline{\omega}) m_H^{(r)}(\mathrm{d}\underline{\omega}) = u\widehat{I}_A(\chi) = \widehat{u}(\chi).$$

Consequently, $I_A(\underline{\omega}) = u$ for almost all $\underline{\omega} \in \Omega^{(r)}$. However, $u = 0$ or $u = 1$, hence, $m_H^{(r)}(A) = 0$ or $m_H^{(r)}(A) = 1$, and the lemma is proved.

Proof of Theorem 2.1. We fix a continuity set of the limit measure $\underline{P}$ in Theorem 2.5. Then the latter theorem, together with Lemma 1.7, yields

$$\lim_{T \to \infty} \nu_T^t\big(\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it, \underline{\omega}) \in A\big) = \underline{P}(A). \tag{2.10}$$

On $(\Omega^{(r)}, \mathcal{B}(\Omega^{(r)}))$ define a random variable ξ by

$$\xi = \xi(\underline{\omega}) = \begin{cases} 1 & \text{if } \underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}) \in A, \\ 0 & \text{if } \underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}) \notin A. \end{cases}$$

By this definition,

$$\mathbb{E}\xi = \int\limits_{\Omega^{(r)}} \xi \mathrm{d} m_H^{(r)} = m_H^{(r)}\big(\underline{\omega} \in \Omega : \underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}) \in A\big) = Q_{\underline{L}}(A). \tag{2.11}$$

In view of Lemma 2.1, the process $\xi(\varphi_{t,\underline{\alpha}}(\underline{\omega}))$ is ergodic . Thus, by Lemma 2.2

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \xi(\varphi_{t,\underline{\alpha}}(\underline{\omega}))\mathrm{d}t = \mathbb{E}\xi \tag{2.12}$$

for almost all $\underline{\omega} \in \Omega^{(r)}$. On the other hand, by the definition of $\varphi_{t,\underline{\alpha}}$ we find that

$$\frac{1}{T} \int\limits_0^T \xi(\varphi_{t,\underline{\alpha}}(\underline{\omega}))\mathrm{d}t = \nu_T^t\big(\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it, \underline{\omega}) \in A\big).$$

This relation together with (2.11) and (2.12) shows that

$$\lim_{T\to\infty} \nu_T^t\big(\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it, \underline{\omega}) \in A\big) = Q_{\underline{L}}(A)$$

for almost all $\underline{\omega} \in \Omega^{(r)}$. Thus, by (2.10) we have that $\underline{P}(A) = Q_{\underline{L}}(A)$ for every continuity set A of the measure $\underline{P}$. Hence, $\underline{P}(A) = Q_{\underline{L}}(A)$ for all $A \in \mathcal{B}(\mathbb{C}^r)$.

The theorem is proved.

Chapter 3

A limit theorem in the space of analytic functions for the Lerch zeta-function with algebraic irrational parameter

3.1. The statement of the limit theorem in the space of analytic functions

We recall, that $D = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$, and $H(D)$ is the space of analytic on D functions equipped with the topology of uniform convergence on compacta. As in Chapter 1, we suppose that $\alpha \in \mathcal{A}$.

For $s \in D$ and $\omega \in \Omega$, define

$$L(\lambda, \alpha, s, \omega) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^s}. \tag{3.1}$$

Then $L(\lambda, \alpha, s, \omega)$ is an $H(D)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Indeed, in view of the orthogonality of the random variables $\omega(m)$, defined in Section 1.1, and the classical Rademacher theorem [54], for every $\sigma > \frac{1}{2}$, the series

$$\sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^\sigma}$$

converges for almost all $\omega \in \Omega$. Therefore, by the well-known property of Dirichlet series, the series(3.1), for almost all $\omega \in \Omega$, converges uniformly on compact subsets of

$$A_r = \left\{ s \in \mathbb{C} : \sigma > \frac{1}{2} + \frac{1}{r} \right\}, \quad r \in \mathbb{N}.$$

Hence, it follows that this series, for almost all $\omega \in \Omega$, converges uniformly on compact subsets of D.

Define

$$P_{T,H}(A) = \nu_T^\tau \left(L(\lambda, \alpha, s + i\tau) \in A \right), \quad A \in \mathcal{B}(H(D)).$$

Theorem 3.1. *Suppose that $\lambda \in (0,1)$ and $\alpha \in \mathcal{A}$. The probability measure $P_{T,H}$ converges weakly to the distribution P_L of the random element $L(\lambda,\alpha,s,\omega)$ as $T \to \infty$.*

Let $D_1 = \{s \in \mathbb{C} : \sigma > 1\}$. Then the condition $\alpha \in \mathcal{A}$ can be removed from Theorem 3.1. Suppose that the $H(D_1)$-valued random element is a restriction of $L(\lambda,\alpha,s,\omega)$ to $H(D_1)$.

Theorem 3.2. *Suppose that $\lambda \in (0,1)$ and α is an algebraic irrational number. Then the probability measure*

$$\nu_T^\tau\left(L(\lambda,\alpha,s+i\tau) \in A\right), A \in \mathcal{B}(H(D_1)),$$

converges weakly to the distribution of the $H(D_1)$-valued random element $L(\lambda,\alpha,s,\omega)$.

3.2. Case of absolutely convergent series

In this section, we prove limit theorems in the space of analytic functions for an absolutely convergent Dirichlet series related to the function $L(\lambda,\alpha,s)$.

Let, as in Chapter 1, for $\sigma_1 > \frac{1}{2}$ be fixed,

$$v_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\sigma_1}\right\}, \quad m,n \in \mathbb{N}_0.$$

In section 1.3, it was obtained that the series

$$L_n(\lambda,\alpha,s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda m} v_n(m,\alpha)}{(m+\alpha)^s}$$

and

$$L_n(\lambda,\alpha,s,\omega_0) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda m} v_n(m,\alpha)\omega_0(m)}{(m+\alpha)^s},$$

where $\omega_0 \in \Omega$, both converge absolutely for $\sigma > \frac{1}{2}$. On $(H(D),\mathcal{B}(H(D)))$, define two probability measures

$$P_{T,n}(A) = \nu_T^\tau\big(L_n(\lambda,\alpha,s+i\tau) \in A\big)$$

and

$$\widehat{P}_{T,n}(A) = \nu_T^\tau\big(L_n(\lambda, \alpha, s + i\tau, \omega_0) \in A\big).$$

Theorem 3.3. *Let $\lambda \in (0,1)$ and $\alpha \in \mathcal{A}$. Then the probability measures $P_{T,n}$ and $\widehat{P}_{T,n}$ both converge weakly to the same probability measure P_n on $(H(D), \mathcal{B}(H(D)))$ as $T \to \infty$.*

Proof. Define the function $u_{n,\alpha} : \Omega \to H(D)$ by the formula

$$u_{n,\alpha}(\omega) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m} v_n(m,\alpha)\omega(m)}{(m+\alpha)^s}.$$

The continuity of the function $u_{n,\alpha}$ follows from the absolute convergence of this series for $\sigma > \frac{1}{2}$. Moreover,

$$u_{n,\alpha}\big(\{(m+\alpha)^{-i\tau} : m \in \mathcal{M}(\alpha)\}\big) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m} v_n(m,\alpha)}{(m+\alpha)^{s+i\tau}} = L_n(\lambda, \alpha, s+i\tau).$$

Thus, we have that the $P_{T,n} = Q_T u_{n,\alpha}^{-1}$, where, for $A \in \mathcal{B}(H(D))$, $Q_T u_{n,\alpha}^{-1}(A) = Q_T(u_{n,\alpha}^{-1}A)$. Therefore, by Theorem 1.2 and Lemma 1.1 we find that the measure $P_{T,n}$ converges weakly to $m_H u_{n,\alpha}^{-1}$ as $T \to \infty$.

The weak convergence of $\widehat{P}_{T,n}$ is obtained similarly. Define $\widehat{u}_{n,\alpha} : \Omega \to H(D)$ by the formula

$$\widehat{u}_{n,\alpha}(\omega) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m} v_n(m,\alpha)\omega_0(m)\omega(m)}{(m+\alpha)^s}.$$

Then, analogically to the case of $P_{T,n}$, we find that the measure $\widehat{P}_{T,n}$ converges weakly to $m_H \widehat{u}_{n,\alpha}^{-1}$ as $T \to \infty$. Thus, it remains to prove that $m_H u_{n,\alpha}^{-1} = m_H \widehat{u}_{n,\alpha}^{-1}$. For this, we define $u_0 : \Omega \to \Omega$ by the formula $u_0(\omega) = \omega\omega_0$. Then $\widehat{u}_{n,\alpha}(\omega) = u_{n,\alpha}(u_0(\omega))$, and the invariance of the Haar measure m_H shows that

$$m_H \widehat{u}_{n,\alpha}^{-1} = m_H(u_{n,\alpha} u_0)^{-1} = (m_H u_0^{-1}) u_{n,\alpha}^{-1} = m_H u_{n,\alpha}^{-1}.$$

Thus, $\widehat{P}_n = m_H u_{n,\alpha}^{-1}$, and the theorem is proved.

3.3. Approximation in the mean

To pass from the measure $P_{T,n}$ to P_T, we need an approximation of the function $L(\lambda, \alpha, s)$ by $L_n(\lambda, \alpha, s)$ in the mean.

Theorem 3.4. *Let $\lambda \in (0,1)$, $\alpha \in \mathcal{A}$ and K be a compact subset of D. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \sup_{s\in K} \left|L(\lambda, \alpha, s+i\tau) - L_n(\lambda, \alpha, s+i\tau)\right| \mathrm{d}\tau = 0.$$

Proof. In [46], Lemma 5.2.11, the equality of the theorem was proved for transcendental α. However, it is easily seen that the proof is independent of the arithmetic of the number α. Therefore, Theorem 3.3 is true for every fixed α, $0 < \alpha < 1$.

The case of the functions $L(\lambda, \alpha, s, \omega)$ and $L_n(\lambda, \alpha, s, \omega)$ is more complicated.

Theorem 3.5. *Suppose that $\lambda \in (0,1)$, $\alpha \in \mathcal{A}$ and that K is a compact subset of D. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \sup_{s\in K} \left|L(\lambda, \alpha, s+i\tau, \omega) - L_n(\lambda, \alpha, s+i\tau, \omega)\right| \mathrm{d}\tau = 0$$

for almost all $\omega \in \Omega$.

Proof. For $\sigma_1 > \frac{1}{2}$ and $n \in \mathbb{N}_0$, define

$$l_n(s, \alpha) = \frac{s}{\sigma_1} \Gamma\left(\frac{s}{\sigma_1}\right) (n+\alpha)^s,$$

where, as usual, $\Gamma(s)$ denotes the Euler gamma-function. Let,

$$b_n(\lambda, \alpha, m, \omega) = \frac{\mathrm{e}^{2\pi i\lambda m}\omega(m)}{2\pi i} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{l_n(z, \alpha)}{(m+\alpha)^z z} \mathrm{d}z.$$

Then the well-known estimates of $\Gamma(s)$ imply

$$b_n(\lambda,\alpha,m,\omega) \ll (m+\alpha)^{-\sigma_1} \int_{-\infty}^{\infty} \frac{l_n(\sigma_1+it,\alpha)}{|\sigma_1+it|} \mathrm{d}t \ll_n (m+\alpha)^{-\sigma_1}.$$

Hence, we obtain that the series

$$\sum_{m=0}^{\infty} \frac{b_n(\lambda,\alpha,m,\omega)}{(m+\alpha)^s}$$

converges absolutely for $\sigma > \frac{1}{2}$. Therefore, the interchange of order of summation and integration yields

$$\sum_{m=0}^{\infty} \frac{b_n(\lambda,\alpha,m,\omega)}{(m+\alpha)^s} = \frac{1}{2\pi i} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \left(\frac{l_n(z,\alpha)}{z} \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^{s+z}} \right) \mathrm{d}z =$$

$$= \frac{1}{2\pi i} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} L(\lambda,\alpha,s+z,\omega) \frac{l_n(z,\alpha)}{z} \mathrm{d}z. \qquad (3.2)$$

On the other hand, the Mellin inversion formula

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(s)\alpha^{-s} \mathrm{d}s = \mathrm{e}^{-\alpha}, \quad a,b>0,$$

and the definitions of $b_n(\lambda,\alpha,m,\omega)$ and $v_n(m,\alpha)$ show that

$$b_n(\lambda,\alpha,m,\omega) = \mathrm{e}^{2\pi i\lambda m} v_n(m,\alpha)\omega(m).$$

From this and (3.1), we deduce that

$$L_n(\lambda,\alpha,s,\omega) = \frac{1}{2\pi i} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} L(\lambda,\alpha,s+z,\omega) \frac{l_n(z,\alpha)}{z} \mathrm{d}z. \qquad (3.3)$$

Since the series (3.1) converges uniformly on compact subsets of D for almost all $\omega \in \Omega$, the function $L(\lambda, \alpha, s, \omega)$ is analytic on D for almost all $\omega \in \Omega$. Suppose that $\sigma_2 > \frac{1}{2}$ and $\sigma_2 < \sigma$. Then the above remark, (3.3), and the residue theorem yield

$$L_n(\lambda, \alpha, s, \omega) = \frac{1}{2\pi i} \int\limits_{\sigma_2-\sigma-i\infty}^{\sigma_2-\sigma+i\infty} L(\lambda, \alpha, s+z, \omega) \frac{l_n(z, \alpha)}{z} \mathrm{d}z + L(\lambda, \alpha, s, \omega). \quad (3.4)$$

Suppose that $\frac{1}{2} + \eta = \min\limits_{s \in K} \mathrm{Re}s$. Clearly, $\eta > 0$. Let L be a simple closed contour enclosing the set K such that $\frac{1}{2} + \frac{3\eta}{4} = \min\limits_{s \in L} \mathrm{Re}s$, $\delta \geq \frac{\eta}{4}$, where δ is the distance of L from the set K. Then by the Cauchy integral formula

$$\sup_{s \in K} \big| L(\lambda, \alpha, s+i\tau, \omega) - L_n(\lambda, \alpha, s+i\tau, \omega) \big| \leq$$

$$\leq \frac{1}{2\pi\delta} \int\limits_L \big| L(\lambda, \alpha, z+i\tau, \omega) - L_n(\lambda, \alpha, z+i\tau, \omega) \big| |\mathrm{d}z|.$$

If $|L|$ denotes the length of the contour L, then this, for sufficiently large T, gives

$$\frac{1}{T} \int\limits_0^T \sup_{s \in K} \big| L(\lambda, \alpha, s+i\tau, \omega) - L_n(\lambda, \alpha, s+i\tau, \omega) \big| \mathrm{d}\tau \ll$$

$$\ll \frac{1}{T\delta} \int\limits_0^T |\mathrm{d}z| \int\limits_{\mathrm{Im}z}^{T+\mathrm{Im}z} \big| L(\lambda, \alpha, \mathrm{Re}z+i\tau, \omega) - L_n(\lambda, \alpha, \mathrm{Re}z+i\tau, \omega) \big| \mathrm{d}\tau \ll$$

$$\ll \frac{|L|}{T\delta} \sup_{s \in L} \int\limits_0^{2T} \big| L(\lambda, \alpha, \sigma+i\tau, \omega) - L_n(\lambda, \alpha, \sigma+i\tau, \omega) \big| \mathrm{d}\tau. \quad (3.5)$$

By (3.4) we have

$$L(\lambda,\alpha,s+i\tau,\omega)-L_n(\lambda,\alpha,s+i\tau,\omega)\ll$$
$$\ll\int\limits_{-\infty}^{\infty}\big|L(\lambda,\alpha,\sigma_2+it+i\tau,\omega)\big|\frac{|l_n(\sigma_2-\sigma+it,\alpha)|}{|\sigma_2-\sigma+it|}\mathrm{d}t.$$

Therefore, taking into account Lemma 1.6, we obtain that

$$\frac{1}{T}\int\limits_{0}^{2T}\big|L(\lambda,\alpha,\sigma+i\tau,\omega)-L_n(\lambda,\alpha,\sigma+i\tau,\omega)\big|\mathrm{d}\tau\ll$$
$$\ll\int\limits_{-\infty}^{\infty}\big|l_n(\sigma_2-\sigma+it,\alpha)\big|\frac{1}{T}\int\limits_{-|t|}^{2T+|t|}\big|L(\lambda,\alpha,\sigma_2+i\tau,\omega)\big|\mathrm{d}\tau\,\mathrm{d}t\ll$$
$$\ll\int\limits_{-\infty}^{\infty}\big|l_n(\sigma_2-\sigma+it,\alpha)\big|\left(\frac{1}{T}\int\limits_{-|t|}^{2T+|t|}\big|L(\lambda,\alpha,\sigma_2+i\tau,\omega)\big|^2\mathrm{d}\tau\right)^{\frac{1}{2}}\mathrm{d}t\ll$$
$$\ll\int\limits_{-\infty}^{\infty}\big|l_n(\sigma_2-\sigma+it,\alpha)\big|\left(1+\frac{|t|}{T}\right)^{\frac{1}{2}}\mathrm{d}t\ll$$
$$\ll\int\limits_{-\infty}^{\infty}\big|l_n(\sigma_2-\sigma+it,\alpha)\big|\,(1+|t|)\,\mathrm{d}t.$$

Combining this with (3.5), and taking $\sigma_2=\frac{1}{2}+\frac{\eta}{2}$, we find that

$$\frac{1}{T}\int\limits_{0}^{T}\sup_{s\in K}\big|L(\lambda,\alpha,s+i\tau,\omega)-L_n(\lambda,\alpha,s+i\tau,\omega)\big|\mathrm{d}\tau\ll$$
$$\ll\sup_{\sigma\leq-\frac{\eta}{4}}\int\limits_{-\infty}^{\infty}\big|l_n(\sigma+it,\alpha)\big|\,(1+|t|)\,\mathrm{d}t.$$

Since, for $\sigma<0$,

$$\lim_{n\to\infty}\int\limits_{-\infty}^{\infty}\big|l_n(\sigma+it,\alpha)\big|\,(1+|t|)\,\mathrm{d}t=0,$$

the theorem is proved.

3.4. Proof of the Theorem 3.1

For $A \in \mathcal{B}(H(D))$ and $\omega \in \Omega$, define the other probability measure

$$\widehat{P}_T(A) = \nu_T^{\tau}\big(L(\lambda, \alpha, s + i\tau, \omega) \in A\big).$$

To prove the weak convergence for the measures P_T and $\widehat{P}_T$, we need a metric on $H(D)$ which induces its topology of uniform convergence on compacta.

It is well known, (see, for example, [9]) that there exists a sequence $\{K_l : l \in \mathbb{N}\}$ of compact subsets of D such that

1) $D = \bigcup\limits_{l=1}^{\infty} K_l$;

2) $K_l \subset K_{l+1}, \quad l \in \mathbb{N}$;

3) If K is a compact subset of D, then $K \subseteq K_l$ for some l.

For $f, g \in H(D)$, define

$$\varrho(f, g) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup\limits_{s \in K_l} |f(s) - g(s)|}{1 + \sup\limits_{s \in K_l} |f(s) - g(s)|}.$$

Then $\varrho(f, g)$ is the desired metric.

Theorem 3.6. *Suppose that $\lambda \in (0,1)$ and $\alpha \in \mathcal{A}$. Then the probability measures P_T and $\widehat{P}_T$ both converge weakly to the same probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $T \to \infty$.*

Proof. We have obtained in Theorem 3.3 that the probability measures $P_{T,n}$ and $\widehat{P}_{T,n}$ both converge weakly to the same probability measure P_n on $(H(D), \mathcal{B}(H(D)))$ as $T \to \infty$. Now we will show that the family of probability measures $\{P_n : n \in \mathbb{N}_0\}$ is tight.

Let θ be a uniformly distributed on some probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \mathbb{P})$. Define

$$X_{T,n} = X_{T,n}(s) = L_n(\lambda, \alpha, s + iT\theta).$$

Then we have by Theorem 3.3 that

$$X_{T,n} \xrightarrow[T \to \infty]{\mathcal{D}} X_n, \tag{3.6}$$

where $X_n = X_n(s)$ is an $H(D)$-valued random element with distribution P_n.

The series for $L_n(\lambda, \alpha, s)$ converges absolutely for $\sigma > \frac{1}{2}$. Therefore, for $\sigma > \frac{1}{2}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|L_n(\lambda, \alpha, \sigma + it)\right|^2 \mathrm{d}t = \\ = \sum_{m=0}^{\infty} \frac{v_n^2(m, \alpha)}{(m+\alpha)^{2\sigma}} \leq \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{2\sigma}}.$$

From this, using the Cauchy integral formula, we deduce that there exists a positive constant C_l such that

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_l} \left|L_n(\lambda, \alpha, s + i\tau)\right|^2 \mathrm{d}t \ll C_l \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{2\sigma_l}}, \quad l \in \mathbb{N}, \tag{3.7}$$

with some $\sigma_l > \frac{1}{2}$. Therefore, there exists a number $0 < R_l < \infty$ such that

$$\sup_{n\in\mathbb{N}_0} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_l} \left|L_n(\lambda, \alpha, s + i\tau)\right| \mathrm{d}\tau \ll$$

$$\ll \sup_{n\in\mathbb{N}_0} \limsup_{T\to\infty} \left(\frac{1}{T} \int_0^T \sup_{s\in K_l} \left|L_n(\lambda, \alpha, s + i\tau)\right|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \leq R_l, \quad l \in \mathbb{N}. \tag{3.8}$$

By (3.7) we can take, for example,

$$R_l = \left(C_l \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{\sigma_l}} \right)^{\frac{1}{2}}, \quad l \in \mathbb{N}.$$

Let ε be an arbitrary positive number, and $M_{l,\varepsilon} = 2^l R_l \varepsilon^{-1}$. Then, in view of (3.8),

$$\limsup_{T\to\infty} P_{T,n}\left(\left\{g\in H(D):\sup_{s\in K_l}|g(s)|>M_{l,\varepsilon}\right\}\right)=$$

$$=\limsup_{T\to\infty}\nu_T^\tau\left(\sup_{s\in K_l}\left|L_n(\lambda,\alpha,s+i\tau)\right|\geq M_{l,\varepsilon}\right)\leq$$

$$\leq\frac{1}{M_{l,\varepsilon}}\int_0^T\sup_{s\in K_l}\left|L_n(\lambda,\alpha,s+i\tau)\right|\mathrm{d}\tau\leq\frac{\varepsilon}{2^l},\quad l\in\mathbb{N}. \tag{3.9}$$

The function $h:H(D)\to\mathbb{R}$ given by the formula $h(g)=\sup_{s\in K_l}|g(s)|$, $g\in H(D)$, is continuous. Therefore, Theorem 3.3 and Lemma 1.2 imply the weak convergence of the probability measure

$$\nu_T^\tau\left(\sup_{s\in K_l}\left|L_n(\lambda,\alpha,s+i\tau)\right|\in A\right),\quad A\in\mathcal{B}(\mathbb{R}),$$

to the measure P_nh^{-1} as $T\to\infty$. Thus, by Theorem 1.8 and (3.9),

$$P_n\left(\left\{g\in H(D):\sup_{s\in K_l}|g(s)|>M_{l,\varepsilon}\right\}\right)\leq$$

$$\leq\liminf_{T\to\infty} P_{T,n}\left(\left\{g\in H(D):\sup_{s\in K_l}|g(s)|>M_{l,\varepsilon}\right\}\right)\leq$$

$$\leq\limsup_{T\to\infty} P_{T,n}\left(\left\{g\in H(D):\sup_{s\in K_l}|g(s)|>M_{l,\varepsilon}\right\}\right)\leq\frac{\varepsilon}{2^l},\quad l\in\mathbb{N}. \tag{3.10}$$

Now let

$$K_\varepsilon=\left\{g\in H(D):\sup_{s\in K_l}|g(s)|\leq M_{l,\varepsilon},\, l\in\mathbb{N}\right\}.$$

Then the set K_ε is uniformly bounded and, therefore, a compact subset of $H(D)$. Moreover, by (3.10),

$$
\begin{aligned}
P_n(K_\varepsilon) = 1 - P_n(K_\varepsilon^C) \quad &\geq \quad 1 - \sum_{l=1}^{\infty} P_n\left(\left\{g \in H(D) : \sup_{s \in K_l} |g(s)| > M_{l,\varepsilon}\right\}\right) \\
&\geq \quad 1 - \varepsilon \sum_{l=1}^{\infty} \frac{1}{2^l} = 1 - \varepsilon
\end{aligned}
$$

for all $n \in \mathbb{N}_0$. By definition, this means that the family $\{P_n : n \in \mathbb{N}_0\}$ is tight. The Prokhorov theorem (see Lemma 1.8) now implies the relative compactness of the family $\{P_n : n \in \mathbb{N}_0\}$. Therefore, there exists $\{P_{n_k}\} \subset \{P_n\}$ such that P_{n_k} converges weakly to some probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $k \to \infty$. Hence, we have the relation

$$
X_{n_k} \xrightarrow[k \to \infty]{\mathcal{D}} P. \tag{3.11}
$$

Now define

$$
X_T = X_T(s) = L(\lambda, \alpha, s + iT\theta).
$$

Then, using Theorem 3.4, we obtain that, for every ε,

$$
\begin{aligned}
&\lim_{n \to \infty} \limsup_{T \to \infty} \mathbb{P}\big(\varrho(X_T, X_{T,n}) \geq \varepsilon\big) = \\
&= \lim_{n \to \infty} \limsup_{T \to \infty} \nu_T^\tau \left(\varrho\big(L(\lambda, \alpha, s + i\tau), L_n(\lambda, \alpha, s + i\tau)\big) \geq \varepsilon\right) \leq \\
&\leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T\varepsilon} \int_0^T \varrho\big(L(\lambda, \alpha, s + i\tau), L_n(\lambda, \alpha, s + i\tau)\big) \mathrm{d}\tau = 0.
\end{aligned}
$$

From this and from (3.6), (3.11) and Lemma 1.9 we deduce that

$$
X_T \xrightarrow[T \to \infty]{\mathcal{D}} P. \tag{3.12}
$$

Thus, we have that the measure P_T converges weakly to P as $T \to \infty$.

Relation (3.12) shows that the limit measure P is independent on the sequence $\{P_{n_k}\}$. Hence, since $\{P_n : n \in \mathbb{N}_0\}$ is relatively compact, we obtain the relation

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{3.13}$$

Now define

$$\widehat{X}_{T,n} = \widehat{X}_{T,n}(s) = L_n(\lambda, \alpha, s + iT\theta, \omega)$$

and

$$\widehat{X}_T = \widehat{X}_T(s) = L(\lambda, \alpha, s + iT\theta, \omega).$$

Then reasoning similarly as above and applying Theorems 3.3 and 3.5 and (3.13), we find that

$$\widehat{X}_T \xrightarrow[T\to\infty]{\mathcal{D}} P,$$

i.e., the measure $\widehat{P}_T$ converges weakly to P as $T \to \infty$.

The theorem is proved.

Proof of Theorem 3.1. By Theorem 3.5, we have to show that the measure P coincides with P_L.

Let $A \in \mathcal{B}(H(D))$ be a fixed continuity set of the measure P. Then Theorem 3.5 and Lemma 1.8 imply the relation

$$\lim_{T\to\infty} \nu_T^{\tau}\big(L(\lambda, \alpha, s + i\tau) \in A\big) = P(A). \tag{3.14}$$

In the sequel, we use some elements of ergodic theory. For $\tau \in \mathbb{R}$, we put

$$a_\tau = \big\{(m+\alpha)^{-i\tau} : m \in \mathcal{M}(\alpha)\big\}$$

and define the family $\{\varphi_\tau : t \in \mathbb{R}\}$ of transformations on Ω defined by $\varphi_\tau(\omega) = a_\tau\omega$, $\omega \in \Omega$. Then $\{\varphi_\tau : t \in \mathbb{R}\}$ is a one-parameter group of measurable measure-preserving transformations on the torus Ω. By Lemma 1.4, the one-parameter group $\{\varphi_\tau : t \in \mathbb{R}\}$ is ergodic.

On $(\Omega, \mathcal{B}(\Omega))$, define the random variable $\xi = \xi(\omega)$ by

$$\xi = \begin{cases} 1 & \text{if } L(\lambda, \alpha, s, \omega) \in A, \\ 0 & \text{if } L(\lambda, \alpha, s, \omega) \notin A. \end{cases}$$

Then we have that

$$\mathbb{E}(\xi) = \int_{\Omega} \xi \mathrm{d} m_H = m_H\big(\omega \in \Omega : L(\lambda, \alpha, s, \omega) \in A\big) = P_L(A). \tag{3.15}$$

In view of ergodicity of the group $\{\varphi_\tau : t \in \mathbb{R}\}$, the random process $\xi(\varphi_\tau(\omega))$ is also ergodic. Therefore, by the classical Birkhoff-Khinchine theorem (see Lemma 1.6),

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \xi(\varphi_\tau(\omega)) \mathrm{d}\tau = \mathbb{E}(\xi) \tag{3.16}$$

for almost all $\omega \in \Omega$. On the other hand, the definitions of ξ and φ_τ show that

$$\frac{1}{T} \int_0^T \xi(\varphi_\tau(\omega)) \mathrm{d}\tau = \nu_T^\tau\big(L(\lambda, \alpha, s + i\tau, \omega) \in A\big).$$

From this and from (3.15) and (3.16) we find that

$$\lim_{T\to\infty} \nu_T^\tau\big(L(\lambda, \alpha, s + i\tau, \omega) \in A\big) = P_L(A)$$

for almost all $\omega \in \Omega$. Therefore, in view of (3.14), $P(A) = P_L(A)$ for every continuity set A of the measure P. Since the continuity sets constitute a determining class, we have that $P(A) = P_L(A)$ for all $A \in \mathcal{B}(H(D))$.

The theorem is proved.

Proof of Theorem 3.2. The theorem follows similarly to Theorem 3.1, but its proof is simpler because the series converge absolutely, and we do not need the orthogonality of random variables $\omega(m)$. Therefore, we can remove the hypothesis $\alpha \in \mathcal{A}$ from Theorem 3.1.

Conclusions

In the thesis, the following statistical properties for the Lerch zeta-function are obtained.

1. For the Lerch zeta-function $L(\lambda, \alpha, s)$ with parameters $\lambda \in (0,1)$ and algebraic irrational parameter α from the class $\mathcal{A}$, a limit theorem in the sense of weak convergence of probability measures on the complex plane is valid.

2. For a collection of Lerch zeta-functions with algebraic irrational parameters from the class $\mathcal{A}$, a joint limit theorem in the sense of weak convergence of probability measures on the complex plane is valid.

3. For the Lerch zeta-function $L(\lambda, \alpha, s)$ with parameters $\lambda \in (0,1)$ and algebraic irrational parameter α from the class $\mathcal{A}$, a limit theorem in the sense of weak convergence of probability measures in the space of analytic functions is valid.

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Notation

$\mathbb{N}_0$	set of all non-negative integers
$\mathbb{N}$	set of all positive integers
$\mathbb{Z}$	set of all integer numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{P}$	set of all prime numbers
$\mathbb{C}$	set of all complex numbers
i	imaginary unity: $i^2 = -1$
$s = \sigma + it$	complex variable
$\text{meas}\{A\}$	Lebesgue measure of the set A
$\mathcal{B}(S)$	class of Borel sets of the space S
$\mathbb{E}(X)$	expectation of the random element X
∂A	boundary of the set A
$\xrightarrow{\mathcal{D}}$	convergence in distribution
A^c	complement of the set A
$L(\lambda, \alpha, s)$	Lerch zeta-function
$\Gamma(s)$	Euler gamma-function