VILNIUS UNIVERSITY

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ASYMPTOTICALLY HOMOGENEOUS MARKOV CHAINS

Summary of Doctoral Dissertation
Physical sciences, mathematics (01P)

Vilnius, 2010

The scientific work was carried out in 2006 – 2010 at Vilnius University

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The dissertation will be defended at the public meeting of the council on December 21, 2010, in Vilnius University Remote Education Study Center at 2 pm.
Address: Šaltinių 1A, 03225, Vilnius, Lithuania.

The summary of the dissertation was distributed in November, 2010.
The dissertation is available at the Library of Vilnius University.

VILNIAUS UNIVERSITETAS

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ASIMPTOTIŠKAI HOMOGENINĖS MARKOVO GRANDINĖS

Daktaro disertacijos santrauka
Fiziniai mokslai, matematika (01P)

Vilnius, 2010

Disertacija rengta 2006 – 2010 metais Vilniaus universitete

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Disertacija ginama Vilniaus universiteto matematikos mokslo krypties taryboje:

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Disertacija bus ginama viešame Matematikos mokslo krypties tarybos posėdyje 2010 m. gruodžio mėn. 21 d. 14 val. Vilniaus universiteto Nuotolinių studijų centre.
Adresas: Šaltinių 1A 9, 03225, Vilnius, Lietuva.

Disertacijos santrauka išsiuntinėta 2010 m. lapkričio mėn.
Su disertacija galima susipažinti Vilniaus universiteto bibliotekoje.

1 General characteristics of the dissertation

Topicality of the problem

Markov chains are very widely used in behavioral sciences to model sequences of dependent observations. Therefore investigations in this field are always relevant.

In this work we explored properties of a chain $(X_n, n \geqslant 0)$ evolving in a closed cone C of a finite dimensional normed space E. We assumed that iterations of (X_n) are given by a function $F : C \times W \to C$ and a sequence $(\varepsilon_n, n \geqslant 1)$ of independent identically distributed random elements of a separable metric space W, i.e.

$$X_n = F(X_{n-1}, \varepsilon_n), n \geqslant 1.$$

The main assumption regarding F was asymptotical homogeneity of $F(\cdot, w)$. To be more precise we assumed that for each $w \in W$ there exists homogeneous function $G(\cdot, w)$ such that

$$t_n \to \infty, x_n \to x \neq 0 \Rightarrow t_n^{-1} F(t_n x_n, w) \to G(x, w); \tag{1.1}$$

$$G(0, w) = 0. \tag{1.2}$$

The class of the described chains covers many models found in practical applications (several examples are provided in section 2.1). Further on we call such chains *asymptotically homogeneous*.

Aim and tasks of the work

In this work we aimed to solve two problems:

- to give conditions for existence of a unique stationary distribution of an asymptotically homogeneous Markov chains;
- to find out when the tail of the stationary distribution is heavy.

Scientific novelty and practical value

Novelty. To our best knowledge the class of asymptotically homogeneous Markov chains was not described elsewhere in the literature in the same fashion as we did this. Therefore suggested approach is essentially new. Considering the solved problems it is worthwhile to mention the sources which gave rise to the obtained results.

Existence of a stationary distribution. In 1992 Bougerol and Picard (see [1]) investigated random linear equation

$$X_n = A_n X_{n-1} + B_n, \tag{1.3}$$

here A_n is a real random $k \times k$ matrix and B_n is a real random $k \times 1$ vector. Under very weak assumptions they have shown that

- with probability 1 there exists limit

  $$\gamma = \lim_{n\to\infty} \frac{1}{n} \log\|A_1 \dots A_n\|,$$

  which is called *the top Lyapunov exponent* of the model;

- the random linear equation admits unique stationary distribution if γ is negative and there is no stationary solution if γ is positive.

One can see that linear random equation is a particular case of asymptotically homogeneous chain correspondig to $F(x,(w_1,w_2)) = w_1x + w_2$ and $C \subset \mathbb{R}^k$. On the other hand there are many models which fit into the scope of the class but are not linear. Therefore it was unclear if one can develop criteria analogous to that of Bougerol and Picard (see theorem 2.1 and remark 2.1) and under certain assumptions applicable not only to the linear model.

Tail index. There are several sources which inspired investigations in this direction.

First of all it is well known result of Kesten (see [7]) on linear random equation (1.3). He considered non-negative linear random equation and has shown that under certain assumptions stationary distribution of linear random equation is heavy tailed, i.e. there exist $\alpha > 0$ and function $c : [0;\infty)^k \to [0;\infty)$ such that

$$t^\alpha \,\mathrm{P}\{x^T X > t\} \xrightarrow[t\to\infty]{} c(x),$$

here X denotes a random vector having stationary distribution of the chain whereas $x \in [0;\infty)^k$ is a fixed non-random vector. As it was mentioned above linear random equation is a particular case of an asymptotically homogeneous Markov chain.

Another general work on heavy tails of stationary solutions of random equations was done by Goldie (see [3]). He considered chains evolving in a line or in a half-line and gave results applicable to a big subset of one dimensional asymptotically homogeneous chains. He actually exploited the property of asymptotical homogeneity however big subset of models was lost because the property was noticed only for chains having special limiting function G introduced by relations (1.1)–(1.2) (for details see original source and remark 2.2).

Finally there was work [9]. The authors of it also dealt with linear random equation and explored the case which was not covered by Kesten.

Practical value. The obtained results give unified approach for investigation of asymptotically homogeneous Markov chains and in many cases *quickly* lead to the results which are not trivial to prove by means of other methods. On the other hand dealing with particular models one can relax certain assumptions put by our theorems. To give deeper insight we provide several comments on each of the solved problems.

Existence of a stationary distribution. Despite it's generality our main theorem on stationarity (see theorem 2.1) imposes irreducibility conditions which may be considered as disadvantage. However analyzing different models met in the literature† we have come to the conclusion that irreducibility assumptions are usually weaker once it is possible to write down implicit stationary solution via the sequence of innovations. In other cases irreducibility assumptions are usually similar to that of theorem 2.1.

Talking on moment conditions we can say that they are usually optimal or better as compared to those met in the literature.

Tail index. Considering applications one can see that our main theorem on tail behaviour (see theorem 2.3) is difficult to apply in practice since it's

†analysis of models can be found in the dissertation conditions involve random element having stationary distribution of the chain. The same can be said about results obtained in [3]. However in contrast to [3] we provide auxiliary propositions (see prop. 2.2, 2.3) which substantially facilitate checking conditions of the main theorem since they rely only on the function ruling iterations of the chain.

The structure of the work

The dissertation is written in lithuanian. There are three chapters, two appendixes and a list of references. The total number of pages is 63.

The first chapter is devoted to rigorous definition of an asymptotically homogenous chain as well as examples of such chains. In the second chapter there are given results on the existence of a stationary distribution meanwhile it the third one can find results on a tail index of one dimensional asymptotically homogeneous chain.

The first appendix is devoted to some mathematical facts. In the second one there are listed abbreviations of financial time series models.

2 Main results

In this section we summarize our main results. If it is not said otherwise (X_n) denotes an asymptotically homogeneous chain, C stands for it's state space† and (ε_n) is a sequence of independent identically distributed random elements of a separable metric space W. There are also two measurable functions F, G from $C \times W$ to C. The first one defines iterations of (X_n):

$$X_n = F(X_{n-1}, \varepsilon_n).$$

The second one is defined by relations (1.1)–(1.2). It is called *homogeneous function corresponding to* F and plays important role in a formulation of the results.

We also use the following notions:

$$R(x,w) = \|G(x,w)\|, H(x,w) = \frac{G(x,w)}{\|G(x,w)\|}, S = \{x \in C \mid \|x\| = 1\}$$

and by ε we denote a random element of W having the same distribution as members of (ε_n).

Formulation of the results requires concepts from the theory of irreducible Markov chains. We do not provide them here referring reader to comprehensive account [10].

2.1 Examples of asymptotically homogeneous chains

G/G/1 queue. Consider $G/G/1$ queue. Let customers arrive at moments $U_1 + \cdots + U_n$ and serving times are V_n (here (U_n) and (V_n) are sequences of independent identically distributed non-negative random variables). If X_n denotes waiting time of a customer number n then (see for example [2], chapter VI, section 9)

$$X_n = (X_{n-1} + \varepsilon_n)^+; \tag{2.1}$$

†recall that C is assumed to be a closed cone of a finite dimensional normed space with $\varepsilon_n = V_{n-1} - U_n$. If X_n denotes total time spent by a customer number n then

$$X_n = V_n + (X_{n-1} - U_n)^+. \tag{2.2}$$

It is not difficult to check out that in both cases (X_n) is an asymptotically homogeneous Markov chain corresponding to state space $C = [0; \infty)$. In the model (2.1) one has $W \subset \mathbb{R}$ and $F(x, w) = (x + w)^+$ meanwhile in the model (2.2) these are $W = (0; \infty)^2$ and $F(x, (u, v)) = v + (x - u)^+$. In both cases corresponding limiting function is $G(x, w) = x$. Consider for example the first model. If $t_n \to \infty$, $x_n \to x \neq 0$ then

$$t_n^{-1} F(t_n x_n, w) = t_n^{-1} (t_n x_n + w)^+ = (x_n + t_n^{-1} w)^+ \to x.$$

AR(1) model with ARCH type errors. Klüppelberg and Borkovec (see [9]) investigated a model defined by a system of equations

$$X_n = aX_{n-1} + \varepsilon_n \sqrt{b + cX_{n-1}^2}; \tag{2.3}$$

here $a \neq 0, b, c > 0$ are fixed parameters of the model and a sequence (ε_n) consists of real valued independent identically distributed random variables. One can see that the chain is asymptotically homogeneous with $C = \mathbb{R}, W \subset \mathbb{R}, F(x, w) = ax + w\sqrt{b + cx^2}$ and $G(x, w) = ax + w\sqrt{c}|x|$.

HARCH process. In 1997 Müller et. al. (see [11]) introduced HARCH(k) process to describe behaviour of financial time series. The model is defined by a system of equations

$$x_n = \sigma_n \varepsilon_n,$$
$$\sigma_n^2 = c_0 + \sum_{j=1}^{k} c_j \left(\sum_{i=1}^{j} x_{n-i} \right)^2;$$

here $c_0, \ldots, c_k$ are fixed non-negative parameters of the model (it is also assumed that $c_0, c_k > 0$) and (ε_n) is a sequence of real valued independent identically distributed random variables. Denote

$$X_n = \begin{pmatrix} x_n \\ \vdots \\ x_{n-k+1} \end{pmatrix},$$

and define function $F\colon \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$ as follows

$$F(x, w) = \begin{pmatrix} \sqrt{c_0 + \sum_{j=1}^{k} c_j \left(\sum_{i=1}^{j} x_i \right)^2} \, w \\ x_1 \\ \vdots \\ x_{k-1} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}.$$

Then $X_n = F(X_{n-1}, \varepsilon_n)$. It is asymptotically homogeneous chain corresponding to $C = \mathbb{R}^k$ and $W \subset \mathbb{R}$. One can also easily check that limiting homogeneous function is given by

$$G(x, w) = \begin{pmatrix} \sqrt{\sum_{j=1}^{k} c_j \left(\sum_{i=1}^{j} x_i\right)^2} w \\ x_1 \\ \vdots \\ x_{k-1} \end{pmatrix}.$$

2.2 Stationarity

Let (X_n) be an asymptotically homogeneous chain and $b : W \to [0; \infty)$ be a Borel function satisfying condition

$$\mathrm{E}\, b(\varepsilon) < \infty.$$

In this subsection we assume that for all $w \in W$ the following relations hold:

(NZ) $G(x, w) \neq 0$ for each $x \neq 0$;

(M1) $M(x, w) = |\log^+ \|F(x, w)\| - \log^+ \|G(x, w)\|| \leqslant b(w)$ for all x;

(M2) $\sup_{\|x\|=1} |\log R(x, w)| \leqslant b(w)$.

Together with the chain (X_n) we will need two auxiliary sequences (Y_n) and (Z_n) which are defined by equalities

$$Y_n = G(Y_{n-1}, \varepsilon_n), \quad n \geqslant 1; \quad Z_n = \frac{Y_n}{\|Y_n\|}, \quad n \geqslant 0. \tag{2.4}$$

Note that (Y_n) is a Markov chain provided Y_0 and (ε_n) are independent. We assume that $Y_0 \neq 0$ a.s. In such case NZ implies that $Y_n \neq 0$ a.s. for each n. Hence the chain (Y_n) evolves in $C_0 = C \setminus \{0\}$ and the sequence (Z_n) is defined correctly. Moreover, by homogeneity

$$Z_n = \frac{Y_n}{\|Y_n\|} = \frac{G(Y_{n-1}\|Y_{n-1}\|^{-1}, \varepsilon_n)}{R(Y_{n-1}\|Y_{n-1}\|^{-1}, \varepsilon_n)} = H(Z_{n-1}, \varepsilon_n).$$

Therefore (Z_n) is also a Markov chain evolving in S. Finally notice that by M2

$$\mathrm{E}|\log R(Z, \varepsilon)| < \infty$$

for each random element Z having support in S.

THEOREM 2.1. *Assume that (X_n) and (Y_n) are irreducible T-chains, (X_n) is aperiodic and conditions NZ, M1–M2 hold. Then (Z_n) is positive Harris recurrent chain. Let Z be a random element of S having it's stationary distribution. Take ε independent of Z and denote*

$$\gamma = \mathrm{E} \log R(Z, \varepsilon).$$

If $\gamma < 0$ then (X_n) is positive Harris recurrent chain; if $\gamma > 0$ it is dissipative or null.

Corollary 2.1. For each $z \in S$

$$\gamma = \lim_{n\to\infty} \sum_{i=1}^{n} \log R(Z_{i-1}(z), \varepsilon_i) \text{ a.s.}; \tag{2.5}$$

here $Z_0(z) = z$, $Z_n(z) = H(Z_{n-1}(z), \varepsilon_n)$.

Remark 2.1. In the rest of the summary we call γ *Lyapunov exponent.* The justification of this name comes from the following proposition which is proved in the dissertation.

Proposition 2.1. *Let the chain (X_n) be defined by (1.3), D_k denotes a set of invertible $k \times k$ matrices and $k \times (k+1)$ matrix $\varepsilon_n = (A_n, B_n)$ is a random element of $W = D_k \times \mathbb{R}^k$. If ε_n has a density which is positive in the whole set W then theorem 2.1 applies to (X_n). Moreover the following equality holds*

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \log\|A_n \cdots A_1\|.$$

2.3 Tail index

In this subsection we assume that $C = [0; \infty)$ or $C = \mathbb{R}$ (i.e. asymptotically homogeneous chain evolves in a half-line or in a line) and investigate heaviness of a tail of a stationary distribution of the chain. We also assume that $G(x, w) \neq 0$ for all $x \neq 0$ and $w \in W$. In what follows we use auxiliary chains introduced in the previous subsection. They are correctly defined. If the chain starts from x we denote this by adding brackets with a starting point inside. For example $(Z_n(x))$ means the chain (Z_n) with $Z_0 = x$ a.s. Since a standard norm on a line is an absolute value function we simplify norm notion and use $|\cdot|$ instead of $\|\cdot\|$.

In the previous subsection we have given conditions under which asymptotically homogeneous chains possesses a unique stationary distribution. They are not necessary in general and there are examples then one can prove uniqueness without our assumptions. Results stated here also do not require these conditions in general. Therefore we simply assume that the chain (X_n) has a unique stationary distribution and by X denote a random variable having this stationary distribution.

Our final assumption is that with probability 1 there exists Lyapunov exponent, i.e. for each $z \in S$ the limit

$$\gamma = \lim_{n\to\infty} \sum_{i=1}^{n} \log R(Z_{i-1}(z), \varepsilon_i)$$

is well defined.

2.3.1 Chains on a half-line

In this case asymptotical homogeneity of F means

$$x^{-1} F(x, w) \xrightarrow[x\to\infty]{} G(1, w),$$

$R(1, \varepsilon) = G(1, \varepsilon)$ and the top Lyapunov exponent $\gamma = \mathrm{E} \log G(1, \varepsilon)$, provided $\log G(1, \varepsilon)$ is integrable. In paper [3] the following theorem was proved (see lemma 2.2, theorem 2.3 and corollary 2.4).

Theorem 2.2. *Suppose the top Lyapunov exponent γ is negative, the distribution of $\log G(1,\varepsilon)$ is non-arithmetic and with some $\alpha > 0$*

$$\mathrm{E}\, G^\alpha(1,\varepsilon) = 1, \quad \mathrm{E}\, G^\alpha(1,\varepsilon)|\log G(1,\varepsilon)| < \infty. \tag{2.6}$$

If, in addition,

$$\mathrm{E}|F^\alpha(X,\varepsilon) - G^\alpha(X,\varepsilon)| < \infty \tag{2.7}$$

then

$$t^\alpha\, \mathrm{P}\{X > t\} \xrightarrow[t\to\infty]{} \kappa$$

with $\kappa = (m\alpha)^{-1}\, \mathrm{E}\,[F^\alpha(X,\varepsilon) - G^\alpha(X,\varepsilon)]$ *and* $m = \mathrm{E}\, G^\alpha(1,\varepsilon)\log G(1,\varepsilon)$.

A sufficient condition for existence of α satisfying (2.6) is

$$\mathrm{E}\, G^\beta(1,\varepsilon) \in (1;\infty) \quad \text{for some } \beta > 0.$$

In this case the function $f(x) = \mathrm{E}\, G^x(1,\varepsilon)$ is continuous on $(0;\beta)$ and take values less than 1 in a neighborhood of 0 (a proof is provided in the dissertation), therefore $f(\alpha) = 1$ for some $\alpha \in (0;\beta)$.

The main disadvantage of condition (2.7) is that it involves the unknown stationary distribution of the model. Therefore we provide some criteria, based only on the known function F, for this condition to hold.

Proposition 2.2. *Suppose that (X_n) is a φ-irreducible T-chain and the top Lyapunov exponent is negative. If, for some $0 < \theta < \theta_1 < \alpha$ and all $\Delta > 0$,*

(F1) $\sup_{x\leqslant\Delta} \mathrm{E}\, F^\alpha(x,\varepsilon) < \infty$,

(F2) $\sup_{x>\Delta} x^{-\theta_1}\, \mathrm{E}\, F^{\theta_1}(x,\varepsilon) < \infty$,

(F3) $\sup_{x>\Delta} x^{-\theta}\, \mathrm{E}\big|F^\alpha(x,\varepsilon) - G^\alpha(x,\varepsilon)\big| < \infty$,

then the chain is positive recurrent and condition (2.7) *holds.*

If $\alpha \leqslant 1$, condition F3 can be replaced by either of the following two conditions:

(F3a) $\sup_{x>\Delta} x^{-\theta}\, \mathrm{E}\big|F(x,\varepsilon) - G(x,\varepsilon)\big|^\alpha < \infty$;

(F3b) $\sup_{x>\Delta} x^{-\theta}\, \mathrm{E}\, G^{\alpha-1}(x,\varepsilon)\big|F(x,\varepsilon) - G(x,\varepsilon)\big| < \infty$.

This follows from inequalities

$$|a^\alpha - b^\alpha| \leqslant |a-b|^\alpha, \quad |a^\alpha - b^\alpha| \leqslant b^{\alpha-1}|a-b|, \tag{2.8}$$

which are valid for all $a, b > 0$. If $\alpha > 1$, we may use another inequality,

$$|a^\alpha - b^\alpha| \leqslant \alpha 2^{\alpha-1}\big(b^{\alpha-1}|a-b| + |a-b|^\alpha\big), \tag{2.9}$$

and replace F3 by both F3a and F3b.

If, for $x > \Delta$,

$$\big|F(x,w) - G(x,w)\big| \leqslant c x^\sigma p(w)$$

with some $\sigma < 1$ and some function p satisfying

$$\mathrm{E}\, p^\alpha(\varepsilon) < \infty \quad \text{and} \quad \mathrm{E}\, G^{\alpha-1}(1,\varepsilon)p(\varepsilon) < \infty$$

(in the case $\alpha \leqslant 1$ only one of these conditions is required), then in both cases F3 holds with some $\theta < \alpha$ (either $\theta = \alpha\sigma$ or $\theta = \alpha - 1 + \sigma$).

Let us return to theorem 2.2. If we want to say that the tail index of the stationary distribution of the model is α, we must show that the limiting constant $\kappa = (m\alpha)^{-1} \mathrm{E}\big[F^\alpha(X,\varepsilon) - G^\alpha(X,\varepsilon)\big]$ is positive. This is not always the case. Consider, for example, the model defined by

$$X_n = \max(1, X_{n-1} - 1)\varepsilon_n. \tag{2.10}$$

If $\mathrm{P}\{\varepsilon \in (0;2)\} = 1$, $\mathrm{E}\log\varepsilon < 0$ and $\mathrm{E}\,\varepsilon = 1$ then the distribution of ε is the unique stationary distribution for the model, i.e. we can take $X = \varepsilon$. All assumptions of theorem 2.2 are satisfied; however, $\kappa = 0$.

We spent some time trying to prove the following conjecture: $\kappa = 0$ if and only if the stationary distribution is concentrated on some compact interval $[a; b]$. However, we neither could prove it, nor could construct a counterexample. Now, we are less convinced that the conjecture is true.

Obviously, $\kappa > 0$ if $F(x, w) - G(x, w) > 0$ for all $x \geqslant 0$ and $w \in W$. More generally, κ is positive if for some $g \in L^1(\pi)$ (where π denotes the stationary distribution of the chain (X_n)) and all x

$$\mathrm{E}[F^\alpha(x,\varepsilon) - G^\alpha(x,\varepsilon)] > \mathrm{E}\, g\big(F(x,\varepsilon)\big) - g(x).$$

Indeed, integrating this inequality with respect to π we get

$$\mathrm{E}\big[F^\alpha(X,\varepsilon) - G^\alpha(X,\varepsilon)\big] > \mathrm{E}\, g(F(X,\varepsilon)) - \mathrm{E}\, g(X) \tag{2.11}$$

and the term in the right hand side equals 0, because $F(X,\varepsilon) \overset{d}{=} X$.

If $\mathrm{E}\big[F^\alpha(x,\varepsilon) - G^\alpha(x,\varepsilon)\big] = O(x^\theta)$, as $x \to \infty$, (cf. condition F3) one can try the function $g(x) = bx^\theta$ with an appropriate $b \in \mathbb{R}$, because typically

$$\mathrm{E}\, F^\theta(x,\varepsilon) - x^\theta \sim \mathrm{E}\, G^\theta(x,\varepsilon) - x^\theta = x^\theta(\mathrm{E}\, G^\theta(1,\varepsilon) - 1).$$

For example, let us turn back to model (2.10) with $\mathrm{E}\log\varepsilon < 0$ and $\mathrm{E}\,\varepsilon = 1$. Assume that $\mathrm{E}\,\varepsilon\log\varepsilon > \log 2$ and denote $q_\theta = \mathrm{E}\,\varepsilon^\theta$. Then one can show that there exists a $\theta \in (0;1)$ with $q_\theta < 2^{\theta-1}$ and that (2.11) holds with $g(x) = bx^\theta$ and any $b \in (2^{1-\theta}; q_\theta^{-1})$. If $\mathrm{E}\,\varepsilon\log\varepsilon \leqslant \log 2$, our method fails, but we do not know if $\kappa = 0$ in this case.

2.3.2 Chains on a line

In this subsection we consider the case $C = \mathbb{R}$, where $S = \{-1, 1\}$ is a two-point set. In this case asymptotical homogeneity means

$$t^{-1}F(tz, w) \xrightarrow[t\to\infty]{} G(z, w), \quad z \in S.$$

We assume that the transition probability matrix of the chain $\big(Z_n(z)\big)$ is positive and denote by Z a random variable, independent of ε and distributed accordingly to the unique stationary distribution of the chain. Then the top Lyapunov exponent equals $\mathrm{E}\log R(Z,\varepsilon)$ provided this mean is well-defined.

Take $\alpha \geqslant 0$. If $\mathrm{E}\, R^\alpha(\pm 1,\varepsilon) < \infty$, define

$$Q_\alpha(z, z') = \mathrm{E}\, R^\alpha(z,\varepsilon) 1_{\{H(z,\varepsilon)=z'\}}, \quad (z, z') \in S \times S.$$

We can think about Q_α as a non-negative 2×2 matrix. For example,

$$Q_0 = \begin{pmatrix} \mathrm{P}\{H(-1,\varepsilon)=-1\} & \mathrm{P}\{H(-1,\varepsilon)=1\} \\ \mathrm{P}\{H(1,\varepsilon)=-1\} & \mathrm{P}\{H(1,\varepsilon)=1\} \end{pmatrix}$$

is the matrix of transition probabilities of the chain $\big(Z_n(z)\big)$. If $Q_0(z,z')=0$ then $Q_\alpha(z,z')=0$ for all α and vice versa. Hence, all Q_α are irreducible matrices.

Denote by ρ_α the spectral radius of the matrix Q_α. Then ρ_α is a simple eigenvalue of Q_α and there exist positive numbers $r_\alpha(\pm 1)$ and $\pi_\alpha\{\pm 1\}$ such that

$$r_\alpha = \begin{pmatrix} r_\alpha(-1) \\ r_\alpha(1) \end{pmatrix} \quad \text{and} \quad \pi_\alpha = \begin{pmatrix} \pi_\alpha\{-1\} & \pi_\alpha\{1\} \end{pmatrix}$$

are, respectively, right and left eigenvectors of Q_α, corresponding to the eigenvalue ρ_α [4, theorem 8.2.11]. We think about r_α and π_α as a positive function on S and a measure on S, respectively. Without loss of generality we can assume that π_α is a probability and $\pi_\alpha r_\alpha = 1$.

THEOREM 2.3. *Suppose that*

(i) there does not exist a $d\in\mathbb{R}$ *such that the distribution of both* $\log R(\pm 1,\varepsilon)$ *is concentrated on the lattice* $\mathbb{Z}d$*;*

(ii) the top Lyapunov exponent is negative;

(iii) for some $\alpha>0$ *the matrix* Q_α *is well defined,* $\rho_\alpha = 1$ *and for all* $z\in S$

$$\mathrm{E}\, R^\alpha(z,\varepsilon)|\log R(z,\varepsilon)| < \infty. \tag{2.12}$$

If, in addition,

(X1) $\mathrm{E}|X|^\theta<\infty$ *for some* $0<\theta<\alpha$*,*

(X2a) $\mathrm{E}\big||F(X,\varepsilon)|^\alpha - |G(X,\varepsilon)|^\alpha\big| < \infty$*,*

(X2b) $\mathrm{E}\big[|F(X,\varepsilon)|^\alpha + |G(X,\varepsilon)|^\alpha\big]1_{\{F(X,\varepsilon)G(X,\varepsilon)<0\}} < \infty$*,*

then for all $z\in S$

$$t^\alpha\, \mathrm{P}\{\operatorname{sign} X = z,\ |X|>t\} \xrightarrow[t\to\infty]{} \kappa(z)$$

with some $\kappa(z)\in[0;\infty)$.

REMARK 2.2. If $G(-1,\varepsilon)\overset{d}{=}-G(1,\varepsilon)$ then by homogeneity and independence of X and ε, for all Borel $A\subset\mathbb{R}$,

$$\begin{aligned}\mathrm{P}\{X<0,\ G(X,\varepsilon)\in A\} &= \mathrm{P}\{X<0,\ |X|G(-1,\varepsilon)\in A\} \\ &= \mathrm{P}\{X<0,\ XG(1,\varepsilon)\in A\}\end{aligned}$$

and also

$$\mathrm{P}\{X>0,\ G(X,\varepsilon)\in A\} = \mathrm{P}\{X>0,\ XG(1,\varepsilon)\in A\}.$$

Summing up we get

$$\mathrm{P}\{G(X,\varepsilon)\in A\} = \mathrm{P}\{XG(1,\varepsilon)\in A\},$$

i.e. $G(X,\varepsilon) \stackrel{d}{=} XG(1,\varepsilon)$. Therefore in this case theorem 2.3 reduces to corollary 2.4 of [3] (which is proved, however, without assumption X1). If distributions of $G(-1,\varepsilon)$ and $-G(1,\varepsilon)$ differ, our theorem is not covered by that corollary. This is seen from the fact that in all cases considered in [3] $\kappa(-1) = \kappa(1)$ and this equality does not hold in general.

For the existence of α satisfying $\rho_\alpha = 1$ and (2.12) it suffices to find a β with $\rho_\beta > 1$. In order to check this inequality we may use any estimate of the spectral radius from below. For example, it is known [4, corollary 8.1.20, theorem 8.1.21] that

$$\rho_\theta \geqslant \max_{z\in S} Q_\theta(z,z) \quad \text{and} \quad \rho_\theta \geqslant \min_{z\in S} \sum_{z'\in S} Q_\theta(z,z').$$

Therefore it suffices to find a β with either $\mathrm{E}\, R^\beta(z,\varepsilon)1_{\{H(z,\varepsilon)=z\}} \in (1;\infty)$ for some $z \in S$ or $\mathrm{E}\, R^\beta(z,\varepsilon) \in (1;\infty)$ for all z.

For checking conditions X1–X2, we provide the following proposition.

PROPOSITION 2.3. *Suppose that (X_n) is a φ-irreducible T-chain and the top Lyapunov exponent is negative. If, for some $\theta < \theta_1 < \alpha$, all $\Delta > 0$ and all $z \in S$,*

(F1) $\sup_{0\leqslant t\leqslant\Delta} \mathrm{E}\big|F(tz,\varepsilon)\big|^\alpha < \infty$,

(F2) $\sup_{t>\Delta} t^{-\theta_1}\, \mathrm{E}\big|F(tz,\varepsilon)\big|^{\theta_1} < \infty$,

(F3a) $\sup_{t>\Delta} t^{-\theta}\, \mathrm{E}\big||F(tz,\varepsilon)|^\alpha - |G(tz,\varepsilon)|^\alpha\big| < \infty$,

(F3b) $\sup_{t>\Delta} t^{-\theta}\, \mathrm{E}\big[|F(tz,\varepsilon)|^\alpha + |G(tz,\varepsilon)|^\alpha\big]1_{\{F(tz,\varepsilon)G(tz,\varepsilon)<0\}} < \infty$,

then the chain is positive recurrent and conditions X1–X2 hold.

The limits $\kappa(z)$ in theorem 2.3 can be explicitly written as

$$\kappa(z) = \frac{\pi_\alpha\{z\}}{m\alpha} \sum_{z'\in S} r_\alpha(z')\, \mathrm{E}\Big(\big[z'F(X,\varepsilon)\big]^{+\alpha} - \big[z'G(X,\varepsilon)\big]^{+\alpha}\Big),$$

where $x^{+\alpha}$ means $(x^+)^\alpha$ and

$$m = \sum_{z\in S} \pi_\alpha\{z\}\, \mathrm{E}\, R^\alpha(z,\varepsilon) r_\alpha(H(z,\varepsilon)) \log R(z,\varepsilon).$$

If $G(-1,\varepsilon) \stackrel{d}{=} -G(1,\varepsilon)$ then $Q_\alpha = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with

$$a = \mathrm{E}\, R^\alpha(1,\varepsilon)1_{\{H(1,\varepsilon)=1\}} \quad \text{and} \quad b = \mathrm{E}\, R^\alpha(1,\varepsilon)1_{\{H(1,\varepsilon)=-1\}}.$$

It is easily seen that then $\rho_\alpha = a+b$, $\pi_\alpha = (\frac{1}{2}\ \frac{1}{2})$ and $r = (1\ 1)^\top$. Therefore in this case

$$\kappa(1) = \kappa(-1) = \frac{1}{2\alpha m}\, \mathrm{E}\big(|F(X,\varepsilon)|^\alpha - |G(X,\varepsilon)|^\alpha\big), \tag{2.13}$$

where $m = \mathrm{E}\, R^\alpha(1,\varepsilon) \log R(1,\varepsilon)$.

If $G(-1,\varepsilon) \stackrel{d}{=} G(1,\varepsilon)$ then

$$Q_\alpha = \begin{pmatrix} b & a \\ b & a \end{pmatrix}, \quad \rho_\alpha = a+b, \quad \pi_\alpha = \begin{pmatrix} \frac{a}{a+b} & \frac{b}{a+b} \end{pmatrix}, \quad r_\alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(with the same a, b as above) and again (2.13) hold. Hence equalities (2.13) are valid in all cases, where $R(-1,\varepsilon) \stackrel{d}{=} R(1,\varepsilon)$.

Now consider general case. Since both probabilities $\pi_\alpha\{z\}$ are positive, either both $\kappa(z)$ are positive, or both equal 0. Since $r_\alpha > 0$, the constants are positive when

$$\sum_{z\in S} r_\alpha(z)\,\mathrm{E}\Big(\big[zF(X,\varepsilon)\big]^{+\alpha} - \big[zG(X,\varepsilon)\big]^{+\alpha}\Big) > 0.$$

To prove this inequality we can use the same method as in the previous subsection: it suffices to show that for some $g \in L^1(\pi)$ and all $x \in \mathbb{R}$

$$\sum_{z\in S} r_\alpha(z)\,\mathrm{E}\Big(\big[zF(x,\varepsilon)\big]^{+\alpha} - \big[zx\big]^{+\alpha}\Big) > \mathrm{E}\, g(F(x,\varepsilon)) - g(x).$$

This time one can try the function $g(x) = b\sum_z r_\theta(z)(zx)^{+\theta}$, because

$$\begin{aligned}
\mathrm{E}\sum_z r_\theta(z)[zG(x,\varepsilon)]^{+\theta} &= |x|^\theta\,\mathrm{E}\sum_z r_\theta(z)[zG(\operatorname{sign} x,\varepsilon)]^{+\theta} \\
&= \sum_{z,z'} 1_{\{\operatorname{sign} x = z'\}}|x|^\theta r_\theta(z)\,\mathrm{E}[zG(z',\varepsilon)]^{+\theta} = \sum_{z,z'}(z'x)^{+\theta} r_\theta(z) Q_\theta(z',z) \\
&= \rho_\theta \sum_{z'}(z'x)^{+\theta} r_\theta(z').
\end{aligned}$$

3 Concluding part

Conclusions

In the dissertation there is investigated a class of Markov chains defined by iterations of a function possessing a property of asymptotical homogeneity. Two problems are solved:

- there are established rather general conditions under which the chain has unique stationary distribution;
- for the chains evolving in a real line there are established conditions under which the stationary distribution of the chain is heavy-tailed.

Published works on the topic

1. V. Kazakevičius and V. Skorniakov. Asymptotically homogeneous iterated random functions with applications to the HARCH process. *Lithuanian Mathematical Journal*, 49(1):26–39, 2009.

2. V. Kazakevičius and V. Skorniakov. Tail index of asymptotically homogeneous Markov chain. *Lithuanian Mathematical Journal*, 2010 (to appear).

Acknowledgement

I am very grateful to my supervisor Assoc. Prof. Dr. Vytautas Kazakevičius. It was a big pleasure for me to learn from him and to work with him. He spared me plenty of time and made every effort to increase the quality of the work done as well as expand my horizon.

I also would like to thank my colleagues from Vilnius University, department of Mathematical Statistics for their support during my study period and mild work atmosphere.

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Reziumė

Mokslo problemos aktualumas. Markovo grandinės plačiai naudojamos įvairiuose praktiniuose taikymuose modeliuojant priklausomų atsitiktinių dydžių sekas, todėl tyrimai šioje srityje visada aktualūs.

Darbe nagrinėtos Markovo grandinės, kurių būsenų aibė C yra baigtiniamatės normuotos erdvės E kūgis. Laikyta, kad grandinės $(X_n, n \geqslant 0)$ iteracijos yra nusakomos mačia funkcija $F : C \times W \to C$ ir nepriklausomų vienodai pasiskirsčiusių separabilios metrinės erdvės W elementų seka $(\varepsilon_n, n \geqslant 1)$:

$$X_n = F(X_{n-1}, \varepsilon_n), n \geqslant 1.$$

Pagrindinė prielaida apie funkciją F buvo $F(\cdot, w)$ asimptotinis homogeniškumas, t.y. laikyta, kad kiekvienam $w \in W$ egzistuoja funkcija $G(\cdot, w)$, tenkinanti sąryšius:

$$t_n \to \infty, x_n \to x \neq 0 \Rightarrow t_n^{-1} F(t_n x_n, w) \to G(x, w); \tag{3.1}$$

$$G(0, w) = 0. \tag{3.2}$$

Darbe tirtų grandinių klasė apima daug praktiniuose taikymuose sutinkamų modelių. Toliau tokios grandinės vadinamos *asimptotiškai homogeninėmis.*

Darbo tikslas ir uždaviniai. Darbe siekėme charakterizuoti asimptotiškai homogeninių Markovo grandinių klasę ir išspręsti du uždavinius:

- pateikti bendras sąlygas, garantuojančias vienintelio stacionaraus grandinės skirstinio egzistavimą;

- pateikti sąlygas, kurioms esant stacionarus grandinės skirstinys turi „sunkias“ uodegas.

Mokslinis naujumas ir praktinė reikšmė. Mūsų žiniomis iki šiol literatūroje asimptotiškai homogeninių grandinių klasė nebuvo charakterizuojama tokiu būdu, kaip tai atlikta disertacijoje, todėl pateikiama medžiaga yra iš esmės nauja.

Gauti rezultatai leidžia nesunkiai tirti asimptotiškai homogenines grandines, kurių tyrimas kitais metodais būtų sudėtingas uždavinys, todėl yra aktualūs ne tik teorine, bet ir praktine prasme.

Darbo struktūra. Disertacija parašyta lietuvių kalba. Ją sudaro trys skyriai ir du priedai. Bendras puslapių skaičius — 63.

Pirmajame skyriuje griežtai apibrėžiama asimptotiškai homogeninės grandinės sąvoka ir pateikiami pavyzdžiai, iliustruojantys kokia plati yra nagrinėjama Markovo grandinių klasė. Antrajame skyriuje pateikiami rezultatai, susiję su stacionaraus skirstinio egzistavimu, o trečiajame — su to skirstinio uodegos svoriu, kai nagrinėjamos grandinės būsenų aibė yra pustiesė arba tiesė.

Pirmasis priedas skirtas matematiniams faktams, kuriais remiamasi įrodinėjant disertacijos teoremas ir teiginius. Antrajame priede pateikiami laiko eilučių teorijos trumpiniai.

Išvados. Disertacijoje tirta Markovo grandinių klasė, kurios iteracijos nusakomos atsitiktinėmis asimptotiškai homogeninėmis funkcijomis, ir išspręsti du uždaviniai:

- surastos bendros sąlygos, kurios garantuoja vienintelio stacionaraus skirstinio egzistavimą;

- vienmatėms grandinėms surastos sąlygos, kurioms esant stacionarus skirstinys turi „sunkias“ uodegas.

Padėka. Nuoširdžiai dėkoju savo vadovui doc. dr. Vytautui Kazakevičiui. Buvo labai malonu pas jį mokytis ir kartu dirbti. Jo skirtas laikas ir įdėtos pastangos leido ženkliai pagerinti disertacijos kokybę ir praplėsti mano akiratį.

Taip pat norėčiau padėkoti savo kolegoms iš Matematinės statistikos katedros už visokeriopą paramą studijų metu ir gerą darbinę atmosferą.

Trumpos žinios apie autorių. Gimimo vieta ir data — Vilnius, 1981-11-16. 2000 – 2004 m. įgytas matematinės statistikos bakalauro laipsnis, 2004 – 2006 m. — matematinės statistikos magistro laipsnis, 2006 – 2010 m. studijuota matematikos krypties doktorantūroje. Visos studijos (nuo 2000 iki 2010) vykdytos Vilniaus universitete, Matematikos ir informatikos fakultete.

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