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PRESSURE BOUNDARY CONDITIONS FOR VISCOUS FLOWS IN THIN TUBE STRUCTURES: STOKES EQUATIONS WITH LOCALLY DISTRIBUTED BRINKMAN TERM

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Abstract. The steady state Stokes-Brinkman equations in a thin tube structure is considered. The Brinkman term differs from zero only in small balls near the ends of the tubes. The boundary conditions are: given pressure at the inflow and outflow of the tube structure and the no slip boundary condition on the lateral boundary. The complete asymptotic expansion of the problem is constructed. The error estimates are proved. The method of partial asymptotic dimension reduction is introduced for the Stokes-Brinkman equations and justified by an error estimate. This method approximates the main problem by a hybrid dimension problem for the Stokes-Brinkman equations in a reduced domain.

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Mathematical models of viscous flows in thin domains have multiple applications. Such domains have one or several dimensions which are much smaller than other ones. In particular, tube structures are some unions of thin cylinders or rectangles (pipes or channels) [28]. This geometry simulates a network of blood vessels as a biological application or pipelines and cooling systems as industrial applications.

Full dimension numerical computations of flows networks of thin tubes require huge computer resources. To reduce these resources and accelerate computations we use asymptotic analysis where the small parameter is the ratio of thickness of pipes or channels to their length. This analysis leads to the construction of asymptotic expansions justified by error estimates [30], [33], [5], [24], [25], [19], [18], [26], [6]. It is also implemented in some special numerical methods combining the description with reduced dimension and full dimension zooms for small zones of singular behavior of the solution. For example, method of asymptotic partial decomposition of domain was introduced for the stationary Navier–Stokes equations in [29] and then developed in [30], [33], [3]. Another practical approach coupling models of different dimension was developed in [11], [36], [37].

Basically, the Newtonian rheology for the fluid motion corresponding to the stationary and nonstationary Navier–Stokes or Stokes equations was considered, while several papers studied non-Newtonian models ([34], [5], [25]). However, the modeling of zones of thrombus formation could be better described by the Brinkman equations combining the Stokes description of the fluid motion with the Darcy filtration law. Indeed, the external

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part of the thrombus behaves as a porous medium, but approaching the surface of the thrombus it corresponds better to the Newtonian fluid.

Also in the present paper we will consider the inflow-outflow boundary conditions involving pressure. These conditions are more convenient from the computational point of view. They also allow to compute the permeability of a piece of tissue containing a network of vessels. An extensive studying of the Stokes and Navier–Stokes equations with boundary conditions involving pressure started in the pioneer paper [8] and since then continues in a vast mathematical literature (see [35], [25], [17], [13], [16] and the bibliography there).

For classical theory of the Navier–Stokes equations see [20], [42], [12].

1. Definition of a thin tube structure

Let us remind the definitions of the tube structure and its graph given in [30].

Definition 1.1. Let $O_1, O_2, ..., O_N$ be N different points in $\mathbb{R}^n, n = 2, 3$, and $e_1, e_2, ..., e_M$ be M closed segments each connecting two of these points (*i.e.* each $e_j = \overline{O_{i_j} O_{k_j}}$, where $i_j, k_j \in \{1, ..., N\}, i_j \neq k_j$). All points O_i are supposed to be the ends of some segments e_j. The segments e_j are called edges of the graph. A point O_i is called a node, if it is the common end of at least two edges and O_i is called a vertex, if it is the end of the only one edge. Any two edges e_j and e_i can intersect only at the common node. The set of vertices is supposed to be non-empty.

Denote $\mathcal{B} = \bigcup\limits_{j=1}^{M} e_j$ the union of edges and assume that $\mathcal{B}$ is a connected set (see Fig. 1). The union of all edges having the same end point O_l is called the bundle $\mathcal{B}_l$. Fig. 1 a) presents the graph as a union of edges $e_1, ..., e_5$, points O_1, O_2, O_3 are the nodes, points O_4, O_5, O_6 are the vertices. Each point O_i, a node or a vertex, with all edges containing O_i as an end point, form bundle $\mathcal{B}_l$, for example, O_1 with edges e_1 and e_5 form bundle $\mathcal{B}_1$. Fig. 1 b) presents the graph as a union of edges $e_1, ..., e_9$, points O_1, O_2, O_3, O_4 are the nodes, points O_5, O_6, O_7 are the vertices.

Let e be some edge, $e = \overline{O_i O_j}$. Consider two Cartesian coordinate systems in $\mathbb{R}^n$. The first one has the origin in O_i and the axis $O_i x_n^{(e)}$ has the direction of the ray $[O_i O_j)$; the second one has the origin in O_j and the opposite direction, *i.e.* $O_j \tilde{x}_n^{(e)}$ is directed over the ray $[O_j O_i)$.

Below in various situations, we choose one or another coordinates system denoting the local variable in both cases by $x^{(e)}$ and pointing out which end is taken as the origin of the coordinate system.

With every edge e_j we associate a bounded domain $\sigma^j \subset \mathbb{R}^{n-1}$ containing the origin and having C^2- smooth boundary $\partial\sigma^j,\ j = 1, ..., M$. For every edge $e_j = e$ and associated $\sigma^j = \sigma^{(e)}$ we denote by $\Pi_\varepsilon^{(e)}$ the cylinder

$$\Pi_\varepsilon^{(e)} = \Big\{ x^{(e)} \in \mathbb{R}^n : x_n^{(e)} \in (0, |e|), \frac{x^{(e)\prime}}{\varepsilon} \in \sigma^{(e)} \Big\},$$

where $x^{(e)\prime} = (x_1^{(e)}, ..., x_{n-1}^{(e)}),\ |e|$ is the length of the edge e and $\varepsilon > 0$ is a small parameter. Notice that the edges e_j and Cartesian coordinates of nodes and vertices O_j, as well as the domains σ^j, do not depend on ε. We will define as well a semi-infinite dilated cylinder $\Pi_\infty^{(e)} = \Big\{ x^{(e)} \in \mathbb{R}^n : x_n^{(e)} \in [0, \infty), x^{(e)\prime} \in \sigma^{(e)} \Big\}$.

Let $O_1, ..., O_{N_1}$ be nodes and $O_{N_1+1}, ..., O_N$ be vertices. Let $\omega^1, ..., \omega^{N_1}$ be bounded independent of ε domains in $\mathbb{R}^n$; introduce the nodal domains $\omega_\varepsilon^j = \{x \in \mathbb{R}^n : \dfrac{x - O_j}{\varepsilon} \in \omega^j\}$.

Every vertex O_j is the end of one and only one edge e_k which will also be denoted as e_{O_j}; we will re-denote as well the domain σ^k associated to this edge as σ^{O_j}. Notice that the subscript k may be different from j.

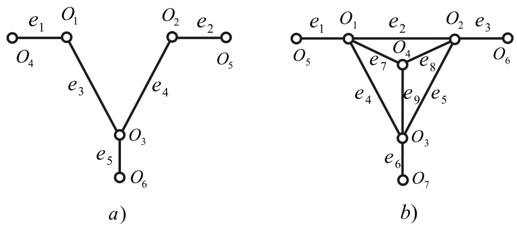


FIGURE 1. Graphs of tube structures.

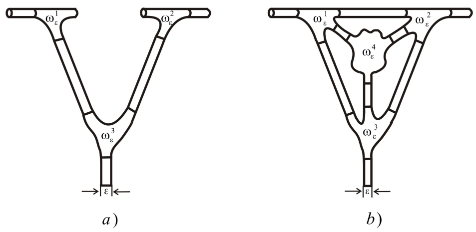


FIGURE 2. Tube structures.

Definition 1.2. By a tube structure, we call the following domain

$$B_\varepsilon = \Big(\bigcup_{j=1}^{M} \Pi_\varepsilon^{(e_j)}\Big) \bigcup \Big(\bigcup_{j=1}^{N_1} \omega_\varepsilon^j\Big).$$

Suppose that it is a connected set and that the boundary ∂B_ε of B_ε is C^2-smooth except for the parts of the boundary which are the bases $\gamma_\varepsilon^j = \{x^{(e)\prime} \in \sigma^{O_j}, x_n^{(e)} = 0\}$ of cylinders $\Pi_\varepsilon^{(e)}$, $j = N_1+1, ..., N$ (see Fig. 2).

Let r_1 be the maximal diameter of domains ω^i, $i = 1, ..., N$, denote $r = r_1 + 1$. Consider a node or a vertex O_l and all edges e_j having O_l as one of their end points. We call the union of all these edges a bundle of edges and denote it $\mathcal{B}_l$, *i.e.*, $\mathcal{B}_l = \bigcup_{j: O_l \in e_j} e_j$. By a bundle of cylinders B_{O_l} we call the union $\omega_\varepsilon^l \cup \Big(\bigcup_{j: O_l \in e_j} \Pi_\varepsilon^{(e_j)}\Big)$. We will consider as well the half-bundle $HB_{O_l} = \omega_\varepsilon^l \cup \Big(\bigcup_{j: O_l \in e_j} \{x \in \Pi_\varepsilon^{(e_j)}, x_n^{(e_j)} \in [0, |e_j|/2]\}\Big)$. We will use also $\Omega_l = \omega^l \cup \Big(\bigcup_{j: O_l \in e_j} \Pi_\infty^{(e_j)}\Big)$, a bundle of dilated semi-infinite cylinders.

2. Formulation of the problem. Existence and uniqueness of a solution

Let $\Gamma = \partial B_\varepsilon \setminus \bigcup_{j=N_1+1}^{N} \gamma_\varepsilon^j$ be the lateral surface of the domain B_ε. In the tube structure B_ε we define the spaces

$$\begin{aligned} \widehat{W}^{1,2}(B_\varepsilon) &= \Big\{\boldsymbol{\eta} \in W^{1,2}(B_\varepsilon) : \boldsymbol{\eta}|_\Gamma = 0,\ \boldsymbol{\eta}_{\boldsymbol{\tau}}|_{\gamma_\varepsilon^j} = 0,\ j = N_1+1, \dots, N\Big\}, \\ \widehat{K}^{1,2}(B_\varepsilon) &= \Big\{\boldsymbol{\eta} \in \widehat{W}^{1,2}(B_\varepsilon) : \mathrm{div}\boldsymbol{\eta} = 0\Big\}, \end{aligned} \tag{2.1}$$

We denote $B(O, R)$ the open ball in $\mathbb{R}^n$ with the center O and the radius R.

Let us consider the following boundary value problem for the steady-state Stokes equations in a tube structure B_ε

$$\begin{cases} -\mathrm{div}\Big(\nu_\varepsilon(x)\Big(\nabla+\nabla^T\Big)\mathbf{u}\Big)+K_\varepsilon(x)\mathbf{u}+\nabla p=0 \ \text{ in } B_\varepsilon, \\ \mathrm{div}\mathbf{u}=0 \ \text{ in } B_\varepsilon, \\ \mathbf{u}=0 \ \text{ on } \partial B_\varepsilon\setminus\cup_{j=N_1+1}^N\gamma_\varepsilon^j, \\ \mathbf{u}_{\boldsymbol{\tau}}=0 \ \text{ on } \gamma_\varepsilon^j, \\ -\nu\partial_n\mathbf{u}\cdot\mathbf{n}+p=c^j/\varepsilon^2 \ \text{ on } \gamma_\varepsilon^j,\ j=N_1+1,...,N, \end{cases} \tag{2.2}$$

where ν_ε is the function (effective dynamical viscosity related to the porosity) equal to the positive constant $\nu^{(0)}$ everywhere except for the balls $B(O_l,r\varepsilon)$ and equal to the given functions $\nu_\varepsilon(x)=\nu^{(l)}\Big(\frac{x-O_l}{\varepsilon}\Big)$ in the balls $B(O_l,r\varepsilon),\ l=1,...,N_1,\ K_\varepsilon$ is the $n\times n$ symmetric matrix-valued function (inverse to effective permeability of porous medium) equal to zero everywhere except for the balls $B(O_l,r\varepsilon)$ and equal to the given functions $K_\varepsilon(x)=K^{(l)}\Big(\frac{x-O_l}{\varepsilon}\Big)$ in the balls $B(O_l,r\varepsilon),\ l=1,...,N_1,\ \nu_\varepsilon,K_\varepsilon\in C^1(B_\varepsilon),\ \nu_\varepsilon$ is greater than some positive constant independent of ε, $K_\varepsilon\geq 0$ (non-negative matrix) ; $\nu^{(l)},K^{(l)}$ are independent of ε; $\mathbf{n}$ is the unit normal vector to $\gamma_\varepsilon^j,\ \mathbf{u}_{\boldsymbol{\tau}}=\mathbf{u}-(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$ is the tangential component of the vector $\mathbf{u},\ \partial_n g=\nabla g\cdot\mathbf{n}$ is the normal derivative of g, c^j are some given constants. This model was rigorously derived from the Navier–Stokes equation in a porous medium in [2] and was extensively studied in fluid mechanics [14]. This model describes the Newtonian flow in the tubes combined with the fluid filtration process through the zones $B(O_l,r\varepsilon)$, simulating the eventual clots or thrombi. In these zones ν_ε stands for the effective dynamical velocity taking into account the porosity of the clot, while K_ε stands for the inverse to the effective permeability of the clot. The coefficients $K_\varepsilon,\nu_\varepsilon$ is supposed to be C^1-smooth function, while ν_ε is continuous in the closure of B_ε.

In this section we prove the existence and uniqueness of the solution to problem (2.2) with the right-hand side $\mathbf{f}\in L^2(B_\varepsilon)$. From the boundary condition $\mathbf{u}_{\boldsymbol{\tau}}|_{\gamma_\varepsilon^j}=0$ and the divergence equation $\mathrm{div}\mathbf{u}=0$, it follows that $-\nu\partial_n\mathbf{u}\cdot\mathbf{n}|_{\gamma_\varepsilon^j}=0$.Thus we can rewrite (2.2) with the right-hand side in the following form

$$\begin{cases} -\mathrm{div}\Big(\nu_\varepsilon(x)D\mathbf{u}\Big)+K_\varepsilon(x)\mathbf{u}+\nabla p=\mathbf{f}(x) \ \text{ in } B_\varepsilon, \\ \mathrm{div}\mathbf{u}=0 \ \text{ in } B_\varepsilon, \\ \mathbf{u}=0 \ \text{ on } \partial B_\varepsilon\setminus\cup_{j=N_1+1}^N\gamma_\varepsilon^j, \\ \mathbf{u}_{\boldsymbol{\tau}}=0 \ \text{ on } \gamma_\varepsilon^j, \\ p=p_j \ \text{ on } \gamma_\varepsilon^j,\ j=N_1+1,...,N, \end{cases} \tag{2.3}$$

where p_j stand for the constants c^j/ε^2, $D=\nabla+\nabla^T$.

Let us define a weak solution of problem (2.3) as a vector field $\mathbf{u}\in\widehat{K}^{1,2}(B_\varepsilon)$ satisfying the integral identity

$$\int\limits_{B_\varepsilon}\Big(\tfrac{1}{2}\nu_\varepsilon(x)D\mathbf{u}\cdot D\boldsymbol{\eta}+K_\varepsilon(x)\mathbf{u}\cdot\boldsymbol{\eta}\Big)\,\mathrm{d}x=-\sum_{j=N_1+1}^N p_j\int\limits_{\gamma_\varepsilon^j}\boldsymbol{\eta}_{\mathbf{n}}\,\mathrm{d}s+\int\limits_{B_\varepsilon}\mathbf{f}\cdot\boldsymbol{\eta}\,\mathrm{d}x \tag{2.4}$$

for every $\boldsymbol{\eta}\in\widehat{K}^{1,2}(B_\varepsilon)$. Here and below for any two $n\times n$ matrices A and B having entries a_{ij} and b_{ij} denote $A\cdot B$ the sum $\sum_{i,j=1}^n a_{ij}b_{ij}$.

Introduce $p_j^*=p_j-p_N,\quad j=N_1,...,N$. Consider an equivalent weak formulation: find a vector field $\mathbf{u}\in\widehat{K}^{1,2}(B_\varepsilon)$ satisfying the integral identity

$$\int\limits_{B_\varepsilon}\Big(\tfrac{1}{2}\nu_\varepsilon(x)D\mathbf{u}\cdot D\boldsymbol{\eta}+K_\varepsilon(x)\mathbf{u}\cdot\boldsymbol{\eta}\Big)\,\mathrm{d}x=-\sum_{j=N_1+1}^{N-1} p_j^*\int\limits_{\gamma_\varepsilon^j}\boldsymbol{\eta}_{\mathbf{n}}\,\mathrm{d}x'+\int\limits_{B_\varepsilon}\mathbf{f}(x)\cdot\boldsymbol{\eta}\,\mathrm{d}x \tag{2.5}$$

for every $\boldsymbol{\eta} \in \widehat{K}^{1,2}(B_\varepsilon)$. The equivalence of these formulations follows from the identity

$$\sum_{j=N_1+1}^{N-1} p_j^* \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}} \,\mathrm{d}x' = \sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}} \,\mathrm{d}x',$$

which is a corollary of the relation

$$\sum_{j=N_1+1}^{N} \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}} \,\mathrm{d}x' = 0$$

for the solenoidal vector-valued function $\boldsymbol{\eta}$.

Let us explain this weak formulation heuristically; the rigorous analysis of the equivalence of the weak formulation and the classical one needs to study the regularity of the weak solution, see [9], [1] for the methods.

Identity (2.4) follows from equations (2.3) after multiplying them by $\boldsymbol{\eta} \in \widehat{K}^{1,2}(B_\varepsilon)$ and integrating by parts in B_ε. On the other hand, for a sufficiently regular solution $\mathbf{u}$ satisfying (2.4) there exists a pressure field p such that the pair $(\mathbf{u}, p)$ satisfies equations $(2.3)_{1,2}$ a.e. in B_ε. Boundary conditions $(2.3)_{3,4,5}$ are satisfied in the sense of traces (see the definition of the space $\widehat{K}^{1,2}(B_\varepsilon)$). More exactly, function p is defined up to an additive constant but this constant can be chosen so that p satisfies $(2.3)_5$. Indeed, take in (2.4) a smooth solenoidal function $\boldsymbol{\eta}$ satisfying the boundary conditions $\boldsymbol{\eta}|_\Gamma = 0,\ \boldsymbol{\eta}_{\boldsymbol{\tau}}|_{\gamma_\varepsilon^j} = 0,\ j = N_1+1, ..., N$. Integrating by parts in (2.4) yields

$$\begin{aligned}&\int\limits_{B_\varepsilon} \Big(-\operatorname{div}\Big(\nu_\varepsilon(x) D\mathbf{u}\Big) + K_\varepsilon(x)\mathbf{u} - \mathbf{f}(x)\Big) \cdot \boldsymbol{\eta}\Big) \,\mathrm{d}x \\ &= -\nu^{(0)} \sum_{j=N_1+1}^{N} \int\limits_{\gamma_\varepsilon^j} \partial_n \mathbf{u} \cdot \boldsymbol{\eta} \,\mathrm{d}s - \sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}} \,\mathrm{d}x' \\ &\qquad = -\sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}} \,\mathrm{d}x'.\end{aligned} \tag{2.6}$$

If $\boldsymbol{\eta} \in J_0^\infty(B_\varepsilon) = \{\boldsymbol{\eta} \in C_0^\infty(B_\varepsilon) : \operatorname{div}\boldsymbol{\eta} = 0\}$, then it follows from (2.6) that

$$\int\limits_{B_\varepsilon} \Big(-\operatorname{div}\Big(\nu_\varepsilon(x) D\mathbf{u} + K_\varepsilon(x)\mathbf{u} - \mathbf{f}(x)\Big) \cdot \boldsymbol{\eta}\Big) \,\mathrm{d}x = 0 \quad \forall \boldsymbol{\eta} \in J_0^\infty(B_\varepsilon).$$

Hence, there exists a pressure function p such that

$$-\operatorname{div}\Big(\nu_\varepsilon(x) D\mathbf{u}\Big) + K_\varepsilon(x)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{a.e. in } B_\varepsilon.$$

Then

$$\int\limits_{B_\varepsilon} \Big(-\operatorname{div}\Big(\nu_\varepsilon(x) D\mathbf{u}\Big) + K_\varepsilon(x)\mathbf{u} - \mathbf{f}(x)\Big) \cdot \boldsymbol{\eta} \,\mathrm{d}x = -\int\limits_{B_\varepsilon} \nabla p \cdot \boldsymbol{\eta} \,\mathrm{d}x = -\sum_{j=N_1+1}^{N} \int\limits_{\gamma_\varepsilon^j} p \cdot \boldsymbol{\eta}_{\mathbf{n}} \,\mathrm{d}x'$$

for every $\boldsymbol{\eta} \in \widehat{K}^{1,2}(B_\varepsilon)$. Therefore,

$$\sum_{j=N_1+1}^{N} \int\limits_{\gamma_\varepsilon^j} p\boldsymbol{\eta}_{\mathbf{n}}\, \mathrm{d}x' = \sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}}\, \mathrm{d}x'.$$

Thus,

$$\sum_{j=N_1+1}^{N} \int\limits_{\gamma_\varepsilon^j} (p - p_j) \cdot \boldsymbol{\eta}_{\mathbf{n}}\, \mathrm{d}x' = 0 \quad \forall \boldsymbol{\eta} \in \widehat{K}^{1,2}(B_\varepsilon). \tag{2.7}$$

Let us fix arbitrary $j \in \{N_1 + 1, \ldots, N\}$. Taking $\boldsymbol{\eta} \in K_0^{1,2}(B_\varepsilon)$ such that $\boldsymbol{\eta}|_{\gamma_\varepsilon^k} = 0$ for $k \neq j$, we get

$$\int\limits_{\gamma_\varepsilon^j} (p - p_j)\, \boldsymbol{\eta} \cdot \mathbf{n}\, \mathrm{d}x^{(e_j)\prime} = 0. \tag{2.8}$$

The normal traces of functions $\boldsymbol{\eta}$ from $K_0^{1,2}(B_\varepsilon)$ satisfying the conditions $\boldsymbol{\eta}|_{\gamma_\varepsilon^k} = 0$ for $k \neq j$ compile the whole space $W^{1/2,2}(\gamma_\varepsilon^j) \cap \widehat{L}^2(\gamma_\varepsilon^j)$.

Since $C_0^\infty(B_\varepsilon) \cap L^2(B_\varepsilon)$ is dense in $W^{1/2,2}(\gamma_\varepsilon^j) \cap \widehat{L}^2(\gamma_\varepsilon^j)$ from the last equality it follows that

$$(p - p_j)|_{\gamma_\varepsilon^j} = c_j,$$

where c_j is a constant (similarly to [16], [17]). Using these relations and taking now in (2.8) a test function $\boldsymbol{\eta} \in \widehat{K}^{1,2}(B_\varepsilon)$ such that $\int\limits_{\gamma_\varepsilon^k} \boldsymbol{\eta} \cdot \mathbf{n}\, \mathrm{d}x' = 0$ for $k \neq j$ and $k \neq N$, $\int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n}\, \mathrm{d}x' = 1$ and $\int\limits_{\gamma_\varepsilon^N} \boldsymbol{\eta} \cdot \mathbf{n}\, \mathrm{d}x' = -1$, we get

$$\sum_{j=N_1+1}^{N} c_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n}\, \mathrm{d}x' = c_j - c_N \Rightarrow c_j = c_N.$$

Thus,

$$c_j = c_N \quad \forall j = N_1 + 1, ..., N. \tag{2.9}$$

Since the pressure p in the weak formulation is defined up to an additive constant, we may set $c_j = c_N = 0, \;\; j = N_1 + 1, ..., N$. Then from (2.9) we have

$$p|_{\gamma_\varepsilon^j} = p_j, \quad j = N_1 + 1, ..., N.$$

These considerations justify the definition of the weak solution.

Theorem 2.1. *For arbitrary $\mathbf{f} \in L^2(B_\varepsilon)$ and $p_j^* \in \mathbb{R}, \; j = N_1 + 1, ..., N - 1$ problem (2.3) admits a unique weak solution $\mathbf{u} \in \widehat{K}^{1,2}(B_\varepsilon)$. There holds the estimate*

$$\|\nabla \mathbf{u}\|_{L^2(B_\varepsilon)} \leq c\Big(\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p_j^*| + \varepsilon \|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big) \tag{2.10}$$

with the constant c independent of ε.

Proof. Define in $\widehat{K}^{1,2}(B_\varepsilon)$ the inner product $[\mathbf{u},\boldsymbol{\eta}] = \int\limits_{B_\varepsilon} \frac{1}{2}\nu_\varepsilon(x)D\mathbf{u}\cdot D\boldsymbol{\eta} + K_\varepsilon(x)\mathbf{u}\cdot\boldsymbol{\eta}\,\mathrm{d}x$. Due to the Poincaré-Friedrichs inequality and the Korn inequality the corresponding norm is equivalent to the Dirichlet norm with a constant independent of ε. Using the Cauchy-Schwarz, the trace theorem and the Poincaré-Friedrichs inequality we derive the estimates

$$\begin{aligned} \Big|\sum_{j=N_1+1}^{N-1} p_j^* \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}}\,\mathrm{d}x'\Big| &\le \sum_{j=N_1+1}^{N-1} |p_j^*|\Big(\int\limits_{\gamma_\varepsilon^j}|\boldsymbol{\eta}|^2\,\mathrm{d}x\Big)^{1/2}|\gamma_\varepsilon^j|^{1/2} \\ &\le c\varepsilon^{n/2}\sum_{j=N_1+1}^{N-1}|p_j^*|\|\nabla\boldsymbol{\eta}\|_{L^2(B_\varepsilon)}. \end{aligned} \tag{2.11}$$

Finally,

$$\begin{aligned} \Big|\int\limits_{B_\varepsilon}\mathbf{f}\cdot\boldsymbol{\eta}\,\mathrm{d}x\Big| &\le \Big(\int\limits_{B_\varepsilon}|\mathbf{f}|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int\limits_{B_\varepsilon}|\boldsymbol{\eta}|^2\,\mathrm{d}x\Big)^{1/2} \\ &\le c\varepsilon\|\mathbf{f}\|_{L^2(B_\varepsilon)}\|\nabla\boldsymbol{\eta}\|_{L^2(B_\varepsilon)}. \end{aligned} \tag{2.12}$$

Consider the linear functional $\Phi : \widehat{K}^{1,2}(B_\varepsilon) \to \mathbb{R}$ defined by

$$\Phi(\boldsymbol{\eta}) = -\sum_{j=N_1+1}^{N-1} p_j^* \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}}\,\mathrm{d}x' + \int\limits_{B_\varepsilon}\mathbf{f}\cdot\boldsymbol{\eta}\,\mathrm{d}x.$$

Due to estimates (2.11) and (2.12) it is bounded in the Hilbert space $\widehat{K}^{1,2}(B_\varepsilon)$ and so, by the Riesz theorem there exists a unique function $\mathbf{u} \in \widehat{K}^{1,2}(B_\varepsilon)$ such that for all $\boldsymbol{\eta} \in \widehat{K}^{1,2}(B_\varepsilon)$,

$$\Phi(\boldsymbol{\eta}) = [\mathbf{u}\cdot\boldsymbol{\eta}].$$

So, this function $\mathbf{u}$ is a unique solution to the weak formulation (2.3). Due to estimates (2.11) and (2.12) it satisfies estimate (2.10). □

Remark 2.2. Notice also that the weak solution $\mathbf{u}$ of problem (2.3) belongs to the space $W^{2,2}(B_\varepsilon)$ whenever $\mathbf{f} \in L^2(B_\varepsilon)$. The corresponding pressure belongs to $W^{1,2}(B_\varepsilon)$. This can be proved extending the solutions and the data by reflection over the sections γ_ε^j to a larger domain (see [9], [1]).

Remark 2.3. We will also consider problem (2.3) with $\mathbf{f} \in W^{-1,2}(B_\varepsilon)$ such that

$$\mathbf{f} = \mathbf{f}_0 - \sum_{i=1}^{n}\frac{\partial \mathbf{f}_i}{\partial x_i}, \quad \mathbf{f}_i \in L^2(B_\varepsilon),\ i = 0,1,...,n, \tag{2.13}$$

$\mathbf{f}_i(x) = 0$ in the neighborhood of bases γ_ε^j. A weak solution is defined as $\mathbf{u} \in \widehat{K}^{1,2}(B_\varepsilon)$ satisfying

$$\begin{aligned} &\int\limits_{B_\varepsilon}\Big(\tfrac{1}{2}\nu_\varepsilon(x)D\mathbf{u}\cdot D\boldsymbol{\eta} + K_\varepsilon(x)\mathbf{u}\cdot\boldsymbol{\eta}\Big)\mathrm{d}x \\ = -&\sum_{j=N_1+1}^{N} p_j\int\limits_{\gamma_\varepsilon^j}\boldsymbol{\eta}_{\mathbf{n}}\,\mathrm{d}s + \int\limits_{B_\varepsilon}\Big(\mathbf{f}_0\cdot\boldsymbol{\eta} + \textstyle\sum_{i=1}^n \mathbf{f}_i\cdot\frac{\partial\boldsymbol{\eta}}{\partial x_i}\Big)\mathrm{d}x \end{aligned} \tag{2.14}$$

for every $\boldsymbol{\eta} \in \widehat{K}^{1,2}(B_\varepsilon)$. In this case Theorem 2.1 can be easily generalized with the estimate

$$\|\nabla \mathbf{u}\|_{L^2(B_\varepsilon)} \leq c\Big(\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p_j^*| + \varepsilon \|\mathbf{f}_0\|_{L^2(B_\varepsilon)} + \sum_{i=1}^{n} \|\mathbf{f}_i\|_{L^2(B_\varepsilon)}\Big) \tag{2.15}$$

with the constant c independent of ε.

Let us now recover and estimate the pressure p corresponding to the weak solution $\mathbf{u} \in \widehat{W}^{1,2}(B_\varepsilon)$ of problem (2.3). To do this, we need to prove that a linear bounded functional $\mathcal{L}$ defined on the space $\widehat{W}^{1,2}(B_\varepsilon)$ and vanishing on the subspace $\widehat{K}^{1,2}(B_\varepsilon)$ can be represented in the form $\mathcal{L}(\eta) = \int_{B_\varepsilon} p(x) \mathrm{div} \boldsymbol{\eta}(x) \mathrm{d}x$, where $p \in L^2(B_\varepsilon)$. First of all, let us study the divergence equation in the space $\widehat{W}^{1,2}(B_\varepsilon)$ with the right-hand side from $L^2(B_\varepsilon)$. First of all recall the well-known result on the divergence equation.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$. Denote

$$\widehat{L}^2(\Omega) = \{h \in L^2(\Omega) : \int\limits_\Omega h(x) \mathrm{d}x = 0\}.$$

Consider the following problem (divergence equation): for given $h \in \widehat{L}^2(\Omega)$, find a vector field $\mathbf{w} \in \mathring{W}^{1,2}(\Omega)$ satisfying the equation

$$\mathrm{div} \mathbf{w} = h, \tag{2.16}$$

and the estimate

$$\|\nabla \mathbf{w}\|_{L^2(\Omega)} \leq c \|h\|_{L^2(\Omega)} \tag{2.17}$$

with some constant c depending only on Ω.

Notice that the condition $\int\limits_\Omega h(x)\, \mathrm{d}x = 0$ is necessary for the solvability of the above problem. Indeed, by the Stokes formula,

$$\int_\Omega h \, \mathrm{d}x = \int_\Omega \mathrm{div} \mathbf{w} \, \mathrm{d}x = \int_{\partial\Omega} \mathbf{w} \cdot \mathbf{n} \, \mathrm{d}s = 0.$$

The following lemma was proved in [21].

Lemma 2.4. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with Lipschitz boundary. Then (2.16) admits at least one solution $\mathbf{w} \in \mathring{W}^{1,2}(\Omega)$ satisfying estimate (2.17).*

The next lemma is proved in [31] (Lem. 3.6).

Lemma 2.5. *Consider equation (2.16) set in $\Omega = \Pi_\varepsilon^{(e)}$. Then (2.16) admits at least one solution $\mathbf{w} \in \mathring{W}^{1,2}(\Pi_\varepsilon^{(e)})$ satisfying estimate*

$$\|\nabla \mathbf{w}\|_{L^2(\Pi_\varepsilon^{(e)})} \leq c\varepsilon^{-1} \|h\|_{L^2(\Pi_\varepsilon^{(e)})}. \tag{2.18}$$

This lemma yields the following assertion.

Lemma 2.6. *Consider equation (2.16) set in $\Omega = B_\varepsilon$. Then (2.16) admits at least one solution $\mathbf{w} \in \mathring{W}^{1,2}(B_\varepsilon)$ satisfying estimate*

$$\|\nabla \mathbf{w}\|_{L^2(B_\varepsilon)} \leq c\varepsilon^{-3/2}\|h\|_{L^2(B_\varepsilon)} \tag{2.19}$$

Its formulation follows [31] (Lem. 3.7). However, in [31] there is a misprint in the constant which we fixed here. Its proof is in Appendix B.

Let $\sigma \subset \mathbb{R}^{n-1}$ be a bounded domain with Lipschitz boundary and $\Pi = \big\{x : x' \in \sigma, 0 < x_n < 1\big\}$, be a cylinder in $\mathbb{R}^n$. First, we consider the divergence equation in the cylinder Π.

Lemma 2.7. *Let $h \in L^2(\Pi)$. Then the divergence equation*

$$\mathrm{div}\mathbf{w}(x) = h(x), \quad x \in \Pi. \tag{2.20}$$

admits at least one solution $\mathbf{w} \in W^{1,2}(\Pi)$ vanishing on the part of the boundary $\partial\Pi \setminus \{x : x_n = 0\}$ and $\mathbf{w}_{\boldsymbol{\tau}}|_{x_n=0} = 0$. The solution satisfies the estimate

$$\|\nabla \mathbf{w}\|_{L^2(\Pi)} \leq c\|h\|_{L^2(\Pi)}. \tag{2.21}$$

Proof. Let us extend h as an odd function to the larger cylinder $\widetilde{\Pi} = \big\{x : x' \in \sigma, -1 < x_n < 1\big\}$ (with respect to x_n). Consider in $\widetilde{\Pi}$ the following divergence equation:

$$\begin{cases} \mathrm{div}\ \mathbf{W} &= \tilde{h}(x), \quad x \in \widetilde{\Pi}, \\ \mathbf{W} &= 0, \qquad x \in \partial\widetilde{\Pi}, \end{cases} \tag{2.22}$$

where

$$\tilde{h}(x) = \begin{cases} h(x), & x \in \Pi, \\ -h(x', -x_n), & x \in \widetilde{\Pi} \setminus \Pi. \end{cases}$$

Note that $\int\limits_{\widetilde{\Pi}} \tilde{h}(x)\mathrm{d}x = 0$.

Then, according to Lemma 2.4, there exists the solution $\mathbf{W} \in \mathring{W}^{1,2}(B_\varepsilon)$ satisfying the estimate

$$\|\mathbf{W}\|_{W^{1,2}(\widetilde{\Pi})} \leq c\|\tilde{h}\|_{L^2(\widetilde{\Pi})} \leq c\|h\|_{L^2(\Pi)}\ .$$

Without loss of generality we can assume that $\mathbf{W}$ has the odd component $\mathbf{W}'$ and even component W_n (with respect to x_n). Indeed, if we have some solution $\mathbf{W}$ to this divergence equation (2.22), we can consider the function $\widetilde{\mathbf{W}}$ defined by relations

$$\widetilde{W}_n(x', x_n) = \frac{1}{2}(W_n(x', x_n) + W_n(x', -x_n)),$$

$$\widetilde{\mathbf{W}}'(x', x_n) = \frac{1}{2}(\mathbf{W}'(x', x_n) - \mathbf{W}'(x', -x_n)).$$

Then $\widetilde{\mathbf{W}}$ is still a solution to the divergence equation (2.22) and it satisfies the above parity conditions with respect to x_n. This implies that $\widetilde{\mathbf{W}}' = 0$ on the section $x_n = 0$, that is $\widetilde{\mathbf{W}}_{\boldsymbol{\tau}} = 0$ for $x_n = 0$.

The restriction $\mathbf{w}$ of $\widetilde{\mathbf{W}}$ on Π is a solution to problem (2.20). □

Lemma 2.8. *Let h be a function from $L^2(B_\varepsilon)$. Then the divergence equation*

$$\mathrm{div}\mathbf{w}(x) = h(x), \quad x \in B_\varepsilon, \tag{2.23}$$

admits at least one solution $\mathbf{w} \in \widehat{W}^{1,2}(B_\varepsilon)$, satisfying the estimate

$$\|\nabla \mathbf{w}\|_{L^2(B_\varepsilon)} \le c\varepsilon^{-3/2} \|h\|_{L^2(B_\varepsilon)}. \tag{2.24}$$

Proof. Let us fix a cylinder $\Pi_\varepsilon^{(e)}$ having the base γ_ε^N. Let us cut this cylinder at the distance ε from the base γ_ε^N. Denote the cut piece by C_ε. Introduce $\langle h \rangle_{B_\varepsilon} = \int\limits_{B_\varepsilon} h(x)\mathrm{d}x/\mathrm{mes}(B_\varepsilon)$ the mean value of h in B_ε. Let χ_A be the characteristic function of the set A. For the function $h_1 = h - \langle h \rangle_{B_\varepsilon} \chi_{C_\varepsilon}(x)$ we construct a solution of the divergence equation

$$\mathrm{div}\mathbf{w}_1(x) = h_1(x), \quad x \in B_\varepsilon, \tag{2.25}$$

from the space $\mathring{W}^{1,2}(B_\varepsilon)$.

According to Lemma 2.6, it satisfies the estimate

$$\|\nabla \mathbf{w}_1\|_{L^2(B_\varepsilon)} \le c\varepsilon^{-3/2} \|h_1\|_{L^2(B_\varepsilon)} \le c\varepsilon^{-3/2} \|h\|_{L^2(B_\varepsilon)}, \tag{2.26}$$

because

$$\|\langle h \rangle_{B_\varepsilon} \chi_{C_\varepsilon}(x)\|_{L^2(B_\varepsilon)} = \frac{1}{\mathrm{mes}B_\varepsilon} \Big| \int\limits_{B_\varepsilon} h\mathrm{d}x \Big| \sqrt{\mathrm{mes}C_\varepsilon} \le c\varepsilon^{1/2} \|h\|_{L^2(B_\varepsilon)}. \tag{2.27}$$

Now, in the cylinder C_ε we construct a solution $\mathbf{w}_0 \in \widehat{W}^{1,2}(C_\varepsilon)$ of the divergence equation

$$\mathrm{div}\mathbf{w}_0(x) = \langle h \rangle_{B_\varepsilon} \chi_{C_\varepsilon}(x), \quad x \in C_\varepsilon, \tag{2.28}$$

satisfying $\mathbf{w}_{\boldsymbol{\tau}}|_{\gamma_\varepsilon^N} = 0$ and the estimate

$$\|\nabla \mathbf{w}_0\|_{L^2(C_\varepsilon)} \le c \|\langle h \rangle_{B_\varepsilon} \chi_{C_\varepsilon}(x)\|_{L^2(C_\varepsilon)} \le c\varepsilon^{1/2} \|h\|_{L^2(B_\varepsilon)}, \tag{2.29}$$

The first inequality in (2.29) follows from Lemma 2.7 contracting $1/\varepsilon$-times the cylinder Π in (2.21), the second inequality follows from (2.27). Since, by the construction of Lemma 2.7, the vector-function $\mathbf{w}_0$ vanishes on the second base of the cylinder C_ε, it can be extended by zero into $B_\varepsilon \setminus C_\varepsilon$. So, $\mathbf{w}_0 \in \widehat{W}^{1,2}(B_\varepsilon)$. Finally, taking $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_0$, we finalize the proof. □

Theorem 2.9. *Let Φ be a linear bounded functional defined on the space $\widehat{W}^{1,2}(B_\varepsilon)$, $\boldsymbol{\eta} \mapsto \Phi(\boldsymbol{\eta})$ vanishing on the subspace $\widehat{K}^{1,2}(B_\varepsilon)$. Then there exists a unique function $p \in \hat{L}^2(B_\varepsilon)$ such that $\Phi(\boldsymbol{\eta})$ can be presented in a form $\int_{B_\varepsilon} p(x)\mathrm{div}\boldsymbol{\eta}(x)\mathrm{d}x$.*

Proof. In the proof of Lemma 2.6 for any $h \in L^2(B_\varepsilon)$ there was constructed a function $\mathbf{w}$, belonging the space $\widehat{W}^{1,2}(B_\varepsilon)$, satisfying the relation

$$\mathrm{div}\mathbf{w}(x) = h(x), \quad x \in B_\varepsilon, \tag{2.30}$$

and estimate (2.19). One can easily check that this construction defines a bounded linear operator M^{-1} from $L^2(B_\varepsilon)$ onto $\widehat{W}^{1,2}(B_\varepsilon)$.

Let us define the equivalence classes in $\widehat{W}^{1,2}(B_\varepsilon)$. We say that two functions $\mathbf{w}$ and $\mathbf{v}$ belong to the same class if they have the same divergence: $\mathrm{div}\mathbf{w} = \mathrm{div}\mathbf{v}$. We will call functions $\mathbf{w}$ belonging to the class W as representatives of this class. Define the sum $W + V$ of two classes W and V as the equivalence class containing the function $\mathbf{w} + \mathbf{v}$, where $\mathbf{w} \in W$ and $\mathbf{v} \in V$ are the representatives of W and V respectively. Also, define the product αW of the equivalence class W by a real number α as the equivalence class containing the function $\alpha\mathbf{w}$, where $\mathbf{w} \in W$ is a representative of the class W. So, we can consider the vector space of the equivalence classes (known in the literature as the quotient space). Introduce the inner product in this space: if W and V are two classes and $\mathbf{w} \in W$, $\mathbf{v} \in V$, then the inner product in this quotient space is defined as

$$(W, V) = \int_{B_\varepsilon} \mathrm{div}\mathbf{w} \cdot \mathrm{div}\mathbf{v}\mathrm{d}x. \tag{2.31}$$

One can easily check that this definition satisfies the axioms of the inner product (bilinearity, symmetry and positivity of the associated norm) and is stable with respect to the choice of the representatives $\mathbf{w}$ and $\mathbf{v}$ of the classes W and V respectively.

Consider the value $\Phi(\mathbf{w})$ for some $\mathbf{w} \in \widehat{W}^{1,2}(B_\varepsilon)$. Let W be its equivalence class corresponding to the divergence $h \in L^2(B_\varepsilon)$, *i.e.* $\mathbf{w} \in W$ and $M^{-1}h \in W$. Then $\Phi(\mathbf{w}) = \Phi(M^{-1}h)$, because $\mathbf{w} - M^{-1}h$ is divergence free, and so $\Phi(\mathbf{w} - M^{-1}h) = 0$. So, $\Phi(\mathbf{w})$ is uniquely defined for all functions of the class W, and so, one can consider Φ as a linear functional on the vector space of equivalence classes. As the functional Φ is bounded with respect to the norm of $\widehat{W}^{1,2}(B_\varepsilon)$ and the operator M^{-1} is also bounded,

$$|\Phi(M^{-1}h)| \leq c_\varepsilon \|h\|_{L^2(B\varepsilon)} = c_\varepsilon \|\mathrm{div}\mathbf{w}\|_{L^2(B\varepsilon)},$$

where c_ε is a constant depending on ε. So, Φ is bounded on the space of equivalence classes with inner product (2.31) and so, according to the Riesz theorem, there exists a unique equivalence class U such that for a representative $\mathbf{u}$ of U and for any $\mathbf{w} \in \widehat{W}^{1,2}(B_\varepsilon)$ it can be represented in a form of an inner product

$$\Phi(\mathbf{w}) = \int_{B_\varepsilon} \mathrm{div}\mathbf{u} \cdot \mathrm{div}\mathbf{w}\mathrm{d}x. \tag{2.32}$$

Taking now $p = \mathrm{div}\mathbf{u}$ we complete the proof of the existence of p. Its uniqueness follows from the uniqueness of the equivalence class U, so that for all $\mathbf{u}$ from U $\mathrm{div}\mathbf{u}$ is the same.

□

Now we can introduce another weak formulation for problem (2.3), namely, formulation "with pressure": find $\mathbf{u} \in \widehat{W}^{1,2}(B_\varepsilon)$ and $p \in L^2(B_\varepsilon)$ such that

$$\int\limits_{B_\varepsilon} \Big(\frac{1}{2}\nu_\varepsilon(x) D\mathbf{u} \cdot D\boldsymbol{\eta} + K_\varepsilon(x)\mathbf{u} \cdot \boldsymbol{\eta}\Big)\,\mathrm{d}x - \int\limits_{B_\varepsilon} p(x)\mathrm{div}\boldsymbol{\eta}\,\mathrm{d}x = -\sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n}\,\mathrm{d}s + \int\limits_{B_\varepsilon} \mathbf{f} \cdot \boldsymbol{\eta}\,\mathrm{d}x \tag{2.33}$$

holds for every $\boldsymbol{\eta} \in \widehat{W}^{1,2}(B_\varepsilon)$.

Theorem 2.10. *For arbitrary* $\mathbf{f} \in L^2(B_\varepsilon)$ *and* $p_j \in \mathbb{R},\ j = N_1+1, ..., N$*, problem* (2.3) *admits a unique weak solution* $(\mathbf{u}, p)$, $\mathbf{u} \in \widehat{W}^{1,2}(B_\varepsilon)$, $p \in L^2(B_\varepsilon)$*. The following estimate*

$$\varepsilon^{-3/2}\|\nabla \mathbf{u}\|_{L^2(B_\varepsilon)} + \|p\|_{L^2(B_\varepsilon)} \le c\varepsilon^{-3/2}\Big(\varepsilon^{n/2}\sum_{j=N_1+1}^{N}|p_j| + \varepsilon\|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big) \tag{2.34}$$

holds with the constant c *independent of* ε*.*

Proof. Applying Theorem 2.9 to the functional

$$\mathcal{L}(\boldsymbol{\eta}) = \int\limits_{B_\varepsilon}\Big(\frac{1}{2}\nu_\varepsilon(x)D\mathbf{u}\cdot D\boldsymbol{\eta} + K_\varepsilon(x)\mathbf{u}\cdot\boldsymbol{\eta}\Big)\,\mathrm{d}x + \sum_{j=N_1+1}^{N} p_j\int\limits_{\gamma_\varepsilon^j}\boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}s - \int\limits_{B_\varepsilon}\mathbf{f}\cdot\boldsymbol{\eta}\,\mathrm{d}x, \tag{2.35}$$

defined on $\boldsymbol{\eta} \in \widehat{W}^{1,2}(B_\varepsilon)$, we get the existence and uniqueness of the pressure $p \in L^2(B_\varepsilon)$ such that

$$\mathcal{L}(\boldsymbol{\eta}) = \int_{B_\varepsilon} p\,\mathrm{div}\,\boldsymbol{\eta}\,\mathrm{d}x.$$

So, $\mathbf{u}$ and p satisfy integral identity (2.33).

Evidently, applying estimates (2.11), (2.12), we get:

$$|\mathcal{L}(\boldsymbol{\eta})| \le \Big(\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)} + \varepsilon^{n/2}\sum_{j=N_1+1}^{N}|p_j| + \varepsilon\|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big)\|\nabla\boldsymbol{\eta}\|_{L^2(B_\varepsilon)}. \tag{2.36}$$

Using Lemma 2.8, we can take in (2.33) the test function $\boldsymbol{\eta}$ such that $\mathrm{div}\boldsymbol{\eta} = p(x)$ and

$$\|\nabla\boldsymbol{\eta}\|_{L^2(B_\varepsilon)} \le c\varepsilon^{-3/2}\|p\|_{L^2(B_\varepsilon)}.$$

Then (2.36) yields:

$$|\mathcal{L}(\boldsymbol{\eta})| = \|p\|^2_{L^2(B_\varepsilon)} \le c\varepsilon^{-3/2}\Big(\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)} + \varepsilon^{n/2}\sum_{j=N_1+1}^{N}|p_j| + \varepsilon\|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big)\|p\|_{L^2(B_\varepsilon)}, \tag{2.37}$$

and from (2.10) we get (2.34). □

Note that $\mathbf{u}$ is the same in both weak formulations.

Remark 2.11. As before if

$$\mathbf{f} = \mathbf{f}_0 - \sum_{i=1}^{n}\frac{\partial\mathbf{f}_i}{\partial x_i},\quad \mathbf{f}_i \in L^2(B_\varepsilon),\ i = 0, 1, ..., n, \tag{2.38}$$

$\mathbf{f}_i(x) = 0$ in the neighborhood of bases γ_ε^j, then the term $\varepsilon\|\mathbf{f}\|_{L^2(B_\varepsilon)}$ in (2.34) is replaced by $\varepsilon\|\mathbf{f}_0\|_{L^2(B_\varepsilon)} + \sum_{l=1}^{n}\|\mathbf{f}_l\|_{L^2(B_\varepsilon)}$.

In this case a weak solution is defined as $\mathbf{u} \in \widehat{K}^{1,2}(B_\varepsilon)$ satisfying

$$\begin{aligned}&\int\limits_{B_\varepsilon}\Big(\tfrac{1}{2}\nu_\varepsilon(x)D\mathbf{u}\cdot D\boldsymbol{\eta}+K_\varepsilon(x)\mathbf{u}\cdot\boldsymbol{\eta}\Big)\,\mathrm{d}x\\&=-\sum_{j=N_1+1}^{N}p_j\int\limits_{\gamma_\varepsilon^j}\boldsymbol{\eta}_{\mathbf{n}}\,\mathrm{d}s+\int\limits_{B_\varepsilon}\Big(\mathbf{f}_0\cdot\boldsymbol{\eta}+\textstyle\sum_{i=1}^{n}\mathbf{f}_i\cdot\frac{\partial\boldsymbol{\eta}}{\partial x_i}\Big)\,\mathrm{d}x\end{aligned} \tag{2.39}$$

for every $\boldsymbol{\eta} \in \widehat{K}^{1,2}(B_\varepsilon)$. Now, Theorem 2.1 can be easily generalized with the estimate

$$\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)}\le c\Big(\varepsilon^{n/2}\sum_{j=N_1+1}^{N-1}|p_j^*|+\varepsilon\|\mathbf{f}_0\|_{L^2(B_\varepsilon)}+\sum_{i=1}^{n}\|\mathbf{f}_i\|_{L^2(B_\varepsilon)}\Big) \tag{2.40}$$

with the constant c independent of ε.

2.1. Asymptotic expansion of the solution

In this section we describe the construction of the asymptotic expansion. Let $\zeta \in C^2(\mathbb{R})$ be even function independent of ε such that, $\zeta(t)=0$ if $|t|\le 1/3$, and $\zeta(t)=1$ if $|t|\ge 2/3$. Denote $e=e_{O_j}$ (the edge with the end O_j) and $x^{(e)}$ the Cartesian coordinates corresponding to the origin O_j and the edge e, *i.e.*, $x^{(e)}=\mathcal{P}^{(e)}(x-O_j)$, $\mathcal{P}^{(e)}$ is the orthogonal matrix relating the global coordinates x with the local ones $x^{(e)}$.

The asymptotic expansion of the velocity field is sought in the form:

$$\begin{aligned}\mathbf{u}^{(J)}(x)&=\textstyle\sum_{O_l,l=N_1+1,\dots,N;e=\overline{O_lO_{i_l}}}\zeta\Big(\dfrac{|e|-x_n^{(e)}}{3r_1\varepsilon}\Big)\mathbf{U}^{[e,J]}\Big(\dfrac{x^{(e)\prime}}{\varepsilon}\Big)\\&+\textstyle\sum_{e=\overline{O_\alpha O_\beta};\alpha,\beta\le N_1}\zeta\Big(\dfrac{x_n^{(e)}}{3r_1\varepsilon}\Big)\zeta\Big(\dfrac{|e|-x_n^{(e)}}{3r_1\varepsilon}\Big)\mathbf{U}^{[e,J]}\Big(\dfrac{x^{(e)\prime}}{\varepsilon}\Big)\\&+\sum_{l=1}^{N}\Big(1-\zeta\Big(\dfrac{|x-O_l|}{|e|_{min}}\Big)\Big)\mathbf{U}^{[BLO_l,J]}\Big(\dfrac{x-O_l}{\varepsilon}\Big),\end{aligned} \tag{2.41}$$

where the first sum is taken over all edges having a vertex as an end point (and with the origin of the local coordinate system at the vertex), the second sum is taken over all remaining edges, and all terms in these sums are extended by zero out of cylinders $\Pi_\varepsilon^{(e)}$; the terms of the third sum are extended by zero out of the corresponding bundles; functions $\mathbf{U}^{[e,J]}$ are the Poiseuille flow's velocities corresponding to the cylinders $\Pi_\varepsilon^{(e)}$ (they will be defined below), they are expanded in powers of ε; functions $\mathbf{U}^{[BLO_l,J]}$ are the boundary layer type functions exponentially decaying at the infinity, they as well are expanded in powers of ε:

$$\begin{cases}\mathbf{U}^{[e,J]}=\big(P^{(e)}\big)^t(0,\dots,0,\widetilde{U}^{[e,J]})^t,\\[1ex]\mathbf{U}_{(j)}^{(e)}=\big(P^{(e)}\big)^t(0,\dots,0,\widetilde{U}_j^{(e)})^t,\quad j=0,1,\dots,J,\\[1ex]\widetilde{U}^{[e,J]}(y^{(e)\prime})=\displaystyle\sum_{j=0}^{J}\varepsilon^j\widetilde{U}_j^{(e)}(y^{(e)\prime}),\\[1ex]\mathbf{U}^{[BLO_l,J]}(y)=\displaystyle\sum_{j=0}^{J}\varepsilon^j\mathbf{U}_j^{[BLO_l]}(y).\end{cases} \tag{2.42}$$

The asymptotic expansion of the pressure for every half-cylinder $\Pi_\varepsilon^{(e)}$, $x_n < |e|/2$, corresponding to the edge $e = \overline{O_l O_{i_l}}$, $l = N_1+1, ..., N$, (O_l is the origin of the local coordinate system) is sought in the form:

$$p^{(J)}(x) = -s^{(e)} x_n^{(e)} + a^{(e)}, \tag{2.43}$$

and on every half-bundle HB_{O_l}, $l = 1, ..., N_1$, (O_l is the origin of the local coordinate system) we define:

$$\begin{aligned} p^{(J)}(x) = \textstyle\sum_{e\subset\mathcal{B}_l} \zeta\Big(\frac{x_n^{(e)}}{3r_1\varepsilon}\Big)\Big(-s^{(e)}x_n^{(e)} + a^{(e)} - a^{(e_s)}\Big) + a^{(e_s)} \\ + \frac{1}{\varepsilon}\Big(1 - \zeta\Big(\frac{|x-O_l|}{|e|_{min}}\Big)\Big) P^{[BLO_l,J]}\Big(\frac{x-O_l}{\varepsilon}\Big), \end{aligned} \tag{2.44}$$

where the terms of the sum are extended by zero out of cylinders $\Pi_\varepsilon^{(e)}$,

$$s^{(e)} = \frac{1}{\varepsilon^2}\sum_{j=0}^J \varepsilon^j s_j^{(e)}, \quad a^{(e)} = \frac{1}{\varepsilon^2}\sum_{j=0}^J \varepsilon^j a_j^{(e)} \tag{2.45}$$

and

$$P^{[BLO_l,J]}(y) = \sum_{j=0}^J \varepsilon^j P_j^{[BLO_l,J]}(y). \tag{2.46}$$

Here e_s is the selected edge of the bundle (arbitrary chosen among edges of the bundle) and the local coordinates $x^{(e)}$ are redefined so that all of them have the same origin O_l.

The algorithm of successive determination of the terms in asymptotic expansions (2.41), (2.43) is as follows.

The base case. For any edge e define $U_0^{(e)}(y^{(e)\prime})$ as the solution of the Dirichlet problem

$$\begin{gathered} -\nu^{(0)}\Delta_{(y^{(e)\prime})} U_0^{(e)}(y^{(e)\prime}) = 1, \quad y^{(e)\prime} \in \sigma^{(e)}; \\ U_0^{(e)}|_{\partial\sigma^{(e)}} = 0, \end{gathered}$$

and define κ_e as the integral $\int_{\sigma^{(e)}} U_0^{(e)}(y^{(e)\prime})\mathrm{d}y^{(e)\prime}$.

Solve the conductivity problem on the graph for the function p_0:

$$\left\{ \begin{aligned} -\kappa_e \frac{\partial^2 p_0^{(e)}}{\partial {x_n^{(e)}}^2}(x_n^{(e)}) = 0, \quad x_n^{(e)} \in (0, |e|), \\ -\sum_{e: O_l\in e} \kappa_e \frac{\partial p_0^{(e)}}{\partial x_n^{(e)}}(0) = 0, \quad l = 1, ..., N_1, \\ p_0^{(e)}(0) = c^l, \quad l = N_1+1, ..., N, \\ p_0^{(e)}(0) = p_0^{(e_s)}(0), \quad \forall e \subset \mathcal{B}_l, \quad l = N_1. \end{aligned} \right. \tag{2.47}$$

Here the local coordinates $x^{(e)}$ are redefined so that all of them have the same origin O_l. So, p_0 is a continuous function on the graph. Indeed, the last condition of this problem means that the values of the function p_0 for the values of local variables $x_n^{(e)} = 0$ associated to all edges e of the bundle $\mathcal{B}_l$ are the same. Note that introducing the weak (variational) formulation in appropriate spaces and applying the Lax-Milgram lemma as in the first part of [32] one can prove the existence and uniqueness of the solution of this problem.

Solving the above conductivity problem, we define for every edge e the constants $s_0^{(e)}$ and $a_0^{(e)}$ such that

$$p_0^{(e)}(x^{(e)}) = -s_0^{(e)} x_n^{(e)} + a_0^{(e)}$$

and the velocity

$$\widetilde{U}_{(0)}^{(e)}(y^{(e)\prime}) = s_0^{(e)} U_0^{(e)}(y^{(e)\prime}), \quad \mathbf{U}_0^{(e)}(y^{(e)\prime}) = \big(\mathcal{P}^{(e)}\big)^t (0,\dots,0,\widetilde{U}_{(0)}^{(e)})^t (y^{(e)\prime}). \tag{2.48}$$

Introduce the notation:

$$\mathcal{D}^{(l)} = -\mathrm{div}_y\Big(\nu^{(l)}(y) D_y \,\cdot\Big),$$

where functions $\nu^{(l)}$ and $K^{(l)}$ are extended out of the ball $B(0,R)$ by $\nu^{(0)}$ and zero respectively. Now to compensate the residual due to the multiplication of the Poiseuille flow by a cut-off function, we consider the boundary layer correctors and problems for these correctors set in a dilated bundle of semi-infinite cylinders Ω_l:

For $l = 1, ..., N_1$ the boundary layer problem for $(\mathbf{U}_0^{[BLO_l]}(y), P_0^{[BLO_l]}(y))$ is:

$$\begin{gathered} \mathcal{D}^{(l)} \mathbf{U}_0^{[BLO_l]} + \nabla_y P_0^{[BLO_l]} = \mathbf{f}_0^{[REGO_l]} + \mathbf{f}_0^{[BLO_l]}, \quad y \in \Omega_l, \\ \mathrm{div}_y \mathbf{U}_0^{[BLO_l]} = h_0^{[REGO_l]}, \quad y \in \Omega_l, \\ \mathbf{U}_0^{[BLO_l]}|_{\partial\Omega_l} = 0, \end{gathered} \tag{2.49}$$

where

$$\begin{aligned} &\mathbf{f}_0^{[REGO_l]}(y) \\ &= -\sum_{e:O_l\in e} \Big\{ s_0^{(e)} \Big(-\nu^{(0)} \Delta_y \Big(\zeta\Big(\frac{y_n^{(e)}}{3r_1}\Big) \big(\mathcal{P}^{(e)}\big)^t (0,\dots,0,U_0^{[e]}(y^{(e)\prime}))^* \Big) \\ &-\nabla_y \Big(\zeta\Big(\frac{y_n^{(e)}}{3r_1}\Big) y_n^{(e)} \Big) \Big) + (a_1^{(e)} - a_1^{(e_s)}) \; \nabla_y \Big(\zeta\Big(\frac{y_n^{(e)}}{3r_1}\Big) \Big) \Big\}, \end{aligned} \tag{2.50}$$

$$\mathbf{f}_0^{[BLO_l]}(y) = 0, \tag{2.51}$$

$$h_0^{[REGO_l]}(y) = \mathrm{div}_y \sum_{e:O_l\in e} \Big\{ s_0^{(e)} \zeta\Big(\frac{y_n^{(e)}}{3r_1}\Big) \big(\mathcal{P}^{(e)}\big)^t (0,\dots,0,U_0^{[e]}(y^{(e)\prime}))^t \Big\}. \tag{2.52}$$

Here the sum $\sum\limits_{e:O_l\in e}$ is taken over all edges e having ends in the node O_l and the terms are extended by zero out of each cylinder $\Pi_\varepsilon^{(e_j)}$. Here we have an unknown quantity in the right-hand side, the constant $a_1^{(e)} - a_1^{(e_s)}$ is unknown. Let us address this term. Denote by $(\mathbf{U}_0^{[BLO_l]}(y), \widehat{P}_0^{[BLO_l]}(y))$ the solution of problem (2.49) without the last term $(a_1^{(e)} - a_1^{(e_s)})\nabla_y\Big(\zeta\Big(\frac{y_n^{(e)}}{3r_1}\Big)\Big)$ in $\mathbf{f}_0^{[REGO_l]}(y)$ (since this term is of gradient form, the solutions differ only by the pressure components). The right-hand sides of system (2.49) have compact supports. Therefore, according to results of Propositions A.1 and A.2 [33], the pressure $\widehat{P}_0^{[BLO_l]}(y)$ exponentially stabilizes in each outlet (corresponding to the edge e) to a constant, say $\widehat{a}_0^{[BLO_l,e]}$, in the sense of integral estimates

$$\lim_{k\to+\infty} \mathrm{e}^{2\beta k} \int_{\{y_n^{(e)}\in(k,k+1)\}\cap\Omega_l} (\widehat{P}_0^{[BLO_l]}(y) - \widehat{a}_0^{[BLO_l,e]})^2 \,\mathrm{d}y = 0, \quad \beta > 0. \tag{2.53}$$

This constant can be set equal to zero at one of the outlets, corresponding to the edge e_s. Then we define the constants $a_1^{(e)} - a_1^{(e_s)} = \widehat{a}_0^{[BLO_l,e]}$ and define the boundary layer corrector for the pressure as

$$P_0^{[BLO_l]}(y) = \widehat{P}_0^{[BLO_l]}(y) - \sum_{e:O_l\in e,\, e\neq e_s} \zeta\Big(\frac{y_n^{(e)}}{3r_1}\Big)\widehat{a}_0^{[BLO_l,e]}.$$

Consider now the conductivity problem of rank 1 on the graph for the function p_1:

$$\begin{cases} -\kappa_e \dfrac{\partial^2 p_1^{(e)}}{\partial {x_n^{(e)}}^2}(x_n^{(e)}) = 0, \quad x_n^{(e)} \in (0,|e|), \\ -\sum\limits_{e:O_l\in e} \kappa_e \dfrac{\partial p_1^{(e)}}{\partial x_n^{(e)}}(0) = 0, \quad l = 1,...,N_1, \\ p_1^{(e)}(x_n^{(e)} = 0) = \widehat{a}_0^{[BLO_{l'},e]}, \quad l = N_1+1,...,N, \\ p_1^{(e)}(0) - p_1^{(e_s)}(0) = \widehat{a}_0^{[BLO_l,e]}, \quad \forall e \subset \mathcal{B}_l, e \neq e_s, \end{cases} \tag{2.54}$$

where e_s is the selected edge of the bundle. Here and below $O_{l'}$ is the node connected by an edge $e = O_{l'}O_l$ with the vertex O_l, $\widehat{a}_0^{[BLO_{l'},e]}$ is the limit of $\widehat{P}_0^{[BLO_{l'},e]}(y)$ at the outlet corresponding to the edge $e = O_{l'}O_l$. So, in this problem on the graph the solution may be discontinuous at the nodes. Namely, at each node O_l there are prescribed jumps of $p_1^{(e)}$ between the edges e and e_s of the bundle. This problem as well has a unique solution p_1. Note that problem (2.54) can be reduced to a problem on the graph with a right-hand side for a continuous unknown function as in [32].

Now, constants $s_1^{(e)}$ and $a_1^{(e)}$ are known: $p_1^{(e)}(x_n^{(e)}) = -s_1^{(e)}x_n^{(e)} + a_1^{(e)}$, and we can completely determine the pressure in the boundary layer problem (2.49):

$$P_0^{[BLO_l]}(y) = \widehat{P}_0^{[BLO_l]}(y) - \sum_{e:O_l\in e,\, e\neq e_s} \zeta\Big(\frac{y_n^{(e)}}{3r_1}\Big)\widehat{a}_0^{[BLO_l,e]}.$$

Suppose that all terms of expansion (2.41)–(2.46) corresponding to the rank less or equal to $j-1$ are known, and that the macroscopic pressure p_j on the graph is known as well. Let us describe the passage from the rank $j-1$ to the rank j.

Step 1. As the macroscopic pressure on the graph p_j is known, define for every edge e constants $s_j^{(e)}$ and $a_j^{(e)}$ such that

$$p_j^{(e)}(x^{(e)}) = -s_j^{(e)}x_n^{(e)} + a_j^{(e)}$$

and

$$\begin{aligned} \widetilde{U}_{(j)}^{(e)}(y^{(e)\prime}) &= s_j^{(e)} U_0^{(e)}(y^{(e)\prime}), \\ \mathbf{U}_j^{(e)} &= \big(\mathcal{P}^{(e)}\big)^t (0,\dots,0,\widetilde{U}_{(j)}^{(e)})^t. \end{aligned} \tag{2.55}$$

Step 2. The boundary layer solution is a pair $\big(\mathbf{U}_j^{[BLO_l]}, P_j^{[BLO_l]}\big)$ solving the following Stokes system in Ω_l, $l = 1, ..., N_1$:

$$\begin{gathered}\mathcal{D}_y^{(l)}\mathbf{U}_j^{[BLO_l]} + \nabla_y P_j^{[BLO_l]} = \mathbf{f}_j^{[REGO_l]} + \mathbf{f}_j^{[BLO_l]},\\ \mathrm{div}_y \mathbf{U}_j^{[BLO_l]} = h_j^{[REGO_l]},\\ \mathbf{U}_j^{[BLO_l]}|_{\partial\Omega_l} = 0, \quad j = 0, ..., J,\end{gathered} \tag{2.56}$$

where

$$\begin{aligned}\mathbf{f}_j^{[REGO_l]}(y^{(e)}) = -\sum_{e:O_l\in e}\Big\{ & -\nu\Delta_y\Big[\zeta\Big(\frac{y_n^{(e)}}{3r_1}\Big)\mathbf{U}_j^{[e]}(y^{(e)\prime})\Big]\\ & +\nabla_y\Big[\zeta\Big(\frac{y_n^{(e)}}{3r_1}\Big)\big(-s_j^{(e)}y_n^{(e)}\big)\Big]\\ & +\widehat{a}_j^{[BLO_l,e]}\ \nabla_y\zeta\Big(\frac{y_n^{(e)}}{3r_1}\Big)\Big\}\end{aligned} \tag{2.57}$$

(for $j = J$ the coefficient $\widehat{a}_j^{[BLO_l,e]}(t)$ is omitted),

$$\mathbf{f}_j^{[BLO_l]}(y^{(e)}) = K^{(l)}\mathbf{U}_{j-2}^{[BLO_l]}, \tag{2.58}$$

$$h_j^{[REGO_l]}(y,t) = -\sum_{e:O_l\in e}\mathrm{div}_y\Big(\zeta\Big(\frac{y_n^{(e)}}{3r_1}\Big)\mathbf{U}_j^{[e]}(y^{(e)\prime},t)\Big). \tag{2.59}$$

Here the sum $\sum\limits_{e:O_l\in e}$ is taken over all edges e having ends in the node O_l, the terms of the sum are extended by zero out of cylinders $\Pi_\varepsilon^{(e)}$ and by convention, the terms with the negative subscripts j are equal to zero. Note that by construction, $\mathrm{supp}\mathbf{f}_j^{[REGO_l]} \cap \mathrm{supp}K^{(l)} = \emptyset$.

First, we find the couple $\big(\mathbf{U}_j^{[BLO_l]}, \widehat{P}_j^{[BLO_l]}\big)$ which is the solution to the same problem (2.56) without the last term in the definition of $\mathbf{f}_j^{[REGO_l]}$ (see (2.57)). It can be proved by induction, using Theorems A.1 and A.2 [33], that $\mathbf{U}_j^{[BLO_l]}$ exponentially tends to zero as $|y| \to +\infty$, while the corresponding pressure function $\widehat{P}_j^{[BLO_l]}$ stabilizes in outlets to infinity to some constants $\widehat{a}_j^{[BLO_l,e]}$ in the sense of (2.53); these constants may be different for different outlets. Since the pressure function is defined up to an additive constant, we can fix the limit constant equal to zero for the outlet corresponding to the selected edge e_s.

Step 3. Solve the conductivity problem on the graph for the function $p_{j+1}^{(e)}$ ($j < J$):

$$\begin{gathered}-\kappa_e\frac{\partial^2 p_{j+1}^{(e)}}{\partial {x_n^{(e)}}^2}(x_n^{(e)}) = 0, \quad x_n^{(e)} \in (0, |e|),\\ -\sum_{e:O_l\in e}\kappa_e\frac{\partial p_{j+1}^{(e)}}{\partial x_n^{(e)}}(0) = 0, \quad l = 1, ..., N_1,\\ p_{j+1}^{(e)}(0) = \widehat{a}_j^{[BLO_{l'},e]}, l = N_1+1, ..., N,\\ p_{j+1}^{(e)}(0) - p_{j+1}^{(e_s)}(0) = \widehat{a}_j^{[BLO_l,e]}, \quad \forall e \subset \mathcal{B}_l, e \neq e_s,\end{gathered}$$

where the local coordinates $x^{(e)}$ are redefined so that all of them have the same origin O_l.

Step 4. Finally, we find the pressure $P_j^{[BLO_l]}(y)$ in boundary layer problem (2.56), (2.57) for $l = 1, ..., N_1$:

$$P_j^{[BLO_l]}(y) = \widehat{P}_j^{[BLO_l]}(y) - \sum_{e:O_l \in e,\ e \neq e_s} \zeta\Big(\frac{y_n^{(e_{\alpha_m})}}{3r_1}\Big)\widehat{a}_j^{[BLO_l,e]}.$$

This step finalizes the passage from j to $j+1$.

2.2. Residual

Consider the asymptotic expansion $\big(\mathbf{u}^{(J)}, p^{(J)}\big)$ of order J (see (2.41), (2.43)). By construction

$$\mathbf{u}^{(J)} \in W^{1,2}(B_\varepsilon), p^{(J)} \in L^2(B_\varepsilon). \tag{2.60}$$

Put $\mathcal{L}(\mathbf{u}, p) = -\mathrm{div}\Big(\nu_\varepsilon(x) D\mathbf{u}\Big) + K_\varepsilon(x)\mathbf{u} + \nabla p$. Let us calculate $\mathcal{L}(\mathbf{u}^{(J)}, p^{(J)})$ in a half-bundle HB_{O_l}, $l = 1, ..., N_1$. We obtain

$$\mathbf{f}^{(J)}(x) = \mathcal{L}(\mathbf{u}^{(J)}, p^{(J)})$$

$$= \varepsilon^{J-1} K^{(l)} \mathbf{U}_{J-1}^{[BLO_l]} + \varepsilon^J K^{(l)} \mathbf{U}_J^{[BLO_l]} + \varepsilon^{J-2} \sum_{e:O_l \in e} \widehat{a}_J^{[BLO_l,e]} \nabla_y \zeta\Big(\frac{y_n^{(e)}}{3r_1}\Big)$$

$$-\Big\{\mathcal{L}\Big(\zeta\Big(\frac{|x-O_l|}{|e|_{min}}\Big)\mathbf{U}^{[BLO_l,J]}(y), \zeta\Big(\frac{|x-O_l|}{|e|_{min}}\Big)P^{[BLO_l,J]}(y)\Big)\chi(x)\Big\}.$$

Here $y = \dfrac{x - O_l}{\varepsilon}$; $y^{(e)} = \dfrac{x^{(e)}}{\varepsilon}$; $\chi = \chi_{\mathrm{supp}\left(1-\zeta\left(\frac{|x-O_l|}{|e|_{min}}\right)\right)}$ is the characteristic function of the set $\mathrm{supp}\Big(1 - \zeta\Big(\frac{|x-O_l|}{|e|_{min}}\Big)\Big)$. As before the terms of the sums $\sum_{e:O_l \in e}$ are extended by zero out of cylinders $\Pi_\varepsilon^{(e)}$.

Here the first line of the right-hand side is the residual of the term $K^{(l)}\mathbf{u}^{(J)}$, the second line of the right-hand side comes from the pressure gradient term; this term is the only one that was not compensated by the boundary layer-in-space problems. The last line is the residual generated by the multiplication of the boundary layer correctors by the cut-off function $\zeta\big(\frac{|x-O_l|}{|e|_{min}}\big)$. Notice that terms appearing in this last line exponentially vanish because in the set $\mathrm{supp}\big(1 - \zeta\big(\frac{|x-O_l|}{|e|_{min}}\big)\big)$ (where $\chi \neq 0$) the order of this term in L^2-norm is $O(\mathrm{e}^{-c_1/\varepsilon})$ with some positive constant c_1 (see the Appendix in [33]). From the obtained formulas it follows that

$$\|\mathbf{f}^{(J)}\|_{L^2(B_\varepsilon)} = \|\mathcal{L}(\mathbf{u}^{(J)}, p^{(J)})\|_{L^2(B_\varepsilon)} = O(\varepsilon^{J-2}). \tag{2.61}$$

In the vertex associated cylinders B_{O_l}, $l = N_1 + 1, ..., N$, the residual is equal to zero.

Let us calculate the divergence of $\mathbf{u}^{(J)}$. In any half-bundle we have

$$\mathrm{div}\mathbf{u}^{(J)} = -\nabla\zeta\Big(\frac{|x-O_l|}{|e|_{min}}\Big) \cdot \mathbf{U}^{[BLO_l,J]}(y) = h^{(J)}(y).$$

Obviously, $h^{(J)} \in W^{1,2}(B_\varepsilon)$. Since the support of the function $\nabla\zeta\left(\frac{|x-O_l|}{|e|_{min}}\right)$ belongs to the middle third of every cylinder, there the relations

$$\|h^{(J)}\|_{W^{1,2}(B_\varepsilon)} = O(\mathrm{e}^{-c_2/\varepsilon}) \tag{2.62}$$

hold for some $c_2 > 0$.

It is easy to see that

$$\int_{B_\varepsilon} h^{(J)}(y)\,\mathrm{d}y = 0.$$

Therefore, by Lemma 2.6, there exists a vector field $\mathbf{w}^{(J)} \in \mathring{W}^{1,2}(B_\varepsilon)$ such that $\mathrm{div}\mathbf{w}^{(J)} = -h^{(J)}$. Moreover, the estimates

$$\|\mathbf{w}^{(J)}\|_{W^{1,2}(B_\varepsilon)} \le \varepsilon^{-3/2} c \|h^{(J)}\|_{L^2(B_\varepsilon)} \tag{2.63}$$

hold.

Set $\widetilde{\mathbf{u}}^{(J)} = \mathbf{u}^{(J)} + \mathbf{w}^{(J)}$. Then $\mathrm{div}\widetilde{\mathbf{u}}^{(J)} = 0$, $\widetilde{\mathbf{u}}^{(J)}$ satisfies the lateral boundary Γ the no-slip boundary conditions without residual, and because of (2.62) we have

$$\|\mathbf{f}_1^{(J)}\|_{L^2(B_\varepsilon)} = O(\varepsilon^{J-2}), \tag{2.64}$$

where $\mathbf{f}_1^{(J)} = \mathcal{L}(\widetilde{\mathbf{u}}^{(J)}, p^{(J)})$.

As for the pressure boundary conditions, the non-compensated values $\widehat{a}_J^{[BLO_{l'},e]}, l = N_1+1, ..., N$ (limits of $\widehat{P}_J^{[BLO_{l'},e]}$ at the outlets corresponding to edges $O_{l'}O_l$, $l = N_1+1, ..., N$ generate a constant residual in the boundary condition on γ_ε^l, that is $\varepsilon^{J-1}\widehat{a}_J^{[BLO_{l'},e]}$.

2.3. Error estimate

Theorem 2.12. *The following error estimates*

$$\|\mathbf{u} - \widetilde{\mathbf{u}}^{(J)}\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J+(n-1)/2}), \quad \|p - p^{(J)}\|_{L^2(B_\varepsilon)} = O(\varepsilon^{J+(n-1)/2-1}) \tag{2.65}$$

hold.

Proof. Let $\mathbf{v} = \mathbf{u} - \widetilde{\mathbf{u}}^{(J)}$, $q = p - p^{(J)}$. Then the integral identity

$$\int\limits_{B_\varepsilon} \tfrac{1}{2}\nu_\varepsilon(x) D\mathbf{u}^{(J)} \cdot D\boldsymbol{\eta} + K_\varepsilon(x)\mathbf{u}^{(J)} \cdot \boldsymbol{\eta}\,\mathrm{d}x - \int\limits_{B_\varepsilon} q(x)\mathrm{div}\boldsymbol{\eta}\,\mathrm{d}x$$
$$= \sum_{j=N_1+1}^{N} \varepsilon^{J-1}\widehat{a}_J^{[BLO_{l'},e]} \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}}\,\mathrm{d}s - \int\limits_{B_\varepsilon} \mathbf{f}_1^{(J)} \cdot \boldsymbol{\eta}\,\mathrm{d}x$$

holds for every $\boldsymbol{\eta} \in \widehat{W}^{1,2}(B_\varepsilon)$.

Applying Theorem 2.10 with estimate (2.34) we get:

$$\|\mathbf{v}\|_{W^{1,2}(B_\varepsilon)} = \|\mathbf{u} - \tilde{\mathbf{u}}^{(J)}\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J-1}) \tag{2.66}$$

and

$$\|q\|_{L^2(B_\varepsilon)} = \|p - p^{(J)}\|_{L^2(B_\varepsilon)} = O(\varepsilon^{J-5/2}). \tag{2.67}$$

Evaluating now the norm of the difference $\mathbf{u}^{(J)}$ and $\mathbf{u}^{(J+2)}$ we obtain:

$$\|\tilde{\mathbf{u}}^{(J)} - \tilde{\mathbf{u}}^{(J+2)}\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J+(n-1)/2}).$$

Replacing J by $J+2$ in (2.66) yields:

$$\|\mathbf{u} - \tilde{\mathbf{u}}^{(J+2)}\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J+1}).$$

So, the triangle inequality gives the first estimate (2.65). Applying the same argument to $p^{(J)}$ and $p^{(J+4)}$, we get the second estimate (2.65). □

Remark 2.13. The asymptotic expansion (2.41)–(2.46) can be slightly modified without loss of the accuracy. Namely, the argument $\dfrac{|x-O_l|}{|e|_{min}}$ of the cut-off function ζ may be replaced by $\dfrac{|x-O_l|}{\delta}$ with $\delta = C_J\varepsilon|\ln\varepsilon||e|_{min}$, where the constant C_J is chosen in such a way that

$$E_{\delta/\varepsilon} = \|\mathbf{U}^{[BLO_l,J]}\|_{W^{2,2}(\Omega_{l,\delta/\varepsilon})} + \|P^{[BLO_l,J]}\|_{W^{1,2}(\Omega_{l,\delta/\varepsilon})} = O(\varepsilon^{J+(n-1)/2}). \tag{2.68}$$

Here $\Omega_{l,R} = \Omega_l \cup \{|y| > R\}$. Indeed, by Theorem A1 [33], $\mathbf{U}^{[BLO_l,J]}(y)$ and $P^{[BLO_l,J]}(y)$ exponentially vanish as $|y| \to \infty$. Thus,

$$E_R \le c_1 \mathrm{e}^{-c_2 R}, c_1, c_2 > 0.$$

So, the estimate

$$E_{\delta/\varepsilon} = O(\varepsilon^{J+(n-1)/2})$$

is true for δ such that

$$c_2\delta/\varepsilon \ge (J + (n-1)/2)|\ln\varepsilon|,$$

i.e. for $\delta = C_J\varepsilon|\ln\varepsilon|$.

Notice that the constant $C_J = \bar{c}_1 J + \bar{c}_2$, where $\bar{c}_1, \bar{c}_2$ are constants independent of J.

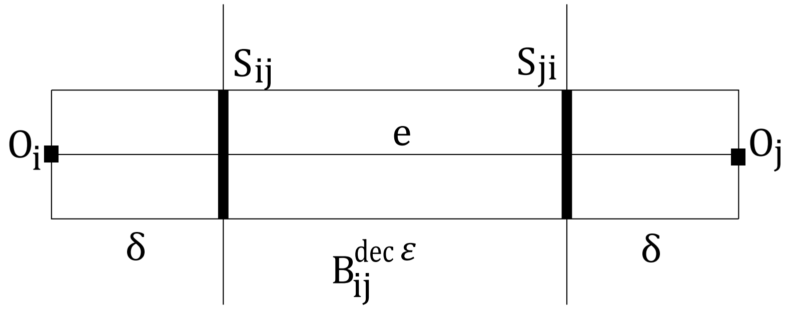
Then the $W^{1,2}(B_\varepsilon)$-norm of the difference of the constructed asymptotic expansion $\mathbf{u}^{(J)}$ and the modified one is of order $O(\varepsilon^{J+(n-1)/2})$.

2.4. Method of asymptotic partial decomposition of the domain for the inflow-outflow boundary condition involving pressure

Using the obtained results we introduce and justify the method of asymptotic partial decomposition of the domain (MAPDD) for problem (2.2). This method first published in [27] reduces the dimension on the main part of B_ε replacing the solution by the Poiseuille type flow and keeps the original full dimension in small neighbourhoods of the nodes and vertices. By this, it reduces the computational costs and accelerates the traditional numerical strategies.

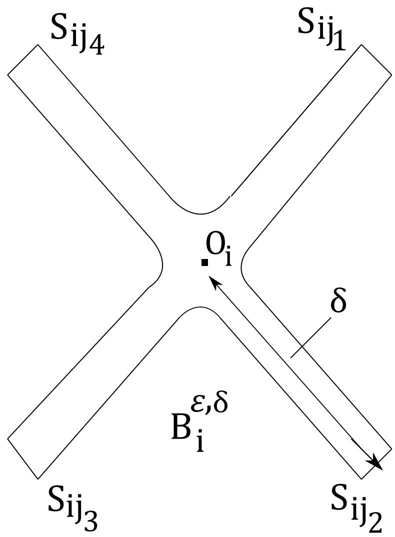
Let us describe the algorithm of the MAPDD for the Stokes problem set in a tube structure B_ε. Let δ be a small positive number much greater than ε (it will be chosen of the order $O(\varepsilon|\ln\varepsilon|)$). For any edge $e = \overline{O_i O_j}$

of the graph (O_i, O_j are two nodes) introduce two hyperplanes orthogonal to this edge and crossing it at the distance δ from its ends. If in the edge $e = \overline{O_iO_j}$ only one end O_i is a node and the second end is a vertex, then we introduce only one hyperplane at the distance δ from the node O_i. Denote the cross-sections of the cylinder $\Pi_\varepsilon^{(e)}$ by these two hyperplanes respectively, by $S_{i,j}$ (the cross-section at the distance δ from O_i) and $S_{j,i}$ (the cross-section at the distance δ from O_j) and denote the part of the cylinder $\Pi_\varepsilon^{(e)}$ between these two cross-sections by $B_{ij}^{\mathrm{dec},\varepsilon}$ (if O_j is a vertex, then $B_{ij}^{\mathrm{dec},\varepsilon}$ stands for the part of the cylinder $\Pi_\varepsilon^{(e)}$ between $S_{i,j}$ and the base of the cylinder γ_ε^j containing the vertex O_j; in this case this base of the cylinder γ_ε^j is denoted $S_{j,i}$).



Truncation of the cylinder $\Pi_\varepsilon^{(e)}$

Let $B_i^{\varepsilon,\delta}$ be the connected, truncated by the cross-sections $S_{i,j}$, part of B_ε which contains the node O_i.



Connected component $B_i^{\varepsilon,\delta}$

Define the subspace $\mathcal{W}_{\mathrm{dec}}(B_\varepsilon, \delta)$ of $\widehat{W}^{1,2}(B_\varepsilon)$ such that on every truncated cylinder $B_{ij}^{\mathrm{dec},\varepsilon}$ its elements (vector-valued functions) have on $B_{ij}^{dec,\varepsilon}$ vanishing tangential entries of the vector-function and independent of the normal variable normal component of the vector-function. Let $\mathcal{H}_{\mathrm{dec}}(B_\varepsilon, \delta)$ be the subspace of $\mathcal{W}_{\mathrm{dec}}(B_\varepsilon, \delta)$ consisting of the divergence free functions. We will consider as well the subspace $L^2_{\mathrm{dec}}(B_\varepsilon, \delta)$ of the space $L^2(B_\varepsilon)$ such that its elements are affine functions of $x_1^{(e)}$ on every truncated cylinder $B_{ij}^{\mathrm{dec},\varepsilon}$. The MAPDD approximation to problem (2.2) is formulated as a projection of the weak formulation (2.4) with $\mathbf{f} = 0$ on the subspace $\mathcal{H}_{\mathrm{dec}}(B_\varepsilon, \delta)$, namely find $\mathbf{v}_d \in \mathcal{H}_{\text{dec}}(B_\varepsilon, \delta)$, such that $\forall \boldsymbol{\eta} \in \mathcal{H}_{\text{dec}}(B_\varepsilon, \delta)$, the following integral identity

$$\int\limits_{B_\varepsilon} \frac{1}{2}\nu_\varepsilon(x) D\mathbf{v}_d \cdot D\boldsymbol{\eta} + K_\varepsilon(x)\mathbf{v}_d \cdot \boldsymbol{\eta} \mathrm{d}x = -\sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}} \, \mathrm{d}s \tag{2.69}$$

holds.

Applying the standard Lax-Milgram lemma arguments one can prove the existence and uniqueness of the solution $\mathbf{v}_d$ to this problem.

Note that the corrected, according to the above Remark 1.1, asymptotic solution $\mathbf{u}^{(J)}$ belongs to the space $\mathcal{H}_{\text{dec}}(B_\varepsilon, \delta)$ and still satisfies the weak formulation (2.4) with the same residual as before, *i.e.*, of order $O(\varepsilon^{J-2})$ in $L^2(B_\varepsilon)$-norm:

$$\int\limits_{B_\varepsilon} \frac{1}{2}\nu_\varepsilon(x) D\mathbf{u}^{(J)} \cdot D\boldsymbol{\eta} + K_\varepsilon(x)\mathbf{u}^{(J)} \cdot \boldsymbol{\eta} \mathrm{d}x = -\sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}} \, \mathrm{d}s + \int\limits_{B_\varepsilon} \mathbf{f}_1^{(J)} \cdot \eta \mathrm{d}x$$

for all $\eta \in \widehat{K}^{1,2}(B_\varepsilon)$, where

$$\|\mathbf{f}_1^{(J)}\|_{L^2(B_\varepsilon)} = O(\varepsilon^{J-2}). \tag{2.70}$$

Respectively, $\mathbf{u}^{(J+2)}$ satisfies

$$\int\limits_{B_\varepsilon} \frac{1}{2}\nu_\varepsilon(x) D\mathbf{u}^{(J+2)} \cdot D\boldsymbol{\eta} + K_\varepsilon(x)\mathbf{u}^{(J+2)} \cdot \boldsymbol{\eta} \mathrm{d}x = -\sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}} \, \mathrm{d}s + \int\limits_{B_\varepsilon} \mathbf{f}_1^{(J)} \cdot \eta \mathrm{d}x,$$

where

$$\|\mathbf{f}_1^{(J+2)}\|_{L^2(B_\varepsilon)} = O(\varepsilon^{J}). \tag{2.71}$$

Of course, this identity is still true for $\eta \in \mathcal{H}_{\text{dec}}(B_\varepsilon, \delta)$, because $\mathcal{H}_{\text{dec}}(B_\varepsilon, \delta)$ is a subspace of $\widehat{K}^{1,2}(B_\varepsilon)$. Evidently, the difference $\mathbf{v}_d - \mathbf{u}^{(J+2)}$ belongs to $\mathcal{H}_{\text{dec}}(B_\varepsilon, \delta)$ and for every $\eta \in \mathcal{H}_{\text{dec}}(B_\varepsilon, \delta)$ satisfies the integral identity

$$\int\limits_{B_\varepsilon} \frac{1}{2}\nu_\varepsilon(x) D(\mathbf{v}_d - \mathbf{u}^{(J+2)}) \cdot D\eta + K_\varepsilon(x)(\mathbf{v}_d - \mathbf{u}^{(J+2)}) \cdot \eta \mathrm{d}x = -\int\limits_{B_\varepsilon} \mathbf{f}_1^{(J+2)} \cdot \eta \mathrm{d}x.$$

Taking $\eta = \mathbf{v}_d - \mathbf{u}^{(J+2)}$ we get the estimate

$$\|\mathbf{v}_d - \mathbf{u}^{(J+2)}\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J+1}). \tag{2.72}$$

Using estimates (2.72), (2.65), and applying the triangle inequality we get

Theorem 2.14. *Let δ be $C_{J+2}\varepsilon|\ln\varepsilon|$ with constant C_{J+2} satisfying (2.68) with J replaced by $J+2$. Then the following error estimate*

$$\|\mathbf{v} - \mathbf{v}_d\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J+(n-1)/2}) \tag{2.73}$$

holds.

This estimate justifies the MAPDD.

Remark 2.15. The variational formulation (2.69) is equivalent to the following one, which uses functions defined only in the "octopus-like" domains $B_i^{\varepsilon,\delta}$. Let $(UB)^{\varepsilon,\delta}=\cup_{i=1}^{N_1}B_i^{\varepsilon,\delta}$. Introduce the space $\widehat{\mathcal{W}}_{\rm dec}((UB)^{\varepsilon,\delta})$ of functions $\boldsymbol{\eta}$ belonging to $W^{1,2}(B_i^{\varepsilon,\delta})$ for all $i=1,...,N_1$, such that $\boldsymbol{\eta}=0$ on $\partial B_\varepsilon\cap\partial B_i^{\varepsilon,\delta}$ and for all $e=O_lO_j$

$$\boldsymbol{\eta}|_{S_{lj}}=\boldsymbol{\eta}|_{S_{jl}}=(\mathcal{P}^{(e)})^*\Big(0,...,0,\tilde{\eta}_n(x^{(e)\prime})\Big),\tilde{\eta}_n\in\mathring{W}^{1,2}(\sigma_\varepsilon^{(e)}). \tag{2.74}$$

Also, introduce the space $\widehat{\mathcal{H}}_{\rm dec}((UB)^{\varepsilon,\delta})$ as the subspace of divergence free functions of the space $\widehat{\mathcal{W}}_{\rm dec}((UB)^{\varepsilon,\delta})$. Then problem (2.69) can be stated as follows: find $\mathbf{v}_d\in\widehat{\mathcal{H}}_{\rm dec}((UB)^{\varepsilon,\delta})$, such that $\forall\boldsymbol{\eta}\in\widehat{\mathcal{H}}_{\rm dec}((UB)^{\varepsilon,\delta})$, the following integral identity

$$\begin{aligned}&\textstyle\sum_{i=1}^N\int_{B_i^{\varepsilon,\delta}}\frac12\nu_\varepsilon(x)D\mathbf{v}_d\cdot D\boldsymbol{\eta}+K_\varepsilon(x)\mathbf{v}_d\cdot\boldsymbol{\eta}\mathrm{d}x\\&+\textstyle\sum_{l=1}^M d_l\int_{\sigma_\varepsilon^{(e_l)}}\nu^{(0)}\nabla_{x^{(e_l)\prime}}\mathbf{v}_d\cdot\nabla_{x^{(e_l)\prime}}\boldsymbol{\eta}dx^{(e_l)\prime}=-\sum_{j=N_1+1}^N p_j\int_{\gamma_\varepsilon^j}\boldsymbol{\eta}_{\mathbf{n}}\,\mathrm{d}s\end{aligned} \tag{2.75}$$

holds. Here $d_l=|e|-2\delta$, the distance between the cross-sections S_{ij} and S_{ji} cutting the segment e (for segments with the both end points which are nodes, e connects two nodes), $d_l=|e|-\delta$ if one of the end points is a vertex, *i.e.* e connects a node and a vertex. In the last case, in (2.74) with $j=N_1+1,...,N$ or $l=N_1+1,...,N$, by convention $S_{lj}=\gamma_\varepsilon^j$ (respectively, $S_{lj}=\gamma_\varepsilon^l$). The advantage of this form of the integral identity is that it uses only functions defined in small truncated domains.

It is possible to "recover" the MAPDD pressure and get an equivalent formulation "with pressure":

find the vector-field $\mathbf{v}_d$ and the "MAPDD pressure" p_d for all $i=1,...,N_1$ belonging to the spaces $\mathbf{v}_d\in\widehat{\mathcal{W}}_{\rm dec}((UB)^{\varepsilon,\delta})$ and $p_d\in L^2(B_i^{\varepsilon,\delta})$, and satisfying for all $\boldsymbol{\eta}\in\widehat{\mathcal{W}}_{\rm dec}((UB)^{\varepsilon,\delta})$ the following integral identity

$$\begin{aligned}&\textstyle\sum_{i=1}^N\int_{B_i^{\varepsilon,\delta}}\frac12\nu_\varepsilon(x)D\mathbf{v}_d\cdot D\boldsymbol{\eta}+K_\varepsilon(x)\mathbf{v}_d\cdot\boldsymbol{\eta}\mathrm{d}x\\&+\textstyle\sum_{l=1}^M d_l\int_{\sigma_\varepsilon^{(e_l)}}\nu^{(0)}\nabla_{x^{(e_l)\prime}}\mathbf{v}_d\cdot\nabla_{x^{(e_l)\prime}}\boldsymbol{\eta}dx^{(e_l)\prime}\\&=\textstyle\sum_{i=1}^N\int_{B_i^{\varepsilon,\delta}}p_d\mathrm{div}\boldsymbol{\eta}\mathrm{d}x-\displaystyle\sum_{j=N_1+1}^N p_j\int_{\gamma_\varepsilon^j}\boldsymbol{\eta}_{\mathbf{n}}\,\mathrm{d}s.\end{aligned} \tag{2.76}$$

Theorem 2.16. *There exists a unique solution to problem (2.76) such that the vector field $\mathbf{v}_d$ is the same as in Theorem 2.14, and the uniquely defined in all domains $B_i^{\varepsilon,\delta}$ pressure p_d can be extended on all cylinders $B_{ij}^{\rm dec,\varepsilon}$ as an affine function of the local variable $x_n^{(e)}$ so that the following error estimate*

$$\|p-p_d\|_{W^{1,2}(B_\varepsilon)}=O(\varepsilon^{J+(n-3)/2}) \tag{2.77}$$

holds. Here $\delta=C_{J+2n+1}\varepsilon|\ln\varepsilon|$ with constant C_{J+2n+1} satisfying (2.68) with J replaced by $J+2n+1$.

The details of the proof of this theorem are presented in the Appendix A.

2.5. Some comments on the mixed boundary conditions

Consider now the case when on some part of the surfaces γ_ε^j, $j=N_1+1,...,N_2$, one sets the pressure conditions $(2.2)_5$ while on the other part ($j=N_2+1,...,N$) the inflow/outflow velocity is given. This case is treated in the same way as above. The boundary layers are constructed as above. There is no need of the compatibility condition on the integrals of the boundary value function $\mathbf{g}\cdot\mathbf{n}$ over γ_ε^j, $j=N_2+1,...,N$ (if $N_2<N_1$). The

equations on the graph have then boundary conditions for fluxes at the vertices O_j, $j = N_2+1, ..., N$, and the boundary conditions for the unknown macroscopic pressure p at the vertices O_j, $j = N_1+1, ..., N_2$.

Also note, that the results can be generalized to the case when ν_ε and K_ε are less regular: boundad measurable functions.

2.6. On the Darcy law for a tissue with network of vessels

The constructed above asymptotic expansion of the solution to problem (2.2) can be applied to the determining the permeability of a piece of tissue containing a network of vessels. The derivation of the Darcy law for flows in porous media with periodic structure from the Stokes and Navier–Stokes equations was introduced in [22], [10], [39]. Its justification was based on the method developed in [38], [40], [41] and the Appendix by L.Tartar in the book [39] (see also [22] for non-stationary setting and [43] and [4] for asymptotic expansions of the solution).

Consider a domain G containing a tube structure B_ε. For simplicity, assume that G is a cube $(0,1)^n$ and that all vertices and the surfaces γ_ε^j, $j = N_1+1, ..., N$ belong to the faces of the cube $S_0 = \{x_1 = 0, (x_2, x_3) \in (0,1)^{n-1}\}$ and $S_1 = \{x_1 = 1, (x_2, x_3) \in (0,1)^{n-1}\}$. Let all constants c_j corresponding to $\gamma_\varepsilon^j \subset S_1$ are equal to ε^2, so that $p_j = 1$, while all remaining constants $c_j = 0$. The cube G can be considered as a porous medium and according to the Darcy's law (confirmed by the above asymptotic expansion) the pressure gap (here equal to one) is proportional to the average velocity in the direction x_1, equal to $\bar{u}_1 = \int_{B_\varepsilon} u_1(x)\mathrm{d}x$. According to the above asymptotic analysis, this integral is of order ε^{n+1} because the velocity magnitude is $O(\varepsilon^2)$ and the measure of the domain B_ε is of order ε^{n-1}. Using the leading term of the asymptotic expansion $\varepsilon^2 \tilde{U}_0^{(e)}(x^{(e)}/\varepsilon)$ in each cylinder $\Pi_\varepsilon^{(e)}$ and replacing the integration over B_ε by the integration over $\left(\bigcup_{j=1}^{M} \Pi_\varepsilon^{(e_j)}\right)$ we modify the average velocity with an error of order ε^{n+2} because the measure of all smoothing domains ω_j^ε is of order ε^n, and the integral of the exponentially decaying boundary layer correctors is of order ε^{n+2}. Therefore, the algorithm to compute the permeability in x_1 direction is as follows: solve the problem on the graph (2.47), compute the approximate average velocity

$$\bar{u}_1^a = \sum_{j=1}^{M} \int_{\Pi_\varepsilon^{(e_j)}} \varepsilon^2 \tilde{U}_0^{(e_j)}(x^{(e_j)}/\varepsilon) \cos(\mathbf{n}, x_1)\mathrm{d}x,$$

where $\cos(\mathbf{n}, x_1)$ is the cosine between the edge e_j and the axis x_1, and define the permeability in x_1 direction as the average velocity divided by the pressure gap (equal to one), so, the permeability is equal to $\bar{u}_1^a$. Note that for $\bar{u}_1^a$ we get the following expression:

$$\bar{u}_1^a = \varepsilon^{n+1} \sum_{j=1}^{M} |e_j| \kappa_{e_j} s_0^{(e_j)} \cos(\mathbf{n}, x_1).$$

Here $s_0^{(e_j)} = -\frac{\partial p_0^{(e_j)}}{\partial x_n^{(e_j)}}$. In the case of the round cross-sections of the tubes $\Pi_\varepsilon^{(e_j)}$, having radius εr_j, we get for the permeability:

$$\bar{u}_1^a = \varepsilon^4 \pi (8\nu^{(0)})^{-1} \sum_{j=1}^{M} |e_j| r_j^4 s_0^{(e_j)} \cos(\mathbf{n}, x_1).$$

3. Conclusion

The stationary Stokes equation with Brinkman term in small parts of the domain is studied in thin tube structures with the pressure condition at the inflows and outflows and no-slip boundary condition on the lateral boundary. This boundary value problem models the blood flow in a network of blood vessels, where the Brinkman flow zones simulate the filtration of the blood through thrombi. Also, it can be used to model the flow through a roll of thin capillaries, part of the network. The leading term of the asymptotic expansion can be used to determine the permeability of the tissue. The obtained MAPDD model of the flow can be coupled with the diffusion-convection equations modeling the transport of cells or substances by the blood in the same way as it was done in [7]. Such coupling can be done as well for the non-Newtonian flows in a network of vessels.

Appendix A. Proof of the recovery of the pressure in the MAPDD problem

Denote $\langle\cdot\rangle = (\text{mes}B_\varepsilon)^{-1}\int_{B_\varepsilon}\cdot\, \mathrm{d}x$, $\langle\cdot\rangle_i = \left(\text{mes}B_i^{\varepsilon,\delta}\right)^{-1}\int_{B_i^{\varepsilon,\delta}}\cdot\, \mathrm{d}x$ and define

$$\mathcal{Q}_i(\mathbf{w}) = \int_{\partial B_i^{\varepsilon,\delta}} \mathbf{w}\cdot\mathbf{n}\, \mathrm{d}s.$$

Lemma A.1. *There exist N_1 vector-valued functions $\mathbf{U}_i \in \mathcal{W}_{\text{dec}}(B_\varepsilon,\delta)$, $i = 1, ..., N_1$, such that $\mathcal{Q}_j(\mathbf{U}_i) = \delta_{ij}$, $j = 1, ..., N_1$, and*

$$\|\mathbf{U}_i\|^2_{L^2(B_\varepsilon)} \le C_*\varepsilon^{-(n-1)}, \quad \|\nabla\mathbf{U}_i\|^2_{L^2(B_\varepsilon)} \le C_*\varepsilon^{-(n+1)} \tag{A.1}$$

with constant C_* independent of ε and δ.

Proof. Consider for any $i = 1, ..., N_1$ the following problem on the graph $\mathcal{B}$: Find a function $q_i \in W^{1,2}(\mathcal{B})$, affine at each edge e of the graph ($q_i(x) = -s_i^{(e)}x_1^{(e)} + a_i^{(e)}$) and satisfying the conditions $q_i(O_j) = 0, j = N_1+1, ..., N$ and

$$-\sum_{e:O_j\in e}\kappa_\varepsilon^{(e)}s_i^{(e)} = \delta_{ij}, \tag{A.2}$$

for each node or vertex O_j, $j = 1, ..., N-1$, where the local coordinate system has the origin O_j, $u_{0,\varepsilon} \in \mathring{W}^{1,2}(\sigma_\varepsilon^{(e)})$ is a solution of the problem

$$\left\{\begin{array}{rcll} -\nu^{(0)}\Delta'_{x^{(e)\prime}}u_{0,\varepsilon}(x^{(e)\prime}) &=& 1, & x^{(e)\prime}\in\sigma_\varepsilon^{(e)}, \\ u_{0,\varepsilon}(x^{(e)\prime}) &=& 0, & x^{(e)\prime}\in\partial\sigma_\varepsilon^{(e)}, \end{array}\right. \tag{A.3}$$

and $\kappa_\varepsilon^{(e)} = \int_{\sigma_\varepsilon^{(e)}} u_{0,\varepsilon}(x^{(e)\prime})dx^{(e)\prime}$.

Clearly, $\kappa_\varepsilon^{(e)} = \varepsilon^{n+1}\kappa^{(e)}$, where $\kappa^{(e)} = \int_{\sigma^{(e)}} u_0(y^{(e)\prime})dy^{(e)\prime}$ and u_0 is a unique solution of the problem

$$\left\{\begin{array}{rcll} -\nu^{(0)}\Delta'_{y^{(e)\prime}}u_0(y^{(e)\prime}) &=& 1, & y^{(e)\prime}\in\sigma^{(e)}, \\ u_0(y^{(e)\prime})|_{\partial\sigma^{(e)}} &=& 0, & y^{(e)\prime}\in\partial\sigma^{(e)}. \end{array}\right. \tag{A.4}$$

Note that $\kappa^{(e)}$ does not depend on ε.

Problem (A.2) is a particular case of problem on the graph

$$\begin{gathered}
-\kappa_e \frac{\partial^2 p^{(e)}}{\partial {x_n^{(e)}}^2}(x_n^{(e)}) = 0, \quad x_n^{(e)} \in (0,|e|), \\
-\sum_{e:O_l\in e} \kappa_e \frac{\partial p^{(e)}}{\partial x_n^{(e)}}(0) = \Psi_l, \quad l=1,...,N_1, \\
p^{(e)}(0) = 0, \quad l = N_1+1,...,N, \\
p^{(e)}(0) - p^{(e_s)}(0) = 0, \quad \forall e \subset \mathcal{B}_l, e \neq e_s,
\end{gathered} \tag{A.5}$$

where Ψ_l are given real numbers. The existence and uniqueness of the solution to this problem is proved as in [32].

Relation between $\kappa_\varepsilon^{(e)}$ and $\kappa^{(e)}$ yields: $s_i^{(e)} = \varepsilon^{-(n+1)}\tilde{s}_i^{(e)}$, where $\tilde{s}_i^{(e)}$ are the scaled pressure slopes and they do not depend on ε. Let us construct now for every cylinder $B_{ij}^{dec,\varepsilon}$, corresponding to the edge e, a function $\mathbf{U}_i$ as Poiseuille velocity equal, in the local variables $x^{(e)}$, to $s_i^{(e)}(\mathcal{P}^{(e)})^*\Big(0,...,0,u_{0,\varepsilon}(x^{(e)\prime})\Big)$. Note that the order of the measure of cross section of the cylinder is ε^{n-1} and the magnitude of $u_{0,\varepsilon}(x^{(e)\prime})$ in the Poiseuille velocity is of order ε^2 (and respectively, ε^4 is the square of the magnitude of $u_{0,\varepsilon}$) while its derivatives are of order ε. So, we have:

$$\|\mathbf{U}_i\|^2_{L^2(B_{ij}^{dec,\varepsilon})} \le c\varepsilon^{-2(n+1)+(n-1)+4} = c\varepsilon^{-n+1}, \quad \|\mathbf{U}_i\|^2_{W^{1,2}(B_{ij}^{dec,\varepsilon})} \le c\varepsilon^{-n-1}.$$

Then we extend $\mathbf{U}_i$ inside the domains $B_k^{\varepsilon,\delta}$ as an arbitrary function from $W^{1,2}(B_k^{\varepsilon,\delta})$ with the given boundary values (we keep the same notation for the extended function). In particular, we can do it just by multiplication of the Poiseuille velocities $\mathbf{U}_i(x^{(e)\prime})$ by cut-off functions $\eta_\varepsilon = 1-\zeta$ depending only on the longitudinal variable $\dfrac{x_1^{(e)}-\delta}{\varepsilon}$, *i.e.* $\eta_\varepsilon(x_1^{(e)}-\delta) = 1-\zeta(|\dfrac{x_1^{(e)}-\delta}{\varepsilon}|)$ and we obtain the same estimates in $B_k^{\varepsilon,\delta}$, namely

$$\|\mathbf{U}_i\|^2_{L^2(B_k^{\varepsilon,\delta})} \le c\varepsilon^{-n+1}, \quad \|\mathbf{U}_i\|^2_{W^{1,2}(B_k^{\varepsilon,\delta})} \le c\varepsilon^{-(n+1)}.$$

In fact, the last estimates will contain an extra factor ε but it doesn't improve the overall result: let us calculate, for example, the norm

$$\|\nabla(\eta_\varepsilon \mathbf{U}_i)\|^2_{L^2(B_k^{\varepsilon,\delta})} = \|\eta_\varepsilon' \mathbf{U}_i\|^2_{L^2(B_k^{\varepsilon,\delta})} + \|\eta\nabla'\mathbf{U}_i)\|^2_{L^2(B_k^{\varepsilon,\delta})}$$

$$= \|\eta_\varepsilon' \mathbf{U}_i\|^2_{L^2(B_k^{\varepsilon,\delta})} + \|\eta\nabla\mathbf{U}_i)\|^2_{L^2(B_k^{\varepsilon,\delta})}$$

$$\le c\|\eta_\varepsilon'\|^2_{L^2((0,\varepsilon))}\|\mathbf{U}_i\|^2_{L^2(B_{ij}^{\varepsilon})} + \|\eta\|^2_{L^2((0,\varepsilon))}\|\nabla\mathbf{U}_i\|^2_{L^2(B_k^{\varepsilon,\delta})}$$

$$\le c(\varepsilon^{-1}\varepsilon^{-n+1} + \varepsilon\varepsilon^{-(n+1)}) = c\varepsilon^{-n}$$

Here we used the estimates $\eta_\varepsilon = O(1)$, $\eta_\varepsilon' = O(\varepsilon^{-1})$, so that $\|\eta_\varepsilon'\|^2_{L^2((0,\varepsilon))} = O(\varepsilon^{-1})$ and $\|\eta_\varepsilon\|^2_{L^2((0,\varepsilon))} = O(\varepsilon)$.

Summing up all these estimates for $\mathbf{U}_i$ we get estimate (A.1). The proof of the Lemma is completed. □

Lemma A.2. *Let p be a function defined in $\bigcup\limits_{i=1}^{N} B_i^{\varepsilon,\delta}$, belonging to $L^2(B_i^{\varepsilon,\delta})$ for all $i=1,...,N_1$. Then there exists a vector-valued function $\mathbf{U}\in\mathcal{W}_{\rm dec}(B_\varepsilon,\delta)$ such that*

$$\operatorname{div}\mathbf{U}(x)=\begin{cases} p(x),\ x\in\bigcup\limits_{i=1}^{N_1}B_i^{\varepsilon,\delta},\\ 0,\ x\in B_\varepsilon\setminus\bigcup\limits_{i=1}^{N_1}B_i^{\varepsilon,\delta}.\end{cases}\tag{A.6}$$

There holds the estimate

$$\|\mathbf{U}\|^2_{W^{1,2}(B_\varepsilon)}\le C\Big(\varepsilon^{-2n}\delta^{-1}+\varepsilon^{-2n-3}\delta^2\Big)\sum_{i=1}^{N_1}\|p\|^2_{L^2(B_i^{\varepsilon,\delta})}\tag{A.7}$$

with some positive constant C independent of ε and δ. For $\delta=O(\varepsilon|\ln\varepsilon|)$ we get

$$\|\mathbf{U}\|^2_{W^{1,2}(B_\varepsilon)}\le C\varepsilon^{-2n-3/2}\sum_{i=1}^{N_1}\|p\|^2_{L^2(B_i^{\varepsilon,\delta})}.\tag{A.8}$$

Proof. Consider the sum

$$\mathbf{\Psi}=\sum_{i=1}^{N_1}\langle p\rangle_i\mathbf{U}_i,$$

where $\mathbf{U}_i$ are functions constructed in the previous lemma. By Stokes formula, $\int_{B_k^{\varepsilon,\delta}}\operatorname{div}\mathbf{U}_k\mathrm{d}x=\mathcal{Q}_k(\mathbf{U}_k)=\delta_{kk}=1$, and so,

$$\int_{B_k^{\varepsilon,\delta}}\operatorname{div}\mathbf{\Psi}\mathrm{d}x=\langle p\rangle_k\int_{B_k^{\varepsilon,\delta}}\operatorname{div}\mathbf{U}_k\mathrm{d}x=\langle p\rangle_k.$$

Thus,

$$\int_{B_k^{\varepsilon,\delta}}(p-\operatorname{div}\mathbf{\Psi})\mathrm{d}x=0.$$

Now we need to solve the divergence equation in $B_k^{\varepsilon,\delta}$, *i.e.* to construct a function $\mathbf{\Theta}\in\mathring{W}^{1,2}(B_k^{\varepsilon,\delta})$ such that

$$\operatorname{div}\mathbf{\Theta}=-\operatorname{div}\mathbf{\Psi}+p.\tag{A.9}$$

The existence of a solution of (A.9) follows from Lemma 2.7. However, in order to obtain an appropriate estimate (with a constant independent of ε and δ), we need to reduce this problem to the same problem in the δ^{-1}-times dilated domain

$$\delta^{-1}B_k^{\varepsilon,\delta}=\big\{x:\delta x+O_k\in B_k^{\varepsilon,\delta}\big\}.$$

This domain is a thin tube structure with ε/δ as a small parameter replacing former small parameter ε (recall that $\delta=c\varepsilon|\ln\varepsilon|$). So, we can apply Lemma 2.7 and construct the solution satisfying the estimate in original variables:

$$\begin{aligned}\|\boldsymbol{\Theta}\|^2_{W^{1,2}(B_k^{\varepsilon,\delta})} &\le c(\varepsilon/\delta)^{-3}\|-\mathrm{div}\boldsymbol{\Psi}+p\|^2_{L^2(B_k^{\varepsilon,\delta})}\\ &\le c(\varepsilon/\delta)^{-3}\left(\|\nabla\boldsymbol{\Psi}\|^2_{L^2(B_k^{\varepsilon,\delta})}+\|p\|^2_{L^2(B_k^{\varepsilon,\delta})}\right).\end{aligned} \tag{A.10}$$

Here and below c is a generic constant independent of small parameters.

Let us evaluate the norm $\|\boldsymbol{\Psi}\|_{W^{1,2}(B_k^{\varepsilon,\delta})}$. It is majorated by the sum

$$\sum_{i=1}^{N-1}|\langle p\rangle_i|\|\mathbf{U}_i\|_{W^{1,2}(B_k^{\varepsilon,\delta})},$$

where

$$|\langle p\rangle_i|\le c\varepsilon^{-(n-1)}\delta^{-1}\Big|\int_{B_k^{\varepsilon,\delta}}p\mathrm{d}x\Big|\le\varepsilon^{-(n-1)/2}\delta^{-1/2}\|p\|_{L^2(B_k^{\varepsilon,\delta})},$$

and

$$\|\mathbf{U}_i\|_{W^{1,2}(B_k^{\varepsilon,\delta})}\le\|\mathbf{U}_i\|_{W^{1,2}(B_\varepsilon)}\le C\varepsilon^{-(n+1)/2}$$

(see the previous lemma). So, finally,

$$\begin{aligned}\|\boldsymbol{\Psi}\|^2_{W^{1,2}(B_k^{\varepsilon,\delta})}\le\|\boldsymbol{\Psi}\|^2_{W^{1,2}(B_\varepsilon)}\le C\varepsilon^{-2n}\delta^{-1}\textstyle\sum_{i=1}^{N_1}\|p\|^2_{L^2(B_k^{\varepsilon,\delta})},\\ \|\boldsymbol{\Theta}\|^2_{W^{1,2}(B_k^{\varepsilon,\delta})}\le C\varepsilon^{-2n-3}\delta^2\textstyle\sum_{i=1}^{N_1}\|p\|^2_{L^2(B_k^{\varepsilon,\delta})}.\end{aligned} \tag{A.11}$$

Let us take $\mathbf{U}=\boldsymbol{\Psi}+\boldsymbol{\Theta}$, where $\boldsymbol{\Theta}$ is extended by zero to the cylinders $B_{ij}^{dec,\varepsilon}$. Then $\mathbf{U}=\boldsymbol{\Psi}$ for the remaining part of the tube structure. Recall that $\mathrm{div}\boldsymbol{\Psi}=0$ for $x\in B_\varepsilon\setminus\bigcup_{i=1}^{N_1}B_i^{\varepsilon,\delta}$. So, using estimate (A.11) we finalize the proof. □

Now we are in position to prove the existence of the solution $(\mathbf{v}_d,p_d)$ to problem (2.76) with the test functions from the space $\mathcal{W}_{\mathrm{dec}}(B_\varepsilon,\delta)$.

Introduce the space $\widehat{L}^2_{dec}(B_\varepsilon)$ as the space of scalar functions $q\in L^2_{dec}(B_\varepsilon)$, equal to zero on all cylinders $B_{ij}^{dec,\varepsilon}$.

Theorem A.3. *There exists a unique function $p_d\in\widehat{L}^2_{dec}(B_\varepsilon)$ satisfying integral identity (2.76).*

Proof. The proof of this theorem repeats the proof of Theorem 2.9 where the spaces $L^2(B_\varepsilon)$, $\widehat{K}^{1,2}(B_\varepsilon)$, and $\widehat{W}^{1,2}(B_\varepsilon)$ are replaced by $\widehat{L}^2_{dec}(B_\varepsilon)$, $\mathcal{H}_{\mathrm{dec}}(B_\varepsilon,\delta)$, and $\mathcal{W}_{\mathrm{dec}}(B_\varepsilon,\delta)$ respectively, while the bounded linear operator M^{-1} from $\widehat{L}^2_{dec}(B_\varepsilon)$ onto $\mathcal{W}_{\mathrm{dec}}(B_\varepsilon,\delta)$ is defined by the formula $\mathbf{w}=M^{-1}p=\boldsymbol{\Psi}+\boldsymbol{\Theta}$, where $\boldsymbol{\Psi}$ and $\boldsymbol{\Theta}$ are defined in the proof of Lemma A.2.

□

Theorem A.4. *For $\delta=C_{J+2n+1}\varepsilon|\ln\varepsilon|$ the estimate*

$$\|p_d-p\|_{L^2(B_k^{\varepsilon,\delta})}=O(\varepsilon^{J+(n-3)/2}) \tag{A.12}$$

holds, and there exists an extension $\widetilde{p}_d \in L^2_{\rm dec}(B_\varepsilon)$ *of* p_d *such that the following estimate*

$$\|\widetilde{p}_d - p\|_{L^2(B_\varepsilon)} = O(\varepsilon^{J+(n-3)/2}) \tag{A.13}$$

holds.

Proof. The asymptotic expansion $(\mathbf{u}^{(J)}, p^{(J)})$ satisfies the following integral identity:

$$\begin{gathered}\int_{B_\varepsilon} \tfrac{1}{2}\nu_\varepsilon(x) D\mathbf{u}^{(J)} \cdot D\boldsymbol{\eta} + K_\varepsilon(x)\mathbf{u}^{(J)} \cdot \boldsymbol{\eta} \mathrm{d}x + \sum_{j=N_1+1}^N p_j \int_{\gamma_\varepsilon^j} \boldsymbol{\eta}_{\mathbf{n}} \,\mathrm{d}s\ , \\ = \int_{B_\varepsilon} p^{(J)} \mathrm{div}\eta \mathrm{d}x + \int_{B_\varepsilon} \mathbf{r}_\varepsilon^{(J)} \cdot \eta \mathrm{d}x\end{gathered} \tag{A.14}$$

for all $\eta \in \mathring{W}^{1,2}(B_\varepsilon)$, where $\mathbf{r}_\varepsilon^{(J)} = \mathbf{f}_1^{(J)} = \mathcal{L}(\tilde{\mathbf{u}}^{(J)}, p^{(J)})$ and, according to (2.64),

$$\|\mathbf{r}_\varepsilon^{(J)}\|_{L^2(B_\varepsilon)} = O(\varepsilon^{J-2}). \tag{A.15}$$

Let us take the approximations $(\mathbf{u}^{(J+2n+1)}, p^{(J+2n+1)})$, modified according to Remark 2.13 and denote $J' = J + 2n + 1$. Choose $\delta = C_{J'}\varepsilon|\ln\varepsilon|$.

Consider the difference $q = p^{(J')} - p_d$.

Applying Lemma A.2, we can construct a function $\mathbf{U} \in \mathcal{W}_{\rm dec}(B_\varepsilon, \delta)$ such that

$$\mathrm{div}\mathbf{U}(x) = \begin{cases} q(x),\ x \in \bigcup\limits_{i=1}^{N_1} B_i^{\varepsilon,\delta}\ , \\ 0,\ x \in B_\varepsilon \setminus \bigcup\limits_{i=1}^{N_1} B_i^{\varepsilon,\delta}, \end{cases} \tag{A.16}$$

and

$$\|\mathbf{U}\|^2_{W^{1,2}(B_\varepsilon)} \le C\varepsilon^{-2n-3/2} \sum_{i=1}^{N_1} \|q\|^2_{L^2(B_i^{\varepsilon,\delta})} \tag{A.17}$$

with some constant C independent of ε, and δ. Taking $\mathbf{U}$ as a test function in (A.14) and in (2.76) with J' instead of J, consider the difference between these integral identities. Denoting $\mathbf{u} = \mathbf{u}^{(J')} - \mathbf{v}_d$, we get

$$\int_{B_\varepsilon} \tfrac{1}{2}\nu_\varepsilon(x) D\mathbf{u} \cdot D\mathbf{U} + K_\varepsilon(x)\mathbf{u} \cdot \mathbf{U} \mathrm{d}x = \sum_{i=1}^{N_1} \int_{B_i^{\varepsilon,\delta}} q(x) \mathrm{div}\mathbf{U} \mathrm{d}x + \int_{B_\varepsilon} \mathbf{r}_\varepsilon^{(J')} \cdot \mathbf{U} \mathrm{d}x, \tag{A.18}$$

and so,

$$\sum_{i=1}^{N_1} \int_{B_i^{\varepsilon,\delta}} q^2 \mathrm{d}x \le\ c\Big(\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)} + \|\mathbf{r}_\varepsilon^{(J')}\|_{L^2(B_\varepsilon)}\Big)\|\nabla\mathbf{U}\|_{L^2(B_\varepsilon)}.$$

So, applying (A.17), (2.73) and (A.15) we get

$$\Big(\sum_{i=1}^{N} \int_{B_i^{\varepsilon,\delta}} q^2 \mathrm{d}x\Big)^{\frac{1}{2}} \le\ c_2\varepsilon^{-(n+3/4)}\Big(\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)} + \|\mathbf{r}_\varepsilon^{(J')}\|_{L^2(B_\varepsilon)}\Big)$$

$$= O(\varepsilon^{-n-3/4+J'-2}) = O(\varepsilon^{J+n-2}) = O(\varepsilon^{J+(n-3)/2}).$$

Applying estimate (2.67) for $p - p^{(J')}$, and then the triangle inequality we prove estimate (A.12).

Let us extend now p_d to the cylinders $B_{ij}^{dec,\varepsilon}$ as an affine function by the following formula:

$$p_d(x_n^{(e)}) = \langle p_d \rangle_{S_{ij}} + \frac{\langle p_d \rangle_{S_{ji}} - \langle p_d \rangle_{S_{ij}}}{|e| - 2\delta + \varepsilon} \left(x_n^{(e)} - \delta + \frac{\varepsilon}{2} \right), \tag{A.19}$$

where for any function $q \in L^2(B_\varepsilon)$ we denote by $\langle q \rangle_{S_{ij}}$ the mean value of q in the cylinder $C_{ij} = \{x_1^{(e)} \in S_{ij} \times (\delta - \varepsilon, \delta)\}$, having one of the bases at the cross section S_{ij} and the height of the length ε. Note that the asymptotic approximation $p^{(J')}$ satisfies the relation analogous to (A.19):

$$p^{(J')}(x_n^{(e)}) = \langle p^{(J')} \rangle_{S_{ij}} + \frac{\langle p^{(J')} \rangle_{S_{ji}} - \langle p^{(J')} \rangle_{S_{ij}}}{|e| - 2\delta + \varepsilon} \left(x_n^{(e)} - \delta + \frac{\varepsilon}{2} \right), \tag{A.20}$$

because it is affine function of $x_n^{(e)}$ within the cylinder $\sigma_\varepsilon^{(e)} \times \{x_n^{(e)} \in (\delta - \varepsilon, |e| - \delta + \varepsilon)\}$. On the other hand,

$$|\langle p_d \rangle_{S_{ij}} - \langle p^{(J')} \rangle_{S_{ij}}|^2 \le c_5 (\text{mes} C_{ij})^{-1} \|p_d - p^{(J')}\|^2_{L^2(B_i^{\varepsilon,\delta})}. \tag{A.21}$$

Applying estimate (A.12) of Theorem A.4, we get:

$$|\langle p_d \rangle_{S_{ij}} - \langle p^{(J')} \rangle_{S_{ij}}|^2 = O(\varepsilon^{-n+2J+2n-2}) = O(\varepsilon^{2J+n-4}). \tag{A.22}$$

So, taking into account that all cylinders have measure of order of $O(\varepsilon^{(n-1)/2})$ we finally have

$$\|p_d - p^{(J')}\|_{L^2(B_\varepsilon)} = O(\varepsilon^{J+(n-3)/2}). \tag{A.23}$$

As in the proof of Theorem A.4, using estimate (2.67) for $p - p^{(J')}$, and then the triangle inequality we get estimate (A.13). □

Appendix B. Proof of the estimate for the divergence equation in thin tube structures

Lemma B.1. *Consider equation (2.16) set in $\Omega = B_\varepsilon$. Then (2.16) admits at least one solution $\mathbf{w} \in \mathring{W}^{1,2}(B_\varepsilon)$ satisfying estimate*

$$\|\nabla \mathbf{w}\|_{L^2(B_\varepsilon)} \le c\varepsilon^{-3/2} \|h\|_{L^2(B_\varepsilon)} \tag{B.1}$$

Proof. Let $\{\varphi_i(x)\}_{i=1}^{M+N_1}$ be a partition of unity subordinated to the covering of B_ε, *i.e.*, $supp\, \varphi_i \subset \bar{\Pi}_\varepsilon^{(e_i)}$, $i = 1, \dots, M$, $supp\, \varphi_{M+k} \subset \omega_\varepsilon^k$, $k = 1, \dots, N_1$. We can take the functions φ_i so that $|\nabla \varphi_i(x)| \le c\varepsilon^{-1}$. We represent the function h in the form

$$h(x) = \sum_{i=1}^{M+N_1} \varphi_i(x) h(x) \quad \forall x \in B_\varepsilon$$

and denote $\mu_i = \int\limits_{B_\varepsilon} \varphi_i(x)h(x)\,\mathrm{d}x$. Evidently

$$|\mu_i| \le c\varepsilon^{(n-1)/2}\|h\|_{L^2(B_\varepsilon)}.$$

Since

$$\int_{B_\varepsilon} h(x)\,\mathrm{d}x = 0,$$

we have

$$\sum_{i=1}^{M+N_1} \mu_i = 0. \tag{B.2}$$

Put $\gamma_i = \Pi_\varepsilon^{(e_i)}$, $i = 1,\dots,M$, $\gamma_{M+k} = \omega_\varepsilon^k$, $k = 1,\dots,N_1$, and $\gamma_{il} = \gamma_i \cap \gamma_l$. Notice that $\operatorname{mes}\gamma_{il}$ is of order $\varepsilon^n, n = 2,3$. For each pair of intersecting domains γ_i and γ_l we introduce a C^2–regular function $\eta_{il}(x)$ defined on B_ε such that $supp\,\eta_{il} \subset \gamma_{il}$, $|\eta_{il}(x)| \le \dfrac{c}{\varepsilon^n}$, $|\nabla\eta_{il}(x)| \le \dfrac{c}{\varepsilon^{n+1}}$ and

$$\int_{\gamma_{il}} \eta_{il}(x)\,\mathrm{d}x = 1. \tag{B.3}$$

Let $\{K_{il}\}_{i,l=1,\dots,M+N_1}$ be a skew-symmetric matrix such that $K_{il} = 0$ if γ_{il} is empty, and

$$\sum_{l=1}^{M+N_1} c_{il}K_{il} = \mu_i,\ \ i = 1,\dots,M+N_1, \tag{B.4}$$

where $\{c_{il}\}_{i,l=1,\dots,M+N_1}$ is the adjacency matrix of the graph of the covering $\{\gamma_i\}_{i=1}^{M+N_1}$ of the tube structure B_ε, *i.e.*, $c_{il} = 1$, if $i \ne l$ and γ_{il} is nonempty, and $c_{il} = 0$ in remaining cases. The existence of the matrix $\{K_{il}\}_{i,l=1,\dots,M+N_1}$ is proved by induction on the number $M+N_1$. If $M+N_1 = 2$, then $c_{12} = c_{21} = 1$ and by virtue of (B.2), one can set $K_{12} = -K_{21} = \mu_1$, $K_{11} = K_{22} = 0$. Assume that the assertion is valid for $M+N_1 = m-1$. We prove it for $M+N_1 = m$. Let us set $K_{il} = 0$ for those i and l for which $c_{il} = 0$. As it is well known (see [15]), from a connected graph with the number of vertices greater than two, one can always discard one vertex so that the remaining graph is connected. From here there follows the existence of an index λ, $1 \le \lambda \le m$, such that the matrix $\{c_{il}\}$, $i \ne \lambda$, $l \ne \lambda$, is the adjacency matrix of some connected graph with $(m-1)$ vertices. Without loss of generality one can assume that $\lambda = m$. We consider the m-th equation from (B.4):

$$\mu_m = \sum_{l=1}^{m-1} c_{ml}K_{ml}. \tag{B.5}$$

Since the initial graph is connected, there exists an index l_1, such that $c_{ml_1} = 1$. We set $K_{ml_1} = -K_{l_1 m} = \mu_m$, while $K_{ml} = -K_{lm} = 0$ for $l \ne l_1$. The remaining equations from (B.4) will be written in the form:

$$\tilde{\mu}_i = \sum_{l=1}^{m-1} c_{il}K_{il}, \qquad i = 1,\dots,m-1, \tag{B.6}$$

where $\tilde{\mu}_i = \mu_i - c_{im}K_{im}$. We note that by virtue of (B.2) and (B.5), we have the equality

$$\sum_{i=1}^{m-1} \tilde{\mu}_i = 0.$$

Indeed,

$$\sum_{i=1}^{m-1} \tilde{\mu}_i = \sum_{i=1}^{m-1} \mu_i - \sum_{i=1}^{m-1} c_{im}K_{im} = \sum_{i=1}^{m-1} \mu_i + \sum_{i=1}^{m-1} c_{mi}K_{mi} = \sum_{i=1}^{m} \mu_i = 0.$$

In addition, as it is mentioned above, the matrix $\{c_{il}\}_{i,l=1,\dots,m-1}$ satisfies the induction hypothesis for $M + N_1 = m-1$. Consequently, system (B.6) has a solution with the required properties.

Note that the matrix $\{c_{il}\}_{i.l=1,\dots,M+N_1}$ is independent of ε, and it can be proved by induction (as before) that

$$|K_{il}| \leq c \max\left\{|\mu_1(t)|, \dots, |\mu_{M+N_1}|\right\}$$

with the constant c independent of ε. Therefore,

$$|K_{il}| \leq c\varepsilon^{(n-1)/2}\|h\|_{L^2(B_\varepsilon)}.$$

Define on B_ε the function

$$\eta_i(x) = \sum_{l=1}^{M+N_1} K_{il}\eta_{il}(x), \qquad i = 1, \dots, M + N_1. \tag{B.7}$$

Using the skew–symmetry of $\{K_{il}\}_{i,l=1,\dots,M+N_1}$, it is easy to see that $supp\,\eta_i \subset \gamma_i$ and

$$\sum_{i=1}^{M+N_1} \eta_i(x) = 0 \qquad \forall x \in B_\varepsilon. \tag{B.8}$$

Set

$$h^{(i)}(x) = \varphi_i(x)h(x) - \eta_i(x) \qquad \text{for} \quad x \in B_\varepsilon. \tag{B.9}$$

The functions $h^{(i)}$ have the following properties:

(i) $h^{(i)}$ have the same regularity as h,
(ii) $supp\, h^{(i)} \subset \gamma_i$,
(iii) $\sum_{i=1}^{M+N_1} h^{(i)}(x) = h(x) \;\; \forall x \in B_\varepsilon$,
(iv) $\int_{\gamma_i} h^{(i)}(x)\,\mathrm{d}x = 0$, $i = 1, \dots M, M+1, \dots, M+N_1$.

Let $\mathbf{w}^{(e_i)}(x), i = 1, \dots, M$, be solutions in $\Pi_\varepsilon^{(e_i)}$ of problem (2.16) with the right–sides $h^{(i)}(x)$, and $\mathbf{w}^{(k)}(x), k = N+1, \dots, M+N_1$, – solutions of (2.16) in ω_ε^k with the right–sides $h^{(k)}(x)$. For $\mathbf{w}^{(e_i)}$ and $\mathbf{w}^{(k)}$ hold estimates of Lemmas 2.5 and 2.4 (for the contracted $1/\varepsilon$ times domain Ω) , respectively. Extend the functions $\mathbf{w}^{(e_i)}$ and $\mathbf{w}^{(k)}$ by zero into B_ε and put

$$\mathbf{w}(x) = \sum_{i=1}^{M} \mathbf{w}^{(e_i)}(x) + \sum_{k=M+1}^{M+N_1} \mathbf{w}^{(k)}(x).$$

Then

$$\text{div}\mathbf{w}(x) = \sum_{i=1}^{M+N_1} h^{(i)}(x) = h(x).$$

According to (B.7),

$$\|\eta_i\|_{L^2(B_\varepsilon)} \leq c\varepsilon^{(n-1)/2}\|h\|_{L^2(B_\varepsilon)} \sum_{l=1}^{M+N_1} \|\eta_{il}\|_{L^2(\gamma_{il})} \leq c\varepsilon^{-1/2}\|h\|_{L^2(B_\varepsilon)}.$$

So,

$$\|h^{(i)}\|_{L^2(B_\varepsilon)} \leq \frac{c}{\sqrt{\varepsilon}}\|h\|_{L^2(B_\varepsilon)}.$$

Therefore, using Lemmas 2.4 and 2.5 we obtain required estimate for $\mathbf{w}$. □

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