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Article

Asymptotic Domain Decomposition Method for Approximation the Spectrum of the Diffusion Operator in a Domain Containing Thin Tubes

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Abstract: The spectral problem for the diffusion operator is considered in a domain containing thin tubes. A new version of the method of partial asymptotic decomposition of the domain is introduced to reduce the dimension inside the tubes. It truncates the tubes at some small distance from the ends of the tubes and replaces the tubes with segments. At the interface of the three-dimensional and one-dimensional subdomains, special junction conditions are set: the pointwise continuity of the flux and the continuity of the average over a cross-section of the eigenfunctions. The existence of the discrete spectrum is proved for this partially reduced problem of the hybrid dimension. The conditions of the closeness of two spectra, i.e., of the diffusion operator in the full-dimensional domain and the partially reduced one, are obtained.

Keywords: asymptotic domain decomposition method; approximation of the spectrum; diffusion operator; thin tubes; junction conditions

MSC: 35J05; 35P05; 35P10; 74K30

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1. Introduction

The method of asymptotic partial decomposition of the domain (MAPDD) was introduced for partial derivative equations set in thin tube structures in [1] (cf. also [2–4]). The thin tube structures (also called thin rod structures) are some unions of thin cylinders. In this case, the method gives an important gain in computational resources reducing the dimension to one everywhere except for small two- or three-dimensional parts of the domain. This method was applied to the spectral problems in [5], where the asymptotic of the spectrum of the Laplacian in two joint thin rectangles with Neumann's boundary conditions on the lateral boundary were considered. The MAPDD was justified for such a structure. The asymptotic behavior of the spectrum of the Laplacian with Dirichlet's or Neumann's boundary conditions in thin domains was considered in a vast body of literature (see [6–13], and the references therein). Furthermore, let us mention some works on the eigenvalues for the Laplacian for several special cases of domains containing thin tubes, described with the help of a small parameter (cf. [14–17] among others); these spectral problems were treated using asymptotic analysis. As a matter of fact, Refs. [6–8,13] address spectral problems in thin planar domains, while Refs. [10–12] deal with three-dimensional domains in very different structures from those considered here. This topic is of great interest due to its multiple applications in scattering theory, wave-guides, etc.

In the present paper, we introduce a new version of the MAPDD with special interface conditions between the full-dimensional and one-dimensional parts, that is, the pointwise continuity of the flux and the continuity of the average over a cross-section of the eigenfunctions. These conditions were used for MAPDD approximation of the heat equation in [18]. In particular, some numerical methods also use junction conditions with jumps of the unknown function (see, e.g., [19,20]). Here, we deal with a spectral problem for the Neumann diffusion operator.

The results obtained in former works on the asymptotic analysis of spectral problems in thin domains mostly use the presence in the model of a small parameter, that is, the ratio of the characteristic sizes in the transverse and longitudinal (or in-plane) directions. The justification for these results was provided via the proof of theorems concerning the convergence of the spectrum of the original problem to that of the spectrum of the reduced problem as the small parameter tends to zero. In the present paper, we use another approach related to the method of justification for the MAPDD in [21], but we introduce different junction conditions. There is no explicit small parameter in the description of the domain, but, implicitly, it is introduced via the assumption that the first positive eigenvalues of the Neumann diffusion operator on the cross-sections of the tubes are sufficiently large. The closeness of the spectra of the original and reduced models is proved under the condition that the eigenvalues of both the original and the reduced models are smaller than these first eigenvalues on the cross-sections of the tubes. In addition, here, we do not assume that the domain is thin everywhere. It may have some thick parts connected by thin tubes. The boundary is assumed to be Lipschitz, and we do not require the regularity of the coefficient in the diffusion operator out of the tubes. The proof of the aforementioned closeness of the spectra also relies on the distance from the ends of the tubes to the one-dimensional domain in the reduced model, a distance that needs to be adjusted to achieve the desired accuracy for the approximation between the spectra.

The main results are as follows. The spectral problem for the diffusion operator is considered in a domain containing thin tubes. The reduced model is obtained from the original one by the truncation of the three-dimensional tubes at some small distance from the ends of the tubes replacing the truncated parts of the tubes by the segments. At the interface of the three-dimensional and one-dimensional subdomains, special junction conditions are set: the pointwise continuity of the flux and continuity of the average of the eigenfunctions over a cross-section. The existence of the discrete spectrum is proved for this partially reduced problem of the hybrid dimension. After prescribing an accurate precision ε, to obtain it, the conditions of the closeness of two spectra (i.e., of the diffusion operator in the full-dimensional domain and the partially reduced one) are obtained.

The structure of the paper is as follows: Section 2 contains some preliminary results, the definition of the domain, and the setting of the spectral problem. Moreover, the required properties on the smoothness of the eigenfunctions are obtained (cf. Section 2.3). In Section 3, the reduced approximate spectral problem is formulated. Because of the hybrid dimension, notations, weak formulation, and properties of spaces become more complicated. The required smoothness results for the eigenfunctions are also proved (cf. Section 3.3). In Section 4, we state the main results on the closeness of the spectra of the original and reduced problems (cf. Theorems 2 and 3). As a consequence, we claim the closeness of the spectra in the case where the tubes are cylinders (cf. Theorems 4 and 5). Sections 5 and 6 contain the proofs of Theorems 2 and 3, respectively. Finally, some concluding remarks are outlined in Section 7.

2. Preliminary Results and Setting of the Problem

In this section, we introduce some notations and preliminary results that will be used throughout the whole paper. Section 2.2 contains the setting of the spectral problem under consideration, and, in Section 2.3, we provide certain smoothness properties for the eigenfunctions.

2.1. Preliminaries

By a tube, we mean a set $\mathcal{T} \subset \mathbb{R}^3$ of the form $\mathcal{T} = \Omega \times (0, \ell)$ in the local coordinates $(y, z) = (y_1, y_2, z)$, where the base Ω of the tube is a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. The set $\partial\mathcal{T} = \partial\Omega \times (0, \ell)$ is called the lateral surface of the tube $\mathcal{T}$, and ℓ stands for its length. Define

$$D_1\varphi = \frac{\partial\varphi}{\partial y_1}, \quad D_2\varphi = \frac{\partial\varphi}{\partial y_2}, \quad D_z\varphi = \frac{\partial\varphi}{\partial z}, \quad \nabla_y\varphi = (D_1\varphi, D_2\varphi), \quad \Delta_y\varphi = D_1^2\varphi + D_2^2\varphi.$$

Let $\mu_1(\Omega)$ be the first positive eigenvalue of the following problem

$$-\Delta_y\varphi = \mu\varphi, \quad y \in \Omega,$$
$$\nabla_y\varphi \cdot \nu = 0, \quad y \in \partial\Omega,$$

where ν is the outward normal to $\partial\Omega$.

Recall that for a disk Ω with a diameter D, $\mu_1(\Omega) = 4p^2/D^2$ where $p = 1.8142\ldots$ is the smallest positive root of the derivative J_1' of the Bessel function J_1. In the general case, we have [22]

$$\mu_1(\Omega) \le \mu_1(\Omega^*) = \frac{\pi p^2}{\text{meas }(\Omega)}, \tag{1}$$

where Ω^* is a disk of the same measure as Ω. Inequality (1) becomes an equality only if Ω is the disk.

In the case where Ω is a convex domain of diameter D, we have the lower estimate [23]:

$$\frac{\pi^2}{2D^2} \le \mu_1(\Omega).$$

Note that $\rho(\Omega) = 1/\sqrt{\mu_1(\Omega)}$ is an optimal constant in the Poincaré inequality

$$\|\varphi\|_{L^2(\Omega)} \le \rho(\Omega)\|\nabla_y\varphi\|_{L^2(\Omega)} \quad \forall\, \varphi \in \widetilde{H}^1(\Omega).$$

Here, $\widetilde{H}^1(\Omega) = \{\varphi \in H^1(\Omega) \mid \int\limits_\Omega \varphi\, dx = 0\}$ is a Hilbert space equipped with the inner product and norm, respectively

$$(\varphi, \psi)_{\widetilde{H}^1(\Omega)} = (\nabla_y\varphi, \nabla_y\psi)_{L^2(\Omega)}, \quad \|\varphi\|_{\widetilde{H}^1(\Omega)} = \|\nabla_y\varphi\|_{L^2(\Omega)}.$$

Obviously, $\mu_1(\epsilon\Omega) = \epsilon^{-2}\mu_1(\Omega)$ for the contracted domain, and $\epsilon\Omega = \{\epsilon y \mid y \in \Omega\}$ for all $0 < \epsilon$.

We see that for a wide class of domains, the value $\rho(\Omega)$ characterizes the size and geometry of Ω. So, we refer to a thin tube $\mathcal{T}$ when $\rho(\Omega) \ll \ell$. For an Ω that is a disk with a radius of ϵ, the tube is thin if $\epsilon \ll \ell$; in this case, $\rho(\Omega) = \epsilon/p$.

2.2. Spectral Problem

Let G be a bounded domain in $\mathbb{R}^3$ with a Lipschitz boundary ∂G. Consider the following spectral problem: find a couple $(\lambda, u) \in \mathbb{R} \times H^1(G)$ such that $\|u\|_{L^2(G)} = 1$, satisfying the equation

$$-\text{div}(K\nabla u) = \lambda u, \quad x \in G$$

and the boundary condition

$$K\nabla u \cdot n = 0, \quad x \in \partial G.$$

Here, $K \in L^\infty(G)$, $\operatorname*{ess\,inf}_{x\in G} K(x) > 0$, n is the outward normal to ∂G.

The weak formulation is given by the following identity:

$$(K\nabla u, \nabla \varphi)_{L^2(G)} = \lambda(u, \varphi)_{L^2(G)} \quad \forall\, \varphi \in H^1(G). \tag{2}$$

It is well known that this problem has a countable set $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ of eigenvalues and that the corresponding set $\{\varphi_k\}_{k=0}^{\infty}$ of eigenfunctions can be chosen to form an orthogonal base in $H^1(G)$ and an orthonormal base in $L^2(G)$.

Assume that G contains a set of disjoint thin tubes $\mathcal{T}_j$, $1 \leq j \leq N$ such that their lateral surfaces $\partial\mathcal{T}_j$ belong to the boundary ∂G. Let each tube $\mathcal{T}_j$ in the local coordinate system (y, z) have the form $\mathcal{T}_j = \Omega_j \times (0, \ell_j)$ and let $\partial\mathcal{T}_j = \partial\Omega_j \times (0, \ell_j)$. Consider the subtube $\mathcal{T}_{j,h_j} = \Omega_j \times (h_j, \ell_j - h_j)$, where $h_j \in (0, \ell_j/2)$, and denote $\mathcal{T}_h = \bigcup\limits_{j=1}^{N} \mathcal{T}_{j,h_j}$, $G_h = G \setminus \overline{\mathcal{T}}_h$.

Define the average of the function $u \in L^2(\Omega_j)$ over the cross-section of the tube $\mathcal{T}_j$

$$[u]_j = \frac{1}{\text{meas }(\Omega_j)} \int\limits_{\Omega_j} u(y)\, dy.$$

Assume that K satisfies the following condition:

$$K(x) = K_j = \text{const} \quad \text{in } \mathcal{T}_j, \quad 1 \leq j \leq N.$$

2.3. Some Properties of Eigenfunctions

Let u be an eigenfunction corresponding to the eigenvalue λ. Denote by u_j the restriction of the function u to $\mathcal{T}_j$, $1 \leq j \leq N$. Let $r_j = u_j - [u_j]_j$. We will consider u_j and r_j as functions on $(0, \ell_j)$ with values in $H^1(\Omega_j)$. Note that $u_j, r_j \in L^2(0, \ell_j; H^1(\Omega_j))$, $D_z u_j, D_z r_j \in L^2(0, \ell_j; L^2(\Omega_j))$. We set $\widetilde{\lambda}_j = \lambda / K_j$.

Lemma 1. *The function $[u_j]_j$ belongs to the space $H^2(0, \ell_j)$, and*

$$-D_z^2[u_j]_j = \widetilde{\lambda}_j [u_j]_j. \tag{3}$$

Proof. It is clear that $[u_j]_j \in H^1(0, \ell_j)$ and $D_z[u_j]_j = [D_z u_j]_j$.

Let $\eta \in C_0^{\infty}(0, \ell_j)$. Substituting into (2) the function φ, equal to $K_j^{-1}\eta(z)$ on $\mathcal{T}_j$ and equal to zero on $G \setminus \mathcal{T}_j$, we obtain the identity

$$\int\limits_0^{\ell_j} D_z[u_j]_j(z)\eta'(z)\, dz = \int\limits_0^{\ell_j} \widetilde{\lambda}_j [u_j]_j(z)\eta(z)\, dz \quad \forall\, \eta \in C_0^{\infty}(0, \ell_j), \tag{4}$$

which implies the existence of the weak derivative $D_z^2[u_j]_j$ by definition and equality (3).

The lemma is proved. □

Lemma 2. *The functions u_j and r_j have the derivatives $D_z^2 u_j$ and $D_z^2 r_j$, which belong to the space $L^2_{loc}(0, \ell_j; L^2(\Omega_j))$.*

Proof. We introduce the finite-difference analogues of the derivative $D_z\Phi$

$$\partial^h \Phi(z) = \frac{\Phi(z+h) - \Phi(z)}{h}, \quad \overline{\partial}^h \Phi(z) = \frac{\Phi(z) - \Phi(z-h)}{h},$$

where $h > 0$.

Assume that $(a, b) \subset (0, \ell_j)$, and let $\zeta \in C_0^\infty(0, \ell_j)$ be a cut-off function, equal to one on (a, b). By substituting into identity (2) the test function φ equal to $-K_j^{-1}\overline{\partial}^h(\zeta^2 \partial^h u_j)$ on $\mathcal{T}_j$ and zero on $G \setminus \mathcal{T}_j$, we obtain

$$-\int_0^{\ell_j} \big(D_z u_j, \overline{\partial}^h(\zeta^2 \partial^h D_z u_j)\big)_{L^2(\Omega_j)}\, dz - \int_0^{\ell_j} \big(D_z u_j, \overline{\partial}^h(2\zeta\zeta' \partial^h u_j)\big)_{L^2(\Omega_j)}\, dz$$
$$-\int_0^{\ell_j} \big(\nabla_y u_j, \overline{\partial}^h(\zeta^2 \partial^h \nabla_y u_j)\big)_{L^2(\Omega_j)}\, dz = -\widetilde{\lambda}_j \int_0^{\ell_j} \big(u_j, \overline{\partial}^h(\zeta^2 \partial^h u_j)\big)_{L^2(\Omega_j)}\, dz.$$

Here, h is small enough.

Using the finite-difference analogue of the formula for integration by parts, we obtain

$$\int_0^{\ell_j} \|\zeta \partial^h D_z u_j\|^2_{L^2(\Omega_j)}\, dz + \int_0^{\ell_j} \|\zeta \partial^h \nabla_y u_j\|^2_{L^2(\Omega_j)}\, dz$$
$$= -\int_0^{\ell_j} (\zeta \partial^h D_z u_j, 2\zeta' \partial^h u_j)_{L^2(\Omega_j)}\, dz + \widetilde{\lambda}_j \int_0^{\ell_j} \|\zeta \partial^h u_j\|^2_{L^2(\Omega_j)}\, dz$$
$$\le \frac{1}{2} \int_0^{\ell_j} \|\zeta \partial^h D_z u_j\|^2_{L^2(\Omega_j)}\, dz + \frac{1}{2} \int_0^{\ell_j} \|2\zeta' \partial^h u_j\|^2_{L^2(\Omega_j)}\, dz + \widetilde{\lambda}_j \int_0^{\ell_j} \|\zeta \partial^h u_j\|^2_{L^2(\Omega_j)}\, dz.$$

From here,

$$\int_0^{\ell_j} \|\zeta \partial^h D_z u_j\|^2_{L^2(\Omega_j)}\, dz \le c_\zeta \|D_z u_j\|^2_{L^2(0,\ell_j;L^2(\Omega_j))}.$$

Thus, we have the following estimate uniform in h

$$\|\partial^h D_z u_j\|^2_{L^2(a,b;L^2(\Omega_j))} \le c_\zeta \|D_z u_j\|^2_{L^2(0,\ell_j;L^2(\Omega_j))},$$

where c_ζ is a constant depending on ζ.

It implies the existence of the derivative $D_z^2 u_j \in L^2(a, b; L^2(\Omega_j))$. Thus, $D_z^2 u_j \in L^2_{loc}(0, \ell_j; L^2(\Omega_j))$. As a consequence, there exists a derivative $D_z^2 r_j \in L^2_{loc}(0, \ell_j; L^2(\Omega_j))$.

The lemma is proved. □

Lemma 3. *For almost all $z \in (0, \ell_j)$ the following identities hold*

$$(D_z^2 u_j(z), \varphi)_{L^2(\Omega_j)} = (\nabla_y u_j(z), \nabla_y \varphi)_{L^2(\Omega_j)} - \widetilde{\lambda}_j (u_j(z), \varphi)_{L^2(\Omega_j)} \quad \forall\, \varphi \in H^1(\Omega_j). \tag{5}$$
$$(D_z^2 r_j(z), \varphi)_{L^2(\Omega_j)} = (\nabla_y r_j(z), \nabla_y \varphi)_{L^2(\Omega_j)} - \widetilde{\lambda}_j (r_j(z), \varphi)_{L^2(\Omega_j)} \quad \forall\, \varphi \in H^1(\Omega_j). \tag{6}$$

Proof. Let $\eta \in C_0^\infty(0, \ell_j)$ and $\varphi \in H^1(\Omega_j)$. Since $D_z^2 u_j \in L^2_{loc}(0, \ell_j; L^2(\Omega_j))$, then the substitution into (2) of the test function equal to $K_j^{-1}\eta(z)\varphi(y)$ on $\mathcal{T}_j$ and zero on $G \setminus \mathcal{T}_j$ and an integration by parts gives

$$-\int_0^{\ell_j} (D_z^2 u_j(z), \varphi)_{L^2(\Omega_j)} \eta(z)\, dz + \int_0^{\ell_j} (\nabla u_j(z), \nabla \varphi)_{L^2(\Omega_j)} \eta(z)\, dz = \widetilde{\lambda}_j \int_0^{\ell_j} (u(z), \varphi)_{L^2(\Omega_j)} \eta(z)\, dz.$$

This implies identity (5). As a consequence (cf. (4)), identity (6) holds.

The lemma is proved. □

3. Approximate Spectral Problem

This section is devoted to the setting of the problem referred to by us as the approximate spectral problem of hybrid dimensions. Section 3.1. contains the notations and preliminary results convenient for the formulation of the problem. In Section 3.2, we formulate the spectral problem along with its variational formulation in the suitable Hilbert spaces and show the discreteness of the spectrum. Moreover, we prove the required smoothness results for the eigenfunctions which, due to the junctions conditions (cf. (15) and (16)), can be weaker than those obtained for the eigenfunctions of the original problem in Section 2.3, but which somehow justify the junction conditions (cf. Corollaries 2 and 3).

3.1. Spaces $\mathcal{L}^2_h(G)$, $\widetilde{\mathcal{L}}^2_h(G)$, $\mathcal{H}^1_h(G)$, and $\widetilde{\mathcal{H}}^1_h(G)$

Henceforth, we will use the notation u_j for the restriction of a function $u \in L^2(G)$ to $\mathcal{T}_j$, $1 \le j \le N$.

Let $\mathcal{L}^2_h(G)$ be the close subspace of $L^2(G)$ of functions u such that $u_j(y,z) = [u_j]_j(z)$ for $x = (y,z) \in \mathcal{T}_{j,h_j}$, $1 \le j \le N$. Introduce also the space

$$\widetilde{\mathcal{L}}^2_h(G) = \{u \in \mathcal{L}^2_h(G) \mid \int\limits_G u\,dx = 0\}.$$

As noted in the introduction, the approximate spectral problem deals with the domain G with truncated tube-like parts. The following lemma pertains to the topological structure of the set $G_h = G \setminus \bigcup\limits_{j=1}^{N} \overline{\mathcal{T}_{j,h_j}}$.

Lemma 4. *The set G_h has the following structure:*

$$G_h = \bigcup_{k=1}^{m} G_{h,k}, \tag{7}$$

where $m \le N+1$ and $G_{h,k}$ are domains with Lipschitz boundaries such that $\overline{G_{h,k}} \cap \overline{G_{h,\ell}} = \emptyset$ for $k \ne \ell$.

Proof. Let $\widetilde{G}$ be an open subset of the set G. We will write that points $x, y \in \widetilde{G}$ are connected in $\widetilde{G}$ if there exists a continuous curve $L \subset \widetilde{G}$ starting at x and ending at y or $y = x$.

We set $G^0_h = G$ and $G^j_h = G^{j-1}_h \setminus \overline{\mathcal{T}_{j,h_j}}$, $1 \le j \le N$. Note that the boundary ∂G^j_h of each of the sets G^j_h satisfies the Lipschitz condition because $\partial G^j_h = \partial G^{j-1}_h \setminus (\partial\Omega_j \times (0,\ell_j)) \cup (\Omega_j \times \{0,\ell_j\})$.

Let us show that for all $1 \le j \le N$ the following equality holds:

$$G^j_h = \bigcup_{k=1}^{m_j} G_{h,k}, \tag{8}$$

where $m_j \le j+1$ and $G_{h,k}$ are domains with Lipschitz boundaries such that $\overline{G_{h,k}} \cap \overline{G_{h,\ell}} = \emptyset$ for $k \ne \ell$.

Introduce the sets $\mathcal{T}^\ell_{j,h_j}$ and $\mathcal{T}^r_{j,h_j}$, $1 \le j \le N$, which in the local coordinate system associated with $\mathcal{T}_j$ have the following form:

$$\mathcal{T}^\ell_{j,h_j} = \Omega_j \times (0,h_j), \quad \mathcal{T}^r_{j,h_j} = \Omega_j \times (\ell_j - h_j, \ell_j).$$

Step 1. Let us show that equality (8) is valid for $j = 1$. If the set $G^1_h = G \setminus \overline{\mathcal{T}_{1,h_1}}$ is connected, then (8) is true with $m_1 = 1$ and $G_{h,1} = G^1_h$.

Assume that G_h^1 is not connected. Fix points $x^\ell \in \mathcal{T}_{1,h_1}^\ell$ and $x^r \in \mathcal{T}_{1,h_1}^r$. Denote by $G_h^{1,\ell}$ the set of points $y \in G_h^1$, connected with x^ℓ in G_h^1. Similarly, denote by $G_h^{1,r}$ the set of points $y \in G_h^1$, connected with x^r in G_h^1. It is clear that $G_h^{1,\ell}$ and $G_h^{1,r}$ are disjoint domains and $\mathcal{T}_{1,h_1}^\ell \subset G_h^{1,\ell}$, $\mathcal{T}_{1,h_1}^r \subset G_h^{1,r}$.

Let us prove that

$$G_h^1 = G_h^{1,\ell} \cup G_h^{1,r} \tag{9}$$

Assume that $y \in G_h^1 \setminus G_h^{1,\ell}$. Since set G is connected, there is a curve $L = \{x = \zeta(t), 0 \le t \le 1\} \subset G$, where the function $\zeta : [0,1] \to G$ is continuous, and $\zeta(0) = y$, $\zeta(1) = x^\ell$.

Since $y \notin G_h^{1,\ell}$, then $L \cap \overline{\mathcal{T}_{1,h_1}} \ne \emptyset$, and there exists $t_* \in (0,1)$ such that $\zeta(t_*) \in \overline{\mathcal{T}}_{1,h_1}$ and $\zeta(t) \in G_h^1$ for $0 \le t < t_*$. Moreover, there exists $t \in (0, t_*)$ such that $\zeta(t) \in \mathcal{T}_{1,h_1}^\ell \cup \mathcal{T}_{1,h_1}^r$. If $\zeta(t) \in \mathcal{T}_{1,h_1}^\ell$ then points y and x^ℓ are connected in G_h^1, which contradicts the assumption $y \notin G_h^{1,\ell}$. Hence, $\zeta(t) \in \mathcal{T}_{1,h_1}^r$, and so $y \in G_h^{1,r}$. Thus, equality (9) is true, i.e., (8) is true with $m_1 = 2$ and $G_{h,1} = G_h^{1,\ell}$, $G_{h,2} = G_h^{1,r}$.

Note also that $\partial G_h^1 = \partial G_h^{1,\ell} \cup \partial G_h^{1,r}$, where $\partial G_h^{1,\ell} \cap \partial G_h^{1,r} = \emptyset$. Since ∂G_h^1 satisfies the Lipschitz condition, then $\partial G_h^{1,\ell}$ and $\partial G_h^{1,r}$ satisfy the Lipschitz condition also.

Step n**.** Assume that (8) is true for $j = n-1$. Since the set $\mathcal{T}_{j,h_j} \subset G_h^{j-1}$ is connected, then $\mathcal{T}_{j,h_j} \subset G_{h,k}$ for some $1 \le k \le m_{n-1}$. If $\widetilde{G}_{h,k} = G_{h,k} \setminus \overline{\mathcal{T}_{j,h_j}}$ is connected then

$$G_h^n = \bigcup_{1 \le i \le m_{n-1},\, i \ne k} G_{h,i} \cup \widetilde{G}_{h,k}.$$

If the set $\widetilde{G}_{h,k}$ is not connected, then repeating the argument at step 1 (with G_h^1 replaced by $\widetilde{G}_{h,k}$ and $\mathcal{T}_{1,h_1}^\ell$, $\mathcal{T}_{1,h_1}^r$ replaced by $\mathcal{T}_{j,h_j}^\ell$, $\mathcal{T}_{j,h_j}^r$), we see that $\widetilde{G}_{h,k}$ is represented as the union of two non-intersecting regions $\widetilde{G}_{h,k}^\ell$ and $\widetilde{G}_{h,k}^r$. In this case

$$G_h^n = \bigcup_{1 \le i \le m_{n-1},\, i \ne k} G_{h,i} \cup \widetilde{G}_{h,k}^\ell \cup \widetilde{G}_{h,k}^r.$$

Thus, representation (8) also holds for $j = n$.

Having done N steps, we claim equality (7).

The lemma is proved. □

Introduce the space $\mathcal{H}_h^1(G)$ consisting of functions $u \in \mathcal{L}_h^2(G)$ such that $u \in H^1(G_{h,k})$ for all $1 \le k \le m$, $u_j \in H^1(\mathcal{T}_{j,h})$ for all $1 \le j \le N$, and the following junction conditions

$$[u]_j|_{z=h_j-0} = u_j|_{z=h_j+0}, \quad u_j|_{z=\ell_j-h_j-0} = [u]_j|_{z=\ell_j-h_j+0} \tag{10}$$

hold.

Naturally, $u_j = [u_j]_j \in H^1(0,\ell_j)$ and $\nabla u_j(x) = (0,0,D_z u_j(z))$ for $x = (y,z) \in \mathcal{T}_{j,h}$, $1 \le j \le N$.

Note that $\mathcal{H}_h^1(G)$ is a Hilbert space with the inner product and the norm

$$(u,v)_{\mathcal{H}_h^1(G)} = (K\nabla u, \nabla v)_{L^2(G)} + (u,v)_{L^2(G)},$$
$$\|u\|_{\mathcal{H}_h^1(G)} = \left(\|\sqrt{K}\nabla u\|_{L^2(G)}^2 + \|u\|_{L^2(G)}^2\right)^{1/2}.$$

Lemma 5. *The embedding of $\mathcal{H}_h^1(G)$ into $\mathcal{L}_h^2(G)$ is compact.*

Proof. Assume that $\{u_n\}_{n=1}^\infty \subset \mathcal{H}_h^1(G)$ is a sequence that converges weakly in $\mathcal{H}_h^1(G)$. As a result, it weakly converges in $H^1(G_{h,k})$ for all $1 \le k \le m$ and in $H^1(\mathcal{T}_{j,h})$ for all

$1 \le j \le N$. So, $\{u_n\}_{n=1}^{\infty}$ converges strongly in $L^2(G_{h,k})$ for all $1 \le k \le m$ and in $L^2(\mathcal{T}_{j,h})$ for all $1 \le j \le N$. Thus, it converges strongly in $\mathcal{L}_h^2(G)$.

The lemma is proved. □

Lemma 6. *Suppose that $u \in \mathcal{H}_h^1(G)$ and $\nabla u = 0$. Then, $u = \text{const}$.*

Proof. It follows from $\nabla u = 0$ that $u(x) = c_{h,k} = \text{const}$ in $G_{h,k}$ for all $1 \le k \le m$ and $u(x) = c_j = \text{const}$ in $\mathcal{T}_{j,h_j}$ for all $1 \le j \le m$. Hence, due to the connectedness of G and condition (10), it implies that $u(x) = \text{const}$ in G.

The lemma is proved. □

The following analogue of the Poincaré inequality holds in $\mathcal{H}_h^1(G)$ (see analogous inequalities in some spaces of discontinuous functions in [19,24]).

Lemma 7. *There exists a constant $C > 0$ such that*

$$\|u\|^2_{\mathcal{H}_h^1(G)} \le C\Big[(K\nabla u, \nabla u)_{L^2(G)} + \big(\int_G u\,dx\big)^2\Big] \quad \forall\, u \in \mathcal{H}_h^1(G). \tag{11}$$

Proof. Assume the opposite. Then, there is a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{H}_h^1(G)$ such that

$$\|u_n\|^2_{\mathcal{H}_h^1(G)} > n\Big[(K\nabla u_n, \nabla u_n)_{L^2(G)} + \big(\int_G u_n\,dx\big)^2\Big] \quad \forall\, n \ge 1.$$

Putting $v_n = \dfrac{1}{\|u_n\|_{\mathcal{H}_h^1(G)}} u_n$, we have $\|v_n\|_{\mathcal{H}_h^1(G)} = 1$ and

$$\frac{1}{n} > (K\nabla v_n, \nabla v_n)_{L^2(G)} + \Big(\int_G v_n\,dx\Big)^2 \quad \forall\, n \ge 1. \tag{12}$$

Using the compactness of embedding of $\mathcal{H}_h^1(G)$ into $\mathcal{L}_h^2(G)$, we choose a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ such that $v_{n_k} \to v$ weakly in $\mathcal{H}_h^1(G)$ and strongly in $\mathcal{L}_h^2(G)$.

It follows from (12) that $\|\nabla v_{n_k}\|_{L^2(G)} \to 0$ and $\int_G v_{n_k}\,dx \to 0$. So, $\nabla v = 0$, $v = c = \text{const}$ and $\int_G v\,dx = c \cdot \text{meas}\, G = 0$. Thus, $v = 0$ and $v_{n_k} \to 0$ in $\mathcal{H}_h^1(G)$ which contradicts the equality $\|v_{n_k}\|_{\mathcal{H}_h^1(G)} = 1$.

The lemma is proved. □

Let us introduce in $\mathcal{H}_h^1(G)$ the closed subspace

$$\widetilde{\mathcal{H}}_h^1(G) = \mathcal{H}_h^1(G) \cap \widetilde{\mathcal{L}}_h^2(G) = \{u \in \mathcal{H}_h^1(G) \mid \int_G u\,dx = 0\}.$$

It follows from (11) that $\widetilde{\mathcal{H}}_h^1(G)$ is a Hilbert space with the following inner product and the norm:

$$(u, v)_{\widetilde{\mathcal{H}}_h^1(G)} = (K\nabla u, \nabla v)_{L^2(G)}, \quad \|u\|_{\widetilde{\mathcal{H}}_h^1(G)} = \|\sqrt{K}\,\nabla u\|_{L^2(G)}.$$

Moreover, it follows from Lemma 5 that the embedding of $\widetilde{\mathcal{H}}_h^1(G)$ into $\widetilde{\mathcal{L}}_h^2(G)$ is compact.

3.2. Setting of the Approximate Spectral Problem

Consider the following spectral problem: find a couple $(\widehat{\lambda}, U) \in \mathbb{R} \times \mathcal{H}_h^1(G)$, such that $\|U\|_{L^2(G)} = 1$, satisfying the equation

$$-\text{div}(K\Delta U) = \widehat{\lambda}\, U, \quad x \in G_h \cup \mathcal{T}_h, \tag{13}$$

the boundary condition

$$K\nabla U \cdot n = 0, \quad x \in \partial G \tag{14}$$

and the following junction conditions:

$$[U_j]|_{z=h_j-0} = U_j|_{z=h_j+0}, \quad U_j|_{z=\ell_j-h_j-0} = [U_j]|_{z=\ell_j-h_j+0}, \quad 1 \le j \le N, \tag{15}$$

$$D_z U_j|_{z=h_j-0} = D_z U_j|_{z=h_j+0}, \quad D_z U_j|_{z=\ell_j-h_j-0} = D_z U_j|_{z=\ell_j-h_j+0}, \quad 1 \le j \le N. \tag{16}$$

where U_j is the restriction of U on $\mathcal{T}_j$.

The weak formulation of this problem is given by the following identity:

$$(K\nabla U, \nabla\varphi)_{L^2(G)} = \widehat{\lambda}(U, \varphi)_{L^2(G)} \quad \forall\, \varphi \in \mathcal{H}^1_h(G). \tag{17}$$

Note that conditions (15) are satisfied since $U \in \mathcal{H}^1_h(G)$. We will show also that (17) implies a validity of junction conditions (16) (see Corollaries 2 and 3).

It is clear that each eigenvalue $\widehat{\lambda}$ is non-negative, the minimal eigenvalue is $\widehat{\lambda}_0 = 0$, and the corresponding eigenfunction $\widehat{U}_0$ is a constant.

Since the embedding of $\widetilde{\mathcal{H}}^1_h(G)$ into $\widetilde{\mathcal{L}}^2_h(G)$ is dense and compact, we are in a classical abstract framework of bilinear, continuous, coercive forms on a couple of Hilbert spaces $\widetilde{\mathcal{H}}^1_h(G)$ and $\widetilde{\mathcal{L}}^2_h(G)$, cf., for instance, Theorem 5.5 of Chapter I in [25]. So, the following theorem is true.

Theorem 1. *There exists a system $\{\widehat{\varphi}_k\}_{k=1}^{\infty}$ of eigenfunctions to problem (17) corresponding to the eigenvalues $0 < \widehat{\lambda}_1 \le \widehat{\lambda}_2 \le \cdots \le \widehat{\lambda}_k \le \ldots$ forming an orthonormal base in $\widetilde{\mathcal{H}}^1_h(G)$ and an orthogonal base in $\widetilde{\mathcal{L}}^2_h(G)$.*

Corollary 1. *There is a system $\{\widehat{\varphi}_k\}_{k=0}^{\infty}$ of eigenfunctions to problem (17) corresponding to the eigenvalues $0 = \widehat{\lambda}_0 < \widehat{\lambda}_1 \le \widehat{\lambda}_2 \le \ldots \widehat{\lambda}_k \le \ldots$, forming an orthonormal base in $\mathcal{L}^2_h(G)$ and an orthogonal base in $\mathcal{H}^1_h(G)$.*

3.3. Some Properties of Eigenfunctions

Henceforth, in this section, $\widehat{\lambda}$ is an eigenvalue and $U \in \mathcal{H}^1_h(G)$ is corresponding eigenfunction such that $\|U\|_{L^2(G)} = 1$. Moreover, $\overline{\lambda}_j = \widehat{\lambda}/K_j$, $1 \le j \le N$.

Lemma 8. *The function U satisfies the identity*

$$(K\nabla U, \nabla\varphi)_{L^2(G)} = \widehat{\lambda}(U, \varphi)_{L^2(G)} \quad \forall\, \varphi \in H^1(G). \tag{18}$$

Proof. Let $\varphi \in H^1(G)$. Note that the function

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x), & x \in G_h, \\ [\varphi]_j(x), & x \in \mathcal{T}_{j,h_j}, \quad 1 \le j \le N \end{cases}$$

belongs to $\mathcal{H}^1_h(G)$ and

$$(K\nabla U, \nabla\varphi)_{L^2(G)} - \widehat{\lambda}(U, \varphi)_{L^2(G)} = (K\nabla U, \nabla\widetilde{\varphi})_{L^2(G)} - \widehat{\lambda}(U, \widetilde{\varphi})_{L^2(G)}$$

So, (18) holds.

The lemma is proved. □

Remember that U_j is the restriction of U on $\mathcal{T}_j$, $1 \le j \le m$. We will consider U_j as a function on $(0, \ell_j)$ with values in $H^1(\Omega_j)$. We put $W_j(z) = D_z U_j(z)$ for $z \in (0, h_j) \cup (\ell_j - h_j, \ell_j)$ and $W_j(z) = D_z[U_j]_j(z)$ for $z \in (h_j, \ell_j - h_j)$. It is clear that $U_j \in L^2(0, \ell_j; H^1(\Omega_j))$ and $W_j \in L^2(0, \ell_j; L^2(\Omega_j))$. Moreover, $[U_j]_j \in L^2(0, \ell_j)$ and $D_z[U_j]_j \in L^2(0, \ell_j)$.

Lemma 9. *The function W_j has a derivative $D_zW_j \in L^2(0,\ell_j;(H^1(\Omega_j))')$.*

Proof. It follows from (17) that

$$\int_0^{\ell_j}(W_j(z),\varphi)_{L^2(\Omega_j)}\eta'(z)\,dz+\int_0^{\ell_j}\Big[(\nabla_y U_j(z),\nabla_y\varphi)_{L^2(\Omega_j)}-\overline{\lambda}_j(U_j(z),\varphi)_{L^2(\Omega_j)}\Big]\eta(z)\,dz=0 \quad (19)$$

for all $\varphi \in H^1(\Omega_j)$ and $\eta \in C_0^\infty(0,\ell_j)$. Remember that $\overline{\lambda}_j = \widehat{\lambda}/K_j$

Therefore, the function W_j has a derivative $D_zW_j : (0,\ell_j) \to (H^1(\Omega_j))'$ such that

$$\langle D_zW_j(z),\varphi\rangle_{(H^1(\Omega_j))'\times H^1(\Omega_j)} = (\nabla_y U_j(z),\nabla_y\varphi)_{L^2(\Omega_j)} - \overline{\lambda}_j(U_j(z),\varphi)_{L^2(\Omega_j)} \quad \forall\,\varphi \in H^1(\Omega_j).$$

for almost all $z \in (0,\ell_j)$.

It follows from the estimate

$$\begin{aligned}\big|\langle D_zW_j(z),\varphi\rangle_{(H^1(\Omega_j))'\times H^1(\Omega_j)}\big| &\le \|\nabla_y U_j(z)\|_{L^2(\Omega_j)}\|\nabla_y\varphi\|_{L^2(\Omega_j)} + \overline{\lambda}_j\|U_j(z)\|_{L^2(\Omega_j)}\|\varphi\|_{L^2(\Omega_j)}\\ &\le \max\{1,\overline{\lambda}_j\}\|U_j(z)\|_{H^1(\Omega_j)}\|\varphi\|_{H^1(\Omega_j)}\end{aligned}$$

that

$$\|D_zW_j\|_{L^2(0,\ell_j;(H^1(\Omega_j))')} \le \max\{1,\overline{\lambda}_j\}\|U_j\|_{L^2(0,\ell_j;H^1(\Omega_j))}.$$

Thus, $D_zW_j \in L^2(0,\ell_j;(H^1(\Omega_j))')$.

The lemma is proved. □

Corollary 2. *$W_j \in C([0,\ell_j];(H^1(\Omega_j))')$, and so, the eigenfunction U satisfies the junction conditions (16) in $(H^1(\Omega_j))'$.*

Lemma 10. *The function $[U_j]_j$ belongs to the space $H^2(0,\ell_j)$, and*

$$-D_z^2[U_j]_j = \overline{\lambda}_j[U_j]_j.$$

Proof. The substitution of $\varphi = K_j^{-1}\eta$ with $\eta \in C_0^\infty(0,\ell_j)$ into (17) gives

$$\int_0^{\ell_j} D_z[U_j]_j(z)\eta'(z)\,dz = \overline{\lambda}_j\int_0^{\ell_j}[U_j]_j(z)\eta(z)\,dz = 0 \quad \forall\,\eta \in C_0^\infty(0,\ell_j). \quad (20)$$

So, $D_z^2[U_j]_j = -\overline{\lambda}_j[U]_j$.

The lemma is proved. □

We put $R_j = U_j - [U_j]_j$, $1 \le j \le N$ and will consider R_j as a function on $(0,\ell_j)$ with values in $H^1(\Omega_j)$. It is clear that $R_j = 0$, $D_zR_j = 0$ on (h_j,ℓ_j-h_j) and $D_zR_j = W_j - D_z[U_j]_j$ on $(0,h_j)\cup(\ell_j-h_j,\ell_j)$.

It follows from Lemmas 9 and 10 that $D_zR_j \in C([0,\ell_j];(H^1(\Omega_j))')$. So $D_zR_j(h_j) = 0$ and $D_zR_j(\ell_j-h_j) = 0$ in $(H^1(\Omega_j))'$.

Lemma 11. *For each $0 < a < h_j$ the function R_j has derivatives $D_z^2R_j \in L^2(a,h_j;L^2(\Omega_j))$ and $D_z^2R_j \in L^2(\ell_j-h_j,\ell_j-a;L^2(\Omega_j))$.*

Proof. We put $\widetilde{R}_j(z) = R_j(z)$ for $0 < z \le h_j$ and $\widetilde{R}_j(z) = R_j(2h_j - z)$ for $h_j \le z < 2h_j$. Note that $\widetilde{R}_j \in C(0,2h_j;L^2(\Omega_j))$ and $D_z\widetilde{R}_j \in L^2(0,2h_j;L^2(\Omega_j))$.

Let $\varphi \in L^2(0, 2h_j; H^1(\Omega_j))$ be an arbitrary function identically equal to zero in some neighborhoods of the points $z = 0$ and $z = 2h_j$ and such that $D_z\varphi \in L^2(0, 2h_j; L^2(\Omega_j))$.

As $R_j(z) = 0$ for $z \in (h_j, \ell_j - h_j)$ then (17) and (20) imply that

$$\int_0^{h_j} (D_z R_j, D_z\varphi)_{L^2(\Omega_j)}\, dz + \int_0^{h_j} (\nabla_y R_j, \nabla_y \varphi)_{L^2(\Omega_j)}\, dz = \overline{\lambda}_j \int_0^{h_j} (R_j, \varphi)_{L^2(\Omega_j)}\, dz.$$

Moreover,

$$-\int_{h_j}^{2h_j} (D_z \widetilde{R}_j, D_z\widetilde{\varphi})_{L^2(\Omega_j)}\, dz + \int_{h_j}^{2h_j} (\nabla_y \widetilde{R}_j, \nabla_y \widetilde{\varphi})_{L^2(\Omega_j)}\, dz = \overline{\lambda}_j \int_{h_j}^{2h_j} (\widetilde{R}_j, \widetilde{\varphi})_{L^2(\Omega_j)}\, dz,$$

where $\widetilde{\varphi}(z) = \varphi(2h_j - z)$. Consequently,

$$\int_0^{2h_j} (D_z \widetilde{R}_j, D_z\varphi)_{L^2(\Omega_j)}\, dz + \int_0^{2h_j} (\nabla_y \widetilde{R}_j, \nabla \varphi)_{L^2(\Omega_j)}\, dz = \overline{\lambda}_j \int_0^{2h_j} (\widetilde{R}_j, \varphi)_{L^2(\Omega_j)}\, dz. \tag{21}$$

Assume that $0 < a < h_j < b < 2h_j$ and $\zeta \in C_0^\infty(0, 2h_j)$ is a cutoff function, equal to one on (a, b). Substituting into (21) the test function $\varphi = -\overline{\partial}^h(\zeta^2 \partial^h \widetilde{R}_j)$, where h is small enough, we have

$$\begin{aligned}
&-\int_0^{2h_j} (D_z \widetilde{R}_j, \overline{\partial}^h(\zeta^2 \partial^h D_z \widetilde{R}_j))_{L^2(\Omega_j)}\, dz - \int_0^{2h_j} (D_z \widetilde{R}_j, \overline{\partial}^h(2\zeta\zeta' \partial^h \widetilde{R}_j))_{L^2(\Omega_j)}\, dz \\
&-\int_0^{2h_j} (\nabla_y \widetilde{R}_j, \nabla_y \overline{\partial}^h(\zeta^2 \partial^h \widetilde{R}_j))_{L^2(\Omega_j)}\, dz = -\overline{\lambda}_j \int_0^{2h_j} (\widetilde{R}_j, \overline{\partial}^h(\zeta^2 \partial^h \widetilde{R}_j))_{L^2(\Omega_j)}\, dz.
\end{aligned}$$

Using the difference analogue of the formula for integration by parts, we obtain

$$\begin{aligned}
&\int_0^{2h_j} \|\zeta \partial^h D_z \widetilde{R}_j\|^2_{L^2(\Omega_j)}\, dz + \int_0^{2h_j} \|\zeta \partial^h \nabla_y \widetilde{R}_j\|^2_{L^2(\Omega_j)}\, dz \\
&= -\int_0^{2h_j} (\zeta \partial^h D_z \widetilde{R}_j, 2\zeta' \partial^h \widetilde{R}_j)_{L^2(\Omega_j)}\, dz + \widetilde{\lambda}_j \int_0^{2h_j} \|\zeta \partial^h \widetilde{R}_j\|^2_{L^2(\Omega_j)}\, dz \\
&\le \frac{1}{2} \int_0^{2h_j} \|\zeta \partial^h D_z \widetilde{R}_j\|^2_{L^2(\Omega_j)}\, dz + \frac{1}{2} \int_0^{2h_j} \|2\zeta' \partial^h \widetilde{R}_j\|^2_{L^2(\Omega_j)}\, dz + \overline{\lambda}_j \int_0^{2h_j} \|\zeta \partial^h \widetilde{R}_j\|^2_{L^2(\Omega_j)}\, dz.
\end{aligned}$$

From here, we have the following estimate uniform in h

$$\|\partial^h D_z \widetilde{R}_j\|^2_{L^2(a,b;L^2(\Omega_j))} \le c_\zeta \|D_z \widetilde{R}_j\|^2_{L^2(0,2h_j;L^2(\Omega_j))} = 2c_\zeta \|D_z R_j\|^2_{L^2(0,h_j;L^2(\Omega_j))}.$$

This implies the existence of the derivative $D_z^2 \widetilde{R}_j \in L^2(a, b; L^2(\Omega_j))$. Thus, $D_z^2 R_j \in L^2(a, h_j; L^2(\Omega_j))$.

The existence of the derivative $D_z^2 R_j \in L^2(\ell_j - h_j, \ell_j - a; L^2(\Omega_j))$ is proved in a similar way.

The lemma is proved. □

Corollary 3. *The following properties hold:*

(1) $D_z R_j \in C((0, h_j]; L^2(\Omega_j))$ *and* $D_z R_j(h_j - 0) = 0$ *in* $L^2(\Omega_j)$.
(2) $D_z R_j \in C([\ell_j - h_j, \ell_j); L^2(\Omega_j))$ *and* $D_z R_j(\ell_j - h_j + 0) = 0$ *in* $L^2(\Omega_j)$.
(3) *The eigenfunction* U *satisfies the junction conditions in (16) in* $L^2(\Omega_j)$.

Proof. To prove properties (1) and (2), it should be taken into account that, due to Corollary 2, $D_z R_j(h_j - 0) = 0$, $D_z R_j(\ell_j - h_j + 0) = 0$ in $(H^1(\Omega_j))'$, and $H^1(\Omega_j)$ is dense in $L^2(\Omega_j)$.

Property (3) holds because $D_z U_j = D_z[U_j]_j + D_z R_j$, where $D_z R_j(z) = 0$ for $z \in (h_j, \ell_j - h_j)$ and $D_z[U_j]_j \in C[0, \ell_j]$.

The corollary is proved. □

Lemma 12. *For almost all* $z \in (0, h_j) \cup (\ell_j - h_j, \ell_j)$, *the following identities hold*

$$(D_z^2 U_j(z), \varphi)_{L^2(\Omega_j)} = (\nabla_y U_j(z), \nabla_y \varphi)_{L^2(\Omega_j)} - \overline{\lambda}_j (U_j(z), \varphi)_{L^2(\Omega_j)} \quad \forall\, \varphi \in H^1(\Omega_j). \tag{22}$$

$$(D_z^2 R_j(z), \varphi)_{L^2(\Omega_j)} = (\nabla_y R_j(z), \nabla_y \varphi)_{L^2(\Omega_j)} - \overline{\lambda}_j (R_j(z), \varphi)_{L^2(\Omega_j)} \quad \forall\, \varphi \in H^1(\Omega_j). \tag{23}$$

The proof is similar to the proof of Lemma 3.

4. Formulation of the Main Results

Let $\{\lambda_k\}_{k=0}^{\infty}$ be the set of eigenvalues to problem (2), and $\{\widehat{\lambda}_k\}_{k=0}^{\infty}$ be the set of eigenvalues to problem (17).

Let ε be the desired accuracy, $0 < \varepsilon < 1$ and $\varepsilon_1 = \dfrac{\varepsilon}{1+\varepsilon}$.

Let $0 = \mu_{0,j} < \mu_{1,j} \le \mu_{2,j} \le \dots$ be the set of eigenvalues to the Neumann problem

$$-\Delta_y \psi = \mu \psi, \quad y \in \Omega_j,$$
$$\nabla_y \psi \cdot \nu = 0, \quad y \in \partial\Omega_j,$$

and $\{\psi_{k,j}\}_{k=0}^{\infty}$ be the set of corresponding eigenfunctions, forming an orthogonal base in $H^1(\Omega_j)$ and an orthonormal base in $L^2(\Omega_j)$.

Theorem 2. *Let* λ *be the eigenvalue of problem (2) and* $\widetilde{\lambda}_j = \lambda / K_j < \mu_{1,j}$ *for all* $1 \le j \le N$. *Let the numbers* h_j *be such that*

$$\frac{1}{\nu_{1,j}} \max\left\{ \ln\frac{2}{\varepsilon_1^2}, \ln\frac{96^2 K_j^2 \nu_{1,j}^4}{\varepsilon_1^2},\ \ln\frac{24^2 K_j \lambda \nu_{1,j}^2}{\varepsilon_1^2} \right\} \le h_j < \ell_j/2, \quad 1 \le j \le N, \tag{24}$$

where $\nu_{1,j} = \sqrt{\mu_{1,j} - \widetilde{\lambda}_j}$. *Then, the following estimate holds:*

$$\min_{k \ge 0} |\widehat{\lambda}_k - \lambda| \le \varepsilon. \tag{25}$$

Theorem 3. *Let* $\widehat{\lambda}$ *be the eigenvalue of problem (17) and* $\overline{\lambda}_j = \widehat{\lambda}/K_j < \mu_{1,j}$ *for all* $1 \le j \le N$. *Let the numbers* h_j *be such that*

$$\frac{1}{\widehat{\nu}_{1,j}} \max\left\{ \ln\frac{4}{\varepsilon_1^2}, \ln\frac{96^2 K_j^2 \widehat{\nu}_{1,j}^4}{\varepsilon_1^2},\ \ln\frac{12^2 K_j \widehat{\lambda} \widehat{\nu}_{1,j}^2}{\varepsilon_1^2} \right\} \le h_j < \ell_j/2, \quad 1 \le j \le N, \tag{26}$$

where $\widehat{\nu}_{1,j} = \sqrt{\mu_{1,j} - \overline{\lambda}_j}$. *Then, the following estimate holds:*

$$\min_{k \ge 0} |\lambda_k - \widehat{\lambda}| \le \varepsilon. \tag{27}$$

Consider a special case where, for each of the tubes $\mathcal{T}_j$, its cross section is a disk with a radius of ϵ_j. Remember that, in this case, $\mu_{1,j} = p^2/\epsilon_j^2$, where $p = 1.8142\ldots$ is the smallest positive root of the derivative J_1' of the Bessel function J_1. So, Theorems 2 and 3 can be reformulated as follows.

Theorem 4. *Let λ be the eigenvalue of problem (2) and $\lambda < K_j p^2/\epsilon_j^2$ for all $1 \le j \le N$. Let the numbers h_j be such that*

$$\frac{\sqrt{K_j}}{\sqrt{\rho_j}} \max\left\{\ln\frac{2}{\varepsilon_1^2}, \ln\frac{96^2\rho_j^2}{\varepsilon_1^2},\ \ln\frac{24^2\lambda\rho_j}{\varepsilon_1^2}\right\} \le h_j < \ell_j/2, \quad 1 \le j \le N,$$

where $\rho_j = K_j p^2/\epsilon_j^2 - \lambda$. Then, the following estimate holds:

$$\min_{k\ge 0} |\widehat{\lambda}_k - \lambda| \le \varepsilon.$$

Theorem 5. *Let $\widehat{\lambda}$ be the eigenvalue of problem (17) and $\widehat{\lambda} < K_j p^2/\epsilon_j^2$ for all $1 \le j \le N$. Let the numbers h_j be such that*

$$\frac{\sqrt{K_j}}{\widehat{\rho}_j} \max\left\{\ln\frac{4}{\varepsilon_1^2}, \ln\frac{96^2\widehat{\rho}_j^2}{\varepsilon_1^2},\ \ln\frac{12^2\widehat{\lambda}\widehat{\rho}_j}{\varepsilon_1^2}\right\} \le h_j < \ell_j/2, \quad 1 \le j \le N,$$

where $\widehat{\rho}_j = \sqrt{K_j p^2/\epsilon_j^2 - \widehat{\lambda}}$. Then, the following estimate holds:

$$\min_{k\ge 0} |\lambda_k - \widehat{\lambda}| \le \varepsilon.$$

5. Proof of Theorem 2

Let $u \in H^1(G)$ be the eigenfunction of problem (2) corresponding to the eigenvalue λ such that $\|u\|_{L^2(G)} = 1$. Recall that according to the hypothesis of the theorem, $\widetilde{\lambda}_j = \lambda/K_j < \mu_{1,j}$ for all $1 \le j \le N$ and $\nu_{k,j} = \sqrt{\mu_{k,j} - \widetilde{\lambda}_j}$ for $k \ge 1$.

Henceforth, we will use the following notations:

$$\mathcal{T}_{j,h_j/2,h_j} = \Omega_j \times [(h_j/2, h_j) \cup (\ell_j - h_j, \ell_j - h_j/2)],$$
$$\mathcal{T}_{j,h_j/4,h_j} = \Omega_j \times [(h_j/4, h_j) \cup (\ell_j - h_j, \ell_j - h_j/4)].$$

We introduce the function $\widehat{u} \in \mathcal{H}_h^1(G)$ by the formula

$$\widehat{u}(x) = \begin{cases} u(x), & x \in G_{h/2} = G \setminus \bigcup_{j=1}^{N} \overline{\mathcal{T}_{j,h_j/2}}, \\ u_j(y,z) - \zeta_j(z) r_j(y,z), & x = (y,z) \in \mathcal{T}_{j,h_j/2}, \quad 1 \le j \le N, \end{cases}$$

where $\quad r_j(y,z) = u_j(y,z) - [u_j]_j(z), \quad (y,z) \in \mathcal{T}_{j,h_j/2}, \quad 1 \le j \le N \quad$ and

$$\zeta_j(z) = \begin{cases} \dfrac{1}{2} + \dfrac{3}{h_j}(z - 3h_j/4) - \dfrac{16}{h_j^3}(z - 3h_j/4)^3, & z \in [h_j/2, h_j], \\ \dfrac{1}{2} - \dfrac{3}{h_j}(z - \ell_j + 3h_j/4) + \dfrac{16}{h_j^3}(z - \ell_j + 3h_j/4)^3, & z \in [\ell_j - h_j, \ell_j - h_j/2], \\ 1, & z \in [h_j, \ell_j - h_j]. \end{cases} \tag{28}$$

Note that $\zeta_j \in H^2(h_j/2, \ell_j - h_j/2)$. Moreover,

$$\zeta_j(h_j/2) = 0, \quad \zeta_j'(h_j/2) = 0, \quad \zeta_j(\ell_j - h_j/2) = 0, \quad \zeta_j'(\ell_j - h_j/2) = 0,$$
$$0 \le \zeta_j \le 1, \quad |\zeta_j'| \le \frac{3}{h_j}, \quad |\zeta_j''| \le \frac{24}{h_j^2}. \tag{29}$$

Let us expand $r_j(z)$ and $D_z r_j(z)$ for almost all $z \in (0, \ell_j)$ into a Fourier series converging in $L^2(\Omega_j)$

$$r_j(z) = \sum_{k=1}^{\infty} c_{k,j}(z)\psi_{k,j}, \quad D_z r_j(z, y) = \sum_{k=1}^{\infty} c'_{k,j}(z)\psi_{k,j},$$

where $c_{k,j}(z) = (r_j(z), \psi_{k,j})_{L^2(\Omega_j)}$, $c'_{k,j}(z) = (D_z r_j(z), \psi_{k,j})_{L^2(\Omega_j)}$. It is clear that $c_{0,j}(z) = 0$, $c'_{0,j}(z) = 0$.

It follows from (2) and (20) that

$$\int_0^{\ell_j} (D_z r_j(z), \eta'(z)\psi_{k,j})_{L^2(\Omega_j)}\, dz + \int_0^{\ell_j} (\nabla_y r_j(z), \eta(z)\nabla_y \psi_{k,j})_{L^2(\Omega_j)}\, dz$$
$$= \widetilde{\lambda}_j \int_0^{\ell_j} (r_j(z), \eta(z)\psi_{k,j})_{L^2(\Omega_j)}\, dz \quad \forall\, \eta \in C_0^\infty(0, \ell_j).$$

Taking into account that

$$(\nabla_y r_j(z), \nabla_y \psi_{k,j})_{L^2(\Omega_j)} = \mu_{k,j}(r_j(z), \psi_{k,j})_{L^2(\Omega_j)},$$

we have

$$\int_0^{\ell_j} c'_{k,j}(z)\eta'(z)\, dz + \int_0^{\ell_j} \nu_{k,j}^2 c_{k,j}(z)\eta(z)\, dz = 0 \quad \forall\, \eta \in C_0^\infty(0, \ell_j).$$

Thus, for all $k \ge 1$,

$$c''_{k,j}(z) = \nu_{k,j}^2 c_{k,j}(z), \quad z \in (0, \ell_j). \tag{30}$$

Lemma 13. *Let $h_j < \ell_j/2$, $1 \le j \le N$. Then, for all $k \ge 1$, the following estimates hold*

$$\|c_{k,j}\|^2_{L^2(h_j/2, h_j)} + \|c_{k,j}\|^2_{L^2(\ell_j - h_j, \ell_j - h_j/2)} \le 4e^{-\nu_{1,j}h_j}\|c_{k,j}\|^2_{L^2(0,\ell_j)}, \tag{31}$$
$$\|c_{k,j}\|^2_{L^2(h_j/2, \ell_j - h_j/2)} \le 2e^{-\nu_{1,j}h_j}\|c_{k,j}\|^2_{L^2(0,\ell_j)}, \tag{32}$$
$$\|c'_{k,j}\|^2_{L^2(h_j/2, h_j)} + \|c'_{k,j}\|^2_{L^2(\ell_j - h_j, \ell_j - h_j/2)} \le 4e^{-\nu_{1,j}h_j}\|c'_{k,j}\|^2_{L^2(0,\ell_j)},. \tag{33}$$

Proof. Since the coefficients $c_{k,j}$ satisfy Equation (30), then for all $0 < \xi < h_j/2 < z < h_j$ the following formulas hold:

$$
\begin{aligned}
&c_{k,j}(z) \\
&= \frac{[e^{-\nu_{k,j}(z-\xi)} - e^{-\nu_{k,j}(2\ell_j-3\xi-z)}]c_{k,j}(\xi) + [e^{-\nu_{k,j}(\ell_j-\xi-z)} - e^{-\nu_{k,j}(\ell_j-3\xi+z)}]c_{k,j}(\ell_j-\xi)}{1-e^{-2\nu_{k,j}(\ell_j-2\xi)}} \\
&= \frac{e^{-\nu_{k,j}(z-\xi)}[1-e^{-2\nu_{k,j}(\ell_j-\xi-z)}]c_{k,j}(\xi) + e^{-\nu_{k,j}(\ell_j-\xi-z)}[1-e^{-2\nu_{k,j}(z-\xi)}]c_{k,j}(\ell_j-\xi)}{1-e^{-2\nu_{k,j}(\ell_j-2\xi)}}, \\
&c_{k,j}(\ell_j-z) \\
&= \frac{[e^{-\nu_{k,j}(\ell_j-z-\xi)} - e^{-\nu_{k,j}(\ell_j-3\xi+z)}]c_{k,j}(\xi) + [e^{-\nu_{k,j}(z-\xi)} - e^{-\nu_{k,j}(2\ell_j-3\xi-z)}]c_{k,j}(\ell_j-\xi)}{1-e^{-2\nu_{k,j}(\ell_j-2\xi)}} \\
&= \frac{e^{-\nu_{k,j}(\ell_j-z-\xi)}[1-e^{-2\nu_{k,j}(z-\xi)}]c_{k,j}(\xi) + e^{-\nu_{k,j}(z-\xi)}[1-e^{-2\nu_{k,j}(\ell_j-z-\xi)}]c_{k,j}(\ell_j-\xi)}{1-e^{-2\nu_{k,j}(\ell_j-2\xi)}},
\end{aligned}
$$

From here,

$$
\begin{aligned}
&c_{k,j}^2(z) \le 2\big[e^{-2\nu_{k,j}(z-\xi)}c_{k,j}^2(\xi) + e^{-2\nu_{k,j}(\ell_j-\xi-z)}c_k^2(\ell_j-\xi)\big], \\
&c_{k,j}^2(\ell_j-z) \le 2\big[e^{-2\nu_{k,j}(\ell_j-\xi-z)}c_{k,j}^2(\xi) + e^{-2\nu_{k,j}(z-\xi)}c_{k,j}^3(\ell_j-\xi)\big].
\end{aligned}
$$

Summing up these estimates, we have

$$
\begin{aligned}
c_{k,j}^2(z) + c_{k,j}^2(\ell_j-z) &\le 2\big[e^{-2\nu_{k,j}(z-\xi)} + e^{-2\nu_{k,j}(\ell_j-\xi-z)}\big][c_{k,j}^2(\xi) + c_{k,j}^2(\ell_j-\xi)] \\
&\le 2\big[e^{-2\nu_{1,j}(z-\xi)} + e^{-2\nu_{1,j}(\ell_j-\xi-z)}\big][c_{k,j}^2(\xi) + c_{k,j}^2(\ell_j-\xi)\big].
\end{aligned}
$$

From here,

$$
e^{-2\nu_{1,j}\xi}\big[c_{k,j}^2(z) + c_{k,j}^2(\ell-z)\big] \le 2\big[e^{-2\nu_{1,j}z} + e^{-2\nu_{1,j}(\ell_j-z)}\big]\big[c_{k,j}^2(\xi) + c_{k,j}^2(\ell-\xi)\big]. \tag{34}
$$

Integrating this inequality over $\xi \in (0, h_j/2)$ and over $z \in (h_j/2, h_j)$, we arrive at the estimate

$$
\begin{aligned}
&\|c_{k,j}\|^2_{L^2(h_j/2,h_j)} + \|c_{k,j}\|^2_{L^2(\ell_j-h_j,\ell_j-h_j/2)} \\
&\le 2\big[e^{-\nu_{1,j}h_j} + e^{-\nu_{1,j}(2\ell_j-3h_j)}\big]\big[\|c_{k,j}\|^2_{L^2(0,h_j/2)} + \|c_{k,j}\|^2_{L^2(\ell_j-h_j/2,\ell_j)}\big] \le 4e^{-\nu_{1,j}h_j}\|c_{k,j}\|^2_{L^2(0,\ell_j)}.
\end{aligned}
$$

Integrating inequality (34) over $\xi \in (0, h_j/2)$ and over $z \in (h_j/2, \ell_j/2)$, we have

$$
\begin{aligned}
&(1-e^{-\nu_{1,j}h_j})\|c_{k,j}\|^2_{L^2(h_j/2,\ell_j-h_j/2)} \\
&\le 2(e^{-\nu_{1,j}h_j} - e^{-2\nu_{1,j}(\ell_j-h_j/2)})\big[\|c_{k,j}\|^2_{L^2(0,h_j/2)} + \|c_{k,j}\|^2_{L^2(\ell_j-h_j/2,\ell_j)}\big] \\
&\le 2e^{-\nu_{1,j}h_j}\big[\|c_{k,j}\|^2_{L^2(0,h_j/2)} + \|c_{k,j}\|^2_{L^2(\ell_j-h_j/2,\ell_j)}\big].
\end{aligned} \tag{35}
$$

From here,

$$
\begin{aligned}
\|c_{k,j}\|^2_{L^2(h_j/2,\ell_j-h_j/2)} &\le 2e^{-\nu_{1,j}h_j}\big[\|c_{k,j}\|^2_{L^2(0,h_j/2)} + \|c_{k,j}\|^2_{L^2(\ell_j-h_j/2,\ell_j)}\big] \\
&\quad + e^{-\nu_{1,j}h_j}\|c_{k,j}\|^2_{L^2(h_j/2,\ell_j-h_j/2)} \le 2e^{-\nu_{1,j}h_j}\|c_{k,j}\|^2_{L^2(0,\ell_j)}.
\end{aligned}
$$

It follows from (30) that the coefficients $c'_{k,j}$ satisfy the equation $(c'_{k,j})'' = \nu_{k,j}^2 c'_{k,j}$. Therefore, estimate (33) is proved in exactly the same way as estimate (31).

The lemma is proved. □

Corollary 4. *Let $h_j < \ell_j/2$, $1 \le j \le N$. Then, the following estimates hold:*

$$\|r_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})} \le 4e^{-\nu_{1,j}h_j}\|u_j\|^2_{L^2(\mathcal{T}_j)}, \tag{36}$$

$$\|D_z r_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})} \le 4e^{-\nu_{1,j}h_j}\|D_z u_j\|^2_{L^2(\mathcal{T}_j)}. \tag{37}$$

Proof. Indeed, it follows from (31) that

$$\|r_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})} = \sum_{k=1}^{\infty}\left[\|c_{k,j}\|^2_{L^2(h_j/2,h_j)} + \|c_{k,j}\|^2_{L^2(\ell_j-h_j,\ell_j-h_j/2)}\right]$$
$$\le 4e^{-\nu_{1,j}h_j}\sum_{k=1}^{\infty}\|c_{k,j}\|^2_{L^2(0,\ell_j)} = 4e^{-\nu_{1,j}h_j}\|u_j-[u_j]_j\|^2_{L^2(\mathcal{T}_j)} \le 4e^{-\nu_{1,j}h_j}\|u_j\|^2_{L^2(\mathcal{T}_j)}.$$

Similarly, estimate (37) follows from (33).

The corollary is proved. □

Lemma 14. *Let $\dfrac{1}{\nu_{1,j}}\ln\dfrac{2}{\varepsilon_1^2} \le h_j \le \ell_j/2$, $1 \le j \le N$. Then, the following estimate holds:*

$$\|\widehat{u}-u\|_{L^2(G)} \le \varepsilon_1. \tag{38}$$

Proof. Using estimate (32) and noting that $2e^{-\nu_{1,j}h_j} \le \varepsilon_1^2$, we have

$$\|\widehat{u}-u\|^2_{L^2(G)} \le \sum_{j=1}^{N}\|r_j\|^2_{L^2(\mathcal{T}_{j,h_j/2})}$$
$$= \sum_{j=1}^{N}\sum_{k=1}^{\infty}\|c_{k,j}\|^2_{L^2(h_j/2,\ell_j-h_j/2)} \le \sum_{j=1}^{N}\sum_{k=1}^{\infty}2e^{-\nu_{1,j}h_j}\|c_{k,j}\|^2_{L^2(0,\ell_j)}$$
$$\le \sum_{j=1}^{N}\|u_j-[u_j]_j\|^2_{L^2(\mathcal{T}_j)}\varepsilon_1^2 \le \sum_{j=1}^{N}\|u_j\|^2_{L^2(\mathcal{T}_j)}\varepsilon_1^2 \le \|u\|^2_{L^2(G)}\varepsilon_1^2 = \varepsilon_1^2.$$

The lemma is proved. □

Lemma 15. *The function $\widehat{u}$ satisfies the identity*

$$(K\nabla\widehat{u},\nabla\varphi)_{L^2(G)} - \lambda(\widehat{u},\varphi)_{L^2(G)} = (\Psi,\varphi)_{L^2(G)} \quad \forall\,\varphi \in \mathcal{H}^1_h(G), \tag{39}$$

where

$$\Psi(x) = \begin{cases} 0, & x \in G \setminus \bigcup_{j=1}^{N}\mathcal{T}_{j,h_j/2,h_j}, \\ \Psi_j(x) = K_j[\zeta_j''(z)r_j(y,z) + 2\zeta_j'(z)D_z r_j(y,z)], & x = (y,z) \in \mathcal{T}_{j,h_j/2,h_j},\ 1 \le j \le N. \end{cases}$$

Proof. Let $\varphi \in \mathcal{H}^1_h(G)$, and let functions $\eta_j \in C^\infty[0,\ell_j]$, where $1 \le j \le N$, be such that $\eta_j(z) = 0$ for $z \in [0,h_j/4] \cup [\ell_j-h_j/4,\ell_j]$, and $\eta_j(z) = 1$ for $z \in [h_j/2,\ell_j-h_j/2]$.

Introduce functions

$$\psi_0(x) = \begin{cases} \varphi(x), & x \in G \setminus \bigcup_{j=1}^{N}\mathcal{T}_{j,h_j/4}, \\ (1-\eta_j(z))\varphi(y,z), & x = (y,z) \in \mathcal{T}_{j,h_j/4}, \quad 1 \le j \le N. \end{cases}$$

$$\psi_j(x) = \begin{cases} 0, & x \in G \setminus \mathcal{T}_{j,h_j/4}, \\ \eta_j(z)\varphi(y,z), & x = (y,z) \in \mathcal{T}_{j,h_j/4}. \end{cases}$$

As $\phi = \sum_{j=0}^{N} \psi_j$ then

$$(K\nabla\widehat{u}, \nabla\varphi)_{L^2(G)} - \lambda(\widehat{u}, \varphi)_{L^2(G)} = (K\nabla\widehat{u}, \nabla\psi_0)_{L^2(G)} - \lambda(\widehat{u}, \psi_0)_{L^2(G)} + \sum_{j=1}^{N} K_j\Big[(\nabla\widehat{u}, \nabla\psi_j)_{L^2(\mathcal{T}_{j,h_j/4})} - \widetilde{\lambda}_j(\widehat{u}, \psi_j)_{L^2(\mathcal{T}_{j,h_j/4})}\Big]$$

Note that $\psi_0 \in H^1(G)$, $\psi_0 = 0$ in $G \setminus G_{h/2}$ and $\widehat{u} = u$ in $G_{h/2}$. So,

$$(K\nabla\widehat{u}, \nabla\psi_0)_{L^2(G)} - \lambda(\widehat{u}, \psi_0)_{L^2(G)} = (K\nabla u, \nabla\psi_0)_{L^2(G)} - \lambda(u, \psi_0)_{L^2(G)} = 0.$$

Note also that

$$\begin{aligned}
&(\nabla\widehat{u}, \nabla\psi_j)_{L^2(\mathcal{T}_{j,h_j/4})} - \widetilde{\lambda}_j(\widehat{u}, \psi_j)_{L^2(\mathcal{T}_{j,h_j/4})}\\
&= (D_z\widehat{u}, D_z\psi_j)_{L^2(\mathcal{T}_{j,h_j/4,h_j})} + (\nabla_y\widehat{u}, \nabla_y\psi_j)_{L^2(\mathcal{T}_{j,h_j/4,h_j})} - \widetilde{\lambda}_j(\widehat{u}, \psi_j)_{L^2(\mathcal{T}_{j,h_j/4,h_j})}\\
&+ (D_z[u_j]_j, D_z\psi_j)_{L^2(\mathcal{T}_{j,h_j})} - \widetilde{\lambda}_j([u_j]_j, \psi_j)_{L^2(\mathcal{T}_{j,h_j})}.
\end{aligned}$$

Integrating by parts over z and taking into account that $\psi_j(h_j/4) = 0$, $\psi_j(\ell_j - h_j/4) = 0$, and $-D_z^2[u_j]_j = \widetilde{\lambda}_j[u_j]_j$ in $\mathcal{T}_{j,h_j}$, we have

$$\begin{aligned}
&(\nabla\widehat{u}, \nabla\psi_j)_{L^2(\mathcal{T}_{j,h_j/4})} - \widetilde{\lambda}_j(\widehat{u}, \psi_j)_{L^2(\mathcal{T}_{j,h_j/4})}\\
&= -(D_z^2\widehat{u} + \widetilde{\lambda}_j\widehat{u}, \psi_j)_{L^2(\mathcal{T}_{j,h_j/4,h_j})} + (\nabla_y\widehat{u}, \nabla_y\psi_j)_{L^2(\mathcal{T}_{j,h_j/4,h_j})}\\
&+ (D_z\widehat{u}, \psi_j)_{L^2(\Omega_j)}|_{z=h_j-0} + (D_z[u_j]_j, \psi_j)_{L^2(\Omega_j)}|_{z=\ell_j-h_j-0}\\
&- (D_z[u_j]_j, \psi_j)_{L^2(\Omega_j)}|_{z=h_j+0} - (D_z\widehat{u}, \psi_j)_{L^2(\Omega_j)}|_{z=\ell_j-h_j-0}.
\end{aligned}$$

Noting that $D_z\widehat{u}(h_j - 0) = D_z[u_j]_j(h_j - 0)$ does not depend on y and $[\psi_j]_j(h_j - 0) = \psi_j(h_j + 0)$, we have

$$\begin{aligned}
&(D_z\widehat{u}, \psi_j)_{L^2(\Omega_j)}|_{z=h_j-0} - (D_z[u_j]_j, \psi_j)_{L^2(\Omega_j)}|_{z=h_j+0}\\
&= (D_z[u_j]_j, [\psi_j]_j)_{L^2(\Omega_j)}|_{z=h_j-0} - (D_z[u_j]_j, \psi_j)_{L^2(\Omega_j)}|_{z=h_j+0} = 0.
\end{aligned}$$

Likewise

$$(D_z\widehat{u}, \psi_j)_{L^2(\Omega_j)}|_{z=\ell_l-h_j-0} - (D_z[u]_j, \psi_j)_{L^2(\Omega_j)}|_{\ell_j-h_j+0} = 0.$$

Thus, using (5) we obtain

$$\begin{aligned}
&(\nabla\widehat{u}, \nabla\psi_j)_{L^2(\mathcal{T}_{j,h_j/4})} - \widetilde{\lambda}_j(\widehat{u}, \psi_j)_{L^2(\mathcal{T}_{j,h_j/4})}\\
&= -(D_z^2\widehat{u} + \widetilde{\lambda}_j\widehat{u}, \psi_j)_{L^2(\mathcal{T}_{j,h_j/4,h_j})} + (\nabla_y\widehat{u}, \nabla_y\psi_j)_{L^2(\mathcal{T}_{j,h_j/4,h_j})}\\
&= -(D_z^2 u + \widetilde{\lambda}_j u, \psi_j)_{L^2(\mathcal{T}_{j,h_j/4,h_j})} + (\nabla_y u, \nabla_y\psi_j)_{L^2(\mathcal{T}_{j,h_j/4,h_j})}\\
&+ (D_z^2(\zeta_j r_j) + \widetilde{\lambda}_j(\zeta_j r_j), \psi_j)_{L^2(\mathcal{T}_{j,h_j/2,h_j})} - (\nabla_y(\zeta_j r_j), \nabla_y\psi_j)_{L^2(\mathcal{T}_{j,h_j/2,h_j})}\\
&= (\zeta_j'' r_j + 2\zeta_j' D_z r_j, \psi_j)_{L^2(\mathcal{T}_{j,h_j/2,h_j})} +\\
&+ (D_z^2 r_j + \widetilde{\lambda}_j r_j, \zeta_j\psi_j)_{L^2(\mathcal{T}_{j,h_j/2,h_j})} - (\nabla_y r_j, \zeta_j\nabla_y\psi_j)_{L^2(\mathcal{T}_{j,h_j/2,h_j})}.
\end{aligned}$$

By virtue of identity (6), the last two terms vanish. Noting that $\psi_j = \varphi$ on $\mathcal{T}_{j,h_j/2,h_j}$, we obtain

$$(K\nabla\widehat{u}, \nabla\varphi)_{L^2(G)} - \lambda(\widehat{u}, \varphi)_{L^2(G)} = \sum_{j=1}^{N} K_j(\zeta_j'' r_j + 2\zeta_j' D_z r_j, \psi_j)_{L^2()} = (\Psi, \varphi)_{L^2(G)}.$$

The lemma is proved. □

Lemma 16. *Assume that the numbers h_j satisfy the conditions in (24). Then, the following inequalities hold:*

$$2K_j^2 \frac{48^2}{h_j^4} e^{-\nu_{1,j}h_j} \leq \frac{1}{2}\varepsilon_1^2, \quad 2K_j \frac{12^2}{h_j^2} e^{-\nu_{1,j}h_j} \leq \frac{1}{2\lambda}\varepsilon_1^2. \tag{40}$$

Proof. Let us set $t = \nu_{1,j}h_j$ and transform the inequalities in (40) to the form

$$t + 4\ln t \geq \alpha_j, \quad t + 2\ln t \geq \beta_j \tag{41}$$

where $\alpha_j = \ln \dfrac{96^2 K_j^2 \nu_{1,j}^4}{\varepsilon_1^2}$, $\beta_j = \ln \dfrac{24^2 K_j \lambda \nu_{1,j}^2}{\varepsilon_1^2}$.

The function $g(t) = t + 4\ln t$ is increasing. If $\alpha_j \leq 1$, then $g(t) \geq g(1) = 1 \geq \alpha_j$ for $t \geq 1$. If $\alpha_j \geq 1$, then $g(t) \geq g(\alpha_j) \geq \alpha_j$ for $t \geq \alpha_j$. Thus, the first inequality in (41) holds for $h_j \geq \dfrac{1}{\nu_{1,j}} \max\{\alpha_j, 1\}$.

Similarly, the second inequality (41) holds for $h_j \geq \dfrac{1}{\nu_{1,j}} \max\{\beta_j, 1\}$.

It should be noted that (24) implies that $h_j\nu_{1,j} \geq \max\{\alpha_j, \beta_j, 1\}$. Furthermore, note that $\nu_{1,j}h_j > \ln(2/\varepsilon_1^2)$ already implies that $\nu_{1,j}h_j > 1$.

The lemma is proved. □

Lemma 17. *Assume that the numbers h_j satisfy the conditions in (24). Then, the following estimate is true:*

$$\|\Psi\|_{L^2(G)} \leq \varepsilon_1. \tag{42}$$

Proof. Using estimates (29), (36), (37), we have

$$\|\Psi\|^2_{L^2(G)} = \sum_{j=1}^{N} \|\Psi_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})} \leq 2\sum_{j=1}^{N} K_j^2 \left[\|\zeta_j'' r_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})} + \|2\zeta_j' D_z r_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})}\right]$$
$$\leq 2\sum_{j=1}^{N} K_j^2 \left[\frac{24^2}{h_j^4}\|r_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})} + \frac{6^2}{h_j^2}\|D_z r_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})}\right]$$
$$\leq 2\sum_{j=1}^{N} K_j^2 e^{-\nu_{1,j}h_j}\left[\frac{48^2}{h_j^4}\|u_j\|^2_{L^2(\mathcal{T}_j)} + \frac{12^2}{h_j^2}\|\nabla u_j\|^2_{L^2(\mathcal{T}_j)}\right].$$

Due to the estimates in (40), we have

$$\|\Psi\|^2_{L^2(G)} \leq \sum_{j=1}^{N}\left[\|u_j\|^2_{L^2(\mathcal{T}_j)} + \frac{1}{\lambda}\|\sqrt{K_j}\nabla u_j\|^2_{L^2(\mathcal{T}_j)}\right]\frac{\varepsilon_1^2}{2} \leq \left[\|u\|_{L^2(G)} + \frac{1}{\lambda}\|\sqrt{K}\nabla u\|^2_{L^2(G)}\right]\frac{\varepsilon_1^2}{2}.$$

Taking into account that $\frac{1}{\lambda}\|\sqrt{K}\nabla u\|^2_{L^2(G)} = \|u\|^2_{L^2(G)} = 1$, we arrive at inequality (42).

The lemma is proved. □

Let us now show that estimate (25) holds under the conditions of Theorem 2.

Let $\{\widehat{\lambda}_k\}_{k=0}^{\infty}$ be the set of eigenvalues to problem (17) and $\{\widehat{\varphi}_k\}_{k=0}^{\infty}$ be the set of corresponding eigenfunctions, forming an orthonormal base in $\mathcal{L}_h^2(G)$.

From (39) and equality

$$(K\nabla\widehat{u}, \nabla\widehat{\varphi}_k)_{L^2(G)} = \widehat{\lambda}_k(\widehat{u}, \widehat{\varphi}_k)_{L^2(G)}, \quad k \geq 0$$

it follows that

$$(\widehat{\lambda}_k - \lambda)(\widehat{u}, \widehat{\varphi}_k)_{L^2(G)} = (\Psi, \widehat{\varphi}_k)_{L^2(G)}, \quad k \geq 0.$$

So,

$$\min_{k\geq 0} (\widehat{\lambda}_k - \lambda)^2 \|\widehat{u}\|^2_{L^2(G)} \leq \sum_{k=0}^{\infty} (\widehat{\lambda}_k - \lambda)^2 (\widehat{u}, \widehat{\varphi}_k)^2_{L^2(G)} \leq \sum_{k=0}^{\infty} (\Psi, \widehat{\varphi}_k)^2_{L^2(G)} \leq \|\Psi\|^2_{L^2(G)}.$$

Hence,

$$\min_{k\geq 0} |\widehat{\lambda}_k - \lambda| (\|u\|_{L^2(G)} - \|\widehat{u} - u\|_{L^2(G)}) \leq \min_{k\geq 0} |\widehat{\lambda}_k - \lambda| \, \|\widehat{u}\|_{L^2(G)} \leq \|\Psi\|_{L^2(G)}.$$

As $\|u\|_{L^2(G)} = 1$, $\|\widehat{u} - u\|_{L^2(G)} \leq \varepsilon_1$ and $\|\Psi\|_{L^2(G)} \leq \varepsilon_1$, we obtain the estimate

$$\min_{k\geq 0} |\widehat{\lambda}_k - \lambda| \, (1 - \varepsilon_1) \leq \varepsilon_1,$$

which is equivalent to estimate (25).

Theorem 2 is proved.

6. Proof of Theorem 3

Let $U \in \mathcal{H}_h^1(G)$ be the eigenfunction of problem (17) corresponding to the eigenvalue $\widehat{\lambda}$ and such that $\|U\|_{L^2(G)} = 1$. Recall that according to the hypothesis of the theorem, $\overline{\lambda}_j = \widehat{\lambda}/K_j < \mu_{1,j}$ for all $1 \leq j \leq N$ and $\widehat{\nu}_{k,j} = \sqrt{\mu_{k,j} - \overline{\lambda}_j}$ for $k \geq 1$.

We introduce the function $\widehat{U} \in H^1(G)$ by the formula

$$\widehat{U}(x) = \begin{cases} U(x), & x \in G \setminus \bigcup_{j=1}^{N} \mathcal{T}_{j,h_j/2,h_j} \\ U_j(y,z) - \zeta_j(z) R_j(y,z), & x = (y,z) \in \mathcal{T}_{j,h_j/2,h_j}, \quad 1 \leq j \leq N, \end{cases}$$

where

$$R_j(y,z) = U_j(y,z) - [U_j]_j(z), \quad (y,z) \in \mathcal{T}_{j,h_j/2,h_j}, \quad 1 \leq j \leq N,$$

and the function ζ_j is defined by Formula (28).

Let us expand $R_j(z)$ and $D_z R_j(z)$ for almost all $z \in (0, h_j) \cup (\ell_j - h_j, \ell_j)$ into a Fourier series converging in $L^2(\Omega_j)$:

$$R_j(z) = \sum_{k=1}^{\infty} c_{k,j}(z)\psi_{k,j}, \quad D_z R_j(z) = \sum_{k=1}^{\infty} c'_{k,j}(z)\psi_{k,j}.$$

where $c_{k,j}(z) = (R_j(z), \psi_{k,j})_{L^2(\Omega_j)}$, $c'_{k,j}(z) = (D_z R_j(z), \psi_{k,j})_{L^2(\Omega_j)}$.

It follows from (17) that

$$\int_0^{h_j} (D_z R_j(z), \eta'(z)\psi_{k,j})_{L^2(\Omega_j)}\, dz + \int_0^{h_j} (\nabla_y R_j(z), \eta(z)\nabla_y \psi_{k,j})_{L^2(\Omega_j)}\, dz$$
$$= \overline{\lambda}_j \int_0^{h_j} (R_j(z), \eta(z)\psi_{k,j})_{L^2(\Omega_j)}\, dz \quad \forall\, \eta \in C_0^\infty(0, h_j).$$

Taking into account that

$$(\nabla_y R_j(z), \nabla_y \psi_{k,j})_{L^2(\Omega_j)} = \mu_{k,j}(R_j(z), \psi_{k,j})_{L^2(\Omega_j)},$$

for almost all $z \in (0, h_j) \cup (\ell_j - h_j, \ell_j)$, we have

$$\int_0^{h_j} c'_{k,j}(z)\eta'(z)\, dz + \int_0^{h_j} \widehat{\nu}^2_{k,j} c_{k,j}(z)\eta(z)\, dz = 0 \quad \forall\, \eta \in C_0^\infty(0, h_j) \cup C_0^\infty(\ell_j - h_j, \ell_j).$$

Thus,

$$c''_{k,j}(z) = \widehat{\nu}^2_{k,j} c_{k,j}(z), \quad z \in (0, h_j) \cup (\ell_j - h_j, \ell_j). \tag{43}$$

Remember that $D_z R_j(h_j) = 0$, $D_z R_j(\ell_j - h_j) = 0$ (cf. Corollary 3.5). Hence,

$$c'_{k,j}(h_j) = 0, \quad c'_{k,j}(\ell_j - h_j) = 0 \quad \forall\, k \geq 1.$$

Lemma 18. *Let $h_j \leq \ell_j/2$, $1 \leq j \leq N$. Then, for all $k \geq 1$, the following estimates hold:*

$$\|c_{k,j}\|^2_{L^2(h_j/2, h_j)} + \|c_{k,j}\|^2_{L^2(\ell_j - h_j, \ell_j - h_j/2)} \leq 4e^{-\widehat{\nu}_{1,j} h_j} \|c_{k,j}\|^2_{L^2(0,\ell_j)}, \tag{44}$$
$$\|c'_{k,j}\|^2_{L^2(h_j/2, h_j)} + \|c'_{k,j}\|^2_{L^2(\ell_j - h_j, \ell_j - h_j/2)} \leq e^{-\widehat{\nu}_{1,j} h_j} \|c'_{k,j}\|^2_{L^2(0,\ell_j)}. \tag{45}$$

Proof. Let $0 < \xi < h_j/2 < z < h_j$. As the coefficients $c_{k,j}$ satisfy equation (43) and the condition $c'_{k,j}(h_j) = 0$, we have

$$c_{k,j}(z) = c_{k,j}(\xi) e^{-\widehat{\nu}_{k,j}(z-\xi)} \frac{1 + e^{-2\widehat{\nu}_{k,j}(h_j - z)}}{1 + e^{-2\widehat{\nu}_{k,j}(h_j - \xi)}}.$$

From here,

$$c^2_{k,j}(z) \leq 4c^2_{k,j}(\xi) e^{-2\widehat{\nu}_{k,j}(z-\xi)} \leq 4c^2_{k,j}(\xi) e^{-2\widehat{\nu}_{1,j}(z-\xi)}.$$

So,

$$c^2_{k,j}(z) e^{-2\widehat{\nu}_{1,j}\xi} \leq 4c^2_{k,j}(\xi) e^{-2\widehat{\nu}_{1,j} z}.$$

Integrating this inequality over $\xi \in (0, h_j/2)$ and $z \in (h_j/2, h_j)$, we come to an inequality

$$\|c_{k,j}\|^2_{L^2(h_j/2, h_j)} \leq 4e^{-\widehat{\nu}_{1,j} h_j} \|c_{k,j}\|^2_{L^2(0, h_j/2)}. \tag{46}$$

The following inequality is proved in a similar way

$$\|c_{k,j}\|^2_{L^2(\ell_j - h_j, \ell_j - h_j/2)} \leq 4e^{-\widehat{\nu}_{1,j} h_j} \|c_k\|^2_{L^2(\ell_j - h_j/2, \ell_j)}. \tag{47}$$

By adding inequalities (46) and (47) and coarsening the result, we arrive at inequality (44).

From (43), it follows that the coefficients $c'_{k,j}$ satisfy the equation $(c'_k)'' = \widehat{\nu}^2_{k,j} c'_k$. Taking into account that $c'_{k,j}(h_j) = 0$, we have for all $0 < \xi < h_j/2 < z < h_j$

$$c'_{k,j}(z) = c'_{k,j}(\xi) e^{-\widehat{\nu}_{k,j}(z-\xi)} \frac{1 - e^{-2\widehat{\nu}_{k,j}(h_j - z)}}{1 - e^{-2\widehat{\nu}_{k,j}(h_j - \xi)}}.$$

From here,

$$|c'_k(z)|^2 \le |c'_k(\xi)|^2 e^{-2\widehat{\nu}_{k,j}(z-\xi)} \le |c'_k(\xi)|^2 e^{-2\widehat{\nu}_{1,j}(z-\xi)}.$$

Hence,

$$e^{-2\widehat{\nu}_{1,j}\xi} |c'_{k,j}(z)|^2 \le |c'_{k,j}(\xi)|^2 e^{-2\widehat{\nu}_{1,j} z}.$$

Integrating this inequality over $\xi \in (0, h_j/2)$ and $z \in (h_j/2, h_j)$, we come to an inequality

$$\|c'_{k,j}\|^2_{L^2(h_j/2, h_j)} \le e^{-\widehat{\nu}_{1,j} h_j} \|c'_{k,j}\|^2_{L^2(0, h_j/2)}. \tag{48}$$

In the same way, we prove the inequality

$$\|c'_{k,j}\|^2_{L^2(\ell_j - h_j, \ell_j - h_j/2)} \le e^{-\widehat{\nu}_{1,j} h_j} \|c'_{k,j}\|^2_{L^2(\ell_j - h_j/2, \ell_j)}. \tag{49}$$

By adding inequalities (48) and (49) and coarsening the result, we arrive at (45).

The lemma is proved. □

Corollary 5. *Let $h_j < \ell_j/2$, $1 \le j \le N$. Then, the following estimates hold:*

$$\|R_j\|^2_{L^2(\mathcal{T}_{j, h_j/2, h_j})} \le 4 e^{-\widehat{\nu}_{1,j} h_j} \|U_j\|^2_{L^2(\mathcal{T}_j)}, \tag{50}$$

$$\|D_z R_j\|^2_{L^2(\mathcal{T}_{j, h_j/2, h_j})} \le e^{-\widehat{\nu}_{1,j} h_j} \|D_z U_j\|^2_{L^2(\mathcal{T}_j)}. \tag{51}$$

Proof. Indeed,

$$\begin{aligned}
&\|R_j\|^2_{L^2(\mathcal{T}_{j, h_j/2, h_j})} = \sum_{k=1}^{\infty} \left[\|c_{k,j}\|^2_{L^2(h_j/2, h_j)} + \|c_{k,j}\|^2_{L^2(\ell_j - h_j, \ell_j - h_j/2)} \right] \\
&\le 4 e^{-\widehat{\nu}_{1,j} h_j} \sum_{k=1}^{\infty} \|c_{k,j}\|^2_{L^2(0, \ell_j)} = 4 e^{-\widehat{\nu}_{1,j} h_j} \|U_j - [U_j]_j\|^2_{L^2(\mathcal{T}_j)} \le 4 e^{-\widehat{\nu}_{1,j} h_j} \|U_j\|^2_{L^2(\mathcal{T}_j)}, \\
&\|D_z R_j\|^2_{L^2(\mathcal{T}_{j, h_j/2, h_j})} = \sum_{k=1}^{\infty} \left[\|c'_{k,j}\|^2_{L^2(h_j/2, h_j)} + \|c'_{k,j}\|^2_{L^2(\ell_j - h_j, \ell_j - h_j/2)} \right] \\
&\le e^{-\widehat{\nu}_{1,j} h_j} \sum_{k=1}^{\infty} \|c'_{k,j}\|^2_{L^2(0, \ell_j)} = e^{-\widehat{\nu}_{1,j} h_j} \|D_z U_j - [D_z U_j]_j\|^2_{L^2(\mathcal{T}_j)} \le e^{-\widehat{\nu}_{1,j} h_j} \|D_z U_j\|^2_{L^2(\mathcal{T}_j)}.
\end{aligned}$$

The lemma is proved. □

Lemma 19. *Let $\dfrac{1}{\widehat{\nu}_{1,j}} \ln \dfrac{4}{\varepsilon_1^2} \le h_j < \ell_j/2$ for all $1 \le j \le N$. Then, the following estimate holds:*

$$\|\widehat{U} - U\|_{L^2(G)} \le \varepsilon_1. \tag{52}$$

Proof. Using estimate (50) and taking into account that $4e^{-\widehat{\nu}_{1,j}h_j} \leq \varepsilon_1^2$, we have

$$\|\widehat{U}-U\|^2_{L^2(G)} = \sum_{j=1}^N \|\zeta_j R_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})} \leq \sum_{j=1}^N \|R_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})}$$
$$\leq \sum_{j=1}^N 4e^{-\widehat{\nu}_{1,j}h_j}\|U_j\|^2_{L^2(\mathcal{T}_j)} \leq \sum_{j=1}^N \|U_j\|^2_{L^2(\mathcal{T}_j)}\varepsilon_1^2 \leq \|U\|^2_{L^2(G)}\varepsilon_1^2 = \varepsilon_1^2.$$

The lemma is proved. □

Lemma 20. *The following identity is true:*

$$(K\nabla\widehat{U},\nabla\varphi)_{L^2(G)} - \widehat{\lambda}(\widehat{U},\varphi)_{L^2(G)} = (\Phi,\varphi)_{L^2(G)} \quad \forall\,\varphi \in H^1(G), \tag{53}$$

where

$$\Phi(x) = \begin{cases} 0, & x \in G \setminus \bigcup_{j=1}^N \mathcal{T}_{j,h_j/2,h_j}, \\ \Phi_j(x) = K_j[\zeta_j''(z)R_j(y,z) + 2\zeta_j'(z)D_zR_j(y,z)], & x = (y,z) \in \mathcal{T}_{j,h_j/2,h_j},\ 1 \leq j \leq N. \end{cases}$$

Proof. From (18), it follows that

$$(K\nabla\widehat{U},\nabla\varphi)_{L^2(G)} - \widehat{\lambda}(\widehat{U},\varphi)_{L^2(G)} = (K\nabla(\widehat{U}-U),\nabla\varphi)_{L^2(G)} - \widehat{\lambda}(\widehat{U}-U,\varphi)_{L^2(G)}$$
$$= -\sum_{j=1}^N K_j\Big[(D_z(\zeta_jR_j),D_z\varphi)_{L^2(\mathcal{T}_{j,h_j/2,h_j})} + (\zeta_j\nabla_yR_j,\nabla_y\varphi)_{L^2(\mathcal{T}_{j,h_j/2,h_j})} - \overline{\lambda}_j(\zeta_jR_j,\varphi)_{L^2(\mathcal{T}_{j,h_j/2,h_j})}\Big].$$

Using the fact that $D_z(\zeta_jR_j)(z) = 0$ in $L^2(\Omega_j)$ for $z = h_j/2, h_j, \ell_j - h_j, \ell_j - h_j/2$, we perform the transformation

$$(D_z(\zeta_jR_j),D_z\varphi)_{L^2(\mathcal{T}_{j,h_j/2,h_j})} = -(D_z^2(\zeta_jR_j),\varphi)_{L^2(\mathcal{T}_{j,h_j/2,h_j})}$$
$$= -(\zeta_jD_z^2R_j,\varphi)_{L^2(\mathcal{T}_{j,h_j/2,h_j})} - (\zeta_j''R_j + 2\zeta_j'D_zR_j,\varphi)_{L^2(\mathcal{T}_{j,h_j/2,h_j})}.$$

As a result, we obtain

$$(K\nabla\widehat{U},\nabla\varphi)_{L^2(G)} - \widehat{\lambda}(\widehat{U},\varphi)_{L^2(G)} = (\Phi,\varphi)_{L^2(G)}$$
$$+ \sum_{j=1}^N K_j\big[(D_z^2R_j + \overline{\lambda}_jR_j,\zeta_j\varphi)_{L^2(\mathcal{T}_{j,h_j/2,h_j})} - (\nabla_yR_j,\zeta_j\nabla_y\varphi)_{L^2(\mathcal{T}_{j,h_j/2,h_j})}\big].$$

Using identity (23), we arrive at (53).
The lemma is proved. □

Lemma 21. *Assume that the numbers h_j satisfy the conditions in (26). Then, the following inequalities hold:*

$$2K_j^2\frac{48^2}{h_j^4}e^{-\widehat{\nu}_{1,j}h_j} \leq \frac{\varepsilon_1^2}{2}, \quad 2K_j\frac{6^2}{h_j^2}e^{-\widehat{\nu}_{1,j}h_j} \leq \frac{\varepsilon_1^2}{2\widehat{\lambda}}. \tag{54}$$

The proof of this lemma repeats the proof of lemma 16.

Lemma 22. *Assume that the numbers h_j satisfy the conditions in (26). Then, the following estimate holds:*

$$\|\Phi\|_{L^2(G)} \leq \varepsilon_1. \tag{55}$$

Proof. Using estimates (29), (50), and (51), we have

$$\|\Phi\|^2_{L^2(G)} = \sum_{j=1}^{N} \|\Phi_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})}$$
$$\leq 2\sum_{j=1}^{N} K_j^2 \left[\|\zeta'' R_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})} + \|2\zeta' D_z R_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})}\right]$$
$$\leq 2\sum_{j=1}^{N} K_j^2 \left[\frac{24^2}{h_j^4}\|R_j\|_{L^2(\mathcal{T}_{j,h_j/2,h_j})} + \frac{6^2}{h_j^2}\|D_z R_j\|^2_{L^2(\mathcal{T}_{j,h_j/2,h_j})}\right]$$
$$\leq 2\sum_{j=1}^{N} K_j^2 e^{-\widehat{\nu}_{1,j}h_j} \left[\frac{48^2}{h_j^4}\|U_j\|^2_{L^2(\mathcal{T}_j)} + \frac{6^2}{h_j^2}\|D_z U_j\|^2_{L^2(\mathcal{T}_j)}\right].$$

Using estimates (54), we have

$$\|\Phi\|^2_{L^2(G)} \leq \sum_{j=1}^{N}\left[\|U_j\|^2_{L^2(\mathcal{T}_j)} + \frac{1}{\lambda}\|\sqrt{K_j} D_z U_j\|^2_{L^2(\mathcal{T}_j)}\right]\frac{\varepsilon_1^2}{2} \leq \left[\|U\|^2_{L^2(G)} + \frac{1}{\lambda}\|\sqrt{K}\nabla U\|^2_{L^2(G)}\right]\frac{\varepsilon_1^2}{2}.$$

Taking into account that $\frac{1}{\lambda}\|\sqrt{K}\nabla U\|^2_{L^2(G)} = \|U\|^2_{L^2(G)} = 1$, we arrive at inequality (55). The lemma is proved. □

Let us now establish the validity of Theorem 3.
Remember that $\widehat{U} \in H^1(G)$. So

$$(K\nabla\widehat{U}, \nabla\varphi_k)_{L^2(G)} = \lambda_k(\widehat{U}, \varphi_k)_{L^2(G)}, \quad k \geq 0 \tag{56}$$

It follows from (53) and (56) that

$$(\lambda_k - \widehat{\lambda})(\widehat{U}, \varphi_k)_{L^2(G)} = (\Phi, \varphi_k)_{L^2(G)}, \quad k \geq 0.$$

So,

$$\min_{k\geq 0}(\lambda_k - \widehat{\lambda})^2\|\widehat{U}\|^2_{L^2(G)} \leq \sum_{k=0}^{\infty}(\lambda_k - \widehat{\lambda})^2(\widehat{U}, \varphi_k)^2_{L^2(G)} \leq \sum_{k=0}^{\infty}(\Phi, \varphi_k)^2_{L^2(G)} = \|\Phi\|^2_{L^2(G)}.$$

Hence,

$$\min_{k\geq 0}|\lambda_k - \widehat{\lambda}|(\|U\|_{L^2(G)} - \|\widehat{U} - U\|_{L^2(G)}) \leq \min_{k\geq 0}|\lambda_k - \widehat{\lambda}|\,\|\widehat{U}\|_{L^2(G)} \leq \|\Phi\|_{L^2(G)}.$$

As $\|U\|_{L^2(G)} = 1$, $\|\widehat{U} - U\|_{L^2(G)} \leq \varepsilon_1$ and $\|\Phi\|_{L^2(G)} \leq \varepsilon_1$, we obtain the estimate

$$\min_{k\geq 0}|\lambda_k - \widehat{\lambda}|\,(1 - \varepsilon_1) \leq \varepsilon_1,$$

which is equivalent to estimate (27).
Theorem 3 is proved.

7. Conclusions

A new method reducing computational resources is introduced to find a set of first eigenvalues of the Neumann diffusion operator in a three-dimensional domain containing thin tubes. The method consists of the truncation of the tubes at some small distance from the ends of the tubes. The truncated parts are replaced by one-dimensional segments, and special junction conditions are stated on the interfaces of the three-dimensional parts and one-dimensional segments: pointwise continuity of the fluxes and continuity of an average of the eigenfunction. The method is justified by the theorems pointing out at what distance we can truncate the tubes while keeping the given accuracy for the approximations of eigenvalues. In the case in which the domain consists of thin tubes only, the method significantly reduces the computational time: if the ratio of the thicknesses of tubes to their lengths is $1/m$, the time is reduced m times. This acceleration brings a significant gain in time and allows the computations of the eigenvalues in domains of complex geometry. Using the technique developed here, the approach to the eigenvalues of the original and reduced problems preserving their multiplicities (up to a prescribed accuracy), and the approaches of the corresponding eigenfunctions will be addressed in a forthcoming paper by the authors. This extension shell likely involve the construction of sets of almost orthogonal eigenfunctions of each problem (cf. [26,27] for an abstract framework and [28] for the technique in a singularly perturbed spectral problem). The analysis of the method will be extended to the comparison of the eigenfunction of the original and partially decomposed problems.

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