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Article

Asymptotic Solution for a Visco-Elastic Thin Plate: Quasistatic and Dynamic Cases

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Abstract: The Kelvin–Voigt model for a thin stratified two-dimensional visco-elastic strip is analyzed both in the quasistatic and in the dynamic cases. The Neumann boundary conditions on the upper and the lower parts of the boundary and periodicity conditions with respect to the longitudinal variable are stated. A complete asymptotic expansion of the solution is constructed in both cases, by using the dimension reduction combined with a homogenization technique. The error between the exact solution and the asymptotic one is evaluated in each case and the obtained results fully justify the asymptotic construction. The results were partially (quasistatic case) announced in the short note in C.R. Acad. Sci. Paris; the present article contains the complete proofs and generalizations in the dynamic case.

Keywords: Kelvin–Voigt visco-elasticity; thin plate; laminate; asymptotic expansion; dimension reduction; homogenization

MSC: 35Q74; 35B27; 74D05

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1. Introduction

The dimension reduction of a thin viscoelastic stratified laminate and the construction of the complete asymptotic solution in the quasistatic and dynamic cases represent the main results of this article. The method of the dimension reduction replaces the two-dimensional models such as the partial differential equations (PDEs) set in a thin rectangle or a union of thin rectangles by a one-dimensional model set in an interval or a graph. In the case of the three-dimensional domain, this method reduces the PDE model to a two-dimensional one (in the case of a plate (thin parallelepiped) or a shell) or to a one-dimensional model (in the case of a "long" parallelepiped simulation, a bar). For thin homogeneous plates and bars, this method was applied starting from the nineteenth century. However, its mathematical justification via the derivation of models with reduced dimension from the elasticity equations by means of the asymptotic analysis started in the twentieth century (see [1–5] and the literature therein). This theory used the ratio of characteristic sizes of the domain in the "transversal" and "longitudinal" directions as a small parameter. Later this approach was applied to the heterogeneous plates and rods. The dimension reduction of thin heterogeneous plates was considered first in [6–8]; in particular, in [7–10], the complete asymptotic expansions were constructed and justified. Such dimension reduction is an important tool which can be applied to the analysis of stresses and strains in thin heterogeneous structures [11].

Earlier, the Kelvin–Voigt model was considered in the homogenization theory for composite materials in [12–14] where the so-called memory effect was found; the asymptotic analysis of the PDE model without non-local terms leads to a limit model containing such terms. In Ref. [15], this effect was obtained for the case of time-depending coefficients.

The presence of the fading memory term in the homogenized model means that the homogenized model becomes nonlocal in time, while the initial Kelvin–Voigt model does not contain any integral terms in time. So, the Kelvin–Voigt model being the short memory model generates the homogenized model, that is, the long memory model. The homogenization of porous two-phase viscoelastic media was recently presented in [16] and the model reduction of viscoelasticity in heterogeneous media is still of great interest in the mechanics of composites (see the recent thesis [17] and review [18]).

The results obtained below for the quasistatic Kelvin–Voigt model were announced in [19] but the proof is given for the first time in the present paper; the dynamic case is also considered here for the first time. In the present paper, the quasistatic and dynamic two-dimensional PDE boundary value problems for the Kelvin–Voigt model are considered in a thin layer simulating a stratified plate. The small parameter ε corresponds to the ratio of the thickness to the characteristic longitudinal size of the plate. The developed asymptotic technique introduces the homogenization combined with the dimension reduction of the two-dimensional plate and derives a one-dimensional model. In this case, the dimension reduction leads to a different problem compared with that obtained in the homogenization of a massive body, because in the dimension reduction, one of the homogenized equations has the fourth order with respect to the space derivative, while in the "massive" case, all the homogenized equations are of the second order. As for the "massive" case, we show that the fading memory effect holds for the plates in the dimension reduction. To our knowledge, this is the first time that the complete asymptotic expansion of the solution to the Kelvin–Voigt model of a thin stratified plate has been constructed and the error estimates of order $O(\varepsilon^J)$ proved for asymptotic approximations of order J with arbitrary J. Discussing strong and weak points of the proposed method, it can be pointed out that it allows us to obtain all terms of the asymptotic expansion and, for any given J, to obtain approximations of the accuracy $O(\varepsilon^J)$, while other methods mainly give the leading term only and do not always give error estimates. Note that these estimates are important to describe the limitations of the theory. Another advantage is that the form of the asymptotic expansion gives the possibility to use the form of the solution in the construction of some subspaces in model reduction via projection, as is done in the method of asymptotic partial decomposition of the domain and in the method of partial homogenization (see [10,20,21]). However, to obtain such high accuracy of the reduced model, the method of asymptotic expansions requires a high regularity of the right-hand side function.

The outlook of this article is as follows. In Section 2, we present the quasistatic Kelvin–Voigt problem with Neumann and periodicity conditions on the boundary. Section 3 deals with the construction of the complete asymptotic expansion of the solution to this problem. As is usual in asymptotic methods, one of the main challenges is to find the **form** of an asymptotic solution as some formal series with undetermined coefficients. Furthermore, this **form** (called **anstaz**) is plugged into the equations and the boundary conditions of the problem and, equating the terms of the same order, one can obtain a recurrent chain of equations for the undetermined coefficients of this ansatz. The ansatz in the present paper generalizes N. Bakhvalov's ansatz [22,23], applied to the elastic composite plates and rods in [9], but additionally it contains the integral terms. A similar ansatz was introduced in [24], where it was applied for the homogenization of the long memory visco-elasticity equations for heterogeneous media. However, we believe that for the short memory viscoelastiity, this ansatz is introduced for the first time.

The justification of asymptotic expansions is provided by a truncation at the high-order terms of the formal series and the evaluation of the residual in the equation and in the boundary conditions after plugging the truncated series into the equations and boundary conditions. Then, usually the a priori estimates of the initial model are applied and the stability argument is used to obtain the accurate error estimate for the difference of the exact solution and the truncated asymptotic approximation. This justification is presented in Section 4 and the error estimates for the difference of the exact and asymptotic solutions are derived. The absence of the non-steady term in the Kelvin–Voight equation imposes an auxiliary construction in order to obtain estimates that contain constants which have known expressions with respect to the small parameter of the problem. In Section 5, we pass to the more general Kelvin–Voigt model, the dynamic one. We present the asymptotic expansion proposed for this case, and we obtain the homogenized equations. We prove the existence of the auxiliary problems for the terms of asymptotic expansion by means of the Galerkin method. The technique used for obtaining the a priori estimates relies on the Gronwall inequality. Finally, in Section 6, we justify the asymptotic construction of the dynamic problem by proving that the error between the exact and asymptotic solutions is small.

2. Quasistatic Visco-Elastic Plates/Rods

Let

$$G_\varepsilon = \mathbb{R} \times (0, \varepsilon) \tag{1}$$

be a thin layer in $\mathbb{R}^2$, modeling a plate/rod. Consider the quasistatic visco-elasticity equations set in this layer with the 1-periodicity condition in the variable x_1 and with Neumann conditions on the other boundaries of the layer:

$$\begin{cases} P_\varepsilon \mathbf{u}_\varepsilon \equiv -\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(B_{ij}(\frac{x_2}{\varepsilon}) \frac{\partial \dot{\mathbf{u}}_\varepsilon}{\partial x_j} \right) - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(A_{ij}(\frac{x_2}{\varepsilon}) \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j} \right) \\ \qquad = \mathbf{f}_\varepsilon(x_1, t) \quad \text{in } G_\varepsilon \times (0,T), \\ \displaystyle\sum_{j=1}^{2} \left(B_{2j} \frac{\partial \dot{\mathbf{u}}_\varepsilon}{\partial x_j} + A_{2j} \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j} \right) = \mathbf{0} \quad \text{on } (\{x_2 = 0\} \cup \{x_2 = \varepsilon\}) \times (0,T), \\ \mathbf{u}_\varepsilon(0) = \mathbf{0} \quad \text{in } G_\varepsilon, \end{cases} \tag{2}$$

where $\dot{f}$ represents the time derivative of the function f.

The coefficients A_{ij}, B_{ij} are 2×2 matrix-valued functions depending on the transversal variable only and having the following form:

$$A_{11} = \begin{pmatrix} \lambda + 2\mu & 0 \\ 0 & \mu \end{pmatrix}, \; A_{12} = \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} 0 & \mu \\ \lambda & 0 \end{pmatrix}, \; A_{22} = \begin{pmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{pmatrix},$$

$$B_{11} = \begin{pmatrix} \hat\lambda + 2\hat\mu & 0 \\ 0 & \hat\mu \end{pmatrix}, \; B_{12} = \begin{pmatrix} 0 & \hat\lambda \\ \hat\mu & 0 \end{pmatrix},$$

$$B_{21} = \begin{pmatrix} 0 & \hat\mu \\ \hat\lambda & 0 \end{pmatrix}, \; B_{22} = \begin{pmatrix} \hat\mu & 0 \\ 0 & \hat\lambda + 2\hat\mu \end{pmatrix},$$

where λ, μ, $\hat\lambda$, $\hat\mu$ are piece-wise smooth positive functions of $\xi_2 = \frac{x_2}{\varepsilon}$; namely, there exist positive numbers $\xi^1 < ... < \xi^N < 1$, such that $\lambda, \mu, \hat\lambda, \hat\mu \in C^1([\xi^i, \xi^{i+1}])$ for all $i = 0, ..., N$ ($\xi^0 = 0, \xi^{N+1} = 1$). Moreover, if we denote $A_{ij} = (a_{ij}^{kl})_{1 \le k,l \le 2}$, the following properties hold:

$$\begin{aligned} &(i)\; a_{ij}^{kl}(\xi_2) = a_{kj}^{il}(\xi_2) = a_{ji}^{lk}(\xi_2), \;\forall\, i,j,k,l \in \{1,2\}, \forall \xi_2 \in [0,1], \\ &(ii)\; (\exists)\, \kappa > 0 \text{ independent of } \varepsilon \text{ such that} \\ &\sum_{i,j,k,l=1}^{2} a_{ij}^{kl}(\xi_2)\eta_j^l \eta_i^k \ge \kappa \sum_{j,l=1}^{2} (\eta_j^l)^2, (\forall)\, \xi_2 \in [0,1], (\forall)\, \eta = (\eta_j^l)_{1 \le j,l \le 2}, \; \eta_j^l = \eta_l^j. \end{aligned} \tag{3}$$

The properties (i) and (ii) are valid for the elements of the matrices B_{ij}, b_{ij}^{kl}, as well. We suppose that the right hand side, $\mathbf{f}_\varepsilon$, depends on the longitudinal space variable x_1 and on the time variable t and is scaled as follows:

$$\mathbf{f}_\varepsilon(x_1,t) = \begin{pmatrix} \varepsilon f_1(x_1,t) \\ \varepsilon^2 f_2(x_1,t) \end{pmatrix}$$

where the two components have the following properties:

$$\begin{aligned} & f_1,\ f_2 \text{ are independent of } \varepsilon; \\ & f_1,\ f_2 \in C^\infty(\mathbb{R}\times[0,T]),\ 1-\text{periodic in } x_1; \\ & (\exists) \text{ a positive number } t^*, \text{ such that } f_j(\cdot,t)=0 \text{ for } t<t^*; \\ & < f_j(x_1,t) >= 0\ \ (\forall)\, t\in[0,T]; \end{aligned} \tag{4}$$

where $< \cdot >= \int_0^1 \cdot\, dx_1$ is the average over the period.

Problem (2) simulates the viscoelastic deformation of a thin stratified plate under a periodic in x_1 mass force; ε is the ratio of the thickness of the plate to the longitudinal period of the applied force and is a small parameter.

Denote as G_ε^1 the rectangle $(0,1)\times(0,\varepsilon)$. Denote as $H^1_{per}(G^1_\varepsilon)$ the space of functions defined on G_ε, 1-periodic in x_1 and such that their restriction to any rectangle $(a,b)\times(0,\varepsilon)$ belongs to $H^1((a,b)\times(0,\varepsilon))$. It is supplied by the inner product of the space $H^1(G^1_\varepsilon)$.

The weak solution to problem (2) is defined as a two-dimensional vector-valued function $\mathbf{u}_\varepsilon$ with $\mathbf{u}_\varepsilon \in H^1(0,T;(H^1_{per}(G^1_\varepsilon))^2)$, satisfying

$$\begin{cases} \displaystyle\int_{G^1_\varepsilon} \sum_{i,j=1}^{2} \left(B_{ij}(\frac{x_2}{\varepsilon}) \frac{\partial \dot{\mathbf{u}}_\varepsilon}{\partial x_j} + A_{ij}(\frac{x_2}{\varepsilon}) \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j} \right) \frac{\partial \mathbf{v}}{\partial x_i} \mathrm{d}x = \int_{G^1_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{v} \mathrm{d}x, \\ \qquad\qquad\qquad\qquad (\forall)\, \mathbf{v} \in (H^1_{per}(G^1_\varepsilon))^2, \text{ a.e. in } (0,T), \\ \mathbf{u}_\varepsilon(0) = \mathbf{0} \text{ in } (L^2_{per}(G^1_\varepsilon))^2. \end{cases} \tag{5}$$

Classical methods allow us to prove the following theorem:

Theorem 1. *There exists a unique solution for (5),* $\mathbf{u}_\varepsilon \in H^1(0,T;(H^1_{per}(G^1_\varepsilon))^2)$ *satisfying the condition*

$$< \mathbf{u}_\varepsilon(\cdot, x_2, t) >= 0. \tag{6}$$

Proof. The existence and the regularity are given by Galerkin's method. The uniqueness follows by taking in (5) a right-hand side equal to zero and the test function $\mathbf{v} = \mathbf{u}_\varepsilon$. □

The next section deals with the construction of the complete asymptotic expansion.

3. Asymptotic Expansion of the Solution

The results of this section were announced in [19]. The asymptotic approximation of order J is sought in the form

$$\begin{aligned} \mathbf{u}_\varepsilon^{(J)}(x,t) = \mathbf{v}_\varepsilon^{(J)}(x_1,t) + \sum_{l=1}^{J} \varepsilon^l \int_0^t \Big(& N_l^V(\frac{x_2}{\varepsilon}, t-t') D^l \dot{\mathbf{v}}_\varepsilon^{(J)}(x_1,t') \\ & + N_l^E(\frac{x_2}{\varepsilon}, t-t') D^l \mathbf{v}_\varepsilon^{(J)}(x_1,t') \Big) dt', \end{aligned} \tag{7}$$

where $D^l = \partial^l/\partial x_1^l$, N_l^V, N_l^E are 2×2 matrix-valued functions, $\mathbf{v}_\varepsilon^{(J)}$ is a two-dimensional vector function, such that its two components have the following expansion with respect to ε:

$$v_{\varepsilon,1}^{(J)}(x_1,t) = \sum_{j=0}^{J} \varepsilon^{j+1} v_{j,1}(x_1,t),\; v_{\varepsilon,2}^{(J)}(x_1,t) = \sum_{j=0}^{J} \varepsilon^{j} v_{j,2}(x_1,t), \tag{8}$$

and the functions $v_{j,1}$, $v_{j,2}$ do not depend on the small parameter.

This ansatz generalizes N. Bakhvalov's ansatz [22,23], applied to the elastic composite plates and rods in [9]. Ansatz (7) contains the integral terms.

To construct the complete asymptotic solution means to determine the matrix-valued coefficients N_l^V, N_l^E and the scalar functions $v_{j,1}$, $v_{j,2}$.

Substituting the ansatz (7) into $(2)_{1,2}$, taking together the terms of the same order with respect to ε, and letting the coefficients of the derivatives $D^l\dot{\mathbf{v}}_\varepsilon^{(J)}$ and $D^l\mathbf{v}_\varepsilon^{(J)}$ be constant, we obtain, as in [9], equations for the matrix-valued coefficients N_l^V, N_l^E. Denote

$$\begin{aligned}
\tilde{F}_l^V(\xi_2,s) &= B_{12}\frac{\partial^2 N_{l-1}^V}{\partial s\partial\xi_2} + A_{12}\frac{\partial N_{l-1}^V}{\partial\xi_2} + B_{11}\frac{\partial N_{l-2}^V}{\partial s} + A_{11}N_{l-2}^V;\\
F_l^V(\xi_2,s) &= \tilde{F}_l^V(\xi_2,s) + \frac{\partial}{\partial\xi_2}\left(B_{21}\frac{\partial N_{l-1}^V}{\partial s} + A_{21}N_{l-1}^V\right);\\
\tilde{F}_l^E(\xi_2,s) &= B_{12}\frac{\partial^2 N_{l-1}^E}{\partial s\partial\xi_2} + A_{12}\frac{\partial N_{l-1}^E}{\partial\xi_2} + B_{11}\frac{\partial N_{l-2}^E}{\partial s} + A_{11}N_{l-2}^E;\\
F_l^E(\xi_2,s) &= \tilde{F}_l^E(\xi_2,s) + \frac{\partial}{\partial\xi_2}\left(B_{21}\frac{\partial N_{l-1}^E}{\partial s} + A_{21}N_{l-1}^E\right);
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
\tilde{G}_l^V(\xi_2) &= B_{12}\frac{\partial N_{l-1}^V}{\partial\xi_2}(\xi_2,0) + B_{11}N_{l-2}^V(\xi_2,0) + B_{11}\delta_{l2};\\
G_l^V(\xi_2) &= \tilde{G}_l^V(\xi_2) + \frac{\partial}{\partial\xi_2}\Big(B_{21}(N_{l-1}^V(\xi_2,0) + I_2\delta_{l1})\Big);\\
\tilde{G}_l^E(\xi_2) &= B_{12}\frac{\partial N_{l-1}^E}{\partial\xi_2}(\xi_2,0) + B_{11}N_{l-2}^E(\xi_2,0) + A_{11}\delta_{l2};\\
G_l^E(\xi_2) &= \tilde{G}_l^E(\xi_2) + \frac{\partial}{\partial\xi_2}\Big(B_{21}N_{l-1}^E(\xi_2,0) + A_{21}\delta_{l1}\Big).
\end{aligned} \tag{10}$$

By means of the previous notation, the boundary value problems for matrices N_l^V, N_l^E can be written as follows:

$$\begin{cases}
-\dfrac{\partial}{\partial\xi_2}\left(B_{22}(\xi_2)\dfrac{\partial^2 N_l^I}{\partial\xi_2\partial s}(\xi_2,s)\right) - \dfrac{\partial}{\partial\xi_2}\left(A_{22}(\xi_2)\dfrac{\partial N_l^I}{\partial\xi_2}(\xi_2,s)\right)\\
\qquad\qquad = F_l^I(\xi_2,s) - \langle\tilde{F}_l^I\rangle(s) \quad \text{in } (0,1)\times(0,T),\\
B_{22}(0)\dfrac{\partial^2 N_l^I}{\partial\xi_2\partial s}(0,s) + A_{22}(0)\dfrac{\partial N_l^I}{\partial\xi_2}(0,s)\\
\qquad\qquad = -B_{21}(0)\dfrac{\partial N_{l-1}^I}{\partial s}(0,s) - A_{21}(0)N_{l-1}^I(0,s) \quad \text{in } (0,T),\\
-\dfrac{\partial}{\partial\xi_2}\left(B_{22}(\xi_2)\dfrac{\partial N_l^I}{\partial\xi_2}(\xi_2,0)\right) = G_l^V(\xi_2) - \langle\tilde{G}_l^I\rangle \quad \text{in } (0,1),\\
B_{22}(0)\dfrac{\partial N_l^I}{\partial\xi_2}(0,0) = -B_{21}(0)N_{l-1}^I(0,0) - N(I)(0)\delta_{l1},\\
\langle N_l^I(\cdot,s)\rangle = 0,
\end{cases} \tag{11}$$

where $I = V$ or $I = E$ and $N(I) = B_{21}$ if $I = V$, $N(I) = A_{21}$ if $I = E$. Note that these problems are non-steady and non-local with respect to the variable s, and that the initial conditions are given by boundary value problems for the ordinary differential

Equation $(11)_{3,4}$. These problems can be solved analytically and so there exist the unique solutions N_l^V and N_l^E. The substitution of the ansatz into the left-hand side of $(2)_1$ gives

$$P_\varepsilon \mathbf{u}_\varepsilon^{(J)} = -\sum_{l=1}^{J} \varepsilon^{l-2} \int_0^t \Big(\Big\langle \tilde{F}_l^V \Big\rangle (t-t') D^l \dot{\mathbf{v}}_\varepsilon^{(J)}(x_1, t') \\ + \Big\langle \tilde{F}_l^E \Big\rangle (t-t') D^l \mathbf{v}_\varepsilon^{(J)}(x_1, t') \Big) dt' - \sum_{l=1}^{J} \varepsilon^{l-2} \Big(\Big\langle \tilde{G}_l^V \Big\rangle D^l \dot{\mathbf{v}}_\varepsilon^{(J)}(x_1, t) \\ + \Big\langle \tilde{G}_l^E \Big\rangle D^l \mathbf{v}_\varepsilon^{(J)}(x_1, t) \Big) + \mathbf{r}_\varepsilon^{(J)}, \tag{12}$$

where the residual $\mathbf{r}_\varepsilon^{(J)}$ can be evaluated and its order is $O(\varepsilon^{J-1}\sqrt{\varepsilon})$ in the norm $L^2(0,T;(L^2(G_\varepsilon^1))^2)$. In order to calculate the coefficients $\langle \tilde{F}_l^I \rangle (s)$, $\langle \tilde{G}_l^I \rangle$, $I \in \{V, E\}$ for $l \in \{1,2,3,4\}$, we introduce the notation

$$M_l^I = \frac{\partial N_l^I}{\partial \xi_2}, \;\; I \in \{V, E\}, \tag{13}$$

and

$$\begin{cases} \overline{F}(x,t) = x\langle F \rangle - \int_0^x F(s,t)\mathrm{d}s, \\ \underline{F}(x,t) = \left\langle \int_0^\theta F(s,t)\mathrm{d}s \right\rangle - \int_0^x F(s,t)\mathrm{d}s, \end{cases} \tag{14}$$

where $F : [0,1] \times [0,T] \mapsto \mathbb{R}$ is an integrable function. In this way, problem (11) leads to

$$\begin{cases} B_{22}\dfrac{\partial M_l^I}{\partial s} + A_{22}M_l^I = \bar{\tilde{F}}_l^I - \left(B_{21}\dfrac{\partial N_{l-1}^I}{\partial s} + A_{21}N_{l-1}^I \right) \text{ in } (0,1)\times(0,T), \\ B_{22}M_l^I(0) = \bar{\tilde{G}}_l^I - B_{21}N_{l-1}^I(0) - N(I)\delta_{l1} \text{ in } (0,1), \;\; I \in \{V, E\}. \end{cases} \tag{15}$$

The unique solution of (15) is given by

$$M_l^I = B_{22}^{-1}\exp(-B_{22}^{-1}A_{22}s)\left(\bar{\tilde{G}}_l^I + \int_0^s \exp(B_{22}^{-1}A_{22}\sigma)\bar{\tilde{F}}_l^I \mathrm{d}\sigma - N(I)\delta_{l1} \right) \\ -B_{22}^{-1}B_{21}N_{l-1}^I + B_{22}^{-1}\exp(-B_{22}^{-1}A_{22}s)C_1\int_0^s \exp(B_{11}^{-1}A_{11}\sigma)N_{l-1}^I \mathrm{d}\sigma, \tag{16}$$

where

$$C_1 = \begin{pmatrix} 0 & 0 \\ \frac{2(\hat{\lambda}\mu - \hat{\mu}\lambda)}{\hat{\lambda}+2\hat{\mu}} & 0 \end{pmatrix}.$$

In addition, from $(15)_1$ and $(9)_1$ (or $(9)_3$), we obtain

$$\tilde{F}_l^I = C_2\bar{\tilde{F}}_{l-1}^I + C_4\frac{\partial N_{l-2}^I}{\partial s} + C_3N_{l-2}^I + C_5M_{l-1}^I, \;\; I \in \{V, E\}, \tag{17}$$

with

$$C_2 = \begin{pmatrix} 0 & \frac{\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}} \\ 1 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} \frac{(\lambda+2\mu)(\hat{\lambda}+2\hat{\mu})-\lambda\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}} & 0 \\ 0 & 0 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} \frac{(\hat{\lambda}+2\hat{\mu})^2-\hat{\lambda}^2}{\hat{\lambda}+2\hat{\mu}} & 0 \\ 0 & 0 \end{pmatrix}, C_5 = \begin{pmatrix} 0 & \frac{2(\hat{\mu}\lambda-\hat{\lambda}\mu)}{\hat{\lambda}+2\hat{\mu}} \\ 0 & 0 \end{pmatrix}.$$

In the same way, from $(15)_2$ and $(10)_1$ (or $(10)_3$), we establish the recurrence relation for $\tilde{G}_l^I$:

$$\tilde{G}_l^I = C_2\bar{\tilde{G}}_{l-1}^I + C_4N_{l-2}^I(0) + C(I)\delta_{l2}, \;\; I \in \{V, E\}, \tag{18}$$

with $C(I) = C_4$ for $I = V$ and $C(I) = C_3$ for $I = E$.

Using next the relations (17) and (18), we calculate the first coefficients, so Equation $(2)_1$, after the substitution of the ansatz, becomes

$$\begin{cases} -\varepsilon\Big(\hat{E}^V D^2\dot{v}_{0,1} + \hat{E}^E D^2 v_{0,1} + \hat{\hat{E}}^V D^3\dot{v}_{0,2} + \hat{\hat{E}}^E D^3 v_{0,2} \\ + \int\limits_0^t \Big(\hat{F}^V(t-t')D^2\dot{v}_{0,1}(t') + \hat{F}^E(t-t')D^2 v_{0,1} \\ + \hat{\hat{F}}^V(t-t')D^3\dot{v}_{0,2} + \hat{\hat{F}}^V(t-t')D^3 v_{0,2}\Big)dt'\Big) \\ +\ldots = \varepsilon f_1 \\ -\varepsilon^2\Big(\hat{\hat{\hat{E}}}^V D^3\dot{v}_{0,1} + \hat{\hat{\hat{E}}}^E D^3 v_{0,1} + \hat{J}^V D^4\dot{v}_{0,2} + \hat{J}^E D^4 v_{0,2} \\ + \int\limits_0^t \Big(\hat{\hat{\hat{F}}}^V(t-t')D^3\dot{v}_{0,1} + \hat{\hat{\hat{F}}}^V(t-t')D^3 v_{0,1} \\ + (\hat{H}^V(t-t')D^4\dot{v}_{0,2} + \hat{H}^E(t-t')D^4 v_{0,2}\Big)dt'\Big) \\ +\ldots = \varepsilon^2 f_2. \end{cases} \tag{19}$$

In the previous relations we wrote explicitly only the leading term, all the other terms being replaced by "...".

The coefficients $\hat{E}^V$, $\hat{E}^E$, $\hat{\hat{E}}^V$, $\hat{\hat{E}}^E$, $\hat{\hat{\hat{E}}}^V$, $\hat{\hat{\hat{E}}}^E$, $\hat{J}^V$, $\hat{J}^E$ and $\hat{F}^V$, $\hat{F}^E$, $\hat{\hat{F}}^V$, $\hat{\hat{F}}^E$, $\hat{\hat{\hat{F}}}^V$, $\hat{\hat{\hat{F}}}^E$, $\hat{H}^V$, $\hat{H}^E$ have the following expressions:

$$\begin{cases} \langle(\tilde{G}_2^V)_{11}\rangle =: \hat{E}^V = \left\langle\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}\right\rangle, \ \langle(\tilde{G}_2^E)_{11}\rangle =: \hat{E}^E = \left\langle\dfrac{(\lambda+2\mu)(\hat{\lambda}+2\hat{\mu})-\lambda\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}}\right\rangle, \\ \langle(\tilde{G}_3^V)_{12}\rangle =: \hat{\hat{E}}^V = \left\langle\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}\left(\dfrac{1}{2}-\xi_2\right)\right\rangle, \ \langle(\tilde{G}_3^E)_{12}\rangle =: \hat{\hat{E}}^E = \left\langle\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}\left(\dfrac{\mu}{\underline{\hat{\mu}}}\right)\right\rangle, \\ \langle(\tilde{G}_3^V)_{21}\rangle =: \hat{\hat{\hat{E}}}^V = \left\langle\overline{\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}}\right\rangle, \ \langle(\tilde{G}_3^E)_{21}\rangle =: \hat{\hat{\hat{E}}}^E = \left\langle\overline{\dfrac{(\lambda+2\mu)(\hat{\lambda}+2\hat{\mu})-\lambda\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}}}\right\rangle, \\ \langle(\tilde{G}_4^V)_{22}\rangle =: \hat{J}^V = \left\langle\overline{\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}\left(\dfrac{1}{2}-\xi_2\right)}\right\rangle, \ \langle(\tilde{G}_4^E)_{22}\rangle =: \hat{J}^E = \left\langle\overline{\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}\left(\dfrac{\mu}{\underline{\hat{\mu}}}\right)}\right\rangle \end{cases} \tag{20}$$

and

$$\begin{cases} \begin{cases} \langle(\tilde{F}_2^V)_{11}\rangle(s) =: \hat{F}^V(s) = \left\langle\dfrac{2\hat{\lambda}(\mu\hat{\lambda}-\lambda\hat{\mu})}{(\hat{\lambda}+2\hat{\mu})^2}\exp\left(-\dfrac{\lambda+2\mu}{\hat{\lambda}+2\hat{\mu}}s\right)\right\rangle, \\ \langle(\tilde{F}_2^E)_{11}\rangle(s) =: \hat{F}^E(s) = \left\langle\dfrac{2\lambda(\mu\hat{\lambda}-\lambda\hat{\mu})}{(\hat{\lambda}+2\hat{\mu})^2}\exp\left(-\dfrac{\lambda+2\mu}{\hat{\lambda}+2\hat{\mu}}s\right)\right\rangle, \end{cases} \\ \langle(\tilde{F}_3^V)_{12}\rangle(s) =: \hat{\hat{F}}^V(s), \ \langle(\tilde{F}_3^E)_{12}\rangle(s) =: \hat{\hat{F}}^E(s), \\ \begin{cases} \langle(\tilde{F}_3^V)_{21}\rangle(s) =: \hat{\hat{\hat{F}}}^V(s) = \left\langle\overline{\dfrac{2\hat{\lambda}(\mu\hat{\lambda}-\lambda\hat{\mu})}{(\hat{\lambda}+2\hat{\mu})^2}\exp\left(-\dfrac{\lambda+2\mu}{\hat{\lambda}+2\hat{\mu}}s\right)}\right\rangle, \\ \langle(\tilde{F}_3^E)_{21}\rangle(s) =: \hat{\hat{\hat{F}}}^E(s) = \left\langle\overline{\dfrac{2\lambda(\mu\hat{\lambda}-\lambda\hat{\mu})}{(\hat{\lambda}+2\hat{\mu})^2}\exp\left(-\dfrac{\lambda+2\mu}{\hat{\lambda}+2\hat{\mu}}s\right)}\right\rangle, \end{cases} \\ \langle(\tilde{F}_4^V)_{22}\rangle(s) =: \hat{H}^V(s) = \left\langle\overline{(\tilde{F}_3^V)_{12}}\right\rangle(s), \langle(\tilde{F}_4^E)_{22}\rangle(s) =: \hat{H}^E(s) = \left\langle\overline{(\tilde{F}_3^E)_{12}}\right\rangle(s), \end{cases} \tag{21}$$

where

$$\begin{cases} \hat{\hat{F}}^V(s) = \\ = -\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}\left(\underline{\dfrac{\mu}{\hat{\mu}}\exp\left(-\dfrac{\mu}{\hat{\mu}}s\right)}\right)+\dfrac{4(\mu\hat{\lambda}(\hat{\lambda}+2\hat{\mu})+\hat{\mu}^2(\lambda+2\mu))}{(\hat{\lambda}+2\hat{\mu})^2}\left(\underline{\exp\left(-\dfrac{\mu}{\hat{\mu}}s\right)}\right) \\ -\dfrac{4(\mu\hat{\lambda}-\lambda\hat{\mu})^2}{(\hat{\lambda}+2\hat{\mu})^3}\exp\left(-\dfrac{\lambda+2\mu}{\hat{\lambda}+2\hat{\mu}}s\right)\int_0^s \exp\left(\dfrac{\lambda+2\mu}{\hat{\lambda}+2\hat{\mu}}\sigma\right)\left(\underline{\exp\left(-\dfrac{\mu}{\hat{\mu}}\sigma\right)}\right)\mathrm{d}\sigma, \\ \hat{\hat{F}}^E(s) = \\ = -\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}\left(\underline{\left(\dfrac{\mu}{\hat{\mu}}\right)^2\exp\left(-\dfrac{\mu}{\hat{\mu}}s\right)}\right)+\dfrac{4(\mu\hat{\lambda}(\hat{\lambda}+2\hat{\mu})+\hat{\mu}^2(\lambda+2\mu))}{(\hat{\lambda}+2\hat{\mu})^2}\left(\underline{\dfrac{\mu}{\hat{\mu}}\exp\left(-\dfrac{\mu}{\hat{\mu}}s\right)}\right) \\ -\dfrac{4(\mu\hat{\lambda}-\lambda\hat{\mu})^2}{(\hat{\lambda}+2\hat{\mu})^3}\exp\left(-\dfrac{\lambda+2\mu}{\hat{\lambda}+2\hat{\mu}}s\right)\int_0^s \exp\left(\dfrac{\lambda+2\mu}{\hat{\lambda}+2\hat{\mu}}\sigma\right)\left(\underline{\dfrac{\mu}{\hat{\mu}}\exp\left(-\dfrac{\mu}{\hat{\mu}}\sigma\right)}\right)\mathrm{d}\sigma. \end{cases}$$

Divide the first equation of (19) by $-\varepsilon$ and the second by $-\varepsilon^2$. From (19), we obtain a recurrent chain of 1-periodic in x_1 problems with the unknowns $v_{j,1}$, $v_{j,2}$, the two components of $\mathbf{v}_j$, introduced in the ansatz (8). For determining $\mathbf{v}_j$ we have to solve in $\mathbb{R}\times(0,T)$

$$\begin{cases} \hat{E}^V D^2\dot{v}_{j,1}(x_1,t)+\hat{E}^E D^2 v_{j,1}(x_1,t)+\hat{\hat{E}}^V D^3\dot{v}_{j,2}(x_1,t)+\hat{\hat{E}}^E D^3 v_{j,2}(x_1,t) \\ = -\displaystyle\int_0^t \Big(\hat{F}^V(t-t')D^2\dot{v}_{j,1}(x_1,t')+\hat{F}^E(t-t')D^2 v_{j,1}(x_1,t') \\ +\hat{\hat{F}}^V(t-t')D^3\dot{v}_{j,2}(x_1,t')+\hat{\hat{F}}^V(t-t')D^3 v_{j,2}(x_1,t')\Big)dt'+F_{j,1}(x_1,t), \\ \hat{\hat{\hat{E}}}^V D^3\dot{v}_{j,1}(x_1,t)+\hat{\hat{\hat{E}}}^E D^3 v_{j,1}(x_1,t)+\hat{J}^V D^4\dot{v}_{j,2}(x_1,t)+\hat{J}^E D^4 v_{j,2}(x_1,t) \\ = -\displaystyle\int_0^t \Big(\hat{\hat{\hat{F}}}^V(t-t')D^3\dot{v}_{j,1}(x_1,t')+\hat{\hat{\hat{F}}}^V(t-t')D^3 v_{j,1}(x_1,t') \\ +(\hat{H}^V(t-t')D^4\dot{v}_{j,2}(x_1,t')+\hat{H}^E(t-t')D^4 v_{j,2}(x_1,t')\Big)dt'+F_{j,2}(x_1,t), \end{cases} \tag{22}$$

with $\mathbf{v}_j(0)=\mathbf{0}$ and $\langle\mathbf{v}_j\rangle=\mathbf{0}$, conseqences of $(4)_{3,4}$.

The function $\mathbf{F}_j=\begin{pmatrix} F_{j,1} \\ F_{j,2} \end{pmatrix}$ appearing in the right-hand side of (22) depends on the values of functions $\mathbf{v}_0,\dots,\mathbf{v}_{j-1}$ and of their derivatives of order 1 with respect to t and of arbitrary order with respect to x_1. For $j=0$ we have $\mathbf{F}_0=-\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. By induction, we prove that

$$<\mathbf{F}_j(\cdot,t)>=\mathbf{0}.$$

For proving the existence and the uniqueness of the solution for (22), we use a version of the fixed point theorem on the reiterate integral operator (see [25]). This idea was used for the long memory viscoelastic equations in a bounded domain in [26]. Towards this end, we keep Equation $(22)_2$ and we replace $(22)_1$ with $D(22)_1$. We introduce the notation

$$\mathbf{W}_j(x_1,t)=\begin{pmatrix} D^3 v_{j,1}(x_1,t) \\ D^4 v_{j,2}(x_1,t) \end{pmatrix} \tag{23}$$

for the new unknown of the problem and

$$B^V=\begin{pmatrix} \hat{E}^V & \hat{\hat{E}}^V \\ \hat{\hat{\hat{E}}}^V & \hat{J}^V \end{pmatrix},\quad B^E=\begin{pmatrix} \hat{E}^E & \hat{\hat{E}}^E \\ \hat{\hat{\hat{E}}}^E & \hat{J}^E \end{pmatrix}, \tag{24}$$

$$K^V(s) = -\begin{pmatrix} \hat{F}^V(s) & \hat{\hat{F}}^V(s) \\ \hat{\hat{\hat{F}}}^V(s) & \hat{H}^V(s) \end{pmatrix},\ K^E(s) = -\begin{pmatrix} \hat{F}^E(s) & \hat{\hat{F}}^E(s) \\ \hat{\hat{\hat{F}}}^E(s) & \hat{H}^E(s) \end{pmatrix}, \tag{25}$$

$$\boldsymbol{\psi}_j(x_1,t) = \begin{pmatrix} DF_{j,1}(x_1,t) \\ F_{j,2}(x_1,t) \end{pmatrix} \tag{26}$$

for the known elements of (22). In this way, we have to solve the following problem:

$$\begin{cases} B^V\dot{\mathbf{W}}_j + B^E\mathbf{W}_j \\ = \int_0^t \Big(K^V(t-t')\dot{\mathbf{W}}_j(x_1,t') + K^E(t-t')\mathbf{W}_j(x_1,t') \Big) \mathrm{d}t' + \boldsymbol{\psi}_j(x_1,t), \\ \mathbf{W}_j(0) = \mathbf{0}, \langle \mathbf{W}_j \rangle = \mathbf{0}. \end{cases} \tag{27}$$

It can be proved as in [27] that $\det(B^V) \neq 0$. Calculating $(B^V)^{-1}(27)_1$ and then integrating from 0 to t the obtained relation using the initial condition $(27)_2$, we obtain

$$\begin{cases} \mathbf{W}_j(x_1,t) \\ = \int_0^t K_1\mathbf{W}_j(x_1,\theta)\mathrm{d}\theta + \int_0^t \Big(\int_0^\theta K_2(\theta-t')\mathbf{W}_j(x_1,t')\mathrm{d}t' \Big) \mathrm{d}\theta + \boldsymbol{\varphi}_j(x_1,t), \\ \mathbf{W}_j(0) = \mathbf{0}, \langle \mathbf{W}_j \rangle = \mathbf{0}, \end{cases} \tag{28}$$

where

$$\begin{cases} K_1 = (B^V)^{-1}(K^V(0) - B^E), \\ K_2(s) = (B^V)^{-1}(K^E(s) + \dot{K}^V(s)), \\ \boldsymbol{\varphi}(x_1,t) = \int_0^t (B^V)^{-1}\boldsymbol{\psi}_j(x_1,\theta)\mathrm{d}\theta. \end{cases} \tag{29}$$

Proposition 1. *Let us suppose that $\boldsymbol{\varphi}_j \in L^\infty(0,T;(L^2(0,1))^2)$. Then every solution $\mathbf{W}_j$ of (28), if any, has the regularity*

$$\mathbf{W}_j \in L^2(0,T;(L^2(0,1))^2). \tag{30}$$

Proof. We obtain by induction the following result:

$$\begin{aligned} P(n) : \int_0^1 \mathbf{W}_j^2(x_1,t)\mathrm{d}x_1 &\le \left(\frac{3M}{2}\right)^n T^{n-1}\frac{t^n}{n!}\|\mathbf{W}_j\|^2_{L^2(0,t;(L^2(0,1))^2)} \\ &+ 3\frac{\left(\frac{3M}{2}T^2\right)^n - 1}{\frac{3M}{2}T^2 - 1}\|\boldsymbol{\varphi}_j\|^2_{L^\infty(0,T;(L^2(0,1))^2)} \text{ a.e. in } (0,T), (\forall)\, n \ge 1, \end{aligned} \tag{31}$$

with

$$\begin{cases} M = 8(M_{K_1} + M_{K_2}T^2), \\ M_{K_1} = \max\{((K_1)_{kl})^2; k,l \in \{1,2\}\}, \\ M_{K_2} = \max\{\max_{s\in[0,T]}((K_2(s))_{kl})^2; k,l \in \{1,2\}\}. \end{cases} \tag{32}$$

Let us denote

$$\begin{cases} \boldsymbol{\alpha}_j(x_1,t) = \int_0^t K_1\mathbf{W}_j(x_1,\theta)\mathrm{d}\theta, \\ \boldsymbol{\beta}_j(x_1,t) = \int_0^t \left(\int_0^\theta K_2(\theta-t')\mathbf{W}_j(x_1,t')\mathrm{d}t' \right)\mathrm{d}\theta. \end{cases} \tag{33}$$

From $\int_0^1 ((33)_1)^2 \mathrm{d}x_1$ it follows that

$$\int_0^1 \boldsymbol{\alpha}_j^2(x_1,t)\mathrm{d}x_1 \le 4M_{K_1} t \|\mathbf{W}_j\|^2_{L^2(0,t;(L^2(0,1))^2)} \tag{34}$$

and from $\int_0^1 ((33)_2)^2 \mathrm{d}x_1$ we obtain

$$\int_0^1 \boldsymbol{\beta}_j^2(x_1,t)\mathrm{d}x_1 \le 4M_{K_2} T^2 t \|\mathbf{W}_j\|^2_{L^2(0,t;(L^2(0,1))^2)}. \tag{35}$$

Introducing (33) in $(28)_1$, calculating $\int_0^1 ((28)_1)^2 \mathrm{d}x_1$ and using the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ and the estimates (34), (35) we obtain a.e. in $(0,T)$

$$\int_0^1 \mathbf{W}_j^2(x_1,t)\mathrm{d}x_1 \le \frac{3M}{2} t \|\mathbf{W}_j\|^2_{L^2(0,t;(L^2(0,1))^2)} + 3\|\boldsymbol{\varphi}_j\|^2_{L^\infty(0,T;(L^2(0,1))^2)}, \tag{36}$$

which represents $P(1)$. We suppose next that $P(n-1)$ is true and, introducing $P(n-1)$ into the first term of the right-hand side of (36), we obtain $P(n)$ as follows:

$$\int_0^1 \mathbf{W}_j^2(x_1,t)\mathrm{d}x_1 \le \left(\frac{3M}{2}\right)^n T^{n-2} t \int_0^t \frac{\theta^{n-1}}{(n-1)!} \|\mathbf{W}_j\|^2_{L^2(0,\theta;(L^2(0,1))^2)} \mathrm{d}\theta$$

$$+3\left(\frac{3M}{2}T^2 \frac{\left(\frac{3M}{2}T^2\right)^{n-1}-1}{\frac{3M}{2}T^2-1} + 1\right) \|\boldsymbol{\varphi}_j\|^2_{L^\infty(0,T;(L^2(0,1))^2)}$$

and we use $\|\mathbf{W}_j\|^2_{L^2(0,\theta;(L^2(0,1))^2)} \le \|\mathbf{W}_j\|^2_{L^2(0,t;(L^2(0,1))^2)}$.

If in the right-hand side of (31) we use the inequality $t \le T$ and then we integrate from 0 to T, it follows that $(\forall)\, n \ge 1$

$$\begin{aligned} &\left(1 - \frac{1}{n!}\left(\frac{3M}{2}T^2\right)^n\right) \|\mathbf{W}_j\|^2_{L^2(0,T;(L^2(0,1))^2)} \\ &\quad \le 3T \frac{\left(\frac{3M}{2}T^2\right)^n - 1}{\frac{3M}{2}T^2 - 1} \|\boldsymbol{\varphi}_j\|^2_{L^\infty(0,T;(L^2(0,1))^2)}, \end{aligned} \tag{37}$$

which completes the proof, since $\lim_{n\to\infty} \frac{1}{n!}\left(\frac{3M}{2}T^2\right)^n = 0$, which means that there exists $n_0 \in \mathbb{N}$ such that for any fixed value $n \ge n_0$, $1 - \frac{1}{n!}\left(\frac{3M}{2}T^2\right)^n \ge \frac{1}{2}$. □

Corollary 1. *Every solution of (28), if any, has the additional regularity*

$$\mathbf{W}_j \in L^\infty(0,T;(L^2(0,1))^2). \tag{38}$$

Proof. This regularity is obtained as a consequence of (30) and (31). □

We introduce the space

$$H = \{\boldsymbol{\eta} \in L^2(0,T;(L^2_{per}(0,1))^2) / \langle \boldsymbol{\eta} \rangle = \mathbf{0}\} \tag{39}$$

and we define the operator $A_j : H \mapsto H$

$$A_j \mathbf{W} = \int_0^t K_1 \mathbf{W}(x_1,\theta)\mathrm{d}\theta + \int_0^t \left(\int_0^\theta K_2(\theta - t')\mathbf{W}(x_1,t')\mathrm{d}t'\right)\mathrm{d}\theta + \boldsymbol{\varphi}_j \tag{40}$$

Proposition 2. *If* $\boldsymbol{\varphi}_j \in L^\infty(0,T;(L^2(0,1))^2)$ *then*
(a) the operator A_j *is continuous from* H *to* H,
(b) there exists $p \in \mathbb{N}$ *such that* A_j^p *is contraction.*

Proof. (a) Let $\mathbf{W}_1, \mathbf{W}_2$ be two elements of the space H; denote

$$\begin{cases} \mathbf{a}(x_1,t) = \displaystyle\int_0^t K_1(\mathbf{W}_1(x_1,\theta) - \mathbf{W}_2(x_1,\theta))\mathrm{d}\theta, \\ \mathbf{b}(x_1,t) = \displaystyle\int_0^t \left(\int_0^\theta K_2(\theta - t')(\mathbf{W}_1(x_1,t') - \mathbf{W}_2(x_1,t'))\mathrm{d}t' \right)\mathrm{d}\theta. \end{cases} \tag{41}$$

Instead of (34) and (35), now we have

$$\begin{cases} \displaystyle\int_0^1 \mathbf{a}^2(x_1,t)\mathrm{d}x_1 \le 4M_{K_1} t \|\mathbf{W}_1 - \mathbf{W}_2\|^2_{L^2(0,t;(L^2(0,1))^2)} \\ \displaystyle\int_0^1 \mathbf{b}^2(x_1,t)\mathrm{d}x_1 \le 4M_{K_2} T^2 t \|\mathbf{W}_1 - \mathbf{W}_2\|^2_{L^2(0,t;(L^2(0,1))^2)}. \end{cases} \tag{42}$$

Calculating $\displaystyle\int_0^T \int_0^1 (A_j\mathbf{W}_1 - A_j\mathbf{W}_2)^2 \mathrm{d}x_1\mathrm{d}t$ by using (42) it follows that

$$\|A_j\mathbf{W}_1 - A_j\mathbf{W}_2\|^2_{L^2(0,T;(L^2(0,1))^2)} \le \frac{MT^2}{2}\|\mathbf{W}_1 - \mathbf{W}_2\|^2_{L^2(0,T;(L^2(0,1))^2)}. \tag{43}$$

(b) In the same way as before we also obtain from (42)

$$\begin{gathered} \|A_j\mathbf{W}_1(t) - A_j\mathbf{W}_2(t)\|^2_{(L^2(0,1))^2} \le Mt\|\mathbf{W}_1 - \mathbf{W}_2\|^2_{L^2(0,t;(L^2(0,1))^2)} \\ (\forall)\, \mathbf{W}_1, \mathbf{W}_2 \in H. \end{gathered} \tag{44}$$

We prove by induction

$$P(n): \begin{gathered} \|A_j^n\mathbf{W}_1(t) - A_j^n\mathbf{W}_2(t)\|^2_{(L^2(0,1))^2} \le M^n T^{n-1}\frac{t^n}{n!}\|\mathbf{W}_1 - \mathbf{W}_2\|^2_{L^2(0,t;(L^2(0,1))^2)}, \\ (\forall)\, n \ge 1, (\forall)\, \mathbf{W}_1, \mathbf{W}_2 \in H, \text{ a.e. in } (0,T). \end{gathered} \tag{45}$$

For $n = 1$ (45) becomes (44). We suppose next that $P(n-1)$ holds and we prove $P(n)$. Let us take in (44) $\mathbf{W}_l \to A_j^{n-1}\mathbf{W}_l, l = 1,2$; it follows that

$$\begin{aligned} \|A_j^n\mathbf{W}_1(t) - A_j^n\mathbf{W}_2(t)\|^2_{(L^2(0,1))^2} &\le MT\|A_j^{n-1}\mathbf{W}_1 - A_j^{n-1}\mathbf{W}_2\|^2_{L^2(0,t;(L^2(0,1))^2)} \\ &\le M^nT^{n-1}\int_0^t \frac{\theta^{n-1}}{(n-1)!}\|\mathbf{W}_1 - \mathbf{W}_2\|^2_{L^2(0,\theta;(L^2(0,1))^2)}\mathrm{d}\theta \\ &\le M^nT^{n-1}\|\mathbf{W}_1 - \mathbf{W}_2\|^2_{L^2(0,t;(L^2(0,1))^2)}\int_0^t \frac{\theta^{n-1}}{(n-1)!}\mathrm{d}\theta \\ &= M^nT^{n-1}\frac{t^n}{n!}\|\mathbf{W}_1 - \mathbf{W}_2\|^2_{L^2(0,t;(L^2(0,1))^2)}, \end{aligned}$$

which represents (45). Using in the right-hand side of (45) the inequality $t \le T$ and then integrating (45) from 0 to T we obtain

$$\begin{gathered} \|A_j^n\mathbf{W}_1 - A_j^n\mathbf{W}_2\|^2_{L^2(0,T;(L^2(0,1))^2)} \le \left(\frac{(MT^2)^n}{n!}\right)^{1/2}\|\mathbf{W}_1 - \mathbf{W}_2\|^2_{L^2(0,T;(L^2(0,1))^2)} \\ (\forall)\, n \ge 1, (\forall)\, \mathbf{W}_1, \mathbf{W}_2 \in H. \end{gathered} \tag{46}$$

Since $\lim_{n\to\infty} \frac{(MT^2)^n}{n!} = 0$, it follows that there exists $p \in \mathbb{N}$ such that $c_p = \frac{(MT^2)^p}{p!} < 1$, which completes the proof. □

Theorem 2. *If $\boldsymbol{\varphi}_j \in L^\infty(0,T;(L^2(0,1))^2)$ then problem (28) has a unique solution.*

Proof. Define the operator $B_j : H \mapsto H$ by

$$B_j = A_j^p. \tag{47}$$

From Proposition 2 it follows that

$$(\forall)\, j \in \{1,...,J\}\ (\exists!)\, \mathbf{W}_j^0 \in H \text{ such that } B_j\mathbf{W}_j^0 = \mathbf{W}_j^0. \tag{48}$$

The relation (48) yields

$$B_j^k\mathbf{W}_j^0 = \mathbf{W}_j^0 \ \ (\forall)\, k \geq 1. \tag{49}$$

Taking next $n = p$ in (46), we obtain

$$\begin{gathered}\|B_j\mathbf{W}_1 - B_j\mathbf{W}_2\|^2_{L^2(0,T;(L^2(0,1))^2)} \leq c_p\|\mathbf{W}_1 - \mathbf{W}_2\|^2_{L^2(0,T;(L^2(0,1))^2)} \\ (\forall)\, \mathbf{W}_1, \mathbf{W}_2 \in H.\end{gathered} \tag{50}$$

Let us consider $k \geq 1$ arbitrary. Applying (50) k-times it follows that

$$\begin{gathered}\|B_j^k\mathbf{W}_1 - B_j^k\mathbf{W}_2\|^2_{L^2(0,T;(L^2(0,1))^2)} \leq c_p^k\|\mathbf{W}_1 - \mathbf{W}_2\|^2_{L^2(0,T;(L^2(0,1))^2)} \\ (\forall)\, k \geq 1, (\forall)\, \mathbf{W}_1, \mathbf{W}_2 \in H.\end{gathered} \tag{51}$$

Taking in (51) $\mathbf{W}_2 = \mathbf{W}_j^0$, using (49) and making $k \to \infty$ we obtain

$$\lim_{k\to\infty} \|B_j^k\mathbf{W}_1 - \mathbf{W}_j^0\|^2_{L^2(0,T;(L^2(0,1))^2)} = 0 \ \ (\forall)\, \mathbf{W}_1 \in H. \tag{52}$$

In order to apply (52) for $\mathbf{W}_1 = A_j\mathbf{W}_j^0$, we calculate $A_j\mathbf{W}_j^0$, by means of (49) and (47), as follows:

$$A_j\mathbf{W}_j^0 = A_j(B_j^k\mathbf{W}_j^0) = A_j^{kp+1}\mathbf{W}_j^0 = A_j^{kp}(A_j\mathbf{W}_j^0) = B_j^k(A_j\mathbf{W}_j^0),$$

which gives the existence result for problem (28).

To complete the proof, it remains to establish the uniqueness of the solution. Let us suppose that problem (28) has two solutions $\mathbf{W}_j^1 \neq \mathbf{W}_j^0$ and let us calculate $B_j\mathbf{W}_j^1 = A_j^{p-1}(A\mathbf{W}_j^1) = A_j^{p-1}\mathbf{W}_j^1 = ... = A_j\mathbf{W}_j^1 = \mathbf{W}_j^1$, which is in contradiction with the uniqueness of the fixed point of B_j. □

Proposition 3. *For any $j \in \{0,1,...,J\}$ we have $\boldsymbol{\varphi}_j \in L^\infty(0,T;(L^2(0,1))^2)$. Then the unique solution of (28), $\mathbf{W}_j^0$, has the regularity*

$$P(j,k) : \mathbf{W}_j^0,\, D^k\mathbf{W}_j^0,\, \dot{\mathbf{W}}_j^0,\, D^k\dot{\mathbf{W}}_j^0 \in L^\infty(0,T;(L^2(0,1))^2),\, k \in \mathbb{N}. \tag{53}$$

Proof. We prove $P(j,k)$ by induction with respect to j and, for any $j \in \{1,...,J\}$, with respect to k. We begin by proving $P(0,k)$. Using the assumption $(4)_2$, it is easy to obtain from the expression of $\mathbf{F}_0$, (26) and $(29)_3$ at least $\boldsymbol{\varphi}_0 \in L^\infty(0,T;(L^2(0,1))^2)$. Applying next Corollary 1, we obtain

$$\mathbf{W}_0^0 \in L^\infty(0,T;(L^2(0,1))^2). \tag{54}$$

Calculating in a distributional sense $D^k(28)_1$ for $j = 0$ and $k \in \mathbb{N}$ arbitrary, it follows that

$$D^k\mathbf{W}_0^0 = \int_0^t K_1 D^k\mathbf{W}_0^0(\theta)\mathrm{d}\theta + \int_0^t \left(\int_0^\theta K_2(\theta - t')D^k\mathbf{W}_0^0(t')\mathrm{d}t' \right)\mathrm{d}\theta + D^k\boldsymbol{\varphi}_0. \tag{55}$$

Due to the regularity $(4)_2$ we have $D^k\boldsymbol{\varphi}_0 \in L^\infty(0,T;(L^2(0,1))^2)$ $(\forall)\, k \in \mathbb{N}$ and, applying again Corollary 1 (for problem (28) with $\boldsymbol{\varphi}_0 \to D^k\boldsymbol{\varphi}_0$), we obtain

$$D^k\mathbf{W}_0^0 \in L^\infty(0,T;(L^2(0,1))^2)\ (\forall)\, k \in \mathbb{N}. \tag{56}$$

Calculating next $\dfrac{\partial}{\partial t}(28)_1$ for $j = 0$ we get a.e. in $(0,1)\times(0,T)$

$$\dot{\mathbf{W}}_0^0 = K_1\mathbf{W}_0^0 + \int_0^t K_2(t-t')\mathbf{W}_0^0(t')\mathrm{d}t' + \dot{\boldsymbol{\varphi}}_0. \tag{57}$$

Computing $\int_0^1 (57)^2 \mathrm{d}x_1$ and using the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ we obtain

$$\begin{aligned} &\int_0^1 \left(\dot{\mathbf{W}}_0^0(x_1,t)\right)^2 \mathrm{d}x_1 \leq 3\Bigg(\int_0^1 \left(K_1\mathbf{W}_0^0(x_1,t)\right)^2 \mathrm{d}x_1 \\ &\int_0^1 \left(\int_0^t K_2(t-t')\mathbf{W}_0^0(x_1,t')\mathrm{d}t' \right)^2 \mathrm{d}x_1 + \int_0^1 (\dot{\boldsymbol{\varphi}}_0(x_1,t))^2 \mathrm{d}x_1 \Bigg). \end{aligned} \tag{58}$$

In order to majorate the right-hand side of (58), we introduce the additional notation

$$M_{(B^V)^{-1}} = \max\{\left((B^V)^{-1}_{kl}\right)^2; k,l \in \{1,2\}\}.$$

With estimates of the same type as those from the proof of Proposition 1 for the first two terms of the right-hand side of (58) and with

$$\int_0^1 (\dot{\boldsymbol{\varphi}}_0(x_1,t))^2 \mathrm{d}x_1 \leq 4M_{(B^V)^{-1}}\|\boldsymbol{\psi}_0\|_{L^\infty(0,T;(L^2(0,1))^2)}$$

for the last term, it follows that

$$\begin{aligned} \|\dot{\mathbf{W}}_0^0\|^2_{L^\infty(0,T;(L^2(0,1))^2)} &\leq 4M_{K_1}\|\mathbf{W}_0^0\|^2_{L^\infty(0,T;(L^2(0,1))^2)} \\ &+ 4TM_{K_2}\|\mathbf{W}_0^0\|^2_{L^2(0,T;(L^2(0,1))^2)} + 4M_{(B^V)^{-1}}\|\boldsymbol{\psi}_0\|_{L^\infty(0,T;(L^2(0,1))^2)}, \end{aligned} \tag{59}$$

which yields

$$\dot{\mathbf{W}}_0^0 \in L^\infty(0,T;(L^2(0,1))^2). \tag{60}$$

Calculating next $D^k(57)$ and proceeding as for the previous estimate, we finally obtain

$$D^k\dot{\mathbf{W}}_0^0 \in L^\infty(0,T;(L^2(0,1))^2)\ \ (\forall)\, k \in \mathbb{N}, \tag{61}$$

which means that $P(0,k)$ (represented by (54), (56), (60) and (61)) was proved.

Suppose next that $P(j',k)$ holds for any $j' < j$ and for any $k \in \mathbb{N}$ and prove $P(j,k)$ for any $k \in \mathbb{N}$. We first notice that $\boldsymbol{\varphi}_j \in L^\infty(0,T;(L^2(0,1))^2)$, as a consequence of (29), (26) and of the expression of $\mathbf{F}_j$. Indeed, as we have already said, the functions $\mathbf{F}_j$ depend on the values of functions $\mathbf{v}_0,\ldots,\mathbf{v}_{j-1}$ and of their derivatives of order 1 with respect to t and of arbitrary order with respect to x_1; so, if we replace in $\mathbf{F}_j$ the functions $\mathbf{v}_0,\ldots,\mathbf{v}_{j-1}$ with $\mathbf{W}_0,\ldots,\mathbf{W}_{j-1}$ from (23) and we use either $P(0,k)$ or the induction assumption, we obtain the desired regularity for $\boldsymbol{\varphi}_j$, as stated above. Proceeding next as for the proof of $P(0,k)$, we obtain step by step the regularities (54), (56), (60) and (61) for $\mathbf{W}_0^0$ replaced with $\mathbf{W}_j^0$, which completes the proof. □

4. Error Estimates

In order to justify the asymptotic construction presented in the previous section, we show that the error between the exact solution and the asymptotic solution of order J is of order $\varepsilon^{\alpha(J)}$, with $\alpha(J) > 0$, which will be given in what follows. From the previous computations, the problem for $\mathbf{u}_\varepsilon^{(J)}$ can be written as

$$\begin{cases} P_\varepsilon \mathbf{u}_\varepsilon^{(J)} = \mathbf{f}_\varepsilon(x_1,t) - \mathbf{r}_\varepsilon^{(J)} \quad \text{in } G_\varepsilon \times (0,T), \\ \sum_{j=1}^{2} \left(B_{2j} \frac{\partial \dot{\mathbf{u}}_\varepsilon^{(J)}}{\partial x_j} + A_{2j} \frac{\partial \mathbf{u}_\varepsilon^{(J)}}{\partial x_j} \right) = -\mathbf{r}_\varepsilon^{(J),b} \text{ on } (\{x_2=0\} \cup \{x_2=\varepsilon\}) \times (0,T), \\ \mathbf{u}_\varepsilon^{(J)} \;\; 1-\text{periodic in } x_1, \\ \langle \mathbf{u}_\varepsilon^{(J)}(\cdot, x_2, t) \rangle = \mathbf{0}, \\ \mathbf{u}_\varepsilon^{(J)}(0) = \mathbf{0} \quad \text{in } G_\varepsilon, \end{cases} \tag{62}$$

with $\mathbf{r}_\varepsilon^{(J),b}$ representing the residual function on the boundaries, with $b = 0$ for $x_2 = 0$ and $b = \varepsilon$ for $x_2 = \varepsilon$. The residual functions can be evaluated and $\mathbf{r}_\varepsilon^{(J)} = \mathcal{O}(\varepsilon^{J-1/2})$ in the norm $L^2(0,T;(L^2(G_\varepsilon^1))^2)$, $\mathbf{r}_\varepsilon^{(J),b} = \mathcal{O}(\varepsilon^J)$ in the norm $L^2((0,1)\times(0,T))$.

For replacing the boundary conditions $(62)_2$ with homogeneous ones, we proceed as in [27] and we introduce a particular function with some desired properties. More precisely, we consider the function $\boldsymbol{\varphi}_\varepsilon^{(J)} : G_\varepsilon \mapsto \mathbb{R}^2$ given by

$$\begin{aligned} \boldsymbol{\varphi}_\varepsilon^{(J)}(x_1,x_2,t) = \frac{x_2(\varepsilon - x_2)}{\varepsilon^2} \Bigg\{ \Bigg(& \frac{x_2-\varepsilon}{\hat{\mu}(0)} \exp\left(-\frac{\mu(0)}{\hat{\mu}(0)} t\right) \int_0^t \exp\left(\frac{\mu(0)}{\hat{\mu}(0)}\theta\right) r_{\varepsilon,1}^{(J),0} \mathrm{d}\theta \\ & + \frac{x_2}{\hat{\mu}(1)} \exp\left(-\frac{\mu(1)}{\hat{\mu}(1)} t\right) \int_0^t \exp\left(\frac{\mu(1)}{\hat{\mu}(1)}\theta\right) r_{\varepsilon,1}^{(J),\varepsilon} \mathrm{d}\theta \Bigg) \mathbf{e}_1 \\ & + \left(\frac{x_2-\varepsilon}{\hat{\lambda}(0)+2\hat{\mu}(0)} \exp\left(-\frac{\lambda(0)+2\mu(0)}{\hat{\lambda}(0)+2\hat{\mu}(0)} t\right) \int_0^t \exp\left(\frac{\lambda(0)+2\mu(0)}{\hat{\lambda}(0)+2\hat{\mu}(0)}\theta\right) r_{\varepsilon,2}^{(J),0} \mathrm{d}\theta \right. \\ & \left. + \frac{x_2}{\hat{\lambda}(1)+2\hat{\mu}(1)} \exp\left(-\frac{\lambda(1)+2\mu(1)}{\hat{\lambda}(1)+2\hat{\mu}(1)} t\right) \int_0^t \exp\left(\frac{\lambda(1)+2\mu(1)}{\hat{\lambda}(1)+2\hat{\mu}(1)}\theta\right) r_{\varepsilon,2}^{(J),\varepsilon} \mathrm{d}\theta \right) \mathbf{e}_2 \Bigg\} \end{aligned} \tag{63}$$

and we show directly that this function has the properties

$$\begin{cases} \sum_{j=1}^{2} \left(B_{2j} \frac{\partial \dot{\boldsymbol{\varphi}}_\varepsilon^{(J)}}{\partial x_j} + A_{2j} \frac{\partial \boldsymbol{\varphi}_\varepsilon^{(J)}}{\partial x_j} \right) = -\mathbf{r}_\varepsilon^{(J),b} \text{ on } (\{x_2=0\} \cup \{x_2=\varepsilon\}) \times (0,T), \\ \boldsymbol{\varphi}_\varepsilon^{(J)} \;\; 1-\text{periodic in } x_1, \\ \langle \boldsymbol{\varphi}_\varepsilon^{(J)}(\cdot, x_2, t) \rangle = \mathbf{0}, \\ \boldsymbol{\varphi}_\varepsilon(0) = \mathbf{0} \quad \text{in } G_\varepsilon. \end{cases} \tag{64}$$

Denoting

$$\mathbf{g}_\varepsilon^{(J)} = \mathbf{r}_\varepsilon^{(J)} + P_\varepsilon \boldsymbol{\varphi}_\varepsilon^{(J)} \tag{65}$$

and then, successively

$$\mathbf{U}_\varepsilon^{(J)} = \mathbf{u}_\varepsilon^{(J)} - \boldsymbol{\varphi}_\varepsilon^{(J)} \tag{66}$$

and

$$\hat{\mathbf{U}}_\varepsilon^{(J)} = \mathbf{u}_\varepsilon - \mathbf{U}_\varepsilon^{(J)}, \tag{67}$$

we obtain from (62), (64) and (2) the problem

$$\begin{cases} P_\varepsilon \hat{\mathbf{U}}_\varepsilon^{(J)} = \mathbf{g}_\varepsilon^{(J)} \quad \text{in } G_\varepsilon \times (0,T), \\ \sum_{j=1}^{2} \left(B_{2j} \dfrac{\partial \dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}}{\partial x_j} + A_{2j} \dfrac{\partial \hat{\mathbf{U}}_\varepsilon^{(J)}}{\partial x_j} \right) = \mathbf{0} \text{ on } (\{x_2=0\} \cup \{x_2=\varepsilon\}) \times (0,T), \\ \hat{\mathbf{U}}_\varepsilon^{(J)} \; 1-\text{periodic in } x_1, \\ \langle \hat{\mathbf{U}}_\varepsilon^{(J)}(\cdot, x_2, t) \rangle = \mathbf{0}, \\ \hat{\mathbf{U}}_\varepsilon^{(J)}(0) = \mathbf{0} \quad \text{in } G_\varepsilon, \end{cases} \tag{68}$$

Let us define the space

$$V_{G_\varepsilon^1} = \{ \mathbf{v} \in (H^1_{per}(G_\varepsilon^1))^2 / \langle \mathbf{v}(\cdot, x_2) \rangle = \mathbf{0} \}. \tag{69}$$

As in Section 2, we obtain the variational problem associated with (68) as follows:

$$\begin{cases} \hat{\mathbf{U}}_\varepsilon^{(J)} \in H^1(0,T; V_{G_\varepsilon^1}), \\ \displaystyle\int_{G_\varepsilon^1} \sum_{i,j=1}^{2} \left(B_{ij} \frac{\partial \dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}}{\partial x_j} + A_{ij} \frac{\partial \hat{\mathbf{U}}_\varepsilon^{(J)}}{\partial x_j} \right) \frac{\partial \mathbf{v}}{\partial x_i} \mathrm{d}x = \int_{G_\varepsilon^1} \mathbf{g}_\varepsilon^{(J)} \cdot \mathbf{v} \mathrm{d}x, \\ \qquad\qquad\qquad\qquad (\forall)\, \mathbf{v} \in V_{G_\varepsilon^1}, \text{ a.e. in } (0,T), \\ \hat{\mathbf{U}}_\varepsilon^{(J)}(0) = \mathbf{0} \text{ in } (L^2_{per}(G_\varepsilon^1))^2. \end{cases} \tag{70}$$

Taking in (70) as test function $\mathbf{v} = \dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}(t)$, integrating the obtained relation from 0 to t, using the equalities

$$\begin{cases} \displaystyle\sum_{i,j=1}^{2} A_{ij} (\text{or } B_{ij}) \frac{\partial \mathbf{u}}{\partial x_j} \cdot \frac{\partial \mathbf{v}}{\partial x_i} = \sum_{i,j=1}^{2} A_{ij} (\text{or } B_{ij}) \frac{\partial \mathbf{v}}{\partial x_j} \cdot \frac{\partial \mathbf{u}}{\partial x_i} \;\; (\forall)\, \mathbf{u}, \mathbf{v} \in (H^1(G_\varepsilon^1))^2, \\ \displaystyle\sum_{i,j=1}^{2} A_{ij} (\text{or } B_{ij}) \frac{\partial \mathbf{u}}{\partial x_j} \cdot \frac{\partial \mathbf{u}}{\partial x_i} = \sum_{i,j,k,l=1}^{2} a_{ij}^{kl} (\text{or } b_{ij}^{kl}) \mathcal{E}_{lj}(\mathbf{u}) \mathcal{E}_{ki}(\mathbf{u}) \;\; (\forall)\, \mathbf{u} \in (H^1(G_\varepsilon^1))^2, \end{cases} \tag{71}$$

and the property (3) (ii) we obtain

$$2\kappa \int_0^t \int_{G_\varepsilon^1} \left(\mathcal{E}(\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}) \right)^2 + \kappa \int_{G_\varepsilon^1} \left(\mathcal{E}(\hat{\mathbf{U}}_\varepsilon^{(J)}(t)) \right)^2 \leq 2 \int_0^t \int_{G_\varepsilon^1} \mathbf{g}_\varepsilon^{(J)} \cdot \dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}. \tag{72}$$

where $\mathcal{E}(\mathbf{u}) = \dfrac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ is the linearized strain tensor.

We notice that the left-hand side of (72) does not contain a term corresponding to $\|\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}\|_{L^2(0,T;(L^2(G_\varepsilon^1))^2)}$, since we deal with the quasistatic case. In order to obtain such a term (with a constant having a known expression with respect to ε) we establish several auxiliary results.

Proposition 4. *Let $\Omega \in \mathbb{R}^2$ a bounded domain with Lipschitz boundary. Then there exists $\alpha(\Omega) > 0$ such that*

$$\int_\Omega (\mathcal{E}(\mathbf{u}))^2 dx \geq \alpha(\Omega) \int_\Omega \mathbf{u}^2 dx \;\; (\forall)\, \mathbf{u} \in H_\Omega, \tag{73}$$

the space H_Ω being defined as in (69).

Proof. The relation (73) is equivalent to

$$\int_\Omega (\mathcal{E}(\mathbf{w}))^2 \mathrm{d}x \geq \alpha(\Omega) \;\; (\forall)\, \mathbf{w} \in H_\Omega,\ \|\mathbf{w}\|_{(L^2(\Omega))^2} = 1. \tag{74}$$

By contradiction, let us suppose that $(\forall)\, n \in \mathbb{N}, (\exists)\, \mathbf{w}_n \in H_\Omega$ with $\|\mathbf{w}\|_{(L^2(\Omega))^2} = 1$ such that $\int_\Omega (\mathcal{E}(\mathbf{w}_n))^2 \mathrm{d}x \le \frac{1}{n}$; hence

$$\lim_{n\to\infty} \int_\Omega (\mathcal{E}(\mathbf{w}_n))^2 \mathrm{d}x = 0. \tag{75}$$

On the other hand, Korn's inequality written for $\mathbf{w}_n$ gives $\|\mathbf{w}_n\|^2_{(H^1(\Omega))^2} \le \frac{1}{c(\Omega)} \left(\int_\Omega (\mathcal{E}(\mathbf{w}_n))^2 \mathrm{d}x + 1 \right)$. This leads to the convergence (on a subsequence) $\mathbf{w}_{n_k} \rightharpoonup \mathbf{w}^*$ weakly in $(H^1(\Omega))^2$ when $k \to \infty$, which yields

$$\|\mathbf{w}^*\|_{(L^2(\Omega))^2} = 1 \tag{76}$$

and, with the weakly lower semicontinuity of the norm, $\mathcal{E}(\mathbf{w}^*) = 0$. This last property, together with the 1-periodicity in x_1 of the function $\mathbf{w}^*$ and with $\langle \mathbf{w}^*(\cdot, x_2) \rangle = \mathbf{0}$, gives $\mathbf{w}^* = \mathbf{0}$, in contradiction with (76). □

Let us define the domain $G = (0,1)^2$, the new variable $(\xi_1, \xi_2), (\xi_1, \xi_2) = (x_1, x_2/\varepsilon)$ and the function $\mathbf{V}_\varepsilon : \bar{G} \times [0, T] \mapsto \mathbb{R}^2$,

$$\begin{cases} V_{\varepsilon,1}(\xi, t) = \varepsilon^{-1} \hat{U}^{(J)}_{\varepsilon,1}(x, t), \\ V_{\varepsilon,2}(\xi, t) = \hat{U}^{(J)}_{\varepsilon,2}(x, t). \end{cases} \tag{77}$$

Using Proposition 4 and the previous definitions we prove that

Proposition 5. *The following estimate holds:*

$$\int_{G^1_\varepsilon} \left(\mathcal{E}(\dot{\hat{\mathbf{U}}}^{(J)}_\varepsilon(t)) \right)^2 dx \ge \alpha(G) \varepsilon^2 \int_{G^1_\varepsilon} (\dot{\hat{\mathbf{U}}}^{(J)}_\varepsilon(t))^2 dx \text{ a.e. in } (0, T). \tag{78}$$

Proof. Taking into account (77) we obtain

$$\begin{aligned} &\int_{G^1_\varepsilon} \left(\mathcal{E}(\dot{\hat{\mathbf{U}}}^{(J)}_\varepsilon(t)) \right)^2 \mathrm{d}x \\ &= \varepsilon \int_G \left(\varepsilon^2 \left(\frac{\partial \dot{V}_{\varepsilon,1}}{\partial \xi_1} \right)^2 + \frac{1}{2} \left(\frac{\partial \dot{V}_{\varepsilon,1}}{\partial \xi_2} + \frac{\partial \dot{V}_{\varepsilon,2}}{\partial \xi_1} \right)^2 + \frac{1}{\varepsilon^2} \left(\frac{\partial \dot{V}_{\varepsilon,2}}{\partial \xi_2} \right)^2 \right)(t) \mathrm{d}\xi \\ &\ge \varepsilon^3 \int_G (\mathcal{E}(\dot{\mathbf{V}}_\varepsilon(t)))^2 \mathrm{d}\xi. \end{aligned} \tag{79}$$

Applying next (73) for $\Omega = G$ and $\mathbf{u} = \dot{\mathbf{V}}_\varepsilon(t)$ we obtain

$$\int_G (\mathcal{E}(\dot{\mathbf{V}}_\varepsilon(t)))^2 \mathrm{d}\xi \ge \alpha(G) \int_G (\dot{\mathbf{V}}_\varepsilon(t))^2 \mathrm{d}\xi. \tag{80}$$

The last relation necessary for obtaining (78) is

$$\int_G (\dot{\mathbf{V}}_\varepsilon(t))^2 \mathrm{d}\xi \ge \frac{1}{\varepsilon} \int_{G^1_\varepsilon} (\dot{\hat{\mathbf{U}}}^{(J)}_\varepsilon(t))^2 \mathrm{d}x \tag{81}$$

and, combining (79), (80), (81) we obtain the estimate (78). □

We are now in a position to establish a first result concerning the error between the exact solution and the asymptotic solution of order J.

Theorem 3. *Let* $\mathbf{u}_\varepsilon$ *be the exact solution of problem* (2) *and* $\mathbf{u}_\varepsilon^{(J)}$ *the asymptotic solution of order* J. *Then*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(J)}\|_{H^1(0,T;(H^1(G_\varepsilon^1)^2))} = \mathcal{O}(\varepsilon^{J-9/2}). \tag{82}$$

Proof. Introducing the estimate (78) into (72) and majorating the right-hand side $2\int_0^t \int_{G_\varepsilon^1} \mathbf{g}_\varepsilon^{(J)} \cdot \dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}$ by $\kappa\alpha(G)\varepsilon^2 \int_0^t \int_{G_\varepsilon^1} \left(\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}\right)^2 + \frac{1}{\kappa\alpha(G)\varepsilon^2} \int_0^t \int_{G_\varepsilon^1} \left(\mathbf{g}_\varepsilon^{(J)}\right)^2$ we obtain a.e. in $(0,T)$

$$\left\{ \begin{array}{l} \displaystyle\int_0^t \int_{G_\varepsilon^1} (\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)})^2 \leq C_1 \varepsilon^{-4} \|\mathbf{g}_\varepsilon^{(J)}\|^2_{L^2(0,T;(L^2(G_\varepsilon^1))^2)}, \\ \displaystyle\int_{G_\varepsilon^1} \left(\mathcal{E}(\hat{\mathbf{U}}_\varepsilon^{(J)}(t))\right)^2 \leq C_2 \varepsilon^{-2} \|\mathbf{g}_\varepsilon^{(J)}\|^2_{L^2(0,T;(L^2(G_\varepsilon^1))^2)}, \end{array} \right. \tag{83}$$

with C_1, C_2 positive constants independent of ε. From (65), (63) and the estimates for $\mathbf{r}_\varepsilon^{(J)}, \mathbf{r}_\varepsilon^{(J),b}$ given at the beginning of this section it follows that

$$\|\mathbf{g}_\varepsilon^{(J)}\|_{L^2(0,T;(L^2(G_\varepsilon^1))^2)} = \mathcal{O}(\varepsilon^{J-1/2}). \tag{84}$$

From (84) and (83) we obtain the first error estimates

$$\left\{ \begin{array}{l} \|\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}\|_{L^2(0,T;(L^2(G_\varepsilon^1))^2)} = \mathcal{O}(\varepsilon^{J-5/2}), \\ \|\mathcal{E}(\hat{\mathbf{U}}_\varepsilon^{(J)})^2\|_{L^\infty(0,T;(L^2(G_\varepsilon^1))^{2\times 2})} = \mathcal{O}(\varepsilon^{J-3/2}). \end{array} \right. \tag{85}$$

Using the initial condition $(68)_5$ we obtain $\hat{\mathbf{U}}_\varepsilon^{(J)}(t) = \displaystyle\int_0^t \frac{\partial \hat{\mathbf{U}}_\varepsilon^{(J)}(s)}{\partial s}(s)\mathrm{d}s$, which yields

$$\|\hat{\mathbf{U}}_\varepsilon^{(J)}\|_{L^2(0,T;(L^2(G_\varepsilon^1))^2)} = \mathcal{O}(\varepsilon^{J-5/2}). \tag{86}$$

An obvious consequence of (72) and (84) is also

$$\|\mathcal{E}(\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)})^2\|_{L^2(0,T;(L^2(G_\varepsilon^1))^{2\times 2})} = \mathcal{O}(\varepsilon^{J-3/2}). \tag{87}$$

Finally, using the estimates (85), (86), (87) and Korn's inequality applied in G (by means of the function $\mathbf{V}_\varepsilon$ defined by (77)), we obtain (82) and the proof is completed. □

As one can see, for values of $J \leq 4$ the error between the exact and the asymptotic solution is not small, while we are interested in constructing an asymptotic solution that represents a good approximation of the exact solution from the first term. In order to improve this error estimate and justify in this way our asymptotic construction, we prove that

Theorem 4. *Let* $\mathbf{u}_\varepsilon$ *be the exact solution of problem* (2) *and* $\mathbf{u}_\varepsilon^{(J)}$ *the asymptotic solution of order* J. *Then*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(J)}\|_{H^1(0,T;(H^1(G_\varepsilon^1)^2))} = \mathcal{O}(\varepsilon^{J+1/2}). \tag{88}$$

Proof. Let us consider $J \geq 1$ and $K \geq J + 5$. We calculate

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(J)}\|_{H^1(0,T;(H^1(G_\varepsilon^1)^2))} \leq \|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(K)}\|_{H^1(0,T;(H^1(G_\varepsilon^1)^2))} + \|\mathbf{u}_\varepsilon^{(K)} - \mathbf{u}_\varepsilon^{(J)}\|_{H^1(0,T;(H^1(G_\varepsilon^1)^2))}.$$

For obtaining (88) we have to estimate the second term of the previous inequality using (7) written for K and for J, which gives $\mathbf{u}_\varepsilon^{(K)} - \mathbf{u}_\varepsilon^{(J)} = \mathcal{O}(\varepsilon^{J+1})$ and $\dfrac{\partial}{\partial x_2}(\mathbf{u}_\varepsilon^{(K)} - \mathbf{u}_\varepsilon^{(J)}) = \mathcal{O}(\varepsilon^J)$. Hence

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(J)}\|_{H^1(0,T;(H^1(G_\varepsilon^1)^2))} = \mathcal{O}(\varepsilon^{K-9/2}) + \mathcal{O}(\varepsilon^{J+1/2}) = \mathcal{O}(\varepsilon^{J+1/2}).$$

□

5. Dynamic Visco-Elastic Plates/Rods: Description of the Problem and Asymptotic Construction

Consider the same thin layer in $\mathbb{R}^2$, G_ε, defined by (1). We analyze in what follows the dynamic visco-elastic problem set in this layer with the same conditions as in the quasistatic case, i.e., the 1-periodicity condition in the variable x_1 and with Neumann conditions on the other boundaries of the layer

$$\begin{cases} P_\varepsilon \mathbf{u}_\varepsilon \equiv \rho \ddot{\mathbf{u}}_\varepsilon - \sum\limits_{i,j=1}^{2} \dfrac{\partial}{\partial x_i}\left(B_{ij}(\dfrac{x_2}{\varepsilon})\dfrac{\partial \dot{\mathbf{u}}_\varepsilon}{\partial x_j}\right) - \sum\limits_{i,j=1}^{2} \dfrac{\partial}{\partial x_i}\left(A_{ij}(\dfrac{x_2}{\varepsilon})\dfrac{\partial \mathbf{u}_\varepsilon}{\partial x_j}\right) \\ \qquad = \mathbf{f}(x_1,t) \quad \text{in } G_\varepsilon \times (0,T), \\ \sum\limits_{j=1}^{2}\left(B_{2j}\dfrac{\partial \dot{\mathbf{u}}_\varepsilon}{\partial x_j} + A_{2j}\dfrac{\partial \mathbf{u}_\varepsilon}{\partial x_j}\right) = \mathbf{0} \quad \text{on } (\{x_2=0\} \cup \{x_2=\varepsilon\}) \times (0,T), \\ \mathbf{u}_\varepsilon(0) = \dot{\mathbf{u}}_\varepsilon(0) = \mathbf{0} \quad \text{in } G_\varepsilon. \end{cases} \tag{89}$$

The coefficients A_{ij}, B_{ij} are those presented in Section 2. Unlike in the quasistatic case, we take a right-hand side, $\mathbf{f}$, independent of ε. The two components of the mass force, $\mathbf{f}$, have the properties (4).

In the dynamic case we define the weak solution for problem (89) as the solution of the variational problem

$$\begin{cases} \mathbf{u}_\varepsilon \in H^1(0,T;(H^1_{per}(G^1_\varepsilon))^2), \\ \displaystyle\int_{G^1_\varepsilon} \ddot{\mathbf{u}}_\varepsilon \cdot \mathbf{v} + \int_{G^1_\varepsilon} \sum_{i,j=1}^{2}\left(B_{ij}(\frac{x_2}{\varepsilon})\frac{\partial \dot{\mathbf{u}}_\varepsilon}{\partial x_j} + A_{ij}(\frac{x_2}{\varepsilon})\frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}\right)\frac{\partial \mathbf{v}}{\partial x_i}\mathrm{d}x \\ \qquad = \displaystyle\int_{G^1_\varepsilon} \mathbf{f}\cdot\mathbf{v}\mathrm{d}x \quad (\forall)\, \mathbf{v} \in (H^1_{per}(G^1_\varepsilon))^2, \text{ a.e. in } (0,T), \\ \mathbf{u}_\varepsilon(0) = \dot{\mathbf{u}}_\varepsilon(0) = \mathbf{0} \text{ in } (L^2_{per}(G^1_\varepsilon))^2. \end{cases} \tag{90}$$

For the dynamic case we propose an asymptotic solution of order J different from the one considered in the quasistatic case, in the form

$$\begin{aligned} \mathbf{u}_\varepsilon^{(J)}(x_1,x_2,t) &= \mathbf{v}_\varepsilon^{(J)}(x_1,t) \\ &+ \sum_{p=-1}^{J}\sum_{q+l=1}^{J} \varepsilon^{q+l+p}\int_0^t N_{q,l,p}(\frac{x_2}{\varepsilon}, t-t')\frac{\partial^{q+l}\mathbf{v}_\varepsilon^{(J)}(x_1,t')}{\partial t'^q \partial x_1^l}dt', \end{aligned} \tag{91}$$

where $N_{q,l,p}$ are 2×2 matrix-valued functions and $\mathbf{v}_\varepsilon^{(J)}$ is a two-dimensional vector function, with the following expansion with respect to ε:

$$\mathbf{v}_\varepsilon^{(J)}(x_1,t) = \sum_{j=0}^{J}\varepsilon^j \mathbf{v}_j(x_1,t), \tag{92}$$

the functions $\mathbf{v}_j$ being independent of the small parameter ε.

To construct the complete asymptotic solution means to determine the matrix-valued coefficients $N_{q,l,p}$ and the scalar functions $v_{j,1}, v_{j,2}$.

Substituting the ansatz (91) into $(89)_{1,2}$, taking together the terms of the same order with respect to ε, and letting the coefficients of the derivatives $\dfrac{\partial^{q+l}\mathbf{v}_\varepsilon^{(J)}(x_1,s)}{\partial s^q \partial x_1^l}$ be constant, we obtain the problems for the matrix-valued coefficients $N_{q,l,p}$. For writing these problems we introduce the following notation:

$$\begin{aligned}\tilde{F}_{q,l,p}(\xi_2,s) &= B_{12}\frac{\partial^2 N_{q,l-1,p}}{\partial s \partial \xi_2} + A_{12}\frac{\partial N_{q,l-1,p}}{\partial \xi_2}\\ &\quad + B_{11}\frac{\partial N_{q,l-2,p}}{\partial s} + A_{11}N_{q,l-2,p} - \rho\frac{\partial^2 N_{q,l,p-2}}{\partial s^2},\\ F_{q,l,p}(\xi_2,s) &= \tilde{F}_{q,l,p}(\xi_2,s) + \frac{\partial}{\partial \xi_2}\left(B_{21}\frac{\partial N_{q,l-1,p}}{\partial s} + A_{21}N_{q,l-1,p}\right)\end{aligned} \tag{93}$$

and

$$\begin{aligned}&\tilde{G}_{q,l,p}(\xi_2) = B_{12}\frac{\partial N_{q,l-1,p}}{\partial \xi_2}(\xi_2,0) + B_{11}N_{q,l-2,p}(\xi_2,0) - \rho\frac{\partial N_{q,l,p-2}}{\partial s}(\xi_2,0)\\ &-\rho N_{q-1,l,p-1}(\xi_2,0) + B_{11}\delta_{q,1}\delta_{l,2}\delta_{p,-1} + A_{11}\delta_{q,0}\delta_{l,2}\delta_{p,0} - \rho I_2\delta_{q,2}\delta_{l,0}\delta_{p,0},\\ &G_{q,l,p}(\xi_2) = \tilde{G}_{q,l,p}(\xi_2) + \frac{\partial}{\partial \xi_2}\big(B_{21}N_{q,l-1,p}(\xi_2,0) + B_{21}\delta_{q,1}\delta_{l,1}\delta_{p,-1}\\ &\qquad + A_{21}\delta_{q,0}\delta_{l,1}\delta_{p,0}\big).\end{aligned} \tag{94}$$

By means of the previous notation, the boundary value problems for matrices $N_{q,l,p}$, similar to those obtained in the quasistatic case, can be written as follows:

$$\begin{cases} -\dfrac{\partial}{\partial \xi_2}\left(B_{22}(\xi_2)\dfrac{\partial^2 N_{q,l,p}}{\partial \xi_2 \partial s}(\xi_2,s)\right) - \dfrac{\partial}{\partial \xi_2}\left(A_{22}(\xi_2)\dfrac{\partial N_{q,l,p}}{\partial \xi_2}(\xi_2,s)\right)\\ \qquad = F_{q,l,p}(\xi_2,s) - \langle \tilde{F}_{q,l,p}\rangle(s) \quad \text{in } (0,1)\times(0,T),\\ B_{22}(0)\dfrac{\partial^2 N_{q,l,p}}{\partial \xi_2 \partial s}(0,s) + A_{22}(0)\dfrac{\partial N_{q,l,p}}{\partial \xi_2}(0,s)\\ \qquad = -B_{21}(0)\dfrac{\partial N_{q,l-1,p}}{\partial s}(0,s) - A_{21}(0)N_{q,l-1,p}(0,s) \text{ in } (0,T),\\ -\dfrac{\partial}{\partial \xi_2}\left(B_{22}(\xi_2)\dfrac{\partial N_{q,l,p}}{\partial \xi_2}(\xi_2,0)\right) = G_{q,l,p}(\xi_2) - \langle \tilde{G}_{q,l,p}\rangle \quad \text{in } (0,1),\\ B_{22}(0)\dfrac{\partial N_{q,l,p}}{\partial \xi_2}(0,0) = -B_{21}(0)\left(N_{q,l-1,p}(0,0) + I_2\delta_{q,1}\delta_{l,1}\delta_{p,-1}\right)\\ \qquad - A_{21}(0)\delta_{q,0}\delta_{l,1}\delta_{p,0},\\ \langle N_{q,l,p}(\cdot,s)\rangle = 0. \end{cases} \tag{95}$$

The unique solution of the previous problem, $N_{q,l,p}$, can be determined by solving analytically the corresponding problem. The substitution of the ansatz (91) into the left-hand side of $(89)_1$ gives

$$\begin{aligned}P_\varepsilon \mathbf{u}_\varepsilon^{(J)} &= -\sum_{p=-1}^{J}\sum_{q+l=1}^{J}\varepsilon^{q+l+p-2}\left\langle \tilde{G}_{q,l,p}\right\rangle \frac{\partial^{q+l}\mathbf{v}_\varepsilon^{(J)}(x_1,t)}{\partial t^q \partial x_1^l}\\ &- \sum_{p=-1}^{J}\sum_{q+l=1}^{J}\varepsilon^{q+l+p-2}\int_0^t \left\langle \tilde{F}_{q,l,p}\right\rangle(t-t')\frac{\partial^{q+l}\mathbf{v}_\varepsilon^{(J)}(x_1,t')}{\partial t'^q \partial x_1^l}\mathrm{d}t' + \mathbf{r}_\varepsilon^{(J)},\end{aligned} \tag{96}$$

where the residual $\mathbf{r}_\varepsilon^{(J)}$ can be evaluated and its order is $O(\varepsilon^{J-3/2})$ in the norm $L^2(0,T;(L^2(G_\varepsilon^1))^2)$.

Introducing next the ansatz (91) into the left-hand side of $(89)_2$ we obtain the boundary conditions

$$\sum_{j=1}^{2}\left(B_{2j}\frac{\partial \dot{\mathbf{u}}_\varepsilon^{(J)}}{\partial x_j}+A_{2j}\frac{\partial \mathbf{u}_\varepsilon^{(J)}}{\partial x_j}\right)=-\mathbf{r}_\varepsilon^{(J),b} \text{ on } (\{x_2=0\}\cup\{x_2=\varepsilon\})\times(0,T), \tag{97}$$

where $\mathbf{r}_\varepsilon^{(J),b}$ (with $b=0$ on $\{x_2=0\}$ and $b=\varepsilon$ on $\{x_2=\varepsilon\}$) is a residual of order $\mathcal{O}(\varepsilon^{J-1})$ in the norm $L^2((0,1)\times(0,T))$.

In order to calculate the coefficients $\left\langle \tilde{F}_{q,l,p}\right\rangle(s)$, $\left\langle \tilde{G}_{q,l,p}\right\rangle$, we introduce the notation

$$M_{q,l,p}=\frac{\partial N_{q,l,p}}{\partial \xi_2} \tag{98}$$

and, proceeding as in the quasistatic case, we obtain the recurrent relations that give the unique matrices $M_{q,l,p}$, $\left\langle \tilde{F}_{q,l,p}\right\rangle(s)$, $\left\langle \tilde{G}_{q,l,p}\right\rangle$:

$$\begin{aligned} M_{q,l,p} =& B_{22}^{-1}\exp(-B_{22}^{-1}A_{22}s)\Big(\bar{\tilde{G}}_{q,l,p}-B_{21}\delta_{q,1}\delta_{l,1}\delta_{p,-1}-A_{21}\delta_{q,0}\delta_{l,1}\delta_{p,0} \\ &+\int_0^s \exp(B_{22}^{-1}A_{22}\sigma)\bar{\tilde{F}}_{q,l,p})\mathrm{d}\sigma\Big)-B_{22}^{-1}B_{21}N_{q,l-1,p} \\ &+B_{22}^{-1}\exp(-B_{22}^{-1}A_{22}s)C_1\int_0^s \exp(B_{11}^{-1}A_{11}\sigma)N_{q,l-1,p}\mathrm{d}\sigma, \end{aligned} \tag{99}$$

$$\begin{aligned} \tilde{F}_{q,l,p} =& C_2\bar{\tilde{F}}_{q,l-1,p}+C_4\frac{\partial N_{q,l-2,p}}{\partial s}+C_3N_{q,l-2,p} \\ &+C_5M_{q,l-1,p}-\rho\frac{\partial^2 N_{q,l,p-2}}{\partial s^2}, \end{aligned} \tag{100}$$

$$\begin{aligned} \tilde{G}_{q,l,p} =& C_2\bar{\tilde{G}}_{q,l-1,p}+C_4N_{q,l-2,p}(0)+C_4\delta_{q,1}\delta_{l,2}\delta_{p,-1}+C_3\delta_{q,0}\delta_{l,2}\delta_{p,0} \\ &-\rho\frac{\partial N_{q,l,p-2}}{\partial s}(0)-\rho N_{q-1,l,p-1}(0)-\rho I_2\delta_{q,2}\delta_{l,0}\delta_{p,0}, \end{aligned} \tag{101}$$

with the matrices $C_1,\ldots,C_5$ defined in Section 3. Moreover, $N_{q,l,p}$ is uniquely determined from (98) and $(95)_5$.

We give below all the matrices that are important in the asymptotic analysis that follows. We begin with the elements corresponding to $q+l+p\in\{0,1\}$:

$$\begin{aligned} &q+l+p=0\begin{cases} \tilde{F}_{1,0,-1}=\tilde{G}_{1,0,-1}=M_{1,0,-1}=N_{1,0,-1}=O_2, \\ \tilde{F}_{0,1,-1}=\tilde{G}_{0,1,-1}=M_{0,1,-1}=N_{0,1,-1}=O_2, \end{cases} \\ &q+l+p=1\begin{cases} \tilde{F}_{2,0,-1}=\tilde{G}_{2,0,-1}=O_2, \\ \tilde{F}_{0,2,-1}=\tilde{G}_{0,2,-1}=O_2, \\ \tilde{F}_{1,1,-1}=\tilde{G}_{1,1,-1}=O_2, \\ \tilde{F}_{0,1,0}=\tilde{G}_{0,1,0}=O_2, \\ \tilde{F}_{1,0,0}=\tilde{G}_{1,0,0}=O_2. \end{cases} \end{aligned} \tag{102}$$

For $q+l+p=2$ we obtain

$$\begin{cases} \tilde{F}_{3,0,-1}=\tilde{G}_{3,0,-1}=\tilde{F}_{0,3,-1}=\tilde{G}_{0,3,-1}=\tilde{F}_{2,1,-1}=\tilde{G}_{2,1,-1}=O_2, \\ \tilde{F}_{1,1,0}=\tilde{G}_{1,1,0}=\tilde{F}_{1,0,1}=\tilde{G}_{1,0,1}=\tilde{F}_{0,1,1}=\tilde{G}_{0,1,1}=O_2, \\ \tilde{F}_{1,2,-1}=\tilde{F}_2^V,\ \tilde{G}_{1,2,-1}=\tilde{G}_2^V, \\ \tilde{F}_{2,0,0}=O_2,\ \tilde{G}_{2,0,0}=-\rho I_2, \\ \tilde{F}_{0,2,0}=\tilde{F}_2^E,\ \tilde{G}_{0,2,0}=\tilde{G}_2^E, \end{cases} \tag{103}$$

where $\tilde{F}_2^V, \tilde{G}_2^V, \tilde{F}_2^E, \tilde{G}_2^E$ have the elements given by (20) and (21). Introducing the expansion (92) into (96) and then taking in $(89)_1$ as left-hand side (96), we obtain the problem for $\mathbf{v}_j$ in $(0,1)\times(0,T)$ as below:

$$\begin{cases} \langle\rho\rangle\ddot{v}_{j,1} - \hat{E}^E\dfrac{\partial^2 v_{j,1}}{\partial x_1^2} - \hat{E}^V\dfrac{\partial^2 \dot{v}_{j,1}}{\partial x_1^2} \\ \quad -\displaystyle\int_0^t\left(\hat{F}^E(t-t')\frac{\partial^2 v_{j,1}}{\partial x_1^2}(t') + \hat{F}^V(t-t')\frac{\partial^2 \dot{v}_{j,1}}{\partial x_1^2}(t')\right)\mathrm{d}t' = F_{j,1}, \\ \langle\rho\rangle\ddot{v}_{j,2} = F_{j,2}, \end{cases} \tag{104}$$

where $\mathbf{F}_j = \mathbf{F}_j(x_1,t)$ depends on $\mathbf{v}_0, \mathbf{v}_1, ..., \mathbf{v}_{j-1}$, $\mathbf{F}_0 = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and the same property as in the quasistatic case holds: $<\mathbf{F}_j(\cdot,t)>=\mathbf{0}$. The coefficients from (104) are given by $(20)_1$ and $(21)_{1,2}$. From (104) one can see that, unlike in the quasistatic case, in the dynamic case the problems for $v_{j,1}$ and $v_{j,2}$ are uncoupled and they contain only second-order derivatives with respect to x_1.

The problem for $v_{j,2}$

$$\begin{cases} \langle\rho\rangle\ddot{v}_{j,2} = F_{j,2} \text{ in } (0,1)\times(0,T), \\ v_{j,2}(0) = \dot{v}_{j,2}(0) = 0 \quad \text{in } (0,1), \end{cases} \tag{105}$$

gives, by integrating twice from 0 to t Equation $(105)_1$ and using the initial conditions $(105)_2$,

$$v_{j,2}(x_1,t) = \frac{1}{\langle\rho\rangle}\int_0^t\int_0^\theta F_{j,2}(x_1,s)\mathrm{d}s\mathrm{d}\theta. \tag{106}$$

Notice that the property $\langle v_{j,2}(\cdot,x_2,t)\rangle = 0$ is obtained as a consequence of the same property of $F_{j,2}$.

In what follows, we study the problem for $v_{j,1}$, completing Equation $(104)_1$ with the other properties of $v_{j,1}$ as below,

$$\begin{cases} \langle\rho\rangle\ddot{v}_{j,1} - \hat{E}^E\dfrac{\partial^2 v_{j,1}}{\partial x_1^2} - \hat{E}^V\dfrac{\partial^2 \dot{v}_{j,1}}{\partial x_1^2} \\ -\displaystyle\int_0^t\left(\hat{F}^E(t-t')\frac{\partial^2 v_{j,1}}{\partial x_1^2}(t') + \hat{F}^V(t-t')\frac{\partial^2 \dot{v}_{j,1}}{\partial x_1^2}(t')\right)\mathrm{d}t' = F_{j,1} \text{ in } (0,1)\times(0,T), \\ v_{j,1} \;\; 1-\text{periodic in } x_1, \\ \langle v_{j,1}(\cdot,t)\rangle = 0, \\ v_{j,1}(x_1,0) = \dot{v}_{j,1}(x_1,0) = 0, \end{cases} \tag{107}$$

by means of its variational analysis. Towards this end, let us define the space

$$V = \{v \in H^1_{per}(0,1) / \langle v\rangle = 0\} \tag{108}$$

and let us compute $\int_0^1 (107)_1\, \varphi\, \mathrm{d}x_1$, for $\varphi \in V$. This yields

$$\begin{aligned} &\langle\rho\rangle\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \dot{v}_{j,1}(t)\varphi + \hat{E}^E\int_0^1\frac{\partial v_{j,1}}{\partial x_1}(t)\frac{\partial\varphi}{\partial x_1} + \hat{E}^V\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\frac{\partial v_{j,1}}{\partial x_1}(t)\frac{\partial\varphi}{\partial x_1} \\ &+\int_0^t\left(\hat{F}^E(t-t')\int_0^1\frac{\partial v_{j,1}}{\partial x_1}(t')\frac{\partial\varphi}{\partial x_1} + \hat{F}^V(t-t')\frac{\mathrm{d}}{\mathrm{d}t'}\int_0^1\frac{\partial v_{j,1}}{\partial x_1}(t')\frac{\partial\varphi}{\partial x_1}\right)\mathrm{d}t' \\ &= \int_0^1 F_{j,1}(t)\varphi \quad (\forall)\, \varphi \in V, \text{ a.e. in } (0,T). \end{aligned} \tag{109}$$

We calculate next using the initial condition

$$\int_0^t \hat{F}^V(t-t')\frac{\mathrm{d}}{\mathrm{d}t'}\int_0^1 \frac{\partial v_{j,1}}{\partial x_1}(t')\frac{\partial \varphi}{\partial x_1}\mathrm{d}t' = \int_0^t \frac{\partial}{\partial t'}\left(\hat{F}^V(t-t')\int_0^1 \frac{\partial v_{j,1}}{\partial x_1}(t')\frac{\partial \varphi}{\partial x_1}\right)\mathrm{d}t'$$
$$-\int_0^t \left(\frac{\partial}{\partial t'}\left(\hat{F}^V(t-t')\right)\int_0^1 \frac{\partial v_{j,1}}{\partial x_1}(t')\frac{\partial \varphi}{\partial x_1}\right)\mathrm{d}t' = \hat{F}^V(0)\int_0^1 \frac{\partial v_{j,1}}{\partial x_1}(t)\frac{\partial \varphi}{\partial x_1}$$
$$-\int_0^t \left(\frac{\partial}{\partial t'}\left(\hat{F}^V(t-t')\right)\int_0^1 \frac{\partial v_{j,1}}{\partial x_1}(t')\frac{\partial \varphi}{\partial x_1}\right)\mathrm{d}t'.$$

Replacing it in (109) and introducing the notation

$$\begin{cases} \hat{C} = 4\left\langle \dfrac{\mu\hat{\lambda}(\hat{\lambda}+2\hat{\mu}) + \hat{\mu}^2(\lambda+2\mu)}{(\hat{\lambda}+2\hat{\mu})^2}\right\rangle, \\ \hat{D}(s) = 4\left\langle \dfrac{(\lambda\hat{\mu}-\mu\hat{\lambda})^2}{(\hat{\lambda}+2\hat{\mu})^3}\exp\left(-\dfrac{\lambda+2\mu}{\hat{\lambda}+2\hat{\mu}}s\right)\right\rangle \end{cases} \tag{110}$$

we obtain the following variational problem associated with (107):

$$\begin{cases} v_{j,1} \in W^{1,\infty}(0,T;V)\cap H^2(0,T;L^2(0,1)), \\ \langle\rho\rangle \displaystyle\int_0^1 \ddot{v}_{j,1}(t)\varphi + \hat{C}\int_0^1 \frac{\partial v_{j,1}}{\partial x_1}(t)\frac{\partial \varphi}{\partial x_1} + \hat{E}^V\int_0^1 \frac{\partial \dot{v}_{j,1}}{\partial x_1}(t)\frac{\partial \varphi}{\partial x_1} \\ -\displaystyle\int_0^t \left(\hat{D}(t-t')\int_0^1 \frac{\partial v_{j,1}}{\partial x_1}(t')\frac{\partial \varphi}{\partial x_1}\right)\mathrm{d}t' = \int_0^1 F_{j,1}(t)\varphi\ (\forall)\ \varphi \in V, \text{a.e. in } (0,T), \\ v_{j,1}(0) = \dot{v}_{j,1}(0) = 0. \end{cases} \tag{111}$$

We prove next

Theorem 5. *There exists a unique function $v_{j,1}$, solution for (111).*

Proof. We begin by proving the uniqueness of the solution that relies on some a priori estimates. Consider $v^1_{j,1}$ and $v^2_{j,1}$, two solutions for (111), and denote $v = v^1_{j,1} - v^2_{j,1}$. Subtracting the corresponding two relations $(111)_2$ written for an arbitrary fixed value $t = \theta$, taking as test function $\varphi = \dot{v}(\theta)$, integrating from 0 to t with respect to θ, and using the initial conditions $(111)_3$, we obtain

$$\begin{aligned} &\langle\rho\rangle\|\dot{v}_{j,1}(t)\|^2_{L^2(0,1)} + \hat{C}\left\|\frac{\partial v_{j,1}}{\partial x_1}(t)\right\|^2_{L^2(0,1)} + 2\hat{E}^V\int_0^t \left\|\frac{\partial \dot{v}_{j,1}}{\partial x_1}(t')\right\|^2_{L^2(0,1)}\mathrm{d}t' \\ &= 2\int_0^t \left(\int_0^\theta \left(\hat{D}(\theta-t')\int_0^1 \frac{\partial v_{j,1}}{\partial x_1}(t')\frac{\partial \dot{v}_{j,1}}{\partial x_1}(\theta)\right)\mathrm{d}t'\right)\mathrm{d}\theta. \end{aligned} \tag{112}$$

The right-hand side of (112) can be written as follows:

$$2\int_0^t \left(\int_0^\theta \left(\hat{D}(\theta-t')\int_0^1 \frac{\partial v_{j,1}}{\partial x_1}(t')\frac{\partial \dot{v}_{j,1}}{\partial x_1}(\theta)\right)\mathrm{d}t'\right)\mathrm{d}\theta$$
$$= 2\int_0^t \left(\hat{D}(t-t')\frac{\partial v_{j,1}}{\partial x_1}(t'), \frac{\partial v_{j,1}}{\partial x_1}(t)\right)_{L^2(0,1)}\mathrm{d}t' - 2\hat{D}(0)\int_0^t \left\|\frac{\partial v_{j,1}}{\partial x_1}(\theta)\right\|^2_{L^2(0,1)}\mathrm{d}\theta$$
$$-2\int_0^t \left(\int_0^\theta \frac{\partial \hat{D}}{\partial \theta}(\theta-t')\frac{\partial v_{j,1}}{\partial x_1}(t')\mathrm{d}t', \frac{\partial v_{j,1}}{\partial x_1}(\theta)\right)_{L^2(0,1)}\mathrm{d}\theta =: I_1 + I_2 + I_3.$$

We majorate next each of the three terms of the previous relation.

$$I_1 \leq 2\int_0^t \left(\left|\hat{D}(t-t')\right| \left\| \frac{\partial v_{j,1}}{\partial x_1}(t') \right\|_{L^2(0,1)} \left\| \frac{\partial v_{j,1}}{\partial x_1}(t) \right\|_{L^2(0,1)} \right) \mathrm{d}t'$$

$$\leq \alpha_1 \left\| \frac{\partial v_{j,1}}{\partial x_1}(t) \right\|_{L^2(0,1)} \int_0^t \left\| \frac{\partial v_{j,1}}{\partial x_1}(t') \right\|_{L^2(0,1)} \mathrm{d}t',$$

with $\alpha_1 = 2 \max_{s\in[0,T]} \left|\hat{D}(s)\right|$.

For the second term we obviously have $I_2 \leq 0$ and for the third one the following estimate holds:

$$-2\int_0^t \left(\int_0^\theta \frac{\partial \hat{D}}{\partial \theta}(\theta - t') \frac{\partial v_{j,1}}{\partial x_1}(t') \mathrm{d}t', \frac{\partial v_{j,1}}{\partial x_1}(\theta) \right)_{L^2(0,1)} \mathrm{d}\theta$$

$$\leq 2\int_0^t \left(\left\| \frac{\partial v_{j,1}}{\partial x_1}(\theta) \right\|_{L^2(0,1)} \int_0^\theta \left| \frac{\partial \hat{D}}{\partial \theta}(\theta - t') \right| \left\| \frac{\partial v_{j,1}}{\partial x_1}(t') \right\|_{L^2(0,1)} \mathrm{d}t' \right) \mathrm{d}\theta$$

$$\leq 2\int_0^t \left(\left\| \frac{\partial v_{j,1}}{\partial x_1}(\theta) \right\|_{L^2(0,1)} \int_0^t \left| \frac{\partial \hat{D}}{\partial \theta}(\theta - t') \right| \left\| \frac{\partial v_{j,1}}{\partial x_1}(t') \right\|_{L^2(0,1)} \mathrm{d}t' \right) \mathrm{d}\theta$$

$$\leq \alpha_2 \int_0^t \left\| \frac{\partial v_{j,1}}{\partial x_1}(t') \right\|^2_{L^2(0,1)} \mathrm{d}t',$$

with $\alpha_2 = 2T \max_{s\in[0,T]} \left|\dot{\hat{D}}(s)\right|$.

Introducing the previous estimates into (112) we obtain

$$\begin{aligned} &\langle \rho \rangle \|\dot{v}_{j,1}(t)\|^2_{L^2(0,1)} + \hat{C} \left\| \frac{\partial v_{j,1}}{\partial x_1}(t) \right\|^2_{L^2(0,1)} + 2\hat{E}^V \int_0^t \left\| \frac{\partial \dot{v}_{j,1}}{\partial x_1}(t') \right\|^2_{L^2(0,1)} \mathrm{d}t' \\ &\leq \alpha_1 \left\| \frac{\partial v_{j,1}}{\partial x_1}(t) \right\|_{L^2(0,1)} \int_0^t \left\| \frac{\partial v_{j,1}}{\partial x_1}(t') \right\|_{L^2(0,1)} \mathrm{d}t' + \alpha_2 \int_0^t \left\| \frac{\partial v_{j,1}}{\partial x_1}(t') \right\|^2_{L^2(0,1)} \mathrm{d}t'. \end{aligned} \tag{113}$$

We majorate next the first term of the right-hand side of (113) as follows:

$$\begin{aligned} \alpha_1 \left\| \frac{\partial v_{j,1}}{\partial x_1}(t) \right\|_{L^2(0,1)} \int_0^t \left\| \frac{\partial v_{j,1}}{\partial x_1}(t') \right\|_{L^2(0,1)} \mathrm{d}t' &\leq \frac{\hat{C}}{2} \left\| \frac{\partial v_{j,1}}{\partial x_1}(t) \right\|^2_{L^2(0,1)} \\ &+ \alpha_3 \int_0^t \left\| \frac{\partial v_{j,1}}{\partial x_1}(t') \right\|^2_{L^2(0,1)} \mathrm{d}t'. \end{aligned}$$

Introducing this estimate into (113) we obtain

$$\|\dot{v}_{j,1}(t)\|^2_{L^2(0,1)} + \left\| \frac{\partial v_{j,1}}{\partial x_1}(t) \right\|^2_{L^2(0,1)} \leq \alpha_4 \int_0^t \left(\|\dot{v}_{j,1}(t')\|^2_{L^2(0,1)} + \left\| \frac{\partial v_{j,1}}{\partial x_1}(t') \right\|^2_{L^2(0,1)} \right) \mathrm{d}t',$$

which gives, applying the Gronwall's inequality,

$$\dot{v}_{j,1} = \frac{\partial v_{j,1}}{\partial x_1} = 0 \quad \text{a.e. in } (0,T).$$

Combining this with the condition $\langle v_{j,1}(\cdot,t) \rangle = 0$ we get to the uniqueness of the solution for problem (111).

For obtaining the existence and regularity results we use Galerkin's method. Consider $\{\varphi_k\}_{k\in\mathbb{N}}$ a basis of the separable space V and define

$$v_n(x_1,t) = \sum_{k=1}^{n} a_k(t)\varphi_k(x_1), \tag{114}$$

with $a_k : [0,T] \mapsto \mathbb{R}$, $a_k(0) = \dot{a}_k(0) = 0$, such that

$$\begin{aligned} &\langle\rho\rangle \int_0^1 \ddot{v}_n(t)\varphi_k + \hat{C}\int_0^1 \frac{\partial v_n}{\partial x_1}(t)\frac{\partial \varphi_k}{\partial x_1} + \hat{E}^V \int_0^1 \frac{\partial \dot{v}_n}{\partial x_1}(t)\frac{\partial \varphi_k}{\partial x_1} \\ &-\int_0^t \left(\hat{D}(t-t')\int_0^1 \frac{\partial v_n}{\partial x_1}(t')\frac{\partial \varphi_k}{\partial x_1}\right)\mathrm{d}t' = \int_0^1 F_{j,1}(t)\varphi_k \quad k=1,\ldots,n, \text{ a.e. in } (0,T). \end{aligned} \tag{115}$$

Introducing (114) into (115) we obtain an $n\times n$ linear system of integro-differential equations of order 2 for the functions $a_1,\ldots,a_n$. Taking into account that the matrix coefficient of $\begin{pmatrix}\ddot{a}_1\\ \ldots\\ \ddot{a}_n\end{pmatrix}$ is $\langle\rho\rangle I_n$ due to the choice of $\{\varphi_k\}_{k\in\mathbb{N}}$ with $\int_0^1 \varphi_l\varphi_k = \delta_{lk}$, we obtain the existence and the uniqueness of the functions $a_1,\ldots,a_n$ satisfying the initial conditions $a_k(0) = \dot{a}_k(0) = 0, k = 1,\ldots,n$.

Calculating next $\sum_{k=1}^{n} \dot{a}_k(t)(115)$, integrating from 0 to t and using the initial conditions, we obtain below the analogous of (113), but with a right-hand side $\neq 0$:

$$\begin{aligned} &\langle\rho\rangle\|\dot{v}_n(t)\|^2_{L^2(0,1)} + \hat{C}\left\|\frac{\partial v_n}{\partial x_1}(t)\right\|^2_{L^2(0,1)} + 2\hat{E}^V\int_0^t \left\|\frac{\partial \dot{v}_n}{\partial x_1}(t')\right\|^2_{L^2(0,1)}\mathrm{d}t' \\ &\leq \alpha_1\left\|\frac{\partial v_n}{\partial x_1}(t)\right\|_{L^2(0,1)}\int_0^t\left\|\frac{\partial v_n}{\partial x_1}(t')\right\|_{L^2(0,1)}\mathrm{d}t' + \alpha_2\int_0^t\left\|\frac{\partial v}{\partial x_1}(t')\right\|^2_{L^2(0,1)}\mathrm{d}t' + 2\int_0^t\int_0^1 F_{j,1}\dot{v}_n. \end{aligned} \tag{116}$$

Finally, majorating the last term of the right-hand side of (116) and using Gronwall's inequality, we obtain the first estimates that give

$$\begin{cases} \{\dot{v}_n\}_n \text{ bounded in } L^\infty(0,T;L^2(0,1)), \\ \left\{\dfrac{\partial v_n}{\partial x_1}\right\}_n \text{ bounded in } L^\infty(0,T;L^2(0,1)), \\ \left\{\dfrac{\partial \dot{v}_n}{\partial x_1}\right\}_n \text{ bounded in } L^2((0,1)\times(0,T)). \end{cases} \tag{117}$$

In order to obtain further estimates, we calculate $\sum_{k=1}^{n} \ddot{a}_k(t)(115)$. Integrating by parts the second term of the right-hand side of the obtained relation and then integrating the equality from 0 to θ with the initial conditions $v_n(0) = \dot{v}_n(0) = 0$ we obtain

$$\begin{aligned} &2\langle\rho\rangle\int_0^\theta \|\ddot{v}_n(t)\|^2_{L^2(0,1)}\mathrm{d}t + \hat{E}^V\left\|\frac{\partial \dot{v}_n}{\partial x_1}(\theta)\right\|^2_{L^2(0,1)} = -2\hat{C}\left(\frac{\partial v_n}{\partial x_1}(\theta),\frac{\partial \dot{v}_n}{\partial x_1}(\theta)\right)_{L^2(0,1)} \\ &+2\hat{C}\left\|\frac{\partial \dot{v}_n}{\partial x_1}\right\|^2_{L^2((0,1)\times(0,T))} - 2\int_0^\theta\left(\int_0^t\left(\hat{D}(t-t')\int_0^1\frac{\partial v_n}{\partial x_1}(t')\frac{\partial \ddot{v}_n}{\partial x_1}(t)\right)\mathrm{d}t'\right)\mathrm{d}t \\ &+2\int_0^\theta\int_0^1 F_{j,1}\ddot{v}_n. \end{aligned} \tag{118}$$

The first term of the right-hand side of (118) is estimated as follows, using $(117)_2$:

$$-2\hat{C}\left(\frac{\partial v_n}{\partial x_1}(\theta),\frac{\partial \dot{v}_n}{\partial x_1}(\theta)\right)_{L^2(0,1)} \leq \frac{\hat{E}^V}{2}\left\|\frac{\partial \dot{v}_n}{\partial x_1}(\theta)\right\|^2_{L^2(0,1)} + \alpha_3;$$

for the second term of the right-hand side of (118) we use $(117)_3$; for the third term we proceed as for obtaining (113); and the last term is majorated by $\langle\rho\rangle\int_0^\theta \|\ddot{v}_n(t)\|^2_{L^2(0,1)}\mathrm{d}t + \frac{1}{\langle\rho\rangle}\|F_{j,1}\|^2_{L^2((0,1)\times(0,T))}$. Introducing these calculations into (118) and using again Gronwall's inequality we obtain

$$\begin{cases} \left\{\dfrac{\partial \dot{v}_n}{\partial x_1}\right\}_n \text{ bounded in } L^\infty(0,T;L^2(0,1)), \\ \{\ddot{v}_n\}_n \text{ bounded in } L^2((0,1)\times(0,T)). \end{cases} \tag{119}$$

Taking into account (117) and (119) we obtain the regularity $(111)_1$ for any weak limit point of the sequence $\{v_n\}_n$ with respect to the spaces that appear in (117) and (119), denoted $v_{j,1}$. Passing to the limit (on a subsequence) with $n \to \infty$ in $\int_0^T \tau(115)$, with τ an arbitrary function of $L^2(0,T)$, it follows that any weak limit point of the sequence $\{v_n\}_n$, $v_{j,1}$, verifies also the equality $(111)_2$. For completing the proof, it remains to show the initial conditions $(111)_3$. For this purpose let us calculate $\int_0^T \tau\,(115)\mathrm{d}t$, with $\tau \in C^1([0,T])$, $\tau(T) = 0$ an arbitrary function of t and (115) written for an arbitrary test function $\varphi \in V$. Integrating by parts the first term and using $\dot{v}_n(0) = 0$, $\tau(T) = 0$ it follows that $(\forall)\,\tau \in C^1([0,T])$, $\tau(T) = 0$, $(\forall)\,\varphi \in V$

$$\begin{aligned} &-\langle\rho\rangle\int_0^T\int_0^1 \dot{v}_n\varphi\dot{\tau}\mathrm{d}x_1\mathrm{d}t + \hat{C}\int_0^T\int_0^1 \frac{\partial v_n}{\partial x_1}\frac{\partial\varphi}{\partial x_1}\tau\mathrm{d}x_1\mathrm{d}t + \hat{E}^V\int_0^T\int_0^1 \frac{\partial \dot{v}_n}{\partial x_1}\frac{\partial\varphi}{\partial x_1}\tau\mathrm{d}x_1\mathrm{d}t \\ &-\int_0^T \tau\left(\int_0^t \hat{D}(t-t')\int_0^1 \frac{\partial v_n}{\partial x_1}(t')\frac{\partial\varphi}{\partial x_1}\mathrm{d}x_1\mathrm{d}t'\right)\mathrm{d}t = \int_0^T\int_0^1 \mathcal{F}_{j,1}\varphi\tau\mathrm{d}x_1\mathrm{d}t. \end{aligned} \tag{120}$$

Performing the same calculation for (111) instead of (115) we obtain

$$\begin{aligned} &-\langle\rho\rangle\tau(0)\int_0^1 \dot{v}_{j,1}\varphi\mathrm{d}x_1 - \langle\rho\rangle\int_0^T\int_0^1 \dot{v}_{j,1}\varphi\dot{\tau}\mathrm{d}x_1\mathrm{d}t + \hat{C}\int_0^T\int_0^1 \frac{\partial v_{j,1}}{\partial x_1}\frac{\partial\varphi}{\partial x_1}\tau\mathrm{d}x_1\mathrm{d}t \\ &+\hat{E}^V\int_0^T\int_0^1 \frac{\partial \dot{v}_{j,1}}{\partial x_1}\frac{\partial\varphi}{\partial x_1}\tau\mathrm{d}x_1\mathrm{d}t - \int_0^T \tau\left(\int_0^t \hat{D}(t-t')\int_0^1 \frac{\partial v_{j,1}}{\partial x_1}(t')\frac{\partial\varphi}{\partial x_1}\mathrm{d}x_1\mathrm{d}t'\right)\mathrm{d}t = \int_0^T\int_0^1 \mathcal{F}_{j,1}\varphi\tau\mathrm{d}x_1\mathrm{d}t. \end{aligned} \tag{121}$$

Passing to the limit in (120) and using (121) we obtain

$$\int_0^1 \dot{v}_{j,1}\varphi\mathrm{d}x_1 = 0 \ \ (\forall)\,\varphi \in V;$$

hence, from the definition of V, it follows that

$$\dot{v}_{j,1}(0) = const. \tag{122}$$

The regularity $(111)_1$ yields $\dot{v}_{j,1} \in C([0,T];L^2(0,1))$, with

$$\langle\dot{v}_{j,1}(t)\rangle\ (\forall)\,t \in [0,T]. \tag{123}$$

Finally, from (122) and (123) we obtain the initial condition $(111)_{3_2}$. In a similar way we obtain $(111)_{3_1}$, which completes the proof. □

6. Error Estimates

In order to justify the asymptotic construction presented in the previous section, we proceed as in the quasistatic case for showing that the error between the exact solution and the asymptotic one is small. The problem for the asymptotic solution of order J is similar to that of the quasistatic case

$$\begin{cases} P_\varepsilon \mathbf{u}_\varepsilon^{(J)} = \mathbf{f}_\varepsilon(x_1,t) - \mathbf{r}_\varepsilon^{(J)} \quad \text{in } G_\varepsilon \times (0,T), \\ \displaystyle\sum_{j=1}^{2} \left(B_{2j} \frac{\partial \dot{\mathbf{u}}_\varepsilon^{(J)}}{\partial x_j} + A_{2j} \frac{\partial \mathbf{u}_\varepsilon^{(J)}}{\partial x_j} \right) = -\mathbf{r}_\varepsilon^{(J),b} \text{ on } (\{x_2=0\} \cup \{x_2=\varepsilon\}) \times (0,T), \\ \mathbf{u}_\varepsilon^{(J)} \;\; 1-\text{periodic in } x_1, \\ \langle \mathbf{u}_\varepsilon^{(J)}(\cdot, x_2, t) \rangle = \mathbf{0}, \\ \mathbf{u}_\varepsilon^{(J)}(0) = \dot{\mathbf{u}}_\varepsilon^{(J)}(0) = \mathbf{0} \quad \text{in } G_\varepsilon, \end{cases} \tag{124}$$

with $P_\varepsilon \mathbf{u}_\varepsilon^{(J)}$ given by $(89)_1$, which means that it contains in addition the term $\ddot{\mathbf{u}}_\varepsilon^{(J)}$. In this case, the residuals have the same meaning as in the quasistatic case, but here their orders are $\mathbf{r}_\varepsilon^{(J)} = \mathcal{O}(\varepsilon^{J-3/2})$ in the norm $L^2(0,T;(L^2(G_\varepsilon^1))^2)$, $\mathbf{r}_\varepsilon^{(J),b} = \mathcal{O}(\varepsilon^{J-1})$ in the norm $L^2((0,1)\times(0,T))$. This is a consequence of the difference between the second term of the ansatz (91) and the second term of the ansatz (7).

With the construction of the same auxiliary function $\boldsymbol{\varphi}_\varepsilon^{(J)}$ as in the quasistatic case and defining in the same way $\mathbf{g}_\varepsilon^{(J)}, \mathbf{U}_\varepsilon^{(J)}, \hat{\mathbf{U}}_\varepsilon^{(J)}$ we obtain for $\hat{\mathbf{U}}_\varepsilon^{(J)}$ the following problem, of the same type as in the quasistatic case:

$$\begin{cases} P_\varepsilon \hat{\mathbf{U}}_\varepsilon^{(J)} = \mathbf{g}_\varepsilon^{(J)} \quad \text{in } G_\varepsilon \times (0,T), \\ \displaystyle\sum_{j=1}^{2} \left(B_{2j} \frac{\partial \dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}}{\partial x_j} + A_{2j} \frac{\partial \hat{\mathbf{U}}_\varepsilon^{(J)}}{\partial x_j} \right) = \mathbf{0} \text{ on } (\{x_2=0\} \cup \{x_2=\varepsilon\}) \times (0,T), \\ \hat{\mathbf{U}}_\varepsilon^{(J)} \;\; 1-\text{periodic in } x_1, \\ \langle \hat{\mathbf{U}}_\varepsilon^{(J)}(\cdot, x_2, t) \rangle = \mathbf{0}, \\ \hat{\mathbf{U}}_\varepsilon^{(J)}(0) = \dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}(0) = \mathbf{0} \quad \text{in } G_\varepsilon. \end{cases} \tag{125}$$

The variational problem associated with (125) is in this case

$$\begin{cases} \hat{\mathbf{U}}_\varepsilon^{(J)} \in W^{1,\infty}(0,T;V_{G_\varepsilon^1}) \cap H^2(0,T;(L^2_{per}(G_\varepsilon^1))^2), \\ \displaystyle \rho \int_{G_\varepsilon^1} \ddot{\hat{\mathbf{U}}}_\varepsilon^{(J)}(t) \cdot \mathbf{v} \mathrm{d}x + \int_{G_\varepsilon^1} \sum_{i,j=1}^{2} \left(B_{ij} \frac{\partial \dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}(t)}{\partial x_j} + A_{ij} \frac{\partial \hat{\mathbf{U}}_\varepsilon^{(J)}(t)}{\partial x_j} \right) \frac{\partial \mathbf{v}}{\partial x_i} \mathrm{d}x \\ \displaystyle \quad = \int_{G_\varepsilon^1} \mathbf{g}_\varepsilon^{(J)}(t) \cdot \mathbf{v} \mathrm{d}x \qquad (\forall)\, \mathbf{v} \in V_{G_\varepsilon^1}, \text{ a.e. in } (0,T), \\ \hat{\mathbf{U}}_\varepsilon^{(J)}(0) = \dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}(0) = \mathbf{0} \text{ in } (L^2_{per}(G_\varepsilon^1))^2, \end{cases} \tag{126}$$

the space $V_{G_\varepsilon^1}$ being the one defined in the quasistatic case. Taking in $(126)_2$ as test function $\mathbf{v} = \dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}(t)$, integrating the obtained relation from 0 to t, using the equalities

$$\begin{cases} \displaystyle\sum_{i,j=1}^{2} A_{ij}(\text{or } B_{ij}) \frac{\partial \mathbf{u}}{\partial x_j} \cdot \frac{\partial \mathbf{v}}{\partial x_i} = \sum_{i,j=1}^{2} A_{ij}(\text{or } B_{ij}) \frac{\partial \mathbf{v}}{\partial x_j} \cdot \frac{\partial \mathbf{u}}{\partial x_i} \;\; (\forall)\, \mathbf{u}, \mathbf{v} \in (H^1(G_\varepsilon^1))^2, \\ \displaystyle\sum_{i,j=1}^{2} A_{ij}(\text{or } B_{ij}) \frac{\partial \mathbf{u}}{\partial x_j} \cdot \frac{\partial \mathbf{u}}{\partial x_i} = \sum_{i,j,k,l=1}^{2} a_{ij}^{kl}(\text{or } b_{ij}^{kl}) \mathcal{E}_{lj}(\mathbf{u}) \mathcal{E}_{ki}(\mathbf{u}) \;\; (\forall)\, \mathbf{u} \in (H^1(G_\varepsilon^1))^2, \end{cases} \tag{127}$$

the coercivity property of the matrices A and B and the initial conditions we obtain a different estimate from that obtained in the quasistatic case:

$$\begin{aligned} \rho \int_{G_\varepsilon^1} \left(\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}(t) \right)^2 + 2\kappa \int_0^t \int_{G_\varepsilon^1} \left(\mathcal{E}(\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}) \right)^2 + \kappa \int_{G_\varepsilon^1} \left(\mathcal{E}(\hat{\mathbf{U}}_\varepsilon^{(J)}(t)) \right)^2 \\ \leq C(\rho,T) \int_0^t \int_{G_\varepsilon^1} \left(\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)} \right)^2 + C_1 \int_0^t \int_{G_\varepsilon^1} \left(\mathbf{g}_\varepsilon^{(J)} \right)^2, \quad \text{a.e. in } (0,T). \end{aligned} \tag{128}$$

with $C(\rho, T)$, C_1 positive constants independent of ε. The first result concerning the error between the exact solution and the asymptotic solution of order J is given below:

Theorem 6. *Let $\mathbf{u}_\varepsilon$ be the exact solution of problem (89) and $\mathbf{u}_\varepsilon^{(J)}$ the asymptotic solution of order J. Then*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(J)}\|_{H^1(0,T;(H^1_{per}(G^1_\varepsilon)^2))} = \mathcal{O}(\varepsilon^{J-7/2}). \tag{129}$$

Proof. From (128) we obtain

$$\rho \int_{G^1_\varepsilon} \left(\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}(t)\right)^2 \leq C(\rho, T) \int_0^t \int_{G^1_\varepsilon} \left(\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}\right)^2 + C_1 \int_0^t \int_{G^1_\varepsilon} \left(\mathbf{g}_\varepsilon^{(J)}\right)^2$$

$$\leq C(\rho, T) \|\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}\|^2_{L^2(0,T;(L^2_{per}(G^1_\varepsilon)^2))} + C_1 \|\mathbf{g}_\varepsilon^{(J)}\|^2_{L^2(0,T;(L^2_{per}(G^1_\varepsilon)^2))}.$$

Since

$$\|h\|_{L^2(0,T;L^2(G^1_\varepsilon))} \leq T^{1/2} \|h\|_{L^\infty(0,T;L^2(G^1_\varepsilon))}$$

for any $h \in L^\infty(0, T; L^2(G^1_\varepsilon))$ with $h(0) = 0$, we obtain from the previous inequality an estimate for $\|\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}\|^2_{L^\infty(0,T;(L^2_{per}(G^1_\varepsilon)^2))}$ by choosing $C(\rho, T) = \dfrac{\rho}{2T^{1/2}}$, which gives, combined with (128),

$$\begin{cases} \max\Big\{\|\hat{\mathbf{U}}_\varepsilon^{(J)}\|_{L^2(0,T;(L^2(G^1_\varepsilon))^2)}, \|\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)}\|_{L^2(0,T;(L^2(G^1_\varepsilon))^2)}, \|\mathcal{E}(\hat{\mathbf{U}}_\varepsilon^{(J)})^2\|_{L^2(0,T;(L^2(G^1_\varepsilon))^{2\times 2})}, \\ \|\mathcal{E}(\dot{\hat{\mathbf{U}}}_\varepsilon^{(J)})^2\|_{L^2(0,T;(L^2(G^1_\varepsilon))^{2\times 2})}\Big\} \leq C_2 \|\mathbf{g}_\varepsilon^{(J)}\|^2_{L^2(0,T;(L^2(G^1_\varepsilon))^2)}, \end{cases} \tag{130}$$

with C_2 a positive constant independent of ε.

We establish by direct calculation the order of $\|\mathbf{g}_\varepsilon^{(J)}\|_{L^2(0,T;(L^2(G^1_\varepsilon))^2)}$ using, as in Section 4, (63), (65) and the estimates for $\mathbf{r}_\varepsilon^{(J)}, \mathbf{r}_\varepsilon^{(J),b}$ given at the beginning of this section as follows:

$$\|\mathbf{g}_\varepsilon^{(J)}\|_{L^2(0,T;(L^2(G^1_\varepsilon))^2)} = \mathcal{O}(\varepsilon^{J-3/2}). \tag{131}$$

For obtaining (129), we proceed as in the quasistatic case by considering the function $\mathbf{V}_\varepsilon$ defined by (77) and, from (130), (131) and Korn's inequality applied in G (by means of the function $\mathbf{V}_\varepsilon$), we obtain (129), which completes the proof. □

As one can see, for values of $J \leq 3$ the error between the exact and the asymptotic solution is not small, while we are interested in constructing an asymptotic solution that represents a good approximation of the exact solution from the first term. The next result improves this error estimate:

Theorem 7. *Let $\mathbf{u}_\varepsilon$ be the exact solution of problem (89) and $\mathbf{u}_\varepsilon^{(J)}$ the asymptotic solution of order J. Then*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(J)}\|_{H^1(0,T;(H^1(G^1_\varepsilon)^2))} = \begin{cases} \mathcal{O}(\varepsilon^{J-1/2}) & \text{if } J \geq 1, \\ \mathcal{O}(\varepsilon^{1/2}) & \text{if } J = 0. \end{cases} \tag{132}$$

Proof. Let us consider $J \geq 0$ and $K \geq J + 4$ and let us calculate

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(J)}\|_{H^1(0,T;(H^1(G^1_\varepsilon)^2))} \leq \|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{(K)}\|_{H^1(0,T;(H^1(G^1_\varepsilon)^2))}$$

$$+ \|\mathbf{u}_\varepsilon^{(K)} - \mathbf{u}_\varepsilon^{(J)}\|_{H^1(0,T;(H^1(G^1_\varepsilon)^2))}.$$

For estimating the second term we analyze separately two cases:
(i) $J = 0$; then

$$\mathbf{u}_\varepsilon^{(K)} - \mathbf{u}_\varepsilon^{(0)} = \mathbf{v}_\varepsilon^{(K)} - \mathbf{v}_\varepsilon^{(0)}$$

$$+\sum_{p=-1}^{K}\sum_{q+l=1}^{K}\varepsilon^{q+l+p}\int_0^t N_{q,l,p}(\frac{x_2}{\varepsilon},t-t')\frac{\partial^{q+l}\mathbf{v}_\varepsilon^{(J)}(x_1,t')}{\partial t'^q\partial x_1^l}dt' = \mathcal{O}(\varepsilon^1).$$

This estimate is a consequence of (102)$_1$ and of (8) and leads to the second estimate of (132) (taking into account the derivation with respect to x_2 that introduces ε^{-1} and the integration over G_ε^1 that introduces $\varepsilon^{1/2}$).

(ii) $J \geq 1$; in this case, using the ansatsz (91), we obtain

$$\mathbf{u}_\varepsilon^{(K)} - \mathbf{u}_\varepsilon^{(J)} = \mathbf{v}_\varepsilon^{(K)} - \mathbf{v}_\varepsilon^{(J)}$$

$$+\sum_{p=J+1}^{K}\sum_{q+l=1}^{J}\varepsilon^{q+l+p}\int_0^t N_{q,l,p}(\frac{x_2}{\varepsilon},t-t')\frac{\partial^{q+l}\mathbf{v}_\varepsilon^{(J)}(x_1,t')}{\partial t'^q\partial x_1^l}dt'$$

$$+\sum_{p=-1}^{K}\sum_{q+l=J+1}^{K}\varepsilon^{q+l+p}\int_0^t N_{q,l,p}(\frac{x_2}{\varepsilon},t-t')\frac{\partial^{q+l}\mathbf{v}_\varepsilon^{(J)}(x_1,t')}{\partial t'^q\partial x_1^l}dt'$$

$$= \mathcal{O}(\varepsilon^{J+1}) + \mathcal{O}(\varepsilon^{J+2}) + \mathcal{O}(\varepsilon^{J}) = \mathcal{O}(\varepsilon^{J}).$$

Taking into account, as before, the derivation with respect to x_2 and the integration over G_ε^1 we obtain the first estimate of (132) . □

7. Conclusions

The asymptotic analysis for the quasistatic Kelvin–Voigt model was announced in [19], but the proofs are given for the first time in the present paper, while the approach corresponding to the dynamic case is considered here for the first time. The asymptotic technique introduces the homogenization combined with the dimension reduction of a two-dimensional plate and derives a one-dimensional model. As far as we know, it is the first time that the complete asymptotic expansion of the solution to the Kelvin–Voigt model of a thin stratified plate has been constructed and the error estimates of order $O(\varepsilon^J)$ established for asymptotic approximations of any order J. The proposed method allows us to obtain all the terms of the asymptotic expansion and, for any given J, to obtain approximations of the accuracy $O(\varepsilon^J)$, while other methods mainly give the leading term only and do not always give error estimates. Note that these estimates are important to describe the limitations of the theory. Another advantage is that the form of the asymptotic expansion gives the possibility to use the form of the solution in the construction of some subspaces in model reduction via projection, as is done in the method of asymptotic partial decomposition of the domain and in the method of partial homogenization.

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