Research Article

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Nonstationary Poiseuille flow of a non-Newtonian fluid with the shear rate-dependent viscosity

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Abstract: A nonstationary Poiseuille flow of a non-Newtonian fluid with the shear rate dependent viscosity is considered. This problem is nonlinear and nonlocal in time and inverse to the nonlinear heat equation. The provided mathematical analysis includes the proof of the existence, uniqueness, regularity, and stability of the velocity and the pressure slope for a given flux carrier and of the exponential decay of the solution as the time variable goes to infinity for the exponentially decaying flux.

Keywords: non-Newtonian flow, strain rate dependent viscosity, Poiseuille flows

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1 Introduction

The Poiseuille flow is defined as a solution of the fluid motion equations in an infinite tube with no slip boundary condition on the lateral boundary, satisfying the following conditions: the pressure is linear with respect to the normal (longitudinal) variable, the tangential velocity is equal to zero, and the normal velocity depends only on the transverse variables. In the stationary case of the Newtonian fluid, the Poiseuille flow's normal velocity is a solution of the Dirichlet problem for the Poisson equation on the cross-section of the tube, while for the nonstationary Navier-Stokes equation, the normal velocity satisfies the Dirichlet problem for the heat equation with the right-hand side equal to the pressure slope depending on the time variable only. The reconstruction of the pressure slope in the case of the prescribed flux is related to the inverse heat equation problem (see [4,8,20,22,23,27]). The time periodic case was studied in [2,7]. The existence of the nonstationary Poiseuille solutions under minimal regularity assumptions on the flow rate is considered in [24]. The asymptotic behavior of the Poiseuille solutions with respect to the small parameter was found in [16]. The stationary and nonstationary Poiseuille flows for different models of the non-Newtonian fluid were studied in [5,6, 12,13,19,25,26].

In the present paper, we consider the non-Newtonian flow with the viscosity depending on the shear rate. The stationary case of this nonlinear problem was considered in [9]. The regularity and stability of the stationary Poiseuille flow for non-Newtonian rheology was considered in [18]. The present paper is devoted to the analysis of the nonstationary Poiseuille flow for the non-Newtonian modification of the Stokes equation. We study the existence, uniqueness, regularity, and stability of the velocity and the pressure slope for a given flux and the exponential decay of the solution as the time variable goes to infinity for the

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exponentially decaying flux. The difference with respect to [5,6] is in the type of the non-Newtonian rheology: in [5], it is a shear thinning one, and in [6], it corresponds to a monotone operator, while in the present paper, the fluid satisfies different conditions. In particular, it can be shear thickening.

Apart from theoretical interest for partial differential equations, this set of questions is important for construction of asymptotic expansions of solutions in thin domains. In particular, the results of the present paper are important for the construction of an asymptotic expansion of a non-Newtonian flow in a network of thin cylinders, modeling blood vessels (see, for example, [14,15,17] for Newtonian flows). The blood flow rheology is described by Carreau's law [10], which is covered by the dependence of the viscosity on the shear rate considered below.

Let us present the main results.

Let $n = 2, 3$, $\nu_0, \lambda > 0$ be positive constants. Let σ be a bounded domain in $\mathbb{R}^{n-1}$. Let ν be a bounded C^4–smooth function $\mathbb{R}^{n(n+1)/2} \to \mathbb{R}$ such that for all $y \in \mathbb{R}^{n(n+1)/2}$,

$$|\nu(y)| \le A, \quad |\nabla_y^k \nu(y)| \le A, \ k = 1, \ldots, 4, \tag{1.1}$$

where A is a positive constant independent of y.

Consider the following boundary value problem in the cylinder $\Pi = \sigma \times \mathbb{R}$, $\partial\sigma \in C^4$, for the non-Newtonian fluid motion equations:

$$\begin{cases} \mathbf{u}_t - \operatorname{div}(\nu_0 + \lambda\nu(\dot{\gamma}(\mathbf{u}))D(\mathbf{u})) + \nabla p = 0, & x \in \Pi, \\ \operatorname{div}\mathbf{u} = 0, & x \in \Pi, \\ \mathbf{u}(x,t) = 0, & x \in \partial\Pi, \\ \mathbf{u}(x,0) = 0, & x \in \Pi, \end{cases} \tag{1.2}$$

where $D(\mathbf{u})$ is the strain rate matrix with the elements $d_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ and $\dot{\gamma}(\mathbf{u}) = (d_{12}, d_{13}, d_{23}, d_{11}, d_{22}, d_{33})$ if $n = 3$ and $\dot{\gamma}(\mathbf{u}) = (d_{12}, d_{11}, d_{22})$ if $n = 2$.

We look for solution $\mathbf{u}$ having prescribed flux F through the cross sections σ of cylinder Π:

$$\int_\sigma \mathbf{u} \cdot \mathbf{n} dS = F(t). \tag{1.3}$$

Define a Poiseuille flow as a solution $(\mathbf{V}_{P_\alpha}, \mathcal{P}_{P_\alpha})$ to this problem having the following structure: $\mathbf{V}_{P_\alpha}(x,t) = (v_{P_\alpha}(x',t), 0, \ldots, 0)^T$, and $\mathcal{P}_{P_\alpha}(x,t) = -\alpha(t)x_1 + \beta(t)$, where $x' = (x_2, \ldots, x_n)$, $\alpha \in W^{1,2}(0,T)$, β is an arbitrary function of t, and v_{P_α} is the solution of the following problem:

$$\begin{cases} v_{P_\alpha,t} - \frac{1}{2}\operatorname{div}_{x'}\Big(\big(\nu_0 + \lambda\nu\big(\dot{\gamma}_P(v_{P_\alpha})\big)\big)\nabla_{x'}v_{P_\alpha}\Big) = \alpha(t), & x' \in \sigma, \\ v_{P_\alpha}(x',t) = 0, & x \in \partial\sigma, \\ v_{P_\alpha}(x',0) = 0, & x' \in \sigma, \end{cases} \tag{1.4}$$

Here, $\dot{\gamma}_P(v_{P_\alpha}) = \left(\frac{1}{2}\nabla_{x'}v_{P_\alpha}, 0, 0\right)$ if $n = 2$, $\dot{\gamma}_P(v_{P_\alpha}) = \left(\frac{1}{2}\nabla_{x'}v_{P_\alpha}, 0, 0, 0\right)$ if $n = 3$, and α is the given pressure slope.

The following inverse problem corresponds to problem (1.4): given $F \in W^{2,2}(0,T)$, such that $F(0) = 0$, find $\alpha(t)$ and $v_{P_\alpha}(x',t)$ satisfying the relations:

$$\begin{cases} v_{P_\alpha,t} - \frac{1}{2}\operatorname{div}_{x'}\Big(\big(\nu_0 + \lambda\nu\big(\dot{\gamma}_P(v_{P_\alpha})\big)\big)\nabla_{x'}v_{P_\alpha}\Big) = \alpha(t), & x' \in \sigma, \\ v_{P_\alpha}(x',t) = 0, & x' \in \sigma, \\ v_{P_\alpha}(x',0) = 0, & x \in \sigma, \end{cases} \tag{1.5}$$

and the additional flux condition

$$\int_\sigma v_{P_\alpha}(x',t)dx' = F(t). \tag{1.6}$$

Define the shear rate $\dot{\gamma}'$ as $|d_{12}|$ in the case $n = 2$, and $\dot{\gamma}' = \sqrt{d_{12}^2 + d_{13}^2 + d_{23}^2}$. Note that Carreau's law reads: the viscosity ν depends on the shear rate $\dot{\gamma}'$ as follows:

$$\nu(\dot{\gamma}') = \nu_0(1 + (k\dot{\gamma}')^2)^{(m-1)/2}.$$

Here, $\nu_0 > 0$, $k > 0$, m are constants. For $m < 1$, this rheology is shear thickening and satisfies condition (1.1) for small k. For the definition of function spaces see Section 2.

The first main result is presented by the following two theorems.

Theorem 1.1. *Let $\partial\sigma \in C^4$. For any $\alpha_0 > 0$, there exist $\lambda_1 = \lambda_1(\alpha_0)$ and $R_0 = R_0(\alpha_0)$ such that for all $\lambda \in (0, \lambda_1]$ and any $\alpha \in W^{1,2}(0, T)$ such that $\alpha(0) = 0$ and $\|\alpha\|_{W^{1,2}(0,T)} \le \alpha_0$, problem* (1.4) *admits a unique*[1] *solution $v_{P_\alpha} \in W^{(4,2),2}_{\partial\sigma}(\sigma^T)$ satisfying the estimate*

$$\|v_{P_\alpha}\|_{W^{(4,2),2}_{\partial\sigma}(\sigma^T)} \le c\|\alpha\|_{W^{1,2}(0,T)}, \tag{1.7}$$

and belonging to the ball $\mathcal{B}_{R_0}$ of space $W^{(4,2),2}_{\partial\sigma}(\sigma^T)$, where the constant c depends only on σ. Here, $\sigma^T = \sigma \times (0, T)$.

Theorem 1.2. *For any $\mathcal{F}_0 > 0$, there exists $\lambda_2 = \lambda_2(\mathcal{F}_0)$ such that for all $\lambda \in (0, \lambda_2]$ and every $F \in W^{2,2}(0, T)$ such that $F(0) = 0, F'(0) = 0$ and $\|F\|_{W^{2,2}(0,T)} \le \mathcal{F}_0$, problem,* (1.5) *and* (1.6) *admit a unique solution $(v_{P_\alpha}, \alpha) \in W^{(4,2),2}_{\partial\sigma}(\sigma^T) \times W^{1,2}(0, T)$. Moreover, $\alpha(0) = 0$, and the following estimate*

$$\|v_{P_\alpha}\|_{W^{(4,2),2}(\sigma^T)} + \|\alpha\|_{W^{1,2}(0,T)} \le c\|F\|_{W^{2,2}(0,T)} \tag{1.8}$$

holds.

The second main result (on the stability of the solution) is presented by the following two theorems.

Theorem 1.3. *For any α_0, there exists $\lambda_3 = \lambda_3(\alpha_0)$ such that for all $\lambda \in (0, \lambda_3]$ and every $\alpha_1, \alpha_2 \in W^{1,2}(0, T)$, with $\alpha_i(0) = 0$ and $\|\alpha_i\|_{W^{1,2}(0,T)} \le \alpha_0$, $i = 1, 2$, there holds the estimate*

$$\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|^2_{W^{(2,1),2}(\sigma^T)} \le c\|\alpha_1 - \alpha_2\|^2_{L^2(0,T)}. \tag{1.9}$$

Theorem 1.4. *For any F_0, there exists $\lambda_4 = \lambda_4(F_0)$ such that for all $\lambda \in (0, \lambda_4]$ and every $F_1, F_2 \in W^{2,2}(0, T)$ with $F_i(0) = 0$, $F_i'(0) = 0$ and $\|F_i\|_{W^{2,2}(0,T)} \le F_0$, $i = 1, 2$, there holds the estimate*

$$\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|_{W^{(2,1),2}(\sigma^T)} + \|\alpha_1 - \alpha_2\|_{L^2(0,T)} \le c\|F_1 - F_2\|_{W^{1,2}(\sigma^T)}. \tag{1.10}$$

Finally, the third main result is on the decay of the Poiseuille flow as the time variable tends to infinity:

Theorem 1.5. *Assume that the right-hand side α of* (1.4) *satisfies the conditions of* Theorem 1.1, *$\alpha \in W^{1,2}(0, \infty)$ and satisfies the additional condition*:

$$\int_0^\infty |\alpha(t)|^2 e^{2\beta t}\mathrm{d}t < +\infty. \tag{1.11}$$

Let $v_{P_\alpha} \in W^{(4,2),2}_{\partial\sigma}(\sigma^\infty)$ be a solution of problem (1.4). *There exists β_0 such that if $\beta \in (0, \beta_0)$, then $v_{P_\alpha} \in W^{(4,2),2}_{\partial\sigma}(\sigma^T)$ satisfies the following estimate*:

$$\int_0^\infty \int_\sigma (|v_{P_\alpha}|^2 + |\nabla_{x'} v_{P_\alpha}|^2) e^{2\beta t}\mathrm{d}x'\mathrm{d}t \le c\int_0^\infty |\alpha(t)|^2 e^{2\beta t}\mathrm{d}t. \tag{1.12}$$

Theorem 1.6. *Assume that the flux F in* (1.6) *satisfies the conditions of* Theorem 1.2, *$F \in W^{2,2}(0, +\infty)$ and satisfies the additional condition*:

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1 Here and below, the uniqueness takes place only in some ball where the contraction principle is applied.

$$\int_0^\infty (|F(t)|^2 + |F'(t)|^2) e^{\frac{7\beta}{2}t} \mathrm{d}t < +\infty. \tag{1.13}$$

Let $v_{P_\alpha} \in W^{(4,2),2}_{\partial\sigma}(\sigma^\infty)$ be a solution of the inverse problems (1.5) *and* (1.6). *There exists β_1 such that if $\beta \in (0, \beta_1)$, then (v_{P_α}, α) satisfies the following estimate*:

$$\int_0^\infty \int_\sigma |(v_{P_\alpha})_t|^2 e^{\beta t} \mathrm{d}x' \mathrm{d}t + \int_0^\infty e^{2\beta t} \int_\sigma |\nabla_{x'} v_{P_\alpha}|^2 \mathrm{d}x' \mathrm{d}t + \int_0^\infty e^{\frac{\beta}{2}t} |\alpha(t)|^2 \mathrm{d}t \le c \int_0^\infty (|F(t)|^2 + |F'(t)|^2) e^{\frac{7\beta}{2}t} \mathrm{d}x' \mathrm{d}t + c\lambda^2 R_0^4. \tag{1.14}$$

Notice that for the Newtonian case, the exponential decay of the Poiseuille solution is proved in [21,23].

These three couples of theorems are proved, respectively, in Sections 3, 4, and 5. Section 2 is devoted to several auxiliary results, in particular to some generalization of the Banach fixed point theorem.

2 Auxiliary theorems

2.1 Notation, function spaces and embedding theorems

Let $G \subset \mathbb{R}^n$ be a bounded domain. Denote $G^T = G \times (0, T)$. Let $l \ge 1$ be an integer. By $L^q(G)$, $q \in [1, \infty]$, and $W^{l,q}(G)$, $\mathring{W}^{1,q}(G)$, we denote the standard Lebesgue and Sobolev spaces of functions defined in G.

Let $m \ge 1$ be an integer. Let us define several spaces of functions depending of x and t. $W^{(2m,m),2}(G^T)$ is the space consisting of functions belonging to $L^2(G^T)$ and having all generalized derivatives of the form $\partial_t^r \partial_x^s$ with arbitrary r and s satisfying the inequality $2r + s \le 2m$. The norm in $W^{(2m,m),2}(G^T)$ is defined by the formula:

$$\|u\|_{W^{(2m,m),2}(G^T)} = \sum_{j=0}^{2m} \sum_{2r+s=j} \|\partial_t^r \partial_x^s u\|_{L^2(G^T)}.$$

In the second sum, the summation is over all nonnegative integers r and s satisfying the condition $2r + s = j$.

We will also need the spaces $W^{(1,1),2}(G^T)$ and $W^{(1,0),2}(G^T)$ consisting of functions having the finite norms:

$$\|u\|_{W^{(1,1),m,2}(G^T)} = \|u\|_{L^2(G^T)} + \|\nabla_x u\|_{L^2(G^T)} + \|\partial_t u\|_{L^2(G^T)},$$
$$\|u\|_{W^{(1,0),2}(G^T)} = \|u\|_{L^2(G^T)} + \|\nabla_x u\|_{L^2(G^T)}.$$

For all spaces $W^{(\dots),2}(G^T)$, we denote by $W^{(\dots),2}_{\partial\sigma}(G^T)$ the subspace of functions satisfying the condition $u|_{\partial G} = 0$.

Below we formulate the embedding inequalities, which are used in the paper.

Lemma 2.1. *Let G be a Lipschitz bounded domain. If $u \in W^{3,2}(G)$, then $u \in W^{1,\infty}(G)$ and $u \in W^{2,4}(G)$. There hold the estimates*

$$\|u\|_{W^{1,\infty}(G)} \le c\|u\|_{W^{3,2}(G)},$$
$$\|u\|_{W^{2,4}(G)} \le c\|u\|_{W^{3,2}(G)}. \tag{2.1}$$

For general Sobolev embedding theorems, see, e.g., [1].

Lemma 2.2. *Let G be a Lipschitz-bounded domain.*

(*i*) *If* $u \in W^{(2m,m),2}(G^T)$ *and*

$$p \geq 2, \quad 2m - 2r - s - \left(\frac{1}{2} - \frac{1}{p}\right)(n+2) \geq 0, \tag{2.2}$$

then $\partial_t^r \partial_x^s u \in L^p(G^T)$ *and*

$$\|\partial_t^r \partial_x^s u\|_{L^p(G^T)} \leq c\|u\|_{W^{(2m,m),2}(G^T)}. \tag{2.3}$$

(*ii*) *If* $u \in W^{(2m,m),2}(G^T)$ *and* $2r + s \leq 2m - 1$, *then the trace* $\partial_t^r \partial_x^s u\,|_{t=t_*} \in W^{2m-2r-s-1,2}(G)$ *and*

$$\|\partial_t^r \partial_x^s u\,|_{t=t_*}\|_{W^{2m-2r-s-1,2}(G)} \leq c\|u\|_{W^{(2m,m),2}(G^T)} \tag{2.4}$$

with the constant c *independent of* $t_* \in [0, T]$.

In particular, if $G \subset \mathbb{R}^n$, $n \leq 2$, *then there hold the inequalities*

$$\begin{aligned} \|\nabla^2 u_t\|_{L^2(G^T)} &\leq c\|u\|_{W^{(4,2),2}(G^T)}, \\ \|\nabla u_t\|_{L^4(G^T)} &\leq c\|u\|_{W^{(4,2),2}(G^T)}, \\ \|\nabla^k u\|_{L^4(G^T)} &\leq c\|u\|_{W^{(4,2),2}(G^T)}, \quad k = 2, 3, \\ \|\nabla^2 u\|_{L^6(G^T)} &\leq c\|u\|_{W^{(4,2),2}(G^T)}, \\ \|\nabla u\|_{L^4(G^T)} &\leq c\|u\|_{W^{(2,1),2}(G^T)}, \end{aligned} \tag{2.5}$$

and

$$\sup_{t \in (0,T)} \|u(\cdot,t)\|_{W^{3,2}(G)} \leq c\|u\|_{W^{(4,2),2}(G^T)}. \tag{2.6}$$

The constants in inequalities (2.3)–(2.6) *are independent of* T.

More information on the embedding tor the spaces of function depending on x and t can be found in [3,11].

2.2 Weak Banach contraction principle

Below we formulate and prove some version of the Banach fixed point theorem. It seems that this theorem is known, but we present the proof for reader's convenience since we do not know the appropriate reference.

Theorem 2.1. *Let* X, Y, and Z, K *be reflexive Banach spaces*, $X \subset Y$, $Z \subset K$,

$$\|x\|_Y \leq \|x\|_X \quad \forall x \in X, \quad \|z\|_Z \leq \|z\|_K \quad \forall z \in Z. \tag{2.7}$$

Suppose that $M \subset X$ *and* $N \subset Z$ *are closed, bounded sets,* $M \neq \emptyset$, $N \neq \emptyset$, *and the mapping* $T : M \longmapsto M \times N$ *satisfies the inequality*

$$\|Tx - Ty\|_{Y\times Z} \leq k\|x - y\|_Y \text{ with } k < 1, \text{ for all } x, y \in M. \tag{2.8}$$

Then there exists exactly one pair $(x_*, z_*) \in M \times N$ *such that*

$$Tx_* = (x_*, z_*).$$

Proof. Let us define a sequence (x_n, z_n) by the recurrent formulas:

$$(x_{n+1}, z_{n+1}) = Tx_n, \quad x_0 \in M. \tag{2.9}$$

Since T maps the bounded set M onto the bounded set $M \times N$, there exists a positive constant c_0 such that $\|x_n\|_X \leq c_0$ and $\|Tx_n\|_{X\times Z} \leq c_0$. Since the spaces X and Z are reflexive, there exists a subsequence $\{x_{n_k}\}$ such that

$$x_{n_k} \overset{X}{\rightharpoonup} x_*, \quad Tx_{n_k} = \left(x_{n_k}, z_{n_k}\right) \overset{X\times Z}{\rightharpoonup} (y_*, z_*), \quad x_*, y_* \in M, \ z_* \in N. \tag{2.10}$$

For simplicity, we will not distinguish in notation the subsequence $\{x_{n_k}\}$, $\{z_{n_k}\}$ and the sequences $\{x_n\}$, $\{z_n\}$. From (2.8), it follows that

$$\|Tx_n - Tx_{n+1}\|_{Y\times Z} \le k^n \|x_0 - x_1\|_Y .$$

Therefore, $\{Tx_n\}$ is a Cauchy sequence and is strongly convergent in $Y \times Z$, i.e. $Tx_n \overset{Y\times Z}{\to} (y_*, z_*)$. From (2.9), we obtain

$$(x_n, z_n) = Tx_{n-1} \overset{Y\times Z}{\to} (y_*, z_*) \overset{(2.10)}{=} (x_*, z_*). \tag{2.11}$$

Thus,

$$\|Tx_n - Tx_*\|_{Y\times Z} \le k\|x_n - x_*\|_Y \to 0 \quad \text{as } n \to +\infty.$$

Hence,

$$Tx_* = (y_*, z_*). \tag{2.12}$$

The relations (2.11) and (2.12) yield

$$Tx_* = (x_*, z_*).$$

The uniqueness is obvious. □

3 Existence of non-Newtonian Poiseuille flow with prescribed pressure slope or prescribed flux

Consider the heat equation

$$\begin{cases} w_t - \nu_0 \Delta' w = f, & x' \in \sigma, \\ w(x', t) = 0, & x' \in \partial\sigma, \\ w(x', 0) = w_0, & x' \in \sigma. \end{cases} \tag{3.1}$$

The following theorem is well known (see, e.g., [11]).

Theorem 3.1. *Let* $\partial\sigma \in C^4$, $f \in W^{(2,1),2}(\sigma^T)$, $\sigma^T = \sigma \times (0, T)$, $w_0 \in W^{3,2}(\sigma)$ *and*

$$w_0^{(0)}(x')|_{\partial\sigma} = 0, \quad w_0^{(1)}(x')|_{\partial\sigma} = 0, \tag{3.2}$$

where $w_0^{(0)}(x') = w_0(x')$, $w_0^{(1)}(x') = \nu_0 \Delta' w_0(x') + f(x', 0)$. *Then problem* (3.1) *admits a unique solution* $w \in W_{\partial\sigma}^{(4,2),2}(\sigma^T)$ *and the following estimate*:

$$\|w\|_{W^{(4,2),2}(\sigma^T)} \le c(\|f\|_{W^{(2,1),2}(\sigma^T)} + \|w_0\|_{W^{3,2}(\sigma)}) \tag{3.3}$$

holds with constant c independent of T.

In particular, if $w_0(x') = 0$ *and* $f(x', 0) = 0$, *then the compatibility condition* (3.2) *are satisfied and the statement of the theorem is true.*

We also need results about the following inverse problem for the heat equation: find $W(x', t)$ and $\alpha(t)$ solving the problem

$$\begin{cases} W_t - \nu_0\Delta' W = \alpha(t) + f, \; x' \in \sigma, \\ \quad W(x', t) = 0, \qquad x' \in \partial\sigma, \\ \quad W(x', 0) = W_0, \qquad x' \in \sigma, \\ \int\limits_\sigma W(x', t)\mathrm{d}x' = F(t). \end{cases} \tag{3.4}$$

Theorem 3.2. *Let* $\partial\sigma \in C^4$, $f \in W^{(2,1),2}(\sigma^T)$, $W_0 \in W^{3,2}(\sigma)$ *and* $F \in W^{2,2}(0, T)$,

$$F(0) = \int\limits_\sigma W_0^{(0)}(x')\mathrm{d}x', \quad F'(0) = \int\limits_\sigma W_0^{(1)}(x')\mathrm{d}x', \quad W_0^{(0)}(x')|_{\partial\sigma} = 0, \quad W_0^{(1)}(x')|_{\partial\sigma} = 0, \tag{3.5}$$

where $W_0^{(0)}(x') = W_0(x')$, $W_0^{(1)}(x') = \nu_0\Delta' W_0(x') + f(x', 0)$. *Then problem* (3.4) *admits a unique solution* $(W, \alpha) \in W_{\partial\sigma}^{(4,2),2}(\sigma^T) \times W^{1,2}(0, T)$ *and the following estimate*

$$\|W\|_{W^{(4,2),2}(\sigma^T)} + \|\alpha\|_{W^{1,2}(0,T)} \le c(\|f\|_{W^{(2,1),2}(\sigma^T)} + \|w_0\|_{W^{3,2}(\sigma)} + \|F\|_{W^{2,2}(0,T)}) \tag{3.6}$$

holds with constant c independent of T.

In particular, if $W_0(x') = 0$, $F(0) = F'(0) = 0$, $f(x', 0) = 0$, *then the compatibility conditions* (3.5) *are satisfied, and the statement of the theorem is true. Moreover, in this case, the pressure slope* α *satisfies the condition* $\alpha(0) = 0$.

The proof of this theorem can be found in [23].

Consider now the nonlinear problem (1.4). Let us prove Theorem 1.1.

Proof. Let $\mathcal{L}$ be an operator $W_{\partial\sigma}^{(4,2),2}(\sigma^T) \to W_{\partial\sigma}^{(4,2),2}(\sigma^T)$ such that for arbitrary fixed $\alpha \in W^{1,2}(0, T)$ with $\alpha(0) = 0$ and a given $v \in W_{\partial\sigma}^{4,2}(\sigma^T)$, satisfying the initial condition $v(x', 0) = 0$, the function $V = \mathcal{L}v$ is a solution of the heat equation:

$$\begin{cases} V_t - \dfrac{\nu_0}{2}\Delta V = h(v) + \alpha, \quad x \in \sigma, \\ \quad V|_{\partial\sigma} = 0, \quad V(x', 0) = 0, \end{cases} \tag{3.7}$$

where

$$h(v) = \frac{1}{2}\lambda \mathrm{div}_{x'}\big(\nu(\dot\gamma_P(v))\nabla_{x'}v\big) = \frac{1}{2}\lambda[\nu(\dot\gamma_P(v))\Delta_{x'}v + (\nabla_y \nu(\dot\gamma_P(v)))^T(\nabla_{x'}(\dot\gamma_P(v)))^T\cdot\nabla_{x'}v]. \tag{3.8}$$

Because of the condition $v(x', 0) = 0$, from the definition (3.8) of $h(v)$, we see that $h(v(x', 0)) = 0$. Thus, $f(x', 0) = \alpha(0) + h(v(x', 0)) = 0$ and the compatibility conditions of Theorem 3.1 are valid.

Using the continuous embeddings (2.1), (2.6), and conditions (1.1), we obtain

$$\begin{aligned} \int\limits_0^T \|h(v)\|^2_{L^2(\sigma)}\mathrm{d}t &\le c\lambda^2\Bigg(\int\limits_0^T \|v\|^2_{W^{2,2}(\sigma)}\mathrm{d}t + \int\limits_0^T \|v\|^2_{W^{3,2}(\sigma)}\|v\|^2_{W^{2,2}(\sigma)}\mathrm{d}t\Bigg) \\ &\le c\lambda^2\Bigg(\int\limits_0^T \|v\|^2_{W^{2,2}(\sigma)}\mathrm{d}t + \sup_{t\in(0,T)}\|v\|^2_{W^{3,2}(\sigma)}\int\limits_0^T \|v\|^2_{W^{2,2}(\sigma)}\mathrm{d}t\Bigg) \\ &\le c\lambda^2(\|v\|^2_{W^{(4,2),2}(\sigma^T)} + \|v\|^4_{W^{(4,2),2}(\sigma^T)}). \end{aligned} \tag{3.9}$$

Analogously,

$$|\nabla_{x'}h(v)| \le c\lambda(|\nabla^3_{x'}v| + |\nabla^2_{x'}v|^2)(1 + |\nabla_{x'}v|),$$

and

$$|\partial_t h(v)| \le c\lambda(|\nabla^2_{x'} v_t| + |\nabla^2_{x'} v||\nabla_{x'} v_t|)(1 + |\nabla_{x'} v|).$$

Using, in addition, embeddings (2.1) and (2.5), we derive the estimates

$$\begin{aligned}\int_0^T \|\nabla_{x'} h(v)\|^2_{L^2(\sigma)}\mathrm{d}t &\le c\lambda^2 \int_0^T\int_\sigma (|\nabla^3_{x'} v|^2 + |\nabla^3_{x'} v|^2|\nabla_{x'} v|^2 + |\nabla^2_{x'} v|^4 + |\nabla^2_{x'} v|^4|\nabla_{x'} v|^2)\mathrm{d}x'\mathrm{d}t \\ &\le c\lambda^2 \int_0^T (\|v\|^2_{W^{3,2}(\sigma)} + \|v\|^4_{W^{3,2}(\sigma)} + \|v\|^6_{W^{3,2}(\sigma)})\mathrm{d}t \\ &\le c\lambda^2(\|v\|^2_{W^{(4,2),2}(\sigma^T)} + \|v\|^4_{W^{(4,2),2}(\sigma^T)} + \|v\|^6_{W^{(4,2),2}(\sigma^T)}),\end{aligned} \tag{3.10}$$

$$\begin{aligned}\int_0^T \|\partial_t h(v)\|^2_{L^2(\sigma)}\mathrm{d}t &\le c\lambda^2 \int_0^T\int_\sigma (|\nabla^2_{x'} v_t|^2 + |\nabla^2_{x'} v|^2|\nabla_{x'} v_t|^2 + |\nabla^2_{x'} v_t|^2|\nabla_{x'} v|^2 + |\nabla_{x'} v_t|^2|\nabla^2_{x'} v|^2|\nabla_{x'} v|^2)\mathrm{d}x'\mathrm{d}t \\ &\le c\lambda^2\Big(\|v\|^2_{W^{(4,2)}(\sigma^T)} + \|\nabla^2_{x'} v\|^2_{L^4(\sigma^T)}\|\nabla_{x'} v_t\|^2_{L^4(\sigma^T)} + \sup_{t\in(0,T)}\|v\|^2_{W^{3,2}(\sigma)}\int_0^T \|\nabla^2_{x'} v_t\|^2_{L^2(\sigma)}\mathrm{d}t \\ &\quad + \sup_{t\in(0,T)}\|v\|^2_{W^{3,2}(\sigma)}\int_0^T \|\nabla_{x'} v_t\|^2_{L^4(\sigma)}\|\nabla^2_{x'} v\|^2_{L^4(\sigma)}\mathrm{d}t\Big) \\ &\le c\lambda^2(\|v\|^2_{W^{(4,2),2}(\sigma^T)} + \|v\|^4_{W^{(4,2),2}(\sigma^T)} + \|v\|^6_{W^{(4,2),2}(\sigma^T)}).\end{aligned} \tag{3.11}$$

Further,

$$|\nabla^2_{x'} h(v)| \le c\lambda(1 + |\nabla_{x'} v|)(|\nabla^2 v|^3 + |\nabla^2_{x'} v||\nabla^3_{x'} v| + |\nabla^4_{x'} v|)$$

and

$$\begin{aligned}\int_0^T \|\nabla^2_{x'} h(v)\|^2_{L^2(\sigma)}\mathrm{d}t &\le c\lambda^2 \int_0^T\int_\sigma (|\nabla^4_{x'} v|^2 + |\nabla^4_{x'} v|^2|\nabla_{x'} v|^2 + |\nabla^2_{x'} v|^6 + |\nabla^2_{x'} v|^6|\nabla_{x'} v|^2 \\ &\quad + |\nabla^2_{x'} v|^2|\nabla^3_{x'} v|^2 + |\nabla^2_{x'} v|^2|\nabla^3_{x'} v|^2|\nabla_{x'} v|^2)\mathrm{d}x'\mathrm{d}t \\ &\le c\lambda^2(\|v\|^2_{W^{(4,2),2}(\sigma^T)} + \|v\|^4_{W^{(4,2),2}(\sigma^T)} + \|v\|^6_{W^{(4,2),2}(\sigma^T)} + \|v\|^8_{W^{(4,2),2}(\sigma^T)}).\end{aligned} \tag{3.12}$$

Define in $W^{(4,2),2}_{\partial\sigma}(\sigma^T)$ a closed bounded set $\mathcal{B}_{R_0} = \{u \in W^{(4,2),2}_{\partial\sigma}(\sigma^T) : \|u\|_{W^{(4,2),2}(\sigma^T)} \le R_0 \text{ and } u(x', 0) = 0\}$. Assume that $v \in \mathcal{B}_{R_0}$. Then (3.9)–(3.12) yield the estimate

$$\|h + \alpha\|^2_{W^{(2,1),2}(\sigma^T)} \le c\lambda^2(R_0^2 + R_0^4 + R_0^6 + R_0^8) + c\|\alpha\|^2_{W^{1,2}(0,T)}. \tag{3.13}$$

By Theorem 3.1, the solution of the heat equation (3.7) admits the estimate

$$\|V\|^2_{W^{(4,2),2}(\sigma^T)} \le c_1\lambda^2(R_0^2 + R_0^4 + R_0^6 + R_0^8) + c_2\|\alpha\|^2_{W^{1,2}(0,T)}. \tag{3.14}$$

Set $M_0^2 = c_2|\alpha_0|^2$ and $R_0^2 = 2M_0^2$ and suppose that

$$\lambda^2 \le \frac{1}{2c_1(1 + 2M_0^2 + 4M_0^4 + 8M_0^6)} = \lambda_*^2. \tag{3.15}$$

Then (3.14) yields

$$\|\mathcal{L}v\|^2_{W^{(4,2),2}(\sigma^T)} \le R_0^2.$$

The last inequality implies that the operator $\mathcal{L}$ maps the closed bounded set $\mathcal{B}_{R_0} \subset W^{(4,2),2}_{\partial\sigma}(\sigma^T)$ onto itself.

Let us show that $\mathcal{L}$ is a contraction in $W^{(1,0),2}_{\partial\sigma}(\sigma^T)$. Multiplying equations (3.7) by an arbitrary $\eta \in W^{(1,1),2}_{\partial\sigma}(\sigma^T)$ and integrating by parts, we obtain

$$\int_\sigma (\mathcal{L}v)_t \eta \mathrm{d}x' + \frac{\nu_0}{2}\int_\sigma \nabla_{x'}(\mathcal{L}v)\cdot\nabla_{x'}\eta \mathrm{d}x' = -\frac{1}{2}\lambda\int_\sigma \nu(\dot{\gamma}_P(v))\nabla_{x'}v\cdot\nabla_{x'}\eta\mathrm{d}x' + \alpha\int_\sigma \eta\mathrm{d}x'. \tag{3.16}$$

Then for any $v_1, v_2 \in \mathcal{B}_{R_0}$, the following equality holds

$$\begin{aligned}\int_\sigma ((\mathcal{L}v_1)_t - (\mathcal{L}v_2)_t)\eta\mathrm{d}x' + \frac{\nu_0}{2}\int_\sigma \nabla_{x'}(\mathcal{L}v_1 - \mathcal{L}v_2)\cdot\nabla_{x'}\eta\mathrm{d}x' \\ = -\frac{1}{2}\lambda\int_\sigma \nu(\dot{\gamma}_P(v_1))(\nabla_{x'}v_1 - \nabla_{x'}v_2)\cdot\nabla_{x'}\eta\mathrm{d}x - \frac{1}{2}\lambda\int_\sigma (\nu(\dot{\gamma}_P(v_1)) - \nu(\dot{\gamma}_P(v_2)))\nabla_{x'}v_2\cdot\nabla_{x'}\eta\mathrm{d}x' = J_1 + J_2.\end{aligned} \tag{3.17}$$

Using (1.1) and the inequality,

$$|\nu(\dot{\gamma}_P(v_1)) - \nu(\dot{\gamma}_P(v_2))|^2 \le \sup_y|\nabla_y\nu(y)|^2|\nabla_{x'}v_1 - \nabla_{x'}v_2|^2 \le c_2|\nabla_{x'}v_1 - \nabla_{x'}v|^2,$$

we have

$$\begin{aligned}|J_2| &\le \frac{\nu_0}{8}\int_\sigma|\nabla_{x'}\eta|^2\mathrm{d}x' + \frac{c_2\lambda^2}{\nu_0}\int_\sigma|\nabla_{x'}v_1 - \nabla_{x'}v_2|^2|\nabla_{x'}v_2|^2\mathrm{d}x' \\ &\le \frac{\nu_0}{8}\int_\sigma|\nabla_{x'}\eta|^2\mathrm{d}x' + \frac{c_2\lambda^2}{\nu_0}\sup_{x'\in\sigma}|\nabla_{x'}v_2|^2\int_\sigma|\nabla_{x'}v_1 - \nabla_{x'}v_2|^2\mathrm{d}x' \\ &\le \frac{\nu_0}{8}\int_\sigma|\nabla_{x'}\eta|^2\mathrm{d}x' + \frac{c_3\lambda^2}{\nu_0}\|v_2\|^2_{W^{3,2}(\sigma)}\int_\sigma|\nabla_{x'}v_1 - \nabla_{x'}v_2|^2\mathrm{d}x'.\end{aligned}$$

By Young's inequality,

$$|J_1| \le \frac{\nu_0}{8}\int_\sigma|\nabla_{x'}\eta|^2\mathrm{d}x + C_3\frac{\lambda^2}{\nu_0}\int_\sigma|\nabla_{x'}v_1 - \nabla_{x'}v_2|^2\mathrm{d}x'.$$

Therefore, taking in (3.16) $\eta = \mathcal{L}v_1 - \mathcal{L}v_2$, we derive the inequality

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{L}v_1 - \mathcal{L}v_2\|^2_{L^2(\sigma)} + \frac{\nu_0}{2}\|\nabla_{x'}(\mathcal{L}v_1 - \mathcal{L}v_2)\|^2_{L^2(\sigma)} \\ &\le \frac{\nu_0}{4}\|\nabla_{x'}(\mathcal{L}v_1 - \mathcal{L}v_2)\|^2_{L^2(\sigma)} + \frac{\lambda^2}{\nu_0}[c_3\|v_2\|^2_{W^{3,2}(\sigma)} + 1]\|\nabla_{x'}(v_1 - v_2)\|^2_{L^2(\sigma)}.\end{aligned}$$

Integrating by t over the interval $(0, T)$, we obtain

$$\begin{aligned}&\frac{1}{2}\|\mathcal{L}v_1(T) - \mathcal{L}v_2(T)\|^2_{L^2(\sigma)} + \frac{\nu_0}{4}\int_0^T\|\nabla_{x'}(\mathcal{L}v_1(t) - \mathcal{L}v_2(t))\|^2_{L^2(\sigma)}\mathrm{d}t \\ &\le \frac{\lambda^2}{\nu_0}\int_0^T\|\nabla_{x'}(v_1(t) - v_2(t))\|^2_{L^2(\sigma)}\mathrm{d}t + \frac{\lambda^2c_3}{\nu_0}\sup_{t\in(0,T)}\|v_2(t)\|^2_{W^{3,2}(\sigma)}\int_0^T\|\nabla_{x'}(v_1(t) - v_2(t))\|^2_{L^2(\sigma)}\mathrm{d}t. \\ &\le \frac{\lambda^2}{\nu_0}(C_3 + c_4\|v_2\|^2_{W^{(4,2),2}(\sigma^T)})\int_0^T\|\nabla_{x'}(v_1(t) - v_2(t))\|^2_{L^2(\sigma)}\mathrm{d}t \\ &\le \frac{\lambda^2}{\nu_0}(C_3 + c_4R_0^2)\int_0^T\|\nabla_{x'}(v_1(t) - v_2(t))\|^2_{L^2(\sigma)}\mathrm{d}t.\end{aligned}$$

Then from the last inequality, it follows that

$$\int_0^T\|\nabla_{x'}(\mathcal{L}v_1(t) - \mathcal{L}v_2(t))\|^2_{L^2(\sigma)}\mathrm{d}t \le q\int_0^T\|\nabla_{x'}(v_1(t) - v_2(t))\|^2_{L^2(\sigma)}\mathrm{d}t,$$

where $q = \lambda^2 \frac{4(C_3 + c_4 R_0^2)}{\nu_0^2}$.

Let

$$\lambda_1^2 = \min\left\{\lambda_*^2, \frac{\nu_0}{4(C_3 + c_4 R_0^2)}\right\}. \tag{3.18}$$

Then for any $\lambda \in (0, \lambda_1)$, the operator $\mathcal{L}$ is a contraction in $W^{(1,0),2}_{\partial\sigma}(\sigma^T)$ with the contraction factor $q < 1$ and, by Theorem 2.1, there exists a unique fixed point v_{P_α} of the operator $\mathcal{L}$, which is a solution of problem (1.4). Estimate (1.7) for v_{P_α} follows from the fact that $v_{P_\alpha} \in \mathcal{B}_{R_0}$ with $R_0^2 = 2c_2|\alpha_0|^2$. Indeed, from estimates (3.9)–(3.12) applied to the fixed point v_α, it follows that

$$\|h + \alpha\|^2_{W^{(2,1),2}(\sigma^T)} \le c_1\lambda^2(1 + R_0^2 + R_0^4 + R_0^6)\|v_{P_\alpha}\|^2_{W^{(4,2),2}(\sigma^T)} + c_2\|\alpha\|^2_{W^{1,2}(0,T)}.$$

Thus, by (3.14),

$$\|v_{P_\alpha}\|^2_{W^{(4,2),2}(\sigma^T)} \le c_1\lambda^2(1 + R_0^2 + R_0^4 + R_0^6)\|v_{P_\alpha}\|^2_{W^{(4,2),2}(\sigma^T)} + c_2\|\alpha\|^2_{W^{1,2}(0,T)}.$$

If $\lambda < \lambda_1$, the last estimate implies (1.7). □

Consider now the inverse problem (1.5). Let us introduce the space $\mathfrak{L}^2(0, T)$ as a space of functions having primitives equal to zero at $t = 0$ and possessing the finite norm

$$\|g\|_{\mathfrak{L}^2(0,T)} = \left(\int_0^T |(Sg)(t)|^2 \mathrm{d}t\right)^{1/2},$$

where $(Sg)(t) = \int_0^t g(\tau)\mathrm{d}\tau$ is the primitive of g vanishing at $t = 0$. Obviously, $\mathfrak{L}^2(0, T)$ is a Hilbert space and $L^2(0, T) \subset \mathfrak{L}^2(0, T)$ for $T < +\infty$.

Let us prove Theorem 1.2.

Proof. Let $\mathcal{I}$ be an operator $W^{(4,2),2}_{\partial\sigma}(\sigma^T) \to W^{(4,2),2}_{\partial\sigma}(\sigma^T) \times W^{1,2}(0, T)$ such that for arbitrary given $w \in W^{(4,2),2}_{\partial\sigma}(\sigma^T)$ with $w(x', 0) = 0$, the pair $(W, \alpha) = \mathcal{I}w$ is a solution of the inverse problem for the heat equation:

$$\begin{cases} W_t - \frac{\nu_0}{2}\Delta W = \alpha(t) + h(w), \quad x' \in \sigma, \\ W|_{\partial\sigma} = 0, \quad W(x', 0) = 0, \\ \int_\sigma W(x', t)\mathrm{d}x' = F(t), \end{cases} \tag{3.19}$$

where $h(w)$ is defined by formula (3.8). Since $F(0) = F'(0) = 0$, we have $h(v(x', 0)) = 0$, and so, the compatibility conditions of Theorem 3.2 are valid and problem (3.19) admits a solution satisfying the properties stated in this theorem.

Let $\mathcal{M}_{R_0} = \{u \in W^{(4,2),2}_{\partial\sigma}(\sigma^T) : \|u\|_{W^{(4,2),2}_{\partial\sigma}(\sigma^T)} \le R_0$ and $u(x', 0) = 0\}$ and $\mathcal{N}_{R_0} = \{s \in W^{1,2}(0, T) : \|s\|_{W^{1,2}(0,T)} \le R_0$ and $s(0) = 0\}$. Assume that $w \in \mathcal{M}_{R_0}$. Then as in Theorem 1.1, we have the estimate

$$\|h\|^2_{W^{(2,1),2}(\sigma^T)} \le c\lambda^2(R_0^2 + R_0^4 + R_0^6 + R_0^8). \tag{3.20}$$

So, by Theorem 1.2,

$$\|W\|^2_{W^{(4,2),2}(\sigma^T)} + \|\alpha\|_{W^{1,2}(0,T)} \le c_5\lambda^2(R_0^2 + R_0^4 + R_0^6 + R_0^8) + c_6\|F\|^2_{W^{2,2}(0,T)}. \tag{3.21}$$

Set $M_0^2 = c_6|F_0|^2$ and $R_0^2 = 2M_0^2$ and suppose that

$$\lambda^2 \le \frac{1}{2c_5(1 + 2M_0^2 + 4M_0^4 + 8M_0^6)} = \lambda_*^2. \tag{3.22}$$

Then (3.21) yields

$$\|W\|^2_{W^{(4,2),2}(\sigma^T)} + \|\alpha\|^2_{W^{1,2}(0,T)} \le R_0^2.$$

The last inequality implies that the operator $\mathcal{I}$ maps the closed bounded set $\mathcal{M}_{R_0} \subset W^{(4,2),2}_{\partial\sigma}(\sigma^T)$ onto closed set $\mathcal{M}_{R_0} \times \mathcal{N}_{R_0} \subset W^{(4,2),2}_{\partial\sigma}(\sigma^T) \times W^{1,2}(0,T)$.

Let us show that $\mathcal{I}$ is a contraction in $W^{(1,0),2}_{\partial\sigma}(\sigma^T) \times \mathfrak{L}^2(0,T)$. Multiplying equations (3.19) by an arbitrary $\eta \in W^{(1,1),2}_{\partial\sigma}(\sigma^T)$ and integrating by parts, we obtain

$$\int_\sigma W_t\eta\mathrm{d}x' + \frac{\nu_0}{2}\int_\sigma \nabla_{x'}W\cdot\nabla_{x'}\eta\mathrm{d}x' = \alpha(t)\int_\sigma \eta\mathrm{d}x' - \frac{1}{2}\lambda\int_\sigma \nu(\dot\gamma_P(w))\nabla_{x'}w\cdot\nabla_{x'}\eta\mathrm{d}x'. \tag{3.23}$$

Then for any $w_1, w_2 \in \mathcal{M}_{R_0}$, the following equality holds

$$\begin{aligned}
&\int_\sigma (W_{1t} - W_{2t})\eta\mathrm{d}x' + \frac{\nu_0}{2}\int_\sigma \nabla_{x'}(W_1 - W_2)\cdot\nabla_{x'}\eta\mathrm{d}x' \\
&\quad = (\alpha_1(t) - \alpha_2(t))\int_\sigma \eta\mathrm{d}x' - \frac{1}{2}\lambda\int_\sigma \nu(\dot\gamma_P(w_1))(\nabla_{x'}w_1 - \nabla_{x'}w_2)\cdot\nabla_{x'}\eta\mathrm{d}x \\
&\qquad - \frac{1}{2}\lambda\int_\sigma (\nu(\dot\gamma_P(w_1)) - \nu(\dot\gamma_P(w_2)))\nabla_{x'}w_2\cdot\nabla_{x'}\eta\mathrm{d}x' \\
&\quad = (\alpha_1(t) - \alpha_2(t))\int_\sigma \eta\mathrm{d}x' + I_1 + I_2,
\end{aligned} \tag{3.24}$$

where $(W_i, \alpha_i) \in \mathcal{M}_{R_0} \times \mathcal{N}_{R_0}$. Taking in (3.24) $\eta = W_1 - W_2$ and using that $\int_\sigma (W_1(x',t) - W_1(x',t))\mathrm{d}x' = F(t) - F(t) = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\sigma |W_1 - W_2|^2\mathrm{d}x' + \frac{\nu_0}{2}\int_\sigma |\nabla_{x'}(W_1 - W_2)|^2\mathrm{d}x' \\
&\quad = -\frac{1}{2}\lambda\int_\sigma \nu(\dot\gamma_P(w_1))(\nabla_{x'}w_1 - \nabla_{x'}w_2)\cdot\nabla_{x'}(W_1 - W_2)\mathrm{d}x \\
&\qquad - \frac{1}{2}\lambda\int_\sigma (\nu(\dot\gamma_P(w_1)) - \nu(\dot\gamma_P(w_2)))\nabla_{x'}w_2\cdot\nabla_{x'}(W_1 - W_2)\mathrm{d}x' \\
&\quad = I_1 + I_2.
\end{aligned} \tag{3.25}$$

Estimating the terms I_1 and I_2 in the right-hand side of (3.25) as in Theorem 1.1 and integrating by t, we arrive to the inequality

$$\begin{aligned}
&\frac{1}{2}\|W_1(\cdot,T) - W_2(\cdot,T)\|^2_{L^2(\sigma)} + \frac{\nu_0}{4}\int_0^T \|\nabla_{x'}(W_1(\cdot,t) - W_2(\cdot,t))\|^2_{L^2(\sigma)}\mathrm{d}t \\
&\le c_7\frac{\lambda^2}{\nu_0}(1 + R_0^2)\int_0^T \|\nabla_{x'}(w_1(\cdot,t) - w_2(\cdot,t))\|^2_{L^2(\sigma)}\mathrm{d}t.
\end{aligned} \tag{3.26}$$

Let $v_0 \in \overset{\circ}{W}{}^{1,2}(\sigma) \cap W^{2,2}(\sigma)$ be the solution of the Poisson problem

$$\begin{cases} -\dfrac{\nu_0}{2}\Delta_{x'}v_0 = 1, & x' \in \sigma, \\ v_0 = 0, & x' \in \partial\sigma. \end{cases} \tag{3.27}$$

Multiplying equations (3.19) by v_0 and integrating by parts yields

$$\begin{aligned}\int\limits_{\sigma}(W_1-W_2)_t v_0 \mathrm{d}x' - \frac{\nu_0}{2}\int\limits_{\sigma}\Delta_{x'}(W_1-W_2)\cdot v_0 \mathrm{d}x' \\ = (\alpha_1(t)-\alpha_2(t))\int\limits_{\sigma} v_0 \mathrm{d}x' - \frac{1}{2}\lambda\int\limits_{\sigma}\nu(\dot{\gamma}_P(w_1))(\nabla_{x'}w_1-\nabla_{x'}w_2)\cdot\nabla_{x'}v_0 \mathrm{d}x \\ - \frac{1}{2}\lambda\int\limits_{\sigma}(\nu(\dot{\gamma}_P(w_1)-\nu(\dot{\gamma}_P(w_2))\nabla_{x'}w_2\cdot\nabla_{x'}v_0 \mathrm{d}x.\end{aligned} \tag{3.28}$$

Note that

$$\int\limits_{\sigma} v_0(x')\mathrm{d}x' = \frac{\nu_0}{2}\int\limits_{\sigma}|\nabla_{x'}v_0(x')|^2\mathrm{d}x = \kappa_0 > 0,$$

$$-\frac{\nu_0}{2}\int\limits_{\sigma}\Delta_{x'}(W_1-W_2)\cdot v_0\mathrm{d}x' = -\frac{\nu_0}{2}\int\limits_{\sigma}(W_1-W_2)\cdot\Delta_{x'}v_0 = \int\limits_{\sigma}(W_1-W_2)\mathrm{d}x = 0,$$

$$\int\limits_0^t\int\limits_{\sigma}(W_{1t}(x',\tau)-W_{2t}(x',\tau))v_0(x')\mathrm{d}x'\mathrm{d}\tau = \int\limits_{\sigma}(W_1(x',t)-W_2(x',t))v_0(x')\mathrm{d}x'.$$

Therefore,

$$\begin{aligned}\kappa_0\int\limits_0^t(\alpha_1(\tau)-\alpha_2(\tau))\mathrm{d}\tau = \kappa_0((S\alpha_1)(t)-(S\alpha_2)(t)) \\ = \int\limits_0^t\int\limits_{\sigma}(W_1-W_2)_t v_0\mathrm{d}x'\mathrm{d}\tau + \frac{\lambda}{2}\int\limits_0^t\int\limits_{\sigma}\nu(\dot{\gamma}_P(w_1))(\nabla_{x'}w_1-\nabla_{x'}w_2)\cdot\nabla_{x'}v_0\mathrm{d}x'\mathrm{d}\tau \\ + \frac{\lambda}{2}\int\limits_0^t\int\limits_{\sigma}(\nu(\dot{\gamma}_P(w_1))-\nu(\dot{\gamma}_P(w_2)))\nabla_{x'}w_2\cdot\nabla_{x'}v_0\mathrm{d}x'\mathrm{d}\tau.\end{aligned}$$

From this equality, it follows that

$$\begin{aligned}\kappa_0^2\int\limits_0^T|(S\alpha_1)(t)-(S\alpha_2)(t)|^2\mathrm{d}t \le c\Bigg(\int\limits_0^T\Bigg|\int\limits_{\sigma}(W_1(x',t)-W_2(x',t))v_0(x')\mathrm{d}x'\Bigg|^2\mathrm{d}t \\ + \frac{\lambda^2}{4}\int\limits_0^T\Bigg|\int\limits_0^t\int\limits_{\sigma}\nu(\dot{\gamma}_P(w_1))(\nabla_{x'}w_1-\nabla_{x'}w_2)\cdot\nabla_{x'}v_0\mathrm{d}x'\mathrm{d}\tau\Bigg|^2\mathrm{d}t \\ + \frac{\lambda^2}{4}\int\limits_0^T\Bigg|\int\limits_0^t\int\limits_{\sigma}(\nu(\dot{\gamma}_P(w_1))-\nu(\dot{\gamma}_P(w_2)))\nabla_{x'}w_2\cdot\nabla_{x'}v_0\mathrm{d}x\mathrm{d}\tau\Bigg|^2\mathrm{d}t\Bigg) \\ \le c\Bigg(\int\limits_0^T\|W_1-W_2\|^2_{L^2(\sigma)}\mathrm{d}t + \lambda^2\int\limits_0^T\|\nu(\dot{\gamma}_P(w_1))(\nabla_{x'}w_1-\nabla_{x'}w_2)\|^2_{L^2(\sigma)}\mathrm{d}t\Bigg) \\ + \lambda^2\int\limits_0^T\|(\nu(\dot{\gamma}_P(w_1))-\nu(\dot{\gamma}_P(w_2)))\nabla_{x'}w_2\|^2_{L^2(\sigma)}\mathrm{d}t \\ \le c\int\limits_0^T\|\nabla_{x'}(W_1-W_2)\|^2_{L^2(\sigma)}\mathrm{d}t + c\lambda^2(1+R_0^2)\int\limits_0^T\|\nabla_{x'}(w_1(t)-w_2(t))\|^2_{L^2(\sigma)}\mathrm{d}t.\end{aligned} \tag{3.29}$$

Inequalities (3.26) and (3.29) yield

$$\frac{\nu_0}{4}\int\limits_0^T \|\nabla_{x'}(W_1(t)-W_2(t))\|^2_{L^2(\sigma)}dt+\kappa_0^2\int\limits_0^T |(S\alpha_1)(t)-(S\alpha_2)(t)|^2dt\le c_8\lambda^2(1+R_0^2)\int\limits_0^T \|\nabla_{x'}(w_1(t)-w_2(t))\|^2_{L^2(\sigma)}dt. \quad (3.30)$$

Let

$$\lambda_2^2=\min\left\{\lambda_*^2,\ \frac{\min\{\nu_0/4,\kappa_0^2\}}{c_8(1+R_0^2)}\right\}. \quad (3.31)$$

Then for any $\lambda\in(0,\lambda_2)$, the operator $\mathcal{I}$ is a contraction (satisfies the condition of Theorem 2.1) in $X\times Z=W^{(1,0),2}_{\partial\sigma}(\sigma^T)\times\mathfrak{L}^2(0,T)$ with the contraction factor $q=\frac{\min\{\nu_0/2,\kappa_0^2\}}{c_8(1+R_0^2)}<1$. Thus, by Theorem 2.1, there exists a unique fixed point (v_{P_α},α) of the operator equation $(v_{\alpha P},\alpha)=\mathcal{I}v_{\alpha P}$, which is the solution of problem (1.4).

Estimate (1.8) for (v_{P_α},α) is proved exactly in the same way as inequality (1.7) in Theorem 1.1. □

Remark 3.1. Since the constants in the embedding inequalities of Lemma 2.2 are independent of T, it is easy to see that also the constants in estimates (1.7) and (1.8) do not depend on T.

3.1 Operator relating the pressure slope and the flux

Let $\alpha\in W^{1,2}(0,T)$, $\alpha(0)=0$, be given and let $v_{P_\alpha}\in W^{(4,2),2}_{\partial\sigma}(\sigma^T)$ be the solution of problem (1.4). Define $F[\alpha](t)=\int_\sigma v_{P_\alpha}(x',t)dx'$ the flux (flow rate) corresponding to the pressure slope $-\alpha(t)$. Note that in the case of the Newtonian flow (problem (3.1) with $f(x',t)=\alpha(t)$, $w_0=0$), $F_0[\alpha](t)$ is related to $\alpha(t)$ via the solution of the heat equation (3.1). This case corresponds to the value of $\lambda=0$, i.e., by Theorem 3.1, there exists a bounded operator $\mathcal{H}: W^{1,2}(0,T)\longmapsto W^{2,2}(0,T)$ such that $F_0[\alpha]=\mathcal{H}\alpha=\int_\sigma w(x',\cdot)dx'$.

We consider as well the operator $G: W^{1,2}(0,T)\longmapsto W^{2,2}(0,T)$ corrector of the non-Newtonian flux with respect to the Newtonian one:

$$G[\alpha]=F[\alpha]-F_0[\alpha],$$

where $F[\alpha]$ is defined over the solution v_{P_α} of nonlinear problem (1.4). Below we prove that for sufficiently small $\lambda>0$, that operator $\mathcal{G}$ is a contraction.

Lemma 3.1. *For any $\alpha_0>0$, there exists a number $\mu_1=\mu_1(\alpha_0)$ such that for any $\lambda\in(0,\mu_1]$ and every $\alpha\in W^{1,2}(0,T)$ with $\|\alpha\|_{W^{1,2}(0,T)}\le\alpha_0$ and $\alpha(0)=0$, the solution v_{P_α} of problem* (1.4) *is a Lipschitz continuous function in the norm $\|\nabla_{x'}\cdot\|_{W^{(1,0),2}(\sigma^T)}$ with respect to α in the $W^{1,2}(0,T)$-norm. Moreover, $F[\alpha]$ is a Lipschitz continuous function in $L^2(0,T)$-norm with respect to α.*

Proof. Let $v_{P_{\alpha_1}}\in W^{(4,2),2}_{\partial\sigma}(\sigma^T)$ and $v_{P_{\alpha_2}}\in W^{(4,2),2}_{\partial\sigma}(\sigma^T)$ be two solutions of problem (1.4) corresponding to $\alpha=\alpha_1\in W^{1,2}(0,T)$ and $\alpha=\alpha_2\in W^{1,2}(0,T)$, respectively. By Theorem 1.1, these solutions exist if $\lambda\in(0,\lambda_1(\alpha_0))$. Moreover, the following estimates hold

$$\|v_{P_{\alpha_i}}\|_{W^{(4,2),2}(\sigma^T)}\le c\|\alpha\|_{W^{1,2}(0,T)},\quad i=1,2.$$

These solutions satisfy the integral identities

$$\begin{aligned}&\int\limits_0^T\int\limits_\sigma \left(v_{P_{\alpha_i}}\right)_t\eta dxdt+\frac{\nu_0}{2}\int\limits_0^T\int\limits_\sigma \nabla_{x'}v_{P_{\alpha_i}}\cdot\nabla_{x'}\eta dx'dt\\ &=-\frac{1}{2}\lambda\int\limits_0^T\int\limits_\sigma \nu\left(\dot{\gamma}_P\left(v_{P_{\alpha_i}}\right)\right)\nabla_{x'}v_{P_{\alpha_i}}\cdot\nabla_{x'}\eta dx'dt+\int\limits_0^T\alpha_i(t)\int\limits_\sigma\eta\ dx'dt,\quad i=1,2,\quad\forall\eta\in W^{(1,1),2}_{\partial\sigma}(\sigma^T).\end{aligned} \quad (3.32)$$

Similar arguments as those at the end of the proof of Theorem 1.1 give the estimate

$$
\begin{aligned}
&\frac{1}{2}\|v_{P_{\alpha_1}}(T) - v_{P_{\alpha_2}}(T)\|^2_{L^2(\sigma)} + \frac{\nu_0}{2}\int\limits_0^T \|\nabla_{x'}v_{P_{\alpha_1}}(t) - \nabla_{x'}v_{P_{\alpha_2}}(t)\|^2_{L^2(\sigma)}\mathrm{d}t \\
&\le c\lambda \int\limits_0^T\int\limits_\sigma |\nu(\dot{\gamma}_P(v_{P_{\alpha_1}})) - \nu(\dot{\gamma}_P(v_{P_{\alpha_2}}))|\, |\nabla_{x'}v_{P_{\alpha_1}}|\, |\nabla_{x'}v_{P_{\alpha_1}} - \nabla_{x'}v_{P_{\alpha_2}}|\mathrm{d}x'\mathrm{d}t + c\lambda\int\limits_0^T\int\limits_\sigma |\nu(\dot{\gamma}_P(v_{P_{\alpha_2}}))|\, |\nabla_{x'}v_{P_{\alpha_1}} \\
&\quad - \nabla_{x'}v_{P_{\alpha_2}}|^2\mathrm{d}x'\mathrm{d}t + \int\limits_0^T |\alpha_1 - \alpha_2| \int\limits_\sigma |v_{P_{\alpha_1}} - v_{P_{\alpha_2}}|\mathrm{d}x'\mathrm{d}t \\
&\le c\lambda(1 + \|v_{P_{\alpha_1}}\|_{W^{(4,2),2}(\sigma^T)})\int\limits_0^T \|\nabla_{x'}v_{P_{\alpha_1}} - \nabla_{x'}v_{P_{\alpha_2}}\|^2_{L^2(\sigma)}\mathrm{d}t \\
&\quad + |\sqrt{|\sigma|}\,\|\alpha_1 - \alpha_2\|_{L^2(0,T)}\Bigg(\int\limits_0^T \|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|^2_{L^2(\sigma)}\mathrm{d}t\Bigg)^{1/2} \\
&\le c_9\lambda(1 + \alpha_0)\int\limits_0^T \|\nabla_{x'}v_{P_{\alpha_1}} - \nabla_{x'}v_{P_{\alpha_2}}\|^2_{L^2(\sigma)}\mathrm{d}t + |\sqrt{|\sigma|}\,\|\alpha_1 - \alpha_2\|_{L^2(0,T)}\Bigg(\int\limits_0^T \|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|^2_{L^2(\sigma)}\mathrm{d}t\Bigg)^{1/2},
\end{aligned}
$$

where $|\sigma| = \operatorname{mes}(\sigma)$. If $\lambda < \min\Big\{\lambda_1(\alpha_0), \frac{\nu_0}{4c_9(1+\alpha_0)}\Big\} \equiv \mu_1(\alpha_0)$, then from the last inequality, it follows that

$$
\int\limits_0^T \|\nabla_{x'}(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\|^2_{L^2(\sigma)}\mathrm{d}t \le c\|\alpha_1 - \alpha_2\|^2_{W^{1,2}(0,T)}. \tag{3.33}
$$

Further,

$$
\begin{aligned}
|F[\alpha_1](t) - F[\alpha_2](t)|^2 &\le \Bigg|\int\limits_\sigma \big(v_{P_{\alpha_1}}(x',t) - v_{P_{\alpha_2}}(x',t)\big)\mathrm{d}x'\Bigg|^2 \\
&\le |\sigma|\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|^2_{L^2(\sigma)} \\
&\le c\|\nabla_{x'}(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\|^2_{L^2(\sigma)} \\
&\le c|\alpha_1(t) - \alpha_2(t)|^2.
\end{aligned}
$$

Therefore,

$$
\int\limits_0^T |F[\alpha_1](t) - F[\alpha_2](t)|^2\mathrm{d}t \le c\int\limits_0^T \|\nabla_{x'}(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\|^2_{L^2(\sigma)}\mathrm{d}t \le c\|\alpha_1 - \alpha_2\|^2_{W^{1,2}(0,T)},
$$

and this estimate completes the proof. □

Lemma 3.2. *For any $\alpha_0 > 0$, there exists a number $\mu_2 = \mu_2(\alpha_0) \le \mu_1(\alpha_0)$ such that for all $\lambda \in (0, \mu_2]$ the operator $G(\alpha)$ is a contraction from $W^{1,2}(0,T)$ to $L^2(0,T)$ within the ball $\|\alpha\|_{W^{1,2}(0,T)} \le \alpha_0$, $\alpha(0) = 0$, i.e., the following inequality holds*:

$$
\|G(\alpha_2) - G(\alpha_1)\|_{L^2(0,T)} \le q\|\alpha_1 - \alpha_2\|_{W^{1,2}(0,T)} \tag{3.34}
$$

with the contraction factor $q < 1$.

Proof. Let $\alpha_1, \alpha_2 \in W^{1,2}(0,T)$, $\alpha_i(0) = 0$, $\|\alpha_i\|_{W^{1,2}(0,T)} \le \alpha_0$ and let $\tilde{v}_{\alpha_1}, \tilde{v}_{\alpha_2} \in W^{(4,2),2}_{\partial\sigma}(\sigma^T)$ be the solutions of problem (3.1) corresponding to α_1 and α_2, respectively (i.e., to $f = \alpha_i(t)$, $i = 1, 2$, $w_0 = 0$). Consider the differences $\tilde{v}_{\alpha_1} - v_{P_{\alpha_1}}$ and $\tilde{v}_{\alpha_2} - v_{P_{\alpha_2}}$, where $v_{P_{\alpha_i}}$, $i = 1, 2$, are the solutions of the nonlinear problems (1.4) with the same right-hand sides $\alpha_i(t)$. These differences satisfy the equations

$$\begin{cases}(\tilde{v}_{\alpha_i} - v_{P_{\alpha_i}})_t - \frac{v_0}{2}\Delta_{x'}\big(\tilde{v}_{\alpha_i} - v_{P_{\alpha_i}}\big)\\ = \frac{\lambda}{2}\operatorname{div}_{x'}\Big(\nu\big(\dot{\gamma}_P\big(v_{P_{\alpha_i}}\big)\big)\nabla_{x'}v_{P_{\alpha_i}}\Big), \quad x' \in \sigma,\\ (\tilde{v}_{\alpha_i} - v_{P_{\alpha_i}})|_{\partial\sigma} = 0, \quad (\tilde{v}_{\alpha_i}(x', 0) - v_{P_{\alpha_i}}(x', 0)) = 0,\end{cases} \tag{3.35}$$

where $i = 1$ and $i = 2$. Subtracting one problem from another, we obtain for $K = (\tilde{v}_{\alpha_1} - v_{P_{\alpha_1}}) - (\tilde{v}_{\alpha_2} - v_{P_{\alpha_2}})$ the following relations:

$$\begin{cases}K_t - \frac{v_0}{2}\Delta_{x'}K\\ = \frac{\lambda}{2}\operatorname{div}_{x'}(\nu\big(\dot{\gamma}_P\big(v_{P\alpha_1}\big)\big)\nabla_{x'}v_{P\alpha_1} - \nu\big(\dot{\gamma}_P\big(v_{P\alpha_2}\big)\big)\nabla_{x'}v_{P\alpha_2}),\\ K\,|_{\partial\sigma} = 0, \quad K(x', 0) = 0.\end{cases} \tag{3.36}$$

Applying a standard for the solution of the heat equation *a priori* estimate, we obtain

$$\|K(T)\|^2_{L^2(\sigma)} + \int_0^T \|\nabla_{x'}K\|^2_{L^2(\sigma)}\mathrm{d}t \le c\lambda^2\int_0^T \|\nu\big(\dot{\gamma}_P\big(v_{P\alpha_1}\big)\big)\nabla_{x'}v_{P\alpha_1} - \nu\big(\dot{\gamma}_P\big(v_{P\alpha_2}\big)\big)\nabla_{x'}v_{P\alpha_2}\|^2_{L^2(\sigma)}\mathrm{d}t$$

and, by using the similar arguments as before, we obtain from inequalities (1.7) and (3.33) for $\lambda < \mu_1(\alpha_0)$ that

$$\int_0^T \|\nabla_{x'}K\|^2_{L^2(\sigma)} \le c\lambda^2(\alpha_0 + 1)\int_0^T \|\nabla_{x'}\big(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\big)\|_{L^2(\sigma)}\mathrm{d}t \le c\lambda^2\|\alpha_1 - \alpha_2\|^2_{W^{1,2}(0,T)}.$$

So, finally,

$$\int_0^T \left|\int_\sigma K\mathrm{d}x'\right|^2 \mathrm{d}t \le c\|K\|^2_{L^2(\sigma^T)} \le c\|\nabla_{x'}K\|^2_{L^2(\sigma^T)} \le c_{10}\lambda^2\|\alpha_1 - \alpha_2\|^2_{L^2(0,T)}.$$

Since

$$\int_\sigma K(x', t)\mathrm{d}x' = (\mathcal{F}_0[\alpha_1](t) - F[\alpha_1](t)) - (\mathcal{F}_0[\alpha_2](t) - F[\alpha_2](t)) = G[\alpha_2](t) - G[\alpha_1](t),$$

we have

$$\|G[\alpha_2] - G[\alpha_1]\|^2_{L^2(0,T)} \le c_{10}\lambda^2\|\alpha_1 - \alpha_2\|^2_{W^{1,2}(0,T)}.$$

If $\lambda^2 < \mu_2^2(\alpha_0) = \min\Big\{\mu_1^2(\alpha_0), \frac{1}{c_{10}}\Big\}$, then $G[\alpha]$ is a contraction from $W^{1,2}(0, T)$ to $L^2(0, T)$ with the contraction factor $q = \sqrt{c_{10}}\lambda < 1$. □

4 Continuity of the non-Newtonian Poiseuille flow

Suppose that α_1 and α_2 satisfy conditions of Theorem 1.1. Denote by $v_{P_{\alpha_1}}$ and $v_{P_{\alpha_2}}$ the two solutions of problem (1.4), corresponding to the right-hand sides α_1 and α_2.

Let us prove Theorem 1.3.

Proof. Let $\alpha_0 > 0$ and let $\lambda_1 = \lambda_1(\alpha_0)$ be the number defined in Theorem 1.1. Then, due to this theorem, for $\lambda \in (0, \lambda_1]$ and every $\alpha_1, \alpha_2 \in W^{1,2}(0, T)$, with $\|\alpha_i\|_{W^{1,2}(0,T)} \le \alpha_0$, $\alpha_i(0) = 0$, there exist solutions $v_{P_{\alpha_i}}$ of problems (1.4), $i = 1, 2$, such that $v_{P_{\alpha_i}} \in W^{(4,2),2}_{\partial\sigma}(\sigma^T)$, and the following estimates

$$\|v_{P_{\alpha_i}}\|_{W^{(4,2),2}(\sigma^T)} \le C\|\alpha_i\|_{W^{1,2}(0,T)}, \quad i = 1, 2, \tag{4.1}$$

hold. The difference $v = v_{P_{\alpha_1}} - v_{P_{\alpha_2}}$ satisfies the equations

$$\begin{cases} \left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)_t - \frac{v_0}{2}\Delta\left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right) \\ = h\left(v_{P_{\alpha_1}}\right) - h\left(v_{P_{\alpha_2}}\right) + (\alpha_1 - \alpha_2), \quad x \in \sigma, \\ \left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)|_{\partial\sigma} = 0, \quad \left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)(x', 0) = 0, \end{cases} \tag{4.2}$$

where

$$h(u) = \frac{1}{2}\lambda \operatorname{div}_{x'}\left(v(\dot\gamma_P(u))\nabla_{x'}u\right) = \frac{1}{2}\lambda[v(\dot\gamma_P(u))\Delta_{x'}u + (\nabla_y v(\dot\gamma_P(u))^T(\nabla_{x'}(\nabla_{x'}u))^T \cdot \nabla_{x'}u],$$

It is easy to calculate that

$$\begin{aligned} |h\left(v_{P_{\alpha_1}}\right) - h\left(v_{P_{\alpha_2}}\right)| \le c\lambda(|\nabla^2\left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)| + \left(|\nabla^2 v_{P_{\alpha_1}}| + |\nabla^2 v_{P_{\alpha_2}}|\right)|\nabla\left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)| + |\nabla^2 v_{P_{\alpha_1}}||\nabla v_{P_{\alpha_1}}||\nabla\left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)| \\ + |\nabla v_{P_{\alpha_1}}||\nabla^2\left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)|). \end{aligned}$$

By embedding theorems (see Lemmas 2.1 and 2.2), we obtain the inequality

$$\begin{aligned} &\int_0^T \|h\left(v_{P_{\alpha_1}}\right) - h\left(v_{P_{\alpha_2}}\right)\|^2_{L^2(\sigma)}\,dt \\ &\le c\lambda^2\left(\int_0^T \|\nabla^2\left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)\|^2_{L^2(\sigma)}\,dt\right) + (\|v_{P_{\alpha_1}}\|^2_{W^{(4,2),2}(\sigma^T)} + \|v_{P_{\alpha_2}}\|^2_{W^{(4,2),2}(\sigma^T)})\left(\int_0^T\int_\sigma |\nabla(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}|^4\,dx'dt\right)^{1/2} \\ &\quad + \|v_{P_{\alpha_2}}\|^4_{W^{(4,2),2}(\sigma^T)}\left(\int_0^T\int_\sigma |\nabla(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}|^4\,dx'dt\right)^{1/2} + \|v_{P_{\alpha_1}}\|^2_{W^{(4,2),2}(\sigma^T)}\int_0^T \|\nabla^2\left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)\|^2_{L^2(\sigma)}\,dt \\ &\le c\lambda^2(\alpha_0^2 + \alpha_0^4)\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|^2_{W^{(2,1),2}(\sigma^T)}. \end{aligned} \tag{4.3}$$

Therefore, the classical estimate for the heat equation (4.2) yields

$$\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|^2_{W^{(2,1),2}(\sigma^T)} \le c_*\lambda^2(\alpha_0^2 + \alpha_0^4)\|\left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)\|^2_{W^{(2,1),2}(\sigma^T)} + c\|\alpha_1 - \alpha_2\|^2_{L^2(0,T)}. \tag{4.4}$$

If $\lambda^2 < \lambda_3(\alpha_0) = \min\left\{\lambda_1(\alpha), \frac{1}{c_*(\alpha_0^2 + \alpha_0^4)}\right\}$, then (4.4) implies

$$\|\left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)\|_{W^{(2,1),2}(\sigma^T)} \le c\|\alpha_1 - \alpha_2\|^2_{L^2(0,T)}. \tag{4.5}$$

□

Now assume that we have two fluxes $F_1(t)$ and $F_2(t)$ satisfying the condition of Theorem 3.2. Denote by $(v_{P_{\alpha_1}}, \alpha_1)$ and $(v_{P_{\alpha_2}}, \alpha_2)$ the two solutions of problems (1.5) and (1.6) corresponding to the given fluxes F_1 and F_2. Let us prove Theorem 1.4.

Proof. Let $\lambda_2 = \lambda_2(F_0)$ be the number defined in Theorem 1.2. Then, due to this theorem, for $\lambda \in (0, \lambda_2]$, the problems (1.5) and (1.6), $i = 1, 2$, admit the solutions $(v_{P_{\alpha_i}}, \alpha_i)$ such that $v_{P_{\alpha_i}} \in W^{(4,2),2}_{\partial\sigma}(\sigma^T)$, $\alpha_i \in W^{1,2}(0, T)$ and $\alpha_i(0) = 0$. Moreover, the following estimates

$$\|v_{P_{\alpha_i}}\|_{W^{(4,2),2}(\sigma^T)} + \|\alpha_i\|_{W^{1,2}(0,T)} \le C\|F_i\|_{W^{2,2}(0,T)}, \quad i = 1, 2 \tag{4.6}$$

hold. The difference $v = v_{P_{\alpha_1}} - v_{P_{\alpha_2}}$ satisfies the equations

$$\begin{cases} \left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)_t - \frac{v_0}{2}\Delta\left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right) \\ = h\left(v_{P_{\alpha_1}}\right) - h\left(v_{P_{\alpha_2}}\right) + (\alpha_1 - \alpha_2), \quad x \in \sigma, \\ \left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)|_{\partial\sigma} = 0, \quad \left(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\right)(x', 0) = 0, \\ \int_\sigma \left(v_{P_{\alpha_1}}(x', t) - v_{P_{\alpha_2}}(x', t)\right)dx' = F_1(t) - F_2(t), \end{cases} \tag{4.7}$$

where the function h is the same as in Theorem 1.3 and

$$\int_0^T \|h(v_{P_{\alpha_1}}) - h(v_{P_{\alpha_2}})\|^2_{L^2(\sigma)} dt \le c\lambda^2(F_0^2 + F_0^4)\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|^2_{W^{(2,1),2}(\sigma^T)}. \tag{4.8}$$

Then, due to the estimate for a solution of the inverse problems (1.5) and (1.6) we have

$$\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|^2_{W^{(2,1),2}(\sigma^T)} + \|\alpha_1 - \alpha_2\|_{L^2(0,T)} \le c_*\lambda^2(F_0^2 + F_0^4)\|(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\|^2_{W^{(2,1),2}(\sigma^T)} + c\|F_1 - F_2\|^2_{W^{1,2}(0,T)}. \tag{4.9}$$

If $\lambda^2 < \lambda_4(F_0) = \min\Big\{\lambda_2(F_0), \frac{1}{c_*(F_0^2+F_0^4)}\Big\}$, then (4.9) implies (1.10). □

5 The decay of the Poiseuille type solutions as t tends to infinity

Consider first problem (1.4) and prove Theorem 1.5.

Proof. Multiplying equations (1.4) by $v_{P_\alpha}e^{2\beta t}$, integrating over σ and t, and then integrating by parts in σ, we obtain

$$\begin{aligned}
\int_0^T\int_\sigma (v_{P_\alpha})_t v_{P_\alpha} e^{2\beta t} dx' dt &+ \frac{\nu_0}{2}\int_0^T\int_\sigma |\nabla_{x'} v_{P_\alpha}|^2 e^{2\beta t} dx' dt + \frac{\lambda}{2}\int_0^T\int_\sigma \nu(\dot{y}_P(v_{P_\alpha}))|\nabla_{x'} v_{P_\alpha}|^2 e^{2\beta t} dx' dt \\
&= \int_0^T\int_\sigma \alpha(t) e^{2\beta t} v_{P_\alpha} dx' dt \\
&\le \frac{|\sigma|}{2\varepsilon}\int_0^\infty |\alpha(t)|^2 e^{2\beta t} dt + \frac{\varepsilon}{2}\int_0^T\int_\sigma |v_{P_\alpha}|^2 e^{2\beta t} dx' dt.
\end{aligned} \tag{5.1}$$

Since

$$\int_0^T\int_\sigma (v_{P_\alpha})_t v_{P_\alpha} e^{2\beta t} dx' dt = \int_0^T e^{2\beta t}\frac{1}{2}\frac{d}{dt}\int_\sigma |v_{P_\alpha}|^2 dx' dt = -\beta\int_0^T e^{2\beta t}\int_\sigma |v_{P_\alpha}|^2 dx' dt + \frac{1}{2}e^{2\beta T}\int_\sigma |v_{P_\alpha}(x', T)|^2 dx',$$

the relation (5.1) yields

$$\begin{aligned}
\frac{\nu_0}{2}\int_0^T\int_\sigma |\nabla_{x'} v_{P_\alpha}|^2 e^{2\beta t} dx' dt &\le \frac{|\sigma|}{2\varepsilon}\int_0^T |\alpha(t)|^2 e^{2\beta t} dt + \Big(\beta + \frac{\varepsilon}{2}\Big)\int_0^T\int_\sigma |v_{P_\alpha}|^2 e^{2\beta t} dx' dt \\
&\le \frac{|\sigma|}{2\varepsilon}\int_0^\infty |\alpha(t)|^2 e^{2\beta t} dt + c_*\Big(\beta + \frac{\varepsilon}{2}\Big)\int_0^T\int_\sigma |\nabla_{x'} v_{P_\alpha}|^2 e^{2\beta t} dx' dt.
\end{aligned}$$

Taking $\varepsilon < \frac{\nu_0}{2c_*}$ and assuming that $\beta < \frac{\nu_0}{4c_*}$ from the last inequality, we obtain

$$\int_0^T\int_\sigma |\nabla_{x'} v_{P_\alpha}|^2 e^{2\beta t} dx' dt \le c\int_0^\infty |\alpha(t)|^2 e^{2\beta t} dt. \tag{5.2}$$

The constant c in (5.2) is independent of T, and we can pass T to infinity. As a result, we obtain

$$\int_0^\infty\int_\sigma |\nabla_{x'} v_{P_\alpha}|^2 e^{2\beta t} dx' dt \le c\int_0^\infty |\alpha(t)|^2 e^{2\beta t} dt.$$

This estimate together with the Poincaré inequality implies (1.12). □

Consider now problems (1.5) and (1.6) and prove Theorem 1.6.

Proof. Arguing as in the proof of Theorem 1.5, we derive

$$
\begin{aligned}
&-\beta\int_0^T e^{2\beta t}\int_\sigma |v_{P_\alpha}|^2\,dx'dt+\frac{1}{2}e^{2\beta T}\int_\sigma |v_{P_\alpha}(x',T)|^2dx+\frac{\nu_0}{2}\int_0^T\int_\sigma |\nabla_{x'}v_{P_\alpha}|^2e^{2\beta t}dx'dt\\
&\quad+\frac{\lambda}{2}\int_0^T\int_\sigma \nu\big(\dot\gamma_P(v_{P_\alpha})\big)|\nabla_{x'}v_{P_\alpha}|^2e^{2\beta t}dx'dt\\
&\quad=\int_0^T\alpha(t)e^{2\beta t}\int_\sigma v_{P_\alpha}dx'dt=\int_0^T\alpha(t)F(t)e^{2\beta t}dt
\end{aligned}
$$

and so, for sufficiently small β, we obtain

$$
\frac{\nu_0}{4}\int_0^T\int_\sigma |\nabla_{x'}v_{P_\alpha}|^2e^{2\beta t}dx'dt\le\frac{\varepsilon}{2}\int_0^T|\alpha(t)|^2e^{\frac{\beta}{2}t}dt+\frac{1}{2\varepsilon}\int_0^T|F(t)|^2e^{\frac{7\beta}{2}t}dx'dt. \tag{5.3}
$$

Let us multiply now equations (1.5) by $(v_{P_\alpha})_t$. Integrating over σ, we obtain

$$
\int_\sigma |(v_{P_\alpha})_t|^2dx'+\frac{\nu_0}{4}\frac{d}{dt}\int_\sigma|\nabla_{x'}v_{P_\alpha}|^2dx'+\frac{\lambda}{4}\int_\sigma\nu\big(\dot\gamma_P(v_{P_\alpha})\big)\frac{d}{dt}|\nabla_{x'}v_{P_\alpha}|^2dx'=\alpha(t)\int_\sigma(v_{P_\alpha})_tdx'=\alpha(t)F'(t).
$$

Multiplying this relation by $e^{\beta t}$ and integrating over $(0,T)$ gives

$$
\begin{aligned}
&\int_0^T\int_\sigma|(v_{P_\alpha})_t|^2e^{\beta t}dx'dt+\frac{\nu_0}{4}e^{\beta T}\int_\sigma|\nabla_{x'}v_{P_\alpha}(x',T)|^2dx'+\frac{\lambda}{4}e^{\beta T}\int_\sigma\nu\big(\dot\gamma_P(v_{P_\alpha})\big)|_{t=T}|\nabla_{x'}v_{P_\alpha}(x',T)|^2dx'\\
&\quad=\int_0^Te^{\beta t}\alpha(t)F'(t)dt+\beta\frac{\nu_0}{4}\int_0^Te^{\beta t}\int_\sigma|\nabla_{x'}v_{P_\alpha}(x',t)|^2dx'dt+\beta\frac{\lambda}{4}\int_0^Te^{\beta t}\int_\sigma\nu\big(\dot\gamma_P(v_{P_\alpha})\big)|\nabla_{x'}v_{P_\alpha}|^2dx'dt\\
&\quad+\frac{\lambda}{4}\int_0^Te^{\beta t}\int_\sigma\nabla_y\nu\big(\dot\gamma_P(v_{P_\alpha})\big)\nabla_{x'}\big(v_{P_\alpha}\big)_t|\nabla_{x'}v_{P_\alpha}|^2dx'dt.
\end{aligned}
$$

Therefore, there holds the inequality

$$
\begin{aligned}
&\int_0^T\int_\sigma|(v_{P_\alpha})_t|^2e^{\beta t}dx'dt\\
&\quad\le c\beta\int_0^Te^{\beta t}\int_\sigma|\nabla_{x'}v_{P_\alpha}|^2dx'dt+c\lambda\sup_{(x',t)\in\sigma^T}|\nabla_{x'}v_{P_\alpha}|\int_0^Te^{\beta t}\int_\sigma|\nabla_{x'}(v_{P_\alpha})_t||\nabla_{x'}v_{P_\alpha}|dx'dt\\
&\qquad+\frac{\varepsilon}{2}\int_0^T|\alpha(t)|^2e^{\frac{\beta}{2}t}dt+\frac{1}{2\varepsilon}\int_0^T|F'(t)|^2e^{\frac{3\beta}{2}t}dx'dt\\
&\quad\le c\beta\int_0^Te^{\beta t}\int_\sigma|\nabla_{x'}v_{P_\alpha}|^2dx'dt+c\lambda R_0^2\left(\int_0^T\int_\sigma e^{2\beta t}\int_\sigma|\nabla_{x'}v_{P_\alpha}|^2dx'dt\right)^{1/2}\\
&\qquad+\frac{\varepsilon}{2}\int_0^T|\alpha(t)|^2e^{\frac{\beta}{2}t}dt+\frac{1}{2\varepsilon}\int_0^T|F'(t)|^2e^{\frac{3\beta}{2}t}dx'dt.
\end{aligned}
\tag{5.4}
$$

Assume that $c\beta\le\nu_0/4$. Then from (5.3) and (5.4), it follows that

$$\int_0^T \int_\sigma |\left(v_{P_\alpha}\right)_t|^2 e^{\beta t} \mathrm{d}x' \mathrm{d}t + \int_0^T e^{2\beta t} \int_\sigma |\nabla_{x'} v_{P_\alpha}|^2 \mathrm{d}x' \mathrm{d}t \leq \varepsilon \int_0^T |\alpha(t)|^2 e^{\frac{\beta}{2}t} \mathrm{d}t + \frac{1}{\varepsilon} \int_0^T (|F(t)|^2 + |F'(t)|)^2 e^{\frac{7\beta}{2}t} \mathrm{d}t + c\lambda^2 R_0^4. \tag{5.5}$$

Let us estimate now the norm of α. Multiplying equations (1.5) by the solution v_0 of the Poisson problem (3.27), we obtain

$$\begin{aligned}\alpha(t)\kappa_0 &= \int_\sigma \left(v_{P_\alpha}\right)_t v_0 \mathrm{d}x' - \frac{v_0}{2} \int_\sigma \Delta_{x'} v_{P_\alpha} v_0 \mathrm{d}x' - \frac{\lambda}{2} \int_\sigma \operatorname{div}\left(v\left(\dot{\gamma}_P\left(v_{P_\alpha}\right)\right)\nabla_{x'} v_{P_\alpha}\right) v_0 \mathrm{d}x' \\ &= \int_\sigma \left(v_{P_\alpha}\right)_t v_0 \mathrm{d}x' + F(t) + \frac{\lambda}{2} \int_\sigma v\left(\dot{\gamma}_P\left(v_{P_\alpha}\right)\right)\nabla_{x'} v_{P_\alpha} \nabla_{x'} v_0 \mathrm{d}x'.\end{aligned}$$

Therefore,

$$\begin{aligned}\kappa_0^2 \int_0^T e^{\frac{\beta}{2}t} |\alpha(t)|^2 \mathrm{d}t &\leq c \int_0^T e^{\frac{\beta}{2}t} \left(\int_\sigma \left(v_{P_\alpha}\right)_t v_0 \mathrm{d}x' \right)^2 \mathrm{d}t + c \int_0^T e^{\frac{\beta}{2}t} |F(t)|^2 \mathrm{d}t \\ &\quad + c\lambda^2 \int_0^T e^{\frac{\beta}{2}t} \left(\int_\sigma v\left(\dot{\gamma}_P\left(v_{P_\alpha}\right)\right)\nabla_{x'} v_{P_\alpha} \nabla_{x'} v_0 \mathrm{d}x' \right)^2 \mathrm{d}t \\ &\leq c \int_0^T \int_\sigma e^{\beta t} |\left(v_{P_\alpha}\right)_t|^2 \mathrm{d}x' \mathrm{d}t + c \int_0^T \int_\sigma e^{\beta t} |\nabla_{x'} v_{P_\alpha}|^2 \mathrm{d}x' \mathrm{d}t \\ &\quad + c \int_0^T e^{\frac{\beta}{2}t} |F(t)|^2 \mathrm{d}t.\end{aligned} \tag{5.6}$$

The sufficiently small ε inequalities (5.5) and (5.6) yield

$$\int_0^T \int_\sigma |\left(v_{P_\alpha}\right)_t|^2 e^{\beta t} \mathrm{d}x' \mathrm{d}t + \int_0^T e^{2\beta t} \int_\sigma |\nabla_{x'} v_{P_\alpha}|^2 \mathrm{d}x' \mathrm{d}t + \int_0^T e^{\frac{\beta}{2}t} |\alpha(t)|^2 \mathrm{d}t \leq c \int_0^T (|F(t)|^2 + |F'(t)|)^2 e^{\frac{7\beta}{2}t} \mathrm{d}x' \mathrm{d}t + c\lambda^2 R_0^4.$$

Since the constant in the last inequality is independent of T, we can pass $T \to +\infty$ and we obtain (1.14). □

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