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LIMIT THEOREMS FOR RANDOM LINEAR FIELDS VIA BEVERIDGE–NELSON DECOMPOSITION

Doctoral Dissertation

Physical Sciences, Mathematics (01 P)

Vilnius, 2011

The scientific work was carried out during 2006–2010 at Vilnius University

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Acknowledgments

I am heartily thankful to my supervisor, Prof. Vygantas Paulauskas, whose encouragement, supervision and support from the preliminary to the concluding level enabled me to develop an understanding of the subject. I thank Prof. Yu. Davydov for his insights and support in delivering the results of the thesis, the lecturers and participants of weakly scientific seminar of the departments of Mathematical Analysis and Econometrical Analysis who supported motivation for moving forward.

Lastly, I offer my regards to all of those who supported me in any respect during the completion of the thesis.

Contents

Abstract: The main objective of this thesis is the extension of limit result for sums of random linear field by using Beveridge–Nelson decomposition.

To achieve this goal, we use Beveridge–Nelson decomposition generalization by V. Paulauskas presented in 2010 and D. Marinucci and S. Poghosyan presented in 2001. These works extend results of P.C.B. Phillips and V. Solo for random linear fields, which were formulated for linear processes. The method enables results proved for random fields of innovations apply to random linear fields, generated by these innovations, under some additional assumption on linear filter.

After the investigation of Beveridge–Nelson decomposition method we came to the conclusion that the best application of the method is for the proof of Central limit theorem. We consider random linear fields on $\mathbb{Z}^d$ generated by different type of innovation.

In Chapter 2 we analyze random linear fields generated by martingale difference innovation. Martingale difference definition in the plane and higher dimension spaces is another important topic analyzed in the thesis, because it exists different ways to define them. We use different definitions of martingale difference presented in the works by D. Tjøstheim, R. Morkvėnas, B. Nahapetian, M. El Machkouri, and prove Central limit theorems for random linear fields with three different types of martingale difference innovations.

In the last chapter we consider random linear fields generated by ergodic or mixing (in particular case, independent identically distributed (i.i.d.)) random variables. There we generalize the classical Strong Law of Large Numbers for multi-indexed sums of i.i.d. random variables. These results are easily obtained using ergodic theory. Also we compare the results for SLLN obtained using ergodic theory and with the help of the Beveridge–Nelson decomposition.

Notation

The following notation is used throughout the thesis.

a.s.	almost surely (that is, with probability one)
i.i.d.	independent and identically distributed
m.d.s.	martingale difference sequence
r.f.	random field
BN	Beveridge–Nelson
CLT	central limit theorem
SLLN	strong law of large number
$\mathbf{t}, \mathbf{k}, \ldots$	bold font letters denote vectors
$\mathbb{Z}$	denotes the set of integers
$\mathbb{N}$	denotes the set of natural numbers
$\mathcal{L}_{q,p}$	denotes that $\sum_{\mathbf{k}\in\mathbb{Z}^d}(\prod_{i=1}^{d}(\lvert k_i\rvert + 1))^q \lvert\varphi_{\mathbf{k}}\rvert^p < \infty$
$X \in L\log L^{d-1}$	denotes that $E\lvert X\rvert(\ln(1+\lvert X\rvert))^{d-1} < \infty$

Chapter 1

Introduction

I would rather have a mind opened by wonder than one closed by belief.

Gerry Spence, 'How to Argue and Win Every Time'

Limit theorems play an important role in probability theory and mathematical statistics.

...the epistemological value of the theory of probability is revealed only by limit theorems. Moreover, without limit theorems it is impossible to understand the real content of the primary concept of all our sciences – the concept of probability.

B.V. Gnedenko and A.N. Kolmogorov

There are several types of limit theorems: Central limit theorems (CLT), Strong law of large numbers (SLLN), Law of iterated logarithm (LIL) and other modifications. They all are about behavior of the sum of random variables with appropriate norming and different strength of convergence statement: almost sure (a.s.), in probability (P) or by distribution (d).

Let us take as starting point the classical (Kolmogorov) Strong law of large numbers (SLLN), which states that if we have X_i, $i = 1, 2, \ldots,$ independent identically distributed (i.i.d.) random variables and $E|X_1| < \infty$, $EX_1 = \mu$, then

$$\frac{1}{n} S_n = \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{\text{a.s.}} \mu.$$

We can interpret CLT and LIL as results on the rate of convergence in SLLN if we know additional information about summands. For example CLT can be written in the form

$$\frac{\sqrt{n}}{\sigma} \left(\frac{S_n}{n} - \mu \right) \xrightarrow{\text{d}} N(0, 1),$$

where $\xrightarrow{\text{d}}$ denotes convergence in distribution, $\sigma^2 = \operatorname{var} X_1 < \infty$, $N(0,1)$ – standard normal distribution. LIL is usually written in the form

$$\frac{S_n}{n} - \mu = \sigma\xi(n)\left(\frac{2}{n}\log\log n\right)^{1/2}$$

where $\limsup_{n\to\infty} \xi(n) = 1$ a.s. and $\liminf_{n\to\infty} \xi(n) = -1$ a.s.

These standard expressions suppress the relation with SLLN. Therefore we rewrite LIL as

$$\frac{1}{x_n}\left(\frac{n}{\log\log n}\right)^{1/2}\left(\frac{S_n}{n} - \mu\right) \xrightarrow{\text{a.s.}} 0.$$

for any sequence of constants $\{x_n, n \in \mathbb{N}\}$, $x_n \to \infty$, and for CLT we can write

$$\frac{\sqrt{n}}{x_n}\left(\frac{S_n}{n} - \mu\right) \xrightarrow{\text{p}} 0. \tag{1.0.1}$$

From the last expressions it can be seen that LIL provides a boundary between convergence in probability to zero and a.s. convergence to zero.

It was an example in the one dimensional i.i.d. case, just to illustrate main principles of limit theorems. When random variables have different characteristics, e.g. random variables are not independent or identically distributed, new methods of proof must be found and condition for convergence analyzed.

Rather complete theory of limit theorems in the case of sequences of random variables has been developed. Our interest is different. We consider random variables $X_{\mathbf{t}}$ with discrete indexing parameter $\mathbf{t}$ varying in spaces $\mathbb{Z}^d$, $d = 2$ or $\mathbb{Z}^d$, $d > 2$. Such objects are called random fields.

Multiparameter processes present a natural extension of time series or processes of continues time. They are important in many theoretical problems and applications. As such examples we can mention relation with mathematical statistics [57], statistical mechanics [35], brain data imaging [13], computer graphics, information extraction from text and labeling [36] and other.

Different classes of random fields are studied, we can mention: Markov random fields, Gibbs random fields, conditional random fields, Gaussian random fields and other. Our thesis is devoted to studies of linear random fields, which can be considered as generalization of linear processes also called time series.

The construction of a linear random field is the same as that of linear random process: we have one random field $\{\varepsilon_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ (which often are called as innovations) and we form a new random field by means of non-random coefficients (which are called a filter) in the following way:

$$X_{\mathbf{t}} = \sum_{\mathbf{k} \geqslant 0} \varphi_{\mathbf{k}} \varepsilon_{\mathbf{t}-\mathbf{k}}, \quad \mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{Z}^d, \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d, \tag{1.0.2}$$

or

$$X_{\mathbf{t}} = \sum_{\mathbf{k} \in \mathbb{Z}^d} \varphi_{\mathbf{k}} \varepsilon_{\mathbf{t}-\mathbf{k}}, \quad \mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{Z}^d, \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d.$$

If $X_{\mathbf{t}}$ is well defined (the series converge almost sure) then $\{X_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ is called a linear random field.

Here and in what follows bold letters stand for vectors (multi-dimensional or infinite-dimensional). Linear operation are defined component-wise, for example if $\mathbf{t}, \mathbf{s} \in \mathbb{Z}^d$ then $\mathbf{t}+\mathbf{s} = (t_1, t_2, \ldots, t_d) + (s_1, s_2, \ldots, s_d) = (t_1+s_1, t_2+s_2, \ldots, t_d+s_d)$. The comparison of elements later will be based on partial order or lexicographical order (the latter defines total order for elements with multivariate indices). The main questions analyzed in the thesis are:

- Application of Beveridge–Nelson (BN) decomposition to the random linear fields;
- Different definitions of martingale difference in the case of discrete multivariate indices;
- Proof of CLT for linear random fields generated by martingale difference innovations, by using BN decomposition;
- Proof of SLLN for linear random fields generated by i.i.d. innovations.

It is possible to say that in the thesis we generalize some results formulated for linear processes in [56]. We present shortly main idea of [56]. Consider a linear process, defined by the formula

$$X_t = \sum_{k=0}^{\infty} c_k \varepsilon_{t-k}, \quad t \in \mathbb{Z}, \tag{1.0.3}$$

where ε_i, $i \in \mathbb{Z}$, are i.i.d. random variables, and c_i and ε_i are such that X_t is correctly defined (series converges a.s.) and is a stationary process. For the investigation of sums of such elements the BN decomposition, which will be introduced in Section 1.3, is used. Note that this decomposition can be applied to an arbitrary sequence $\{\varepsilon_i\}$; namely, if $X_t = C(L)\varepsilon_t$ is well defined then

$$a_n^{-1} \sum_{t=1}^{n} X_t = C(1) a_n^{-1} \sum_{t=1}^{n} \varepsilon_t + R_n, \tag{1.0.4}$$

where R_n has a very simple structure: $R_n = a_n^{-1}(\tilde{\varepsilon}_0 - \tilde{\varepsilon}_n)$, $\tilde{\varepsilon}_t = \sum_{k=0}^{\infty} \tilde{c}_k \varepsilon_{t-k}$. Having (1.0.4), the next step is to prove that, under appropriate moment conditions on $\{\varepsilon_i\}$ (which usually are assumed to be i.i.d. or martingale differences) and on the coefficients c_i, $R_n \to 0$ in probability or a.s. Thus, limit theorems for $\sum_{t=1}^{n} X_t$ are reduced to the corresponding limit theorems for $\sum_{t=1}^{n} \varepsilon_t$. Using this approach, it is possible to prove the Law of Large Numbers (LLN), Strong LLN (SLLN), Central Limit Theorem (CLT), and Invariance Principle (IP). The existence of variances of ε_t and X_t is not essential, and it is possible to investigate the case where ε_i's are heavy-tailed. All these possibilities are demonstrated in the fundamental paper [56] by Phillips and Solo. Also it is necessary to mention that the same idea, that relation between limit behavior of sums of innovations and sums of linear processes reflect the dependence structure (short or long memory) of linear processes, is realized in much more general setting of Banach space valued innovations and operator normalization in the paper [58].

In the paper [55], it was demonstrated that BN decomposition is useful when proving limit theorems for sums $\sum_{\mathbf{t} \in D_{\mathbf{n}}} X_{\mathbf{t}}$, where $D_{\mathbf{n}}$ is some subset of $\mathbb{Z}^d$ and $X_{\mathbf{t}}$ is of the form (1.0.2) with i.i.d. innovations $\{\varepsilon_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$. BN decomposition in the case of rectangles $D_{\mathbf{n}} = \{\mathbf{i} \in \mathbb{Z}^2 \colon \mathbf{1} \leqslant \mathbf{i} \leqslant \mathbf{n}\}$, allows to write,

$$\sum_{\mathbf{t} \in D_{\mathbf{n}}} X_{\mathbf{t}} = \mu_1 \sum_{\mathbf{t} \in D_{\mathbf{n}}} \varepsilon_{\mathbf{t}} + R_{\mathbf{n}}, \tag{1.0.5}$$

where $\mu_1 = \sum_{\mathbf{k} \geqslant 0} \varphi_{\mathbf{k}}$, and $R_{\mathbf{n}}$ has not complicated form.

In the paper [55] SLLN and CLT were proved using BN decomposition.

It must be stressed that in [3] it was shown that application of the ergodic theory to prove SLLN gives more general results comparing with ones obtained by using BN decomposition. One of the advantages of BN decomposition mentioned in [56] was its simplicity of application, but the proofs based on applications of ergodic theory are also very simple.

In [43] it was proved the IP for linear random fields generated by i.i.d. innovations, but the moment conditions on innovations were dependent on dimension of indices (the existence of higher moments was required for higher dimension). Therefore, taking the above presented remark on SLLN, in [3] it was noted that the most successful application of BN decomposition is the CLT. In our work the main attention is devoted to the CLT for linear random fields with innovations forming martingale differences.

In the papers [55], [3], [43] innovations were assumed to be i.i.d. Here we consider innovation forming a field of martingale differences. Martingale differences on the line are well studied and there is a lot of results on CLT for martingales. Different situation is in the case where indices are in $\mathbb{Z}^d$ with $d \geqslant 2$ since there are several ways to define a martingale and martingale differences. Random fields with such type of dependence are less investigated.

The strategy of the proofs is the same as in [56]: to take CLT result for sums of innovations ($\sum_{\mathbf{t} \in D_{\mathbf{n}}} \varepsilon_{\mathbf{t}}$) and to prove that the remainder $R_{\mathbf{n}}$ as in (1.0.5) with an appropriate normalization tends to zero.

In [56] the BN decomposition was applied for linear random process in two different ways – the so-called direct and indirect methods of application. By both methods we must prove that $R_{\mathbf{n}}$ tends to zero (after appropriate normalization and in the appropriate sense), only in the direct method we use the explicit form of $R_{\mathbf{n}}$ while in the indirect method we just use the fact that sum $\sum_{\mathbf{t} \in D_{\mathbf{n}}} X_{\mathbf{t}}$ is well approximated by $\mu_1 \sum_{\mathbf{t} \in D_{\mathbf{n}}} \varepsilon_{\mathbf{t}}$. As in the papers [56] and [55] we show that indirect method gives better results (see Remark 2.2.1), namely, by indirect method and under the same moment conditions the CLT is proved under weaker conditions on the coefficients $\{\varphi_{\mathbf{k}}\}$. Although one can think that always indirect method of application of BN decomposition is preferable against direct one, it is not the case: in the paper [45] it is demonstrated that better rates of convergence are obtained by using direct method.

The doctoral thesis is arranged as follows. In Chapter 1 we give overview of the analyzed topic, history of CLT for martingales and main method, which will be applied in the proofs of main results. In Chapter 2 we introduce three different types of martingale differences and prove CLT for random linear fields generated by those innovations. In Section 2.2 we consider innovations for which CLT is proved in [67] and modified in [28]. In Section 2.3 we use the approach, based on [49] and in 2.4 we are dealing with martingale differences defined in [16]. In the last third chapter we prove SLLN for random linear fields generated by i.i.d. innovations and compare this respect with results proved by using ergodic theory.

Each section of Chapters 2 and 3 begins with a short introduction of the problem, definitions and the formulation of known results. Then main new results with proofs follows.

1.1 Martingale

Martingale theory illustrates the history of mathematical probability: the basic definitions are inspired by crude notions of gambling, but the theory has become a sophisticated tool of modern abstract mathematics. . . .

J.L. Doob [19]

Speaking about martingale we do not keep in mind known definition of martingale like tack that is used on horses to control head carriage or a type of dog collar that provides more control over an animal and not even betting system which is of course closely related to our topic. For us martingale or martingale differences is useful as a tool in probability theory.

English word martingale represents *martegal* (French dialect word meaning inhabitant of Martigues; Martigues is or was a village in France). The best known meaning for martingales is a betting strategy. In the gambling context it is a strategy when a gambler doubles his bet after every loss. In that way the first win would recover all previous losses plus win profit equal to original stake.

From mathematical point of view martingale is a stochastic process for which conditional expectation of the next value, given current and preceding values is the current value. It is a “fair game” where nobody wins and nobody losses. We only can guess if inhabitants of village Martigues where considered to be eccentric and venturesome. The Oxford English dictionary the martingale as betting strategy dates back to 1815. This strategy will not be helpful in nowadays casino. In order to apply it you must have very deep pockets, besides casino put max sum for betting, so you are limited in number of games. In some games there is not only two possibilities of wining or losing like throwing a coin, so calculation becomes not so simple.

The first steps to use martingale in modern way can be attributed to early works of Bachelier [2], but martingale as concept of modern probabilistic literature was introduced by Ville (1939) [69] and later developed in Doob works [17], [18]. Papers by Bernstein [5], [6], [7] and Levy [38], [39], [40] predates the use of the name martingale. They used martingale as consecutive sums of i.i.d. random variables to generalize limit results. The martingale convergence theorem proved by Doob completely changed the direction of martingale theory development. His book [18] is one of fundamental work on martingales. The book of Hall and Heyde [26] is another important work summarizing works on martingales and their application in probability theory until 1980 years.

After development of the theory of martingales with one-dimensional indices the natural step is an extension to the case of multivariate indices. But it turns out that it is not an easy task, since at first one must answer the following questions:

- how elements should be ordered,
- how to define stopping time,
- how to define σ-algebras and history.

Depending on chosen answers different definitions of martingales can be formulated.

The first fundamental paper devoted to martingales in the plane is written by Cairoli and Walsh [12]. Main martingale notations and important theoretical results were formulated there. Another important work by Wong and Zakai [72] developed further the plane martingale theory. Among other works we can mention papers by Ledoux [37], Cairoli and Gabriel [11], Walsh [70], Morkvėnas [47], Tjøstheim [67]. One of the latest works in that area is a book of Khoshnevisan (2002) [32]. Theory for set indexed martingales was summarized in [29].

1.1.1 Martingale definition

The definition of a martingale in one dimensional case is quite simple. Let $(\Omega, \mathcal{F}, P)$ be a probability space: Ω is a set, $\mathcal{F}$ a σ-field of subsets of Ω, and P is a probability measure defined on $\mathcal{F}$. Let I be any interval of the form $[a, b), (a, b), (a, b], [a, b]$ of the ordered sets $\{-\infty, \dots, -1, 0, 1, \dots, \infty\}$. Let $\{\mathcal{F}_n, n \in I\}$ be an increasing sequence of σ-fields $\mathcal{F}_n \subset \mathcal{F}$, $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Suppose that $\{Z_n, n \in I\}$ is a sequence of random variables on Ω satisfying

(a) Z_n is measurable with respect to $\mathcal{F}_n$,

(b) $E|Z_n| < \infty$,

(c) $E(Z_n|\mathcal{F}_m) = Z_m$ a.s. for all $m < n$, $m, n \in I$.

Then, the sequence $\{Z_n, n \in I\}$ is said to be martingale with respect to $\{\mathcal{F}_n, n \in I\}$. If (a) and (b) are retained and (c) is replaced by the inequality $E(Z_n|\mathcal{F}_m) \geqslant Z_m$ a.s. ($E(Z_n|\mathcal{F}_m) \leqslant Z_m$ a.s.), then $\{Z_n, n \in I\}$ is called a submartingale (supermartingale).

Example 1 (Branching Processes)**.** Write $X_0 = 1$ and define $X_{n+1} = \sum_{i=1}^{X_n} \varepsilon_{i,n}$, $n = (0, 1, \ldots)$ where $\varepsilon_{i,n}$ are i.i.d., integrable random variables that take values in $\mathbb{N} \cup \{0\}$. $\mathcal{F}_m$ is σ-field generated by $X_0, \ldots, X_m$, $m = (0, 1, \ldots)$, then $\{\mu^{-n} X_n, n \geqslant 0\}$ is a martingale with respect to $\{\mathcal{F}_n, n \geqslant 0\}$, where $\mu = E(\varepsilon_{1,1})$. Since $\mathcal{F}_n$ is generated by $X_0, \ldots, X_m$ then $\mu^{-m} X_m$ is $\mathcal{F}_m$ measurable. $EX_{n+1} = E \sum_{i=1}^{X_n} \varepsilon_{i,n} = \mu E X_n = \cdots = \mu^{n+1} < \infty \Rightarrow E(\mu^{-(n+1)} X_n) = 1$ and finally

$$E\left(\frac{X_{n+1}}{\mu^{n+1}} \,\middle|\, \mathcal{F}_n\right) = \frac{1}{\mu^{n+1}} E\left(\sum_{i=1}^{X_n} \varepsilon_{i,n} \,\middle|\, \mathcal{F}_n\right) = \frac{1}{\mu^{n+1}} \mu E(X_n|\mathcal{F}_n) = \frac{X_n}{\mu^n}. \tag{1.1.1}$$

A reverse martingale or backwards martingale $\{Z_n, n \in I\}$ is defined with respect to decreasing sequence of σ-fields $\{\mathcal{F}_n, n \in I\}$, $\mathcal{F}_n \supset \mathcal{F}_{n+1}$. It satisfies the condition (a) and (b) and the following condition instead of (c),

(c′) $E(Z_n|\mathcal{F}_m) = Z_m$ a.s. for all $m > n, m, n \in I$.

Theory of finite reverse martingales is just a dual of finite martingales i.e. $\{Z_i, 1 \leqslant i \leqslant n\}$ is a reverse martingale with respect to $\{\mathcal{F}_i, 1 \leqslant i \leqslant n\}$ if and only if $\{Z_{n-i+1}, 1 \leqslant i \leqslant n\}$ is martingale with respect $\{\mathcal{F}_{n-i+1}, 1 \leqslant i \leqslant n\}$.

Passing to higher dimension spaces, as it was mentioned in the previous section, we must overcome two main difficulties: to define order between elements,[1] choose one of the possible martingale definitions for processes indexed by several parameters. In [12] the partial order was chosen as an answer to the first question, that is for two elements $\mathbf{t} = (t_1, t_2)$, $\mathbf{t}' = (t'_1, t'_2) \in \mathbb{N}_+^2$ the relation

[1] Well-ordering theorem of E. Zermelo, published in 1904, states that, depending on the chosen axioms, every set can be well-ordered. However, the structure of this well-ordering is not usually known and/or in line with the stochastic structure of the problem at hand.

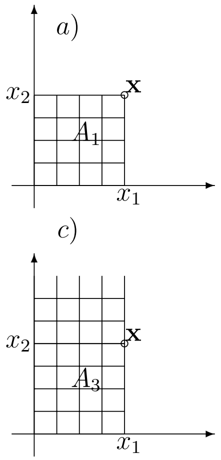

Figure 1.1: The sets, generating σ-algebras.

$\mathbf{t} \leqslant \mathbf{t}'$ means that $t_1 \leqslant t'_1$ and $t_2 \leqslant t'_2$. Then four different types of martingales were introduced: weak martingales, 1-martingale, 2-martingale, strong martingale. These four definitions are based on different σ-algebras. We won't formulate the exact definitions, only illustrate the set of indexes by which the random field elements which generate σ-algebra are indexed. Four different sets are presented in Figure 1.1. If we take $\mathbf{x} = (x_1, x_2) \in \mathbb{N}^2_+$ then in the picture a) set $A_1 = \{\mathbf{y} = (y_1, y_2) \in \mathbb{N}^2_+ \colon y_1 \leqslant x_1, y_2 \leqslant x_2\}$ generate σ-algebra for weak martingale. In the picture b) set $A_2 = \{\mathbf{y} = (y_1, y_2) \in \mathbb{N}^2_+ \colon y_2 \leqslant x_2\}$ generate σ-algebra for 2-martingale. The set $A_3 = \{\mathbf{y} = (y_1, y_2) \in \mathbb{N}^2_+ \colon y_1 \leqslant x_1\}$ is analogous to the set A_2 only the restrictions are applied to the firs coordinate and we get sets generating σ-algebra for 1-martingale. The last set $A_4 = \{\mathbf{y} = (y_1, y_2) \in \mathbb{N}^2_+ \colon \{y_1 \leqslant x_1\} \cup \{y_2 \leqslant x_2\}\}$ illustrate restrictions for the set generating strong martingale σ-algebra.

In our work for introduction of multiparameter martingales we used D. Khoshnevisan book [32] where aspects of Cairoli–Walsh theory were analyzed and new definitions of discrete index martingales were presented.

At first we introduce d-parameter martingales which are the most natural extension of 1-parameter martingales. Suppose $\mathcal{F}^* = \{\mathcal{F}_{\mathbf{t}}, \mathbf{t} \in \mathbb{N}_0^d\}$ is a collection of sub σ-fields of $\mathcal{F}$. We say that $\mathcal{F}^*$ is a filtration if $\mathbf{s} \leqslant \mathbf{t}$ ($\mathbf{t} = (t_1, \ldots, t_d), \mathbf{s} = (s_1, \ldots, s_d)$, we recall that $\mathbf{s} \leqslant \mathbf{t}$ means that $s_i \leqslant t_i$ for $i = 1, \ldots, d$) implies that $\mathcal{F}_{\mathbf{s}} \subset \mathcal{F}_{\mathbf{t}}$. A d-parameter stochastic process $X = \{X_{\mathbf{t}}, \mathbf{t} \in \mathbb{N}_0^d\}$ is adapted to the filtration $\mathcal{F}^*$ if for all $\mathbf{t} \in \mathbb{N}_0^d$, $X_{\mathbf{t}}$ is $\mathcal{F}_{\mathbf{t}}$ measurable.

Definition 1. The process X is d-parameter submartingale (with respect to $\mathcal{F}^*$) if it is adapted (to $\mathcal{F}^*$), for all $\mathbf{t} \in \mathbb{N}_0^d$, $E|X_{\mathbf{t}}| < \infty$ and for all $\mathbf{t} \geqslant \mathbf{s}$, $E(X_{\mathbf{t}}|\mathcal{F}_{\mathbf{s}}) \geqslant X_{\mathbf{s}}$ a.s.

A stochastic process X is a supermartingale if $-X$ is a submartingale. It is a martingale if it is both a supermartingale and a submartingale.

The notion of an orthomartingale is another approach to extend 1-parameter martingale to the multiparameter case. Although, we will not use it while proving main results, we present it as an illustration of the first step of transition from 1-parameter martingale to the multiparameter case. Consider d (one-parameter) filtrations $\mathcal{F}^1, \ldots, \mathcal{F}^d$, where $\mathcal{F}^i = \{\mathcal{F}_k^i, k \geqslant 0\}$, $1 \leqslant i \leqslant d$.

Definition 2. A stochastic process $X = \{X_{\mathbf{t}}, \mathbf{t} \in \mathbb{N}_0^d\}$ is an orthosubmartingale if for each $1 \leqslant i \leqslant d$, and all non negative integers $(t_j, 1 \leqslant j \leqslant d, j \neq i)$, $t_i \to X_{\mathbf{t}}$ is a one-parameter submartingale with respect to one parameter filtration $\mathcal{F}^i$.

A stochastic process X is an orthosupermartingale if $-X$ is an orthosubmartingale. If X is both orthosubmartingale and orthosupermartingale, it is then an orthomartingale.

For example, let us consider the case $d = 2$ and write the process X as $X = (X_{i,j}, i, j \geqslant 0)$. Then, X is orthosubmartingale if

- for all $i, j \geqslant 0$, $E|X_{i,j}| < \infty$,
- for all $j \geqslant 0$, the one-parameter process $i \to X_{i,j}$ is adapted to the filtration $\mathcal{F}^1$, while for each $i \geqslant 0$, $j \to X_{i,j}$ is adapted to $\mathcal{F}^2$, and

- for all $i, j \geqslant 0$, $E(X_{i+1,j}|\mathcal{F}_i^1) \geqslant X_{i,j}$, a.s., and $E(X_{i,j+1}|\mathcal{F}_i^2) \geqslant X_{i,j}$.

Before showing the relation between martingales and orthomartingales defined above we must define marginal filtration and commuting filtration.

Suppose $X = \{X_{\mathbf{t}}, t \in \mathbb{N}_0^d\}$ is a d-parameter random process that is adapted to the d-parameter filtration $\mathcal{F}^* = \{\mathcal{F}_{\mathbf{t}}, \mathbf{t} \in \mathbb{N}_0^d\}$. For all $1 \leqslant i \leqslant d$, define

$$\mathcal{F}_k^j = \bigvee_{\mathbf{t} \in \mathbb{N}_0^d,\ t_j = k} \mathcal{F}_{\mathbf{t}}, \quad k \geqslant 0.$$

We define $\mathcal{F}^j = \{\mathcal{F}_k^j, k \geqslant 0\}$, $1 \leqslant j \leqslant d$, and we call the σ-fields $\mathcal{F}^1, \dots, \mathcal{F}^d$ as marginal filtrations of $\mathcal{F}^*$.

A d-parameter filtration $\mathcal{F}^* = \{\mathcal{F}_{\mathbf{t}}, \mathbf{t} \in \mathbb{N}_0^d\}$ is commuting if for every $\mathbf{s}, \mathbf{t} \in \mathbb{N}_0^d$ and for all bounded $\mathcal{F}_{\mathbf{t}}$-measurable random variables Y

$$E(Y|\mathcal{F}_{\mathbf{s}}) = E(Y|\mathcal{F}_{\mathbf{s} \curlywedge \mathbf{t}}) \quad \text{a.s.} \tag{1.1.2}$$

($s \curlywedge t$ denote the vector whose the ith coordinate is $s_i \wedge t_i$ for all $i = 1, \dots, d$.)

Proposition 1.1.1. *Suppose that $\mathcal{F}^*$ is a d-parameter commuting filtration and that $X = \{X_{\mathbf{t}}, t \in \mathbb{N}_0^d\}$ is adapted to $\mathcal{F}^*$. Then the following is equivalent*

- *X is orthosubmartingale with respect to the marginals of $\mathcal{F}^*$; and*
- *X is submartingale with respect to $\mathcal{F}^*$.*

The exact definition of martingales, used in this thesis, will be presented in Chapter 2, before formulating main results.

1.1.2 Martingale differences

Most often in the applications martingale differences, but not martingales are used. On $\mathbb{Z}$ a definition is quite simple. A sequence $\{\varepsilon_n, n \in \mathbb{N}\}$, $E|\varepsilon_n| < \infty$, is called a martingale difference sequence (m.d.s.) if its expectation with respect to increasing sub σ-field $\mathcal{F}_n$ ($\mathcal{F}_n \subset \mathcal{F}_{n+1}$, $\mathcal{F}_n \subset \mathcal{F}$) is zero:

$$E(\varepsilon_n|\mathcal{F}_n) = 0 \quad \text{for all } n. \tag{1.1.3}$$

For example, if ξ is a martingale, then $\varepsilon_t = \xi_t - \xi_{t-1}$ is m.d.s.

Considering random variables with indices from $\mathbb{Z}^d$, $d \geqslant 2$ the same idea is used: martingale differences are defined requiring that conditional expectation with respect to σ-algebra must be equal to 0. But since in spaces of dimension higher than 1 the notion of the past can be defined in various ways, this gives different definitions of martingale differences. The main definitions of martingale differences, used in this work, will be introduced in Sections 2.2, 2.3, 2.4.

1.2 Central limit theorems for martingales

The central limit theorem (CLT) is one of the most remarkable results of the theory of probability. Some authors say that "Nowadays, the central limit theorem is, considered to be the unofficial sovereign of probability theory" [66]. This theorem says that a sum of a large number of random and not strongly related summands, each of them having only small impact to the sum, has universal law of distribution, not depending on the distribution of summands, and this distribution is the so-called Gaussian or normal law, denoted by $N(\mu, \sigma^2)$, with the density function given by formula:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \tag{1.2.1}$$

where parameter μ is the mean (location of the peak) and σ^2 is the variance (the measure of the width of the distribution).

History of CLT goes to the eighteenth century and is connected with names of Abraham de Moivre (1733) and Pierre-Simon Laplace (1785, 1812). Later the rigorous proofs where given by so called "St. Petersburg school" – Chebyshev, Lyapunov, Markov. The final shape of the CLT for independent summands was formed in works by Lindeberg, Levy and Feller.

At the same time CLT was generalized to several directions: relaxing condition of independence, considering random variables in vector spaces, including general topological vector spaces [68], martingales [26].

Martingales can be considered as the first step from independence to dependent random variables. First relation between martingale theory and CLT can be found in works of Berstein in 1927 [5] and Levy in 1937 [40].

Let $(S_n, \mathcal{F}_n, n \geqslant 1)$ be zero mean, square-integrable martingale and let $X_n = S_n - S_{n-1}$, $n \geqslant 2$ and $X_1 = S_1$ denote martingale difference. Levy's proof involved a direct estimation of difference between martingale distribution and standard normal distribution. The concept of conditional variance V_n^2 plays an important role in limit theorems for martingales

$$V_n^2 = \sum_{i=1}^{n} E(X_i^2 | \mathcal{F}_{i-1}). \tag{1.2.2}$$

In Levy's works it was assumed to be constant a.s. for every n.

Later in 1953 Doob [18] gave the proof of Levy's result based on characteristic function and independently Ibragimov in 1963 [27] established CLT for martingales with stationary and ergodic differences. In the latter work conditional variance is assumed to be asymptotical constant

$$\frac{V_n^2}{s_n^2} \xrightarrow{\text{P}} 1 \tag{1.2.3}$$

where $s_n^2 = E(V_n^2) = E(S_n^2)$.

Rosén in 1967 [61], [60], Dvoretzky in 1971 [20], Loynes (1970) [41] provided some extensions of the theory. Brown in 1971 [10] established a martingale analog of the Lindeberg–Feller theorem and showed that condition (1.2.3), but not stationarity or ergodicity is crucial. The Brown's technique was later developed by Gänssler in 1978 [22]. Scott in 1973 [62] gave an alternative proof together with applications and extensions. McLeish in 1974 [44] introduced an elegant method of proof which provided new CLTs and invariance principles. Later this method was used by Morkvėnas [47] and Tjøstheim [67] to prove theorems for martingales indexed by multi indices.

CLT for random fields was analyzed by many authors. This topic is interesting because the main methods applied in one dimensional indices case face some difficulties in higher dimension. The σ-algebras which naturally appear are not nested as it is in the one-dimensional case. This question has been partially answered first by considering martingale type conditions in [49] and then by studying conditionally centered random fields [31], [30]. In [16] Dedecker proved more general result, getting the CLT for stationary random fields with lexicographical order.

1.3 Beveridge–Nelson decomposition

It is common sense to take a method and try it. If it fails, admit it frankly and try another. But above all, try something.

Franklin D. Roosevelt (1882–1945)

There are several approaches to work with random linear field. One of the methods used is the so-called Beveridge–Nelson (BN) decomposition, which allows to transfer the results obtained for random fields of innovations to linear random field. Most of the results of the thesis are obtained in this way using direct and indirect BN decomposition. At the end of this chapter we shall discuss the recent result of Gordin [25].

The BN decomposition is a model based method for decomposing time series into permanent and transitory component. The origination of the method is modelling and analysis of GDP (Gross domestic product) in economics. Mostly analysing short run and long run aspects of the economy authors use different models. Often this mean distinguishing between cycles in the economy and long term trends. The key idea in terms of trend and cycle decomposition is to allow for permanent shocks to a time series to represent a part of what is meant by trend.

The BN decomposition was first introduced by Stephen Beveridge and Charles R. Nelson in 1981 [8] in time series context. In [56] it is said that such identity in the context of time series was known and used (before the paper [8]) in [21] and [9] for finite lag polynomials. But, as often happens, “In science the credit goes to the man who convinces the world, not the man to whom the idea first occurs. (Sir Francis Darwin)". Now the term Beveridge–Nelson decomposition is so widely spread in economic literature (Google scholar gives more than 1400 entries connected with the term) that to coin a new name for the simple algebraic identity would be impossible [15].

Historically the first draft of the paper that finally became [8] was dated July 1972 and was presented at the Western Economic Association Meeting. The paper appeared when authors rose the question: "What does the trend mean for time series which are not deterministic in the long run, but nevertheless is 'trending' in the sense that it grows over time?" [52]. This seemed an obvious question in 1970 by using Box and Jenkins strategy to model trending economic time series as ARMA in their first difference. These models implied that the future diverge from any pre-specified path. S. Beveridge and R. Nelson thought that satisfactory definition of trend for 'I(1)'[2] time series would preserve the property of trend that is the best estimate of where the variable will be in the distant future. So way not to define trend as long-horizon forecast. Rather than being fixed and pre-determined this trend will shift as new data points reveals new information about the future. That implies that trend is the source of stochastic variation and it is meaningful to think of parsing its fluctuation into part due to trend and a part due to business cycles. Later they showed that the trend is always a random walk with stationary drift and deviation from the trend.

Since the first introduction, this decomposition has proven its usefulness for both theoretical and empirical reasons. The generalization was made by Stock and Watson [65] for multivariate case and other authors [14], [46], [53], [1] offering various ways for infinite sums evaluation. In the work [48] by Morley general unified framework for exact calculation of the BN trend and cycle components for both univariate and multivariate linear processes is provided.

[2] Integrated of order 1 time series. It means that if we take first difference of time series Y_t: $(1-L)Y_t = Y_t - Y_{t-1} = \Delta Y$ we get stationary process.

In the literature the validity of BN decomposition is analyzed in several ways. One of them can be found in [64] where assumptions are made about Wold decomposition[3] and linear filter properties. Another more convenient and simpler method is based on factorization of linear filter, presented in Preposition 1.3.1. From mathematical point of view very important work is paper by Phillips and Solo [56], where BN decomposition is systematically used for limit theorems in univariate case. This work gave an inspiration to other works generalizing results to the multiindex case.

Let us consider a linear process X_t, defined by formula (1.0.3). There are several approaches to investigate sums of (dependent) random variables $\sum_{t=1}^{n} X_t$. One of them is based on BN decomposition of linear processes. It was systematically used in [56] to prove various limit theorems for sums of linear processes. This decomposition is a simple algebraic identity and can be easily formulated. Let, as usual, L denote the lag operator ($L\varepsilon_i = \varepsilon_{i-1}$). Then BN decomposition can be formulated as follows.

Proposition 1.3.1 (([8] and [56])). *Let* $C(L) = \sum_{k=0}^{\infty} c_k L^k$. *Then*

$$C(L) = C(1) - (1 - L)\tilde{C}(L), \tag{1.3.1}$$

where $\tilde{C}(L) = \sum_{k=0}^{\infty} \tilde{c}_k L^k$, $\tilde{c}_k = \sum_{j=k+1}^{\infty} c_j$. *If* $p \geqslant 1$, *then*

$$\sum_{j=1}^{\infty} j^p |c_j|^p < \infty \;\Rightarrow\; \sum_{k=0}^{\infty} |\tilde{c}_k|^p < \infty \quad \text{and} \quad |C(1)| < \infty.$$

If $0 < p < 1$, *then*

$$\sum_{j=1}^{\infty} j |c_j|^p < \infty \;\Rightarrow\; \sum_{k=0}^{\infty} |\tilde{c}_k|^p < \infty.$$

Remark 1.3.1. For linear processes such as (1.0.3), BN decomposition (1.3.1) yields directly the martingale approximation to the partial sum process of stationary time series [26].

[3] In time series analysis, Wold decomposition theorem implies that any stationary discrete time stochastic process can be decomposed into a pair of uncorrelated processes, one deterministic, and the other being a moving average process.

Remark 1.3.2. In [8] the algebraic decomposition (1.3.1) was used explicitly to decompose aggregate time series, but the relations between coefficients c_j and $\tilde{c}_j$ were obtained in [56].

1.3.1 BN decomposition generalization in [55]

Here we will introduce BN decomposition, formulated by V. Paulauskas in [55]. We consider a linear random field (1.0.2) with $d = 2$. Let $\mathbf{L}_2 = (L_1, L_2)$ be the lag operator defined by $L_1 \varepsilon_{t,s} = \varepsilon_{t-1,s}$, $L_2 \varepsilon_{t,s} = \varepsilon_{t,s-1}$, then linear field can be written as $X_{\mathbf{t}} = \Phi(\mathbf{L}_2)\varepsilon_{\mathbf{t}}$ with

$$\Phi(\mathbf{L}_2) = \sum_{k,l \geqslant 0} \varphi_{k,l} L_1^k L_2^l.$$

To formulate the BN decomposition, we need the following notation:

$$\mu_1 = \Phi(1,1) = \sum_{k,l \geqslant 0} \varphi_{k,l}, \tag{1.3.2}$$

$$A_2(\mathbf{L}_2) = \Phi^*(\mathbf{L}_2)\Delta_2(\mathbf{L}_2), \quad \Delta_2(\mathbf{L}_2) = (1 - L_1)(1 - L_2),$$

$$\Phi^*(\mathbf{L}_2) = \sum_{k,l \geqslant 0} \varphi^*_{k,l} L_1^k L_2^l, \quad \varphi^*_{k,l} = \sum_{i \geqslant k+1,\ j \geqslant l+1} \varphi_{i,j},$$

$$A_1(\mathbf{L}_2) = B(L_1)\Delta_1(L_1) + D(L_2)\Delta_1(L_2), \quad \Delta_1(L_i) = (1 - L_i),$$

$$B(L_1) = \sum_{j \geqslant 0} b_j L_1^j, \quad b_j = \varphi^*_{j,-1} = \sum_{i \geqslant j+1,\ k \geqslant 0} \varphi_{i,k},$$

$$D(L_2) = \sum_{j \geqslant 0} d_j L_2^j, \quad d_j = \varphi^*_{-1,j} = \sum_{i \geqslant 0,\ k \geqslant j+1} \varphi_{i,k}.$$

We denote by $\mathcal{L}_{q,p}$ the condition

$$\sum_{k,l \geqslant 0} (k^* l^*)^q |\varphi_{k,l}|^p < \infty,$$

where $i^* = i$ for $i \geqslant 1$ and $0^* = 1$.

Theorem 1.3.2 ([55], [43])**.** *The following identity holds:*

$$\Phi(\mathbf{L}_2) = \mu_1 + A_2(\mathbf{L}_2) - A_1(\mathbf{L}_2). \tag{1.3.3}$$

The relations

$$\sum_{k,l\geqslant 0}|\varphi^*_{k,l}|^p<\infty,\ \sum_{j\geqslant 0}|b_j|^p<\infty,\ \sum_{j\geqslant 0}|d_j|^p<\infty,\ \mu_1<\infty \tag{1.3.4}$$

hold if either condition $\mathcal{L}_{p,p}$ in the case $1\leqslant p<\infty$ or condition $\mathcal{L}_{1,p}$ in the case $0<p<1$ is satisfied.

From (1.3.3), if we have summing sets rectangles $D_{\mathbf{n}}=\{\mathbf{i}=(i_1,i_2)\in\mathbb{Z}^2\colon 1\leqslant i_1\leqslant n_1, 1\leqslant i_2\leqslant n_2\}$ we get (1.0.5) with the remainder term, given by formula where

$$\begin{aligned} R_{\mathbf{n}} &= \xi_{n_1,n_2}-\xi_{n_1,0}-\xi_{0,n_2}+\xi_{0,0} \\ &\quad +\eta_{n_1,n_2}-\eta_{0,n_2}+\zeta_{n_1,n_2}-\zeta_{n_1,0}, \end{aligned} \tag{1.3.5}$$

and

$$\xi_{t,s}=\Phi^*(\mathbf{L}_2)\varepsilon_{t,s}=\sum_{k,l\geqslant 0}\varphi^*_{k,l}\varepsilon_{t-k,s-l},$$

$$\eta_{t,n_2}=\sum_{s=1}^{n_2}\tilde{\varepsilon}_{t,s},\quad \tilde{\varepsilon}_{t,s}=B(L_1)\varepsilon_{t,s}=\sum_{j\geqslant 0}b_j\varepsilon_{t-j,s},$$

$$\zeta_{n_1,s}=\sum_{t=1}^{n_1}\hat{\varepsilon}_{t,s},\quad \hat{\varepsilon}_{t,s}=D(L_2)\varepsilon_{t,s}=\sum_{j\geqslant 0}d_j\varepsilon_{t,s-j}.$$

If the set $D_{\mathbf{n}}$ is a square we can replace n_1, n_2 in (1.3.5) by n and get the expression of remainder. Although, the remainder term in both cases (rectangles and squares) have the same form, the proof of limit theorems, as can be seen in [55], meets bigger difficulties in rectangles case.

1.3.2 BN decomposition generalization in [43]

Earlier than the paper [55] appeared, the BN decomposition in multivariate case (that is (1.3.3) in case $d=2$), but without relation (1.3.4) was given in [43]. Now we formulate this result. A random linear field given by (1.0.2) can be written as $X_{\mathbf{t}}=\Phi(\mathbf{L}_d)\varepsilon_{\mathbf{t}}$ (d denote dimension of lag operator $\mathbf{L}_d=(L_1,\ldots,L_d)$) with

$$\Phi(L_1,\ldots,L_d)=\sum_{i_1=0}^{\infty}\cdots\sum_{i_d=0}^{\infty}\varphi(i_1,\ldots,i_d)L_1^{i_1}\ldots L_d^{i_d}. \tag{1.3.6}$$

Assume that

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} \sum_{k_1=i_1+1}^{\infty} \cdots \sum_{k_d=i_d+1}^{\infty} |\varphi(k_1, \ldots, k_d)| < \infty.$$

For example, this condition is satisfied if

$$|\varphi(i_1, \ldots, i_d)| < C(i_1 \times \ldots \times i_d)^{-2-\varepsilon}$$

for some $\varepsilon > 0$ and

$$\Phi(1, \ldots, 1) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} \varphi(i_1, \ldots, i_d) < \infty.$$

The following lemma represent BN decomposition in d-dimensional case.

Lemma 1.3.3. *Let Γ_d be the class of all 2^d subsets γ of $\{1, \ldots, d\}$. Let $B_j = L_j$ if $j \in \gamma$ and $B_j = 1$ if $j \notin \gamma$; we have*

$$\Phi(L_1, \ldots, L_d) = \sum_{\gamma \in \Gamma_d} \left\{ \prod_{j \in \gamma} (L_j - 1) \right\} \Phi_\gamma(B_1, \ldots, B_d), \tag{1.3.7}$$

where it is assumed that $\prod_{j \in \emptyset} = 1$, and

$$\Phi_\gamma(B_1, \ldots, B_d) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} \varphi_\gamma(i_1, \ldots, i_d) B_1^{i_1} \ldots B_d^{i_d}, \tag{1.3.8}$$

$$\varphi_\gamma(i_1, \ldots, i_d) = \sum_{s_1=i_1+1}^{\infty} \cdots \sum_{s_d=i_d+1}^{\infty} \varphi(s_1, \ldots, s_d) \tag{1.3.9}$$

where sums go over indexes s_j, $j \in \gamma$, whereas $s_j = i_j$ if $j \notin \gamma$.

Remark 1.3.3. This lemma was generalized in [45] by letting summation indices $i_j, j = 1, \ldots, d$ vary not from 0, but from $-\infty$.

1.3.3 Martingale-coboundary representation

In this subsection we present another method for proving limit theorems for stationary random sequence introduced in [24] and then extended for random fields in [25]. The idea is that any stationary random sequence can be written as the sum of martingale difference sequence and a coboundary sequence. This means that the problem is reduced to limit theorem for sums of martingale differences and the proof that coboundary sequence is negligible. But since the author in [25] wrote that application of this representation for limit theorems is postponed for the future research, at present we are not able to comment what relation is between martingale-coboundary representation and the indirect application of BN decomposition, which is used in the thesis.

We formulate martingale-coboundary representation in its simple one-dimensional form. Let $\xi = \{\xi_n, n \in \mathbb{Z}\}$ be a stationary (in the strict sense) random sequence. Under certain assumptions [24] it can be represented in the form

$$\xi_n = \eta_n + \zeta_n,$$

where $\eta = \{\eta_n, n \in \mathbb{Z}\}$ is a stationary sequence of martingale differences (this means that $E(\eta_n|\eta_{n-1}, \eta_{n-2}, \dots) = 0$ for all $n \in \mathbb{Z}$), and $\zeta = \{\zeta_n, n \in \mathbb{Z}\}$ is the so-called coboundary (or coboundary sequence) which can be written as $\zeta_n = \theta_n - \theta_{n-1}$, $n \in \mathbb{Z}$, by means of a certain stationary sequence $\theta = \{\theta_n, n \in \mathbb{Z}\}$. The random sequences ξ, η, θ in this representation are stationary connected, that is, the sequence $\{(\xi_n, \eta_n, \theta_n), n \in \mathbb{Z}\}$ of random vectors is stationary. While studying the asymptotic distributions of $\sum \xi_n$, in many cases the contribution of the $\sum \zeta_n$ into the sum can be neglected and extended to the limit theorems originally known for $\sum \eta_n$ of martingale differences only (notice that the limit theory for martingale differences is well developed). Sometimes the sequence ζ is negligible even if it doesn't satisfy coboundary definition.

Clearly the method proposed in [25] is rather general and effective but there are some merits for BN decomposition which is used in this work. BN decomposition method can be applied not necessarily for stationary sequence, does not depend on martingale definition and can give the exact expression for simple set of summation like rectangles.

Chapter 2

CLT for linear martingale difference random fields

2.1 Introduction

In the paper [55] some results of [56] were generalized to linear random fields with innovations which are i.i.d. Here, we take random field $X_{\mathbf{t}}$ defined by (1.0.2) and write it in the case $d = 2$. Let

$$X_{\mathbf{t}} = \sum_{\mathbf{k} \geqslant 0} \varphi_{\mathbf{k}} \varepsilon_{\mathbf{t}-\mathbf{k}}, \quad \mathbf{t} = \{t_1, t_2\} \in \mathbb{Z}^2,\ \mathbf{k} = \{k_1, k_2\} \in \mathbb{Z}^2 \tag{2.1.1}$$

and let coefficients $\{\varphi_{\mathbf{k}}, \mathbf{k} \geqslant 0,\ \mathbf{k} \in \mathbb{Z}^2\}$ and a random field of innovations $\{\varepsilon_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^2\}$ be such that a random field $\{X_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^2\}$ is well defined and stationary. For partial order between vectors in Sections 2.2 and 2.3 symbols $\leqslant$ and $<$ will be used: $\mathbf{t} \leqslant \mathbf{s}$ will mean $t_1 \leqslant s_1$ and $t_2 \leqslant s_2$, while $\mathbf{t} < \mathbf{s}$ will mean $t_1 < s_1$ and $t_2 < s_2$. In Section 2.4 we will use lexicographical order, which will be introduced later.

In the above cited paper [55] it was demonstrated that the BN decomposition for linear random fields (exact formulation in the case $d = 2$ is presented in Section 1.3.1) is useful when proving limit theorems for sums $\sum_{\mathbf{t} \in D_{\mathbf{n}}} X_{\mathbf{t}}$, where $D_{\mathbf{n}}$ is some subset of $\mathbb{Z}^d$. Namely, this representation in the case of rectangles $D_{\mathbf{n}} = \{\mathbf{i} \in \mathbb{Z}^2 \colon \mathbf{1} \leqslant \mathbf{i} \leqslant \mathbf{n}, \}$ allows to write sum of random linear field (2.1.1) in the form (1.0.5) which is handy for proving limit theorems.

In this chapter we continue the program started in [55] to generalize the results from [56] to linear fields. If in all above mentioned papers [43], [55], and [3] innovations were assumed independent, now we consider innovations forming a field of martingale differences. It is well known that generally dependence structure is more complicated for random fields comparing with random processes and the same can be said, in particular, about martingale type dependence. If martingales and martingale differences on line are well studied and there is a lot of results on the CLT for martingales, different situation is on plane (or in spaces with higher dimensions), there are several ways to define martingales and martingale differences and random fields with such type of dependence are less investigated. In Sections 2.2 and 2.3 we present result published in [4] where CLT was proved for random linear fields with martingale differences innovations defined in [49] and [67]. In Section 2.4 we also take the innovations of linear random field martingale differences, but the difference is that, they satisfy the requirements of CLT for random fields in [16]. Our strategy is the same as in [56]: to take a result with the CLT proved for $\sum_{\mathbf{t}\in D_{\mathbf{n}}} \varepsilon_{\mathbf{t}}$ in (1.0.5) and to prove that $R_{\mathbf{n}}$ with an appropriate normalization tends in probability to zero.

The chapter is organized as follows. In Section 2.2 we consider innovations, for which CLT is proved in [67] and modified in [28]. Next Section 2.3 is based on the approach used in [50]. In the last Section 2.4 we consider innovation which satisfy results formulated in [16].

2.2 CLT for linear random fields with martingale differences defined in [67]

As it was mentioned in the introduction, there are several ways to define martingale differences on the plane or spaces with higher dimension. In this section we shall follow the approach from the papers [67] and [47], where CLT for lattice martingale arrays was proved. These results can be considered as generalization of classical results of Hall and Heyde, see [26]. Later in [28], CLT for lattice martingale arrays under conditional Lindeberg condition was established.

2.2.1 Definitions and auxiliary results

The framework will be essentially the same as in [29] and [28], now we shall present the main definitions, referring to these sources for the details.

Although we shall deal mainly with martingales on plane, as in [28] let us consider random variables indexed by multi-indices from $\mathbb{Z}^d$. Let us denote $T_{\mathbf{a}} := \{\mathbf{x} \in \mathbb{Z}^d \colon \mathbf{a} \leqslant \mathbf{x}\}$, $\mathbf{a} \in \mathbb{Z}^d$ and $V[\mathbf{a}, \mathbf{x}] := \{\mathbf{y} \in \mathbb{Z}^d \colon \mathbf{a} \leqslant \mathbf{y} \leqslant \mathbf{x}\}$, $\mathbf{a}, \mathbf{x} \in \mathbb{Z}^d$. A random field will be denoted by $Y = \{Y_{\mathbf{x}}, \mathbf{x} \in T_{\mathbf{a}}\}$. It can be easily seen that there exists one to one correspondence between elements from $T_{\mathbf{a}}$ and rectangles $V[\mathbf{a}, \mathbf{x}]$, therefore a process Y can be analyzed as indexed by rectangles or by the elements of indexing set $\mathcal{A}$ where

$$\mathcal{A} = \{V[\mathbf{a}, \mathbf{x}] \colon \mathbf{x} \in T_{\mathbf{a}}\}, \tag{2.2.1}$$

i.e. $Y_{\mathbf{x}} = Y_{V[\mathbf{a},\mathbf{x}]}$ for $V[\mathbf{a}, \mathbf{x}] \in \mathcal{A}$. This is a particular example of indexing collection, in [29] one can find a general definition of formal indexing collection $\mathcal{A}$. We define a semi-algebra $\mathcal{C}$ to be the class of all subsets of $T_{\mathbf{a}}$ having form:

$$C = A \backslash B, \quad A \in \mathcal{A}, \quad B \in \mathcal{A}(u),$$

where $\mathcal{A}(u)$ is the class of finite unions of sets from $\mathcal{A}$.

Now, let $(\Omega, \mathcal{F}, P)$ be any complete probability space. A filtration (indexed by $\mathcal{A}$) is a class of complete sub-σ-fields of $\mathcal{F}$ $\{\mathcal{F}_A, A \in \mathcal{A}\}$, which satisfies the following conditions:

- If $A, B \in \mathcal{A}$, and $A \subseteq B$ then $\mathcal{F}_A \subseteq \mathcal{F}_B$.
- Monotone outer-continuity: $\mathcal{F}_{\bigcap A_i} = \bigcap \mathcal{F}_{A_i}$ for all decreasing sequence (A_i) in $\mathcal{A}$.

For consistency of definitions, in what follows, if $T \notin \mathcal{A}$, we define $\mathcal{F}_T = \mathcal{F}$.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two σ-algebras from $\mathcal{F}$, then σ-algebra $\mathcal{F}_1 \vee \mathcal{F}_2$ is generated by $\mathcal{F}_1 \cup \mathcal{F}_2$. Thus, if $B \subset \mathcal{A}(u)$ then $\mathcal{F}_B = \bigvee_{A \in \mathcal{A},\ A \subseteq B} \mathcal{F}_A$. Let us define the so-called strong past σ-field (this term is used in [28], in [29] it is called strong history of C)

$$\mathcal{G}_C^* = \bigvee_{B \in \mathcal{A}(u),\, B \cap C = \emptyset} \mathcal{F}_B, \quad \text{when } C \in \mathcal{C} \backslash \mathcal{A}.$$

In some cases $\mathcal{G}_C^*$ has rather simple structure, for example, if $C = \{\mathbf{x}\}$, where $\mathbf{x} \in T_{\mathbf{a}} \backslash \{\mathbf{a}\}$, then $\mathcal{G}_{\{\mathbf{x}\}}^* = \bigvee_{i=1}^d \mathcal{F}^i(x_i - 1)$. Here $\mathcal{F}^i(t) = \bigvee_{\mathbf{z} \in T_{\mathbf{a}},\ z_i \leqslant t} \mathcal{F}_{V[\mathbf{a},\mathbf{z}]}$ for $t \in \mathbb{Z}$ and $i = 1, \ldots, d$.

Definition 3. A stochastic process $Y = \{Y_A,\ A \in \mathcal{A}\}$ is a collection of random variables indexed by $\mathcal{A}$ and is said to be adapted if Y_A is $\mathcal{F}_A$ measurable, for every $A \in \mathcal{A}$. Y is said to be integrable if $E|Y_A| < \infty$ for every $A \in \mathcal{A}$.

For stochastic processes under consideration, additivity will be imposed in an almost sure sense, not path-wise. A stochastic process $Y = \{Y_A, A \in \mathcal{A}\}$ is additive (see [29]) if it has an (almost sure) additive extension to $\mathcal{C}$ (that is, if $C, C_1, C_2 \in \mathcal{C}$ with $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$ then almost surely

$$Y_{C_1} + Y_{C_2} = Y_C \tag{2.2.2}$$

and $Y_\emptyset = 0$). We recall that $\mathcal{A} \subset \mathcal{C}$.

Here is an example demonstrating the usefulness of the additivity property for a process $Y = (Y_A, A \in \mathcal{A})$. Let us take a set $(\mathbf{y}, \mathbf{x}] = \prod_{i=1}^d (y_i, x_i]$ where $\mathbf{x}, \mathbf{y} \in T_{\mathbf{a}}$. Clearly, this set is from $\mathcal{C}$ and the process Y can be presented as

$$Y_{(\mathbf{y},\mathbf{x}]} = \sum_{i=1\ldots d,\, \varepsilon_i = 0,1} (-1)^{d - \sum_{i=1}^d \varepsilon_i} Y_{V[\mathbf{a}, (y_1 + \varepsilon_1(x_1 - y_1), \ldots, y_d + \varepsilon_d(x_d - y_d))]}. \tag{2.2.3}$$

It can be easily seen that this representation satisfies additivity property (2.2.2) and the process Y is additive.

Using the additivity property sums of $\mathcal{A}$-indexed processes can be written more conveniently. Here and in what follow we denote $Y_{(\mathbf{y},\mathbf{y}]} = Y_{\{\mathbf{y}\}}$. Since $\mathcal{A}$ consists of rectangles and they can be written as finite union of disjoint elements from $\mathcal{C}$, $V[\mathbf{a}, \mathbf{x}] = \bigcup_{\mathbf{y} \in V[\mathbf{a},\mathbf{x}]} \{\mathbf{y}\}$. Using the additivity property one can write

$$Y_{V[\mathbf{a},\mathbf{x}]} = \sum_{\mathbf{y} \in V[\mathbf{a},\mathbf{x}]} Y_{\{\mathbf{y}\}}. \tag{2.2.4}$$

Moreover, from (2.2.4) it follows that for any $\mathbf{a} \leqslant \mathbf{y} \leqslant \mathbf{x}$ random variable $Y_{(\mathbf{y},\mathbf{x}]}$ given in (2.2.3) is $\mathcal{F}_{V[\mathbf{a},\mathbf{x}]}$ measurable. Now we can proceed to definitions of a lattice martingale (LMG) and a strong lattice martingale.

Definition 4. Let $Y = \{Y_A, A \in \mathcal{A}\}$ be an additive, adapted, integrable, $\mathcal{A}$-indexed process. We say that

- Y is LMG if for all A and B in $\mathcal{A}$, $A \subseteq B$ implies $E(Y_B|\mathcal{F}_A) = Y_A$;
- Y is a strong LMG if for all $C \in \mathcal{C}$ $E(Y_C|\mathcal{G}_C^*) = 0$.

In our case, it is more important to have the definition of LMG for processes indexed by points, but not by sets, therefore the following Proposition 2.2.1 is useful.

Proposition 2.2.1 ([28]). *Let $Y = \{Y_A, A \in \mathcal{A}\}$ be an $\mathcal{A}$-indexed process. If, additionally, the process Y is integrable, adapted and additive, then Y is strong LMG if and only if* $\forall \mathbf{x} \in T_\mathbf{a}$, $E(Y_\mathbf{x}|\mathcal{G}_\mathbf{x}^*) = 0$.

When C consists of points $\mathcal{C} = \{C\colon C = \{\mathbf{x}\}, \mathbf{x} \in T_\mathbf{a}\}$ then in [47], [33] strong LMG are called martingale differences. In this section martingale differences will be called strong LMG.

CLT for lattice martingales $(Y_\mathbf{x}, \mathbf{x} \in V[\mathbf{1}, \infty])$ was proved in [67]. Those martingales are strong LMG in terminology formulated above.

Let a countable indexing collection $\mathcal{A}$ be as in (2.2.1). The sequence of strong LMG can be defined as follows. Let $(D_n), n \in \mathbb{N}$, be a sequence of finite increasing sets from $\mathcal{A}$, $D_n \subset D_{n+1}$, $n \geqslant 1$, and $\mathcal{A}_n := \{A\colon A \subset \mathcal{A},\ A \in D_n\}$, $\mathcal{C}_n$ is the class defined in the same way as $\mathcal{C}$, only in the definition instead of class $\mathcal{A}(u)$ we use the class $\mathcal{A}_n(u)$ – finite union elements from $\mathcal{A}_n$. Thus, $\mathcal{C}_n$ is the class of all subsets from $T_\mathbf{a}$, having the form $A\backslash B$, where $A \in \mathcal{A}_n$, $B \in \mathcal{A}_n(u)$.

Definition 5. $(Y_A^n, \mathcal{F}_A^n\colon A \in \mathcal{A}, A \subseteq D_n, n \geqslant 1)$ is called a strong lattice martingale array if for each $n \geqslant 1$ $(\mathcal{F}_A^n)_{A \in \mathcal{A}_n}$ is a filtration (i.e. it satisfies condition

$A, B \in \mathcal{A}_n, A \subseteq B \Rightarrow \mathcal{F}_A^n \subseteq \mathcal{F}_B^n$) and $(Y_A^n)_{A \in \mathcal{A}_n}$ is a strong LMG relative to $(\mathcal{F}_A^n)_{A \in \mathcal{A}_n}$.

It is quite easy to construct a strong LMG array while having strong LMG on some set. If a process $(Y_A)_{A \in \mathcal{A}}$ is a strong LMG associated with filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$, then for all $n \geqslant 1$, $Y_A^n = n^{-d/2} Y_A$ is a strong LMG associated with filtration $\mathcal{F}_A^n = \mathcal{F}_A$ when $A \subseteq D_n$ and $\mathcal{F}_A^n = \emptyset$ when $A \nsubseteq D_n$. Thus $(Y_A^n, \mathcal{F}_A^n \colon A \in \mathcal{A}, A \subseteq D_n, n \geqslant 1)$ is a strong lattice martingale array.

In what follows, sets D_n will be taken of the form

$$D_n = \{\mathbf{x} \in \mathbb{Z}^d, \mathbf{a} \leqslant \mathbf{x} \leqslant \mathbf{k}(n)\}, \quad \mathbf{k}(n) = (k_1(n), k_2(n), \ldots, k_d(n)), \tag{2.2.5}$$

where $k_i(n)$ are non-decreasing functions of n and $\min_{1 \leqslant i \leqslant d}\{k_i(n)\} \to \infty$ as $n \to \infty$. Denote $|D_n| := k_1(n)k_2(n) \cdots k_d(n)$. Thus, sets D_n are rectangles and the lengths of sides are functions of n. More general case, where the sets $D_{\mathbf{n}}$ are indexed by $\mathbf{n} \in \mathbb{Z}^d$ will not be analyzed in this chapter. Sometimes, to emphasize the upper right corner, we use notation $D_{k_1 \ldots k_d}$, which denote rectangle with upper right corner at the point $(k_1, \ldots, k_d)$. Let us denote

$$Z_n = \sum_{\mathbf{x} \in D_n} Y_{\mathbf{x}}^n.$$

For our formulations we need the notion of stable convergence.

Definition 6. We say that a sequence of random variable (Y_n) converges to a random variable Y stably if there exists Y' with the same distribution as Y and such that $\exp(itY_n)$ weakly converge to $\exp(itY') = Z(t)$ in L^1 and $E(Z(t)\mathbf{1}_E)$, as a function of t, is continuous $\forall E \in \mathcal{F}$. Here L^1 denotes the space of random variables Y with $E|Y| < \infty$.

We say that $(\mathcal{F}_A, A \in \mathcal{A})$ satisfies conditional independence property if

$$E(E(\cdot|\mathcal{F}_A)|\mathcal{F}_B) = E(\cdot|\mathcal{F}_{A \cap B}) \quad \text{for } A, B \in \mathcal{A}. \tag{2.2.6}$$

In [26], [44] the CLT for martingale differences in one-dimensional case is considered. The results were extended to multidimensional case in [47], [67]. As it was noted above, martingale differences used in [67] are strong LMG, therefore the result from [67] can be formulated as follows.

Theorem 2.2.2. *([67]) Let $(Y_A^n, \mathcal{F}_A^n\colon A \in \mathcal{A}, A \subseteq D_n, n \geqslant 1)$ be a strong LMG array and let η^2 be some bounded random variable. Suppose that the following conditions are satisfied*

$$\max_{\mathbf{x}\in D_n} |Y_{\mathbf{x}}^n| \xrightarrow[n\to\infty]{\mathrm{P}} 0, \tag{2.2.7}$$

$$\sum_{\mathbf{x}\in D_n} (Y_{\mathbf{x}}^n)^2 \xrightarrow[n\to\infty]{\mathrm{P}} \eta^2, \tag{2.2.8}$$

$$E\Big(\max_{\mathbf{x}\in D_n} (Y_{\mathbf{x}}^n)^2\Big) \text{ is bounded,} \tag{2.2.9}$$

$$\forall n \geqslant 1\ \forall A \in \mathcal{A}_n, \quad \mathcal{F}_A^n \subseteq \mathcal{F}_A^{n+1}. \tag{2.2.10}$$

Then $Z_n \xrightarrow[n\to\infty]{\mathcal{L}} Z$ stably where the random variable Z has the characteristic function $E(\exp(-\frac{1}{2}t^2\eta^2))$.

Similar result (only using different notations) was proved in [47].

Checking of conditions (2.2.7), (2.2.9) is not an easy task. Therefore in [28] the conditional Lindeberg condition was introduced. Let us denote by $\mathcal{G}_{\mathbf{x}}^{n*}$ a sequence of strong past σ-fields $(\mathcal{G}_{\mathbf{x}}^n)^*$ of $\mathbf{x}$ associated with the filtration $(\mathcal{F}_A^n)_{A\in\mathcal{A}_n}$. Introduce the following conditions: for all $\mathbf{x} \in \mathcal{C}_n$ and for some bounded random variable η^2

$$(V_{D_n}^n)^2 := \sum_{\mathbf{x}\in D_n} E((Y_{\mathbf{x}}^n)^2 | \mathcal{G}_{\mathbf{x}}^{n*}) \xrightarrow[n\to\infty]{\mathrm{P}} \eta^2, \tag{2.2.11}$$

$$\forall \varepsilon > 0, \quad \sum_{\mathbf{x}\in D_n} E((Y_{\mathbf{x}}^n)^2 1_{\{|Y_{\mathbf{x}}^n|>\varepsilon\}} | \mathcal{G}_{\mathbf{x}}^{n*}) \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{2.2.12}$$

(2.2.12) is called conditional Lindeberg condition. Conditions (2.2.7), (2.2.8), (2.2.9) of Theorem 2.2.2 can be changed by (2.2.11) and (2.2.12) and the following theorem was proved in [28].

Theorem 2.2.3 ([28])**.** *Let $(Y_A^n, \mathcal{F}_A^n\colon A \in \mathcal{A}, A \subseteq D_n, n \geqslant 1)$ be a strong LMG array. Assume that (2.2.10) holds and that for all $n \geqslant 1$ $(\mathcal{F}_A)_{A\in\mathcal{A}}$ satisfies conditional independence property. Under assumption (2.2.11), (2.2.12) we* *have* $Z_n \xrightarrow[n\to\infty]{\mathcal{L}} Z$ *stably where the random variable* Z *has the characteristic function* $E(\exp(-\frac{1}{2}t^2\eta^2))$.

2.2.2 Results and proofs

Now we are able to formulate our result for linear fields. Denote

$$S_n = \frac{1}{\sqrt{|D_n|}} \sum_{\mathbf{t}\in D_n} X_{\mathbf{t}}. \tag{2.2.13}$$

Theorem 2.2.4. *Let* $\{X_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^2\}$ *be a linear random field defined in* (2.1.1), *coefficients* $\varphi_{\mathbf{k}}$ *satisfy* $\mathcal{L}_{2,2}$. *Suppose that* $\{\varepsilon_A, A \in \mathcal{A}\}$, *where* $\varepsilon_A = \sum_{\mathbf{t}\in A}\varepsilon_{\mathbf{t}}$, *is a strong LMG,* $E\varepsilon_{\mathbf{t}}^2 = 1$ *and the following conditions are satisfied:*

$$\sum_{\mathbf{x}\in D_n} E\left(\frac{\varepsilon_{\mathbf{x}}^2}{|D_n|} \,\middle|\, \mathcal{G}_{\mathbf{x}}^{n*}\right) \xrightarrow[n\to\infty]{\mathrm{P}} \eta^2, \tag{2.2.14}$$

$$\forall\varepsilon > 0, \sum_{\mathbf{x}\in D_n} E\left(\frac{\varepsilon_{\mathbf{x}}^2}{|D_n|}\mathbf{1}_{\{|\varepsilon_{\mathbf{x}}^n|>\varepsilon\}} \,\middle|\, \mathcal{G}_{\mathbf{x}}^{n*}\right) \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{2.2.15}$$

Here for $\mathbf{x} \in \mathcal{C}_n$ $\mathcal{G}_{\mathbf{x}}^{n*}$ *is a strong past* σ*-field of* $\mathbf{x}$ *associated with the filtration* $(\mathcal{F}_A^n)_{A\in\mathcal{A}_n}$ *and* η^2 *is some bounded random variable. Also assume that* $\forall n \geqslant 1$ $(\mathcal{F}_A^n)_{A\in\mathcal{A}_n}$ *satisfies the conditional independence property* (2.2.6) *and* $\forall n \geqslant 1, \forall A \in \mathcal{A}_n$, $\mathcal{F}_A^n \subseteq \mathcal{F}_A^{n+1}$. *Then* $S_n \xrightarrow[n\to\infty]{\mathcal{L}} S$ *stably where the random variable* S *has the characteristic function* $E(\exp(-\frac{1}{2}t^2\eta^2\mu_1^2))$.

Proof. The main tool in the proof is the BN decomposition explained in Section 1.3.1. Using this decomposition we have

$$\sum_{\mathbf{t}\in D_n} X_{\mathbf{t}} = \mu_1 Z_n + R_n, \tag{2.2.16}$$

where μ_1 is defined in (1.3.2),

$$Z_n = \sum_{\mathbf{t}\in D_n} \varepsilon_{\mathbf{t}} \tag{2.2.17}$$

and R_n has the form as in (1.3.5)

$$\begin{aligned} R_n &= \xi_{k_1(n),k_2(n)} - \xi_{k_1(n),0} - \xi_{0,k_2(n)} + \xi_{0,0} \\ &\quad + \eta_{k_1(n),k_2(n)} - \eta_{0,k_2(n)} + \zeta_{k_1(n),k_2(n)} - \zeta_{k_1(n),0}. \end{aligned}$$

We use normalization by $|D_n|^{-1/2}$ with $|D_n| = k_1(n)k_n(n)$ and get new variables $\bar{\varepsilon}_{\mathbf{x}}^n = |D_n|^{-1/2}\varepsilon_{\mathbf{x}}$. Thus, from (2.2.16) we have

$$S_n = \frac{1}{\sqrt{|D_n|}}\sum_{\mathbf{t}\in D_n} X_{\mathbf{t}} = \mu_1\bar{Z}_n + \bar{R}_n, \quad \bar{Z}_n = \frac{1}{\sqrt{|D_n|}}Z_n, \; \bar{R}_n = \frac{1}{\sqrt{|D_n|}}R_n.$$

The proof of the theorem consists of two steps: (i) the proof of CLT for $\bar{Z}_n$, (ii) the proof that $\bar{R}_n \to 0$.

(i) The CLT result for $\bar{Z}_n$ will be proved by using Theorem 2.2.3. Let us consider the sum $\bar{Z}_n = \sum_{\mathbf{x}\in D_n}\bar{\varepsilon}_{\mathbf{x}}^n$ and check the assumptions of Theorem 2.2.3. Conditions (2.2.14) and (2.2.15) ensure that (2.2.11) and (2.2.12) are satisfied. Note that, if $\{\varepsilon_A, A\in\mathcal{A}\}$ is a strong LMG then the normalized process formed by variables $\bar{\varepsilon}_{\mathbf{x}}^n$ will also have the same property with the filtration defined as follows: $\mathcal{F}_{V[\mathbf{a},\mathbf{x}]}^n = \mathcal{F}_{V[\mathbf{a},\mathbf{x}]}$, for all $V[\mathbf{a},\mathbf{x}]\subset D_n, V[\mathbf{a},\mathbf{x}]\in\mathcal{A}$. All conditions of Theorem 2.2.3 are satisfied, therefore we get the CLT for $\bar{Z}_n$.

(ii) Next step is to show that $|D_n|^{-1/2}R_n \xrightarrow{\mathrm{P}} 0$. We shall prove this relation separately for each term $\xi_{\mathbf{t}}$, $\eta_{\mathbf{t}}$, $\zeta_{\mathbf{t}}$ with $\mathbf{t} = (t_1, t_2)$ and $t_1 \in \{0, k_1(n)\}$, $t_2 \in \{0, k_2(n)\}$.

From the definition of ξ_{t_1,t_2} using Chebyshev inequality we can write

$$P\left(\Big|\sum_{s_1,s_2\geqslant 0}\varphi^*_{s_1,s_2}(\bar{\varepsilon}^n_{t_1-s_1,t_2-s_2})\Big| \geqslant \varepsilon\right) \leqslant \frac{1}{\varepsilon^2}E\left(\sum_{s_1,s_2\geqslant 0}\varphi^*_{s_1,s_2}\frac{\varepsilon_{t_1-s_1,t_2-s_2}}{\sqrt{|D_n|}}\right)^2$$
$$\leqslant \frac{1}{\varepsilon^2|D_n|}\sum_{s_1,s_2\geqslant 0}\varphi^{*2}_{s_1,s_2}E\varepsilon^2_{t_1-s_1,t_2-s_2} \leqslant \frac{1}{\varepsilon^2|D_n|}E\varepsilon^2_{t_1,t_2}\sum_{s_1,s_2\geqslant 0}\varphi^{*2}_{s_1,s_2} \to 0,$$

as $n\to\infty$, $|D_n|\to\infty$. In the second inequality mixed products disappear due to the strong LMG property. Namely, we use

$$E(\varepsilon_{\mathbf{t-1}}\varepsilon_{\mathbf{t}}) = E(\varepsilon_{\mathbf{t-1}}E(\varepsilon_{\mathbf{t}}|\mathcal{G}^*_{\mathbf{t}})).$$

Let $\varepsilon_{\mathbf{t}}$ be a strong LMG, $A = \{\mathbf{t}, \mathbf{t}' \in \mathrm{Z}^2 \colon t_1 \neq t_1', \, t_2 \neq t_2'\}$. Consider sums

$$J = E\sum_A c_{\mathbf{t}}c_{\mathbf{t}'}\varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{t}'},$$

where $c_{\mathbf{t}}$ is some constant which depends on $\mathbf{t}$.

Take σ-algebras $\mathcal{G}^*_{\mathbf{t}} = \mathcal{F}^1(t_1-1) \vee \mathcal{F}^2(t_2-1)$ and sets

$$A_1 = \{\mathbf{t}, \mathbf{t}' \in \mathrm{Z}^2 \colon \{t_1 > t_1'\} \cup \{t_1 = t_1', t_2' < t_2\}\},$$

$$A_2 = \{\mathbf{t}, \mathbf{t}' \in \mathrm{Z}^2 \colon \{t_1 < t_1'\} \cup \{t_1 = t_1', t_2' > t_2\}\},$$

$$A = A_1 \cup A_2, \qquad A_1 \cap A_2 = \emptyset,$$

then

$$J = J_1 + J_2,$$

where

$$J_1 = \sum_{A_1} c_{\mathbf{t}} c_{\mathbf{t}'} E(\varepsilon_{\mathbf{t}'} E(\varepsilon_{\mathbf{t}} | \mathcal{G}^*_{\mathbf{t}})), \quad J_2 = \sum_{A_2} c_{\mathbf{t}} c_{\mathbf{t}'} E(\varepsilon_{\mathbf{t}} E(\varepsilon_{\mathbf{t}'} | \mathcal{G}^*_{\mathbf{t}'})).$$

Actually there are 8 different relations between indexes t_1', t_2', t_1, t_2, but it is easy to see that only two different σ-algebra $\mathcal{G}^*_{\mathbf{t}'}$, $\mathcal{G}^*_{\mathbf{t}}$ are needed. After applying Proposition 2.2.1 we get that all terms in J_1 and J_2 are equal 0 and $J = 0$.

Now let us consider $\eta_{\mathbf{t}}$ with $t_1 \in \{0, k_1(n)\}$, $t_2 = k_2(n)$.

$$P\left(\left|\sum_{t_2=1}^{k_2(n)} \sum_{s_1 \geqslant 0} b_{s_1} \bar{\varepsilon}^n_{t_1-s_1,t_2}\right| \geqslant \varepsilon\right) \leqslant \frac{1}{\varepsilon^2 |D_n|} E\left(\sum_{t_2=1}^{k_2(n)} \sum_{s_1 \geqslant 0} b_{s_1} \varepsilon_{t_1-s_1,t_2}\right)^2$$
$$= \frac{1}{\varepsilon^2 |D_n|} \sum_{s_1 \geqslant 0} b_{s_1}^2 \sum_{t_2=1}^{k_2(n)} E\varepsilon^2_{t_1-s_1,t_2} \leqslant \frac{1}{\varepsilon^2 k_1(n)} E\varepsilon^2_{t_1,k_2(n)} \sum_{s_1 \geqslant 0} b_{s_1}^2 \to 0,$$

when $n \to \infty$. Mixed products disappear because of strong LMG properties by using the same consideration as above.

The proof for $\zeta_{\mathbf{t}}$ is the same as for $\eta_{\mathbf{t}}$.

Thus, we get that $\bar{R}_{\mathbf{n}} \xrightarrow{\mathrm{P}} 0$, and the proof of the theorem is complete.

It was shown in [28] that in the case of strong LMG the conditions (2.2.11), (2.2.12) imply (2.2.7), (2.2.8), (2.2.9), therefore the following result can be formulated:

Theorem 2.2.5. *Suppose that a random field $\{X_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^2\}$ is of the form (2.1.1), coefficients $\varphi_{\mathbf{k}}$ satisfy $\mathcal{L}_{2,2}$, $\{\varepsilon_A, A \in \mathcal{A}\}$, and $\varepsilon_A = \sum_{\mathbf{t} \in A} \varepsilon_{\mathbf{t}}$ are strong LMG,*

$E\varepsilon_{\mathbf{t}}^2 = 1$ *and the following conditions are satisfied:*

$$\max_{\mathbf{x}\in D_n} \frac{|\varepsilon_{\mathbf{x}}|}{|D_n|^{\frac{1}{2}}} \xrightarrow[n\to\infty]{\text{P}} 0,$$

$$\sum_{\mathbf{x}\in D_n} \frac{\varepsilon_{\mathbf{x}}^2}{|D_n|} \xrightarrow[n\to\infty]{\text{P}} \eta^2,$$

$$E\left(\max_{\mathbf{x}\in D_n} \frac{\varepsilon_{\mathbf{x}}^2}{|D_n|}\right) \text{ is bounded,}$$

η^2 *is some bounded random variable. Additionally, for* $\forall n \geqslant 1$, $(\mathcal{F}_A^n)_{A\in\mathcal{A}_n}$ *conditional independence condition* (2.2.6) *is satisfied and* $\forall n \geqslant 1, \forall A \in \mathcal{A}_n, \quad \mathcal{F}_A^n \subseteq \mathcal{F}_A^{n+1}$. *Then* $S_n \xrightarrow[n\to\infty]{\mathcal{L}} S$ *stably where the random variable* S *has the characteristic function* $E(\exp(-\frac{1}{2}t^2\eta^2\mu_1^2))$.

Proof. The proof is similar to the proof of Theorem 2.2.4. Recalling the direct application of BN decomposition and constructing the sequence $\bar{\varepsilon}_{\mathbf{x}}^n$ as in the proof of Theorem 2.2.4, we apply Theorem 2.2.2 and obtain that CLT is valid for $\bar{Z}_n$. The expression of residual R_n does not depend on additional assumptions for $\bar{\varepsilon}_{\mathbf{x}}$. Since they are martingale differences we apply the proof of Theorem 2.2.4.

Remark 2.2.1. As it was noted in the Introduction, using direct method of application of BN decomposition in the proof of the CLT we need stronger conditions on the coefficients $\{\varphi_{\mathbf{k}}\}$ comparing with indirect method of BN decomposition. One can check that applying indirect method and Lemma 2.3.4 one can prove Theorems 2.2.4 and 2.2.5 under weaker condition $\mathcal{L}_{0,1}$ instead of $\mathcal{L}_{2,2}$. Thus essentially the results of Section 3 demonstrate that like in the case of i.i.d. innovations the indirect method of application of BN decomposition is better in the proof of the CLT.

2.3 CLT for linear random fields with martingale differences defined in [50]

Another approach to define martingale differences on the plain was proposed in [49], [50]. Nahapetian and Petrosian [50] derived an asymptotic normality result for martingales on the lattice $\mathbb{Z}^d$ directly from the CLT in Dvoretsky [20] for discrete time martingales.

2.3.1 Definitions and auxiliary results

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{W}$ be a space of all finite subsets of $\mathbb{Z}^d$, $d \geqslant 1$. By S_V, $V \in \mathcal{W}$, we denote random variables and $\mathcal{F}_V$, $V \in \mathcal{W}$, stands for a set of σ-algebra with the partial order:

$$\mathcal{F}_V \subset \mathcal{F}, \quad V, \bar{V} \in \mathcal{W}; \bar{V} \subset V \quad \Rightarrow \quad \mathcal{F}_{\bar{V}} \subset \mathcal{F}_V, \mathcal{F}_\emptyset = \{\emptyset, \Omega\}.$$

If for all $V \in \mathcal{W}$ random variables S_V are $\mathcal{F}_V$ measurable, then the family $S = (S_V, \mathcal{F}_V), V \in \mathcal{W}$, is called a stochastic family.

Definition 7. A stochastic family $S = (S_V, \mathcal{F}_V)$ is called a martingale if for any $\bar{V}, V \in \mathcal{W}$, $\bar{V} \subset V$, the following relations hold

$$E|S_V| < \infty \quad \text{and} \quad E(S_V|\mathcal{F}_{\bar{V}}) = S_{\bar{V}} \text{ a.s.} \tag{2.3.1}$$

A special case of a martingale is a martingale differences field.

Definition 8. A random field $\xi_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$, is called a martingale difference random field if for all $\mathbf{t} \in \mathbb{Z}^d$

$$E|\xi_{\mathbf{t}}| < \infty \quad \text{and} \quad E(\xi_{\mathbf{t}}|\xi_{\mathbf{s}}, \mathbf{s} \in \mathbb{Z}^d \backslash \{\mathbf{t}\}) = 0 \text{ a.s.}$$

An example of martingale difference random field is given in Example 2.

Example 2. Let X be a finite subset of $\mathbb{R}^1$, $X = \bigcup_{i=1}^N X_i$, $X_i \cap X_j = \emptyset$, when $i \neq j$ and $\sum_{x \in X_i} x = 0$, $i = 1, 2, 3, \ldots, N$. Consider a random field $\{\xi_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$, such that conditional probability

$$q_{\mathbf{t}}^{\bar{x}}(x) = P(\xi_{\mathbf{t}} = x \mid \xi_{\mathbf{s}} = \bar{x}_{\mathbf{s}}, \mathbf{s} \in \mathbb{Z}^d \backslash \{\mathbf{t}\}),$$
$$\mathbf{t} \in \mathbb{Z}^d, \quad x \in X, \quad \bar{x} = (\bar{x}_{\mathbf{s}}, \mathbf{s} \in \mathbb{Z}^d \backslash \{\mathbf{t}\})$$

takes the constant value $q_{\mathbf{t},j}^{\bar{x}}$, when $x \in X_j$, $j = 1, \ldots, N$, i.e.

$$q_{\mathbf{t}}^{\bar{x}}(x) = q_{\mathbf{t},j}^{\bar{x}}, \quad x \in X_j,\ j = 1, \ldots, N.$$

This random field $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ is a martingale difference random field, because

$$E(\xi_{\mathbf{t}} | \bar{x}) = \sum_{x \in X} x q_{\mathbf{t}}^{\bar{x}}(x) = \sum_{i=1}^N \sum_{x \in X_i} x q_{\mathbf{t}}^{\bar{x}}(x) = \sum_{i=1}^N q_{\mathbf{t},i}^{\bar{x}}(x) \sum_{x \in X_i} x = 0.$$

More general definition of martingale difference field can be given.

Definition 9. A random field $\{\xi_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ is called a martingale difference random field with respect to some partially ordered set of σ-algebras $\{\mathcal{F}_V,\ V \in \mathcal{U}\}, \mathcal{U} \subset \mathcal{W}$, if

$$E|\xi_{\mathbf{t}}| < \infty \quad \text{and} \quad \text{for all } V \in \mathcal{U}, \mathbf{t} \notin V \ \ E(\xi_{\mathbf{t}} | \mathcal{F}_V) = 0 \text{ a.s.}$$

Remark 2.3.1. We have defined a strong LMG as a function indexed by the sets in Section 2.2. In this section martingale differences are random variables indexed by points. If we take $V[\mathbf{a}, \mathbf{x}] \subset \mathcal{A}$ and strong LMG $Y_{V[\mathbf{a},\mathbf{x}]}$, from (2.2.4), $Y_{\mathbf{x}}$ where $\mathbf{x} \in C$ satisfy condition $E(Y_{\mathbf{x}} | \mathcal{G}_{\mathbf{x}}^*) = 0$. According to Proposition 2.2.1 if $\{\varepsilon_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ are martingale differences as defined in his section or i.i.d., then

$$Y_{V[\mathbf{a},\mathbf{x}]} = \sum_{\mathbf{t} \in V[\mathbf{a},\mathbf{x}]} \varepsilon_{\mathbf{t}}, \tag{2.3.2}$$

is a strong LMG.

Proposition 2.3.1 ([50])**.** *Let $\{\xi_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ be a random field, $E|\xi_{\mathbf{t}}| < \infty$, $\mathbf{t} \in \mathbb{Z}^d$ and let $S_V = \sum_{\mathbf{t} \in V} \xi_{\mathbf{t}}$, $V \in \mathcal{W}$. If a stochastic family $(S_V, \sigma(\xi_{\mathbf{t}}, \mathbf{t} \in V))$, $V \in \mathcal{W}$, forms a martingale then the random field $\{\xi_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ is a martingale difference random field. If a random field $\{\xi_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ is a martingale difference random field then the stochastic family $(S_V, \sigma(\xi_{\mathbf{t}}, \mathbf{t} \in V))$, $V \in \mathcal{W}$, represents a martingale.*

Let $\Omega = \{(x_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d)\}$ be a set of all functions defined on $\mathbb{Z}^d$ and $x_{\mathbf{t}} \in X$, $X \subseteq \mathbb{R}^1$. $\mathcal{B}$ is σ-algebra generated by the cylindrical subsets of Ω. For any $V \in \mathcal{W}$ denote by $\mathcal{B}^V$ the σ-algebra of cylindrical subsets with the base X^V.

Denote by $\tau_{\mathbf{h}}, \mathbf{h} \in \mathbb{Z}^d$ the group of translations,

$$(\tau_{\mathbf{h}} x)_{\mathbf{t}} = x_{\mathbf{t}+\mathbf{h}}, \quad \mathbf{h} \in \mathbb{Z}^d, \ x \in \Omega.$$

Let $\mathcal{T}$ be σ-algebra of invariant Ω subsets: $\mathcal{T} = \{A \in \mathcal{B} \colon \tau_{\mathbf{h}} A = A\}$.

A random field $\{\xi_{\mathbf{t}}, \ \mathbf{t} \in \mathbb{Z}^d\}$ with distribution P is called a stationary if for any $A \in \mathcal{T}$ and $\mathbf{h} \in \mathbb{Z}^d$

$$P(\tau_{\mathbf{h}} A) = P(A).$$

(In [49] such field is called homogeneous.)

A random field $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ is ergodic, if $\forall A \in \mathcal{T}, P(A) = 0$ or $P(A) = 1$.

We say that the CLT holds for a random field $\{X_{\mathbf{t}}, \ \mathbf{t} \in \mathbb{Z}^d, E|X_{\mathbf{t}}|^2 < \infty\}$ if

$$\lim_{n\to\infty} P\left(\frac{S_{D_n} - ES_{D_n}}{(\mathrm{Var}(S_{D_n}))^{\frac{1}{2}}} < z\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{u^2}{2}} \mathrm{d}u, \tag{2.3.3}$$

where

$$S_{D_n} = \sum_{\mathbf{t} \in D_n} X_{\mathbf{t}}, \tag{2.3.4}$$

and D_n is defined in (2.2.5).

The next theorem is proved in [50].

Theorem 2.3.2. *Let $\{\xi_{\mathbf{t}}, \ \mathbf{t} \in \mathbb{Z}^d\}$ be a stationary, ergodic martingale difference random field and $0 < \sigma^2 = E\xi_0^2 < \infty$, then CLT for this field holds.*

Remark 2.3.2. In [50] CLT is proved in the case where sets D_n are squares with the side of length n. After inspection of the proof one can see that the result is true for rectangles as defined in (2.2.5).

2.3.2 Results and proofs

Using Theorem 2.3.2 and Remark 2.3.2 the following theorem for linear field will be proved.

Theorem 2.3.3. *Suppose that a random field is of the form* (2.1.1) *with* $d = 2$ *and* $\{\varepsilon_{\mathbf{t}}\}$ *form a stationary ergodic martingale difference random field and* $0 < \sigma^2 = E\varepsilon_{\mathbf{0}}^2 < \infty$. *If coefficients* $\varphi_{\mathbf{k}}$ *satisfy* $\mathcal{L}_{0,1}$ *then the CLT for the linear random field* (2.1.1) *is valid.*

Remark 2.3.3. One could ask a question if a linear field (2.1.1) with increments $\{\varepsilon_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^2\}$ which are martingale differences is a martingale? The answer is no, and it can be seen from the relation

$$\begin{aligned} X_{t+1,s+1} &= \sum_{k,l\geqslant 0} \varphi_{k,l}\varepsilon_{t+1-k,s+1-l} = \sum_{k,l\geqslant 1} \varphi_{k,l}\varepsilon_{t+1-k,s+1-l} + \varphi_{0,0}\varepsilon_{t+1,s+1} \\ &\quad + \sum_{k=0,l>0} \varphi_{k,l}\varepsilon_{t+1,s+1-l} + \sum_{k>0,l=0} \varphi_{k,l}\varepsilon_{t+1-k,s+1}. \end{aligned} \tag{2.3.5}$$

The most important is the first sum, since the other three terms after applying conditional mean will be equal to 0. Therefore,

$$E(X_{t+1,s+1}|\mathcal{F}_{V_{t,s}}) = \sum_{k,l\geqslant 1} \varphi_{k,l}\varepsilon_{t+1-k,s+1-l} = \sum_{i,j\geqslant 0} \varphi_{j+1,i+l}\varepsilon_{t-j,s-i} \neq X_{t,s},$$

where $V_{t,s} = \{(i,j)\colon (i,j) \in \mathrm{Z}^2, i \leqslant t, j \leqslant s\}$.

The following lemma will be used in the proof of the theorem. For $\mathbf{n} \in \mathbb{Z}^2$ denote $|\mathbf{n}| = n_1 n_2$.

Lemma 2.3.4 ([51],[55])**.** *Let* $\{b_{\mathbf{j}}, \mathbf{j} \in \mathbb{Z}^2\}$ *be real numbers such that*

$$\sum_{\mathbf{j}\in\mathbb{Z}^2} |b_{\mathbf{j}}| < \infty, \quad \sum_{\mathbf{j}\in\mathbb{Z}^2} b_{\mathbf{j}} = 0.$$

Then for any $1 < p \leqslant 2$

$$\frac{1}{|\mathbf{n}|}\sum_{\mathbf{j}\in\mathbb{Z}^2}\Big|\sum_{\mathbf{1}-\mathbf{j}\leqslant\mathbf{i}\leqslant\mathbf{n}-\mathbf{j}} b_{\mathbf{i}}\Big|^p \to 0, \quad \textit{when} \ \min\{n_1, n_2\} \to \infty.$$

Remark 2.3.4. In [15] was shown that requirement in Lemma 2.3.4 for p, $p \leqslant 2$ is superfluous. We need only $p > 1$.

Proof of Theorem 2.3.3. The main tool in the proof, once again, will be BN decomposition, that is, we use formula (2.2.16). Only now the so-called indirect

BN decomposition will be used, when instead of using explicit expression of R_n we will use only the fact that $\mu_1 Z_n$ (μ_1 and Z_n as in (2.2.17)) well approximate S_{D_n} from (2.3.4).

The requirements for $\{\varepsilon_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^2\}$ are the same as in Theorem 2.3.2, therefore after proper normalization the CLT is valid for Z_n. Let us define:

$$\omega_n = S_{D_n} - \mu_1 Z_n.$$

It is necessary to show that

$$\frac{\omega_n}{\mu_1 \sigma \sqrt{|D_n|}} \xrightarrow{\mathrm{P}} 0$$

and this will follow from the relation

$$P(\omega_n > \varepsilon \mu_1 \sigma \sqrt{|D_n|}) \leqslant \frac{1}{\varepsilon^2 \mu_1^2 \sigma^2 |D_n|} E\omega_n^2 \to 0. \tag{2.3.6}$$

For this purpose Lemma 2.3.4 will be used. In order to apply it, we set $\varphi_{k,l} = 0$ if $k < 0$ or $l < 0$. Denote $b_{0,0} = \varphi_{0,0} - \mu_1$, $b_{i,j} = \varphi_{i,j}$ when $(i,j) \neq (0,0)$. Then

$$S_{D_n} = \sum_{t,s \in D_n} \sum_{(i,j) \in \mathbb{Z}^2} \varphi_{i,j} \varepsilon_{t-i,s-j} = \sum_{(k,l) \in \mathbb{Z}^2} \left(\sum_{t,s \in D_n} \varphi_{t-k,s-l} \right) \varepsilon_{k,l},$$

and

$$\begin{aligned}
\omega_n &= S_{D_n} - \mu_1 Z_n = \sum_{(k,l) \in \mathbb{Z}^2} \left(\sum_{(t,s) \in D_n} \varphi_{t-k,s-l} \right) \varepsilon_{k,l} - \mu_1 \sum_{(k,l) \in D_n} \varepsilon_{k,l} \\
&= \sum_{(k,l) \in D_n} \left(\sum_{(t,s) \in D_n} \varphi_{t-k,s-l} - \mu_1 \right) \varepsilon_{k,l} + \sum_{(k,l) \notin D_n} \left(\sum_{(t,s) \in D_n} \varphi_{t-k,s-l} \right) \varepsilon_{k,l} \\
&= \sum_{(k,l) \in \mathbb{Z}^2} \left(\sum_{(t,s) \in D_n} b_{t-k,s-l} \right) \varepsilon_{k,l}.
\end{aligned} \tag{2.3.7}$$

Taking into account the definition of D_n in (2.2.5) we get

$$\omega_n = \sum_{(i,j) \in \mathbb{Z}^2} \left(\sum_{t=1}^{k_1(n)} \sum_{s=1}^{k_2(n)} b_{t-i,s-j} \right) \varepsilon_{i,j}, \tag{2.3.8}$$

Coefficients $b_{i,j}$ satisfy conditions of Lemma 2.3.4.

Now we evaluate $E\omega_n^2$:

$$\begin{aligned} E\omega_n^2 &= E\Big(\sum_{(k,l)\in\mathbb{Z}^2}\Big(\sum_{(t,s)\in D_n} b_{t-k,s-l}\Big)\varepsilon_{k,l}\Big)^2 \leqslant \sum_{(k,l)\in\mathbb{Z}^2}\Big(\sum_{(t,s)\in D_n} b_{t-k,s-l}\Big)^2 E\varepsilon_{k,l}^2 \\ &+ \sum_{\substack{(k,l)\in\mathbb{Z}^2,\,(p,r)\in\mathbb{Z}^2,\\ p\neq k, l\neq r}}\Big(\sum_{(t,s)\in D_n} b_{t-k,s-l}\Big)\Big(\sum_{(i,j)\in D_n} b_{i-p,j-r}\Big)E\varepsilon_{k,l}\varepsilon_{p,r}. \quad (2.3.9)\end{aligned}$$

At first we show, that by using martingale difference properties mixed products are equal to 0.

The main idea is to take σ-algebra $\mathcal{F}_{D_n}$ or, more precisely, the regions D_n in such a way that one multiplier of the product $\varepsilon_{k,l}\varepsilon_{p,r}$ would be measurable with respect to this σ-algebra and for the another one properties of martingale differences could be applied. For example $E(\varepsilon_{k,l}\varepsilon_{p,r}) = E(E(\varepsilon_{k,l}\varepsilon_{p,r}|\mathcal{F}_{D_{p-1,r-1}})) = E(\varepsilon_{k,l}E(\varepsilon_{p,r}|\mathcal{F}_{D_{p-1,r-1}})) = 0$, when $k<p, l<r$, where $D_{i,j}$ is the rectangle with the upper right corner at (i,j). Denote $c_{p,r}^n := \sum_{i,j\in D_n} b_{i-p,j-r}$.

Consider the sum

$$I = \sum_A c_{k,l}^n c_{p,r}^n \varepsilon_{k,l}\varepsilon_{p,r}, \quad (2.3.10)$$

where $A = \{(k,l),(p,r)\in\mathbb{Z}^2 \colon p\neq k, l\neq r\}$. We decompose this sum $I=\sum_{i=1}^8 I_i$, where $I_i = \sum_{A_i} c_{k,l}^n c_{p,r}^n \varepsilon_{k,l}\varepsilon_{p,r}$, $i=1,\ldots,8$, over sets A_i which do not overlap:

$$\begin{aligned} A_1 &= \{(k,l),(p,r)\in\mathbb{Z}^2, k>p, l>r\}, \quad A_2 = \{(k,l),(p,r)\in\mathbb{Z}^2, k<p, l>r\},\\ A_3 &= \{(k,l),(p,r)\in\mathbb{Z}^2, k<p, l<r\}, \quad A_4 = \{(k,l),(p,r)\in\mathbb{Z}^2, k<p, l=r\},\\ A_5 &= \{(k,l),(p,r)\in\mathbb{Z}^2, k=p, l>r\}, \quad A_6 = \{(k,l),(p,r)\in\mathbb{Z}^2, k=p, l<r\},\\ A_7 &= \{(k,l),(p,r)\in\mathbb{Z}^2, k>p, l<r\}, \quad A_8 = \{(k,l),(p,r)\in\mathbb{Z}^2, k>p, l=r\}.\end{aligned}$$

The decomposition into sets $A_i, i=1,\ldots,8$, was caused by different possible positions between two points in the plain, which are illustrated in Figure 2.1. For different regions A_i we choose different regions $D^{(i)}$ for corresponding σ-algebras.

$$\begin{gathered} D^{(1)} = D_{k-1,l-1}, D^{(2)} = D_{p,l-1}, D^{(3)} = D_{p-1,r-1}, D^{(4)} = D_{p-1,r},\\ D^{(5)} = D_{k,l-1}, D^{(6)} = D_{p,r-1}, D^{(7)} = D_{k,r-1}, D^{(8)} = D_{k-1,l}.\end{gathered}$$

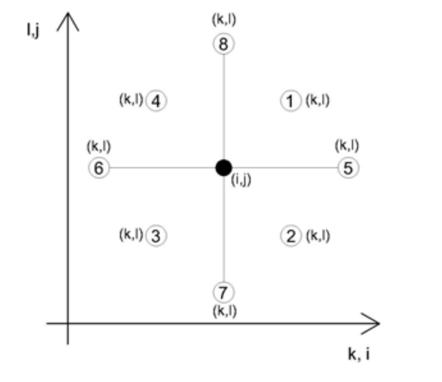

Figure 2.1: Possible different positions between two points in the plain.

Such σ-algebras guarantee that after taking conditional expectation mixed products are equal 0.

If $\{\xi_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^2\}$ is a martingale difference field with respect to the increasing sequence of subsets

$$D_n = \{(i,j), 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n\},$$

then it will be a martingale difference random field with respect to the sequences of subsets

$$D'_n = \{(i,j), 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n\} \text{ and } D''_n = \{(i,j), 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1\}.$$

Arguments presented above allow to make conclusion that mixed product are equal to 0 and

$$\begin{aligned} E\omega_n^2 &= E\bigg(\sum_{(k,l)\in\mathbb{Z}^2}\Big(\sum_{t,s\in D_n} b_{t-k,s-l}\Big)^2 \varepsilon_{k,l}^2\bigg) = \sum_{(k,l)\in\mathbb{Z}^2}\Big(\sum_{t,s\in D_n} b_{t-k,s-l}\Big)^2 E\varepsilon_{k,l}^2 \\ &= E\varepsilon_{0,0}^2\bigg(\sum_{(k,l)\in\mathbb{Z}^2}\Big(\sum_{t,s\in D_n} b_{t-k,s-l}\Big)^2\bigg). \end{aligned} \tag{2.3.11}$$

In the last equality the stationarity of the random field is used: $0 < \sigma^2 = E\varepsilon_{0,0}^2 < \infty$. Applying Lemma 2.3.4 with $d = 2$ we get

$$\frac{E\omega_n^2}{|D_n|} \to 0$$

and (2.3.6) is proved. The theorem is proved.

Now we consider a linear random field with non-stationary increments. We need the notion of strong mixing for random fields .

Definition 10. Let $\{\varepsilon_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ be a random field with distribution P. It satisfies strong mixing condition if

$$\sup\{|P(AB) - P(A)P(B)|, A \in \mathcal{B}^I, B \in \mathcal{B}^V, |I| < n, |V| < m\}$$
$$\leqslant \alpha_{m,n}(d(I,V)) \quad \text{for all } I \in \mathcal{W}, V \in \mathcal{W},\ m,n \in \mathrm{N}.$$

and $\alpha_{m,n}(d) \to 0$ when $d \to \infty$, m, n fixed. Here

$$d(I,V) = \inf_{t \in I, s \in V}\{|t-s|\}, \quad |t| = \max_{1 \leqslant i \leqslant d}\{|t^{(i)}|\}.$$

In [49] CLT for strong mixing martingale difference random field was formulated.

Theorem 2.3.5 ([49])**.** *Let $\{\varepsilon_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ is a martingale difference random field such that $E|\varepsilon_{\mathbf{t}}|^\gamma < C$, $C < \infty$, $\mathbf{t} \in \mathbb{Z}^d$, $\gamma > 2$ and $\inf_t \operatorname{var} \varepsilon_{\mathbf{t}} = \sigma^2 > 0$. If strong mixing condition is valid for $\{\varepsilon_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ and*

$$\alpha_{m,n}(r) < f(m)\alpha(r), \tag{2.3.12}$$

where $\alpha(r) \to 0$, when $r \to \infty$ and $f(m)$, $m \in \mathbb{N}$ is some arbitrary function, then CLT for $\{\varepsilon_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ is valid.

For the linear random field generated by a strong mixing martingale difference random field the following result can be formulated.

Theorem 2.3.6. *If a random field is of the form (2.1.1), conditions of Theorem 2.3.5 are fulfilled when $d = 2$ and coefficients $\varphi_{\mathbf{k}}$ satisfies $\mathcal{L}_{0,1}$ then CLT for (2.1.1) is valid.*

Proof. The proof is almost the same as in the case of a stationary random field. Only the proof of (2.3.11) differs. Since we have the finiteness of moments of $\varepsilon_{\mathbf{t}}$ of the order γ with $\gamma > 2$ (this condition is needed to prove CLT for Z_n) we easily get

$$(E\varepsilon_{\mathbf{t}}^2)^{\frac{1}{2}} \leqslant (E|\varepsilon_{\mathbf{t}}|^{\gamma})^{\frac{1}{\gamma}} \quad \Rightarrow \quad E\varepsilon_{\mathbf{t}}^2 \leqslant (E|\varepsilon_{\mathbf{t}}|^{\gamma})^{\frac{2}{\gamma}} < C^{\frac{2}{\gamma}} = C_1, \quad C < \infty,\ \gamma > 2,$$

then

$$\begin{aligned} E\omega_n^2 &= E\Bigg(\sum_{k,l\in\mathbb{Z}}\bigg(\sum_{t,s\in D_n} b_{t-k,s-l}\bigg)^2\varepsilon_{k,l}^2\Bigg) = \sum_{k,l\in\mathbb{Z}}\bigg(\sum_{t,s\in D_n} b_{t-k,s-l}\bigg)^2 E\varepsilon_{k,l}^2 \\ &\leqslant C_1\Bigg(\sum_{k,l\in\mathbb{Z}}\bigg(\sum_{t,s\in D_n} b_{t-k,s-l}\bigg)^2\Bigg). \end{aligned}$$

The rest part of the proof is the same as in the proof of Theorem 2.3.5. The theorem is proved.

2.4 CLT for linear random fields with martingale differences defined in [16]

In this section we are interested in CLT for linear random fields (2.1.1) on the lattice $\mathbb{Z}^d$ $d \geqslant 2$ with lexicographical order. The main result which will be used in this section is the CLT proved in [16].

The proof is based on the idea, as in the previous section, that sum $\sum_{\mathbf{t}\in\Gamma_n} X_{\mathbf{t}}$ of random linear fields (2.1.1), where Γ_n is some set, can be well approximated by the sum of innovations $Z_n = \sum_{t\in\Gamma_n} \varepsilon_{\mathbf{t}}$ multiplied by $\mu_1 = \sum_{\mathbf{k}\geqslant\mathbf{0}} \varphi_{\mathbf{k}}$.

2.4.1 Definitions and auxiliary results

A real random field will be defined as probability space $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{A}^{\mathbb{Z}^d}, P)$ where $\mathcal{A}$ is the Borel σ-algebra of $\mathbb{R}$. By $x_{\mathbf{t}} \in X$, $X \subseteq \mathbb{R}$, we denote the projection from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}$ defined by $x_{\mathbf{i}}(\omega) = \omega_{\mathbf{i}}$ for any ω in $\mathbb{R}^{\mathbb{Z}^d}$. From now on the field of all projections $\{x_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ will designate the whole random field $(\Omega, \mathcal{B}, P) := (\mathbb{R}^{\mathbb{Z}^d}, \mathcal{A}^{\mathbb{Z}^d}, P)$.

Denote $\tau_{\mathbf{h}}, \mathbf{h} \in \mathbb{Z}^d$ the group of translations,

$$(\tau_{\mathbf{h}} x)_{\mathbf{t}} = x_{\mathbf{t}+\mathbf{h}}, \quad \mathbf{h} \in \mathbb{Z}^d,\ x \in \Omega.$$

Let $\mathcal{I}$ be σ-algebra of translation invariant Ω subsets: $\mathcal{I} = \{A \in \mathcal{B} \colon \tau_{\mathbf{h}} A = A\}$.

A random field $\{\xi_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ with distribution P is called a stationary if for any $A \in \mathcal{I}$ and $\mathbf{h} \in \mathbb{Z}^d$

$$P(\tau_{\mathbf{h}} A) = P(A).$$

We define the lexicographical order on $\mathbb{Z}^d$ as follows: if $\mathbf{i} = (i_1, i_2, \ldots, i_d)$ and $\mathbf{j} = (j_1, j_2, \ldots, j_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $\mathbf{i} <_{\text{lex}} \mathbf{j}$ means that either $i_1 < j_1$ or for some p in $\{2, 3, \ldots, d\}$, $i_p < j_p$ and $i_q = j_q$ for $1 \leqslant q < p$. The lexicographical order provides a total ordering of $\mathbb{Z}^d$.

Let the sets $\{V_{\mathbf{i}}^k \colon \mathbf{i} \in \mathbb{Z}^d, k \in \mathbb{N}\}$ be defined as follows:

$$V_{\mathbf{i}}^1 = \{\mathbf{j} \in \mathbb{Z}^d \colon \mathbf{j} <_{\text{lex}} \mathbf{i}\},$$

and for $k \geqslant 2$:

$$V_{\mathbf{i}}^k = V_{\mathbf{i}}^1 \cap \{\mathbf{j} \in \mathbb{Z}^d \colon |\mathbf{i}-\mathbf{j}| \geqslant k\} \quad \text{where } |\mathbf{i}-\mathbf{j}| = \max_{1 \leqslant k \leqslant d} |i_k - j_k|.$$

For any Γ in $\mathbb{Z}^d$, let $\mathcal{F}_\Gamma$ be the σ-algebra defined by $\mathcal{F}_\Gamma = \sigma(x_{\mathbf{i}} : \mathbf{i} \in \Gamma)$ and set

$$E_{|\mathbf{k}|}(x_{\mathbf{i}}) = E(x_{\mathbf{i}} | \mathcal{F}_{V_{\mathbf{i}}^{|\mathbf{k}|}}), \quad \mathbf{k} \in V_{\mathbf{i}}^1.$$

Let Γ be any finite subset of $\mathbb{Z}^d$. Cardinality of Γ will be denoted by $|\Gamma|$ and

$$\partial\Gamma = \{\mathbf{i} \in \Gamma \colon \exists \mathbf{j} \notin \Gamma \text{ such that } |\mathbf{i}-\mathbf{j}| = 1\}.$$

For a finite subset Γ of $\mathbb{Z}^d$ we define partial sum of random field by

$$S_\Gamma = \sum_{\mathbf{i} \in \Gamma} X_{\mathbf{i}}. \tag{2.4.1}$$

A sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of finite subsets of $\mathbb{Z}^d$, throughout the paper will satisfy

$$\lim_{n\to\infty} |\Gamma_n| = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{|\partial\Gamma_n|}{|\Gamma_n|} = 0. \tag{2.4.2}$$

In this paper we shall consider Γ_n only of the form

$$\Gamma_n = [-n, n]^d \cap \mathbb{Z}^d. \tag{2.4.3}$$

This set clearly satisfy condition (2.4.2).

Dedecker [16] established the central limit theorem for any stationary square-integrable random field $(\varepsilon_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ which satisfy the condition:

$$\sum_{\mathbf{k} \in V_{\mathbf{0}}^1} |\varepsilon_{\mathbf{k}} E_{|\mathbf{k}|}(\varepsilon_{\mathbf{0}})| \in \mathbb{L}^1. \tag{2.4.4}$$

We are interested in special case of a stationary random field: martingale difference random field. In this section we shall use the following definition of a martingale difference random field.

Definition 11. A real random field $(\varepsilon_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ is said to be martingale difference random field if $E|\varepsilon_{\mathbf{k}}| < \infty$, $\mathbf{k} \in \mathbb{Z}^d$ and for any $\mathbf{n} \in \mathbb{Z}^d$, $E(\varepsilon_{\mathbf{n}} | \sigma(\varepsilon_{\mathbf{k}}, \mathbf{k} <_{\text{lex}} \mathbf{n})) = 0$ a.s.

As it was mentioned in [42] martingale difference fields satisfy (2.4.4).

The Lemma 2.3.4 will be used in the proof of the main result of this section, for solving problems, which rises because of the linear filter.

In this section again we use BN decomposition introduced in Section 1.3.1 which enables sum $S_n = \sum_{\mathbf{i}\in\Gamma_n} X_{\mathbf{i}}$ of random fields (1.0.2) analyze as

$$S_n = \mu_1 \sum_{\mathbf{i}\in\Gamma_n} \varepsilon_{\mathbf{i}} + R_n, \tag{2.4.5}$$

where $\mu_1 = \sum_{\mathbf{k}\geqslant 0} \varphi_{\mathbf{k}}$, and $R_{\mathbf{n}}$ is a remainder which exact expression depend on region Γ_n. We use indirect method of application of BN decomposition, that is we shall show that $\lim_{n\to\infty} \omega_n = 0$ in probability where

$$\omega_n = S_n - \mu_1 \sum_{\mathbf{i}\in\Gamma_n} \varepsilon_{\mathbf{i}} = S_n - \mu_1 Z_n. \tag{2.4.6}$$

2.4.2 Results and proofs

The main result of the work is based on Dedecker [16] result. In order to formulate it we need to introduce the notation of stable convergence.

Definition 12. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of real random variables, and let Y be defined on some extension of the underlying probability space $(\Omega, \mathcal{A}, P)$. Let $\mathcal{U} \subset \mathcal{A}$ be a σ-algebra of $\mathcal{A}$. Then $(Y_n)_{n\in\mathbb{N}}$ is said to converge $\mathcal{U}$-stably to Y if for any continuous bounded function ψ and any bounded and $\mathcal{U}$-measurable variable Z, $\lim_{n\to\infty} E(\psi(Y_n)Z) = E(\psi(Y)Z)$.

Remark 2.4.1. Definition 12 is equivalent to 6. We presented 12 there because it was used in [16].

Theorem 2.4.1. *([16]) Let $(\varepsilon_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ be strictly stationary random field with $E\varepsilon_{\mathbf{0}} = 0$, $E\varepsilon_{\mathbf{0}}^2 < +\infty$, $(\Gamma_n)_{n\in N}$ is a sequence of finite subsets of $\mathbb{Z}^d$ satisfying (2.4.2). Assume that condition (2.4.4) is satisfied and set*

$$\eta = \sum_{\mathbf{k}\in\mathbb{Z}^d} E(\varepsilon_{\mathbf{0}}\varepsilon_{\mathbf{k}}|\mathcal{I}).$$

The following results hold

- *The random variable* $|\Gamma_n|^{-1/2} S_{\Gamma_n}$ *converges* $\mathcal{I}$*-stably to* $\zeta\eta^{1/2}$*, where* ζ *is a standard Gaussian random variable independent of* η*.*
- *For any* N *in* $\mathbb{N}$*, denote* $G_N = \{(\mathbf{i}, \mathbf{j}) \in \Gamma_n \times \Gamma_n \colon |\mathbf{i} - \mathbf{j}| \leqslant N\}$*. Let* ρ_n *be a sequence of positive integers satisfying* $\lim_{n\to\infty} \rho_n = +\infty$ *and* $\lim_{n\to\infty} \rho_n^{3d} E(X_{\mathbf{0}}^2 (1 \wedge |\Gamma_n|^{-1} X_{\mathbf{0}}^2)) = 0$*. Then*
$$A_n = \frac{1}{|\Gamma_n|} \max\left(1, \sum_{(\mathbf{i},\mathbf{j}) \in G_{\rho_n}} \varepsilon_{\mathbf{i}} \varepsilon_{\mathbf{j}}\right) \to^{\mathrm{P}} \eta.$$

From this result the following corollary can be derived:

Corollary 2.4.2. *Assume that condition* (2.4.4) *is satisfied. Then with the same notation as in Theorem* 2.4.1*,* $(|\Gamma_n|^{-1/2} S_{\Gamma_n}, A_n)$ *converges in distribution to* $(\zeta\eta^{1/2}, \eta)$*. Assume moreover that* $P(\eta > 0) = 1$*. Then* $(A_n |\Gamma_n|)^{-1/2} S_{\Gamma_n}$ *converges in distribution to* ζ*.*

Using Corollary 2.4.2 we prove the following theorem for linear random fields.

Theorem 2.4.3. *Let* $(\varepsilon_{\mathbf{t}})_{t\in\mathbb{Z}^d}$ *be strictly stationary martingale difference random field* $E\varepsilon_{\mathbf{0}} = 0$*,* $E\varepsilon_{\mathbf{0}}^2 < \infty$*,* $(\Gamma_n)_{n\in\mathbb{N}}$ *is a sequence of finite subset of the form* (2.4.3)*.*

Assume that condition (2.4.4) *is satisfied* $\{\varphi_{\mathbf{k}}\}$ *satisfies* $\mathcal{L}_{2,2}$*. Denote*

$$\eta = \sum_{\mathbf{k}\in\mathbb{Z}^d} E(\varepsilon_{\mathbf{0}} \varepsilon_{\mathbf{k}} | \mathcal{I}).$$

Then the sequence of random variables $S_{\Gamma_n} / (\mu_1 \sqrt{|\Gamma_n|})$ *converges in distribution to* $\zeta\eta^{1/2}$*, where* ζ *is a standard Gaussian random variable independent of* η*.*

Proof. The proof is based on showing that ω_n, defined in (2.4.6), tends to 0 when $n \to \infty$.

The random linear field $\{\varepsilon_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ satisfy Theorem 2.4.1. Therefore, after normalization, according corollary (2.4.2), we have convergence in distribution and CLT is valid for Z_n.

We need to show that
$$\frac{\omega_n}{\mu_1 \sqrt{|\Gamma_n|}} \xrightarrow{\mathrm{P}} 0.$$

This will follow from the relation

$$P(\omega_n > \varepsilon\mu_1\sqrt{|\Gamma_n|}) \leqslant \frac{1}{\varepsilon^2\mu_1^2|\Gamma_n|}E\omega_n^2 \to 0. \tag{2.4.7}$$

Here we will use Lemma 2.3.4. In order to apply it we set $\varphi_{\mathbf{k}} = 0$ when $\min_{1\leqslant i\leqslant d}(k_i) < 0$ and $b_{\mathbf{0}} = \varphi_{\mathbf{0}} - \mu_1$, $b_{\mathbf{j}} = \varphi_{\mathbf{j}}$ when $\mathbf{i} = (i_1, \dots, i_d) \neq (0, \dots, 0) = \mathbf{0}$. We have

$$S_{\Gamma_n} = \sum_{\mathbf{t}\in\Gamma_n}\sum_{\mathbf{k}\in\mathbb{Z}^d}\varphi_{\mathbf{k}}\varepsilon_{\mathbf{t}-\mathbf{k}} = \sum_{\mathbf{m}\in\mathbb{Z}^d}\left(\sum_{\mathbf{t}\in\Gamma_n}\varphi_{\mathbf{t}-\mathbf{m}}\right)\varepsilon_{\mathbf{m}},$$

and

$$\begin{aligned}\omega_n &= S_{\Gamma_n} - \mu_1 Z_n = \sum_{\mathbf{m}\in\mathbb{Z}^d}\left(\sum_{\mathbf{t}\in\Gamma_n}\varphi_{\mathbf{t}-\mathbf{m}}\right)\varepsilon_{\mathbf{m}} - \mu_1\sum_{\mathbf{m}\in\Gamma_n}\varepsilon_{\mathbf{m}} \\ &= \sum_{\mathbf{m}\in\Gamma_n}\left(\sum_{\mathbf{t}\in\Gamma_n}\varphi_{\mathbf{t}-\mathbf{m}} - \mu_1\right)\varepsilon_{\mathbf{m}} + \sum_{\mathbf{m}\notin\Gamma_n}\left(\sum_{\mathbf{t}\in\Gamma_n}\varphi_{\mathbf{t}-\mathbf{m}}\right)\varepsilon_{\mathbf{m}} \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^d}\left(\sum_{\mathbf{t}\in\Gamma_n}b_{\mathbf{t}-\mathbf{m}}\right)\varepsilon_{\mathbf{m}}.\end{aligned}$$

Thus we get

$$\omega_n = \sum_{\mathbf{i}\in\mathbb{Z}^d}\sum_{\mathbf{t}\in\Gamma_n}b_{\mathbf{t}-\mathbf{i}}\varepsilon_{\mathbf{i}}.$$

Coefficients $b_{\mathbf{j}}$ satisfy conditions of Lemma 2.3.4.

Now we evaluate $E\omega_n^2$:

$$\begin{aligned}E\omega_n^2 &= E\left(\sum_{\mathbf{k}\in\mathbb{Z}^d}\left(\sum_{\mathbf{t}\in\Gamma_n}b_{\mathbf{t}-\mathbf{k}}\right)\varepsilon_{\mathbf{k}}\right)^2 \leqslant \sum_{\mathbf{k}\in\mathbb{Z}^d}\left(\sum_{\mathbf{t}\in\Gamma_n}b_{\mathbf{t}-\mathbf{k}}\right)^2 E\varepsilon_{\mathbf{k}}^2 \\ &\quad + \sum_{\mathbf{k},\ \mathbf{m}\in\mathbb{Z}^d,\,\mathbf{m}\neq\mathbf{k}}\left(\sum_{\mathbf{t}\in\Gamma_n}b_{\mathbf{t}-\mathbf{k}}\right)\left(\sum_{\mathbf{s}\in\Gamma_n}b_{\mathbf{s}-\mathbf{m}}\right)E\varepsilon_{\mathbf{k}}\varepsilon_{\mathbf{m}}.\end{aligned} \tag{2.4.8}$$

Separately we consider second member of the sum (2.4.8) and denote

$$A = \sum_{\mathbf{k},\mathbf{m}\in\mathbb{Z}^d,\,\mathbf{m}\neq\mathbf{k}} c_1^{(\mathbf{k})}c_2^{(\mathbf{m})}E\varepsilon_{\mathbf{k}}\varepsilon_{\mathbf{m}}, \tag{2.4.9}$$

where $c_1^{(\mathbf{k})}c_2^{(\mathbf{m})}$ are constants which depend on $\mathbf{k}$, $\mathbf{m}$ and have the form respectively $\sum_{\mathbf{t}\in\Gamma_n}b_{\mathbf{t}-\mathbf{k}}$ and $\sum_{\mathbf{s}\in\Gamma_n}b_{\mathbf{s}-\mathbf{m}}$.

Now we have

$$E\varepsilon_{\mathbf{k}}\varepsilon_{\mathbf{m}} = EE_{|\mathbf{k}-\mathbf{m}|}(\varepsilon_{\mathbf{k}}\varepsilon_{\mathbf{m}}) = E(\varepsilon_{\mathbf{k}}(E_{|\mathbf{k}-\mathbf{m}|}\varepsilon_{\mathbf{m}})), \tag{2.4.10}$$

where $E_{|\mathbf{k}-\mathbf{m}|}\varepsilon_{\mathbf{m}} = E(\varepsilon_{\mathbf{m}}|\mathcal{F}_{V_{\mathbf{m}}^{|\mathbf{k}-\mathbf{m}|}})$. If $\mathbf{k} <_{\text{lex}} \mathbf{m}$ then we take $E_{|\mathbf{k}-\mathbf{m}|}\varepsilon_{\mathbf{m}}$ and when $\mathbf{m} <_{\text{lex}} \mathbf{k}$ then $E_{|\mathbf{k}-\mathbf{m}|}\varepsilon_{\mathbf{k}}$. By using martingale differences property we get that $E_{|\mathbf{k}-\mathbf{m}|}\varepsilon_{\mathbf{m}} = 0$ in (2.4.10). Coefficients c_i^n, $i = \{1, 2\}$, are finite because $\sum_{\mathbf{k}\geqslant\mathbf{0}} \varphi_{\mathbf{k}} < \infty$. Thus we get that $A = 0$ in (2.4.9). Taking into account that innovations of random field are stationary, the expression (2.4.8) can be written as:

$$E\omega_n^2 = E\varepsilon_{\mathbf{0}}^2\Bigg(\sum_{\mathbf{k}\in\mathbb{Z}^d}\bigg(\sum_{\mathbf{t}\in\Gamma_n} b_{\mathbf{t}-\mathbf{m}}\bigg)^2\Bigg).$$

Applying Lemma 2.3.4 we get that

$$\frac{E\omega_n^2}{|\Gamma_n|} \to 0.$$

and (2.4.7) is true. The theorem is proved.

Chapter 3

Some remarks on SLLN for random linear fields

3.1 Introduction

In this chapter, once again, we consider the random linear field $X_{\mathbf{t}}$ defined by (1.0.2). We assume that random linear field is generated by i.i.d. random variables $\{\varepsilon_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$. Our goal is to prove SLLN for sums $\sum_{\mathbf{t}\in D_{\mathbf{n}}} X_{\mathbf{t}}$, where $D_{\mathbf{n}}$ is some subset of $\mathbb{Z}^d$ by using ergodic theory and to compare with results, which give the application of BN decomposition. In the application of ergodic theory the key result is classical ergodic theorem of Zygmund–Fava [34].

The BN decomposition for linear random fields was applied in [55] to prove SLLN. In [55] it was stressed that in the case $d = 2$ and sets $D_n = [1, n]^2 \cap \mathbb{Z}^2$ this approach leads to very simple proofs, but at the same time moment conditions for innovations $\varepsilon_{\mathbf{t}}$ and conditions for coefficients $\varphi_{\mathbf{k}}$ are not optimal (this is the price which we pay for simplicity of proofs). In this chapter we consider SLLN on rectangles

$$D_{\mathbf{n}} = \{\mathbf{t} \in \mathbb{Z}^d \colon 1 \leqslant t_i \leqslant n_i,\ i = 1, 2, \ldots, d\}. \tag{3.1.1}$$

and our goal is to obtain generalization of the classical SLLN for multi-indexed sums of i.i.d. random variables (see [63]). The main message is that the application of the ergodic theory to prove SLLN for linear fields gives much more general and stronger results comparing with ones obtained by using BN decomposition and even the proofs, based on application of ergodic theorems, are very simple.

Before formulation of our results we introduce some notions from the ergodic theory. Let $Y = \{Y_{\mathbf{t}},\ \mathbf{t} \in \mathbb{Z}^d\}$ be a strictly stationary random field. Let $\mathrm{H} = R^{\mathbb{Z}^d}$ denote a space of all real-valued functions on $\mathbb{Z}^d$ with a σ-algebra $\mathcal{H}$, generated by cylindrical sets. Denote by $\{U_{\mathbf{h}},\ \mathbf{h} \in \mathbb{Z}^d\}$ the group of translations:

$$U_{\mathbf{h}} x_{\mathbf{t}} = x_{\mathbf{t}-\mathbf{h}}, \quad x \in \mathrm{H},\ \mathbf{t}, \mathbf{h} \in \mathbb{Z}^d.$$

Let $\mathbf{P}$ stand for a distribution of a random field Y in H. Strict stationarity of Y means that $\mathbf{P}$ is invariant with respect to translations:

$$\mathbf{P} U_{\mathbf{h}}^{-1} = \mathbf{P}.$$

Let $\mathcal{T}$ denote σ-algebra of invariant sets:

$$\mathcal{T} = \{A \in \mathcal{H} :\ U_{\mathbf{h}}(A) = A,\ \forall \mathbf{h} \in \mathbb{Z}^d\}.$$

A random field is *ergodic* if σ-algebra $\mathcal{T}$ is trivial:

$$\forall A \in \mathcal{T} \ \ \mathbf{P}(A) = 0 \text{ or } 1. \tag{3.1.2}$$

From the ergodic theory it follows that relation (3.1.2) is equivalent to the following relation:

$$\forall\ A, B \in \mathcal{H} \quad n^{-d} \sum_{\mathbf{0} \leqslant \mathbf{h} \leqslant \bar{\mathbf{n}} - \mathbf{1}} \mathbf{P}(A \cap U_{\mathbf{h}}^{-1}(B)) \to \mathbf{P}(A)\mathbf{P}(B), \tag{3.1.3}$$

as $n \to \infty$, here $\bar{\mathbf{n}} = (n, \dots, n)$. For more information on ergodic theory see [23] or [71].

We say that a random field Y is *mixing* if

$$\forall A, B \in \mathcal{H} \quad \mathbf{P}(A \cap U_{\mathbf{h}}^{-1}(B)) \to \mathbf{P}(A)\mathbf{P}(B), \tag{3.1.4}$$

as $||\mathbf{h}|| \to \infty$, here $|| \cdot ||$ stands for any norm in R^d. It is clear that (3.1.4) implies (3.1.3), thus, mixing implies ergodicity of a random field. Such definitions for measure-preserving transformations (but in case $d = 1$) can be found in [71].

While using BN decomposition approach we consider linear fields of the form (1.0.2), the ergodic theory approach allows us to consider more general linear random fields with summation extended not over positive quadrant but over all $\mathbb{Z}^d$. Namely, let

$$Y_{\mathbf{t}} = \sum_{\mathbf{k}\in\mathbb{Z}^d} \varphi_{\mathbf{k}}\varepsilon_{\mathbf{t}-\mathbf{k}}, \quad \mathbf{t} \in \mathbb{Z}^d. \tag{3.1.5}$$

We consider sums

$$S_{\mathbf{n}} = \sum_{\mathbf{t}\in D_{\mathbf{n}}} Y_{\mathbf{t}}, \tag{3.1.6}$$

and let us denote $|\mathbf{n}| := \prod_{i=1}^d n_i$. We say that SLLN holds for $S_{\mathbf{n}}$, if

$$|\mathbf{n}|^{-1} S_{\mathbf{n}} \xrightarrow{\text{a.s.}} 0, \tag{3.1.7}$$

when $\mathbf{n}$ tends to infinity. There are several interpretations of the growth of $\mathbf{n}$. In our paper we shall use two possibilities. We shall write $\mathbf{n} \to \infty$ if

$$n_i \to \infty, \quad i = 1, \dots, d. \tag{3.1.8}$$

The second possibility of growth of $\mathbf{n}$ is to assume that

$$|\mathbf{n}| \to \infty. \tag{3.1.9}$$

Evidently, (3.1.9) follows from (3.1.8), but not converse.

Let us denote by $\mathcal{L}_{q,p}$ condition

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\prod_{i=1}^d (|k_i| + 1) \right)^q |\varphi_{\mathbf{k}}|^p < \infty.$$

If a random variable X satisfies

$$E\,|X|\,(\ln(1 + |X|))^{d-1} < \infty, \tag{3.1.10}$$

we shall write $X \in L\log L^{d-1}$.

3.2 Convergence theorems

3.2.1 Results and comments

Theorem 3.2.1. *Let $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$, be a strictly stationary ergodic random field with $E\varepsilon_{\mathbf{0}} = 0$ and $\varepsilon_{\mathbf{0}} \in L\log L^{d-1}$. Suppose that condition $\mathcal{L}_{0,1}$ holds and $S_{\mathbf{n}}$ is defined in (3.1.6). Then, if $\mathbf{n} \to \infty$, the relation (3.1.7) holds.*

In particular, the result holds for the case where $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$, are i.i.d. random variables satisfying the same moment conditions. Therefore in the case where $Y_{\mathbf{t}} = \varepsilon_{\mathbf{t}}$ (this will be if $\varphi_{\mathbf{0}} = 1$, $\varphi_{\mathbf{k}} = 0$ for all $\mathbf{k} \neq \mathbf{0}$) we get the generalization of the result of [63] where it is shown that if $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$, are i.i.d. random variables, then SLLN for $S_{\mathbf{n}} = \sum_{\mathbf{t} \in D_{\mathbf{n}}} \varepsilon_{\mathbf{t}}$ holds if and only if (3.1.10) is satisfied. If we consider the class of all strictly stationary ergodic random fields, which includes a random field $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$, with i.i.d. random variables $\varepsilon_{\mathbf{t}}$, therefore moment condition $\varepsilon_{\mathbf{0}} \in L \log L^{d-1}$ in our theorem is necessary, too. But as results of [59] show, for particular types of dependent stationary sequences situation with necessary conditions can be different, the same can be said about random fields.

If we require stronger condition on initial random field $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$, we get a stronger result.

Theorem 3.2.2. *Let $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$, be a strictly stationary mixing random field with $E\varepsilon_{\mathbf{0}} = 0$ and $\varepsilon_{\mathbf{0}} \in L \log L^{d-1}$. Suppose that condition $\mathcal{L}_{0,1}$ holds. Then the relation (3.1.7) holds if $|\mathbf{n}| \to \infty$.*

Both formulated theorems rely on the classical ergodic theorem of Zygmund–Fava.

Theorem A. *Let $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^d$ be a strictly stationary ergodic random field with $E\varepsilon_{\mathbf{0}} = 0$ and $\varepsilon_{\mathbf{0}} \in L \log L^{d-1}$. Then*

$$V_{\mathbf{n}} = |\mathbf{n}|^{-1} \sum_{\mathbf{t} \in D_{\mathbf{n}}} \varepsilon_{\mathbf{t}} \xrightarrow{\text{a.s.}} 0,$$

as $\mathbf{n} \to \infty$.

As it is formulated here, Theorem A is an easy corollary from Theorem 1.1 in [34], p. 196, where it is formulated in more general setting – for some operators (contractions), acting on finite measure space. Reduction to probability space and shift operators is standard, and that the limit is zero (in [34] the existence of a limit is stated) follows from ergodicity of the random field under consideration.

Having these two results on SLLN for linear random fields it is clear that by using BN decomposition we cannot get such general and strong results. Thus, application of BN decomposition to prove SLLN for linear random fields (the same remark can be applied for linear processes, too) can be justified only from methodological point of view: if one wants (for some reasons) to avoid ergodic theory, one can reduce the problem of SLLN for linear random fields to the case of SLLN for multi-indexed sums of i.i.d. random variables and use (comparatively elementary) [63] result and elementary moment inequalities to estimate remainder term appearing in BN decomposition (as it was done in [55]). We formulate one more such result in the case $d = 2$ in the next proposition. In all aspects this result is weaker comparing with theorems formulated above, only the moment condition is very little stronger than (3.1.10), but it is necessary to note that to achieve such condition instead of the moment inequality we used a result from ergodic theory. The obtained result is stronger than results in [55], and, in a sense, it demonstrates the limits of the BN decomposition approach in the problem. Now we return to a linear random field defined in (1.0.2) and we denote

$$S_{\mathbf{n}}^{(1)} = \sum_{\mathbf{t}\in D_{\mathbf{n}}} X_{\mathbf{t}}. \tag{3.2.1}$$

Proposition 3.2.3. *Let $\varepsilon_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{Z}^2$, be i.i.d. random variables with $E\varepsilon_{\mathbf{0}} = 0$. Suppose that for some $1 < p \leqslant 2$ moment condition $E\left|\varepsilon_{\mathbf{0}}\right|^p < \infty$ and condition $\mathcal{L}_{p,p}$ (with summation only over positive quadrant) holds. Then, under (3.1.9), SLLN for $S_{\mathbf{n}}^{(1)}$ holds.*

3.2.2 Proofs

The main tool in the proofs of theorems will be BN decomposition formulated in Section 1.3.1. Also we shall need the following simple result.

Lemma 3.2.4. *Let $Y_{\mathbf{t}}$ be a random field defined in (3.1.5), $d \geqslant 2$, a random field $\varepsilon_{\mathbf{t}}$ satisfies conditions of Theorem 3.2.1 and the condition $\mathcal{L}_{0,1}$ holds. Then for all $\mathbf{t} \in \mathbb{Z}^d$*

$$E\left|Y_{\mathbf{t}}\right| (\ln(1 + |Y_{\mathbf{t}}|))^{d-1} < \infty. \tag{3.2.2}$$

Proof Let us denote $g(x) = |x|(\ln(1+|x|))^{d-1}$. This function is convex and increasing for $x \geqslant 0$, therefore using obvious inequality $(1+cx) \leqslant (1+x)(1+c)$ we have for all $c > 0,\ x > 0$

$$g(cx) \leqslant 2^{d-2}(cg(x) + xg(c)).$$

We must prove the boundedness of $Eg(Y_{\mathbf{t}})$. Let us denote $c = \sum_{\mathbf{k}\in\mathbb{Z}^d}|\varphi_{\mathbf{k}}|$, then, using the fact that g is convex and increasing, we have

$$\begin{aligned}
Eg(Y_{\mathbf{t}}) &= Eg\left(\left|\sum_{\mathbf{k}\in\mathbb{Z}^d}\varphi_{\mathbf{k}}\varepsilon_{\mathbf{t}-\mathbf{k}}\right|\right) \leqslant Eg\left(\sum_{\mathbf{k}\in\mathbb{Z}^d}|\varphi_{\mathbf{k}}||\varepsilon_{\mathbf{t}-\mathbf{k}}|\right)\\
&= Eg\left(c\sum_{\mathbf{k}\in\mathbb{Z}^d}\frac{|\varphi_{\mathbf{k}}|}{c}|\varepsilon_{\mathbf{t}-\mathbf{k}}|\right)\\
&\leqslant 2^{d-2}\left\{cEg\left(\sum_{\mathbf{k}\in\mathbb{Z}^d}\frac{|\varphi_{\mathbf{k}}|}{c}|\varepsilon_{\mathbf{t}-\mathbf{k}}|\right) + g(c)E\sum_{\mathbf{k}\in\mathbb{Z}^d}\frac{|\varphi_{\mathbf{k}}|}{c}|\varepsilon_{\mathbf{t}-\mathbf{k}}|\right\}\\
&\leqslant 2^{d-2}\left\{cE\sum_{\mathbf{k}\in\mathbb{Z}^d}\frac{|\varphi_{\mathbf{k}}|}{c}g(|\varepsilon_{\mathbf{t}-\mathbf{k}}|) + g(c)\sum_{\mathbf{k}\in\mathbb{Z}^d}\frac{|\varphi_{\mathbf{k}}|}{c}E|\varepsilon_{\mathbf{t}-\mathbf{k}}|\right\}\\
&\leqslant 2^{d-2}\{cEg(|\varepsilon_{\mathbf{0}}|) + g(c)E|\varepsilon_{\mathbf{0}}|\} < \infty.
\end{aligned}$$

The lemma is proved.

Proof of Theorem 3.2.1. Theorem 3.2.1 directly follows from Theorem A. Namely, the random field $\{Y_{\mathbf{t}}\}$ is strictly stationary and ergodic (as a function of a random field $\{\varepsilon_{\mathbf{t}}\}$), $EY_{\mathbf{0}} = 0$, from Lemma 3.2.4 it follows that $Y_{\mathbf{0}} \in L\log L^{d-1}$. Hence we can apply Theorem A.

Proof of Theorem 3.2.2. Let, for simplicity, agree to write $[\mathbf{0},\mathbf{n}]^c$ for a set of those $\mathbf{k} \in (\mathbb{Z}^+)^d$, which satisfy $\mathbf{k} \not\leqslant \mathbf{n}$. It is easy to see that the relation

$$T_{\mathbf{n}} := |\mathbf{n}|^{-1}S_{\mathbf{n}} \xrightarrow{\text{a.s.}} 0,$$

when $|\mathbf{n}| \to \infty$, is equivalent to the following condition: for each $\varepsilon > 0$ there exists $\mathbf{n}^{(\varepsilon)} = (n_1^{(\varepsilon)}, \ldots, n_d^{(\varepsilon)})$ such that

$$P\{|T_{\mathbf{n}}| < \varepsilon, \forall \mathbf{n} \in [\mathbf{0},\mathbf{n}^{(\varepsilon)}]^c\} = 1.$$

At first we prove the theorem in the case $d = 2$. From Theorem 3.2.1 it follows that

$$T_{\mathbf{n}} \xrightarrow{\text{a.s.}} 0,$$

as $\mathbf{n} \to \infty$, that is, $n_i \to \infty,\ i = 1,2$. Let $\Omega_0,\ P(\Omega_0) = 1$, be a set of those ω, for which this relation holds. (Here $(\Omega, \mathcal{F}, P)$ is a probability space on which all random variables under consideration are defined.) Let us fix $\varepsilon > 0$. Then $\forall \omega \in \Omega_0$, there exists $\mathbf{n}^{(\varepsilon)} = (n_1^{(\varepsilon)}, n_2^{(\varepsilon)})$ such that, $\forall \mathbf{n} \geqslant \mathbf{n}^{(\varepsilon)}$, $|T_{\mathbf{n}}| < \varepsilon$. Let us take some $1 \leqslant m \leqslant n_2^{(\varepsilon)} - 1$ and consider sums

$$T_{(n_1,m)} = \frac{1}{n_1 m} \sum_{\mathbf{k} \leqslant (n_1,m)} Y_{\mathbf{k}} = \frac{1}{m} \sum_{j=1}^{m} \left(\frac{1}{n_1} \sum_{k=1}^{n_1} Y_{(k,j)} \right).$$

The random field $\{Y_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}^2\}$, as a function (under very mild conditions on a function; in our case it is linear function) of a mixing field $\{\varepsilon_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^2\}$, is mixing, too. Therefore for any fixed j a random process $\{Y_{(k,j)}, k \in \mathbb{Z}\}$ is also mixing, hence it is ergodic, and we get

$$T_{(n_1,m)} \xrightarrow{\text{a.s.}} 0 \quad \text{as } n_1 \to \infty.$$

Let $\Omega_{1,m},\ P(\Omega_{1,m}) = 1$ be a set of those ω for which this relation holds. Taking the same $\varepsilon > 0$ we can find $n_{\varepsilon,m}$ such that for $n_1 \geqslant n_{\varepsilon,m}$ and all $\omega \in \Omega_{(1,m)}$, we have $|T_{(n_1,m)}| < \varepsilon$. Now for some $1 \leqslant r \leqslant n_1^{(\varepsilon)} - 1$ in a similar way we consider sums $T_{(r,n_2)}$, and (for the same ε) we introduce a set $\Omega_{2,r},\ P(\Omega_{2,r}) = 1$, and number $n_{r,\varepsilon}$ such that, for all $\omega \in \Omega_{(2,r)}$, and all $n_2 \geqslant n_{r,\varepsilon}$, we have $|T_{(r,n_2)}| < \varepsilon$. Let us take $\bar{\mathbf{n}}^{(\varepsilon)} = (\bar{n}_1^{(\varepsilon)}, \bar{n}_2^{(\varepsilon)})$ with

$$\bar{n}_1^{(\varepsilon)} = \max\{n_1^{(\varepsilon)}, n_{\varepsilon,m}, m = 1, \ldots, n_2^{(\varepsilon)} - 1\},$$

$$\bar{n}_2^{(\varepsilon)} = \max\{n_2^{(\varepsilon)}, n_{r,\varepsilon}, r = 1, \ldots, n_1^{(\varepsilon)} - 1\}.$$

Let us denote

$$\bar{\Omega} = \Omega_0 \cap \left(\bigcap_{m=1}^{n_2^{(\varepsilon)}-1} \Omega_{1,m} \right) \cap \left(\bigcap_{r=1}^{n_1^{(\varepsilon)}-1} \Omega_{2,r} \right).$$

It is clear that $P(\bar{\Omega}) = 1$, and from construction it follows that for all $\omega \in \bar{\Omega}$ and for all $\mathbf{n} \in [\mathbf{0}, \bar{\mathbf{n}}^{(\varepsilon)}]^c$,

$$|T_{\mathbf{n}}| < \varepsilon.$$

Thus, we have proved the theorem in the case $d = 2$.

Now, using mathematical induction, we shall prove the general case $d > 2$. Let us assume that the statement of the theorem is true for all dimensions $k \leqslant d-1$, $d \geqslant 3$. We prove then that the statement of the theorem holds for dimension d. The proof is similar to that in the case $d = 2$, only notations are more complicated. Let us denote $\mathbf{n}(k) = (n_1, \ldots, n_{k-1}, n_{k+1}, \ldots, n_d)$ and $(\mathbf{n}(k), m) = (n_1, \ldots, n_{k-1}, m, n_{k+1}, \ldots, n_d)$, $k = 1, \ldots, d$. Again, by using Theorem 3.2.1, we have that SLLN is valid for $S_{\mathbf{n}}$ as $\mathbf{n} \to \infty$. Let Ω_0, $P(\Omega_0) = 1$, be a set of those ω, for which this relation holds. Let us fix $\varepsilon > 0$. Then, $\forall\, \omega \in \Omega_0$, there exists $\mathbf{n}^{(\varepsilon)} = (n_1^{(\varepsilon)}, \ldots, n_d^{(\varepsilon)})$ such that, $\forall\, \mathbf{n} \geqslant \mathbf{n}^{(\varepsilon)}$,

$$|T_{\mathbf{n}}| < \varepsilon.$$

Fix some $1 \leqslant k \leqslant d$ and take some $1 \leqslant m \leqslant n_k^{(\varepsilon)} - 1$. Consider sums

$$\begin{aligned} T_{(\mathbf{n}(k),m)} &= \frac{1}{|\mathbf{n}(k)|m} \sum_{\mathbf{r} \leqslant (\mathbf{n}(k),m)} Y_{\mathbf{r}} \\ &= \frac{1}{m} \sum_{j=1}^{m} \frac{1}{|\mathbf{n}(k)|} \sum_{\mathbf{r}(k) \leqslant \mathbf{n}(k)} Y_{(\mathbf{r}(k),j)}, \end{aligned}$$

where $|\mathbf{n}(k)| = |\mathbf{n}|/n_k$. Then, by using the same argument as in the proof of the case $d = 2$, we get that $\{Y_{(\mathbf{n}(k),j)}, \mathbf{n}(k) \in \mathbb{Z}^{d-1}\}$, for a fixed j, is also mixing. Therefore, from the induction assumption for a fixed m we get,

$$T_{(\mathbf{n}(k),m)} \xrightarrow{\text{a.s.}} 0 \quad \text{as } |\mathbf{n}(k)| \to \infty.$$

Denote by $\Omega_{(k,m)}$, $P(\Omega_{(k,m)}) = 1$, the set where this convergence for $T_{(\mathbf{n}(k),m)}$ holds. Taking the same $\varepsilon > 0$, which we fixed at the beginning of the proof, we will find $\mathbf{n}(k)^{(\varepsilon,m)} = (n_1^{(\varepsilon,m)}, \ldots, n_{k-1}^{(\varepsilon,m)}, n_{k+1}^{(\varepsilon,m)}, \ldots, n_d^{(\varepsilon,m)})$ such that for all $\mathbf{n}(k) \in [\mathbf{0}, \mathbf{n}(k)^{(\varepsilon,m)}]^c$ and for all $\omega \in \Omega_{(k,m)}$, $|T_{(\mathbf{n}(k),m)}| < \varepsilon$. Then by applying the same argument to the other coordinates we will get the set of pairs:

$$\{(\Omega_{(1,m_1)}, \mathbf{n}(1)^{(\varepsilon,m_1)}) \ldots, (\Omega_{(d,m_d)}, \mathbf{n}(d)^{(\varepsilon,m_d)}), \quad 1 \leqslant m_k \leqslant n_k^{(\varepsilon)} - 1\}.$$

To finish the proof we must find $\bar{\Omega}$, for which $P(\bar{\Omega}) = 1$, and $\bar{\mathbf{n}}^\varepsilon$ such that, for all $\mathbf{n} \in [0, \bar{\mathbf{n}}^\varepsilon]^c$ and $\omega \in \bar{\Omega}$, $|T_\mathbf{n}| < \varepsilon$. This can be easily done, namely

$$\bar{\Omega} = \Omega_\mathbf{0} \bigcap_{k=1}^{d} \left(\bigcap_{m_k=1}^{n_k^\varepsilon - 1} \Omega_{(k,m_k)} \right).$$

and

$$\bar{\mathbf{n}}^\varepsilon = (\bar{n}_1^\varepsilon, \ldots, \bar{n}_d^\varepsilon), \quad \bar{n}_k^\varepsilon = \max\Big(n_k^\varepsilon, \max_{j \neq k} n_{j,k}^\varepsilon \Big),$$

$$n_{j,k}^\varepsilon = \max_{m \leqslant n_j^{(\varepsilon)} - 1} n_j^{(\varepsilon,m)}.$$

The theorem is proved.

Proof of Proposition 3.2.3. Applying (1.3.3) to the sum in (3.2.14) we get

$$S_\mathbf{n}^{(1)} = \mu_1 Z_\mathbf{n} + R_\mathbf{n}, \quad Z_\mathbf{n} = \sum_{\mathbf{t} \in D_\mathbf{n}} \varepsilon_\mathbf{t}, \tag{3.2.3}$$

where

$$\begin{aligned} R_\mathbf{n} \;\; = \;\; & \xi_{n_1,n_2} - \xi_{n_1,0} - \xi_{0,n_2} + \xi_{0,0} \\ & + \eta_{n_1,n_2} - \eta_{0,n_2} + \zeta_{n_1,n_2} - \zeta_{n_1,0}, \end{aligned} \tag{3.2.4}$$

$$\xi_\mathbf{t} = \Phi^*(\mathbf{L})\varepsilon_\mathbf{t} = \sum_{\mathbf{k} \geqslant 0} \varphi_\mathbf{k}^* \varepsilon_{\mathbf{t}-\mathbf{k}},$$

$$\eta_{t_1,n_2} = \sum_{t_1=1}^{n_2} \bar{\varepsilon}_{t_1,t_2}, \quad \bar{\varepsilon}_{t_1,t_2} = B(L_1)\varepsilon_{t_1,t_2} = \sum_{l \geqslant 0} b_l \varepsilon_{t_1-l,t_2},$$

$$\zeta_{n_1,t_2} = \sum_{t_1=1}^{n_1} \hat{\varepsilon}_{t_1,t_2}, \quad \hat{\varepsilon}_{t_1,t_2} = D(L_2)\varepsilon_{t_1,t_2} = \sum_{l \geqslant 0} d_l \varepsilon_{t_1,t_2-l}.$$

$Z_\mathbf{n}$ is a sum of i.i.d. mean zero random variables $\varepsilon_\mathbf{t}$ with $E\,|\varepsilon_\mathbf{0}|^p < \infty$, for some $1 < p \leqslant 2$, therefore SLLN for $Z_\mathbf{n}$ holds. Then, in order to prove the proposition, we must prove that

$$|\mathbf{n}|^{-1} R_\mathbf{n} \xrightarrow{\text{a.s.}} 0 \quad \text{as} \;\; |\mathbf{n}| \to \infty. \tag{3.2.5}$$

We have

$$R_{\mathbf{n}} = R^1_{\mathbf{n}} + R^2_{\mathbf{n}} + R^3_{\mathbf{n}}, \tag{3.2.6}$$

where $R^1_{\mathbf{n}} = \xi_{n_1,n_2} - \xi_{n_1,0} - \xi_{0,n_2} + \xi_{0,0}$, $R^2_{\mathbf{n}} = \eta_{n_1,n_2} - \eta_{0,n_2}$, $R^3_{\mathbf{n}} = \zeta_{n_1,n_2} - \zeta_{n_1,0}$, therefore we must prove the relation (3.2.5) for each $R^i_{\mathbf{n}}$, $i = 1,2,3$. We start with $R^1_{\mathbf{n}}$ and prove that

$$\sum_{\mathbf{n}\geqslant\mathbf{1}} P(||\mathbf{n}|^{-1}\xi_{\mathbf{n}}| > \varepsilon) < \infty. \tag{3.2.7}$$

Since $\varepsilon_{\mathbf{t}}$ are i.i.d. and $E\varepsilon_{\mathbf{t}} = 0$, the moment inequality for $\xi_{\mathbf{t}} = \sum_{\mathbf{k}\geqslant 0} \varphi^*_{\mathbf{k}}\varepsilon_{\mathbf{t}-\mathbf{k}}$ yields

$$E\,|\xi_{\mathbf{t}}|^p \leqslant C\sum_{\mathbf{k}\geqslant 0} |\varphi^*_{\mathbf{k}}|^p\, E\,|\varepsilon_{\mathbf{t}-\mathbf{k}}|^p \leqslant C\sum_{\mathbf{k}\geqslant 0} |\varphi^*_{\mathbf{k}}|^p < \infty. \tag{3.2.8}$$

Therefore,

$$\sum_{\mathbf{n}\geqslant\mathbf{1}} P(||\mathbf{n}|^{-1}\xi_{\mathbf{n}}| > \varepsilon) \leqslant E|\xi_{\mathbf{1}}|^p \sum_{\mathbf{n}\geqslant\mathbf{1}} \varepsilon^{-p}|\mathbf{n}|^{-p} < \infty, \tag{3.2.9}$$

thus we get

$$|\mathbf{n}|^{-1}\xi_{\mathbf{n}} \xrightarrow{\text{a.s.}} 0. \tag{3.2.10}$$

Clearly, the same relation holds for $\xi_{n_1,0}$, ξ_{0,n_2}, $\xi_{0,0}$, and we get that

$$R^1_{\mathbf{n}} \xrightarrow{\text{a.s.}} 0. \tag{3.2.11}$$

Unfortunately, the moment inequality is too rough for other two terms $R^2_{\mathbf{n}}$ and $R^3_{\mathbf{n}}$ (this was noted in [55]), since, using low order moments for $\zeta_{\mathbf{t}}$ and $\eta_{\mathbf{t}}$, we get divergent series in (3.2.7), or we must require existence of moment of the order $2+\delta$, $\delta > 0$. Therefore we shall use one result from ergodic theory. Let us consider random variable $\eta_{\mathbf{t}}$ and let us denote

$$K_{\mathbf{n}} = \frac{1}{|\mathbf{n}|}\eta_{\mathbf{n}} = \frac{1}{n_1 n_2}\sum_{t_2=1}^{n_2} \bar{\varepsilon}_{n_1,t_2}. \tag{3.2.12}$$

Taking into account the definition of $\bar{\varepsilon}_{\mathbf{t}}$, we have

$$\begin{aligned} K_{\mathbf{n}} &= \frac{1}{n_1 n_2}\sum_{t_2=1}^{n_2}\left(\sum_{l\geqslant 0} b_l \varepsilon_{n_1-l,t_2}\right) = \frac{1}{n_1 n_2}\sum_{t_2=1}^{n_2}\left(\sum_{k=-\infty}^{n_1} b_{n_1-k}\varepsilon_{k,t_2}\right) \\ &= \frac{1}{n_1}\sum_{k=-\infty}^{n_1} b_{n_1-k}\left(\frac{1}{n_2}\sum_{t_2=1}^{n_2}\varepsilon_{k,t_2}\right). \end{aligned} \tag{3.2.13}$$

Therefore, denoting $f_k = \sup_{n_2} |n_2^{-1} \sum_{t_2=1}^{n_2} \varepsilon_{k,t_2}|$, we get

$$|K_{\mathbf{n}}| \leqslant \frac{1}{n_1} \sum_{k=-\infty}^{n_1} |b_{n_1-k}| \sup_{n_2} \left| \frac{1}{n_2} \sum_{t_2=1}^{n_2} \varepsilon_{k,t_2} \right| = \frac{1}{n_1} \sum_{k=-\infty}^{n_1} |b_{n_1-k}| \, f_k. \tag{3.2.14}$$

Applying Proposition 50.2 from [54] we get $E\,|f_k| < \infty$. Here it is worth to note that the same conclusion we can get under weaker moment condition $\varepsilon_{\mathbf{0}} \in L \log L$ using Exercise 50.4 from the same book, but this moment condition is insufficient in order to get (3.2.10) by means of the moment inequality. Note that (f_k) is a sequence of i.i.d. random variables, since for different k sequences $(\varepsilon_{k,t_2},\ t_2 \in \mathbb{Z})$ are independent. Let us denote $X_{n_1} = \sum_{k=-\infty}^{n_1} |b_{n_1-k}| \, f_k$. From Lemma 11 in [55] we know that $\sum_{l \geqslant 0} |b_l| < \infty$, therefore

$$E\,|X_l| \leqslant \sum_{k=-\infty}^{l} |b_{l-k}| \, E\,|f_k| \leqslant C \sum_{k=-\infty}^{l} |b_{l-k}| < \infty. \tag{3.2.15}$$

Sequence $\{X_n, n \geqslant 1\}$ is a stationary, ergodic (since it is generated by a sequence of i.i.d. random variables) and with finite mean, thus we can write

$$\frac{1}{n_1} X_{n_1} = \frac{1}{n_1} \left(\sum_{1}^{n_1} X_l - \sum_{1}^{n_1-1} X_l \right) = \frac{1}{n_1} \sum_{1}^{n_1} X_l - \frac{1}{n_1-1} \sum_{1}^{n_1-1} X_l \frac{n_1-1}{n_1} \xrightarrow{\text{a.s.}} 0, \tag{3.2.16}$$

when $n_1 \to \infty$. From (3.2.14) we get that, for all n_2,

$$K_{\mathbf{n}} \xrightarrow{\text{a.s.}} 0 \quad \text{as } n_1 \to \infty. \tag{3.2.17}$$

Since we need to show that $K_{\mathbf{n}} \xrightarrow{\text{a.s.}} 0$ as $|\mathbf{n}| \to \infty$, it remains to consider the case, where $n_2 \to \infty$ and n_1 stays bounded. If $|\mathbf{n}| \to \infty$ in a such way that n_1 is fixed and $n_2 \to \infty$, then

$$\frac{1}{n_2} \sum_{t_2=1}^{n_2} \bar{\varepsilon}_{n_1,t_2} \xrightarrow{\text{a.s.}} 0.$$

If a sequence $\mathbf{n}_k = (n_{1,k}, n_{2,k}),\ k \geqslant 1$, is such that $1 \leqslant n_{1,k} \leqslant k_0$, k_0 is some fixed number and $n_{2,k} \to \infty$, then the sequence $K_{\mathbf{n}_k}$ can be divided into k_0 subsequences in a such way, that in each subsequence the first index of summands is fixed. Then we apply the argument used above and get the convergence to zero a.s. of each subsequence, therefore the sequence itself also converges to zero a.s. Thus we have proved (3.2.17), if $|\mathbf{n}| \to \infty$.

It is easy to see that the convergence to zero a.s. of $(n_1 n_2)^{-1}\eta_{0,n_2}$ can be proved in the same way, therefore we have

$$R^2_{\mathbf{n}} \xrightarrow{\text{a.s.}} 0. \tag{3.2.18}$$

Due to symmetry the same consideration could be applied to $\zeta_{\mathbf{t}}$ and we get

$$R^3_{\mathbf{n}} \overset{a.s.}{\to} 0. \tag{3.2.19}$$

Collecting (3.2.6), (3.2.11), (3.2.18), (3.2.19) we get (3.2.5), and the proposition is proved.

Conclusions

The aims of the thesis were formulated in the introduction. In general, it is the extension of limit results for sums of random linear fields using BN decomposition. At the end we can state the following findings:

- The known results of application of BN decomposition to discrete index random linear fields were analyzed and the conclusion that the best application of BN decomposition is for the proof of CLT for sums of random linear fields has been made.

- The main advantage of BN decomposition is that the limit results proved for the sums of random fields of innovations can be easily applied to the sums of random linear fields. The only required condition is that the expression of the remainders are not very complicated. The complexity of the remainder term in BN decomposition depends on the summation sets and properties of innovations.

- Although BN decomposition method is quite flexible method it does not allow to analyze long memory processes since the essential conditions on the linear filter restrict the investigation of short processes only.

- Different discrete index martingale definitions in the plane and higher dimensions were analyzed. The definitions are caused by choice of σ-algebras. Three types of martingale differences were analyzed in more details.

- Based on the choice of martingale difference definition three CLT were formulated and proved for sums of random linear field generated by martingale difference innovations, by using BN decomposition

- The poofs via BN decomposition are quite simple, but, as it was shown in Chapter 3, ergodic theory for SLLN gives stronger results and the proofs are also not complicated.

In the future it would be interesting to analyze and to prove limit theorems for sums of random linear fields by using M. Gordin martingale-coboundary representation.

Appendix

A.1 Publication of the results

The main results of the thesis are published in the following papers:

1. P. Banys, Yu. Davydov, and V. Paulauskas. Remarks on the SLLN for linear random fields. Statist. Probab. Lett., 80:489–496, 2010.

2. P. Banys and V. Paulauskas. CLT for linear random fields with martingale increments. Lith. Math. J., accepted, to appear in 2011(4).

3. P. Banys. CLT for linear random fields with stationary martingale-difference innovations. Lith. Math. J., accepted, to appear in 2011(3).

A.2 Presentation of the results

Several presentations at conferences were given on the topics of the thesis

1. P. Banys. CLT for linear random fields with martingale increments. International Conference "Modern Stochastics: Theory and Applications II" held at Kiev University on 7–11 September, 2010 in Kiev, Ukraine.

2. P. Banys. CLT for linear random fields with martingale increments. 10th International Vilnius Conference on Probability and Mathematical Statistics held at Vilnius University on 28 June – 2 July, 2010 in Vilnius, Lithuania.

3. P. Banys, V. Paulauskas. CLT for linear random fields with martingale increments. LI Conference of the Lithuanian Mathematical Society held at Šiaulių University on 17–18 June 2010 in Šiauliai, Lithuania.

4. P. Banys, V. Paulauskas. Remarks on the SLLN for linear random fields. L Conference of the Lithuanian Mathematical Society held at Institute of Mathematics and Informatics on 18–19 June 2009 in Vilnius, Lithuania.

Also at the seminar of Departments of Financial Insurance Mathematics and Econometrical Analysis at Vilnius University:

1. Remarks on the SLLN for linear random fields, 28 April 2009,
2. CRT tiesiniams laukams su martingaliniais prieaugiais, 9 September 2010,
3. Limit theorems for linear random fields, 12 April 2011.

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