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Randomly Stopped Minimum, Maximum, Minimum of Sums and Maximum of Sums with Generalized Subexponential Distributions

Jūratė Karasevičienė [ID] and Jonas Šiaulys *[ID]

Institute of Mathematics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania;
jurate.karaseviciene@mif.vu.lt
* Correspondence: jonas.siaulys@mif.vu.lt

Abstract: In this paper, we find conditions under which distribution functions of randomly stopped minimum, maximum, minimum of sums and maximum of sums belong to the class of generalized subexponential distributions. The results presented in this article complement the closure properties of randomly stopped sums considered in the authors' previous work. In this work, as in the previous one, the primary random variables are supposed to be independent and real-valued, but not necessarily identically distributed. The counting random variable describing the stopping moment of random structures is supposed to be nonnegative, integer-valued and not degenerate at zero. In addition, it is supposed that counting random variable and the sequence of the primary random variables are independent. At the end of the paper, it is demonstrated how randomly stopped structures can be applied to the construction of new generalized subexponential distributions.

1. Introduction

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of real-valued and independent random variables (r.v.s) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$. Let η be a counting random variable, that is, an r.v. that is nonnegative, integer-valued and not degenerate at zero r.v. In addition, we suppose that the r.v. η and the sequence $\{\xi_1, \xi_2, \ldots\}$ are independent.

At first, let us define the main random objects under consideration: randomly stopped minimum, randomly stopped maximum, randomly stopped minimum of sums and randomly stopped maximum of sums.

- Let $\xi_{(0)} = 0$, $\xi_{(n)} = \min\{\xi_1, \ldots, \xi_n\}$ for $n \in \mathbb{N}$, and let

$$\xi_{(\eta)} = \begin{cases} 0 & \text{if } \eta = 0, \\ \min\{\xi_1, \ldots, \xi_\eta\} & \text{if } \eta \geqslant 1 \end{cases}$$

  be the *randomly stopped minimum* of r.v.s $\{\xi_1, \xi_2, \ldots\}$.
- Let $\xi^{(0)} = 0$, $\xi^{(n)} = \max\{\xi_1, \ldots, \xi_n\}$ for $n \in \mathbb{N}$, and let

$$\xi^{(\eta)} = \begin{cases} 0 & \text{if } \eta = 0, \\ \max\{\xi_1, \ldots, \xi_\eta\} & \text{if } \eta \geqslant 1 \end{cases}$$

  be the *randomly stopped maximum* of r.v.s $\{\xi_1, \xi_2, \ldots\}$.

- For the sequence of r.v.s $\{\xi_1, \xi_2, \ldots\}$ and the counting r.v. η, let $S_0 := 0$, $S_n := \xi_1 + \ldots + \xi_n$ if $n \in \mathbb{N}$, and in addition, let

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

  be the *randomly stopped sum* of r.v.s $\{\xi_1, \xi_2, \ldots\}$,

$$S_{(\eta)} = \begin{cases} 0 & \text{if } \eta = 0, \\ \min\{S_1, \ldots, S_\eta\} & \text{if } \eta \geqslant 1 \end{cases}$$

  be the *minimum of randomly stopped sums* of r.v.s $\{\xi_1, \xi_2, \ldots\}$, and

$$S^{(\eta)} = \begin{cases} 0 & \text{if } \eta = 0, \\ \max\{S_1, \ldots, S_\eta\} & \text{if } \eta \geqslant 1 \end{cases}$$

  be the *maximum of randomly stopped sums* of r.v.s $\{\xi_1, \xi_2, \ldots\}$.

By F_X, we denote the d.f. of an r.v. X, and by $\overline{F}$, we denote the tail function (t.f.) of a d.f. F, that is, $\overline{F}(x) = 1 - F(x)$ for $x \in \mathbb{R}$. We observe that the following equalities hold for positive x:

$$\overline{F}_{\xi_{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\xi_{(n)} > x)\mathbb{P}(\eta = n),$$

$$\overline{F}_{\xi^{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\xi^{(n)} > x)\mathbb{P}(\eta = n),$$

$$\overline{F}_{S_{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(S_{(n)} > x)\mathbb{P}(\eta = n),$$

$$\overline{F}_{S^{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(S^{(n)} > x)\mathbb{P}(\eta = n).$$

In this paper, we consider a sequence $\{\xi_1, \xi_2, \ldots\}$ of independent, real-valued and possibly non-identically distributed r.v.s. We suppose that some of the d.f.s of these r.v.s belong to the class of generalized subexponential distributions $\mathcal{OS}$, and we find conditions under which the d.f.s $F_{\xi_{(\eta)}}$, $F_{\xi^{(\eta)}}$, $F_{S_{(\eta)}}$ and $F_{S^{(\eta)}}$ remain in this class. The definition of class $\mathcal{OS}$ and some of its properties are described in Section 2.

We use the following notations for the asymptotic relations of arbitrary positive functions f and g: $f(x) = o(g(x))$ means that $\lim_{x \to \infty} f(x)/g(x) = 0$; $f(x) \underset{x \to \infty}{\sim} cg(x)$ with $c > 0$ means that $\lim_{x \to \infty} f(x)/g(x) = c$; $f(x) = O(g(x))$ means that $\limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty$; and $f(x) \underset{x \to \infty}{\asymp} g(x)$ means that

$$0 < \liminf_{x \to \infty} \frac{f(x)}{g(x)} \leqslant \limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty.$$

The motivation for investigation of randomly stopped structures comes mainly from insurance and finance, where questions related to extremal or rare events are traditionally considered, see e.g., [1–3]. In particular, exponential, Pareto, gamma, lognormal and loggamma distributions are extremely popular in actuarial mathematics. Mathematical aspects of risk theory related to calculation of ruin probabilities are considered in a large number of works; see [2–7] and references therein. From the mathematical point of view, the success of any insurance business depends on the asymptotic behavior of the distribution of S_η, $S^{(\eta)}$ and $S_{(\eta)}$. If the distribution of individual claim size Z is light-tailed, i.e.,

$$\mathbb{E}e^{\gamma Z} < \infty$$

for some $\gamma > 0$, then the corresponding ruin probability is also small for large values of the initial surplus and usually decreases with an exponential rate; see, e.g., [2,3,6,7]. If the individual claim size is heavy-tailed, i.e.,

$$\mathbb{E}e^{\gamma Z} = \infty$$

for all $\gamma > 0$, then the ruin probability decreases much more slowly with increasing initial surplus; see, e.g., [7]. Therefore, it is worth finding out at the beginning of the investigation whether the distribution of individual claim sizes is light-tailed or heavy-tailed. One of the most significant research directions in risk theory is the investigation of the ruin probability when the distribution of claim sizes is heavy-tailed. In this paper, we consider the class of generalized subexponential distribution $\mathcal{OS}$. From the description of this class given in Section 2, it follows that part of the distributions of this class have heavy tails, and the other part has light tails. However, even in the event that the random variable generating the claim flow has the $\mathcal{OS}$-class or similar regularity, it is possible to provide some considerable information about ruin probability of the model [8–13]. Results on asymptotic behavior of the ruin probability typically turn out to be different for different classes.

The closure properties in probability theory have a long history, back from the middle of the previous century. They appear as substantial supports in reliability theory, queuing theory, branching processes, risk theory, stochastic control, asset pricing and others. Bingham, Goldie and Teugels [14], Seneta [15] and Resnick [16] were among the first researchers to study closure problems. In the mentioned monographs, these authors fully explored the properties of slowly varying, regularly varying, and O-regularly varying functions, which are closely related to the closure properties of distribution functions. It is worth mentioning that the majority of the initial results related to the closure problems were obtained for the d.f.s of identically distributed r.v.s. Some of such results we describe in Section 4. A detailed analysis of closure problems is given in the book by Leipus et al. [17]. The main novelty of this paper is that not only identically distributed r.v.s are considered.

The rest of this paper is organized as follows. In Section 2, we describe a class of generalized subexponential distributions. The main results of this paper are formulated in Section 3. Section 4 consists of some results on closure under randomly stopped structures for regularity classes related with generalized subexponential distributions. The proofs of the main results are given in Sections 5 and 6. In Section 7, we present two examples to expose the analytical usefulness of our results; and finally, we provide some concluding remarks in Section 8.

2. Generalized Subexponentiality

Let ξ be an r.v. defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a d.f. F_ξ.

- *A d.f. F_ξ of a real-valued r.v. is said to be generalized subexponential, denoted $F_\xi \in \mathcal{OS}$, if*

$$\limsup_{x \to \infty} \frac{\overline{F_\xi * F_\xi}(x)}{\overline{F_\xi}(x)} < \infty,$$

where $F_\xi * F_\xi$ denotes the convolution of d.f. F_ξ with itself, i.e.,

$$F_\xi * F_\xi(x) = F_\xi^{*2}(x) := \int_{-\infty}^{\infty} F_\xi(x-y)\mathrm{d}F_\xi(y), \ x \in \mathbb{R},$$

and $\overline{F_\xi * F_\xi}$ denotes the t.f. of $F_\xi * F_\xi$, i.e.,

$$\overline{F_\xi * F_\xi}(x) = 1 - F_\xi * F_\xi(x), \ x \in \mathbb{R}.$$

For distributions of the nonnegative r.v.s, class $\mathcal{OS}$ was introduced by Klüppelberg [18], and later, for real-valued r.v.s, was studied by Yu and Wang [10], Shimura and

Watanabe [9], Baltrūnas et al. [19], Watanabe and Yamamuro [20], Cheng and Wang [21], Lin and Wang [22], Hägele and Lehtomaa [23], Konstantinides et al. [24], Yao and Taimre [25] and Mikutavičius and Šiaulys [26], among others.

In [9], the class of distributions $\mathcal{OS}$ is considered together with other distribution regularity classes. In that paper, several closedness properties of the class $\mathcal{OS}$ were proved. For example, it is shown that class $\mathcal{OS}$ is not closed under convolution roots. This means that there exists r.v. ξ such that n-fold convolution $F_\xi^{*n} \in \mathcal{OS}$ for all $n \geqslant 2$, but $F_\xi \notin \mathcal{OS}$. In [19], the simple conditions are provided under which the d.f. of the special form

$$F_\xi(x) = 1 - \exp\left\{ -\int_0^x q(u)\mathrm{d}u \right\}$$

belongs to the class $\mathcal{OS}$, where q is some integrable hazard rate function. For distributions of class $\mathcal{OS}$, the closure under tail-equivalence and the closure under convolution are established in [20]. The detailed proofs of these closures for nonnegative r.v.s are presented in [18] and for real-valued r.v.s in [10]. The closure under convolution means that in the case of independent r.v.s ξ_1, ξ_2 conditions $F_{\xi_1} \in \mathcal{OS}$, $F_{\xi_2} \in \mathcal{OS}$ imply that $F_{\xi_1} * F_{\xi_2} = F_{\xi_1+\xi_2} \in \mathcal{OS}$. The closure under tail-equivalence means that conditions $F_{\xi_1} \in \mathcal{OS}$, $\overline{F}_{\xi_1}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_2}(x)$ imply $F_{\xi_2} \in \mathcal{OS}$.

A counterexample, showing that $F_{\xi_1}, F_{\xi_2} \in \mathcal{OS}$ for independent r.v.s ξ_1, ξ_2 does not imply that $F_{\xi_1 \vee \xi_2} \in \mathcal{OS}$ can be found in [22]. Moreover, in that paper, the closure under minimum is established, which means that $F_{\xi_1}, F_{\xi_2} \in \mathcal{OS}$ for independent r.v.s ξ_1, ξ_2 imply $F_{\xi_1 \wedge \xi_2} \in \mathcal{OS}$. The authors of articles [24,26] consider when the distribution of the product of two independent random variables ξ, θ belongs to the class $\mathcal{OS}$. For instance, in [26], it is proved that d.f. $F_{\xi\theta}$ is generalized subexponential if $F_\xi \in \mathcal{OS}$ and θ is independent of ξ, nonnegative and not degenerated at zero.

3. Main Results

In this section, we formulate three theorems, which are the main assertions of this paper. It is easy to see that the results of this article complement the results obtained in [27]. The first theorem deals with the closure under the randomly stopped minimum.

Theorem 1. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued r.v.s, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If $F_{\xi_k} \in \mathcal{OS}$ for each k, then $F_{\xi_{(\eta)}}$ and $F_{S_{(\eta)}}$ belong to the class $\mathcal{OS}$, and it holds the following asymptotic relations:*

$$\overline{F}_{\xi_{(\eta)}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_{(\varkappa)}}(x) = \prod_{k=1}^{\varkappa} \overline{F}_{\xi_k}(x), \tag{1}$$

$$\overline{F}_{S_{(\eta)}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{S_{(\varkappa)}}(x) = \prod_{k=1}^{\varkappa} \overline{F}_{S_k}(x), \tag{2}$$

where $\varkappa = \min\{k \geqslant 1 : \mathbb{P}(\eta = k) > 0\}$.

The second theorem states the conditions for the maximum of randomly stopped, possibly differently distributed random variables to belong to the class $\mathcal{OS}$.

Theorem 2. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued r.v.s such that $F_{\xi_1} \in \mathcal{OS}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ with finite expectation $\mathbb{E}\eta$. If*

$$0 < \liminf_{x\to\infty} \inf_{n\geqslant 1} \frac{1}{n}\sum_{k=1}^n \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant \limsup_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n}\sum_{k=1}^n \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty, \tag{3}$$

then $F_{\xi^{(\eta)}} \in \mathcal{OS}$ and $\overline{F}_{\xi^{(\eta)}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$.

The third-last theorem describes the conditions under which the d.f. of the randomly stopped maximum of sums is generalized subexponential.

Theorem 3. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued r.v.s such that $F_{\xi_1} \in \mathcal{OS}$ and conditions* (3) *are satisfied. Let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ such that $\mathbb{E}e^{\lambda\eta} < \infty$ for all $\lambda > 0$. Then $F_{S(\eta)} \in \mathcal{OS}$ and $\overline{F}_{S(\eta)}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$.*

We will present the proofs of all theorems in Section 6. According to the statements of these theorems, many random variables with generalized subexponential distributions can be constructed. We will demonstrate such constructions in Section 7.

4. Results for Other Regularity Classes

In this section, we will describe classes of distributions related to the class $\mathcal{OS}$. For the described classes, we will present some results on their closure with respect to a randomly stopped structures. Section 4 of paper [27] lists many results on the closure of regularity classes with respect to a randomly stopped sum. Meanwhile, there are only a few similar results about the properties of other randomly stopped structures. We present below some of such results.

- *A d.f. F_ξ of a real-valued r.v. ξ is said to be dominatedly varying, denoted $F_\xi \in \mathcal{D}$, if*

$$\limsup_{x\to\infty} \frac{\overline{F}_\xi(yx)}{\overline{F}_\xi(x)} < \infty$$

  for all (or, equivalently, for some) $y \in (0, 1)$.

The class of dominatedly varying d.f.s $\mathcal{D}$ was introduced by Feller [28] and later considered in [14,20,29–32], among others. We remark only that $\mathcal{D} \subset \mathcal{OS}$. The following assertion on $F_{S_\eta} \in \mathcal{D}$ is presented in Theorem 4 of [33].

Theorem 4. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued r.v.s with common d.f. $F_\xi \in \mathcal{D}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then $F_{S_\eta} \in \mathcal{D}$ if $\mathbb{E}\eta^{p+1} < \infty$ for some*

$$p > J_{F_\xi}^+ := -\lim_{y\to\infty} \frac{1}{\log y} \log \liminf_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)}.$$

In the inhomogeneous case, when summands are not necessary identically distributed, the following statement is obtained in Theorem 2.1 of [34].

Theorem 5. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent nonnegative r.v.s, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then, $F_{S_\eta} \in \mathcal{D}$ if the following three conditions are satisfied:*

(i)   *$F_{\xi_\varkappa} \in \mathcal{D}$ for some $\varkappa \in \mathrm{supp}(\eta) := \{n \in \mathbb{N}_0 : \mathbb{P}(\eta = n) > 0\}$,*

(ii) *$\displaystyle\limsup_{x\to\infty} \sup_{n>\varkappa} \frac{1}{n\overline{F}_{\xi_\varkappa}(x)} \sum_{i=1}^n \overline{F}_{\xi_i}(x) < \infty$,*

(iii) *$\mathbb{E}\eta^{p+1} < \infty$ for some $p > J_{F_{\xi_\varkappa}}^+$.*

- *A d.f. F_ξ of a real-valued r.v. ξ is said to be consistently varying, denoted $F_\xi \in \mathcal{C}$, if*

$$\lim_{y\nearrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1, \quad \textit{or, equivalently,} \quad \lim_{y\searrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

A class of consistently varying distributions was introduced by Cline [35] as a generalization of regularly varying distributions, and subsequently has been considered in the various contexts; see, for instance, [31,36–45]. It follows from definitions that $\mathcal{C} \subset \mathcal{D} \subset \mathcal{OS}$.

The following two assertions on $F_{\xi(\eta)} \in \mathcal{C}$ and $F_{S(\eta)} \in \mathcal{C}$ are presented in Theorem 4 and 5 of [43].

Theorem 6. *Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent real-valued r.v.s, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Then, d.f. $F_{\xi(\eta)}$ belongs to the class $\mathcal{C}$ if the following three conditions hold:*

(i) $F_{\xi_\varkappa} \in \mathcal{C}$ *for some $\varkappa \in \mathrm{supp}(\eta)$,*

(ii) *for each $k \neq \varkappa$, either $F_{\xi_k} \in \mathcal{C}$ or $\overline{F}_{\xi_k}(x) = o\big(\overline{F}_{\xi_\varkappa}(x)\big)$,*

(iii) $\displaystyle\limsup_{x \to \infty} \sup_{n \geqslant \varkappa} \frac{1}{\varphi(n)\overline{F}_{\xi_\varkappa}(x)} \sum_{k=1}^{n} \overline{F}_{\xi_k}(x) < \infty$,

where $\big\{\varphi(n)\big\}_{n=1}^{\infty}$ is a positive sequence such that $\mathbb{E}\big(\varphi(\eta)\mathbb{I}_{[1,\infty)}(\eta)\big) < \infty$.

Theorem 7. *Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent real-valued r.v.s, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Then, $F_{S(\eta)} \in \mathcal{C}$ if the following three conditions are satisfied:*

(i) $F_{\xi_k} \in \mathcal{C}$ *for each $k \in \mathbb{N}$,*

(ii) $\displaystyle\limsup_{x \to \infty} \sup_{n \geqslant 1} \frac{1}{n\overline{F}_{\xi_1}(x)} \sum_{k=1}^{n} \overline{F}_{\xi_k}(x) < \infty$,

(iii) $\mathbb{E}\eta^{p+1} < \infty$ *for some $p > J_{F_{\xi_1}}^{+}$.*

- *A d.f. F_ξ of a real-valued r.v. ξ is said to be regularly varying with index $\alpha \geqslant 0$, denoted $F_\xi \in \mathcal{R}_\alpha$, if*

$$\limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha} \quad \text{for any } y > 0.$$

  By $R := \bigcup_{\alpha \geqslant 0} \mathcal{R}_\alpha$ we denote all regularly varying d.f.s.

The standard properties of regularly varying d.f.s and some historical notes on such functions can be found in [14,16,46–48]. We remark only that $\mathcal{R} \subset \mathcal{C} \subset \mathcal{D} \subset \mathcal{OS}$. The following assertion on d.f. $F_{\xi(\eta)}$ is presented in Theorem 5.1(i) of [49].

Theorem 8. *Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent real-valued r.v.s, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$ such that $\mathbb{E}\eta < \infty$. Then, $\overline{F}_{\xi(\eta)}(x) \underset{x \to \infty}{\sim} \mathbb{E}\eta\overline{F}_{\xi_1}(x)$, and hence, $F_{\xi(\eta)} \in \mathcal{R}_\alpha$ if and only if $F_{\xi_1} \in \mathcal{R}_\alpha$.*

The assertion bellow on d.f. $F_{\xi_{(\eta)}}$ can be found in Theorem 4 of [50].

Theorem 9. *Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent real-valued r.v.s. Then, d.f. $\overline{F}_{\xi_k}$ belongs to the class $\mathcal{R}$ for all $k \in \mathbb{N}$ if and only if $F_{\xi_{(\eta)}} \in \mathcal{R}$ for every counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$.*

- *A d.f. F_ξ of a real-valued r.v. ξ is said to belong to the class of generalized long-tailed distributions $\mathcal{OL}$ if for any (equivalently, for some) $y > 0$*

$$\limsup_{x \to \infty} \frac{\overline{F}(x - y)}{\overline{F}(x)} < \infty.$$

The class of d.f.s $\mathcal{OL}$ was proposed by Shimura and Watanabe in [9]. The main properties of the functions from this class are discussed in [9,51–57], among others. We note only that $\mathcal{OS} \subset \mathcal{OL}$. In Theorem 2.4 of [54], the assertion below on $F_{S(\eta)} \in \mathcal{OS}$ is presented.

Theorem 10. *Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of independent real-valued r.v.s, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. Then, the d.f. of the randomly stopped maximum of*

random sums $F_{S(\eta)}$ belongs to the class $\mathcal{OL}$ if $F_{\xi_1} \in \mathcal{OL}$ and r.v. η satisfies the following two requirements:

$$\limsup_{n\to\infty} \frac{\mathbb{P}(\eta > n)}{\max_{1\leqslant k\leqslant n} \mathbb{P}(\eta = k)} < \infty, \quad \limsup_{n\to\infty} \frac{\mathbb{P}(\eta \geqslant n)}{\mathbb{P}(\eta = n)} < \infty.$$

5. Auxiliary Lemmas

In this section, we will present some auxiliary lemmas that will be applied to the derivations of the main Theorems 1–3. The first lemma is a collection of the basic properties of d.f.s from class $\mathcal{OS}$.

Lemma 1. *Let X and Y be two real-valued r.v.s with corresponding d.f.s F_X and F_Y. Then, the following statements are correct:*

(i) *$F_X \in \mathcal{OS}$ if and only if $\sup\limits_{x\in\mathbb{R}} \dfrac{\overline{F_X * F_X}(x)}{\overline{F}_X(x)} < \infty$.*

(ii) *If $F_X \in \mathcal{OS}$ and $\overline{F}_Y(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$, then $F_Y \in \mathcal{OS}$.*

(iii) *If $F_X \in \mathcal{OS}$ and $F_Y \in \mathcal{OS}$, then $F_X * F_Y \in \mathcal{OS}$.*

(iv) *If $F_X \in \mathcal{OS}$, then $F_X \in \mathcal{OL}$ i.e., $\limsup\limits_{x\to\infty} \dfrac{\overline{F}_X(x-1)}{\overline{F}_X(x)} < \infty$.*

(v) *If $F_X \in \mathcal{OS}$ and $\overline{F}_Y(x) = O(\overline{F}_X(x))$, then $F_X * F_Y \in \mathcal{OS}$ and $\overline{F_X * F_Y}(x) \underset{x\to\infty}{\asymp} \overline{F}_X(x)$.*

The complete proof of the lemma is presented in Lemma 1 of [27], and proofs of the separate parts of the lemma can be found in [9,10,18,20].

The next lemma is a refined version of Lemma 2 from [27]. As a result, we present this lemma together with a modified proof.

Lemma 2. *Let $\{\xi_1, \xi_2 \ldots\}$ be a sequence of independent real-valued r.v.s, for which $F_{\xi_1} \in \mathcal{OS}$, and for other indices $k \geqslant 2$, either $F_{\xi_k} \in \mathcal{OS}$ or $\overline{F}_{\xi_k}(x) = O(\overline{F}_{\xi_1}(x))$. Then,*

$$\overline{F}_{S_n}(x) \underset{x\to\infty}{\asymp} \overline{\underset{k\in\mathcal{A}_n}{\circledast} F_{\xi_k}}(x) \tag{4}$$

where $\mathcal{A}_n := \{k \in \{1, 2, \ldots, n\} : F_{\xi_k} \in \mathcal{OS}\}$, and hence, $F_{S_n} \in \mathcal{OS}$ for all $n \in \mathbb{N}$.

Proof. If $n = 1$, then the statement is obvious because $S_1 = \xi_1$. If $n = 2$, then two options are possible: $F_{\xi_2} \in \mathcal{OS}$ or $\overline{F}_{\xi_2}(x) = O(\overline{F}_{\xi_1}(x))$. In the first case, $F_{S_2} = F_{\xi_1} * F_{\xi_2}$ In the second case, $\overline{F}_{S_2}(x) \underset{x\to\infty}{\asymp} \overline{F}_{\xi_1}(x)$ by part (v) of the Lemma 1. The asymptotic relation of the lemma holds for both cases.

Let us suppose that the asymptotic relation (4) is valid for some $n = N$, i.e.,

$$\overline{F}_{S_N}(x) \underset{x\to\infty}{\asymp} \overline{\underset{k\in\mathcal{A}_N}{\circledast} F_{\xi_k}}(x) = \overline{F_{\xi_1} * F_{\xi_{k_1}} * \ldots F_{\xi_{k_r}}}(x) \tag{5}$$

where $\mathcal{A}_N = \{1, k_1, \ldots, k_r\} = \{k \in \{1, 2, \ldots, N\} : F_{\xi_k} \in \mathcal{OS}\}$.

The above relation and parts (ii), (iii) of Lemma 1 imply that $F_{S_N} \in \mathcal{OS}$. The tail of any d.f. from class $\mathcal{OS}$ remains positive throughout the set of real numbers. Hence, relation (5) implies that

$$0 < \inf_{x\in\mathbb{R}} \frac{\overline{F}_{S_N}(x)}{\overline{\underset{k\in\mathcal{A}_N}{\circledast} F_{\xi_k}}(x)} \leqslant \sup_{x\in\mathbb{R}} \frac{\overline{F}_{S_N}(x)}{\overline{\underset{k\in\mathcal{A}_N}{\circledast} F_{\xi_k}}(x)} < \infty. \tag{6}$$

For $n = N + 1$, we have two possibilities: either $F_{\xi_{N+1}} \in \mathcal{OS}$, or $\overline{F}_{\xi_{N+1}}(x) = O(\overline{F}_{\xi_1}(x))$. If $F_{\xi_{N+1}} \in \mathcal{OS}$, then according to (6), it holds that

$$\overline{F}_{S_{N+1}}(x) = \int_{-\infty}^{\infty} \overline{F}_{S_N}(x-y)\mathrm{d}F_{\xi_{N+1}}(y)$$

$$\leqslant c_{1N} \int_{-\infty}^{\infty} \overline{\underset{k\in\mathcal{A}_N}{\text{\Large *}} F_{\xi_k}}(x-y)\mathrm{d}F_{\xi_{N+1}}(y)$$

$$= c_{1N} \overline{\underset{k\in\mathcal{A}_{N+1}}{\text{\Large *}} F_{\xi_k}}(x), x \in \mathbb{R}$$

for some positive quantity c_{1N} not depending on x and $\mathcal{A}_{N+1} = \mathcal{A}_N \cup \{N+1\}$. Similarly,

$$\overline{F}_{S_{N+1}}(x) \geqslant c_{2N} \overline{\underset{k\in\mathcal{A}_{N+1}}{\text{\Large *}} F_{\xi_k}}(x), x \in \mathbb{R}$$

for some positive quantity c_{2N}.

The above two estimates imply the asymptotic relation (4) in the case $n = N+1$.

If $\overline{F}_{\xi_{N+1}} = O\big(\overline{F}_{\xi_1}(x)\big)$, then according to (6), for positive x, we have that

$$\frac{\overline{F}_{\xi_{N+1}}(x)}{\overline{F}_{S_N}(x)} \leqslant \frac{1}{c_{2N}} \frac{\overline{F}_{\xi_{N+1}}(x)}{\overline{\underset{k\in\mathcal{A}_N}{\text{\Large *}} F_{\xi_k}}(x)} \leqslant \frac{1}{c_{2N} \prod_{k\in\mathcal{A}_N\setminus\{1\}} \overline{F}_{\xi_k}(0)} \frac{\overline{F}_{\xi_{N+1}}(x)}{\overline{F}_{\xi_1}(x)}$$

because

$$\overline{F}_{\xi_1}(x) \prod_{k\in\mathcal{A}_N\setminus\{1\}} \overline{F}_{\xi_k}(0) = \mathbb{P}\big(\xi_1 > x, \xi_{k_1} > 0, \ldots, \xi_{k_r} > 0\big) \leqslant \mathbb{P}\big(\xi_1 + \xi_{k_1} + \ldots + \xi_{k_r} > x\big),$$

where $\{k_1, \ldots, k_r\} = \mathcal{A}_N \setminus \{1\}$. Consequently, $\overline{F}_{\xi_{N+1}}(x) = O\big(\overline{F}_{S_N}(x)\big)$, and according to the part (v) of Lemma 1, we obtain that

$$\overline{F}_{S_{N+1}}(x) = \overline{F_{S_N} * F_{\xi_{N+1}}}(x) \underset{x\to\infty}{\asymp} \overline{F}_{S_N}(x) \underset{x\to\infty}{\asymp} \overline{\underset{k\in\mathcal{A}_{N+1}}{\text{\Large *}} F_{\xi_k}}(x)$$

with $\mathcal{A}_{N+1} = \mathcal{A}_N$ in the case.

We derived the asymptotic relation (4) for $n = N+1$ by supposing that this relation holds for $n = N$. Due to the induction principle, the asymptotic relation (4) is valid for all $n \in \mathbb{N}$. The assertion $F_{S_n} \in \mathcal{OS}$ follows from (4) after using part (ii) and (iii) of Lemma 1. This finishes the proof of the lemma. $\square$

The following technical assertion is Lemma 3 in [27].

Lemma 3. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued r.v.s, for which $F_{\xi_1} \in \mathcal{OS}$ and*

$$\limsup_{x\to\infty} \sup_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} < \infty$$

Then, there exists a constant c_3, for which

$$\overline{F}_{S_n}(x) \leqslant c_3^{n-1} \overline{F}_{\xi_1}(x) \tag{7}$$

for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$.

For maximum of sums $S^{(n)} = \max\{S_1, S_2, \ldots, S_n\}$ it holds the following similar statement.

Lemma 4. *Under condition of Lemma 3, there exists a constant c_4 such that*

$$\overline{F}_{S^{(n)}}(x) \leqslant c_4^n \overline{F}_{\xi_1}(x)$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proof. If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$\overline{F}_{S^{(n)}}(x) = \mathbb{P}\big(\max\{S_1, S_2, \ldots S_n\} > x \big)$$
$$= \mathbb{P}\Big(\bigcup_{k=1}^{n} \{S_k > x\} \Big)$$
$$\leqslant \sum_{k=1}^{n} \mathbb{P}\{S_k > x\}.$$

Let us suppose $c_4 = \max\{c_3, 2\}$. According to estimate (7) from Lemma 3, we have that

$$\overline{F}_{S^{(n)}}(x) \leqslant \sum_{k=1}^{n} c_3^{k-1} \overline{F}_{\xi_1}(x) \leqslant \overline{F}_{\xi_1}(x) \frac{c_4^n - 1}{c_4 - 1}$$
$$\leqslant c_4^n \overline{F}_{\xi_1}(x).$$

The estimate of the lemma is proved. $\square$

The next lemma is on the minimum $X_{(n)} = \min\{X_1, \ldots, X_n\}$ of a collection of independent r.v.s $\{X_1, \ldots, X_n\}$. The similar assertion is presented in Lemma 3.1 of [22]. Unfortunately, the proof presented there is suitable for nonnegative absolutely continuous r.v.s. Hence, we present the detailed proof of the lemma below.

Lemma 5. *Let $\{X_1, X_2, \ldots X_n\}$, $n \in \mathbb{N}$ be a collection of independent real-valued r.v.s with d.f.s $\{F_{X_1}, F_{X_2}, \ldots F_{X_n}\}$. If $F_{X_k} \in \mathcal{OS}$ for all $k \in \{1, 2, \ldots n\}$, then d.f. $F_{X_{(n)}}$ of $X_{(n)} = \min\{X_1, X_2, \ldots, X_n\}$ belongs to the class $\mathcal{OS}$ as well.*

Proof. If $n = 1$, then assertion of the lemma is trivial. If $n \geqslant 2$, then

$$\min\{X_1, X_2, \ldots, X_n\} = \min\big\{ \min\{X_1, X_2, \ldots, X_{n-1}\}, X_n \big\}.$$

It follows from this equality that it is sufficient to prove the statement of the lemma for the case $n = 2$, i.e., it is sufficient for us to prove such a statement:

$$X, Y \text{ independent r.v.s } F_X \in \mathcal{OS}, F_Y \in \mathcal{OS} \ \Rightarrow \ F_{X \wedge Y} \in \mathcal{OS}. \tag{8}$$

- At first, let us suppose that X and Y are absolutely continuous r.v.s. In such a case,

$$\overline{F}_{\min(X,Y)}(x) = \overline{F}_{X \wedge Y}(x) = \overline{F}_X(x)\overline{F}_Y(x)$$
$$= \int_{x}^{\infty} f_X(y)\mathrm{d}y \int_{x}^{\infty} f_Y(y)\mathrm{d}y$$

with density functions f_X and f_Y. For $x \in \mathbb{R}$, we have

$$\frac{\overline{F_{X \wedge Y}^{*2}}(x)}{\overline{F}_{X \wedge Y}(x)} = \frac{1}{\overline{F}_{X \wedge Y}(x)} \int\limits_{-\infty}^{\infty} \overline{F}_{X \wedge Y}(x-y) \mathrm{d}F_{X \wedge Y}(y)$$

$$= -\frac{1}{\overline{F}_X(x)\overline{F}_Y(x)} \int\limits_{-\infty}^{\infty} \overline{F}_X(x-y)\overline{F}_Y(x-y) \mathrm{d}\overline{F}_X(y)\overline{F}_Y(y)$$

$$= \int\limits_{-\infty}^{\infty} \frac{\overline{F}_X(x-y)\overline{F}_Y(x-y)\overline{F}_Y(y)f_X(y)\mathrm{d}y}{\overline{F}_X(x)\overline{F}_Y(x)}$$

$$+ \int\limits_{-\infty}^{\infty} \frac{\overline{F}_X(x-y)\overline{F}_Y(x-y)\overline{F}_X(y)f_Y(y)\mathrm{d}y}{\overline{F}_X(x)\overline{F}_Y(x)}. \tag{9}$$

If Y_1 and Y_2 are independent copies of Y, then

$$\overline{F}_Y(x-y)\overline{F}_Y(y) = \mathbb{P}(Y_1 > x-y)\mathbb{P}(Y_2 > y) \leqslant \mathbb{P}(Y_1 + Y_2 > x) = \overline{F_Y^{*2}}(x).$$

for all $x, y \in \mathbb{R}$. Hence, condition $F_Y \in \mathcal{OS}$ implies that

$$\sup_{x,y \in \mathbb{R}} \frac{\overline{F}_Y(x-y)\overline{F}_Y(y)}{\overline{F}_Y(x)} \leqslant \sup_{x \in \mathbb{R}} \frac{\overline{F_Y^{*2}}(x)}{\overline{F}_Y(x)} \leqslant c_5$$

for some constant c_5 according to Lemma 1(i). Similarly for r.v. X, we obtain that

$$\sup_{x,y \in \mathbb{R}} \frac{\overline{F}_X(x-y)\overline{F}_X(y)}{\overline{F}_X(x)} \leqslant \sup_{x \in \mathbb{R}} \frac{\overline{F_X^{*2}}(x)}{\overline{F}_X(x)} \leqslant c_6$$

for some another constant c_6. Hence, according to the decomposition (9),

$$\sup_{x \in \mathbb{R}} \frac{\overline{F_{X \wedge Y}^{*2}}(x)}{\overline{F}_{X \wedge Y}(x)} \leqslant \max\{c_5, c_6\} \sup_{x \in \mathbb{R}} \left(\int\limits_{-\infty}^{\infty} \frac{\overline{F}_X(x-y)f_X(y)\mathrm{d}y}{\overline{F}_X(x)} + \int\limits_{-\infty}^{\infty} \frac{\overline{F}_Y(x-y)f_Y(y)\mathrm{d}y}{\overline{F}_Y(x)} \right)$$

$$= \max\{c_5, c_6\} \sup_{x \in \mathbb{R}} \left(\frac{\overline{F_X^{*2}}(x)}{\overline{F}_X(x)} + \frac{\overline{F_Y^{*2}}(x)}{\overline{F}_Y(x)} \right)$$

$$\leqslant \max\{c_5, c_6\}(c_5 + c_6),$$

which implies that $F_{X \wedge Y} \in \mathcal{OS}$ due to Lemma 1(i) again.

- Now, let us suppose that r.v.s X and Y are not necessarily absolutely continuous. At first, let us consider r.v. X. Since $\mathcal{OS} \subset \mathcal{OL}$ (see Lemma 1(iv)), we have that $F_X \in \mathcal{OL}$. If function F_X belongs to $\mathcal{OL}$, then the function $F_X(\log x)$, $x > 0$, is nonincreasing O-regularly varying, according to Bingham [14], because

$$\limsup_{x \to \infty} \frac{\overline{F}_X(\log xy)}{\overline{F}_X(\log x)} = \limsup_{x \to \infty} \frac{\overline{F}_X(\log x + \log y)}{\overline{F}_X(\log x)} < \infty.$$

for an arbitrary $y > 0$.
From the representation Theorem-see Theorem 2.2.7 in [14], or Theorem A.1 together with Definition A.4 and Remark in page 100 of [15]-we have that

$$\overline{F}_X(\log x) = \exp\left\{ a(x) - \int\limits_{A}^{x} \frac{b(t)}{t} \mathrm{d}t \right\}$$

for all $x \geqslant A$. Here, A is a positive number, a and $b \geqslant 0$ are bounded integrable functions. Therefore, for all $x \geqslant \log A$

$$\overline{F}_X(x) = \exp\left\{a^*(x) - \int\limits_{\log A}^{x} b^*(u)\mathrm{d}u\right\} \tag{10}$$

with bounded and integrable functions a^* and $b^* \geqslant 0$. Since b^* is a positive-bounded and integrable function, function

$$G(x) = \left(1 - \exp\left\{-\int\limits_{\log A}^{x} b^*(u)\mathrm{d}u\right\}\right)\mathbb{I}_{[\log A,\infty)}(x)$$

is an absolutely continuous d.f. from class $\mathcal{OL}$ with tail function

$$\overline{G}(x) = \mathbb{I}_{(-\infty,\log A)}(x) + \exp\left\{-\int\limits_{\log A}^{x} b^*(u)\mathrm{d}u\right\}\mathbb{I}_{[\log A,\infty)}(x)$$

In addition, the boundedness of function a^* in (10) implies that

$$\overline{F}_X(x) \underset{x\to\infty}{\asymp} \overline{G}(x)$$

In a similar way, we derive that

$$\overline{F}_Y(x) \underset{x\to\infty}{\asymp} \overline{H}(x)$$

for some absolutely continuous d.f. $H \in \mathcal{OL}$. According to Lemma 1(ii) and the first part of the proof d.f., $1 - \overline{GH}$ belongs to the class $\mathcal{OS}$, and from Lemma 1(ii), again, we obtain $F_{X \wedge Y} \in \mathcal{OS}$ because

$$\overline{F_{X \wedge Y}}(x) = \overline{F}_X(x)\overline{F}_Y(x) \underset{x\to\infty}{\asymp} \overline{G}(x)\overline{H}(x).$$

It follows from both parts of the proof that relation (8) holds. At the same time, the assertion of the lemma is proved. □

The last technical lemma is necessary for the examination of randomly stopped maximum of sums. For the proof of the lemma below, we use the revised episodes of proof of Theorem 2 from article [27].

Lemma 6. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued r.v.s such that d.f. $F_{\xi_1} \in \mathcal{OS}$ and*

$$\liminf_{x\to\infty} \inf_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} > 0,$$

Then,

$$\inf_{x\in\mathbb{R}} \frac{\overline{F}_{S^{(n)}}(x)}{\overline{F}_{\xi_1}(x)} \geqslant \inf_{x\in\mathbb{R}} \frac{\overline{F}_{S_n}(x)}{\overline{F}_{\xi_1}(x)} \geqslant c_7^{n-1} \tag{11}$$

for all $n \geqslant 1$ and some $c_7 > 0$, where $S_n = \xi_1 + \ldots + \xi_n$ and $S^{(n)} = \max\{S_1, S_2, \ldots S_n\}$.

Proof. Conditions of the lemma give that

$$\inf_{k\geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \geqslant \Delta \tag{12}$$

for all $x \geqslant x_\Delta$ and some positive Δ. If $x < x_\Delta$, then

$$\inf_{k \geqslant 1} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \geqslant \inf_{k \geqslant 1} \overline{F}_{\xi_k}(x_\Delta) = \inf_{k \geqslant 1} \frac{\overline{F}_{\xi_k}(x_\Delta)}{\overline{F}_{\xi_1}(x_\Delta)} \, \overline{F}_{\xi_1}(x_\Delta)$$

$$\geqslant \Delta \, \overline{F}_{\xi_1}(x_\Delta) := c_8 > 0$$

due to the assumption $F_{\xi_1} \in \mathcal{OS}$. The derived inequalities imply that

$$\overline{F}_{\xi_k}(x) \geqslant c_8 \, \overline{F}_{\xi_1}(x)$$

for all $x \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$. Using the last estimate, we obtain

$$\begin{aligned}
\overline{F}_{S_2}(x) &= \int_{-\infty}^{\infty} \frac{\overline{F}_{\xi_2}(x-y)}{\overline{F}_{\xi_1}(x-y)} \, \overline{F}_{\xi_1}(x-y) \, \mathrm{d}F_{\xi_1}(y) \\
&\geqslant c_8 \, \overline{F_{\xi_1} * F_{\xi_1}}(x) \\
&\geqslant c_8 \, \overline{F}_{\xi_1}(0) \overline{F}_{\xi_1}(x), \ x \in \mathbb{R}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\overline{F}_{S_3}(x) &= \int_{-\infty}^{\infty} \frac{\overline{F}_{S_2}(x-y)}{\overline{F}_{\xi_1}(x-y)} \, \overline{F}_{\xi_1}(x-y) \, \mathrm{d}F_{\xi_3}(y) \geqslant c_8 \overline{F}_{\xi_1}(0) \int_{-\infty}^{\infty} \overline{F}_{\xi_1}(x-y) \, \mathrm{d}F_{\xi_3}(y) \\
&= c_8 \, \overline{F}_{\xi_1}(0) \int_{-\infty}^{\infty} \overline{F}_{\xi_3}(x-y) \, \mathrm{d}F_{\xi_1}(y) \geqslant c_8^2 \, \overline{F}_{\xi_1}(0) \int_{-\infty}^{\infty} \overline{F}_{\xi_1}(x-y) \, \mathrm{d}F_{\xi_1}(y) \\
&= c_8^2 \, \overline{F}_{\xi_1}(0) \overline{F_{\xi_1} * F_{\xi_1}}(x) \geqslant c_8^2 \left(\overline{F}_{\xi_1}(0)\right)^2 \overline{F}_{\xi_1}(x), \ x \in \mathbb{R}.
\end{aligned}$$

Continuing tbe process, we obtain

$$\overline{F}_{S_n}(x) \geqslant \left(c_8 \overline{F}_{\xi_1}(0)\right)^{n-1} \overline{F}_{\xi_1}(x)$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Hence, the second estimate in (11) holds with $c_7 = c_8 \, \overline{F}_{\xi_1}(0)$. The first inequality in (11) is obvious because

$$\overline{F}_{S_n}(x) = \mathbb{P}(S_n > x) \leqslant \mathbb{P}(\max(S_1, S_2, ..S_n) > x) = \overline{F}_{S^{(n)}}(x)$$

for an arbitrary real number x. Lemma 6 is proved. □

6. Proofs of the Main Results

In this section, we prove all main results of the paper.

6.1. Proof of Theorem 1

Proof. At first, let us consider the first part of the theorem. Due to Lemma 1(ii), it is enough to prove the asymptotic relations (1) because $F_{\xi_{(\varkappa)}} \in \mathcal{OS}$ by Lemma 5. By definition of the randomly stopped minimum for positive x, we have

$$\overline{F}_{\xi_{(\eta)}}(x) = \sum_{n=1}^{\infty} \overline{F}_{\xi_{(n)}}(x)\mathbb{P}(\eta = n)$$

$$= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa) + \sum_{n=\varkappa+1}^{\infty} \overline{F}_{\xi_{(n)}}(x)\mathbb{P}(\eta = n)$$

$$= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa) + \overline{F}_{\xi_{(\varkappa)}}(x)\sum_{n=\varkappa+1}^{\infty} \mathbb{P}(\eta = n)\prod_{k=\varkappa+1}^{n}\overline{F}_{\xi_k}(x)$$

$$= \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa)\left(1 + \frac{1}{\mathbb{P}(\eta = \varkappa)}\sum_{n=\varkappa+1}^{\infty}\mathbb{P}(\eta = n)\prod_{k=\varkappa+1}^{n}\overline{F}_{\xi_k}(x)\right)$$

$$\leqslant \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa)\left(1 + \overline{F}_{\xi_{\varkappa+1}}(x)\frac{\mathbb{P}(\eta \geqslant \varkappa + 1)}{\mathbb{P}(\eta = \varkappa)}\right).$$

On the other hand, for all $x > 0$,

$$\overline{F}_{\xi_{(\eta)}}(x) \geqslant \overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa).$$

On this basis, we can assert that

$$0 < \liminf_{x\to\infty}\frac{\overline{F}_{\xi_{(\eta)}}(x)}{\overline{F}_{\xi_{(\varkappa)}}(x)} \leqslant \limsup_{x\to\infty}\frac{\overline{F}_{\xi_{(\eta)}}(x)}{\overline{F}_{\xi_{(\varkappa)}}(x)} < \infty.$$

Hence, relation (1) holds, and by Lemma 5, it follows that $\overline{F}_{\xi_{(\eta)}} \in \mathcal{OS}$.

Let us consider the second part of the theorem. By Lemma 2, we have that $F_{S_k} \in \mathcal{OS}$ for each $k \in \mathbb{N}$, and by Lemma 5, we have that $F_{S_{(\varkappa)}} \in \mathcal{OS}$. Hence, it suffices to prove the asymptotic relation (2) in order to obtain $F_{S_{(\eta)}} \in \mathcal{OS}$. Similar to the first part of the proof, we obtain that

$$\overline{F}_{S_{(\eta)}}(x) \leqslant \overline{F}_{S_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa)\left(1 + \overline{F}_{S_{\varkappa+1}}(x)\frac{\mathbb{P}(\eta \geqslant \varkappa + 1)}{\mathbb{P}(\eta = \varkappa)}\right),$$

and

$$\overline{F}_{S_{(\eta)}}(x) \geqslant \overline{F}_{S_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa)$$

for all positive x. Therefore,

$$0 < \liminf_{x\to\infty}\frac{\overline{F}_{S_{(\eta)}}(x)}{\overline{F}_{S_{(\varkappa)}}(x)} \leqslant \limsup_{x\to\infty}\frac{\overline{F}_{S_{(\eta)}}(x)}{\overline{F}_{S_{(\varkappa)}}(x)} < \infty,$$

and the desired relation (2) follows. This finishes the proof of the theorem. □

6.2. Proof of Theorem 2

Proof. For $x > 0$, we have

$$\overline{F}_{\xi^{(\eta)}}(x) = \sum_{n=1}^{\infty}\mathbb{P}(\xi^{(n)} > x)\mathbb{P}(\eta = n)$$

$$= \sum_{n=1}^{\infty}\mathbb{P}\left(\bigcup_{k=1}^{n}\{\xi_k > x\}\right)\mathbb{P}(\eta = n)$$

$$= \sum_{n=1}^{\infty}\mathbb{P}\left(\bigcup_{k=1}^{n}\left(\xi_k > x, \bigcap_{j=1}^{k-1}\{\xi_j \leq x\}\right)\right)\mathbb{P}(\eta = n).$$

Therefore,

$$\frac{\overline{F}_{\xi^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} = \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \prod_{j=1}^{k-1} F_{\xi_j}(x). \tag{13}$$

Condition (3) implies that

$$\sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \leqslant c_9 n$$

for all $n \in \mathbb{N}$, some c_9 and sufficiently large x, say $x > x_1$. Hence,

$$\limsup_{x \to \infty} \frac{\overline{F}_{\xi^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} \leqslant c_9 \sum_{n=1}^{\infty} \mathbb{P}(\eta = n)n = c_9 \mathbb{E}\eta. \tag{14}$$

In a similar way from (3), we derive that there exists $c_{10} > 0$, such that

$$\sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \geqslant c_{10} n$$

for all n and sufficiently large x, say $x > x_2$. Decomposition (13) implies that for each $N \geqslant 1$,

$$\liminf_{x \to \infty} \frac{\overline{F}_{\xi^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} \geqslant \liminf_{x \to \infty} \sum_{n=1}^{N} \mathbb{P}(\eta = n) \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \prod_{j=1}^{k-1} F_{\xi_j}(x)$$

$$= \sum_{n=1}^{N} \mathbb{P}(\eta = n) \liminf_{x \to \infty} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \prod_{j=1}^{k-1} F_{\xi_j}(x)$$

$$\geqslant \sum_{n=1}^{N} \mathbb{P}(\eta = n) \liminf_{x \to \infty} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} \liminf_{x \to \infty} \prod_{j=1}^{n-1} F_{\xi_j}(x)$$

$$\geqslant c_{10} \sum_{n=1}^{N} n \mathbb{P}(\eta = n).$$

By passing N to infinity, we obtain

$$\liminf_{x \to \infty} \frac{\overline{F}_{\xi^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} \geqslant c_{10} \mathbb{E}\eta. \tag{15}$$

Since $0 < \mathbb{E}\eta < \infty$, the derived relations (14) and (15) imply that $\overline{F}_{\xi^{(\eta)}}(x) \underset{x \to \infty}{\asymp} \overline{F}_{\xi_1}(x)$, and hence, $F_{\xi^{(\eta)}} \in \mathcal{OS}$ due to Lemma 1(ii). Theorem 2 is proved. □

6.3. Proof of Theorem 3

Proof. For $x > 0$, we have

$$\overline{F}_{S^{(\eta)}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(S^{(n)} > x) \mathbb{P}(\eta = n).$$

Therefore, for such x,

$$\frac{\overline{F}_{S^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} = \sum_{n=1}^{\infty} \frac{\overline{F}_{S^{(n)}}(x)}{\overline{F}_{\xi_1}(x)} \mathbb{P}(\eta = n)$$

$$\leqslant \sum_{n=1}^{\infty} \sup_{x>0} \frac{\overline{F}_{S^{(n)}}(x)}{\overline{F}_{\xi_1}(x)} \mathbb{P}(\eta = n)$$

$$\leqslant \sum_{n=1}^{\infty} (c_{11})^n \mathbb{P}(\eta = n)$$

$$\leqslant \mathbb{E}(c_{11})^{\eta} < \infty$$

with $c_{11} \geqslant 2$ due to conditions of the theorem and Lemma 4. Consequently,

$$\limsup_{x \to \infty} \frac{\overline{F}_{S^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} < \infty. \tag{16}$$

Now, we check if

$$\liminf_{x \to \infty} \frac{\overline{F}_{S^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} > 0. \tag{17}$$

For positive x, it holds that

$$\frac{\overline{F}_{S^{(\eta)}}(x)}{\overline{F}_{\xi_1}(x)} = \sum_{n=1}^{\infty} \frac{\overline{F}_{S^{(n)}}(x)}{\overline{F}_{\xi_1}(x)} \mathbb{P}(\eta = n)$$

$$\geqslant \sum_{n=1}^{\infty} (c_{12})^{n-1} \mathbb{P}(\eta = n) > 0.$$

with a positive constant c_{12} according to conditions of the theorem and Lemma 6.

The derived estimates (16) and (17) imply that $\overline{F}_{S^{(\eta)}}(x) \underset{x \to \infty}{\asymp} \overline{F}_{\xi_1}(x)$. Hence, $F_{S^{(\eta)}} \in \mathcal{OS}$ due to Lemma 1(ii). Theorem 3 is proved. $\quad\square$

7. Construction of Generalized Subexponential Distributions

In this section, we present two examples showing how using Theorems 1, 2 and 3 it is possible to construct new distributions belonging to the class $\mathcal{OS}$. It is quite difficult to write the analytical expression of d.f.s $F_{\xi_{(\eta)}}, F_{\xi^{(\eta)}}, F_{S_{(\eta)}}$ and $F_{S^{(\eta)}}$ in the general case, but according to Theorems 1–3, we can establish whether the constructed distributions are generalized subexponential.

Example 1. *Let $\{\xi_1, \xi_2, \ldots, \}$ be a sequence of independent and identically distributed r.v.s such that*

$$\overline{F}_{\xi_k}(x) = \mathbb{I}_{(-\infty,0)}(x) + \frac{e^{-x}}{(1+x)^3}\left(1 + \frac{\sin x}{3}\right)\mathbb{I}_{[0,\infty)}(x), \; k \in \mathbb{N}.$$

According to the results of [55], the d.f. F_{ξ_1} belongs to the edge of the class $\mathcal{OS}$. In addition, requirement (3) is certainly satisfied. Therefore, Theorems 1–3 can be applied for sequence of independent and identically distributed r.v.s $\{\xi_1, \xi_2, \ldots, \}$.

Theorem 1 gives that d.f.s $F_{\xi_{(\eta)}}$ and $F_{S_{(\eta)}}$ belong to $\mathcal{OS}$ for each counting r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$. In particular, if

$$\mathbb{P}(\eta = n) = (1-p)p^{n-2}, \; n \in \{2, 3, \ldots\}, \; p \in (0, 1), \tag{18}$$

then d.f.s $F_{\xi_{(\eta)}}$ and $F_{S_{(\eta)}}$ with tails

$$\overline{F}_{\xi_{(\eta)}}(x) = \frac{(1-p)\left(\overline{F}_{\xi_1}(x)\right)^2}{1 - p\overline{F}_{\xi_1}(x)}, \; \overline{F}_{S_{(\eta)}}(x) = \overline{F}_{\xi_1}(x), \; x \geqslant 0,$$

belong to the class of generalized subexponential distributions.

Theorem 2 implies that d.f. $F_{\check{\xi}(\eta)}$ belongs to the class $\mathcal{OS}$ for each counting r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$ such that $\mathbb{E}\eta < \infty$. In a special case, when the counting random variable is defined by the equality (18), according to Theorem 2, we obtain that the distribution function with tail

$$\overline{F}_{\check{\xi}(\eta)}(x) = \frac{1-p}{p^2}\,\overline{F}_{\xi_1}(x) \sum_{n=2}^{\infty} p^n \sum_{k=1}^{n} \left(F_{\xi_1}(x)\right)^{k-1}, \ x \geqslant 0,$$

belongs to the class $\mathcal{OS}$.

Finally, by Theorem 3, we obtain that d.f. $F_{S(\eta)}$ is generalized subexponential if counting r.v. η is independent of $\{\xi_1, \xi_2, \ldots\}$ and $\mathbb{E}\,e^{\lambda\eta} < \infty$ for each $\lambda > 0$. In particular, if

$$\mathbb{P}(\eta = n) = \frac{1}{c_{13}}\,e^{-n^2}, \ n \in \mathbb{N}, \ c_{13} = \sum_{n=1}^{\infty} e^{-n^2},$$

then d.f. $F_{S(\eta)}$ with tail

$$\overline{F}_{S(\eta)}(x) = \frac{1}{c_{13}} \sum_{n=1}^{\infty} e^{-n^2} \overline{\underset{k=1}{\overset{n}{\circledast}} F_{\xi_k}}(x), \ x \geqslant 0,$$

belongs to the class $\mathcal{OS}$.

Example 2. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.s such that*

$$\overline{F}_{\xi_1}(x) = \mathbb{I}_{(-\infty,1)}(x) + \frac{e^{1-x}}{x^2}\,\mathbb{I}_{[1,\infty)}(x),$$

$$\overline{F}_{\xi_k}(x) = \mathbb{I}_{(-\infty,1)}(x) + \left(e - \frac{1}{k-1}\right)\frac{e^{-x}}{x^2}\,\mathbb{I}_{[1,\infty)}(x), \ k \in \{2,3,\ldots\}.$$

According to the results of [58,59] d.f.s F_{ξ_k} belongs to the class $\mathcal{OS}$ for all $k \in \mathbb{N}$. In addition,

$$\limsup_{x \to \infty} \sup_{n \geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} = 1,$$

and

$$\liminf_{x \to \infty} \inf_{n \geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_1}(x)} = 1 - \frac{1}{2e}.$$

Hence, the sequence of r.v.s $\{\xi_1, \xi_2, \ldots, \}$ also satisfies the conditions of Theorems 1–3.

Theorem 1 gives that d.f.s $F_{\check{\xi}_{(\eta)}}$ and $F_{S_{(\eta)}}$ belong to $\mathcal{OS}$ for each counting r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$. In a particular case, when r.v. η is distributed according to the geometric law (18) d.f.s $F_{\check{\xi}_{(\eta)}}$ and $F_{S_{(\eta)}}$ with tails

$$\overline{F}_{\check{\xi}_{(\eta)}}(x) = \mathbb{I}_{(-\infty,1)}(x) + \frac{(1-p)\,e}{p^2} \sum_{n=2}^{\infty} \left(\frac{e^{-x}p}{x^2}\right)^n \prod_{k=2}^{n} \left(e - \frac{1}{k-1}\right)\mathbb{I}_{[1,\infty)}(x),$$

$$\overline{F}_{S_{(\eta)}}(x) = \overline{F}_{\xi_1}(x)$$

belongs to the class $\mathcal{OS}$.

Similarly, Theorem 2 implies that d.f. $F_{\check{\xi}(\eta)}$ belongs to class $\mathcal{OS}$ for each counting r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$ such that $\mathbb{E}\eta < \infty$. If the counting random variable is defined by equality (18), then by Theorem 2, we obtain that the distribution function with tail

$$\overline{F}_{\zeta^{(\eta)}}(x) = \mathbb{I}_{(-\infty,1)}(x) + \frac{1-p}{p^2} \sum_{n=2}^{\infty} p^n \sum_{k=1}^{n} \overline{F}_{\xi_k}(x) \prod_{j=1}^{k-1} F_{\xi_j}(x) \, \mathbb{I}_{[1,\infty)}(x)$$

belongs to the class $\mathcal{OS}$.

Theorem 3 gives that d.f. $F_{S(\eta)}$ belongs to class $\mathcal{OS}$ if the counting r.v. η is independent of $\{\xi_1, \xi_2, \ldots\}$ and $\mathbb{E}\, e^{\lambda \eta} < \infty$ for each $\lambda > 0$. In particular, if η is distributed according to the Poisson law

$$\mathbb{P}(\eta = n) = \frac{\mu^n}{n!}\, e^{-\mu}, \ n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \ \mu > 0,$$

then d.f. $F_{S(\eta)}$ with tail

$$\overline{F}_{S(\eta)}(x) = e^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \overline{\underset{k=1}{\overset{n}{*}} F_{\xi_k}}(x), \ x \geqslant 1,$$

belongs to class $\mathcal{OS}$.

8. Concluding Remarks

In this paper, we study distribution functions obtained by randomly stopping minimum, maximum, minimum of sums and maximum of sums of random variables. Primary random variables are considered to be real-valued, independent and possibly differently distributed. The random variable defining the stopping moment is integer-valued, non-negative and not degenerate at zero. We find conditions when the distribution functions of these randomly stopped structures belong to the class of generalized subexponential distributions. The belonging of the distributions of randomly stopped structures to the class of generalized subexponential distributions can be determined either by primary random variables or by counting random variables. In this paper, we consider the case where a set of primary random variables has a decisive value. Our main results are formulated in Theorems 1–3. The primary random variables considered in all theorems can be differently distributed. But the additional conditions of all theorems are satisfied in the case where the primary random variables are identically distributed.

In the future, it would be interesting to study the case when some randomly stopped structure belongs to the class of generalized subexponential distributions due to the specific properties of the counting random variable. In this case, primary random variables should probably have significantly lighter tails compared to the counting random variable tail.

On the other hand, it would be interesting to study the closure properties of randomly stopped structures that are not related to the sum or maximum, but related to the product of random variables, as was performed in paper [60], for instance. In the class of generalized subexponential distributions, this would be easier compared to other classes due to the results obtained in article [26].

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