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RENORMALIZACIJOS GRUPĖS LYGTYS GRIMUS-NEUFELD MODELIUI

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RENORMALIZATION GROUP EQUATIONS FOR THE GRIMUS-NEUFELD MODEL

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Introduction

The complexity of quantum field theory often forces one to work in a perturbation theoretic framework, that is, to approximate desired quantities of one's model through truncated asymptotic expansions. Taking more terms of this expansion into account can be understood as accounting for higher order quantum corrections. Naive calculations of these corrections imply infinitely large model parameters. Through the so-called renormalization procedure, these infinities are controlled, and the formally divergent quantities are replaced with finite renormalized parameters. Then models can be connected with observations by fixing these parameters at specific experimentally determined values. A concrete value of a given renormalized parameters is defined at a specific energy scale. Since energy scales between experiments differ significantly, there is no way to simultaneously determine the value of each parameter at a single energy scale. Furthermore, one wishes for a model to be valid, if possible, at all energy scales, as otherwise most of its predictive power is lost. These issues are addressed within the framework of the Renormalization Group (RG).

Renormalization group equations (RGEs) describe the energy scale dependence of renormalized parameters, allowing one to test and analyze one's model at different energy scales. The derivation of RGEs for a specific model is an arduous problem. The calculations are error-prone and, when working with a variety of models, inefficient. In 1970s Machacek and Vaughn launched a programme to derive general RGEs, applicable to all renormalizable field theories, *once and for all* [1–3]. The continuation of this programme remains an active field of research to this day [4–11].

With these general results at hand one is able, in principle, to concretized them for a specific model, albeit this still remains a difficult. For this tasks automated tool [12–16] are widely employed. In our work we made use of `PyR@TE 3` [14], a `Python` package for symbolic and numerical calculations of RGEs, to investigate the Grimus-Neufeld model [17].

The GNM minimally extends the Standard Model (SM) with a second Higgs doublet and a sterile Majorana neutrino. It has received a lot of attention in recent years [18–23] as a promising candidate for explaining the existence of neutrino masses.

The goal of this thesis is to investigate and present the RGEs for the Grimus-Neufeld model (GNM) at 2-loop order and to examine the effects of the renormalization group (RG) evolution on the neutrino mass parameter. To this end we utilize the aforementioned `Python` package `PyR@TE 3` [14], that is based on the general expression for RGEs, derived within the programme started by Machacek and Vaughn [1–11]. This approach allows us to conduct efficient and reproducible analysis of the RGEs of the GNM.

The thesis is structured as follows. In Sec. 1 we provide a review of RGEs and of general β-function. In Sec. 2 we present the Grimus-Neufeld model (GNM) and its place within modern high-energy physics. Finally, in Sec. 3 we summarize our analysis of the RG evolution in GNM.

1 Renormalization group and β-functions for general quantum field theories

In this section we attempt to condense into a concise self-contained presentation multiple themes: the theory of the renormalization group (RG) and renormalization group equations (RGEs), the study of β-functions of general renormalizable field theory within the perturbative approach, and their application to specific theories.

We also hope that the presented overview will supplement the currently accessible material on the aforementioned themes. Present day literature is full of great quality material on the these themes. However, a novice may often fail to find the presentation of all of them in a single source, or in concise form. For example, the study of RG and RGEs is often either presented through worked examples, or in a partial manner. That is, the focus is usually either, so to speak, on the *micro* level, i.e. on the explicit perturbative calculations, or on the *macro* level, i.e. on general considerations about the energy scale dependence of observables. When it comes to the study of general β-functions, excellent reviews of the topic exists, such as Ref. [7][1]. But these reviews do not include some of the more recent developments, such as the study of general β-functions of vacuum expectation values (VEVs) [8, 9] or the systematized approach to applying general β-functions to specific models [10].

With the goals outlined above in mind we structure this section as follows. We begin by defining the Lagrangian of a general renormalizable gauge theory in Subsec. 1.1. We follow this with a high-level overview of the renormalization group theory in Subsec. 1.2. We present dimensional regularization and the modified minimal subtraction prescription (often abbreviated as $DR/\overline{MS}$) with an example in Subsec. 1.3. We then move to a discussion of general β-functions. We introduce the well known results for the general β-functions of dimensionless and dimensionful couplings, describe their derivation and application to specific models in Subsec. 1.4. We close the section by presenting the relatively recent results regarding renormalization of vacuum expectation values (VEVs) in the generalized approach within 1.5.

1.1 The Lagrangian density for a general renormalizable gauge theory

We begin by defining a general Lagrangian of a renormalizable quantum field theory. The presentation closely follows Ref. [14, 25]. The Lagrangian density must be Poincare and gauge invariant when neglecting gauge fixing terms, ghosts terms (that arise due to quantization) and terms appearing due to spontaneous symmetry breaking.

In what follows we use the following notation: $\{\mu, \nu \ldots\}$ are spacetime indices, $\{A, B \ldots\}$ will index the gauge generators, $\{j, k \ldots\}$ index the different fermions, and $\{a, b \ldots\}$ index the different scalars.

We now define the symmetries of a general theory. Let $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_M$ be a semi-simple gauge group under which our theory is invariant. We assume that $\mathcal{G}$ includes at most one $U(1)$

[1] See also Ref. [24] for a more comprehensive reevaluation of the seminal papers by Machacek and Vaughn [1–3].

factor as otherwise additional complexities (e.g., Abelian kinetic mixing [6, 26]), that are outside the scope of this thesis, are introduced. The vector gauge fields associated with the gauge factor $\mathcal{G}_p$ are denoted by $V^{(p)A_p}$. The gauge vector fields $V^{(p)A_p}$ are elements of the adjoint representation of $\mathcal{G}_p$. The standard procedure for dealing with multiple gauge factors $\mathcal{G}_p$ each with an associated gauge coupling constant $g_{(p)}$ is to gather all gauge vector fields into a single multiplet $(V^{(1)1} \dots V^{(1)\dim(\mathcal{G}_1)} \dots V^{(M)1} \dots V^{(M)\dim(\mathcal{G}_M)})$ denoted as V^A, where now A runs from 1 to $\Sigma_{i=1}^{M} \dim \mathcal{G}_i$. The gauge couplings now become indexed and associated with each vector field as well, i.e. $g_A \in \{g_{(1)} \dots g_{(1)} \dots g_{(M)} \dots g_{(M)}\}$. A further simplification is achieved by absorbing the gauge couplings into the vector fields $g_A \cdot V^A \to V^A$. The impact of this redefinition on the Lagrangian is presented below. We note that in this construction, in many regards, all calculations can be treated as if $\mathcal{G}$ was a simple gauge group. We note how to recover the dependencies of the general β-function in Subsec. 1.4.

The matter field content of a general theory is defined as follows. All scalars are gathered into a single multiplet, likewise for spinor fields. Let ϕ_a be real scalar fields, and let ψ_j be the two-component Weyl spinors, i.e. fermion fields. Multiplets of scalar fields transform under the representation of the gauge group $\mathcal{G}$ conventionally denoted as S, and all spinor fields transform under the representation F. As in the original publications [1–3], and later automations [14], it is standard to assume that S is a real representation.

A word about the choice of fields made above: a given theory may contain either real, or complex scalar fields, or a combination of both. Any complex scalar field may be expressed via two real scalar fields, one constituting the real component, the other constituting the imaginary component. Any Dirac spinor may be decomposed into two 2-component Weyl spinors.

With the definitions and considerations above in mind the most general renormalizable Lagrangian $\mathscr{L}$ can be expressed as a sum of three terms:

$$\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_\star, \tag{1.1}$$

where $\mathscr{L}_1$ contains only dimensionless parameters (and Kinetic terms), $\mathscr{L}_2$ contain dimensionful parameters, the gauge fixing and ghost terms are separated from the rest of the Lagrangian and contained in $\mathscr{L}_\star$ [2]. We describe each term below.

The first term in Eq. (1.1) is given by

$$\begin{aligned}\mathscr{L}_1 = &-\frac{1}{4} G_{AB}{}^{-2} F_A{}^{\mu\nu} F^B{}_{\mu\nu} + \frac{1}{2}(D^\mu \phi)_a (D_\mu \phi)_a + i\psi_j^\dagger \bar{\sigma}^\mu D_\mu \psi_j \\ &- \frac{1}{2}\left(Y_{aij}\, \psi_i \psi_j + \text{h.c.}\right) \phi_a - \frac{1}{4!} \lambda_{abcd} \phi_a \phi_b \phi_c \phi_d \,.\end{aligned} \tag{1.2}$$

Hereafter, unless indicated otherwise, summation over repeated indices is implied. The gauge field strength tensor in Eq. (1.2) is given by

$$F^A{}_{\mu\nu} = \partial_\mu A^A{}_\nu - \partial_\nu A^A{}_\mu + f^{ABC} V^B{}_\mu V^C{}_\nu \,, \tag{1.3}$$

[2] The possibility of spontaneous symmetry breaking presents additional complications that are neglected here. We address spontaneous symmetry breaking and VEV renormalization in Subsec. 1.5.

where f^{ABC} are the structure constants of the group $\mathcal{G}$. The covariant derivatives acting on the matter fields read

$$D_\mu \phi_a = \partial_\mu \phi_a - i\theta^A{}_{ab} V^A_\mu \, \phi_b \tag{1.4}$$

$$D_\mu \psi_i = \partial_\mu \psi_i - i t^A{}_{ij} V^A_\mu \, \psi_i \,, \tag{1.5}$$

where $t^A{}_{jk}$ and $\theta^A{}_{ab}$ are the generators of the gauge group $\mathcal{G}$, acting on the spinor and scalar fields, respectively. Since scalars are assumed to transform under a real representation S, the generators $\theta^A{}_{ab}$ are taken to be imaginary and anti-symmetric Hermitian matrices. Y_{aij} are the Yukawa couplings between fields ψ_i, ψ_j and ϕ_a, and λ_{abcd} is the scalar quartic coupling. $G_{AB} = \delta_{AB} g_A$ is a diagonal matrix of coupling constants arising due to the redefinition of gauge vector fields.

The second term in Eq. (1.1) is given by

$$\mathscr{L}_2 = -\frac{1}{2}\left(M_{ij}\psi_i\psi_j + \text{h.c.}\right) - \frac{1}{2}\mu_{ab}\phi_a\phi_b - \frac{1}{3!}t_{abc}\phi_a\phi_b\phi_c \,, \tag{1.6}$$

where M_{ij} are the fermion mass parameters, μ_{ab} are the scalar mass parameters, and t_{abc} are the scalar trilinear couplings.

We note the following about the third term in Eq. (1.1). Gauge fixing is needed for quantization, while ghost fields in the ghost part of $\mathscr{L}_\star$ arise as unphysical artifacts of non-Abelian gauge field quantization. Gauge fixing and ghost Lagrangian $\mathscr{L}_\star$ is only relevant for the renormalization of vacuum expectation values (VEVs), as presented in Subsec. 1.5. The gauge fixing terms used for VEV renormalization in the general approach [8, 9] are constructed in a non-standard way, while the discussion of ghosts fields is outside the scope of this thesis. The reader is therefore directed to Refs. [8, 9] and references therein for a presentation of these constructions; a rough overview is provided in Subsec. 1.5 .

1.2 Renormalization group and renormalization group equations

Before moving to concrete renormalization calculations, we present a high-level overview of the theory of RG and RGEs. Specifically, the presentation is given on the level of partition functions and n-point correlation functions[3]. Already at this level we can gain insights into the dependence on the energy scale of a given system. Furthermore, we are able to introduce key concepts of RGE theory, such as the anomalous dimensions γ_φ describing the scaling properties of fields due to quantum corrections, and the β-functions describing evolution of couplings with the energy scale.

Let μ_0 be some reference energy scale. Having defined a theory up to a particular scale μ_0 one may consider two options. On the one hand, one may derive an effective theory with higher energy modes integrated out. This theory would be defined up to a lower energy scale $\mu' < \mu^0$. Through such calculations one may introduce the concept of running couplings together with the so-called β-functions that determine how couplings change with the energy scale. On the other hand, one

[3]The partition functions can be conceived as a QFT analogue of the partition function in statistical mechanics, only defined on Minkowski spacetime, while the n-point correlation functions play a crucial role in describing experimental observations.

may consider the original theory as an effective theory, that comes from a theory that is defined at an even higher energy scale; ultimately, the full theory would be obtained by taking the limit $\mu_0 \to \infty$. Such considerations motivate the introduction of perturbative renormalization.

For this section, the exact field content is not relevant. Therefore, for the sake of brevity, we introduce *all encompassing fields* $\varphi_i \in \{\phi_a, \psi_j, V^A\}$, where the index $i \in \mathcal{I}$ runs through all scalar, spinor, and gauge fields. We also employ the shorthand notation $\boldsymbol{\varphi} = \{\phi_i\}_{i\in\mathcal{I}}$.

1.2.1 RG for a theory with a UV cut-off

We begin by considering the first case: a theory defined up to a cut-off $\mu < \mu_0$. Let $\mathcal{Z}_{\mu_0,}[\boldsymbol{\varphi};\ \boldsymbol{\Theta}]$ be the partition function for a theory defined up to the reference energy scale μ_0 and dependent on the fields $\boldsymbol{\varphi}$, and on a number of parameters $\boldsymbol{\Theta} = \{\Theta_j\}_{j\in\mathcal{J}}$, that are to be understood as coupling constants. Explicitly, $\mathcal{Z}_{\mu_0,}[\boldsymbol{\varphi};\ \boldsymbol{\Theta}]$ reads as

$$\mathcal{Z}_{\mu_0,}[\boldsymbol{\varphi};\ \boldsymbol{\Theta}] = \int_{\mu\leq\mu^0} \mathcal{D}\varphi\ e^{-iS[\boldsymbol{\varphi};\ \boldsymbol{\Theta}]}. \tag{1.7}$$

Often one defines the partition function by also including source terms that modify the action $-S[\varphi;\ \Theta] \to -iS[\boldsymbol{\varphi};\ \boldsymbol{\Theta}] + i\varphi_i J_i$, in which case it is refered to as the generating functional. For the present discussion this is not needed. $\mathcal{Z}_{\mu_0,}[\boldsymbol{\varphi};\ \boldsymbol{\Theta}]$ as defined in Eq. (1.7) is known as the vacuum amplitude [27]. We also abuse the notation slightly to keep the expressions reasonably short: the domain of integration in Eq. (1.7) is a subspace of a functional manifold (manifold of functions of fields, to be precise) $\mathscr{M}$, namely the subspace $C^\infty_{\mu\leq\mu^0}(\mathscr{M})$ of smooth functions whose energy is at most μ^0.

To define a theory at a lower cut-off μ', we simply integrate out fields with energies $\mu' < \mu \leq \mu^0$. This is done in two steps. First we split the fields into lower energy modes, and higher energy modes. The calculation is more convenient in momentum space

$$\begin{aligned}\varphi_i(x) =& \int_{|p|<\mu_0} d^dp \frac{e^{ip\cdot x}}{(2\pi)^d}\tilde{\varphi}(p) = \int_{|p|\leq\mu'} d^dp \frac{e^{ip\cdot x}}{(2\pi)^d}\tilde{\varphi}(p) + \int_{\mu'<|p|\leq\mu^0} d^dp \frac{e^{ip\cdot x}}{(2\pi)^d}\tilde{\varphi}(p)\\ =& \varphi'_i(x) + \varphi^0_i(x)\,,\end{aligned} \tag{1.8}$$

where $\varphi' \in C^\infty_{\mu\leq\mu'}(\mathscr{M})$, and $\varphi^0 \in C^\infty_{\mu'<\mu\leq\mu^0}(\mathscr{M})$. Eq. (1.8) is justified by $C^\infty_{\mu'<\mu\leq\mu^0}(\mathscr{M})$ being a vector space with addition, defined via pointwise addition on $\mathscr{M}$. The measure factorizes as expected

$$\mathcal{D}\varphi = \mathcal{D}\varphi'\mathcal{D}\varphi^0. \tag{1.9}$$

By construction, the partition functions of the initial theory (1.7) and the effective theory (one defined up to μ') are equal

$$\mathcal{Z}_{\mu_0}[\boldsymbol{\varphi};\boldsymbol{\Theta}] = \mathcal{Z}_{\mu'}[\boldsymbol{\varphi};\ \boldsymbol{\Theta}(\mu')]'. \tag{1.10}$$

The effective action $S_{\mu'}[\boldsymbol{\varphi}';\ \boldsymbol{\Theta}(\mu')]$ for a theory defined up to the cut-off μ' is then given by the

relation

$$\int_{\mu<\mu_0} \mathscr{D}\varphi' \mathscr{D}\varphi^0 \; e^{-iS[\boldsymbol{\varphi};\, \boldsymbol{\Theta}]} = \int_{\mu<\mu'} \mathscr{D}\varphi' \; e^{-iS_{\mu'}[\boldsymbol{\varphi}';\, \boldsymbol{\Theta}(\mu')]} \,. \tag{1.11}$$

A note here is in order. By integrating out higher energy modes, we added an explicit dependence on the energy scale μ', thus the dependence of $\boldsymbol{\Theta}(\mu')$ on the energy scale μ' is introduced.

The construction above can be repeated by integrating out lower and lower energy modes to produce a theory defined at a cut-off μ''

$$\mathfrak{Z}_{\mu'}[\boldsymbol{\varphi}';\; \boldsymbol{\Theta}(\mu')] = \mathfrak{Z}_{\mu''}[\boldsymbol{\varphi}'';\; \boldsymbol{\Theta}(\mu'')] \,. \tag{1.12}$$

If we consider the energy scale being lowered infinitesimally we can infer from Eq. (1.12) the following relation

$$\frac{\partial \mathfrak{Z}}{\partial \ln \mu} = \left.\frac{\partial \mathfrak{Z}}{\partial \ln \mu}\right|_{\Theta_i} + \left.\frac{\partial \Theta_i}{\partial \ln \mu}\frac{\partial \mathfrak{Z}}{\partial \Theta_i}\right|_{\mu} = 0 \;\; . \tag{1.13}$$

Eq. (1.13) is known as the Callan-Symanzik equation for the vacuum amplitude.

Eq. (1.13) can be also generalized to n-point functions $\langle \varphi_{i_1} \dots \varphi_{i_n} \rangle$. To achieve the generalization one must allow for the redefinition of fields at different energy scales. Let $\hat{\varphi}_i$ denote the renormalized field, and Z_i the renormalization constant. Then

$$\hat{\varphi}_i = Z_i^{\frac{1}{2}} \varphi_i \;\; . \tag{1.14}$$

The n-point functions are defined as

$$\Gamma = \left\langle \prod_{j\in\hat{\mathcal{J}}} \hat{\varphi}_j \right\rangle = \frac{1}{\mathfrak{Z}_{\mu_0}[\hat{\varphi};\; \Theta]} \int_{\mu<\mu_0} \mathscr{D}\hat{\varphi} \; e^{-iS[\hat{\varphi};\, \Theta]} \prod_{j\in\hat{\mathcal{J}}} \hat{\varphi}_j \;\; ,$$

where $\hat{\mathcal{J}}$ is a set of (possibly repeating) field indices. Through similar arguments as above the Callan-Symanzik equation defined for n-point functions reads as

$$\frac{\partial \Gamma}{\partial \ln \mu} = \left.\frac{\partial \Gamma}{\partial \ln \mu}\right|_{\Theta_i} + \left.\frac{\partial \Theta_i}{\partial \ln \mu}\frac{\partial \mathfrak{Z}}{\partial \Theta_i}\right|_{\mu} + \frac{1}{2}\sum_u \frac{\partial \ln Z_{\varphi_i}}{\partial \ln \mu} = 0 \;\; . \tag{1.15}$$

Important quantities appearing in the Callan-Symanzik equation are the β-functions β_i and the anomalous dimensions γ_{φ_i}[4]. They are defined as

$$\beta_{\Theta_i} = \frac{\partial \Theta_i}{\partial \ln \mu} \,, \tag{1.16a}$$

$$\gamma_{\varphi_i} = -\frac{1}{2}\frac{\partial \ln Z_{\varphi_i}}{\partial \ln \mu} \tag{1.16b}$$

The β-function and the anomalous dimension are related concepts. The β-functions encode the

[4] In some publications an alternative convention is used in defining the anomalous dimension, namely, anomalous dimension is sometimes defined as $\gamma_{\varphi_i} = \frac{1}{2}\frac{\partial \ln Z_{\varphi_i}}{\partial \ln \mu}$.

dependence of the various parameters on the energy scale μ; to obtain the explicit dependence one needs to solve a system of N differential equations of the form (1.16a), where N is the number of parameters of the theory. The anomalous dimension encodes the effect of quantum correction to a field's mass dimension. We note that usually the β-functions and the anomalous dimensions are expressed as matrices, although in our notation this is suppressed.

Remark 1.1 *We note the interpretation of the Callan-Symanzik equation* (1.15)*: the n-point functions are RG invariant. What is meant by this, is that the physical predictions of a theory defined at one energy scale, can be derived for any other energy scale, or, more succinctly, the physical predictions of a theory do not depend on the scale at which the theory is probed.*

1.2.2 RG for a full theory and RG flow

In practicle calculation it is often more convenient to consider the 'full' theory, i.e. the limit $\mu_0 \to \infty$. In this subsection we observe how this motivates the renormalization procedure.

Let us consider a generalized of $d \in \mathbb{C}$ dimensions. We require the action to remain dimensionless. To enforce this we add additional factors in front of dimensionful couplings: mass dimension 1 parameters, known as 't Hooft masses $\tilde{\mu}$, raised to an appropriate power. The effective action, expressed as a sum of terms appearing in the Lagrangian can then be generally expressed as

$$S_\mu[\boldsymbol{\varphi}';\ \boldsymbol{\Theta}(\mu')] = \int d^d x \left(\sum_j \tilde{\mu}^{d-d_j} \Theta_j(\mu) \mathcal{O}_j(\boldsymbol{\varphi}) \right) \tag{1.17}$$

where $d_j = \dim \Theta_j \mathcal{O}_j$ is the mass dimension of the operator $\mathcal{O}_j$.

By considering the scaling properties of the theory we can extract the relations between the mass dimensions of the various couplings as they evolve with the RG flow. Based on this the couplings can be broadly categorized as relevant, irrelevant, and marginal. To elaborate:

- A coupling is called **relevant**, if $d_j < d$. In this case the coupling will grow together with the energy scale towards infinity.
- A coupling is called **irrelevant**, if $d_j > d$. In this case the coupling is suppressed at high energies, and tends towards zero.
- A coupling is called **marginal**, if $d_j = d$. In this case, the scaling properties are indeterminate by dimensional/scaling arguments alone.

Taking the limit $\mu_0 \to \infty$ results in divergencies. The solution to this problem is to work perturbatively: one considers the n-point functions as an asymptotic series and works term-by-term within this series. The contributions coming from each term in this series have a nice interpretation, namely, as loop orders of the so-called Feynman diagrams. These can be understood as representations of various processes, e.g. exchanges of virtual particles, contributing to the overall n-point function. Often at the zeroth order, i.e. at tree-level, one is able to obtain good qualitative, and sometimes quantitative predictions.

Within the perturbative approach, at 1-loop and higher orders, a different problem arises: integrals corresponding to Feynman diagrams with loops become divergent. As it turns out these divergences arise for four dimensional theories, however not for lower dimensional theories. This motivates the so-called dimensional regularization by which a theory is generalized to an arbitrary number of dimension, in practice considered as *just below* 4. By doing so the divergent parts of the loop integrals can be extracted to separate terms. Divergent terms can be canceled out by introducing so-called counterterms into the Lagrangian; this is known as the renormalization procedure. Finite results can therefore be recovered. This is discussed in more detail in Subsec. 1.3.

We finish this subsection by discussing, how the renormalization procedure is to be understood. We may interpret the Lagrangian defined by Eq. (1.1) as the bare Lagrangian containing parameters that formally include contributions from all loop orders. These bare parameters cannot be probed experimentally in principle. Through the renormalization procedure on the bare Lagrangian, up to some finite loop order, the parameters of the theory are made finite, experimentally accessible and dependent on the energy scale; this Lagrangian is known as the renormalized Lagrangian. In Wilsonian terms: experimentally we are always subject to a cut-off, and as such the couplings are defined at this cut-off. Moving to higher cut-offs forces us to introduce some corrections to our Lagrangian $\Delta\mathscr{L}$. As long as the cut-off is finite these corrections are finite. The divergent terms arise when considering the cut-off at $\mu_0 = \infty$ in the perturbative expansion. Counterterms are then needed to compensate the induced RG flow.

1.3 Dimensional regularization and minimal subtraction scheme

We now shift our focus to the *micro* level, and discuss the details of evaluating loop integrals that arise in perturbative QFT calculations. At a given loop level the contributions to a given n-point function can be understood as a sum of relevant Feynman diagrams. As described in the previous section, starting at 1-loop level divergent terms begin to appear. The process of absorbing divergent terms into redefined coupling constants and field renormalization constants is called *the renormalization procedure*. The casting of a divergent integral into a form that is analytically tractable is known as *regularization*.

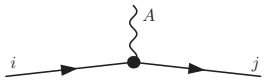
There are multiple regularization prescriptions and renormalization schemes, that yield different analytical results. One often tailors the regularization prescription and renormalization scheme for the problem at hand. In the context of RGEs for general theories, *á la* Machacek and Vaughn [1–3], the $DR/\overline{MS}$ scheme [28, 29] is used. It is a robust and relatively easily applicable scheme apt for work within a general framework and for automation.

In this section we demonstrate the prescriptions of the $DR/\overline{MS}$ scheme by an example. Additionally, we explain how certain group invariants and other tensor contractions arise in the expressions of β-functions for general theories.

1.3.1 Technical overview

We sketch the rough strategy for evaluating loop integrals and renormalizing parameters and fields in the $DR/\overline{MS}$ scheme below:

1. Extract the relevant Feynman rules from the Lagrangian Eq. 1.1 (general guidelines for doing this are outlined in most QFT textbooks, e.g. Ref. [27].) For this presentation, it will suffice to take the Feynman rules in momentum space as already given. For the problem at hand we will need the following set of Feynman rules, corresponding to vertices and edges of the Feynman diagram

$\psi_i\bar{\psi}_j V^A$ vertex 

$= it^A{}_{jk}\gamma^\mu\,\delta_{ij}$ (1.18)

ψ_i propagator

$= \dfrac{i\delta_{ij}}{\not{p} - m + i\epsilon}$ (1.19)

V^A propagator 

$= \dfrac{i\delta^{AB}}{p^2 + i\epsilon}\left(-g^{\mu\nu} + (\xi - 1)\dfrac{\ell^\mu \ell^\nu}{\ell^2}\right)$ (1.20)

The Kronecker deltas δ_{ij} and δ^{AB} appearing in expressions above, are to ensure the conservation of non-Abelian charges (in the case when $\mathcal{G} = SU(3)$, that is, in the case of QCD, these would be referred to as colors); $t^A{}_{jk}$ are the gauge generators for the fermions, as defined in Subsec. 1.1. We note that the vertex terms lacks a coupling constant factor, because, as per the prescription of Subsec. 1.1, the gauge coupling constants have been absorbed into the definition of the gauge vector fields V^A. This allows us to treat the theory (at least in this particular instance) as if its gauge group was simple. The vector boson propagator is expressed in the general R_ξ gauge, with ξ being the gauge parameter. The more familiar Feynman gauge is recovered by setting $\xi = 1$. Hereafter, we will keep the $+i\epsilon$ implicit, as they will eventually drop out, and until then are irrelevant to the presentation. Loop integrals in momentum space can then be constructed by compositions of these rules.

2. Write down the loop integral by composing Feynman rules. Generalize the loop integral from 4 dimensions to $d = 4 - 2\varepsilon$ dimensions, where $|\varepsilon| \ll 1$. The integral should be modified by a factor of $\tilde{\mu}^{4-d}$, where $\tilde{\mu}$ is the 't Hooft mass also appearing in Eq. (1.17). This step is known as dimensional regularization. Divergences then appear as poles in ε. We note that ε should not be confused with ϵ that appears in the fermion (1.19) and scalar (1.20) propagators.

3. Parametrize the integral using either Feynman or Schwinger parameters to bring them into a more suitable form for integration using the Wick rotation. For a pedagogical exposition of the various parameterizations see Appendix B of Ref. [27]. In this subsection we will utilize the following Feynman parametrization

$$\frac{1}{AB^n} = \int_0^1 dz \frac{n(1-z)^{n-1}}{[Az + B(1-z)]^{n+1}} \tag{1.21}$$

The obtained integrals may need to be further manipulated. This is often done by applying a

shift to the loop momentum.

4. Integrate out the internal momentum ℓ, using the Wick rotation. Standard formulas are usually at one's disposal. For the purposes of this chapter we will use

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{[\ell^2 - \Delta + i\epsilon]^2} = \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Delta^{2-\frac{d}{2}}} \Gamma\left(\frac{4-d}{2}\right) \tag{1.22}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{[\ell^2 - \Delta + i\epsilon]^3} = \frac{d}{4} \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Delta^{2-\frac{d}{2}}} \Gamma\left(\frac{4-d}{2}\right) \tag{1.23}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{[\ell^2 - \Delta + i\epsilon]^3} = -\frac{1}{2} \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{i}{\Delta^{3-\frac{d}{2}}} \Gamma\left(\frac{6-d}{2}\right) \tag{1.24}$$

   For a more detailed exposition of the Wick rotation, and other useful formulas see Appendix B of Ref. [27]. The Feynman parameter integral can usually be evaluated by more standard means.

5. The final expression then appears as a sum of terms, of which some are finite while some are divergent as $\varepsilon \to 0$, i.e. terms with a factor of ε^{-n}, where $n \in \mathbb{N}_{>0}$. The counterterms can then be read off from this expression. We note that in the (non-modified) minimal subtractions scheme (MS) the divergent terms are taken to be proportional to $\frac{1}{\varepsilon}$, while in the modified minimal subtraction $(\overline{MS})$ scheme divergent terms are understood as being proportional to

$$\frac{1}{\eta} = \frac{1}{\varepsilon} + \ln 4\pi - \gamma_E \,, \tag{1.25}$$

   where γ_E is the Euler-Mascheroni constant.

Remark 1.2 *Here we would like to remark that while the process outlined above is valid, there are a few subtleties for multi-loop integrals. Firstly, steps* 3 *and* 4 *need to be repeated separately for each loop momentum ℓ_i. In addition to these subtleties the calculations at higher loop orders become increasingly longer and the number of diagrams contributing to a given n-point function become much larger. Therefore, additional techniques are often employed to deals with the increased complexity. For an example of a two-loop self-energy calculation in an Abelian theory see Ref. [30].*

1.3.2 Calculating loop integrals

For a clearer presentation, in what follows we assume the gauge group of the theory to be simple, as such the calculations presented below are equivalent to those of QCD. We also note that, in the language of QCD, the mass parameter m does not depend on the color state, this is an important simplifying feature of our example, as technically this means that the mass parameter does not carry an index. For the detailed presentation of the steps outlined above we take a perennial example: the

1-loop contribution to $\langle \psi_i \bar{\psi}_j \rangle$ Fig. 1, which is given by

$$\langle \psi_i \bar{\psi}_j \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-p(x-y)} i G_{ij}(\not{p}) \tag{1.26}$$

where the $iG(\not{p})$ is the Green's function, and p is the external momentum. Hereafter we use the Feynman slash shorthand $\not{p} = \gamma^\mu p_\mu$. Up to 1-loop order $iG(\not{p})$ can be approximated as

$$iG_{ij}(\not{p}) = iG^{(0)}_{ij}(\not{p}) + iG^{(0)}_{ik}(\not{p})\, i\Sigma_{kl}(\not{p})\, iG^{(0)}_{lj}(\not{p}). \tag{1.27}$$

In Eq. (1.27) the first term is the tree-level Green's function – the fermion propagator (1.19), the second term is a correction due to the so-called self energy Feynman diagram. We focus on $i\Sigma_{ij}(\not{p})$, which can be expressed as a loop integral over internal momentum ℓ. This term corresponds to the amputated Feynman diagram Fig. 1. To develop some intuition on how the $i\Sigma_{ij}(\not{p})$ looks in general

Fig. 1 Feynman diagram of a one-loop contribution to $\langle \psi_i \bar{\psi}_j \rangle$ corresponding to self-energy $i\Sigma(\not{p})$ term.

let us first consider a more specific scenario: a contribution $i\bar{\Sigma}_{ij}(\not{p})$ to $i\Sigma_{ij}(\not{p})$ corresponding to the emission and reabsorption of a specific vector boson $V^{\bar{A}}$, where $\bar{A}$ is understood as denoting a specific field rather than a general index. Let the internal fermion line $\psi_{\bar{k}}$ be fixed as well. By composition of the Feynman rules (1.18) – (1.20) $i\Sigma_{ij}(\not{p})$ is obtained as

$$\begin{aligned} i\bar{\Sigma}_{ij}(\not{p}) =& \delta_{ij} \int \frac{d^4\ell}{(2\pi)^4} \left(i t^{\bar{A}}{}_{i\bar{k}} \gamma_\mu \right) \frac{i}{\not{p} - \not{\ell} - m} \left(i t^{\bar{A}}{}_{\bar{k}j} \gamma_\nu \right) \left(\frac{i}{\ell^2 + i\epsilon} \left[-g^{\mu\nu} + (1-\xi) \frac{\ell^\mu \ell^\nu}{\ell^2} \right] \right) \\ =& (i)^2 t^{\bar{A}}{}_{i\bar{k}} t^{\bar{A}}{}_{\bar{k}j} \int \frac{d^4\ell}{(2\pi)^4} \gamma_\mu \frac{i}{\not{p} - \not{\ell} - m} \gamma_\nu \left(\frac{i}{\ell^2 + i\epsilon} \left[-g^{\mu\nu} + (1-\xi) \frac{\ell^\mu \ell^\nu}{\ell^2} \right] \right) \quad . \end{aligned} \tag{1.28}$$

Let the emission and absorption of $V^{\bar{A}}$ be allowed by the symmetries of the theory. If we were to evaluate an integral in which the emission and absorption of some $V^{\bar{B}}$ would be prohibited by the symmetry of the theory, the integral would simply evaluate to 0, and the same holds for internal fermion lines. Therefore, we can confidently sum over all possibilities and since the only difference between all these diagrams is the factor of $t^{\bar{A}}{}_{i\bar{k}} t^{\bar{A}}{}_{\bar{k}j}$ we can move to the more general case by making the substitution

$$t^{\bar{A}}{}_{i\bar{k}} t^{\bar{A}}{}_{\bar{k}j} \Rightarrow t^A{}_{ik} t^A{}_{kj} = [t^A t^A]_{ij} = [C(F)]_{ij} \tag{1.29}$$

where the factor $C(F)$ is an invariant of the model's group $\mathcal{G}$ in representation F, and is known as the Casimir element. As the Casimir element [31] is diagonal we may drop the Kronecker delta. We may also suppress indices keeping in mind that that only $\Sigma_i(\not{p}) = \Sigma_{ii}(\not{p})$ will be non-zero. Roughly speaking, this is how all group invariants and other tensor contractions appear in the expressions of

the β-functions.

The self energy integral then reads

$$i\Sigma(\not{p}) = i^2 C(F) \int \frac{d^4\ell}{(2\pi)^4} \gamma_\mu \frac{i}{\not{p} - \not{\ell} - m} \gamma_\nu \left(\frac{i}{\ell^2} \left[-g^{\mu\nu} + (1-\xi) \frac{\ell^\mu \ell^\nu}{\ell^2} \right] \right) \tag{1.30}$$

In 4 dimensions the integral is divergent. In the dimensional reglarization perscription, the integral is casted into a more tractable form by generalizing it to $d \in \mathbb{C}$ dimensions. Eq. (1.30) then becomes

$$i\Sigma(\not{p}) = i^2 C(F) \tilde{\mu}^{4-d} \int \frac{d^d\ell}{(2\pi)^d} \gamma_\mu \frac{i}{\not{p} - \not{\ell} - m} \gamma_\nu \left(\frac{i}{\ell^2} \left[-g^{\mu\nu} + (1-\xi) \frac{\ell^\mu \ell^\nu}{\ell^2} \right] \right) \tag{1.31}$$

The factor $\tilde{\mu}^{4-d}$ is introduced to recover the correct dimensionality of the final expression. We split the integral into two parts $i\Sigma(\not{p}) = i\Sigma'(\not{p}) + i\Sigma''(\not{p})$: one not dependent on the gauge parameter

$$\begin{aligned} i\Sigma'(\not{p}) &= -i^2 C(F) \tilde{\mu}^{4-d} \int \frac{d^d\ell}{(2\pi)^d} \gamma_\mu \frac{i}{\not{p} - \not{\ell} - m} \gamma_\nu \frac{i g^{\mu\nu}}{\ell^2} \\ &= i^2 C(F) \tilde{\mu}^{4-d} \int \frac{d^d\ell}{(2\pi)^d} \gamma_\mu \frac{1}{\not{p} - \not{\ell} - m} \gamma^\mu \frac{1}{\ell^2} \\ &= i^2 C(F) \tilde{\mu}^{4-d} \int \frac{d^d\ell}{(2\pi)^d} \gamma_\mu \frac{\not{p} - \not{\ell} + m}{(p-\ell)^2 - m^2} \gamma^\mu \frac{1}{\ell^2} \ , \end{aligned} \tag{1.32}$$

and the other dependent on the gauge parameter

$$i\Sigma''(\not{p}) = (1-\xi) i^2 C(F) \tilde{\mu}^{4-d} \int \frac{d^d\ell}{(2\pi)^d} \gamma_\mu \frac{i}{\not{p} - \not{\ell} - m} \gamma_\nu \frac{i}{\ell^2} \frac{\ell^\mu \ell^\nu}{\ell^2} \ . \tag{1.33}$$

The final result will be the sum of these two parts. We first focus on the gauge parameter independent integral (1.32). Applying Feynman parametrization (1.21) to (1.32), with $n = 1$, $A = (p-\ell)^2 - m^2$, and $B = \ell^2$, yields

$$i\Sigma'(\not{p}) = i^2 C(F) \tilde{\mu}^{4-d} \int_0^1 dz \int \frac{d^d\ell}{(2\pi)^d} \gamma_\mu \frac{\not{p} - \not{\ell} + m}{[(p-\ell)^2 z - m^2 z + \ell^2 (1-z)]^2} \gamma^\mu \ . \tag{1.34}$$

We further manipulate Eq. (1.34) by performing a change of variables $\ell' = \ell - pz$. This simplifies the integral to

$$\begin{aligned} i\Sigma'(\not{p}) &= i^2 C(F) \tilde{\mu}^{4-d} \int_0^1 dz \int \frac{d^d\ell'}{(2\pi)^d} \gamma_\mu \frac{\not{p} - \not{p} z - \not{\ell}' + m}{[(p - \ell' - pz)^2 z - m^2 z + (\ell' + pz)^2 (1-z)]^2} \gamma^\mu \\ &= i^2 C(F) \tilde{\mu}^{4-d} \int_0^1 dz \int \frac{d^d\ell'}{(2\pi)^d} \gamma_\mu \frac{\not{p} - \not{p} z + m}{[\ell'^2 - m^2 z + p^2 z (1-z)]^2} \gamma^\mu \ . \end{aligned} \tag{1.35}$$

In the second line of Eq. (1.35) we were able to eliminate $\not{\ell}'$ from the enumerator, since $\not{\ell}' / [\ell'^2 - m^2 z + p^2 z(1-z)]^2$ is an odd function in ℓ' which vanishes when integrated over the full domain

of ℓ'. The integral over ℓ' can then be evaluated by applying (1.22):

$$\begin{aligned} i\Sigma'(\not{p}) &= i^2 C(F)\tilde{\mu}^{4-d}\int_0^1 dz\,\gamma_\mu(\not{p}-\not{p}z+m)\gamma^\mu\int\frac{d^d\ell'}{(2\pi)^d}\frac{1}{[\ell'^2-m^2z+p^2z(1-z)^2]} \\ &= i^2 C(F)\tilde{\mu}^{4-d}\frac{i}{(4\pi)^{\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dz\,\gamma_\mu(\not{p}-\not{p}z+m)\gamma^\mu\left[m^2z-p^2z(1-z)\right]^{-(2-\frac{d}{2})} \quad . \end{aligned} \tag{1.36}$$

It is now convenient to introduce $\varepsilon = 2-\frac{d}{2}$, which allows us to approximate the Γ function as $\Gamma(\varepsilon)\approx\frac{1}{\varepsilon}-\gamma_{\rm E}$. With this (1.36) reads as

$$\begin{aligned} i\Sigma'(\not{p}) &= i^2 C(F)\tilde{\mu}^{2\varepsilon}\frac{i}{(4\pi)^{2-\varepsilon}}\left(\frac{1}{\epsilon}-\gamma_E\right)\int_0^1 dz\,\gamma_\mu(\not{p}-\not{p}z+m)\gamma^\mu\left[m^2z-p^2z(1-z)\right]^{-\varepsilon} \\ &= i^2 C(F)\frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma_E\right)\int_0^1 dz\,[(2-d)\not{p}(1-z)+dm]\left[\frac{m^2z-p^2z(1-z)}{4\pi\tilde{\mu}^2}\right]^{-\varepsilon} \\ &= i^2 C(F)\frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma_E\right)\int_0^1 dz\,[4m-2\not{p}(1-z)+2\varepsilon(\not{p}(1-z)-m)] \\ &\quad\times\left[1-\epsilon\ln\frac{m^2z-p^2z(1-z)}{4\pi\tilde{\mu}^2}\right] \quad . \end{aligned} \tag{1.37}$$

In the second line of Eq. (1.37) we utilized the following identities:

$$\gamma_\mu\gamma^\mu = d \qquad \text{and} \qquad \gamma_\mu\not{p}\gamma^\mu = (2-d)\not{p} \quad . \tag{1.38}$$

From this point on the calculations continue in a relatively straightforward manner. After performing the integration over z and extracting the relevant terms we obtain the results for the dimensionally regularized theory

$$i\Sigma'(\not{p}) = -i\frac{C(F)}{16\pi^2}(\not{p}-2m)+i\frac{C(F)}{16\pi^2}\frac{1}{\eta}(\not{p}-4m)-i\frac{C(F)}{16\pi^2}I+O(\varepsilon) \tag{1.39}$$

Where η is defined by Eq. (1.25) and

$$I=\int_0^1 dz\,[4m-2\not{p}(1-z)]\,\ln\frac{m^2z-p^2z(1-z)}{\tilde{\mu}^2} \tag{1.40}$$

All terms in $O(\varepsilon)$ in Eq. (1.37) will vanish as we take the limit $\epsilon\to 0$ to recover the expressions for the 4 dimensional theory, so we need not worry about them. The first and third terms in Eq. (1.37) are not dependent on ϵ and are finite. So the divergence is contained in the second term of Eq.(1.37) from which we will be able to infer the needed counter terms for the renormalization procedure.

For the sake of brevity, the full calculation of gauge parameter dependent terms is presented in

Appendix A. $i\Sigma(\not{p})$ then evaluates to

$$i\Sigma(\not{p}) = -i\frac{C(F)}{16\pi^2}\frac{1}{\eta}\left(\not{p} - 4m + (1-\xi)\left[4\not{p} - 4m\right]\right) + \text{Finite Terms} + O(\varepsilon) \ . \tag{1.41}$$

where Finite Terms do not contribute to the renormalization constants and thus are excluded from the expression.

1.3.3 Renormalization

The self energy $\Sigma(\not{p})$ of the fermion ψ_i can be understood as modifying the electron propagator in the following way

$$(\tilde{\Pi}_i)^{-1}(\not{p}) = (\Pi_i)^{-1}(\not{p}) - \Sigma_i(\not{p}) \ , \tag{1.42}$$

where $\tilde{\Pi}(\not{p})$ is to be understood as the effective fermion propagator, and $\Pi(\not{p})$ as the initial fermion propagator equal to Eq. (1.19). To remove the divergences we introduce the so-called counterterms that are equal to the divergent terms in Eq. (1.41) but with the opposite sign:

$$(\tilde{\Pi}_i)^{-1}(\not{p}) = (\Pi_i)^{-1}(\not{p}) - \Sigma_i(\not{p}) - \Sigma_i^c(\not{p}) \ . \tag{1.43}$$

The counterterms correspond to additional terms in the (now effective) Lagrangian, which can be nicely interpreted as modifications of the renormalization constants of the fields or in other cases as couplings.

In the presented case the counterterms $\Sigma_i^c(\not{p})$ take the form

$$\Sigma_i^c(\not{p}) = \frac{[C(F)]_{ii}}{16\pi^2}\frac{1}{\eta}\left((5-4\xi)\not{p} - (8-4\xi)m\right) \ . \tag{1.44}$$

The field renormalizes multiplicatively as

$$\psi_i \Rightarrow Z_i^{\frac{1}{2}}\psi_i = \left(1 + \frac{B_i}{\eta}\right)^{\frac{1}{2}}\psi_i \, , \tag{1.45}$$

where in anticipation of the next subsection we defined B_i as

$$B_i = (5-4\xi)\frac{[C(F)]_i}{16\pi^2} . \tag{1.46}$$

The fermion mass renormalization proceeds as

$$m \Rightarrow Z_i^{-1}\left(M_i - (8-4\xi)\frac{C(F)}{16\pi^2}\frac{1}{\eta}m\right) = Z_{Mi}m \, , \tag{1.47}$$

where Z_{Mi} is the mass renormalization constant.

The anomalous dimension of the fields can then be extracted via Eq. (1.16b). Here we take the opportunity to define the anomalous dimension of the coupling constant (or, equivalently, the

anomalous dimension of the corresponding operator) as

$$\gamma_{M_i} = \frac{\partial \ln Z_{Mi}}{\partial \ln \mu}\,. \tag{1.48}$$

We note that at higher loop orders a sum of multiple Feynman diagrams will contribute to the renormalization constant, see e.g. Table 3. in Ref. [1], or for an updated presentation Table 2.2 in Ref. [24]. The counter terms, and the renormalization constants for couplings can be calculated in an analogous fashion once contributing Feynman diagrams are determined.

1.4 Two-loop order β-functions for a general theory

In previous subsections we discussed in some generality the theory of renormalization and renormalization group equations. We now discuss the concrete results for 2-loop level β-functions for general theories, on which modern automation packages are based, and methods for obtaining them.

β-functions play an important role in precision calculations and are often the starting point for phenomenological analyses. The derivation of them can be an arduous and repetitive task. Therefore, a programme to derive general β-functions for a wide class of QFT models *once and for all* was initiated in the 1980s by Machacek and Vaughn [1–3]. This resulted in a series of publication in which the general renormalization group equations for dimensionless couplings were derived [1–3]. General β-functions were later derived for dimensionful couplings [4, 5]. Kinetic mixing, a complication not addressed in the seminal papers [1–3] as also been taken into account in later works [6, 26]. Errors and typos in the original publications [1–3] have since been corrected in subsequent works [5, 24].

In recent years a novel presentation of β-fuctions based on the so-called basis of tensor structures was developed [11]. A basis of tensor structures is, effectively, a set of all possible tensor contractions that may appear in the expression of the β-function. This novel approach enables a cleaner and more structured presentation, as β-functions can then be expressed as a linear combination of tensor structures. This formalism thus was adopted in developing the latest iterations of automation packages, e.g. Refs. [14]. Furthermore, it allowed the uncovering of remarkable relations between β-functions of different couplings, namely the so-called Weyl consistency conditions [11, 32]. These in turn allow the derivation of 3-loop order β-functions for gauge couplings from the lower loop level β-functions of Yukawa and Quartic couplings. These results have since been incorporated into automation packages such as Ref. [14], and further elaborated for dimensionful couplings [4].

In this subsection we present some main aspects of general β-functions and their derivation. We briefly describe some preparatory results used for their derivations. We also describe the application of the general results to particular theories.

1.4.1 The pole equations

We present a useful result allowing for a sort of shortcut in evaluating β-functions, that is utilized in many publications dealing with general β-functions, the so-called pole equations [33]. Namely, we show that the β-function is completely determined by the residue of the first order poles in ε appearing in the renormalization constants. We present the intermediate steps often omitted in the aforementioned publications. The following presentation also connects the β-functions for a $d = 4 - 2\varepsilon$ dimensions theory with the standard β-functions for 4 dimensional theories.

Let $\Theta_i^{(B)}$ be a bare coupling constant. Its dependence on a renormalized coupling constant Θ_i in $d = 4 - 2\varepsilon$ dimensions to some arbitrary loop order can be expressed as

$$\Theta_i^{(B)}\mu^{-\rho_i\varepsilon} = \Theta_i + \sum_{n=1}^{\infty} a_i^{(n)}(\boldsymbol{\Theta})\frac{1}{\varepsilon^n} \tag{1.49}$$

where for gauge and Yukawa couplings $\rho = 1$, and quartic scalar couplings $\rho = 2$ for (this follows from considering the mass dimension of the corresponding operators); $a^{(n)}$'s are to be determined at some loop order. We note that, strictly speaking that n does not denote a loop order, rather it indexes the coefficients of the formal series that have to be determined at some finite loop order. We also note that at any finite loop order the series in (1.49) will not be infinite, rather it will truncate after at some power of epsilon ϵ^{-N}, that is we will have $a_i^{(n')} = 0$ for all $n' > 0$. In the example given in Sec. 1.3 the series truncates at ε^{-1}, see Eq. 1.41. We will keep this truncation implicit and work with the infininte series. In this subsection we also employ the shorthand $\boldsymbol{\Theta} = \{\Theta_j\}_{j\in\mathcal{J}}$ from Subsec. 1.2.

We denote the β-function of the coupling Θ_i defined in a $(4 - 2\varepsilon)$-dimensional theory as $\beta_{\Theta_i}(\boldsymbol{\Theta},\ \epsilon)$. This function depends on ϵ and is such that $\lim_{\varepsilon\to 0}\beta_{\Theta_i}(\boldsymbol{\Theta},\ \epsilon) = \beta_{\Theta_i}$, where β_{Θ_i} is the beta function in a standard 4-dimensional theory. Since β_{Θ_i} is required to be finite, $\beta_{\Theta_i}(\boldsymbol{\Theta},\ \epsilon)$ cannot have any poles in ε. The dependence of $\beta_{\Theta_i}(\boldsymbol{\Theta},\ \epsilon)$ on ε can, therefore, be expressed as a power-series in ε

$$\beta_{\Theta_i}(\boldsymbol{\Theta},\ \epsilon) = \beta_{\Theta_i}(\boldsymbol{\Theta}) + \sum_{v=1}^{B}\beta_{\Theta_i}^{(v)}\epsilon^v\,, \tag{1.50}$$

where we have defined $\beta_{\Theta_i}^{(0)} = \beta_{\Theta_i}(\boldsymbol{\Theta})$. The relation between $\beta_{\Theta_i}(\boldsymbol{\Theta},\ \epsilon)$ and Θ_i in a $(4 - 2\varepsilon)$-dimensional theory is as expected

$$\beta_{\Theta_i}(\boldsymbol{\Theta},\ \epsilon) = \mu\frac{\partial\Theta_i}{\partial\mu} \tag{1.51}$$

The series in Eq. (1.51) is explicitly truncated at ε^B, again, since all calculations one performs are up to some finite loop order. The index i in Eq. (1.51) should also not be understood as indicating a loop-order, however, in general $\beta_{\Theta_i}(\boldsymbol{\Theta},\ \epsilon)$ will be different at different loop-orders.

We now procede with the derivation of the so-called pole equations. Applying the definition

Eq. (1.51) to Eq. (1.49) yields

$$\beta_{\Theta_i}(\mathbf{\Theta},\varepsilon) = -\Theta_i^{(B)}\rho_i\epsilon\mu^{-\rho_i\varepsilon} - \sum_{n=1}^{\infty}\sum_{l}\mu\frac{d\Theta_l}{d\mu}\frac{\partial a_i^{(n)}(\mathbf{\Theta})}{\partial\Theta_l}\frac{1}{\varepsilon^n}, \tag{1.52}$$

where the index l runs through all coupling parameters $\mathbf{\Theta}$. The factor of $\Theta_i^{(B)}\mu^{-\rho_i\varepsilon}$ appearing in the first term of Eq. (1.52) can be substituted for the right-hand side of Eq. (1.49). This yields

$$\beta_{\Theta_i}(\mathbf{\Theta},\varepsilon) = -\rho_i\varepsilon\Theta_i - \sum_{n=1}^{\infty}\rho_i a_i^{(n)}(\mathbf{\Theta})\frac{1}{\varepsilon^{n-1}} - \sum_{n=1}^{\infty}\sum_{l}\beta_{\Theta_l}(\mathbf{\Theta},\varepsilon)\frac{\partial a_i^{(n)}}{\partial\Theta_l}\frac{1}{\varepsilon^n}, \tag{1.53}$$

In Eq. (1.53) we also recognized that the factor $\mu\frac{d\Theta_l}{d\mu}$ appearing in the second term can be replaced by $\beta_{\Theta_i}(\mathbf{\Theta},\varepsilon)$ (see Eq 1.51). By expanding $\beta_{\Theta_i}(\mathbf{\Theta},\varepsilon)$ and $\beta_{\Theta_l}(\mathbf{\Theta},\varepsilon)$ in Eq. (1.53) as a power series (1.50) one can obtains the following relation

$$\sum_{v=0}^{B}\beta_{\Theta_i}^{(v)}\epsilon^v = -\rho_i\varepsilon\Theta_i - \sum_{n=1}^{\infty}\rho_i a_i^{(n)}(\mathbf{\Theta})\frac{1}{\varepsilon^{n-1}} - \sum_{n=1}^{\infty}\sum_{l}\sum_{v=0}^{B}\beta_{\Theta_l}^{(v)}(\mathbf{\Theta},\varepsilon)\frac{\partial a_i^{(n)}}{\partial\Theta_l}\epsilon^{v-n} \tag{1.54}$$

Eq. (1.54) needs to hold for each power ε: by collecting the terms with ε^0, ε^1 and ε^N, where $B \geq N \geq 2$, one obtains

$$\beta_{\Theta_i}^{(0)} = -\rho_i a_i^{(1)}(\mathbf{\Theta}) - \sum_{m=1}^{B}\sum_{l}\beta_{\Theta_l}^{(m)}(\mathbf{\Theta},\varepsilon)\frac{\partial a_i^{(m)}(\mathbf{\Theta})}{\partial\Theta_l} \tag{1.55a}$$

$$\beta_{\Theta_i}^{(1)} = -\rho\Theta_i - \sum_{l}\sum_{m=2}^{B}\beta_{\Theta_l}^{(m)}(\mathbf{\Theta},\varepsilon)\frac{\partial a_i^{(m-1)}(\mathbf{\Theta})}{\partial\Theta_l} \tag{1.55b}$$

$$\vdots$$

$$\beta_{\Theta_i}^{(N')} + \sum_{l}\sum_{m=N'+1}^{B}\beta_{\Theta_l}^{(m)}(\mathbf{\Theta},\varepsilon)\frac{\partial a_i^{(m-N')}(\mathbf{\Theta})}{\partial\Theta_l} = 0, \quad \text{for} \quad \mathrm{B} > \mathrm{N}' \geq 2 \tag{1.55c}$$

$$\vdots$$

$$\beta_{\Theta_i}^{(B-1)} + \sum_{l}\beta_{\Theta_l}^{(B)}(\mathbf{\Theta},\varepsilon)\frac{\partial a_i^{(1)}(\mathbf{\Theta})}{\partial\Theta_l} = 0 \tag{1.55d}$$

$$\beta_{\Theta_i}^{(B)} = 0 \tag{1.55e}$$

From the set of equations Eqs. (1.55c)–(1.55e), one can show that $\beta_{\Theta_i}^{(n')} = 0$ for $n' \geq 2$: one can show that $\beta_{\Theta_i}^{(B-1)} = 0$, by inserting Eq. (1.55e) into Eq. (1.55d); it is easy to see that the process can then be repeated between all pairs of equations corresponding to $\varepsilon^{N'}$ and $\epsilon^{N'-1}$ until one reaches reaches $\beta_{\Theta_i}^{2} = 0$. This then allows one to simplify Eqs. (1.55a) and (1.55b) to

$$\beta_{\Theta_i}^{(0)} = -\rho_i a_i^{(1)}(\mathbf{\Theta}) - \sum_{l}\beta_{\Theta_l}^{(1)}(\mathbf{\Theta},\varepsilon)\frac{\partial a_i^{(1)}(\mathbf{\Theta})}{\partial\Theta_l} \tag{1.56a}$$

$$\beta_{\Theta_i}^{(1)} = -\rho_i\Theta_i \tag{1.56b}$$

Taken together Eqs. (1.56a) and (1.56b) yield

$$\beta^{(0)}_{\Theta_i} = \beta_{\Theta_i}(\boldsymbol{\Theta}) = \sum_l \rho_l \Theta_l \frac{\partial a_i^{(1)}}{\partial \Theta_l} - \rho_i a_i^{(1)}\,, \tag{1.57}$$

Eq. (1.57) implies that the β-functions are completely determined by the residues of the first-order poles in ε of Eq. (1.49), and is known as the pole equations.

Remark 1.3 *We remark that an alternative derivation of the pole equation is possible, (see, e.g. Ref. [34] and references therein) wherein one requires the explicit relation*

$$\beta_{\Theta_i}(\boldsymbol{\Theta}, \varepsilon) = -\rho_i \varepsilon \Theta_i + \beta_{\Theta_i}(\boldsymbol{\Theta})\,.$$

to hold. Alternativelly, one may consider the truncation of the series (1.50) *as a "pole cancellation condition".*

The same set of arguments applied to the multiplicative field renormalization constants yield

$$\gamma_{\varphi_i} = \frac{1}{2} \sum_l \rho_l \Theta_l \frac{\partial a_\Gamma^{(1)}}{\partial \Theta_l}\ , \tag{1.58}$$

where $a_\Gamma^{(1)}$ is the singular part of the first order pole of the field's renormalization constant associated to the irreducible self-energy part of φ_i; γ_{φ_i} is the anomalous dimension of the field as defined by Eq. (1.16b). The same calculation carried out for the multiplicative renormalization constant of a coupling constant (or, an operator) (1.48) yields

$$\gamma_{\Theta_i} = -\sum_l \rho_l \Theta_l \frac{\partial a_{\Gamma'}^{(1)}}{\partial \Theta_l}\ , \tag{1.59}$$

where $a_{\Gamma'}^{(1)}$ is the residuum of the first order pole of the corresponding proper vertex corrections; γ_{Θ_i} is the anomalous dimension of an operator (1.48). In Ref. [1] an additional useful identity is presented: for the N^{th} loop contribution to $a^{(1)}$ the following relation holds

$$\sum_l \rho_l \Theta_l \frac{\partial a^{(1)}}{\partial \Theta_l}\bigg|_{N^{\text{th}}\text{ loop contribution}} = 2N a^{(1,\ N^{\text{th}}\text{ loop})}\ . \tag{1.60}$$

The relations outlined above structure the calculations of general β-functions.

1.4.2 Two-loop order β-functions

Here we present the standard strategy for obtaining general two-loop order β-functions. This strategy was employed in the original publications [1–3], and in later contributions to this research programme [4–11]. For the sake of clarity, in this subsection we denote matrix quantities with a circumflex $\hat{}$).

The expressions for general β-functions are obtained through their relation to anomalous dimensions, therefore the later need to be obtained first. To obtain the anomalous dimensions of fields one evaluates the contributing Feynman diagrams, and extracts from them their contributions to the singular parts of the wave function renormalization constants. As described in Ref. [1], for fermion, scalar and gauge vector boson fields the contribution due to a given Feynman diagram to the singular part of the wave function renormalization matrix can be expressed as

$$[Z_S^{-1}]_{ab} = \frac{1}{(4\pi)^2} S_{ab} \frac{B^{(1)}}{\eta} + \frac{1}{(4\pi)^4} S_{ab} \left(\frac{A^{(2)}}{\eta^2} + \frac{B^{(2)}}{\eta} \right) \quad , \tag{1.61}$$

where η is defined by Eq. (1.25), $A^{(i)}$ and $B^{(i)}$ are the poles of the singular terms arising in the renormalization matrix (superscripts in parantheses indicate terms arising at different loop orders) and S_{ab} is a group theoretic factor [5] extracted from contributing Feynman diagram. The anomalous dimension can then by obtained by applying the relations (1.58), (1.58) and (1.60) to Eq. (1.61), keeping in mind that one should sum over all contributions of relevant Feynman diagrams. The anomalous dimension matrices $\hat{\gamma}_\varphi$ at two-loop order then read

$$\hat{\gamma}_{\varphi_i} = \frac{1}{(4\pi)^2} \sum_{\text{diagrams}} \hat{S}^{(1)} B^{(1)} + \frac{2}{(4\pi)^4} \sum_{\text{diagrams}} \hat{S}^{(2)} B^{(2)} \quad . \tag{1.62}$$

The anomalous dimensions of operators $\bar{\psi}_i \psi_j \phi_a$ and $\phi_a \phi_b \phi_c \phi_d$, below denoted as $\hat{\gamma}_{\mathbb{6}}$, corresponding to Yukawa and scalar quartic couplings, are computed equivalently (see Sec. 2 of [1]) and read

$$\hat{\gamma}_{\mathbb{6}} = -\frac{2}{(4\pi)^2} \sum_{\text{diagrams}} \hat{S}^{(1)} B^{(1)} - \frac{4}{(4\pi)^4} \sum_{\text{diagrams}} \hat{S}^{(2)} B^{(2)} \quad . \tag{1.63}$$

We note that Eq. (1.61) and Eq. (1.62) the differ by a factor of -2 due to the different definitions of anomalous dimensions for fields Eq. (1.16b) and couplings (1.48), this difference is then carried over to Eq. (1.58) and Eq. (1.59).

One then obtains the corresponding β-function by the following relations presented throughout the series of the seminal papers [1–3]. The relations between the anomalous dimensions of the gauge vector bosons γ_A and the β-functions for gauge couplings β_g is given by (see Eq. 5.2 in Ref. [1])

$$\beta_g = g \gamma_A \, . \tag{1.64}$$

The relation between the anomalous dimensions of scalar fields $\hat{\gamma}^S$, the anomalous dimensions of operators $\phi_a \phi_b \phi_c \phi_d$, here denoted as γ_{abcd}, and the β-function for scalar quartic couplings β_{abcd} is given by (see Eq. 4.2 in Ref. [2])

$$\beta_{abcd} = \gamma_{abcd} + \sum_e \gamma^S(e) \lambda_{abcd} \tag{1.65}$$

[5] We note that the term *group theoretic factor* is somewhat of a misnomer. While it is indeed true that some of these factors are interpretable as group theoretic invariants, in general they can be any contractions of tensors. We nonetheless use the term *group theoretic factor* to maintain consistency between this thesis and the original publications.

where $\gamma^S(e)$ is the anomalous dimension of the e^{th} scalar field. The relation between the anomalous dimensions of scalar and fermion fields, here denoted as $\hat{\gamma}^S$ and $\hat{\gamma}^F$ respectively, the anomalous dimensions of operators $\bar{\psi}_i\psi_j\phi_a$, here denoted as $\hat{\gamma}^a_Y$, and the β-functions for the Yukawa coupling matrix $\hat{\beta}^a_Y$ is given by (see Eq. 3.2 in Ref. [3])

$$\hat{\beta}^a_Y = \hat{\gamma}^a_Y + \hat{\gamma}^{\dagger\,F}\hat{Y}^a + \hat{Y}^a\hat{\gamma}^F + \gamma^S_{ab}\hat{Y}^b \tag{1.66}$$

For the set of general β-functions for dimensionless coupling, with errors present in the original publication corrected see Ref. [5], and for the presentation of general beta functions in the modern Tensor Structure basis see Ref. [11].

1.4.3 Dimensionful couplings and the dummy field method

We now turn our attention to dimensionless couplings. A convenient way has been introduced in Ref. [5] to read off β-functions for dimensionful couplings from β-functions for dimensionless ones. It was shown that by extending the theory with non-propogating so-called *dummy* scalar fields dimensionful couplings appearing in Lagrangian (1.6) can be represented as dimensionless ones, thus allowing one to infer their β-functions.

The dummy scalar $\phi_{\hat{d}}$ field satisfies the following equation

$$D_\mu\phi_{\hat{d}} = 0 \ . \tag{1.67}$$

The dimensionful couplings appearing in the Lagrangian density Eq. (1.6) can then be equated to the following factors

$$Y_{\hat{d}ij}\phi_{\hat{d}} = m_{ij}\,, \quad \lambda_{ab\hat{d}\hat{d}}\phi_{\hat{d}}\phi_{\hat{d}} = 2\mu_{ab}\,, \quad \lambda_{abc\hat{d}}\phi_{\hat{d}} = t_{abc} \ . \tag{1.68}$$

With the identifications of Eq. (1.68) the Lagrangian (1.6) can be interpreted as only having dimensionless couplings. The β-functions for the fermion mass, scalar mass, and scalar trilinear couplings are then recovered from the β-functions for Yukawa and quartic scalar couplings, accordingly.

For a set of general β-functions for dimensionless couplings see Ref. [5]. For a presentation of β-functions for dimensionless couplings in the modern Tersor Structure basis see Ref. [4]. Also, for a convenenient set of substitution rules for converting β-functions of dimensionless couplings to β-functions for dimensionless couplings see Sec. 5 of Ref. [7].

1.4.4 Application to specific theories

General expressions for β-functions greatly simplify precision calculations. However translation of general expressions to ones suited for specific models can be difficult. Two complications arise when translating general β-functions to specific ones: one relating to the recovery of the gauge coupling constant, and one relating to the reconstruction of specific field (e.g., electron doublets or quark singlets) from the general multiplets.

The first complication is related to the fact that in the standard presentations of the general β-

functions (e.g., as in the seminal e.g., [1–3]) assume a simple Gauge group. Alternativelly, under the construction of the Lagrangian as in Subsec. 1.1, the dependence on gauge coupling constants is made implicit altogether. The issue is resolved by carefully examining the contributing Feynman dagrams, and constructing substitution rules for terms appearing in the expressions for general β-functions. For example, given the construction of the general Lagrangian in Subsec. 1.1, the relevant substitution rule that is to be applied in the β-function for the Yukawa coupling, is

$$C_2(R) \ \rightarrow \ \sum_k g_k^2 C_2^{(k)}(R) \tag{1.69}$$

where k indexes the gauge factor, and $R \in \{S,\ F\}$ is either the representation of the group for scalar field S or for fermion fields F. For more substitution Rules see Section VI in Ref. [5].

Regarding the second complication, in principle, one could explicitly construct the fields presented in Subsec. (1.1), however such procedure is error-prone and unsustainable when working with a variety of different models. A systematic approach known as the Structure Δ formalism approach was proposed in Ref. [10], and later expanded in Ref. [12], to deal with this issue. In this formalism most of the labour of constructing the mapping between a general and a specific theory is made implicit by the introduction of a projection operator, the so-called structure Δ. This projection operator is constructed to cast general multiplets described in Subsec. 1.1 into specific fields. For the an in-depth review of the Structure Δ formalism see Refs. [10] and [12].

1.5 Renormalization group equations for vacuum expectation values

We close off this section with a discussion on the renormalization of vacuum expectation values (VEV). One often considers models with spontaneously broken gauge symmetry, such as in the case of the GNM, therefore the renormalization of the VEV should be part of the discussion of general RGEs.

The exact method of VEV renormalization is not uniquely determined by the overall renormalization scheme, like the $DR/\overline{MS}$ scheme described in Subsec. 1.3. There are multiple ways to renormalize the VEV (see Ref. [35] and references therein for an overview of renormalization schemes not covered in this subsection).

In the context of general β-functions, the renormalization group equations for VEVs were derived in [8, 9], using a method based on background fields. These results have been adopted by the developers of the automated package [14], which we use. Therefore we choose to present it.

For VEV renormalization, once one determines the contributing Feynman diagrams, from which the relevant counterterms can be extracted, VEV renormalization proceeds like the general strategy outlined in Subsec. 1.3. The setup, however, for an efficient derivation of general β-functions for a VEV is rather technical. Here we present only some aspects of the VEV renormalization as detailed in [8, 9].

In spontaneously broken gauge theories the scalar field ϕ_a is shifted by a constant v_a, the so-called VEV, that is determined by the minimum of the scalar potential [27]. Gauge invariance is preserved as long as gauge transformations are applied to the shifted field $\phi_a + v_a$ as a whole.

However, gauge-fixing breaks local and, depending on the gauge, global gauge invariance. In the latter case the VEV requires additional corrections separate from those coming from the scalar field renormalization.

Within the background field method the VEV is promoted to a constant field $\hat{v}_a$ and an additional classical background field $\hat{\phi}_a$ is introduced. In this set up, following spontaneous symmetry breaking, the scalar field can be expressed as

$$\phi_a \to \phi_a^{\text{eff.}} = \phi_a + \hat{\phi}_a + \hat{v}_a \ . \tag{1.70}$$

Without the background field the most general renormalization structure for the scalar field and the VEV is

$$(\phi + v) \to \sqrt{Z}\,(\phi_a + v) + \delta v \ , \tag{1.71}$$

Where $\sqrt{Z}$ is the scalar field's renormalization constant, and δv is the additive renormalization constant for the VEV. While in certain gauges (e.g., in Landau gauge with $\xi = 0$), the additive constant is prohibited by a residual global gauge symmetry, δv is non-vanishing in a general R_ξ gauge, that breaks the global gauge symmetry. Without the background field the calculation of the renormalization constant for the VEV becomes somewhat intractable. With the help of the background field one is able to enforce, *á la* Fadeev and Popov [27], global gauge invariance. With this the VEV renormalizes as

$$(\phi + \hat{\phi} + \hat{v}) \to \sqrt{Z}\left(\phi_a + \sqrt{\hat{Z}}(\hat{\phi} + \hat{v})\right) \ , \tag{1.72}$$

as the additive renormalization constant δv in Eq. (1.71) is prohibited by symmetry arguments.

The contributing Feynman diagrams are determined as follows: the gauge-fixed Lagrangian retains a residual symmetry – BRST invariance [27]. For the purposes of this thesis it suffices to say that BRST invariance can be understood as a rigid symmetry parametrized by Grassmann numbers. A nilpotent operator s can be constructed such that its action preserves BRST invariance; it is called the Slavnov or the BRST operator [27]. It transforms the background field in the following way

$$s\hat{\phi}_a = \hat{q}_a \ . \tag{1.73}$$

By nipotency of s we have

$$s\hat{q}_a = 0 \ . \tag{1.74}$$

The renormalization of $\hat{q}$ follows from Eq. (1.72) and Eq. (1.73)

$$\hat{q}_a \to \sqrt{Z}_{ab}\sqrt{\hat{Z}}_{bc}\hat{q}_c \ . \tag{1.75}$$

The transformation of the scalar fields can be inferred from Eq. (1.72), and all other BRST transformations are standard [8]. The final piece of the setup is the introduction of external sources K_{ϕ_a}

for $\hat{q}$ (formally analogous to ones described under Eq. (1.7)) into the Lagrangian

$$\mathscr{L}_{\text{ext}} \ni K_{\phi_a}\hat{q}_a = K_{\phi_a} s\phi_a \ . \tag{1.76}$$

Finally, we note that the source K_{ϕ_a} transforms with the inverse renormalization constant of the scalar field

$$K_{\phi_a} \to \left(\sqrt{Z}^{-1}\right)_{ab} K_{\phi_b} \ . \tag{1.77}$$

The total Lagrangian, which has been massaged into the needed form, reads as

$$\mathscr{L}' = (\mathscr{L}_1 + \mathscr{L}_2)\big|_{\phi\to\phi^{\text{eff.}}} + \mathscr{L}_\star + \mathscr{L}_{\text{ext}} \ , \tag{1.78}$$

where $\mathscr{L}_1$ and $\mathscr{L}_2$ are given in Subsec. (1.1), $\mathscr{L}_\star$ contains gauge fixing and ghost terms (see Eq. 9b in Ref. [8]).

As demonstrated in Ref. [8] all counterterms contributing to the renormalization constant $\sqrt{\hat{Z}}$ can be extracted from the two-point function $\langle K_{\phi_a}\hat{q}_b\rangle$. The number of diagrams contributing to this function at one- and two-loop orders is comparatively small. It is worth noting that the two-point function $\langle K_{\phi_a}\hat{q}_b\rangle$ is manifestly unphysical, and is to be regarded as a technical tool only. For contributing diagrams see Refs. [8, 9].

Renormalization then proceeds in the standard way as described in Subsec. 1.3 with the relation between the anomalous dimensions of the fields and the β-function for the VEV being

$$\beta_{v_a} = \gamma_{ab} v_b + \hat{\gamma}_{ab} v_b \tag{1.79}$$

where γ_{ab} are the anomalous dimensions of the scalar fields and are associated with $\sqrt{Z}_{ab}$, and $\hat{\gamma}_{ab}$ are the anomalous dimensions associated with $\sqrt{\hat{Z}}_{ab}$. The full β-function for VEVs is given by Eq. of Ref. [9]. It is worth noting, that unlike in the case of β-functions for dimensionless couplings, the gauge dependence does not cancel out in the β-function for VEVs.

Remark 1.4 *We remark that a similar construction as described in this section can be used for a sort of shortcut for extracting the counterterms for gauge boson renormalization, and by extension deriving the β-functions for gauge coupling constants [36].*

2 Grimus-Neufeld model, seesaw mechanism and neutrino oscillations

The Grimus-Neufeld Model (GNM) [17] is a Two Higgs Doublet Model (2HDM) [37] extended with a sterile right-handed massive neutrino, that enjoys a Yukawa coupling and a Majorana mass term in the model's Lagrangian. In this section we present this model in more detail.

We begin by presenting the Lagrangian of the GNM term-by-term as this allows us to give a greater context to the model and have this section self-contained as far as model building is concerned. Next we describe the role of the GNM in modern high-energy physics and discuss the seesaw mechanism through which explains the smallness of observed neutrino masses.

2.1 The Bare Grimus-Neufeld Model Lagrangian

The defining features of any quantum field theoretic model are the gauge symmetries of its Lagrangian density. As such we begin the description of the GNM by stating the gauge symmetry group. The gauge group $\mathscr{G}_{GN}$ of the GNM is the same as that of the Standard Model (SM) or the 2HDM, namely, it is the semi-simple group

$$\mathscr{G}_{GN} = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \quad . \tag{2.1}$$

One can roughly think of the last two subgroups $SU(2)_L$ and $U(1)_Y$ of $\mathscr{G}_{GN}$, as constituting the electroweak symmetry, which is broken by the Higgs mechanism, leaving the residual symmetry $U(1)_Y \otimes SU(2)_L \to U(1)_Q$. The first subgroup $SU(3)_C$ of $\mathscr{G}_{GN}$ represents the color symmetry of the quark sector.

The matter content of SM-like models is given by specifying for each matter field the dimensions of the representations of the groups $SU(3)$ and $SU(2)$ in which it transforms, and the hypercharge Y associated with the group $U(1)$. The matter content data of the GNM is summarized in Table 1.

The chirality subscripts R and L, and the flavour indices will be suppressed hereafter.

We begin the construction of the bare GNM Lagrangian from the gauge sector which is determined by the gauge group $\mathscr{G}_{GN}$. The gauge sector Lagrangian reads

$$\mathscr{L}_G = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W^a_{\ \mu\nu} W^{a\mu\nu} - \frac{1}{4} G^b_{\ \mu\nu} G^{b\mu\nu} \,. \tag{2.2}$$

where $B_{\mu\nu}$, $W^a_{\ \mu\nu}$ and $G^b_{\ \mu\nu}$ are the gauge field strenght tensors, corresponding to the Abelian gauge fields B_μ, and the two non-Abelian gauge fields $W^a_{\ \mu}$ and $G^b_{\ \mu}$ accordingly. The exact expressions for the gauge field strenght tensors read

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \,, \tag{2.3}$$

$$W^i_{\ \mu\nu} = \partial_\mu W^i_{\ \nu} - \partial_\nu W^i_{\ \mu} - g_2 w^{ijk} W^j_{\ \mu} W^k_{\ \nu} \,, \tag{2.4}$$

$$G^i_{\ \mu\nu} = \partial_\mu G^i_{\ \nu} - \partial_\nu G^i_{\ \mu} - g_3 g^{ijk} G^j_{\ \mu} G^k_{\ \nu} \,, \tag{2.5}$$

Table. 1 Particle content of the GNM. The symbol $\dim\rho_G$ should be understood as the dimension of the representation ρ_G of group G. The descriptions are given in the context of $SU(2)$ representations. One should keep in mind that the quark sector could primarily be grouped into color triplets.

label	families	$\left(\dim\rho_{SU(3)}, \dim\rho_{SU(2)}, Y\right)$	description	chirality
$\ell_L = \begin{pmatrix}\nu_L \\ e_L\end{pmatrix}$	3	$\left(1, 2, -\frac{1}{2}\right)$	lepton doublet	left-handed
N_R	1	$(1, 1, 0)$	neutrino singlet	Majorana
e_R	3	$(1, 1, -1)$	electron singlet	right-handed
$Q = \begin{pmatrix}u_L \\ e_L\end{pmatrix}$	3	$\left(3, 2, \frac{1}{6}\right)$	quark doublet	left-handed
u_R	3	$\left(3, 1, \frac{2}{3}\right)$	up quark singlet	right-handed
d_R	3	$\left(3, 1, -\frac{2}{3}\right)$	down quark singlet	right-handed
$\phi = \begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix}$	2	$\left(1, 2, \frac{1}{2}\right)$	Higgs doublet	n/a

where w^{ijk} and g^{ijk} are the group structure constants, and g_2 and g_3 are the gauge coupling constants of the two non-Abelian groups $SU(2)_L$ and $SU(3)_C$, respectively. The gauge fields B_μ, $W^i_{\ \mu}$ and $G^i_{\ \mu}$ transform under the adjoint representation of the groups $U(1)_Y$, $SU(2)_L$, and $SU(3)_C$ accordingly.

Having defined the gauge fields we are in the position to define the covariant derivative of $\mathscr{D}_\mu$ of the GNM. The full covariant derivate reads

$$\mathscr{D}_\mu = \partial_\mu + i\frac{1}{2}g_1 Y B_\mu + i\frac{1}{2}g_2\sigma^k W^k_{\ \mu} + i\frac{1}{2}g_3\lambda^k W^k_{\ \mu}\,, \tag{2.6}$$

where $\frac{1}{2}\sigma^k$ are the standard generators of $SU(2)_L$, i.e. the Pauli matrices, and λ^k are the equivalents for $SU(3)_C$, the Gell-Mann matrices. It is worth noting that each term of the covariant derivative acts on each field in accordance with the representation that field is in: the hypercharge Y_f is different for each field f (as given in Table 1), and other terms may drop out, for example, for fields without color charge, like one of the Higgs doublets, the final term of Eq. (2.6) drops out:

$$\mathscr{D}\phi_a = \partial_\mu\phi_a + i\frac{1}{2}g_1 Y_{\phi_a} B_\mu\psi_a + i\frac{1}{2}g_2\sigma^k W^k_{\ \mu}\,\phi\,. \tag{2.7}$$

The gauge-fermion Lagrangian, that includes the kinetic terms for the fermions, is given by

$$\mathscr{L}_F = \sum_\psi \bar{\psi} i\gamma^\mu \mathscr{D}_\mu\psi\,, \tag{2.8}$$

where the summation runs over all fermion fields ψ given in Table 1. The Higgs-gauge sector is given by

$$\mathscr{L}_H = \left(\mathscr{D}^\mu\phi^a\right)^\dagger\left(\mathscr{D}_\mu\phi^a\right) - V(\phi)\,, \tag{2.9}$$

where $V(\phi^a)$ is the scalar potential of the 2HDM. In its most general form it is given as

$$V(\boldsymbol{\phi}) = \mu_{ab}^2 \phi_a^\dagger \phi_b + \frac{1}{2}\lambda_{abcd}\left(\phi_a^\dagger \phi_b\right)\left(\phi_c^\dagger \phi_d\right) . \tag{2.10}$$

For practical purposes Eq. (2.10) is more often expressed as

$$\begin{aligned} V = \; & m_{11}^2 \phi_1^\dagger \phi_1 + m_{22}^2 \phi_2^\dagger \phi_1 \\ & + \frac{1}{2}\lambda_1 (\phi_1^\dagger \phi_1)^2 + \frac{1}{2}\lambda_2 (\phi_2^\dagger \phi_2)^2 + \lambda_3 (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + \lambda_4 (\phi_2^\dagger \phi_1)(\phi_1^\dagger \phi_2) \\ & + \Big[- m_{12}^2 \phi_1^\dagger \phi_2 + \frac{1}{2}\lambda_5 (\phi_2^\dagger \phi_1)(\phi_2^\dagger \phi_1) + \lambda_6 (\phi_1^\dagger \phi_1)(\phi_1^\dagger \phi_2) + \lambda_7 (\phi_2^\dagger \phi_2)(\phi_2^\dagger \phi_1) + \text{h.c.} \Big] \; . \end{aligned} \tag{2.11}$$

In the CP symmetric case all parameters m_{ij} and λ_k are real [38]. One may choose to work with a Z_2 symmetric potential in which case the terms that break the Z_2 symmetry (i.e.: $(\phi_1^\dagger \phi_1)(\phi_1^\dagger \phi_2)$, and $(\phi_2^\dagger \phi_2)(\phi_2^\dagger \phi_1)$, and $m_{12}^2 \phi_1^\dagger \phi_2$, together with their hermitian conjugates) should be dropped [37, 38].

The Yukawa sector of the GNM is given by the Lagrangian density

$$\mathcal{L}_{\text{Yuk}} = -\bar{\ell}_f \phi_a Y^{(e)}{}_{af}\, e_g - \bar{\ell}_f \tilde{\phi}_a Y^{(N)}{}_{af}\, N - \bar{Q}_f \phi_a Y^{(d)}{}_{afg}\, d_g - \bar{Q}_f \tilde{\phi}_a Y^{(u)}{}_{afg}\, u_g + \text{h.c.}\,, \tag{2.12}$$

where the indices f and g run over all flavours. The conjugate Higgs doublet $\tilde{\phi}_a$ is defined as

$$\tilde{\phi}_a = i\sigma_2 \phi_a^*, \tag{2.13}$$

with σ_2 being the second Pauli matrix.

Finally, the Majorana mass term is given by

$$\mathscr{L}_{Maj} = -\frac{1}{2} m_N N^T C^{-1} N + \text{h.c.} \tag{2.14}$$

where C is the so-called charge conjugation matrix [39].

The full Lagrangian density then reads

$$\mathscr{L}_{\text{GNM}} = \mathscr{L}_G + \mathscr{L}_F + \mathscr{L}_H + \mathscr{L}_{Yuk} + \mathscr{L}_{Maj}\,, \tag{2.15}$$

2.2 The Seesaw Mechanism and Neutrino Oscillations

In the SM neutrinos are massless. However, we know from neutrino oscillation experiments that neutrinos are massive. To be precise, we know that at least two of the three SM neutrinos are massive, since from present-day observations we can only determine the differences between masses squared [40]. Furthermore, we know that the masses of the neutrinos are much smaller than other mass scales in the SM. This presents two problems. Firstly, a mechanism by which neutrinos obtain masses needs to be determined. Secondly, it would be nice to explain the smallness of neutrino masses so as to preserve the so-called naturalness of the theory.

One possible solution is given by the type-I seesaw [41]. In this approach sterile right-handed

neutrinos are introduces, with masses far above the electroweak symmetry breaking (EWSB) scale. These sterile neutrinos are singlets under the gauge group of the theory, thus permitting Majorana mass terms. Allowing them to participate in Yukawa interactions with left-handed lepton doublets leads to the apparance of Majorana mass terms following EWSB. The name *seesaw* is due to the fact that in this construction the masses of the left-handed neutrinos are suppressed by the masses of the right-handed neutrinos: the larger the masses of the right-handed neutrinos, the smaller the masses of the left-handed neutrinos. As such, the smallness of neutrino masses is also explained.

One of the drawbacks of seesaw-type models is the large number of free parameters introduced. In the GNM [17] only one right-handed sterile neutrino is introduced, thus limiting the number of free parameters to the most economic case. In this model only one neutrino obtains a mass at tree-level through the seesaw mechanism. However, at one-loop level a second neutrino obtains a mass through interactions with the second scalar doublet. As such, the GNM provides a viable option for explaining neutrino masses in accordance with experimental data. The wide appeal of the 2HDM-like [37] models, also makes the GNM an attractive model to study.

We now take a more technical view at the seesaw mechanism present in the GNM. In the most general case, following EWSB the two scalar fields will aquire VEVs. However, we may rotate the two scalar doublets into the so-called Higgs basis [38], in which only the first scalar doublet aquires a non-vanishing VEV. We may therefore parametrize the two scalar doublets as

$$\phi_1 = \begin{pmatrix} \chi_W^+ \\ \frac{1}{\sqrt{2}}\left(v + H_1 + i\chi_Z\right) \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}\left(H_2 + iA\right) \end{pmatrix} \tag{2.16}$$

where ϕ_1 is known as the SM-like Higgs doublet. In Eq. (2.16) v is the VEV, χ_W^+ is the W Goldstone boson, χ_Z is the Z Goldstone boson; A and H^+ are the so-called pseudoscalar Higgs boson and charged Higgs boson, respectivelly; H_1 is the SM-like Higgs boson, and H_2 is the so-called heavy Higgs boson.

In the parametrization (2.16) and in the neutrino mass eigenbasis the Lagrangian (2.15) leads to the following Majorana Mass terms [18]

$$\mathscr{L} \ni -\frac{1}{2} m_3 \nu_3' \nu_3' - \frac{1}{2} m_4 \nu_4' \nu_4' \tag{2.17}$$

where ν_i' are the neutrinos in the mass eigenbasis. The following seesaw constrains apply

$$\sum_i |Y_i^{(1)}| = \frac{2 m_3 m_4}{v^2}, \qquad M = m_4 - m_3 \ . \tag{2.18}$$

Where m_3 and m_4 the neutrino masses in the mass eigenbasis.

3 Two-loop Renormalization Group evolution of Grimus-Neufeld model's Majorana mass parameter

3.1 Automation tools: PyR@TE 3

The automation tool we use for this section's analysis is PyR@TE 3 [14–16]. It is a Python tool for the computing and solving of renormalization group equations for renormalizable quantum field theories. Like other similar automation tools (e.g. SARAH [13] or RGBeta [12]) it is based on the general solutions for β-functions discussed in Sec. 1.4. The tool provides an automated way of deriving the β-functions for dimensionless and dimensionful couplings, the 2-point anomalous dimensions of scalars and fermions, and the β-functions for the VEVs. Additionally, it generates solvers for the RG evolution of the couplings constants. It has been validated against its previous versions and against results found in peer reviewed literature.

The latest iteration of the tool is PyR@TE 3 [14]. In addition to technical enhancements for improved performance, the most significant new feature of PyR@TE 3 is the updated mathematical core of the tool that is now based on the relatively recent results by Poole and Thomsen [11]. The newly adapted formalism allows for the automated computation of 3-loop β-functions for gauge couplings, while also taking into account kinetic mixing [6].

The documentation for the tool is given in Ref. [14]. For notes on the technical implementation and the architecture of the tool see also [25].

Remark 3.1 *We note that other automation tools are available. The most notable is the Mathematica tool SARAH [13], which in addition to β-functions calculation engine, also includes a very powerful array of functionalities for building and analyzing supersymmetric and non-supersymmetric models. We also make note of RGBeta [12] – a dedicated Mathematica tool for extracting β-functions for a wide class of models. For the purpose of this thesis we chose to work with $PyR@TE$, because its single-purpose nature means it is easily applicable and debuggable. The author also finds the Python code more transparent, and if needed, more easy to modify.*

3.1.1 PyR@TE 3 setup

The latest version of the tool can be downloaded from PyR@TE 3's GitHub repository:

```
https://github.com/LSartore/pyrate
```

The dependencies of the tool are the following. It requires Python v3.6 or higher to be installed on the user's machine. The following Python modules need to be installed as well (lowest required versions are listed): PyYAML v5.3, Sympy v1.5, h5py v2.10, Numpy v1.18, Scipy v1.4 and Matplotlib v3.1. No installation is required: the source code of the tool may simply be placed in a directory of the user's choice, and the functions of tool can be called directly.

3.1.2 Running PyR@TE 3

The starting point of any computations performed with PyR@TE 3 is the *model* file[6]. It should be populated by the user with the nescessary information about the model. In the model file the gauge group of the model needs to be provided in terms of gauge factors. The model file needs to contain the matter field content of the theory. Three kinds of matter fields are allowed: fermions, real scalar, and complex scalar. Fields are defined per species: for each species the number of gerations needs to be defined; additionally the dimension of the representation of each gauge factor under which the field transforms needs to be provided (for Abelian gauge factors this is replaced by hypercharges). The interaction terms are defined as explicit expressions with contracted indices. Yukawa, scalar quartic, scalar trilinear, scalar mass and fermion mass terms are suported.

Vector bosons, and the kinetic terms together with the covariant derivatives are defined automatically from the definitions of the gauge group and the matter field content. Additionally, numerous definitions and substitutions can be defined and different assumptions can be declared in the model file to aid the expression of the various interaction terms. Finally, we note that PyR@TE 3 supports the derivation β-functions for VEVs. For this the user needs to identify the real scalar component that develops a non-zero VEV and the R_ξ gauge parameter ξ as the VEVs break the gauge symmetry. A PyR@TE 3 model file constructed for the GNM is provided in Appendix B. For a more detailed exposition of the model file construction we refer to Ref. [14].

We now describe the process of running the tool with the default settings, for a more detailed exposition see Ref. [14]. Text in square brackets e.g., [TEXT], refers to model or machine specific inputs or arguments. Once the model file is constructed, the tool can be run either from the console or, more conveniently, from an IPython or Jupyter notebook. In the latter case, to run the tool one should first navigate to the folder in which PyR@TE 3 is located

```
%cd [PATH TO THE PyR@TE FOLDER]
```

and then run the code in the pyR@TE.py file with the command

```
%run pyR@TE.py -m [PATH TO MODEL FILE] -res [PATH TO THE OUTPUT FOLDER]
```

In addition, optional arguments can be passed that modify the behaviour of the tool, see Ref. [14]. Outputs are written to the provided output folder. The two most significant outputs are the following. A LaTeX file is generated with the summary of the model and the computed β-functions. Second, a set of Python files is generated: one contains the definitions of the β-functions for use in subsequent calculations; one contains a solver of the β-functions; and one exemplary Python for running the solver.

One can run the solver from within an IPhyton or Jupyter notebook. To do so, one needs to navigate to the PyR@TE 3 folder, add the output directory to Python's module list, and import the RGSolver:

[6]The model file is essentially a YAML file. YAML is data serialization format that is designed to be human-readable. The scope of its application is similar to that of XML or JSON. An easily interpretable model file template is provided within PyR@TE 3's source code, therefore nuanced understanding of YAML is not needed.

```
% cd [PATH TO PyR@TE FOLDER]
sys.path.append('[PATH TO THE OUTPUT FOLDER]/PythonOuput')
from [MODEL NAME] import RGSolver
```

Then one needs to instantiate the `RGSolver` class by providing it with a name, an initial scale at which initial values are defined, and start- and end-points for the running of the couplings

```
rge = RGEsolver('[NAME]', tmin = [STARTING SCALE],
    tmax = [END SCALE], initialScale = [INITIAL SCALE])
```

All initial values for the couplings then need to be set as

```
rge.[NAME OF THE COUPLING].initialValue
    = [VALUE OF THE COUPLING AT INITIAL SCALE]
```

The system of β-functions can then be solved by calling the solve method

```
rge.solve()
```

The solutions are given in the dictionary `rge.solutions`: the keys of this dictionary correspond to names of couplings, and values correspond to arrays of solutions for the particular couplings. The RG times corresponding to these solutions are given in the list `rge.tList`.

Remark 3.2 *We note that since the `Python` files that are part of `PyR@TE 3` are auto-generated user review and corrections might be needed. Furthermore, additional modifications or overall rewrites are, in many cases, needed for adapting the output to one's particular needs.*

3.2 Running Grimus-Neufeld model parameters

We now turn our attention to the running of the GNM parameters. The full set of β-functions of a given model, in our case GNM, constitute a system of differential equations. Therefore, to obtain the RG evolution of the model's parameters one needs to solve for all couplings simultaneously. To this end, we utilize `PyR@TE 3` [14] to obtain the full set of two-loop order β-functions for the GNM parameters. The numerical solutions for their running are obtained with the `PyR@TE 3` module `RGSolver` as described in Subsubsec. 3.1.2.

Since neither the second Higgs, nor the heavy sterile neutrino have been observed experimentally, we cannot take experimentally measured values as initial conditions for our numerical calculations. We therefore run the analysis in a benchmark setup: for SM-like particles we take experimentally available values, while for the as-of-yet unknown parameters we take benchmark values. The Yukawa matrix ${Y^{(N)}}^1$ that couples the SM Higgs doublet, lepton doublets and the sterile neutrino can be obtained from the PMNS matrix [42] through the Grimus-Neufeld construction [23].

For the initial scale, at which all initial parameter values for our analysis are defined, we take the mass of the Z boson $m_Z = 91.1876$ GeV. The full set of initial values we used is summarized in Table 2[7]. When taking parameters of the 2HDM scalar field potential (2.11) we make the following

[7]For the sake of reproducibility the initial values for the couplings in Fig. 5 are given in the exact precision that was used for the numerical analysis.

Table. 2 Initial values used for the coupling constant running defined at $\mu_0 = m_Z = 91.1876$ GeV.

Parameter	Initial value
$\{g_1, g_2, g_3\}$	$\{0.459867, 0.662165, 1.21565\}$
${Y^{(u)}}^1$	$\text{diag}(0.00000729450804767608 84, 0.0035535470985157 35, 0.9838970303676488)$
${Y^{(d)}}^1$	$\begin{pmatrix} 0.00001622932015185 2735 & 0.00007109701641623267 & -0.00005785230857860736 \\ -0.00000374328670587796 04 & 0.00030754968987529264 & 0.00068187471386640 59 \\ 0.00000021051032579315994 & -0.00001239642636846 7326 & 0.0165852015846811 \end{pmatrix}$
${Y^{(N)}}^1$,${Y^{(N)}}^2$	$\begin{pmatrix} 0.0000000000250218 \\ -0.0000000000276767 \\ 0.0000000000332853 \end{pmatrix}, \begin{pmatrix} 0.00000208808 \\ 0.00000062674 \\ 0.0000139731 \end{pmatrix}$
${Y^{(e)}}^1$	$\text{diag}(0.00000279471557539980 67, 0.00058998210837889 17, 0.01002989112848338)$
${Y^{(u)}}^2 = {Y^{(d)}}^2 = {Y^{(e)}}^2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$	$\{1.09731, 1.07132, 5.29197, -3.08684, -0.87161, 0, 0\}$
$\{m_{11}^2, m_{22}^2, m_{12}^2\}$	$\{-77843.9 \text{ GeV}, 1749.52 \text{ GeV}, -1241.41 \text{ GeV}\}$
m_N	10 GeV
VEV	246 GeV

assumptions: we assume a CP conserving potential, so that m_{ij}^2 and λ_k are real; we also assume an almost Z_2 symmetric potential, with a non-zero m_{12}^2.

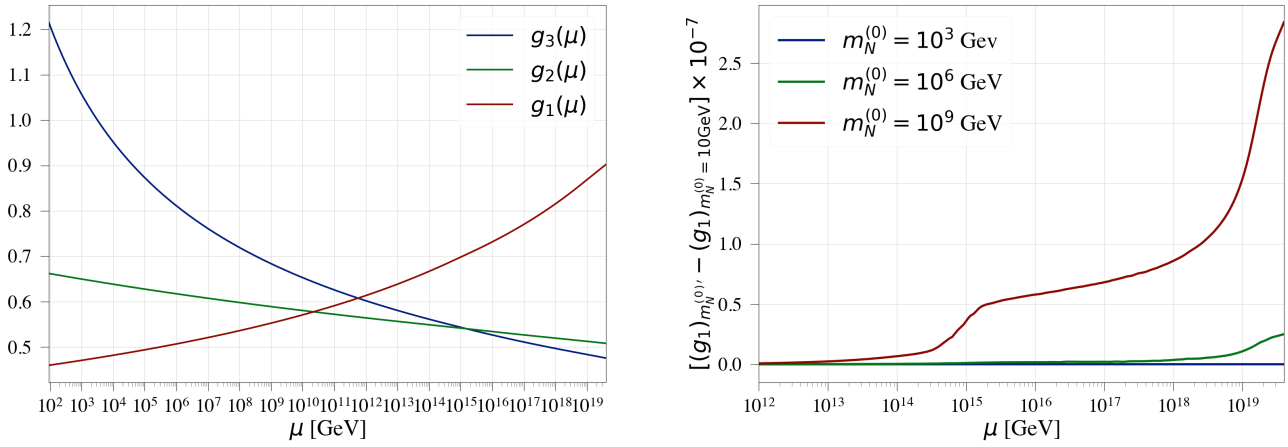
Before proceeding to the analysis of the Majorana mass parameter's m_N dependence on the energy scale μ, we first investigate our implementation of GNM in PyR@TE 3. To this end we check the energy scale dependence of parameters that GNM formally shares with other models, namely SM and 2HDM [37]. In Subsec. 3.2.1 we check the energy scale dependence of the gauge coupling parameters $\{g_1,\ g_2,\ g_3\}$, mass parameters $\{m_{11}^2,\ m_{22}^2\}$ of the scalar field potential, and charged lepton mass parameters $\{m_e, m_\mu, m_\tau\}$; the sensitivity of their RG evolution to initial Majorana mass parameter's value is also checked. Then, in Subsec. 3.2.2, we examine the running of the Majorana mass parameter.

3.2.1 Running of the gauge coupling, fermion mass and scalar mass parameters

We describe the check on gauge coupling parameters of our implementation of GNM in `PyR@TE 3`. The running of gauge coupling constants $\{g_1, g_2, g_3\}$ is presented in Fig. 2a. The results meet our expectations. As seen from Fig. 2b, the running of gauge couplings is dependent on the Majorana mass parameter m_N, however this dependence is extremely small: the difference between $(g_1)(\mu)$ with $m_N^{(0)} = 10^9$ and $m_N^{(0)} = 10^9$ is on the order of 10^{-7}.

The running of the charged lepton mass parameters $\{m_e, m_\mu, m_\tau\}$ is presented in Fig. 3. We note here, that these are the charged lepton mass parameters appearing after spontaneous symmetry breaking, and for charged leptons in their mass eigenbasis. The charged lepton mass parameters are relatively stable with the change of energy scale, and exhibit minor monotonic decrease. These results qualitatively match our expectations, and further point to the validity of our implementation of GNM in `PyR@TE 3`.

Finally, the running of scalar mass parameters $\{m_{11}^2,\ m_{22}^2\}$ is presented in Figs. 4a and 4b. Here we observe more interesting RG evolution and dependence on $m^{(0)}$. We note that the initial value



(a) (b)

Fig. 2 RG evolution of gauge coupling parameters for initial values given in Table 2. **(a)** Gauge coupling parameters' g_1, g_2 and g_3 dependence on the energy scale μ **(b)** The difference between $(g_1)_{m_N^{(0)\prime}}$ and $(g_1)_{m_N^{(0)}=10GeV}$, where $(g_1)_{m_N^{(0)\prime}}$ is the gauge coupling parameter's dependence on μ expressed as a function of the initial value of the Majorana mass parameter $m_N^{(0)\prime}$, $(g_1)_{m_N^{(0)}=10GeV}$ is the gauge coupling parameter's dependence on the energy scale μ for a fixed initial value of the Majorana mass parameter $m_N^{(0)} = 10\,GeV$. We can see from this plot that the RG evolution of g_1 only minimally depends on the initial value of m_N.

of m_{22}^2 is positive (see Table 2), and it changes monotonically with the energy scale. To clearly see seperation in the running of m_{22}^2 for different values of $m_N^{(0)}$ we present the plots on a double-log scale, this forces us to take the absolute value of m_{22}^2. As an artifact of this set up we see *kinks* around 10^2 GeV scale in Fig. 4b. These should be interpreted as point at which the sign of m_{22}^2. Note that the change of a parameter's sign may leads to interesting physical consequences, while this is outside the scope of this thesis it is something that suggests future investigation. The starting value of m_{11}^2 is negative and it decreases monotonically. We also make note that counterintuitively going from $m_N^0 = 10^8$ GeV to $m_N^0 = 10^{10}$ GeV has a much greater impact on the runing of m_{11}^2 and m_{22}^2 than $m_N^0 = 10$ GeV to $m_N^0 = 10^8$. This also invites future investigations.

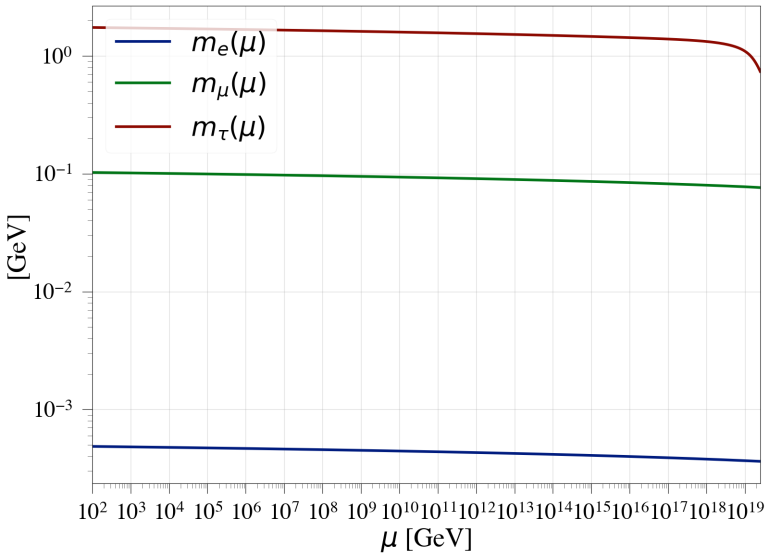
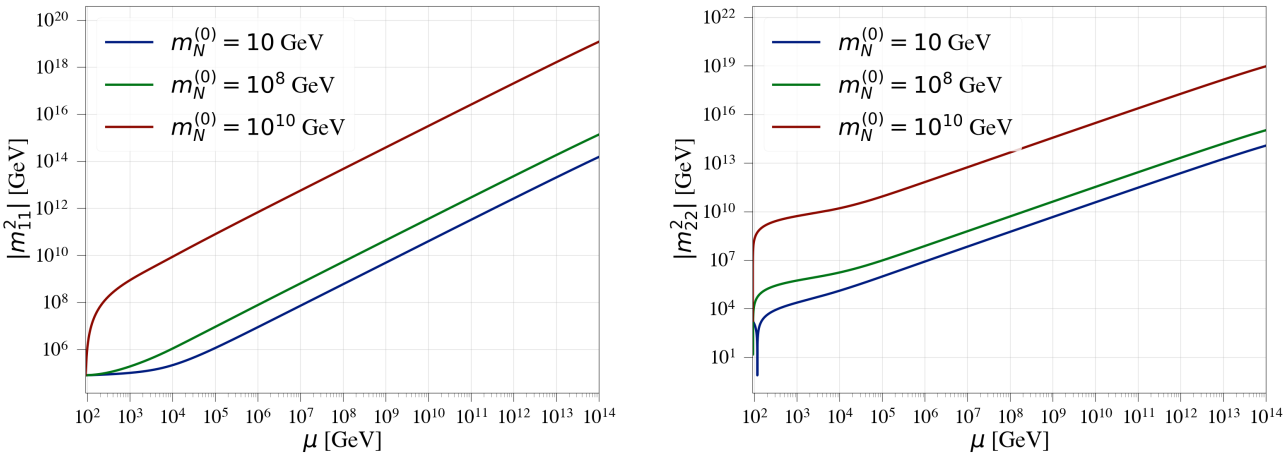


Fig. 3 RG evolution of the charged lepton mass parameter. Initial values for the RG evolution are given in Table 2.



(a) (b)

Fig. 4 RG evolution of mass parameters of the scalar field potential (2.11) for different initial values of the Majorana mass parameter $m_N^{(0)}$. Other initial values given in Table 2. **(a)** Mass parameter's m_{11}^2 dependence on the energy scale μ for different $m_N^{(0)}$. **(b)** Mass parameter's m_{22}^2 dependence on the energy scale μ for different $m_N^{(0)}$

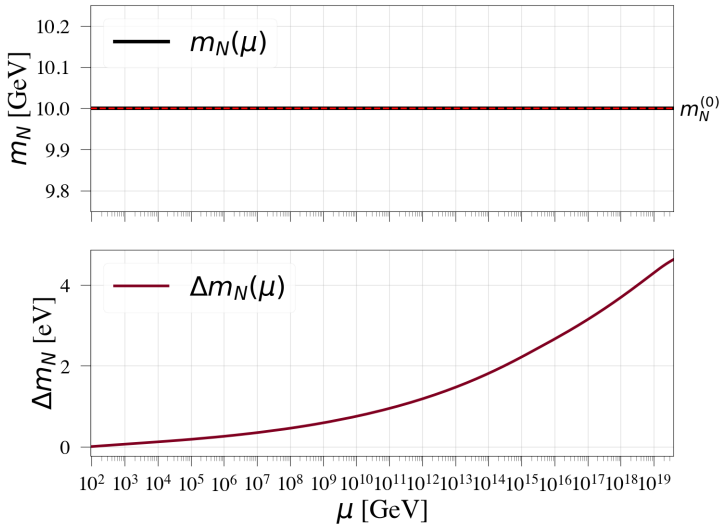


Fig. 5 RG evolution of the the Majorana mass parameter m_N. Initial values for the RG evolution are given Table 2. **(upper figure)** Majorana mass parameter's dependence on the energy scale μ; initial value for the Majorana mass parameter $m_N^{(0)}$ is indicated with a dashed red line. **(lower figure)** deviation of m_N from its initial value with RG evolution; here $\Delta m_N(\mu) = m_n(\mu) - m_N^{(0)}$.

3.2.2 Running of the Majorana mass parameter

We now turn to the analysis of the sterile neutrino's mass parameter's m_N dependence on the energy scale. Before discussing numerical results we present the β-functions for m_N. Up to two-loop order the β-function for m_N takes the following form

$$\beta_{m_N} = \frac{dm_N}{dt} = \frac{1}{2}\left(\frac{1}{(4\pi)^2}\beta^{(1)}_{m_N} + \frac{1}{(4\pi)^4}\beta^{(2)}_{m_N}\right) \quad , \tag{3.1}$$

where, in Eq. (3.1), we define the RG time as $t = \ln \mu$, and the one- and two-loop order contribution to β_{m_N} are denoted as $\beta^{(1)}_{m_N}$ and $\beta^{(2)}_{m_N}$ accordingly. The expressions for $\beta^{(1)}_{m_N}$ is

$$\beta^{(1)}_{m_N} = m_N \sum_{a=1}^{2} \left(Y^{(N)^{a\dagger}} Y^{(N)^a} + Y^{(N)^{a\mathsf{T}}} Y^{(N)^{a*}} \right) . \tag{3.2}$$

We now turn our attention to the numerical solutions for $m_N(\mu)$. The RG evolution of m_N is presented in Fig. 5. We observe that with increasing energy scale μ the Majorana mass parameter m_N deviates from its initial value $m^{(0)}$ by only a few eV. This behaviour can be understood by examining the one-loop order contribution to the Majorana mass parameter's β-function. One can see from Eq. (3.2) that only Yukawa matrices $Y^{(N)^a}$ and m_N affect the RG evolution of m_N. It is expected that the Yukawa couplings for the sterile neurino N are relatively small. Therefore, β_{m_N} gets suppressed, and the value of m_N, even at extremely high energy scales, remains stable. We note that while for high initial values of m_N the deviation from the initial value at extremely high energies is proportionally larger it remains relatively suppressed.

Summary and conclusions

- In this thesis we have adapted an automated tool for symbolic and numerical computation of RGEs, i.e., `PyR@TE 3` [14], for the GNM, and showed that it describes the running of GNM parameters.
- We showed that the two-loop renormalized Majorana mass changes minimally with increasing scale (the change is on the order of a few eV at energy scale of 10^{19} GeV as compared to the initial value, for our benchmark scenario summarized in Table. 2). This is in line with the analytical consideration of the one-loop order β-function for the Majorana mass: the RG evolution of the Majorana mass is suppressed by the Yukawa matrices coupling the sterile neutrino with other particles.

A Calculation of the gauge dependent part of the fermion self energy

We begin evaluating the gauge dependent term, by applying to Eq. (1.33) Feynman parametrization, with $n = 2$, $A = (p-\ell)^2 - m^2$, and $B = \ell^2$, (1.21)

$$i\Sigma''(p) = -2(1-\xi)i^2 C(F)\mu^{4-d}\int_0^1 dz \int \frac{d^d\ell}{(2\pi)^d}\gamma_\mu \frac{(1-z)(\not{p}-\not{\ell}+m)}{[(p-\ell)^2 z - m^2 z + \ell^2(1-z)]^3}\gamma_\nu \cdot \ell^\mu \ell^\nu \tag{A.1}$$

The ℓ term drops out from the enumerator of the fraction under the integral sign in Eq. (A.2) due to integration of an odd function over its full domain. After applying the change of variables $\ell' = \ell + pz$, we obtain

$$\begin{aligned} i\Sigma''(p) = &-2(1-\xi)i^2 C(F)\mu^{4-d}\int_0^1 dz \int \frac{d^d\ell'}{(2\pi)^d}\Bigg[\gamma_\mu \frac{(1-z^2)\not{p} - (1-z)\not{\ell}' + (1-z)m)}{[\ell'^2 - m^2 z + p^2 z(1-z)]^3}\gamma_\nu \\ &\times (\ell'^\mu \ell'^\nu + \ell'^\nu p^\mu z + \ell'^\nu p^\mu z + p^\mu p^\nu z^2)\Bigg] \end{aligned} \tag{A.2}$$

To keep later expressions managable we split this integral into two parts

$$i\Sigma''(p) = i\Sigma''_{[1]}(p) + i\Sigma''_{[1]}(p)\,, \tag{A.3}$$

where

$$\begin{aligned} i\Sigma''_{[1]}(p) = &-2(1-\xi)i^2 C(F)\mu^{4-d}\int_0^1 dz \int \frac{d^d\ell'}{(2\pi)^d}\Bigg[\gamma_\mu \frac{(1-z^2)\not{p} + (1-z)m}{[\ell'^2 - m^2 z + p^2 z(1-z)]^3}\gamma_\nu \\ &\times (\ell'^\mu \ell'^\nu + \ell'^\nu p^\mu z + \ell'^\mu p^\nu z + p^\mu p^\nu z^2)\Bigg] \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} i\Sigma''_{[2]}(p) = &-2(1-\xi)i^2 C(F)\mu^{4-d}\int_0^1 dz \int \frac{d^d\ell'}{(2\pi)^d}\Bigg[\gamma_\mu \frac{-(\not{\ell}')}{[\ell'^2 - m^2 z + p^2 z(1-z)]^3}\gamma_\nu \\ &\times (\ell'^\mu \ell'^\nu + \ell'^\nu p^\mu z + \ell'^\mu p^\nu z + p^\mu p^\nu z^2)\Bigg] \end{aligned} \tag{A.5}$$

We first calculate $i\Sigma''_{[1]}(p)$.

Calculation of $i\Sigma''_{[1]}(p)$ In Eq. (A.4), terms with factors of $\ell'^\nu p^\mu z$ and $\ell'^\mu p^\nu z$ in the enumerator will be odd functions of ℓ' and so will vanish as we integrate over the full domain of ℓ'. Apply the following identity

$$\int d^d\ell'\; f(\ell^2)\; \ell'^\mu \ell'^\nu = \frac{1}{d}\int d^d\ell'\; f(\ell'^2)\; g^{\mu\nu}\ell'^2 \tag{A.6}$$

to Eq. (A.2) yields

$$i\Sigma''_{[1]}(p) = -2(1-\xi)i^2C(F)\mu^{4-d}\int_0^1 dz\int\frac{d^d\ell'}{(2\pi)^d}\left[\frac{d^{-1}\gamma_\mu[(1-z^2)\not{p}+(1-z)m]\gamma_\nu g^{\mu\nu}\ell^2}{[\ell'^2-m^2z+p^2z(1-z)]^3}\right.$$
$$\left.+\frac{\gamma_\mu[(1-z^2)\not{p}+(1-z)m]\gamma_\nu p^\mu p^\nu z^2}{[\ell'^2-m^2z+p^2z(1-z)]^3}\right] \tag{A.7}$$

By utilizing the identities Eq. (1.38) and noting that $\gamma_\mu\gamma_\nu p^\mu p^\nu = \not{p}\not{p} = p^2$ one can rewrite (A.7) to obtain

$$i\Sigma''_{[1]}(p) = I_1 + I_2 \tag{A.8}$$

where

$$I_1 = -2(1-\xi)i^2C(F)\mu^{4-d}\int_0^1 dz\int\frac{d^d\ell'}{(2\pi)^d}\frac{d^{-1}[(1-z^2)(2-d)\not{p}-d(1-z)m]\ell^2}{[\ell'^2-m^2z+p^2z(1-z)]^3} \tag{A.9}$$

$$I_2 = -2(1-\xi)i^2C(F)\mu^{4-d}\int_0^1 dz\int\frac{d^d\ell'}{(2\pi)^d}\frac{[(1-z^2)\not{p}+(1-z)m]p^2z^2}{[\ell'^2-m^2z+p^2z(1-z)]^3} \tag{A.10}$$

We will tackle integrals I_1 and I_2 seperatelly. We start with I_1. Applying Eq. (1.23) to Eq. (A.9) yields

$$I_1 = -i^2C(F)\mu^{4-d}\frac{i}{2}\frac{(1-\xi)}{(4\pi)^{\frac{d}{2}}}\Gamma\left(\frac{4-d}{2}\right)\int_0^1 dz[(1-z^2)(2-d)\not{p}+d(1-z)m] \tag{A.11}$$
$$\times\left[m^2z-p^2z(1-z)\right]^{\frac{d}{2}-2} \tag{A.12}$$

Introducing the parameter $\epsilon = 2-\frac{d}{2}$ into Eq. (A.11) yields

$$I_1 = -\frac{i}{2}i^2C(F)\frac{(1-\xi)}{(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma_E\right)\int_0^1 dz\left[[4(1-z)m+2\varepsilon((1-z^2)\not{p}-(1-z)m)-2(1-z^2)\not{p}]\right.$$
$$\left.\times\left[1-\epsilon\ln\frac{m^2z-p^2z(1-z)}{4\pi\mu^2}\right]\right] \tag{A.13}$$

Integrating Eq. (A.13) over z yields

$$I_1 = -\frac{i}{2}\frac{1}{\eta}\frac{C(F)}{16\pi^2}(1-\xi)\left(\frac{4}{3}\not{p}-2m\right)+\frac{i}{2}\frac{C(F)}{16\pi^2}(1-\xi)\left(\frac{4}{3}\not{p}-m\right)$$
$$-\frac{i}{2}\frac{C(F)}{16\pi^2}(1-\xi)I'+O(\varepsilon) \tag{A.14}$$

where

$$I' = \int_0^1 dz[4(1-z)m + 2\varepsilon((1-z^2)\not{p} - (1-z)m) - 2(1-z^2)\not{p}] \ln \frac{m^2 z - p^2 z(1-z)}{\mu^2} \tag{A.15}$$

We now tackle I_2 (A.10). Applying Eq. (1.24) to Eq. (A.10) yields

$$I_2 = \frac{i(1-\xi)}{(4\pi)^{\frac{d}{2}}} \Gamma\left(\frac{6-d}{2}\right) i^2 C(F) \mu^{4-d} \int_0^1 dz[(1-z^2)\not{p} + (1-z)m] p^2 z^2 \left[m^2 z - p^2 z(1-z)\right]^{-(3-\frac{d}{2})} \tag{A.16}$$

Introducing the parameter $\epsilon = 2 - \frac{d}{2}$ into Eq. (A.16) yields

$$\begin{aligned} I_2 =& \frac{i(1-\xi)}{(4\pi)^{\frac{d}{2}}} \Gamma\left(\frac{6-d}{2}\right) i^2 C(F) \mu^{4-d} \int_0^1 dz[(1-z)^2\not{p} + (1-z)m] p^2 z^2 \left[m^2 z - p^2 z(1-z)\right]^{-\epsilon-1} \\ =& \frac{i(1-\xi)}{(4\pi)^2} \Gamma(1+\epsilon)\, i^2 C(F) \\ & \times \int_0^1 dz \frac{[(1-z)^2\not{p} + (1-z)m] p^2 z^2}{m^2 z - p^2 z(1-z)} \left(1 - \epsilon \ln \frac{m^2 z - p^2 z(1-z)}{4\pi\mu^2}\right) \end{aligned} \tag{A.17}$$

$\Gamma(1+\xi)$ can be approximated as $\Gamma(\epsilon+1) = \epsilon\Gamma(\epsilon) = 1 + \epsilon\gamma_E$. This yields

$$I_2 = -i(1-\xi)\frac{C(F)}{16\pi^2} I'' + O(\varepsilon) \tag{A.18}$$

where

$$I'' = \int_0^1 dz \frac{[(1-z)^2\not{p} + (1-z)m] p^2 z^2}{m^2 z - p^2 z(1-z)} \tag{A.19}$$

From Eq. (A.10) we see that I_2 will be finite in the limit $\epsilon \to 0$, and will not contribute to renormalization constants. $i\Sigma''_{[1]}(\not{p})$ then reads

$$\begin{aligned} i\Sigma''_{[1]}(\not{p}) =& -\frac{i}{2}\frac{1}{\eta}\frac{C(F)}{16\pi^2}(1-\xi)\left(\frac{4}{3}\not{p} - 2m\right) + \frac{i}{2}\frac{C(F)}{16\pi^2}(1-\xi)\left(\frac{4}{3}\not{p} - m\right) \\ & - \frac{i}{2}\frac{1}{\eta}\frac{C(F)}{16\pi^2}(1-\xi)(I' + 2I'') + O(\epsilon)\,. \end{aligned} \tag{A.20}$$

Calculation of $i\Sigma''_{[2]}(p)$ We now turn our attention to $\Sigma''_{[2]}(p)$ (A.5). By dropping off terms corresponding to odd functions under the integral sign we obtain the following:

$$i\Sigma''_{[2]}(p) = 2(1-\xi) i^2 C(F) \mu^{4-d} \int_0^1 dz \int \frac{d^d \ell'}{(2\pi)^d} \left[\gamma_\mu \frac{(1-z)(\not{\ell}')}{[\ell'^2 - m^2 z + p^2 z(1-z)]^3} \gamma_\nu \times (\ell'^\nu p^\mu z + \ell'^\mu p^\nu z)\right], \tag{A.21}$$

In Eq. (A.21) Contracting ℓ'^{μ}'s and p^{μ}'s with γ_{μ}'s we get:

$$i\Sigma''_{[2]}(p) = 2(1-\xi)i^2C(F)\mu^{4-d}\int_0^1 dz \int \frac{d^d\ell'}{(2\pi)^d}\left[\frac{(z-z^2)\ell'^2\not{p}}{[\ell'^2 - m^2z + p^2z(1-z)]^3}\right], \tag{A.22}$$

By applying identity Eq. (1.23) to Eq. (A.22) one obtain

$$i\Sigma''_{[2]}(p) = \frac{d}{2}\frac{i}{(4\pi)^{\frac{d}{2}}}\Gamma\left(\frac{4-d}{2}\right)(1-\xi)i^2C(F)\mu^{4-d}\int_0^1 dz(z-z^2)\not{p}\left[m^2z - p^2z(1-z)\right]^{\frac{d}{2}-2}, \tag{A.23}$$

Setting $\varepsilon = 2 - \frac{d}{2}$ as before we get

$$\begin{aligned} i\Sigma''_{[2]}(p) = &-\frac{i}{(4\pi)^2}(2-\varepsilon)\left(\frac{2}{\varepsilon - (2\gamma_E - 1) + \gamma\epsilon}\right)(1-\xi)C(F)\int_0^1 dz(z-z^2)\not{p} \\ &\times\left[1 - \epsilon\ln\frac{m^2z - p^2z(1-z)}{4\pi\tilde{\mu}^2}\right] \end{aligned} \tag{A.24}$$

Evaluating Eq. (A.24) through we get

$$i\Sigma''_{[2]}(p) = -i\frac{1}{\eta}\frac{C(F)}{(4\pi)^2}(1-\xi)\left(\frac{1}{3}\not{p}\right) + i\frac{C(F)}{(4\pi)^2}(1-\xi)I''' + O(\varepsilon) \tag{A.25}$$

where

$$I''' = 2i\frac{C(F)}{(4\pi)^2}(1-\xi)\int_0^1 dz(z-z^2)\not{p}\ln\frac{m^2z - p^2z(1-z)}{\tilde{\mu}^2}$$

We obtain the full result for the gauge dependent part $i\Sigma''$ by suming Eq. (A.20) and Eq. (A.25), it reads

$$\begin{aligned} i\Sigma''(\not{p}) = &-\frac{i}{2}\frac{1}{\eta}\frac{C(F)}{16\pi^2}(1-\xi)\left(2\not{p} - 2m\right) + \frac{i}{2}\frac{C(F)}{16\pi^2}(1-\xi)\left(\frac{4}{3}\not{p} - m\right) \\ &-\frac{i}{2}\frac{C(F)}{16\pi^2}(1-\xi)(I' + 2I'' - 2I''') \end{aligned} \tag{A.26}$$

B PyR@TE 3 model file for the Grimus-Neufeld model

```
# YAML 1.1
---
Author: Povilas Rackauskas
Date: 04.04.2023
Name: GNM_VEV
Groups: {U1Y: U1, SU2L: SU2, SU3C: SU3}

Fermions: {
```

```
    QL: {Gen: 3, Qnb:{U1Y: 1/6, SU2L: 2, SU3C: 3}},
    LL: {Gen: 3, Qnb:{U1Y: -1/2, SU2L: 2, SU3C: 1}},
    uR: {Gen: 3, Qnb:{U1Y: 2/3, SU2L: 1, SU3C: 3}},
    dR: {Gen: 3, Qnb:{U1Y: -1/3, SU2L: 1, SU3C: 3}},
    eR: {Gen: 3, Qnb:{U1Y: -1, SU2L: 1, SU3C: 1}},
    NR: {Gen: 1, Qnb:{U1Y: 0, SU2L: 1, SU3C: 1}}
}

RealScalars: {
}

ComplexScalars: {
    Phi1 : {RealFields: [Pi1, Sigma1], Norm: 1/sqrt(2), Qnb:{U1Y: 1/2, SU2L: 2, SU3
    Phi2 : {RealFields: [Pi2, Sigma2], Norm: 1/sqrt(2), Qnb:{U1Y: 1/2, SU2L: 2, SU3
}


Potential: {

    Definitions: {
        Phi1tilde[i] : Eps[i,j] Phi1bar[j],
        Phi2tilde[i] : Eps[i,j] Phi2bar[j]
    },

    Yukawas:{
        Y1u   : QLbar[i,a] Phi1tilde[i] uR[a],
        Y1d   : QLbar[i,a] Phi1[i] dR[a],

        Y1e   : LLbar[i] Phi1[i] eR,
        Y1N   : LLbar[i] Phi1tilde[i] NR,

        Y2u : QLbar[i,a] Phi2tilde[i] uR[a],
        Y2d : QLbar[i,a] Phi2[i] dR[a],

        Y2e : LLbar[i] Phi2[i] eR,
        Y2N : LLbar[i] Phi2tilde[i] NR,
    },

    QuarticTerms: {
        l1 : 1/2(Phi1bar[i] Phi1[i] Phi1bar[j] Phi1[j]),
```

```
        l2 : 1/2(Phi2bar[i] Phi2[i] Phi2bar[j] Phi2[j]),
        l3 : Phi1bar[i] Phi1[i] Phi2bar[j] Phi2[j],
        l4 : Phi1bar[i] Phi2[i] Phi2bar[j] Phi1[j],

        l5 : 1/2(Phi1bar[i] Phi2[i] Phi1bar[j] Phi2[j]),
        l6 : Phi1bar[i] Phi2[i] Phi1bar[j] Phi1[j],
        l7 : Phi1bar[i] Phi2[i] Phi2bar[j] Phi2[j]
    },

    ScalarMasses: {
        m1  : Phi1bar[i] Phi1[i],
        m2  : Phi2bar[i] Phi2[i],
        m12 : -Phi1bar[i] Phi2[i]
    },

    FermionMasses: {
        mN : NR NR
    }
}

Vevs:{
    v : Pi1[2]
}

#GaugeParameter : 1


Substitutions: {
    # Rename the gauge coupling constants
    g_U1Y :  g1,
    g_SU2L : g2,
    g_SU3C : g3,

# Z2 symmetry in the scalar sector
    l6 : 0,
    l7 : 0,

}

Latex: {
```

```
QL : Q_L,
LL : L_L,
uR : u_R,
dR : d_R,
eR : e_R,
NR : N_R,

mN : m_N,


Phi1 : \phi_1,
Phi2 : \phi_2,

Phi1tilde : \tilde{\phi}_1,
Phi2tilde : \tilde{\phi}_2,

g1 : g_1,
g2 : g_2,
g3 : g_3,

Y1u : Y^{(1)}_u,
Y1d : Y^{(1)}_d,

Y1e : Y^{(1)}_e,

Y2u : Y^{(2)}_u,
Y2d : Y^{(2)}_d,

Y2e : Y^{(2)}_e,

Y1N : Y^{(1)}_N,
Y2N : Y^{(2)}_N,

l1     : \lambda_1,
l2     : \lambda_2,
l3     : \lambda_3,
l4     : \lambda_4,
l5     : \lambda_5,
l6     : \lambda_6,
l7     : \lambda_7,
```

```
    m1      : m_1^2,
    m2      : m_2^2,
    m12     : m_{12}^2,
}
```

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Renormalizacijos grupės lygtys Grimus-Neufeld modeliui

Povilas Račkauskas

Santrauka

Kvantinių laukų teorijoje dažnai tenka dirbti trigdžių teorijos kontekste, t.y., ieškomi dydžiai yra skleidžiami asimptoninėmis eilutėmis. Aukštesni šių eilučių nariai atitinka aukštesnės eilės kvantines korekcijas. Naiviai įvertinant šias korekcijas gaunami begaliniai dydžiai, taigi modelio parametrai irgi įgauna begalines reikšmes. Šios begalybės yra *sutramdomos* renormalizuojant tiriamąjį modelį: formaliai begalinio dydžio parametrai yra pakeičiami baigtiniais renormalizuotais parametrais. Šie renormalizuoti parametrai priklauso nuo energijos skalės. Jų priklausomybė nuo energijos skalės yra nagrinėjama renormalizacijos grupės kontekste.

Renormalizacijos grupės lygtys nusako kaip tiriamojo modelio parametrai priklauso nup energijos skalės. Deja renormalizacijos lygčių išvedimas konkrečiam modeliui yra itin sudėtingas uždavinys. Dėl to, jau 1970-aisiais Machacek ir Vaughn pradėjo tyrimų programą bendrųjų renormalizacijos grupės lygčių, kurias būtų galima taikyti visiems renormalizuojamiems modeliams, išvedimui [1, 3, 9]. Bendrųjų renormalizacijos grupės lygčių tyrimas vis dar yra aktyvi teorinės fizikos sritis [4–11].

Deja, bendrų lygčių konkretinimas tiriamajam modeliui nėra lengvas uždavinys, todėl dažnai yra taikomi automatizuoti įrankiai [12–16]. Šiame darbe Grimus-Neufeld modelio [17] tyrimui mes pritaikėme `PyR@TE 3` [14–16], kuris yra automatizuotas įrankis simboliniam ir skaitmeniniam renormalizacijos lygčių sprendimui.

Grimus-Neufeld modelis [17] yra minimalus standartinio modelio išplėtimas steriliu neutrinu ir antru Higso dubletu. Šis modelis yra neutrinų masių paaiškinimo kandidatas, dėl to per pastaruosius metus jis susilaukė daug dėmesio [18–23].

Šio darbo tikslas buvo ištirti ir pristatyti renormalizacijos grupės lygtis Grimus-Neufeld modeliui. Šį tikslą pasiekiau. Darbo rezultatai ir išvados yra

- Sėkmingai pritaikiau automatizuotą įrankį simboliniam ir skaitmeniniam renormalizacijos lygčių sprendimui `PyR@TE 3` [14–16] Grimus-Neufeld modelio analizei.
- Taikant `PyR@TE 3` [14–16] nustačiau, kad Grimus-Neufeld modelio Majorana masės parametro priklausomybė nuo energijos skalės yra silpna. Silpna Majorana masės parametro priklausomybė nuo energijos skalės gali būti paaiškinta Jukava sąveikos, kuri sukabina sterilų neutriną su leptonų dubletais ir Higso dubletu silpnumu: pirmos eilės renormalizacijos grupės lygtys priklauso nuo Majorana masės parametro ir Jukava sąveikos matricų, Jukava prametrai, kurie yra itin maži, slopina Majorana masės parametro priklausomybę nuo energijos skalės.