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Rymano hipotezei ekvivalentus teiginys funkcijoms iš Selbergo klasės

Equivalent for the Riemann Hypothesis in the Selberg Class

Baigiamasis magistro darbas

Atliko: Jokūbas Putrius

VU el. p.: jokubas.putrius@mif.vu.lt

Vadovas: prof. dr. Ramūnas Garunkštis

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Introduction

We write $s = \sigma + it$ for a complex number. We denote a zero of a function $F(s)$ by $\rho = \beta + i\gamma$ and by p a prime number.

In 1734 Euler solved the Basel problem, that is he showed that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Specifically, he evaluated the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

at positive even integers. Additionally, in 1737, he showed that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1-p^{-s}},$$

thus giving an alternative proof for the infinitude of the prime numbers. The products of the type as the one on the right hand side of the above equation are called Euler products.

The series (1) and its Euler product converges when $\sigma > 1$. In 1859 Riemann found a meromorphic continuation of this series to the entire complex plane. It is now called the Riemann zeta function and is denoted by $\zeta(s)$. He also proved the following functional equation for $\zeta(s)$:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s),$$

where $\Gamma(s)$ is the gamma function (for definition and properties see [14, Chapter 1.86]).

From this functional equation it is easy to see that $\zeta(s)$ has zeros for negative even integers. These zeros are called trivial and all other zeros of $\zeta(s)$ are called non-trivial. The functional equation also implies that the non-trivial zeros of $\zeta(s)$ are symmetric across the line $\sigma = 1/2$. Remembering the Euler product, one can then easily deduce that all non-trivial zeros of $\zeta(s)$ must be in the strip $0 \leq \sigma \leq 1$.

The yet unresolved Riemann Hypothesis (RH) states that all non-trivial zeros of $\zeta(s)$ are on the line $\sigma = 1/2$. RH is one of the most sought after statements in all of contemporary mathematics due to the fact that it has implications on the distribution of prime numbers. For example, von Koch in 1901 [8] proved that the following asymptotic formula is equivalent to RH:

$$\pi(x) = \int_0^x \frac{du}{\log u} + O(\sqrt{x}\log(x)),\ x \geq 2,$$

where $\pi(x)$ is the prime counting function.

There are many other equivalent statements to the Riemann Hypothesis. A famous example would be Speiser's criterion [12] (also see [9] for quantitative result) which states that RH is equivalent to the non-vanishing of $\zeta'(s)$ in the strip $0 < \sigma < 1/2$. A comprehensive list of other equivalents can be found in [3, Chapter 5].

Recently, Gonek, Graham and Lee [10] formulated a new equivalent for RH. They proved that RH is equivalent to the following relation

$$\sum_{p\leq x} p^{-it} = \int_2^x \frac{u^{-it}}{\log u}du + O(x^{1/2}|t|^{\varepsilon}),$$

for all $\varepsilon, B > 0$ and $2 \leq x \leq |t|^B$. For further development see Banks [2]. In [5, Corollary 6] a similar equivalent was considered in the case of the Lindelöf hypothesis for the Lerch zeta-function.

The goal of this thesis is to extend the above result to a wider class of L-functions which we now present. A function $F(s)$ belongs to the Selberg class S if it has the following properties:

1. For $\sigma > 1$, $F(s)$ is an absolutely convergent Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

2. For some integer $m \geq 0$, $(s-1)^m F(s)$ is an entire function of finite order.

3. $F(s)$ satisfies a functional equation of the form

$$\Phi(s) = \omega\overline{\Phi(1-\overline{s})}$$

where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s),$$

with $Q > 0$, $\lambda_j > 0$, $\Re\mu_j \geq 0$ and $|\omega| = 1$.

4. (Ramanujan hypothesis) For every $\varepsilon > 0$, $a(n) \ll n^{\varepsilon}$.

5. (Euler product) For σ sufficiently large,

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

where $b_n = 0$ unless $n = p^k$ for $k \in \mathbb{N}$ and $b_n \ll n^{\theta}$ for some $\theta < 1/2$.

The data Q, λ_j, μ_j and ω do not determine $F(s)$ uniquely, however $d_F = 2\sum_{j=1}^{r} \lambda_j$ is an invariant called the degree of $F(s)$. Let m_F be the order of the pole of $F(s)$ at $s = 1$. We say that a function from the Selberg class satisfies RH if it does not vanish when $\sigma > 1/2$.

Selberg introduced this class of functions in [11] to study L-functions axiomatically. The defining properties were picked such that there would be no known examples of functions which belong to S but are known to not satisfy RH. It is still not known whether or not all functions in S satisfy RH. Good resources for learning about functions from S are [6], [7] and [13].

The main result of this thesis is the following theorem.

Theorem 1. *Let $F(s) \in S$. Then $F(s)$ satisfies RH if and only if*

$$\sum_{n \le x} b_n n^{-it} = m_F \int_2^x \frac{u^{-it}}{\log u} du + O(x^{1/2}|t|^{\varepsilon}), \tag{2}$$

for all $\varepsilon, B > 0$ and $2 \le x \le |t|^B$.

The proof of this theorem is given in Section 3. Section 2 is devoted to several lemmas which will be needed in the proof of the main result.

Lemmata

Let $F(s) \in S$ and denote $\Lambda_F(n) = b_n \log n$, then

$$-\frac{F'}{F}(s) = \sum_{n=1} \frac{\Lambda_F(n)}{n^s}.$$

Lemma 2. *The following equalities are equivalent*

$$\sum_{n \le x} \Lambda_F(n) n^{-it} = m_F \frac{x^{1-it}}{1-it} + O(x^{1/2}|t|^\varepsilon)$$

and

$$\sum_{n \le x} b_n n^{-it} = m_F \int_2^x \frac{u^{-it}}{\log u} du + O(x^{1/2}|t|^\varepsilon),$$

for all $\varepsilon, B > 0$ *and* $2 \le x \le |t|^B$.

Proof. Suppose the second equality holds, then, using Abel's summation formula (see [1, Theorem 4.2]) we obtain

$$\sum_{n \le x} \Lambda_F(n) n^{-it} = \log x \sum_{n \le x} b_n n^{-it} - \int_2^x \frac{\sum_{n \le u} b_n n^{-it}}{u} du =$$
$$\log x \int_2^x \frac{m_F u^{-it}}{\log u} du - \int_2^x \frac{1}{u} \int_2^u \frac{m_F w^{-it}}{\log w} dw du + O(x^{1/2}|t|^\varepsilon) =$$
$$\int_2^x m_F u^{-it} du + O(x^{1/2}|t|^\varepsilon) = m_F \frac{x^{1-it}}{1-it} + O(x^{1/2}|t|^\varepsilon).$$

Conversely,

$$\sum_{n \le x} b_n n^{-it} = \sum_{n \le x} \frac{\Lambda_F(n) n^{-it}}{\log n} = \frac{1}{\log x} \sum_{n \le x} \Lambda_F(n) n^{-it} - \int_2^x - \frac{\sum_{n \le u} \Lambda_F(n) n^{-it}}{\sqrt{u} \log^2 u} du =$$
$$\frac{m_F}{\log x} \frac{x^{1-it}}{1-it} - \int_2^x - \frac{m_F u^{1-it}}{(1-it)(\sqrt{u} \log^2 u)} du + O(x^{1/2}|t|^\varepsilon) =$$
$$m_F \int_2^x \frac{u^{-it}}{\log u} du + O(x^{1/2}|t|^\varepsilon).$$

□

The next lemma is an adaptation of Lemma 3.12 of Titchmarsh [15].

Lemma 3. *Let $T > 0$ and suppose that $x > 0$ is half an odd integer. Then,*

$$\sum_{n\leq x} \Lambda_F(n)n^{-it} = \frac{1}{2\pi i}\int_{2-iT}^{2+iT} -\frac{F'}{F}(w+it)\frac{x^w}{w}dw + O\left(\frac{x^2}{T}+1\right).$$

Proof. Suppose $n < x$, then

$$\frac{1}{2\pi i}\left(\int_{-\infty-iT}^{2-iT} + \int_{2-iT}^{2+iT} + \int_{2+iT}^{-\infty+iT}\right)\frac{(x/n)^w}{w}dw = 1.$$

Integrating by parts we obtain

$$\int_{-\infty-iT}^{2-iT}\frac{(x/n)^w}{w}dw = \frac{(x/n)^{2-iT}}{(2-iT)\log(x/n)} + \frac{1}{\log(x/n)}\int_{-\infty-iT}^{2-iT}\frac{(x/n)^w}{w^2}dw \ll$$
$$\frac{(x/n)^2}{T\log(x/n)} + \frac{(x/n)^2}{\log(x/n)}\int_{-\infty}^{\infty}\frac{dw}{w^2+T^2} \ll \frac{(x/n)^2}{T\log(x/n)},$$

and we also get the same estimate for the integral from $2+iT$ to $-\infty+iT$. Hence,

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{(x/n)^w}{w}dw = 1 + O\left(\frac{(x/n)^2}{T\log(x/n)}\right).$$

If $n > x$ we get the same expression without the term 1 by arguing similarly with $-\infty$ replaced by ∞.

Multiplying by $\Lambda_F(n)n^{-it}$ and summing we obtain

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT} -\frac{F'}{F}(w+it)\frac{x^w}{w}dw = \sum_{n\leq x}\Lambda_F(n)n^{-it} + O\left(\frac{x^2}{T}\sum_{n=1}^{\infty}\frac{|\Lambda_F(n)|}{n^2|\log(x/n)|}\right).$$

If $n < 1/2x$ or $n > 2x$ then $|\log(x/n)| > \log 2$, thus these parts of the sum in the error term are

$$\ll \sum_{n=1}^{\infty}\frac{\log n}{n^{3/2}}.$$

If $\lceil x\rceil < n \leq 2x$, let $n = \lceil x\rceil + r$, then

$$\log(n/x) \geq \log\frac{\lceil x\rceil + r}{\lceil x\rceil} \gg r/\lceil x\rceil \gg r/x.$$

Hence this part of the sum in the error term is

$$\ll \frac{x^{3/2}\log x}{x^2}\sum_{1\leq r\leq x}\frac{1}{r} \ll \frac{\log^2 x}{x^{1/2}},$$

same for the terms with $1/2x \leq n < \lceil x\rceil$. Finally,

$$\frac{\Lambda_F(\lceil x\rceil)}{\lceil x\rceil^2|\log(x/\lceil x\rceil)|} \ll \frac{\log x}{x^{3/2}|\log(1+(2x)^{-1})|} \ll \frac{\log(x)}{x^{1/2}},$$

thus the lemma follows. □

The next two lemmas are classical results in Riemann zeta function theory generalised to the Selberg class. The proofs were adapted from those found in [4].

Lemma 4. *Let $0 \le \delta < 1/4$ be such that $F(s) \in S$ has no trivial zeros with $\sigma = -1+\delta$. Then*

$$\int_{-1+\delta-iT}^{-1+\delta+iT} -\frac{F'}{F}(w)\frac{x^w}{w}dw = O\left(\frac{\log^2 T}{x^{1-\delta}}\right)$$

for any $T > 0$

Proof. Recall,

$$\frac{\Gamma'}{\Gamma}(s) = \log(s) + O(|s|^{-1}),$$

when s is away from poles of $\Gamma(s)$, and

$$\cot s = 1 + O(e^{-|\Im(s)|}).$$

Using logarithmic differentiation on the functional equation of $F(s)$ we can derive

$$\frac{F'}{F}(s) = -2\log Q + G(s) - \overline{\frac{F'}{F}(1-\overline{s})},$$

where

$$G(s) = \sum_{j=1}^{r} \lambda_j(\pi\cot(\pi(\lambda_j s + \mu_j)) - \frac{\Gamma'}{\Gamma}(1-\lambda_j s - \mu_j) - \frac{\Gamma'}{\Gamma}(\lambda_j(1-s) + \overline{\mu_j})) \ll \log t,$$

when $\sigma = -1+\delta$.

Thus,

$$\frac{F'}{F}(s) = O(\log t),$$

when $\sigma = -1+\delta$. Then,

$$\int_{-1+\delta-iT}^{-1+\delta+iT} -\frac{F'}{F}(w)\frac{x^w}{w}dw \ll x^{-1+\delta}\int_{-1+\delta-iT}^{-1+\delta+iT} \frac{\log w}{w}dw \ll \frac{\log^2(T)}{x^{1-\delta}}.$$

□

Lemma 5. *Let $0 \le \delta < 1/4$ be such that $F(s) \in S$ has no trivial zeros with $\sigma = -1+\delta$. Then*

$$\int_{-1+\delta+iT}^{2+iT} -\frac{F'}{F}(w)\frac{x^w}{w}dw = O\left(\frac{x^2\log^2 T}{T}\right)$$

for any $T > 0$ such that it is not an ordinate of a non-trivial zero of $F(s)$.

Proof. In this proof $\rho = \beta + i\gamma$ denotes a non-trivial zero of $F(s)$.

By Hadamard theory, we can write

$$s^{m_F}(1-s)^{m_F}\Phi(s) = e^{a+bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}.$$

Taking logarithmic derivatives of both sides we obtain

$$\frac{F'}{F}(s) = -\frac{m_F}{s} - \frac{m_F}{s-1} - \log Q - \sum_j \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j s + \mu_j) + b + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

Remembering the estimate $\Gamma'/\Gamma(s) = \log s + O(1/|s|)$ we obtain

$$\frac{F'}{F}(s) = \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + O(\log T),$$

for $-1+\delta \le \sigma \le 2$.

Now applying this formula at s and $2+it$ and subtracting we see that

$$\frac{F'}{F}(s) = \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) + O(\log T).$$

Now for those ρ for which $|\gamma - T| \ge 1$, we have

$$\left| \frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right| = \frac{2-\sigma}{|2+it-\rho||s-\rho|} \le \frac{3}{|\gamma - t|^2}.$$

Then, for $T > 2$,

$$\sum_{|\gamma - T|>1} \frac{1}{|\gamma - T|^2} = \sum_{k=1}^{\infty} \sum_{k<|\gamma-T|\le k+1} \frac{1}{|\gamma - T|^2} \ll \sum_{k=1}^{\infty} \frac{\log(T+k)}{k^2} \ll \log T.$$

Thus,

$$\frac{F'}{F}(s) = \sum_{|\gamma - T|<1} \frac{1}{s-\rho} + O(\log T),$$

due to $|2+iT-\rho| > 1$, when $|\gamma - T| < 1$.

There are $O(\log T)$ zeros of $F(s)$ with $|\gamma - T| < 1$, thus by moving by a bounded amount we can pick T such that $|s-\rho| \gg 1/\log T$. Then with such a choice of T we obtain

$$\int_{-1+\delta+iT}^{2+iT} -\frac{F'}{F}(w)\frac{x^w}{w}dw = O\left(\frac{x^2 \log^2 T}{T} \right).$$

Moving the line of integration by a bounded amount we may cross at most $O(\log T)$ zeros $F(s)$ (counting with multiplicities) and they will contribute residues of total size at most $O\left(x^2 \log^2 T/T\right)$. Then, noting Lemma 4, we obtain

$$\int_{-1+\delta+iT}^{2+iT} -\frac{F'}{F}(w)\frac{x^w}{w}dw = O\left(\frac{x^2 \log^2 T}{T} \right)$$

for any $T > 0$ such that it is not an ordinate of a non-trivial zero of $F(s)$. $\square$

Lemma 6. *Let $F(s) \in S$, $\varepsilon > 0$ and let $\mu_F : \mathbb{R} \to \mathbb{R}$ be such that*

$$F(\sigma + it) \ll |t|^{\mu_F(\sigma)+\varepsilon}.$$

Then

$$\mu_F(\sigma) = \begin{cases} 0, \ \ \textit{if } \sigma > 1, \\ (1/2)d_F(1-\sigma), \ \ \textit{if } 0 \le \sigma \le 1, \\ (1/2-\sigma)d_F, \ \ \textit{if } 0 < \sigma. \end{cases}$$

Proof. See [13, Theorem 6.8] □

Proof of main result

By Lemma 2, it is enough to prove that RH for $F(s)$ is equivalent to

$$\sum_{n\le x}\Lambda_F(n)n^{-it}=m_F\frac{x^{1-it}}{1-it}+O(x^{1/2}|t|^{\varepsilon}) \tag{3}$$

for all $\varepsilon,B>0$ and $2\le x\le |t|^B$.

Suppose $F(s)$ satisfies RH. Let $x\ge 5/2$ be half an odd integer and $T=|t|^C$, where $C>1$ will be chosen later. By Lemma 3 we have

$$\sum_{n\le x}\Lambda_F(n)n^{-it}=\frac{1}{2\pi i}\int_{2-iT}^{2+iT}-\frac{F'}{F}(w+it)\frac{x^w}{w}dw+O\left(\frac{x^2}{T}+1\right). \tag{4}$$

Choose $0\le\delta<1/4$ such that $F(s)$ would have no trivial zeros with $\sigma=-1+\delta$. Replacing the line of integration in (4) by one consisting of the three leftmost sides of the rectangle with vertices $2-iT$, $-1+\delta-iT$, $-1+\delta+iT$ and $2+iT$ and using Lemmas 4 and 5 we see that

$$\begin{aligned}\sum_{n\le x}\Lambda_F(n)n^{-it}&=m_F\frac{x^{1-it}}{1-it}-\sum_{|\gamma-t|<T}\frac{x^{\rho-it}}{\rho-it}\\&\quad+O\left(\frac{x^2}{T}+\frac{x^2\log^2(|t|+T)}{T}+\frac{\log^2(|t|+T)}{x^{1-\delta}}+1\right).\end{aligned}$$

Note that the sum is over the non-trivial zeros of $F(s)$. It might happen that we pass over trivial zeros of $F(s)$, however there are only finitely many of them in $\sigma>-1+\delta$ and for each of them we have $\beta\le 0$, thus they contribute a term of size $O(1)$.

By RH, using Abel's summation formula, we obtain

$$\sum_{|\gamma-t|<T}\frac{x^{\rho-it}}{\rho-it}\ll x^{1/2}\sum_{\substack{|\gamma-t|<T\\ \beta=1/2}}\frac{1}{1+|t-\gamma|}\ll x^{1/2}\log^2(|t|+T).$$

Now, suppose that $5/2\le x\le |t|^B$ and choose $C>\max(1,3B/2)$. Then $T=|t|^C\ge\max(|t|,x^{3/2})$ and we obtain

$$\sum_{n\le x}\Lambda_F(n)n^{-it}=m_F\frac{x^{1-it}}{1-it}+O(x^{1/2}|t|^{\varepsilon}). \tag{5}$$

We have assumed until now that $x \geq 5/2$ is half an odd integer. If we relax this condition and just assume that $x \geq 2$, then such x is always within $O(1)$ of half an odd integer. Changing x by this amount in (5) changes the left-hand side by no more than $O(x^{1/2}\log x)$ and the right-hand side by at most $O(|t|^{\varepsilon})$. Since $x^{1/2}\log x \ll x^{1/2}|t|^{\varepsilon}$, (5) holds for $2 \leq x \leq |t|^B$.

Next we prove that (3) implies RH for $F(s)$.

Write

$$\psi(x,t) = \sum_{n \leq x} \Lambda_F(n) n^{-it}$$

and

$$R(x,t) = \psi(x,t) - m_F \frac{x^{1-it}}{1-it}.$$

Then by our assumption

$$R(x,t) \ll x^{1/2}|t|^{\varepsilon} \tag{6}$$

for $2 \leq x \leq |t|^B$, where $\varepsilon > 0$ and B is arbitrarily large but fixed.

First we show that for all $s \neq 1$

$$\int_1^\infty \frac{R(x,t)}{x^s} dx = -\left(\frac{1}{s-1} \frac{F'}{F}(s+it-1) + \frac{m_F}{(1-it)(s+it-2)} \right). \tag{7}$$

Suppose that $\sigma > 2$. Then we see that

$$\int_1^\infty \frac{\psi(x,t)}{x^s} dx = \sum_{n=2}^\infty \frac{\Lambda_F(n)}{n^{it}} \int_n^\infty x^{-s} dx = -\frac{1}{s-1} \frac{F'}{F}(s+it-1).$$

Integrating the other term and combining we get (7) for $\sigma > 2$, the right hand side of which defines a meromorphic continuation of the left hand side which has a simple pole at $s = 1$.

Define

$$\begin{aligned} H(s) = \int_1^\infty \frac{R(x,t)}{x^s} dx = \\ -\left(\frac{(1-it)(s+it-2)F'(s+it-1) + m_F(s-1)F(s+it-1)}{(s-1)(1-it)(s+it-2)F(s+it-1)} \right). \end{aligned}$$

Assume, by way of contradiction, that $\rho_0 = \beta_0 + i\gamma_0$ is a zero of $F(s)$ with $\beta_0 > 1/2$. Let m be the multiplicity of ρ_0, and define

$$h(s) = \frac{(s+it-2)F(s+it-1)}{(s+it-\rho_0-1)^m (s+it+1)^{4d_F}}.$$

For real u, define

$$w(u) = \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} h(s) e^{us} ds$$

and consider the integral

$$\frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} h(s)H(s)e^{s\log x}ds = \int_1^\infty R(y,t)w(\log x - \log y)dy. \tag{8}$$

We move the line of integration in the integral of the left hand side to left to $\sigma = 5/4$ and pass two poles at $s = 2 - it$ and $s = \rho_0 + 1 - it$. The residue at $s = 2 - it$ is equal to 0 and the other residue is

$$-x^{\rho_0+1-it}\frac{(\rho_0-1)F^{(m)}(\rho_0)}{(m-1)!(\rho_0-it)(\rho_0+2)^{4d_F}}.$$

Using the bounds $F(1/4+iv) \ll (1+|v|^{1/2d_F})$ and $F'(1/4+iv) \ll (1+|v|^{1/2d_F})\log(2+|v|)$, the left hand side is

$$\begin{aligned}
&= x^{\rho_0+1-it}\frac{(\rho_0-1)F^{(m)}(\rho_0)}{(m-1)!(\rho_0-it)(\rho_0+2)^{4d_F}} \\
&+ O\left(x^{5/4}\int_{-\infty}^{\infty}\frac{(1+|t+v|^{1/2d_F})\log(2+|v|)}{(1+|v+t-\gamma_0|)^m(1+|v+t|)^{4d_F})}dv\right) \\
&= x^{\rho_0+1-it}\frac{(\rho_0-1)F^{(m)}(\rho_0)}{(m-1)!(\rho_0-it)(\rho_0+2)^{4d_F}} + O\left(x^{5/4}\right).
\end{aligned}$$

Next we estimate $w(u)$. If $u \le 0$ we pull the contour right to ∞. Since

$$h(s)e^{us} \ll \frac{e^{u\sigma}}{|s+it-\rho_0|^m|s+it|^{4d_F-1}}$$

for $\sigma \ge 3$, we see that $w(u) = 0$. If $u > 0$, we pull the contour left to $-5/4$. We pass a pole of $h(s)$ at $s = -1 - it$ of order $4d_F$ which contributes a residue of size $O(1)$. The integral on the new line is

$$\begin{aligned}
\int_{-5/4-i\infty}^{-5/4+i\infty} h(s)e^{us}ds &\ll \int_{-\infty}^{\infty} e^{-5/4u}\frac{(1+|v+t|)|F(-9/4+i(v+t))|}{|1+(v+t-\gamma_0)|^m|1+(v+t)|^{4d_F}}dv \\
&\ll \int_{-\infty}^{\infty} e^{-5/4u}\frac{(1+|v+t|)|(1+|v+t|^{23/8d_F})}{|1+(v+t-\gamma_0)|^m|1+(v+t)|^{4d_F}}dv \ll 1
\end{aligned}$$

Thus,

$$w(u) = \begin{cases} 0 \text{ if } u \le 0, \\ O(1) \text{ if } u > 0. \end{cases}$$

Collecting the estimates in the previous discussion and applying them to (8) we see that for ρ_0 fixed

$$x^{\beta_0+1} \ll_{\rho_0} \int_1^x |R(y,t)|dy + x^{5/4}.$$

Then, by assumption, setting $B = 2/(\beta_0 - 1/2)$ we get

$$x^{\beta_0+1} \ll_{\rho_0} x^{3/2}|t|^{\varepsilon},$$

for $2 \leq x \leq |t|^{2/(\beta_0-1/2)}$. In other words,

$$x \ll_{\rho_0} |t|^{(1+\varepsilon)/(\beta_0-1/2)}.$$

This contradiction implies that $\beta_0 = 1/2$. This completes the proof of the theorem.

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