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GRIMUS NEUFELD MODELIO RENORMALIZACIJOS SRAUTAS VIENOS KILPOS LYGMENY

Magistro studijų baigiamasis darbas

Studijų programa - teorinė fizika ir astrofizika

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Vilnius 2022

Vilnius university
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RENORMALIZATION GROUP FLOW OF THE GRIMUS-NEUFELD MODEL AT ONE LOOP LEVEL

Magister thesis

Study program - theorethical physics and astrophysics

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Vilnius 2022

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Conventions

As is usual in particle physics Minkowski metric $g_{\mu\nu}$ definition with negative signature is used throughout this thesis. Where possible Einstein summation convention is used and, unless stated otherwise, Greek indices refer to four vector Minkowski products, whereas Latin indices refer to three vector products in Euclidean space. The convention is expanded to include two upper or lower indices too, but in this case it's referring to an arbitrary sum.

Feynman slash notation is used to denote contractions with gamma matrices: $\gamma^\mu F_\mu \equiv \not{F}$.

Feynman $i\epsilon$ prescription is omitted in many Feynman propagators due to brevity as after Wick rotating the momentum integrals to Euclidean plane (at least at one loop level) all the terms proportional to ϵ can be set to 0.

Glossary

EW - Electroweak
GNM - Grimus-Neufeld model
MS ($\overline{MS}$) - (modified) minimal subtraction scheme
QFT - Quantum Field Theory
QED - Quantum Electrodynamics
RG - Renormalization Group
RGE - Renormalization Group Equation
SM - Standard Model (of particle physics)
SSB - spantaneous symmetry breaking

Introduction

Quantum field theories (QFTs henceforth) are one of the most abstract areas of theoretical physics that have defined the frontiers of it for the entire XXth century and still do now. This introduces a steep learning curve for anyone trying to get into the field but the efforts are worth it. On the one hand, these theories are known for the elegance of their mathematical formalism. On the other hand, the methodologies developed in this domain give the most precise theoretical values for various parameters that can be verified by experiments. A good example of this is the electron anomalous magnetic moment predicted by Quantum Electrodynamics (QED) whose current value agrees with the experimental to the eleventh digit ([1]). In fact due to the precision of these results the 3.5 sigma discrepancy between the predicted muon anomalous moment and the experimental value ([2]) is still considered to be a strong hint of physics beyond the Standard Model (SM) which incorporates QED and is the current orthodox theory explaining the interactions of subatomic particles. Another empirical fact that is still to be explained by some SM extensions are the masses of neutrinos. The requirement that they have masses is due to observed neutrino oscillations (see [3]). This phenomenon requires neutrinos to have non-zero mass terms.

One minimal extension of the SM that attempts to explain the neutrino masses is the Grimus-Neufeld model (GNM, [4]) which introduces a heavy Majorana neutrino and an additional Higgs doublet into the picture of the SM. These additions via the seesaw mechanism and radiative corrections account for two of the three neutrino masses at one loop level. The last mass term can be achieved at higher order analysis.

One specific area of exploration in QFTs is the renormalization group (RG) equations which describe how behavior of a given model changes under the change of scale. Originally renormalization in the domain of QFTs was understood as reformulation of the theory in a way that various UV and IR divergences coming from loop diagrams evaluated at infinite energies and/or containing massles particles drop out from any calculations that yield physical observable parameters. This is a huge success on its own as these divergences hindered the penetration of QFTs into mainstream physics in the early days. But even more importantly some mathematical methods devised in the latter part of the XXth century allow the description of QFTs as defined on a certain scale to be extended to an arbitrary scale. This constitutes the basis of our understanding of the phenomenon of running coupling constants and serves as a tool to explore the fundamentals of all forces except gravity (as of yet).

In this thesis RG flow analysis of the important GNM parameters are described. Namely the derivation of counterterm for the field and mass of Majorana neutrino and the fermionic Yukawa couplings can be found in the third section. In the first part, on the other hand, the building blocks of the GNM are listed and introduced and the second section is focused on the methodolgy used for the analysis. Deriving the aforementioned counterterms and describing the RG flow of these couplings via RG equations was the main goal of this thesis.

1 The Grimus-Neufeld model

As mentioned in the introduction the SM is very good at accurately describing particle physics in its domain: 19 parameters are used to calibrate it and after that it yields a huge amount of predictions for other parameters. The key words still being “its domain”. The SM is still a work in progress and many extensions of it are currently actively researched or waiting for experimental confirmation. One area to expand the SM is the description of neutrino masses. As data on neutrino oscillations accumulated over the years (again, see [3]), since this phenomenon requires neutrinos to have masses, many SM extensions were proposed that try to derive them from theoretical arguments. One such so called minimal extension is the GNM. In this model the SM is expanded with a single Majorana neutrino and an additional Higgs doublet. These additions via the seesaw mechanism and radiative mass generation yield analytical expressions for the two measured neutrino mass square differences at one loop level (see [4]). At higher order analysis all the neutrino masses are accounted for by this model.

Since the GNM expands on the electroweak (EW henceforth) sector of the SM and the most important implications of it are contained within it, the analysis in the third section will be focused on solely this sector. This section, on the other hand, is an attempt of an introduction to the main concepts of the EW sector of the standard model and the extensions of it proposed by the GNM.

1.1 Electroweak sector of the Standard Model

To emphasize the additions to the SM coming from the GNM a short overview of Glashow-Salam-Weinberg model ([5–7]) that essentially constitutes the EW sector of the SM is briefly presented in this subsection. This model can be arbitrarily separated into 4 parts: Yang-Mills, Higgs, fermion and Yukawa.

$$\mathcal{L}_{EW} = \mathcal{L}_{YM} + \mathcal{L}_{H} + \mathcal{L}_{F} + \mathcal{L}_{Y}. \tag{1}$$

The Yang-Mills part describes the gauge fields transforming under $SU(2) \times U(1)$. The Lagrangian (“Lagrangian” ≡ ”Lagrangian density” here and henceforth) of this sector reads:

$$\begin{aligned}\mathcal{L}_{YM} &= -\frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\\ &= \frac{1}{2}W^a_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)W^a_\nu + \frac{1}{2}B_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)B_\nu\\ &+ g\epsilon^{abc}(\partial^\mu W^{a\nu})W^b_\mu W^c_\nu - \frac{1}{4}g^2(W^{a\mu}W^{b\nu}W^a_\mu W^b_\nu - W^{a\mu}W^{a\nu}W^b_\mu W^b_\nu),\end{aligned} \tag{2}$$

here $W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\epsilon^{abc}W^b_\mu W^c_\nu$ and $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ are the field strengths of the gauge fields W^a_ν and B_μ. W^a_ν is the isotriplet corresponding to the generators of the weak isospin group $SU(2)$ and B_μ is the isosinglet of the group $U(1)$. ϵ^{abc} are the totally antisymmetric structure constants that define the $SU(2)$ Lie algebra and g, g' are the gauge coupling constants.

Higgs part is written as follows:

$$\mathcal{L}_H = (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi), \tag{3}$$

here $\Phi(x)$=$\begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix}$ is the Higgs doublet and $\phi^+(x)$, $\phi^0(x)$ are complex scalar fields. $\Phi(x)$ is coupled to the gauge fields via the covariant derivative:

$$D_\mu = \partial_\mu - igT^i W^i_\mu - ig'\frac{Y_W}{2}B_\mu, \tag{4}$$

$T^i = \frac{\sigma^i}{2}$ are the $SU(2)$ group generators, with σ^i being the Pauli matrices: $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Y_W is the weak hypercharge of the group $U(1)$. The potential term in eq. (3), with $\mu^2 < 0$ being the tachyonic mass and $\lambda > 0$, describes the famous Mexican hat potential responsible for the spontaneous symmetry breaking (SSB henceforth):

$$V(\Phi) = \mu^2\Phi^\dagger\Phi + \lambda\left(\Phi^\dagger\Phi\right)^2. \tag{5}$$

The fermionic part is as follows:

$$\begin{aligned}\mathcal{L}_F = &\overline{L}_j i\not{D} L_j + \overline{Q}_j i\not{D} Q_j \\ &+ \overline{l}^R_j i\not{D} l^R_j + \overline{u}^R_j i\not{D} u^R_j + \overline{d}^R_j i\not{D} d^R_j,\end{aligned} \tag{6}$$

here the left handed fermions and quarks are grouped into $SU(2)$ doublets:

$$L_j = \begin{pmatrix} \nu^L_j \\ l^L_j \end{pmatrix}, \; Q_j = \begin{pmatrix} u^L_j \\ d^L_j \end{pmatrix}, \; x^L = \omega_- x \tag{7}$$

and right handed fermions (except for neutrinos: for them $\nu^R_j = \omega_+\nu_j \equiv 0$) and quarks into singlets:

$$x^R = \omega_+ x. \tag{8}$$

In these equations $\omega_\pm = \frac{1\pm\gamma_5}{2}$ are the projection operators that bring Dirac spinors into their left or right handed representations, $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 0$ and γ^μ are the 4 dimensional Dirac matrices, that can be summed up as $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ in Dirac representation, with I being the unit matrix. The index j runs over the 3 generations and for fermionic fields $\overline{\psi} \equiv \psi^\dagger\gamma_0$. Henceforth, hopefully without loss of clarity, the superscripts R in the Lagrangians will be dropped.

The fermionic part involves no mass terms as those are not invariant under $SU(2)$ transformations. Instead they are generated via the Yukawa couplings through SSB. The Yukawa part of the Lagrangian describes this:

$$\mathcal{L}_Y = -\overline{L}_j Y^l_{jx} l_x \Phi - \overline{Q}_j Y^u_{jx} u_x \tilde{\Phi} - \overline{Q}_j Y^d_{jx} d_x \Phi + h.c., \tag{9}$$

here Y^l_{jx}, Y^u_{jx}, Y^d_{jx} are the Yukawa coupling matrices and $\tilde{\Phi} = iT^2\Phi^\dagger$ to keep the second term hypercharge neutral.

Under this formalism no mass terms are generated for the SM neutrinos. This is due to the experimental absence of right handed neutrinos and hence Yukawa couplings in the theory that would generate these masses.

The outlines of SSB, which generates the masses for the SM matter fields, are presented in the next subsection and in the latter subsections the EW sector is expanded with some additions that allow to remedy the absence of neutrino masses.

1.2 Spontaneous symmetry breaking

SSB, as opposed to explicit symmetry breaking, implies that the theory (described by its Lagrangian) is symmetric, but the ground state it occupies is not. The form of the Higgs doublet potential (see eq. (5)), so long as $\mu^2 < 0$ and $\lambda > 0$, implies that the minima of the Higgs fields lie on a circle:

$$\frac{\partial}{\partial\left(\Phi^\dagger\Phi\right)}V\left(\Phi^\dagger\Phi\right) = \mu^2 + 2\lambda\Phi^\dagger\Phi => \ \Phi^\dagger\Phi|_{min} = \left(\phi^0_{min}\right)^2 + \left(\phi^+_{min}\right)^2 = \frac{-\mu^2}{2\lambda}. \tag{10}$$

Choosing to expand the fields around any one particular minimum (for simplicity one can as well choose $\phi^0_{min} \equiv \frac{v}{\sqrt{2}} = \sqrt{\frac{-\mu^2}{2\lambda}}$ and $\phi^+_{min} = 0$) breaks the $SU(2) \times U(1)$ gauge symmetry. In the unitary gauge, where all the goldstone modes in the Higgs doublet are set to 0, this yields:

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \tag{11}$$

where $h(x)$ is now a perturbation from the true ground state. Plugging this into eq. (3), but leaving out any constant terms, one gets:

$$\begin{aligned}\mathcal{L}_H &= \frac{1}{2}\partial_\mu h\partial^\mu h + V(v,h) \\ &+ \frac{g^2}{8}\begin{pmatrix} 0 & v+h \end{pmatrix}\begin{pmatrix} \frac{g'}{g}B_\mu + W^3_\mu & W^1_\mu - iW^2_\mu \\ W^1_\mu + iW^2_\mu & \frac{g'}{g}B_\mu - W^3_\mu \end{pmatrix}\begin{pmatrix} \frac{g'}{g}B^\mu + W^{3\mu} & W^{1\mu} - iW^{2\mu} \\ W^{1\mu} + iW^{2\mu} & \frac{g'}{g}B^\mu - W^{3\mu} \end{pmatrix}\begin{pmatrix} 0 \\ v+h \end{pmatrix} \\ &= \frac{1}{2}\partial_\mu h\partial^\mu h + V(v,h) + \frac{g^2}{8}(v+h)^2\left(\left|W^1_\mu + iW^2_\mu\right|^2 + \left(W^3_\mu - \frac{g'}{g}B_\mu\right)^2\right).\end{aligned} \tag{12}$$

Redefining these mixed states:

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right),\ Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(gW^3_\mu - g'B_\mu\right),\ A_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(g'W^3_\mu + gB_\mu\right) \tag{13}$$

one can see that, as promised, through SSB the bosons responsible for weak interactions obtain mass terms equal to $m_{W^\pm} = \frac{v}{2}g$ and $m_Z = \frac{v}{2}\sqrt{g^2+g'^2}$, whereas the photon field remains massless.

Plugging this same Higgs field into the Yukawa Lagrangian eq. (9) and restricting our attention to leptonic fields only:

$$
\begin{aligned}
\mathcal{L}_{Ylep} = -\overline{L}_j Y^l_{jx} l_x \Phi = \frac{Y^l_{jx}}{\sqrt{2}} \left[\begin{pmatrix} \overline{\nu}^L_j & \overline{l}^L_j \end{pmatrix} \begin{pmatrix} 0 \\ v+h \end{pmatrix} l^R_x + \overline{l}^R_j \begin{pmatrix} 0 & v+h \end{pmatrix} \begin{pmatrix} \nu^L_x \\ l^L_x \end{pmatrix} \right] \\
= -\frac{Y^l_{jx}}{\sqrt{2}} (v+h) \left(\overline{l}^L_j l^R_x + \overline{l}^R_j l^L_x \right),
\end{aligned} \tag{14}
$$

here Y^l_{jx} is a complex non-diagonal $3x3$ matrix, but it can be shown using singular value decomposition that, with a rotation of the leptonic fields, it becomes real and diagonal. Then the fermionic mass of j-th generation lepton $m_x = \frac{M^l_{xx}}{\sqrt{2}} v$, with $Y^l = UM^l V^\dagger$ and neutrinos remain massless. On the other hand, going beyond this simple analysis it is known that neutrinos, like quarks, can oscillate between flavour eigenstates. The PMNS (CKM in the case of quarks) matrix describes this mixing. However, these topics are beyond this thesis and so next we turn to the extensions of the SM proposed by the GNM.

1.3 The Majorana neutrino

One of the additions proposed by the GNM is the Majorana neutrino. Majorana fermions are special solutions of the Dirac equation:

$$
\left(i\partial\!\!\!/ - m \right) \Psi = 0. \tag{15}
$$

In the Majorana representation of gamma matrices these solutions are strictly real $\widetilde{\Psi} = \widetilde{\Psi}^*$. But in general various matrices satisfying eq. (15) can be related by a similarity transformation:

$$
\widetilde{\gamma}^\mu = U \gamma^\mu U^\dagger, \tag{16}
$$

so more generally the reality condition looks like:

$$
U^\dagger \Psi = \left(U^\dagger \Psi \right)^* \Longrightarrow \Psi = UU^T \Psi^*. \tag{17}
$$

A more comfortable parametrization is $UU^T = \gamma_0 C$ where C is the charge conjugation matrix which varies for different representations of γ_μ matrices and is defined by $C\gamma_\mu C^{-1} = \left(-\gamma_\mu\right)^T$. This parametrization accentuates an alternative definition of the Majorana field, i. e. a field which is its own charge conjugate. Defining $\hat{\Psi} \equiv \gamma_0 C \Psi^*$ the most general reality condition is as follows:

$$
\Psi = e^{ia} \hat{\Psi}, \tag{18}
$$

where the phase a is irrelevant for the current discussion but can be important when considering the mixing of SM neutrinos with the Majorana neutrino.

From this definition it follows that Majorana fermions are their own antiparticles and this in turn means that by definition they are chargeless. This signifies that they are inert under all gauge interactions of the SM and hence are sometimes called sterile. Another important consequence of this is that the mass term that is forbidden by the EW gauge symmetry for particles that have charge under weak isospin is left intact. So the Majorana part of the Lagrangian can be written as:

$$\mathcal{L}_M = \frac{1}{2}\left(\overline{\Psi} i \partial\!\!\!/ \Psi - m\overline{\Psi}\Psi\right). \tag{19}$$

Additionally extra terms have to be added to the Yukawa sector and this addition has non trivial consequences. The next subsection covers these.

A very intuitive and enlightening introduction to the construction of the Dirac and Majorana spinors from the fundamental building blocks - Weyl spinors - is presented in [8].

1.4 The additional Higgs Doublet and the seesaw mechanism

The second proposal of the GNM is to expand the Higgs sector. Multiple Higgs doublet schemes are pretty trivial expansions of the SM. Arbitrary amounts of doublets can be postulated without any symmetries of the SM being violated, albeit at a steep cost in terms of complexity of the model. In the GNM only one additional Higgs doublet is added:

$$\Phi_2 \equiv \begin{pmatrix} \phi_2^+ \\ \phi_2^0 \end{pmatrix}, \tag{20}$$

Regardless of the simplicity of the addition itself it forces the consideration of a lot of new parameters in the Lagrangian. The most general Higgs part of the Lagrangian (eq. 3) gets expanded by an extra kinetic and a batch of potential terms:

$$\mathcal{L}_H = \left(D_\mu \Phi_1\right)^\dagger \left(D^\mu \Phi_1\right) + \left(D_\mu \Phi_2\right)^\dagger \left(D^\mu \Phi_2\right) - A_{ab}\Phi_a^\dagger \Phi_b - B_{abcd}\left(\Phi_a^\dagger \Phi_b\right)\left(\Phi_c^\dagger \Phi_d\right), \tag{21}$$

where $a, b, c, d = 1, 2$. Following arguments in [9] one can show that only 14 of all A and B parameters are unrelated and of those 14 three more can be ruled out by symmetry arguments. The implications of this and more Higgs doublets including potentials is still an active field of research.

The Yukawa part of the Lagrangian (eq. 9) gets trivially doubled by considering all terms for both Higgs doublets separately, but there are also extra terms coming from the Higgs doublets and Majorana neutrino coupling together:

$$\mathcal{L}_{Y_{additional}} = -Y_{ja}\overline{L}_j \Psi \Phi_a + h.c. \tag{22}$$

The most interesting couplings are the ones that involve neutrinos. The mixing of the 3 SM neutrinos with the Majorana neutrino under specific parametrization leads to one non-vanishing mass term upon transformation from flavor eigenstates to mass eigenstates by say singular value decomposition. This is the so called seesaw mechanism. It it trivial to exemplify its principle with a 2D matrix:

$$A = \left(\begin{array}{cc} 0 & M \\ M & m \end{array}\right). \tag{23}$$

The eigenvalues of this matrix are $\lambda_\pm = \frac{m\pm\sqrt{m^2+4M^2}}{2}$. Then since

$$\lambda_+\lambda_- = |A| = -M^2 \tag{24}$$

if one of the eigenvalues gets bigger the other gets smaller. In the limit $m \gg M$ one can approximate the eigenvalues as $\lambda_+ \approx m$ and $\lambda_- \approx \frac{M^2}{m}$. Assuming that the mass of the Majorana neutrino m is large this grants that the generated mass of the neutrino is small.

In the GNM to distinguish between the 2 neutrinos that remain massless at the tree level an assumption is made that only one of them couples to the second Higgs doublet. This assumption generates an additional mass term for one of the neutrinos at one loop level via the radiative corrections.

This concludes the discussion of the GNM, but in the following subsection one additional important feature - gauge fixing via explicit gauge breaking terms in the Lagrangian - will be introduced. Also, significantly more in depth analysis of the GNM concepts including most of the technical details can be found in [10].

1.5 Feynman-'t Hooft gauge

Higher order analysis in QFTs is significantly simplified if one fixes the gauge degrees of freedom. Specific problems benefit most from specific gauge choices. Most computations in QFT are simplest in the Feynman-'t Hooft gauge so it was used for the analysis to be described in the third section. Customary rather than writing the constraints in auxiliary equations in particle physics they are included in the Lagrangian as additional gauge breaking terms. Following [11] these terms for the specific gauge in question can be summed up as follows:

$$\mathcal{L}_{g.fix} = -\frac{1}{2}\sum_{\alpha=0}^{3}(\mathcal{F}^\alpha)^2, \tag{25}$$

$$\mathcal{F}^0 = \partial^\mu B_\mu - \frac{1}{2}g'vG^0, \tag{26}$$

$$\mathcal{F}^3 = \partial^\mu W^3_\mu + \frac{1}{2}gvG^0, \tag{27}$$

$$\mathcal{F}^1 = \partial^\mu W^1_\mu + \frac{1}{2}gvG^2, \tag{28}$$

$$\mathcal{F}^2 = \partial^\mu W^2_\mu - \frac{1}{2}gvG^1. \tag{29}$$

In these equations v is the vacuum expectation value of the SM Higgs field, $G^1 = \frac{(G^-+G^+)}{\sqrt{2}}$, $G^2 = -\frac{i(G^--G^+)}{\sqrt{2}}$ are linear superpositions of $G^\pm$ and $G^\pm$, G^0 are the Goldstone bosons that become the longitudinal components of $W^\pm$ and Z^0 after SSB. The SM Higgs doublet can be parametrized in terms of these and an additional neutral field h^0 as $\Phi=\begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v+h^0+iG^0) \end{pmatrix}$. Writing $\mathcal{L}_{g.fix}$ explicitly after integrating $\mathcal{F}^\alpha$ by parts:

$$\mathcal{L}_{g.fix} = \frac{1}{2}[B_\mu\partial^\mu\partial^\nu B_\nu + W^j_\mu\partial^\mu\partial^\nu W^j_\nu - \frac{v^2}{4}(G^0(g'^2+g^2) + G^-G^+g^2)$$
$$+v(gW^3_\mu - g'B_\mu)\partial^\mu G^0 + gv(W^1_\mu\partial^\mu G^2 + W^2_\mu\partial^\mu G^1)]. \tag{30}$$

Introducing these gauge fixing terms forces one to also include the Faddeev-Popov ghosts ([12]) into the Lagrangian. These terms are constructed using the Faddeev-Popov determinant which is defined in terms of the variations of gauge fixing terms $\delta\mathcal{F}^\mu$ and the variations of the infinitesimal gauge transformations $\delta\Theta^\beta$ ($U(\Theta) = 1 - iT^\alpha\Theta^\alpha$, where $T^0 = Y_W, T^i = \frac{\sigma^i}{2}$ and $\Theta^\alpha \ll 1$) that caused them:

$$M^{\alpha\beta} = \frac{\delta\mathcal{F}^\alpha}{\delta\Theta^\beta} \tag{31}$$

Applying the infinitesimal gauge transformations to the fields of interest one can write:

$$\delta\Phi = -ig'\delta\Theta^0 Y_W\Phi - ig\delta\Theta^i T^i\Phi, \tag{32}$$

$$\delta\Phi^\dagger = ig'\Phi^\dagger\delta\Theta^0 Y_W + ig\Phi^\dagger\delta\Theta^i T^i, \tag{33}$$

$$\delta B_\mu = \partial_\mu\delta\Theta^0, \tag{34}$$

$$\delta W^a_\mu = \partial_\mu\delta\Theta^a - g\epsilon^{abc}W^b_\mu\delta\Theta^c. \tag{35}$$

With that in mind the ghost field Lagrangian looks like:

$$\mathcal{L}_{ghost} = \bar{\eta}^\alpha M^{\alpha\beta}\eta^\beta = \bar{\eta}^0(\partial^2 + g'^2(v^2+h^0v))\eta^0 + \sum_{j=1}^{3}\bar{\eta}^j(\partial^2 + g^2(v^2+h^0v))\eta^j$$
$$+\bar{\eta}^1(\partial^2 - g\partial^\mu W^3_\mu - ig^2(v^2+h^0v))\eta^2 + \bar{\eta}^1(\partial^2 + g\partial^\mu W^2_\mu - g^2iG^2v)\eta^3$$
$$+\bar{\eta}^2(\partial^2 + g\partial^\mu W^3_\mu + ig^2(v^2+h^0v))\eta^1 + \bar{\eta}^2(\partial^2 - g\partial^\mu W^1_\mu + g^2iG^1v)\eta^3$$
$$+\bar{\eta}^3(\partial^2 - g\partial^\mu W^2_\mu + g^2iG^2v)\eta^1 + \bar{\eta}^3(\partial^2 + g\partial^\mu W^1_\mu - g^2iG^1v)\eta^2$$
$$+\frac{gg'}{2}(\bar{\eta}^0G^1v\eta^1 + \bar{\eta}^0G^2v\eta^2 - \bar{\eta}^0(v^2+h^0v)\eta^3 + \bar{\eta}^1G^1v\eta^0 + \bar{\eta}^2G^2v\eta^0 - \bar{\eta}^3(v^2+h^0v)\eta^0) \tag{36}$$

With all these parts the EW sector of GNM Lagrangian can be summarized as:

$$\mathcal{L}_{GNM} = \mathcal{L}_{EW+\Phi_2+M} + \mathcal{L}_{g.fix} + \mathcal{L}_{ghost}. \tag{37}$$

This concludes the overview of the GNM and in the following section the focus lies on the techniques that were employed to derive the renormalization constants of the aforementioned GNM parameters.

2 The renormalization procedure

Beyond the tree level in QFTs it is common to encounter integrals that diverge in the infrared (IR) or the ultraviolet (UV) limit. IR divergences have to do with vanishing masses of particles in question and they won't be explored in this thesis. UV divergences, on the other hand, are encountered when integrating propagator factors that (following [13]) can be written in the form:

$$T^N_{\mu_1 \ldots \mu_p}(p_1, \ldots, p_{N-1}, m_0, \ldots, m_{N-1}) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q_{\mu_1} \ldots q_{\mu_P}}{D_0 D_1 \ldots D_{N-1}} \tag{38}$$

here D is the dimension number, N is the number of propagator factors, P is the number of integration momenta in the numerator and $D_i = (q + p_i)^2 - m_i^2 + i\epsilon$ are the propagators, $p_0 = 0$ and $i = 1, \ldots, N-1$. In general for $P + D - 2N \geq 0$ these integrals are UV divergent. Since these integrals come from Feynman diagrams and play a role in computation of Green's functions and S matrix elements which in turn are connected to observable quantities one might conclude that this methodology is flawed. Alternatively, the problem might be the parametrization of the theory. For the so called renormalizable theories (QED is one example) the latter is indeed the case and one can reformulate the parameters of the theory in a way that the aforementioned divergences cancel out to any order of analysis. For this the diverging part of eq. (38) has to be extracted. There are multiple ways to achieve this and a brief overview of them is presented in the following subsection.

2.1 The regularization schemes

The first and most straightforward regularization scheme is the momentum cutoff. In it the upper bound of integrals like in eq. (38) is taken to be equal to an arbitrary finite constant Λ^2 rather than infinity. This allows to achieve a finite result with some parts depending on Λ and some not. Then Λ can be taken back to infinity. One issue with this methodology is that it violates Lorentz symmetry in the process, unless specifically accounted for.

Another method that shares one essential detail with the momentum cutoff is the Pauli-Villars regularization. Here additional terms are added to the propagators in a way that they vanish fast enough for large momenta:

$$\frac{1}{q^2 + m^2} \to \frac{1}{q^2 + m^2} - \frac{1}{q^2 + \Lambda^2} = \frac{q^2 + \Lambda^2 - q^2 - m^2}{\left(q^2 + m^2\right)\left(q^2 + \Lambda^2\right)} = \frac{\Lambda^2 - m^2}{\left(q^2 + m^2\right)\left(q^2 + \Lambda^2\right)}, \tag{39}$$

this parameter Λ is unrelated to the momentum cutoff but serves a similar purpose - after the finite result for the integral is achieved and parts with Λ are isolated it can be sent to infinity. This method too happens to violate symmetries inherent in QFTs in the process.

Third option is defining QFTs on a lattice. Due to the discrete spacing of lattice points the momenta will be limited to be less than $\frac{\pi}{a}$, where a is the size of the gap between the lattice points. Taking $a \to 0$ sends the momentum cap back to infinity. The problem with these theories if that they struggle with describing scattering events.

Finally, the most popular method is the dimensional regularization proposed in [14]. Under this methodology the Feynman integrals are formally continued to an arbitrary dimension $D = 4 - 2\epsilon$ (sometimes simply ϵ). The diverging parts for regular QFTs with 4 spacetime dimensions then show up as poles $\frac{1}{\epsilon^n}$ for the n-th order loop diagrams. The reason why this method is so popular is that it avoids interfering with the integral bounds or the integrand itself and hence symmetry violations. For these reasons this method was chosen for the analysis presented in this thesis.

After the diverging integrals are evaluated and the troublemaker terms isolated next step is to reparametrize the model. The main approaches to this are summarized in the next subsection.

2.2 Approaches to the renormalization

The usual next step after regularization is the redefinition of the bare parameters of the theory introducing counterterms that are fixed through the renormalization conditions. It's an arbitrary choice how much of the finite part is included in the counterterm so consistency is important.

One way to define the counterterms is such that the renormalized parameters are equal to the measurable physical parameters to all orders of perturbation theory. This is the so called on-shell scheme. For example in the SM a popular choice of such parameters are the masses of physical particles $M_W, M_Z, M_H, m_{f,}$ charge of the electron q_e and the quark mixing matrix V_{ij}.

A simpler alternative is the Minimal Subtraction (MS) scheme which only includes the divergent term in the counterterm or the modified Minimal Subtraction ($\overline{MS}$, spelled MS-bar) scheme which additionally includes a few constants that come up during the dimensional regularization procedure: the Euler-Mascheroni constant γ_E and $\ln(4\pi)$. This approach has been chosen for the work described in this thesis due to its simplicity and the fact that the masses and couplings of the particles proposed by the GNM are still free parameters.

The additions of all the counterterms in a multiplicative form ($P \rightarrow ZP$, $Z = 1 + \delta Z$) result in Lagrangian of the form:

$$\mathcal{L} = \mathcal{L}_0 - \delta\mathcal{L} \tag{40}$$

where $\mathcal{L}_0$ stands for the original (bare) Lagrangian that we started with and under $\delta\mathcal{L}$ are all the counterterms. For renormalizable theories the newly formulated Lagrangian is divergence free, but to achieve this one has to take the new terms seriously and include Feynman diagrams stemming from them into the analysis.

In the following subsection the dimensional regularization and $\overline{MS}$ scheme will be applied to the simplest possible exemplary integral.

2.3 Dimensional regularization and $\overline{\text{MS}}$ scheme in action

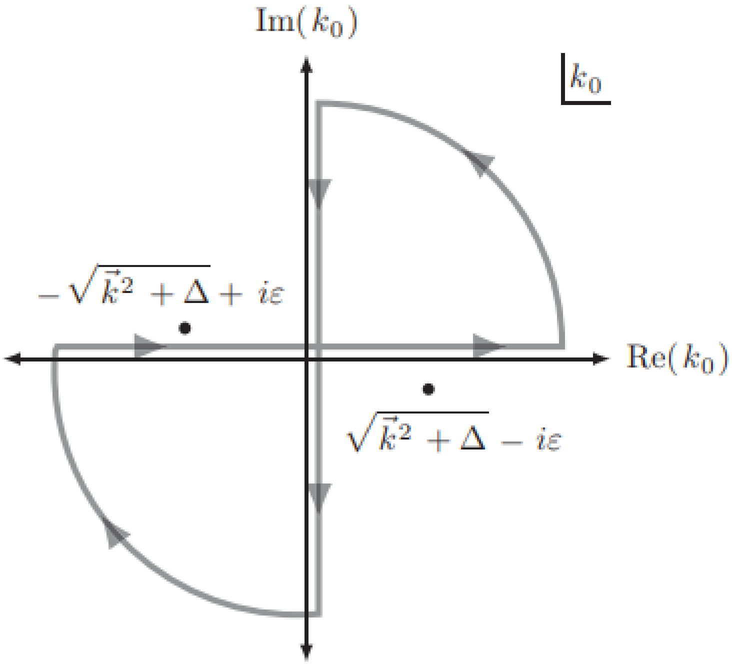
Starting off simple the chosen methods can be applied to a Feynman integral of the form:

$$I = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon}. \tag{41}$$

To begin with, such integrals are significantly easier to evaluate in Euclidean rather than Minkowski space. Hence a trick called Wick rotation is usually performed. Since Feynman integrals usually have their poles lie in the opposite quadrants, integrating over a contour shown in fig. 1, so long as the integrand vanishes at $|k_0| \to \infty$, implies that the integrals over the real and the imaginary axes are identical and opposite, i. e. $I_R + I_{Im} = 0$. Practically Wick rotation is performed by sending $p_0 \to ip_0$. This change makes the aforementioned integral purely Euclidean:

$$I \to i \int \frac{d^4 p}{(2\pi)^4} \frac{1}{-p^2 - m^2}. \tag{42}$$

Note that the $i\epsilon$ term is no longer needed as now one can "cheat" and instead evaluate this integral in, for example, the spherical coordinate system by making use of the spherical symmetry.



1 Fig. The contour of integration that justifies Wick rotation, i. e. the exchange of integration over the real axis to integration over the imginary axis, for Feynman integrals that have their poles in opposite quadrants (taken from [15])

Since this integral is divergent in four dimensions, before shifting to spherical coordinates, the integral in question is generalised to arbitrary dimension D.

$$I \to \mu^{4-D} i \int \frac{d^D p}{(2\pi)^D} \frac{1}{-p^2 - m^2}, \tag{43}$$

here μ is the usual factor that keeps track of the mass dimensions. Now one can finally abuse the spherical symmetry of the integrand and carry out the integration over the polar coordinates:

$$I = i\mu^{4-D} \int d\Omega \int_0^\infty \frac{dp}{(2\pi)^D} \frac{p^{D-1}}{-p^2 - m^2} = -i\mu^{4-D} \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty \frac{dp}{(2\pi)^D} \frac{p^{D-1}}{p^2 + m^2}. \tag{44}$$

The numerical prefactor is just the surface area of a unit length D dimensional sphere:

$$\int d\Omega = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)}. \tag{45}$$

Massaging the integrand a little yields the following:

$$\begin{aligned}
I &= -i\mu^{4-D} \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \frac{m^{D-2}}{(2\pi)^D} \int_0^\infty \frac{1}{2} d\left(\frac{p^2}{m^2}\right) \frac{\left(\frac{p^2}{m^2}\right)^{\frac{D-2}{2}}}{\frac{p^2}{m^2}+1} \\
&= -i\mu^{4-D} \frac{m^{D-2}}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)} \int_0^\infty d\left(\frac{p^2}{m^2}\right) \frac{\left(\frac{p^2}{m^2}\right)^{\frac{D}{2}-1}}{\frac{p^2}{m^2}+1} \\
&= -i\mu^{4-D} \frac{m^{D-2}}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)} B\left(\frac{D}{2}, 1-\frac{D}{2}\right) \\
&= -i\mu^{4-D} \frac{m^{D-2}}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)} \frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(1-\frac{D}{2}\right)}{\Gamma(1)} \\
&= -i\mu^{4-D} \frac{m^{D-2}}{(4\pi)^{\frac{D}{2}}} \Gamma\left(1-\frac{D}{2}\right) = -i\left(\frac{m}{4\pi}\right)^2 \left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}} \Gamma\left(1-\frac{D}{2}\right).
\end{aligned} \tag{46}$$

Here one of the expressions for the beta function, also known as the Euler integral, $B(x,y) = \int_0^\infty dt \frac{t^{x-1}}{(1+t)^{x+y}}$ and its relation to the gamma function $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ were used. From the last expression one can see that the integral that we started with diverges for all even dimensions $D > 0$. Expanding the gamma function around $D = 4 - 2\epsilon$ ($\epsilon \ll 1$) in Laurent series:

$$\Gamma(-1+\epsilon) = -\frac{1}{\epsilon} + \gamma_E - 1 - \left(1 - \gamma_E + \frac{\gamma_E^2}{2} + \frac{\pi^2}{12}\right)\epsilon + O\left(\epsilon^2\right), \tag{47}$$

the remaining factors in Taylor series around the same point:

$$\left(\frac{m^2}{\mu^2 4\pi}\right)^{-\epsilon} = 1 - \ln\left(\frac{m^2}{4\pi\mu^2}\right)\epsilon + O\left(\epsilon^2\right) \tag{48}$$

and plugging these into eq. (46) one gets:

$$I = i\frac{m^2}{(4\pi)^2}\left[1 + \frac{1}{\epsilon} - \gamma_E - \ln\left(\frac{m^2}{4\pi\mu^2}\right)\right] + O(\epsilon). \tag{49}$$

Grouping the pole and some of the constants under $\Delta_{\overline{MS}} \equiv \frac{1}{\epsilon} - \gamma_E + \ln(4\pi)$ and introducing this parameter into the theory in an appropriate way one can make sure that the diverging part always cancels out. In the next section this methodology is showcased by renormalizing some GNM parameters.

3 Renormalization of the relevant GNM parameters

Following [15] for each field and coupling constant (masses included) in the Lagrangian multiplicative counterterms of the form

$$\begin{aligned}
\Psi_B &= Z_\Psi^{\frac{1}{2}} \Psi_R, \; Z_\Psi = 1 + \delta_\Psi \\
m_B &= Z_m m_R, \; Z_m = 1 + \delta_m \\
g_B &= Z_g g_R, \; Z_g = 1 + \delta_g
\end{aligned} \tag{50}$$

are defined. Indexes B and R stand for bare and renormalized quantities. Henceforth the subscripts R will be dropped to avoid confusion. These new terms involving δ_x result in additional Feynman diagrams and rules and these diagrams, on their own accord, are supposed to cancel any effects of UV divergences from the regular diagrams. This is what defines the δ_x and allows one to calculate them to arbitrary precision. Note that this means that δ_x is by definition of one or higher loop order.

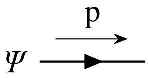
3.1 The renormalization procedure applied to the Majorana neutrino field and mass

One advantage of the MS or $\overline{MS}$ schemes is that one doesn't need to specify renormalization conditions to fix the values of the counterterms as they are constant. With this in mind, the simplest GNM specific parameters to renormalize are the Majorana neutrino mass and field. The renormalization constants for them can be derived by considering the Majorana neutrino two point Green's function, also known as the two point correlation function, the propagator or the self-energy. At one loop level the self-energy gets corrections from processes formally depicted in fig. 2. In these loops neutral parts of both Higgs doublets can couple to the SM neutrino fields and the charged parts to the remaining lepton fields. Here I should briefly mention that there are some subtleties with the Feynman rules for Majorana neutrino fields that stem from the property of self-conjugacy. For example one can formulate 4 different, but related Majorana propagators. For the purposes of RG analysis any one of them is as good as any other, so we will not worry about that, but for further discussion the reader is directed to [16].

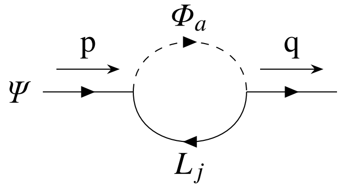
Including the multiplicative counterterms into the Majorana Lagrangian eq. (19) yields 2 additional terms:

$$\begin{aligned}
\mathcal{L}_M &= \frac{1}{2}\left(Z_\Psi \overline{\Psi} i \partial\!\!\!/ \Psi - Z_\Psi Z_m m \overline{\Psi}\Psi\right) \\
&= \frac{1}{2}\left(\overline{\Psi} i \partial\!\!\!/ \Psi - m\overline{\Psi}\Psi\right) + \frac{1}{2}\left(\delta_\Psi \overline{\Psi} i \partial\!\!\!/ \Psi - \left(\delta_m + \delta_\Psi\right) m \overline{\Psi}\Psi\right)
\end{aligned} \tag{51}$$

and, since both of them couple the Majorana field to itself, one additional Feynman diagram (see fig. 3). Note that $\delta_m \delta_\Psi$ contribution is of 2 loop order and hence can be ignored.

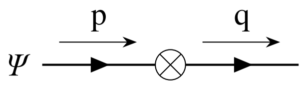


$$\equiv iG_{\Psi\Psi}(p) = \frac{i}{\not{p}-m}$$



$$\equiv iG_{\Psi\Psi}(p)\, i\Sigma_{\Psi\Psi}(p)\, iG_{\Psi\Psi}(q)$$

2 Fig. One of the four possible Majorana neutrino tree level propagators and one loop corrections stemming from the additional Yukawa couplings introduced in eq. (22). This Feynman diagram stands for many identical diagrams in which the neutral parts of the Higgs doublets couple to the SM neutrinos and the charged parts to the charged leptons.



$$\equiv iG_{\Psi\Psi}(p)\, i\Sigma_{c.t.} iG_{\Psi\Psi}(q)$$

3 Fig. Majorana neutrino counterterm diagram and coupling. These stem from rewriting the Lagrangian in terms of renormalized parameters (see eq. (51)) and noting that additional terms look very much like regular couplings and hence yield additional Feynman diagrams.

Applying Feynman rules to these proceses yields the following:

$$\begin{aligned} i(2\pi)^4 \delta^4(p-q)\Sigma_{\Psi\Psi}(p) &= \sum_{j=1}^{3}\sum_{a=1}^{2}\sum_{v=1}^{2}\int \frac{d^4p'}{(2\pi)^4}\int \frac{d^4p''}{(2\pi)^4}\left(-iY_{ja}\right) i\frac{\not{p}'+m_{jv}}{p'^2-m_{jv}^2}\left(-iY_{ja}^*\right)\frac{i}{p''^2-m_{av}^2} \\ &\times (2\pi)^4\delta^4(p-p'+p'')(2\pi)^4\delta^4(-q+p'-p'') \\ &= \sum_{jav}\left|Y_{ja}\right|^2\int \frac{d^4p'}{(2\pi)^4}\frac{\not{p}'+m_{jv}}{p'^2-m_{jv}^2}\frac{1}{(p'-p)^2-m_{av}^2}(2\pi)^4\delta^4(p-q) \end{aligned} \tag{52}$$

$$i\Sigma_{c.t.} = i\left(\delta_\Psi \not{p} - (\delta_m + \delta_\Psi)\, m\right) \tag{53}$$

here and in next subsection indices $j,\ k$ are used to track the left handed fermion doublet generation number and indices $a,\ b$ are used to track whether Higgs fields come from the first or the second doublet. The index v is needed to sum over both the neutral and charged contributions coming from the multiplication of the doublets. Note that any additional counterterms, for example of Yukawa couplings, can be disregarded as they push the analysis beyond 1 loop order. If one demands that the sum $\Sigma_{\Psi\Psi}(p) + \Sigma_{c.t.}$ be finite, one gets an equation for the counterterms. For this the integral in $\Sigma_{\Psi\Psi}(p)$

$$I = \int \frac{d^4p'}{(2\pi)^4}\frac{\not{p}'+m_{jv}}{p'^2-m_{jv}^2}\frac{1}{(p'-p)^2-m_{av}^2} \equiv I_p + I_m \tag{54}$$

has to be regularized. Note that this is a tensor integral, as the denominator is a $4x4$ matrix $\not{p}' + I_{4x4}m_{jv}$. There are some clever ways to reduce the tensor integrals to scalar ones. They don't change the matrix nature of such Green functions, but at least move the matrices in front of the integrals. In this case to reduce I_p one can make use of identities:

$$pp' = -\frac{1}{2}\left((p'-p)^2 - p'^2 - p^2\right) = -\frac{1}{2}\left(\left[(p'-p)^2 - m_{av}^2\right] - \left[p'^2 - m_{jv}^2\right] - \left[p^2 - m_{av}^2 + m_{jv}^2\right]\right). \tag{55}$$

$$\begin{aligned} I_p &= \frac{p^2}{p^2}\int \frac{d^4p'}{(2\pi)^4}\frac{\not{p}'}{p'^2 - m_{jv}^2}\frac{1}{(p'-p)^2 - m_{av}^2} \\ &= -\frac{\not{p}}{2p^2}\int \frac{d^4p'}{(2\pi)^4}\left(\frac{1}{p'^2 - m_{jv}^2} - \frac{1}{(p'-p)^2 - m_{av}^2} - \frac{p^2 - m_{av}^2 + m_{jv}^2}{\left(p'^2 - m_{jv}^2\right)\left((p'-p)^2 - m_{av}^2\right)}\right) \end{aligned} \tag{56}$$

More examples of such tensor integral reductions and a general case proof can be found in [13].

With this the original integral reduces to 3 pieces:

$$\begin{aligned} I &= \frac{\not{p}}{2p^2}\int \frac{d^4p'}{(2\pi)^4}\frac{1}{(p'-p)^2 - m_{av}^2} - \frac{\not{p}}{2p^2}\int \frac{d^4p'}{(2\pi)^4}\frac{1}{p'^2 - m_{jv}^2} \\ &+ \left(\frac{\not{p}}{2p^2}\left(p^2 + m_{jv}^2 - m_{av}^2\right) + m_{jv}\right)\int \frac{d^4p'}{(2\pi)^4}\frac{1}{\left(p'^2 - m_{jv}^2\right)\left((p'-p)^2 - m_{av}^2\right)}. \end{aligned} \tag{57}$$

The first 2 integrals are identical to the one in eq. (41). The third integral is usually rewritten using specific parametrization attributed to Feynman:

$$\begin{aligned} I_3 &= \int \frac{d^4p'}{(2\pi)^4}\frac{1}{\left(p'^2 - m_{jv}^2\right)\left((p'-p)^{\,2} - m_{av}^2\right)} \\ &= \int \frac{d^4p'}{(2\pi)^4}\int_0^1 dx \frac{1}{\left((1-x)\left(p'^2 - m_{jv}^2\right) + x\left((p'-p)^{\,2} - m_{av}^2\right)\right)^2} \\ &= \int \frac{d^4p'}{(2\pi)^4}\int_0^1 dx \frac{1}{\left((p'-px)^{\,2} + x(1-x)\,p^2 - xm_{av}^2 - (1-x)\,m_{jv}^2\right)^2} \\ &= \int_0^1 dx \int \frac{d^4q}{(2\pi)^4}\frac{1}{\left(q^2 - Q\right)^2}, \end{aligned} \tag{58}$$

here $Q = xm_{av}^2 + (1-x)m_{jv}^2 - x(1-x)p^2$ and $q = (p' - px)$. The internal integral is now spherically symmetric and after Wick rotating the zeroth component of the shifted momentum q one can generalize this integral to an arbitrary dimension:

$$I_3 = i\int_0^1 dx \int \frac{d^4q}{(2\pi)^4}\frac{1}{\left(-q^2 - Q\right)^2} \to i\mu^{4-D}\int_0^1 dx \int \frac{d^Dq}{(2\pi)^D}\frac{1}{\left(q^2 + Q\right)^2} \tag{59}$$

Making use of the spherical symmetry again:

$$
\begin{aligned}
I_3 &= i\mu^{4-D}\frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)}\int_0^1 dx\int_0^\infty \frac{dq}{(2\pi)^D}\frac{q^{D-1}}{\left(q^2+Q\right)^2} \\
&= i\mu^{4-D}\frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)}\int_0^1 dx\int_0^\infty \frac{\frac{1}{2}d\frac{q^2}{Q}}{(2\pi)^D}\frac{\left(\frac{q^2}{Q}\right)^{\frac{D-2}{2}}Q^{\frac{D}{2}}}{Q^2\left(\frac{q^2}{Q}+1\right)^2} - \\
&= i\mu^{4-D}\frac{1}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)}\int_0^1 dxQ^{\frac{D}{2}-2}B\left(\frac{D}{2},2-\frac{D}{2}\right) \\
&= i\mu^{4-D}\frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(2-\frac{D}{2}\right)}{(4\pi)^{\frac{D}{2}}\Gamma(2)\Gamma\left(\frac{D}{2}\right)}\int_0^1 dxQ^{\frac{D}{2}-2} = i\mu^{4-D}\frac{\Gamma\left(2-\frac{D}{2}\right)}{(4\pi)^{\frac{D}{2}}}\int_0^1 dxQ^{\frac{D}{2}-2} \\
&= \frac{i\Gamma\left(2-\frac{D}{2}\right)}{(4\pi)^2}\int_0^1 dx\left(\frac{Q}{\mu^2 4\pi}\right)^{\frac{D-4}{2}}
\end{aligned} \tag{60}
$$

Here as in subsection 2.3 the same beta function expression and it's relation to the gamma functions was used. Again, next step is to expand this result around $D = 4-2\epsilon$, $\epsilon \ll 1$ and again Laurent series is used to expand the gamma function due to the pole at $\Gamma(0)$:

$$
\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \frac{1}{2}(\gamma_E^2 + \frac{\pi^2}{6})\epsilon + O\left(\epsilon^2\right), \tag{61}
$$

$$
\left(\frac{Q}{\mu^2 4\pi}\right)^{-\epsilon} = 1 - \ln\left(\frac{Q}{\mu^2 4\pi}\right)\epsilon + O\left(\epsilon^2\right), \tag{62}
$$

Multiplying these and keeping only non vanishing terms:

$$
I_3 = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon} - \gamma_E - \int_0^1 dx \ln\left(\frac{Q}{\mu^2 4\pi}\right)\right] + O(\epsilon). \tag{63}
$$

The integral in eq. (63) is analitically solved in Appendix A but the result isn't very enlightening or relevant for further discussion. Plugging everything back to eq. (52) and denoting $I_x = \int_0^1 dx \ln\left(\frac{Q}{x^2}\right)$ as well as grouping $\Delta_{\overline{MS}} = \frac{1}{\epsilon} - \gamma_E + \ln(4\pi)$ like in the previous section:

$$
\begin{aligned}
\Sigma_{\Psi\Psi}(p) &= \sum_{jav}\frac{\left|Y_{ja}\right|^2}{(4\pi)^2}\left(\frac{\not{p}m_{av}^2}{2p^2}\left[1+\Delta_{\overline{MS}} - \ln\left(\frac{m_{av}^2}{\mu^2}\right)\right] - \frac{\not{p}m_{jv}^2}{2p^2}\left[1+\Delta_{\overline{MS}} - \ln\left(\frac{m_{jv}^2}{\mu^2}\right)\right]\right. \\
&\left.+\left(\frac{\not{p}}{2p^2}\left(p^2+m_{jv}^2-m_{av}^2\right)+m_{jv}\right)\left[\Delta_{\overline{MS}} - \int_0^1 dx\ln\left(\frac{Q}{\mu^2}\right)\right]\right) \\
&= \sum_{jav}\frac{\left|Y_{ja}\right|^2}{(4\pi)^2}\left(\frac{\not{p}}{2p^2}\left(m_{av}^2\left[1+I_{m_{av}}\right] - m_{jv}^2\left[1+I_{m_{jv}}\right]\right) - \left(\frac{\not{p}}{2}+m_{jv}\right)\left[\Delta_{\overline{MS}} - I_\mu\right]\right).
\end{aligned} \tag{64}
$$

As one can see, the UV divergent part is isolated solely in the third term of eq. (64). By demanding that the terms in $\Sigma_{c.t.}$ would cancel it one gets the following equation:

$$\delta_\Psi \not{p} - (\delta_m + \delta_\Psi)\, m + \sum_{jav} \frac{\left|Y_{ja}\right|^2}{(4\pi)^2} \left(\frac{\not{p}}{2} + m_{jv}\right) \Delta_{\overline{MS}} = 0, \tag{65}$$

from which the values for the counterterms can be read off:

$$\delta_\Psi = -\sum_{jav} \frac{\left|Y_{ja}\right|^2}{(4\pi)^2} \frac{\Delta_{\overline{MS}}}{2}, \tag{66}$$

$$\delta_m + \delta_\Psi = \sum_{jav} \frac{\left|Y_{ja}\right|^2}{(4\pi)^2} \frac{m_{jv}}{m} \Delta_{\overline{MS}} \Rightarrow \delta_m = \sum_{jav} \frac{\left|Y_{ja}\right|^2}{(4\pi)^2} \left[\frac{m_{jv}}{m} + \frac{1}{2}\right] \Delta_{\overline{MS}}. \tag{67}$$

These definitions of the counterterms achieve what we aimed for - removing the infinity from the 1 loop Majorana propagator correction. Yet there is additional information one can gather from this analysis - there is this one term I_μ left in eq. (64) that depends on the arbitrary parameter of mass dimension μ. In the domain of RG μ is called the renormalization scale. One would prefer any physical results to be independent of this arbitrary parameter, hence additionaly it is demanded that the renormalized couplings covariate with it in a way as to cancel any μ dependence in $\Sigma_{\Psi\Psi}(p)$. One way to figure this dependence out is to note that the bare parameters of the model are just numbers and hence:

$$0 = \mu \frac{d}{d\mu} m_B = \mu \frac{d}{d\mu} (Z_m m) = \mu m \left(\frac{dZ_m}{d\mu} + \frac{Z_m}{m}\frac{dm}{d\mu}\right). \tag{68}$$

This is the so called Renormalization Group Equation (RGE) for the mass of the Majorana neutrino. From it, by noting that $Z_m = 1 + \delta_m$, taking the sum over all fields in front of the counterterm $\delta_m \to \sum_{jav} \delta_m$ and using the chain rule, one can arrive at:

$$\mu m \left(\sum_{jav} \left(\frac{d\delta_m}{dY_{ja}}\frac{dY_{ja}}{d\mu} + \frac{d\delta_m}{dY^*_{ja}}\frac{dY^*_{ja}}{d\mu} + \frac{d\delta_m}{dm_{jv}}\frac{dm_{jv}}{d\mu}\right) + \sum_{jav} \frac{d\delta_m}{dm}\frac{dm}{d\mu} + \frac{Z_m}{m}\frac{dm}{d\mu}\right) = 0 \tag{69}$$

and from this the equation for the function that describes how couplings of the theory change with the scale - the beta function - of the Majorana mass can be read off:

$$\begin{aligned}
\beta_m \equiv \mu\frac{dm}{d\mu} &= -\sum_{jav}\left(\frac{d\delta_m}{dY_{jav}}\mu\frac{dY_{ja}}{d\mu} + \frac{d\delta_m}{dY^*_{jav}}\mu\frac{dY^*_{ja}}{d\mu} + \frac{d\delta_m}{dm_{jv}}\mu\frac{dm_{jv}}{d\mu}\right)\left(\sum_{jav}\frac{d\delta_m}{dm} + \frac{Z_m}{m}\right)^{-1} \\
&= -m\frac{\Delta_{\overline{MS}}}{(4\pi)^2}\sum_{jav}\left(\left(Y^*_{ja}\beta_{Y_{ja}} + Y_{ja}\beta_{Y^*_{ja}}\right)\left[\frac{m_{jv}}{m} + \frac{1}{2}\right] + \left|Y_{ja}\right|^2\frac{m_{jv}}{m}\gamma_{m_{jv}}\right)\left(1 + \frac{1}{2}\frac{\Delta_{\overline{MS}}}{(4\pi)^2}\sum_{jav}\left|Y_{ja}\right|^2\right)^{-1} \\
&- \frac{2m}{Y}\sum_{jav}\left(\left(Y^*_{ja}\beta_{Y_{ja}} + Y_{ja}\beta_{Y^*_{ja}}\right)\left(\frac{m_{jv}}{m} + \frac{1}{2}\right) + \left|Y_{ja}\right|^2\frac{m_{jv}}{m}\gamma_{m_{jv}}\right)\left(2\frac{(4\pi)^2}{Y\Delta_{\overline{MS}}} + 1\right)^{-1},
\end{aligned} \tag{70}$$

with $\gamma_{m_{j\nu}} \equiv \frac{\mu}{m_{j\nu}} \frac{dm_{j\nu}}{d\mu} = \frac{\beta_{m_{j\nu}}}{m_j}$ being an analogue of the beta function - the so called anomalous dimension - and $Y \equiv \sum_{ja\nu} \left|Y_{ja}\right|^2 = 2\sum_{ja} \left|Y_{ja}\right|^2$. As $D \to 4$, $\Delta_{\overline{MS}} \to \infty$ leaving:

$$\beta_m \overset{D=4}{=} -\frac{2m}{Y} \sum_{ja\nu} \left(\left(Y^*_{ja} \beta_{Y_{ja}} + Y_{ja} \beta_{Y^*_{ja}} \right) \left(\frac{m_{j\nu}}{m} + \frac{1}{2} \right) + \left|Y_{ja}\right|^2 \frac{m_{j\nu}}{m} \gamma_{m_{j\nu}} \right). \tag{71}$$

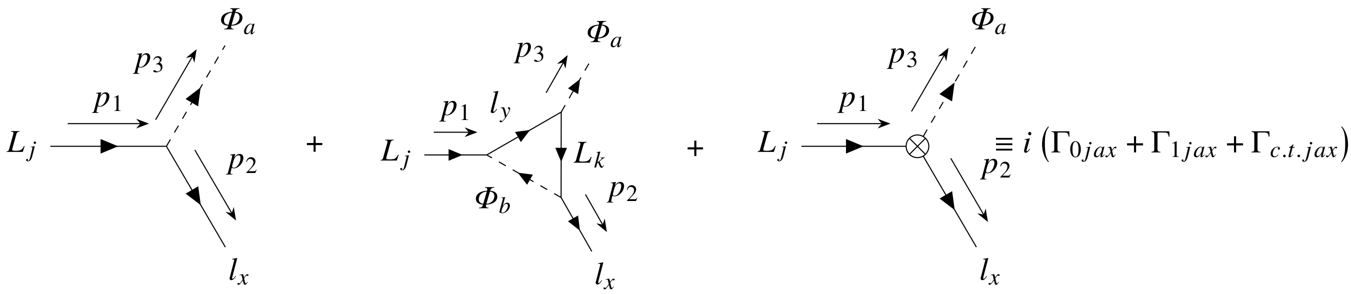
Eq. (71) relates the RG flow of the mass of the Majorana neutrino to the flows of leptonic Yukawa couplings and leptonic masses. Hence in the next subsection similar analysis is applied to the leptonic Yukawa couplings.

3.2 The renormalization procedure applied to leptonic Yukawa couplings

For the purposes of this thesis only leptonic part of the Yukawa sector will be considered. With this in mind, grouping the Majorana neutrino with the right handed lepton fields, dropping the unnecessary indices and, as in previous section, replacing the bare parameters with renormalized ones and the counterterms:

$$\mathcal{L}_Y = -Z_{L_j}^{\frac{1}{2}} Z_{\Phi_a}^{\frac{1}{2}} Z_{l_x}^{\frac{1}{2}} Z_{Y_{jax}} \overline{L}_j Y_{jax} \Phi_a l_x, \tag{72}$$

here $l_1,\ l_2,\ l_3$ are the 3 generations of right handed leptonic fields and l_4 stands for the majorana field Ψ. As in the previous section to track indices easier the indices $j,\ k$ will be used solely for left handed doublets, $a,\ b$ for Higgs doublets and $x,\ y \in \{1,2,3,4\}$ for the singlets.



4 Fig. Schematic diagrams of the tree level and the 1 particle irreducible (1PI) 1 loop corrections to Yukawa vertices plus the counterterm diagram. To avoid cumbersome notation the external propagators were not explicitly written out this time. The field counterterms in eq. (73) are responsible for renormalizing the propagators of external fields, so demanding $\Gamma_{1jax} + \Gamma_{c.t.jax}$ to be finite yields an equation for the counterterms of Yukawa couplings.

One way to look at the counterterms is as perturbative expansions, with δ_x being the second to leading order term, so one is justified in using $Z_x^{\frac{1}{2}} \approx 1 + \frac{\delta_x}{2}$ as terms of $O\left(\delta^2\right)$ are beyond 1 loop level. Hence keeping only the linear counterterm terms one arrives at:

$$\mathcal{L}_Y = -\overline{L}_j Y_{jax} \Phi_a l_x - \left(\frac{\delta_{Lj}}{2} + \frac{\delta_{\Phi a}}{2} + \frac{\delta_{lx}}{2} + \delta_{Yjax} \right) \overline{L}_j Y_{jax} \Phi_a l_x + h.c. + O\left(\delta^2\right). \tag{73}$$

On the other hand, the field counterterms remove the infinities arrising from corrections to field propagators, hence focusing only on the 1 particle irreducible (1PI - meaning those diagrams that cannot be split into 2 by cutting a single line) diagrams these terms can be dropped. All the 1PI diagrams allowed by the Lagrangian of the theory up to 1 loop level can be seen in fig. 4. Tracking the fermion line in reverse order one can contruct Γ_{1jax} using Feynman rules:

$$\begin{aligned} i\,(2\pi)^4\,\delta^4\,(p_1 - p_2 - p_3)\,\Gamma_{1jax} = \sum_{k=1}^{3}\sum_{b=1}^{2}\sum_{y=1}^{4}\sum_{v=1}^{2}\int\frac{d^4p'}{(2\pi)^4}\int\frac{d^4p''}{(2\pi)^4}\int\frac{d^4p'''}{(2\pi)^4}\,(-iY_{kbx})\,i\frac{\not{p}' + m_{kv}}{p'^2 - m_{kv}^2} \\ \times\left(-iY^*_{kay}\right) i\frac{\not{p}'' + m_y}{p''^2 - m_y^2}\left(-iY_{jby}\right)\frac{i}{p'''^2 - m_{bv}^2} \\ \times (2\pi)^4\,\delta^4\,(p' - p''' - p_2)\,(2\pi)^4\,\delta^4\,(p'' - p_3 - p')\,(2\pi)^4\,\delta^4\,(p_1 - p'' + p'' \end{aligned} \tag{74}$$

Note that here, as before, the momenta of the intermediate fields point in the same direction as the field lines. Carrying out the trivial integrals and dropping the $(2\pi)^4\,\delta^4\left(p_i - p_f\right)$ factors:

$$\begin{aligned} i\Gamma_{1jax} &= \sum_{kbyv} Y_{kbx}Y^*_{kay}Y_{jby}\int\frac{d^4p}{(2\pi)^4}\frac{\not{p} + m_{kv}}{p^2 - m_{kv}^2}\frac{\not{p}_3 + \not{p} + m_y}{(p_3 + p)^2 - m_y^2}\frac{1}{(p_3 - p_1 + p)^2 - m_{bv}^2} \\ &= \sum_{kbyv} Y_{kbx}Y^*_{kay}Y_{jby}\int\frac{d^4p}{(2\pi)^4}\left(f\left(p^2\right) + f\,(p) + f\,(C)\right). \end{aligned} \tag{75}$$

Making use of the identity $\not{p}^2 = p^2 I$ and tricks from the previous subsection (see eq. (55)) the integrals can be simplified:

$$\begin{aligned} f\left(p^2\right) &= \frac{p^2 - m_{kv}^2 + m_{kv}^2}{p^2 - m_{kv}^2}\frac{1}{(p_3 + p)^2 - m_y^2}\frac{1}{(p_3 - p_1 + p)^2 - m_{bv}^2} \\ &= \left(1 + \frac{m_{kv}^2}{p^2 - m_{kv}^2}\right)\frac{1}{(p_3 + p)^2 - m_y^2}\frac{1}{(p_3 - p_1 + p)^2 - m_{bv}^2} \end{aligned} \tag{76}$$

$$\begin{aligned} f\,(p) &= \frac{p_3^2}{p_3^2}\frac{\not{p}}{p^2 - m_x^2}\frac{1}{(p_3 + p)^2 - m_y^2}\frac{m_{kv} + \not{p}_3 + m_y}{(p_3 - p_1 + p)^2 - m_{bv}^2} \\ &= \frac{\not{p}_3}{2p_3^2}\frac{\left[(p_3 + p)^2 - m_y^2\right] - \left[p^2 - m_{kv}^2\right] - p_3^2 - m_{kv}^2 + m_y^2}{\left[p^2 - m_{kv}^2\right]\left[(p_3 + p)^2 - m_y^2\right]}\frac{m_x + \not{p}_3 + m_y}{(p_3 - p_1 + p)^2 - m_{bv}^2} \\ &= \frac{\not{p}_3}{2p_3^2}\frac{m_{kv} + \not{p}_3 + m_y}{(p_3 - p_1 + p)^2 - m_{bv}^2}\left(\frac{1}{p^2 - m_{kv}^2} - \frac{1}{(p_3 + p)^2 - m_y^2} + \frac{-p_3^2 - m_{kv}^2 + m_y^2}{\left[p^2 - m_{kv}^2\right]\left[(p_3 + p)^2 - m_y^2\right]}\right), \end{aligned} \tag{77}$$

$$f\,(C) = \frac{m_{kv}\left(\not{p}_3 + m_y\right)}{p^2 - m_{kv}^2}\frac{1}{(p_3 + p)^2 - m_y^2}\frac{1}{(p_3 - p_1 + p)^2 - m_{bv}^2}. \tag{78}$$

For terms with 2 propagators results from the previous subsection can be used by making a few adjustments where needed. On the other hand, the terms with three propagators and without any integration momenta factors in the numerator are not divergent anymore in $D = 4$ (to convince oneself, one can look up eq. 38), so they are irrelevant for further discussion, but for completeness sake the solution of the 3 propagator Feynman integrals is given in Appendix B.

With the 3 propagator terms dropped:

$$\int \frac{d^4p}{(2\pi)^4} f\left(p^2\right) = \frac{i}{(4\pi)^2}\left[\Delta_{\overline{MS}} - \int_0^1 dx \ln\left(\frac{Q_1}{\mu^2}\right)\right] + O(\epsilon), \tag{79}$$

$$\begin{aligned}\int \frac{d^4p}{(2\pi)^4} f(p) &= \frac{\not{p_3}}{2p_3^2}\left(m_{kv} + \not{p_3} + m_y\right) \frac{i}{(4\pi)^2}\left[\Delta_{\overline{MS}} - \int_0^1 dx \ln\left(\frac{Q_2}{\mu^2}\right) - \Delta_{\overline{MS}} + \int_0^1 dx \ln\left(\frac{Q_3}{\mu^2}\right)\right] + O(\epsilon) \\ &= \left(\frac{\not{p_3}\left(m_{kv} + m_y\right)}{2p_3^2} + \frac{1}{2}\right) \frac{i}{(4\pi)^2} \int_0^1 dx \ln\left(\frac{Q_3}{Q_2}\right) + O(\epsilon),\end{aligned} \tag{80}$$

here

$$\begin{aligned} Q_1 = Q_3 &= (1-x)\, m_y^2 + x\left(m_{bv}^2 + p_1^2\right) + p_1^2 x^2, \\ Q_2 &= (1-x)\, m_{kv}^2 + x\left(m_{bv}^2 - (p_3 - p_1)^2\right) + (p_3 - p_1)^2 x^2 \end{aligned} \tag{81}$$

are the constant remains left after assembling full square in the denominator in the same way as in previous subsection. Given that there is only one contribution diverging at $D = 4$ and demanding $\Gamma_{1jax} + \Gamma_{c.t.jax}$ to be finite, one can finally express the counterterms of the Yukawa couplings:

$$\delta_{Yjax} = \sum_{kbyv} Y_{kbx} Y^*_{kay} Y_{jby} \frac{\Delta_{\overline{MS}}}{(4\pi)^2} = 2\sum_{kby} Y_{kbx} Y^*_{kay} Y_{jby} \frac{\Delta_{\overline{MS}}}{(4\pi)^2}, \tag{82}$$

The complex conjugated coupling counterterm can be computed either by retracing all the steps for Feynman diagrams with field lines pointing in the opposite direction or alternatively by taking complex conjugate of eq. (82), which yields:

$$\delta_{Y^*jax} = 2\sum_{kby} Y^*_{kbx} Y_{kay} Y^*_{jby} \frac{\Delta_{\overline{MS}}}{(4\pi)^2}. \tag{83}$$

With the counterterms expressed one can finally perform the RG analysis. The RGEs for Yukawa couplings stem from the same consideration of bare parameter independence from the scale parameter μ:

$$0 = \mu \frac{d}{d\mu} Y_{Bjax} = Y_{jax} \sum_{kby}\left(\frac{d\delta_{Yjax}}{dY_{kbx}} \beta_{Y_{kbx}} + \frac{d\delta_{Yjax}}{dY^*_{kay}} \beta_{Y^*_{kay}} + \frac{d\delta_{Yjax}}{dY_{jby}} \beta_{Y_{jby}}\right) + Z_{Yjax} \beta_{Yjax}. \tag{84}$$

Here the sum over fields has again been moved in front of $\delta_{Y_{jax}}$. Denoting $S_{jax} \equiv \sum\limits_{kby} Y_{kbx} Y^*_{kay} Y_{jby}$:

$$\beta_{Y jax} = -\frac{Y_{jax}}{S_{jax}} \sum_{kby} \left(Y^*_{kay} Y_{jby} \beta_{Y kbx} + Y_{kbx} Y_{jby} \beta_{Y^* kay} + Y_{kbx} Y^*_{kay} \beta_{Y jby} \right) \left(1 + \left(\sum_{kby} \delta_{Y jax} \right)^{-1} \right)^{-1} \tag{85}$$

and letting $D \to 4$ and hence $\delta_{Y jax} \to \infty$:

$$\beta_{Y jax} \overset{D=4}{=} -\frac{Y_{jax}}{S_{jax}} \sum_{kby} \left(Y^*_{kay} Y_{jby} \beta_{Y kbx} + Y_{kbx} Y_{jby} \beta_{Y^* kay} + Y_{kbx} Y^*_{kay} \beta_{Y jby} \right) . \tag{86}$$

The equation for complex conjugated piece is again trivially a complex conjugate of eq. (86). These sets of coupled linear differential equations describe the interconnected flow of leptonic Yukawa couplings. Apart from the trivial solution $\beta_{Y xyz} = 0$ and $\beta_{Y^* xyz} = 0$ for all x, y and z, more fixed point solutions are possible but impractical to solve for analitically.

Results and conclusions

- Using dimensional regularization Majorana neutrino field, mass and leptonic Yukawa couplings have been renormalized at one loop order. The revelevant counterterms can be found in eqs. (66, 67, 82, 83). Additionally, the Beta functions for the mass and couplings have been computed, see eqs. (71, 86). Looking for non-trivial fixed point solutions of these coupled sets of linear differential equations analitically is impractical.
- The methodology of dimensional regularization indeed works flawlessly for regularizing UV divergences in Feynman integrals and multiplicative counterterm method for formulating counterterms indeed yields simple algebraic equations for them. It was found that Passarino-Veltman type tricks for reducing tensor integrals into scalar and sometimes lower order Feynman integrals works faster than applying Feynman parametrization directly.
- For a full description and numerical analysis of the RG flow of the GNM specific parameters this work should be expanded to include at least whole EW sector renormalization and some estimates for Yukawa couplings involving second Higgs doublet.

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Rokas Garbačauskas

RYŠYS TARP RENORMALIZACIJOS SRAUTO, RENORMALIZACIJOS KONSTANTŲ IR SĄLYGŲ. GRIMUS NEUFELD MODELIO RENORMALIZACIJOS KONSTANTOS VIENOS KILPOS LYGMENY

Santrauka

Kvantinio Lauko Teorijos (angl. QFTs) ir jose plėtojami matematiniai metodai yra viena abstrakčiausių fizikos šakų. Tai kelia nemažai iššūkių studentams bandantiems įsigilinti į šią sritį. Bet pastangos atsiperka. Viena vertus, šių teorijų formalizmas yra intelektualiai elegantiškas. Kita vertus, matematiniai metodai, naudojami šiose srityse, leidžia gauti vienas tiksliausių teorinių verčių įvairiems parametrams. Standartinis pavyzdys yra elektrono anomalus magnetinis momentas, kurio vertės gautos iš teorinių skaičiavimų pasinaudojant Kvantinės Elektrodinamikos (angl. QED) formalizmu ir iš atliktų eksperimentų sutampa iki 11 skaitmens po kablelio ([1]). Dėl šitokio metodų tikslumo 3,5 sigmų nesutikimas tarp teorinės ir eksperimentinės anomalaus magnetinio momento verčių miuono atveju ([2]) yra laikomas stipriu įrodymu, kad Standartinis Modelis (SM) - teorija inkorporuojanti QED ir aprašanti subatominių dalelių dinamiką - dar turi būti praplėstas. Tarp empirinių faktų, kuriuos yra bandoma paaiškinti SM plėtiniais, yra ir daugybė eksperimentų, tyrinėjančių neutrinų osciliacijas ([3]). Šiam reiškiniui vykti reikalingos nenykstančios neutrinų masės. Tuo tarpu SM dėl dešiniarankių neutrinų neegzistavimo šioms dalelėms neįmanoma suformuluti masę aprašančių narių.

Vienas iš pasiūlymų, kaip SM galėtų būti minimaliai praplėstas, yra Grimus-Neufeld modelis (GNM, [4]), kuriame prie SM esančių dalelių pridedamas dar vienas papildomas Higgs'o dubletas bei vienas sunkus Majorana neutrinas. Šie priedai sverto mechanizmu bei per pirmos kilpos lygmens pataisas aprašo 2 iš trijų neutrinų masių. Dviejų ir aukštesnių kilpų pataisos leidžia aprašyti visas neutrinų mases.

Viena specifinė QFTs tyrinėjimų sritis yra Renormalizacijos Grupė (RG). RG lygtys aprašo kaip specifinis modelis kinta keičiantis mąsteliui. Iš pradžių renormalizacija QFTs buvo suprantama kaip parametrų reformulavimas teorijos aprašyme siekiant išvengti įvairių diverguojančių integralų indėlio į fizinius parametrus. Šie UV ir/ar IR diverguojantys integralai yra sutinkami Feynman'o formalizme dėl begalinių integravimo rėžių bei masės neturinčių dalelių, kai bandomos įskaičiuoti aukštesnės nei kamieno lygmens pataisos. Šie metodai yra nemažas pasiekimas, bet ne ką mažiau svarbus atradimas yra minėtos RG kygtys, kurios aprašo, kaip specifinė teorija elgiasi skirtinguose mąsteliuose. Taip pat ji susieja šiuos teorijų skirtinguose mąsteliuose aprašymus. Tai sudaro mūsų supratimo apie kintančias sąveikos konstantas pagrindą bei leidžia suprasti visų jėgų, išskyrus (bent jau kol kas) gravitaciją, prigimtį.

Šiame darbe aprašyta RG srauto analizė svarbiausiems GNM modelio parametrams. Trečioje dalyje Majorana neutrino laukui bei masei ir leptoninėnims Yukawa sąveikos konstantoms izoliavus diverguojančius narius iš atitinkamų Feynmano integralų suformuluoti priešiniai nariai (angl. counterterms) bei aprašyti šių parametrų RG srautai. Tuo tarpu pirmame skyrelyje nuosekliai pristatytas GNM, o sekančiame su pavyzdžiais aprašyti naudoti metodai. Gauti minėtų priešinių narių analizines išraiškas bei aprašyti tų parametrų RG srautą ir buvo pagrindinis šio darbo tikslas.

Appendix A

Here the integral from subsection 3.1 eq. (63) is solved analitically.

$$I = \int_0^1 dx \ln\left(xm_k^2 + (1-x)m_j^2 - x(1-x)p^2\right) = \int_0^1 dx \ln\left(ax^2 + 2bx + c\right), \tag{87}$$

with $a = p^2$, $b = \frac{1}{2}\left(m_k^2 - m_j^2 - p^2\right)$, $c = m_j^2$. Partial integration yields:

$$\begin{aligned} I &= \int_0^1 dx \ln\left(ax^2 + 2bx + c\right) = x \ln\left(ax^2 + 2bx + c\right)\Big|_0^1 - \int_0^1 d\left(\ln\left(ax^2 + 2bx + c\right)\right) x \\ &= \ln\left(a + 2b + c\right) - \int_0^1 dx \frac{x\left(2ax + 2b\right)}{ax^2 + 2bx + c} = \ln\left(a + 2b + c\right) - \int_0^1 dx \frac{2\left(ax^2 + 2bx + c\right) - 2bx - 2c}{ax^2 + 2bx + c} \end{aligned} \tag{88}$$

$$= \ln\left(a + 2b + c\right) - 2 + 2\int_0^1 dx \frac{bx + c}{ax^2 + 2bx + c} = \ln\left(a + 2b + c\right) - 2 + 2I_p. \tag{89}$$

Second order polynomials with complex coefficients have 2 different roots when $D \equiv \sqrt{b^2 - ac} \neq 0$ and 2 identical roots when $D = 0$. In either case the denominator can be rewritten in terms of these roots:

$$\begin{aligned} I_p &= \int_0^1 dx \frac{bx + c}{a(x + \frac{b+D}{a})(x + \frac{b-D}{a})} = \frac{1}{a}\int_0^1 dx \left(\frac{\frac{b+D}{2}}{(x + \frac{b+D}{a})} + \frac{\frac{b-D}{2}}{(x + \frac{b-D}{a})}\right) \\ &= \frac{1}{a}\left(\frac{b+D}{2}\ln\left(x + \frac{b+D}{a}\right) + \frac{b-D}{2}\ln\left(x + \frac{b-D}{a}\right)\right)\Bigg|_0^1 \\ &= \frac{b+D}{2a}\ln\left(\frac{a}{b+D} + 1\right) + \frac{b-D}{2a}\ln\left(\frac{a}{b-D} + 1\right). \end{aligned} \tag{90}$$

Plugging this result back to eq. (88) but grouping some terms under $D = \sqrt{\frac{1}{4}\left(m_k^2 - m_j^2 - p^2\right)^2 - p^2 m_j^2}$, $A = b + D$ and $B = b - D$:

$$I = \ln\left(m_k^2\right) - 2 + \frac{A}{p^2}\ln\left(\frac{p^2}{A} + 1\right) + \frac{B}{p^2}\ln\left(\frac{p^2}{B} + 1\right). \tag{91}$$

Appendix B

Here regularization techniques are applied to solve the convergent integrals from subsection 3.2:

$$I = C\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2-m_x^2}\frac{1}{(p_3+p)^2-m_z^2}\frac{1}{(p_3-p_1+p)^2-m_y^2}. \tag{92}$$

By making use of a trick with Feynman variables this can be rewritten as:

$$\begin{aligned} I = C\int \frac{d^4p}{(2\pi)^4}\int_0^1\int_0^1\int_0^1 dvdudn\delta\left(v+u+n-1\right) \\ \times \frac{2}{\left(\left[p^2-m_x^2\right]v+\left[(p_3+p)^2-m_z^2\right]u+\left[(p_3-p_1+p)^2-m_y^2\right]n\right)^3}. \end{aligned} \tag{93}$$

Let $\int df \equiv 2\int_0^1\int_0^1\int_0^1 dvdudn\delta\left(v+u+n-1\right)$ and assembling full integration momentum square in the denominator D:

$$D = \left((p+p_3\left(u+n\right)-p_1n)^2-\left(p_3\left(u+n\right)-p_1n\right)^2-m_x^2v+\left(p_3^2-m_z^2\right)u+\left((p_1-p_3)^2-m_y^2\right)n\right)^3. \tag{94}$$

With an integration variable shift $p+p_3\left(u+n\right)-p_1j \longrightarrow p$, and defining $Q=\left(p_3\left(u+n\right)-p_1n\right)^2+m_x^2v-\left(p_3^2-m_z^2\right)u-\left((p_1-p_3)^2-m_y^2\right)n$, we are in a position to Wick rotate:

$$I \longrightarrow -iC\int df\int \frac{d^4p}{(2\pi)^4}\frac{1}{\left(p^2+Q\right)^3}. \tag{95}$$

Making use of the spherical symmetry and the same beta and gamma function properties:

$$I = -iC\pi^2\int dfQ^{-1}\int \frac{d\left(\frac{p^2}{Q}\right)}{(2\pi)^4}\frac{\frac{p^2}{Q}}{\left(\frac{p^2}{Q}+1\right)^3} = -iC\frac{1}{2}\frac{1}{(4\pi)^2}\int dfQ^{-1}. \tag{96}$$

The remaining integral over Feynman variables is convergent, but involves many algebraic manipulations to solve and isn't very enlightening.