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METHODS OF CALCULATING THE PHYSICAL EVOLUTION IN THE EARLY UNIVERSE

Mokslo tiriamasis darbas

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1 Introduction

In our universe, baryonic (visible) matter only comprises $\sim 4\%$ of the universe's energy content, while around 70% is attributed to the unknown dark energy [1, 2]. The rest is considered to be very weakly interacting particles, mostly interacting with matter through gravity and not interacting electromagnetically, known as dark matter. However, there are possible candidate particles, like sterile neutrinos, that could be dark matter.

In the Standard Model of particle physics, neutrinos do not have a mass term. That is in contradiction with experiments, where, by observing neutrino oscillations, it was shown that neutrinos do have mass [3]. In the Standard Model, particles acquire mass through the interaction with the Higgs field with their right-(left-)handed counterparts. However, in the Standard Model, the neutrinos are only left-handed. Which means that there should be right-handed neutrinos.

These right-handed or sterile neutrinos are weakly interacting - they only allow left-handed neutrinos (the ones we know) to have mass. One possible extension to the Standard Model, called the Grimus-Neufeld model not only adds a heavy sterile neutrino, it also states that neutrinos are Majorana neutrinos and also adds a second Higgs doublet.

Because the sterile neutrinos that come from the Grimus-Neufeld model are weakly interacting, mainly through the Higgs field and gravity, one can say that these particles could be dark matter. This is because we assume them to be heavy - heavier than the left-handed neutrinos and maybe heavier than other particles. And of course because they are so weakly interacting.

In this work, we look over the possibility of sterile neutrinos being dark matter. We begin by presenting the basics of cosmology. Namely by showing how the Boltzmann and Einstein's equations are used to calculate the evolution of all particle densities, including dark matter. This is done in the case of non-thermal equilibrium.

We then look over possible sterile neutrino generation methods: the Dodelson-Widrow or the thermal model [4] and the heavy scalar decay model [5]. For both of these models we derive the neutrino number density n_S formulas. In the heavy scalar decay, we derive two formulas for two periods of sterile neutrino production - early scalar production/decay and late scalar decay.

Using these formulas, we calculate the density parameter of sterile neutrinos Ω_S for both cases and a third combined case and show at what sterile neutrino mass does the sterile neutrino density parameter resemble that of the dark matter density parameter. For the heavy scalar decay we also look at how the additional parameters that come from this model, like the coupling between the scalar and neutrino fields and the scalar mass affect the mass of the sterile neutrinos.

Finally, we look at the results and the possible implications of these models. We compare the two models between themselves. And lastly, we list the possible shortcomings of the models.

2 Theory

To understand how and why the sterile neutrino particle could be a dark matter candidate, one must have an understanding of Einstein's relativity theory and of quantum field theory (QFT). For readers, who have no knowledge on these subjects, I refer you to appendices A and B, which have short introductions into the basics of the theories. For those, who are interested in a deeper dive of the subjects, I refer you to the great books in refs. [1, 6–9].

2.1 On sterile neutrinos

Before we begin talking about the sterile neutrinos, we need to talk about the neutrinos that we already know of. In the Standard Model (SM), the neutrinos are massless. This comes from the fact that in the weak interactions, we only have left-handed neutrinos. However, this „masslessness“ is in contradiction with experimental results, which state that neutrinos oscillate, which means that they should have a mass [3].

What this means in terms of QFT is that we should have not only left-handed neutrinos, but also right-handed neutrinos. This is apparent when one considers the construction of the mass term in a Lagrangian, which is shown here

$$\mathcal{L}_{mass} = \mu H \bar{l}_L l_R, \tag{1}$$

where μ is the Dirac mass (note, this is not the left-handed neutrino mass; the Dirac mass is related to it via the seesaw mechanism $m_\nu = \frac{\mu^2}{M}$ [10], where M is the Majorana mass), H is the Higgs doublet and $l_{L,R}$ is the left- or right-handed lepton. As was noted, the SM only has left-handed neutrinos - there are no interactions in the SM, which include the right-handed neutrinos. And since the SM is the most accurate particle physics model to date, this means that we need an extension to the SM, which does not contradict it.

One of these extension is the Grimus-Neufeld model (GNM), which adds a heavier Majorana neutrino and a second Higgs doublet. The added Majorana neutrino is right-handed. But since it doesn't appear in the SM, that means that the added right-handed neutrino should not interact in any way with other particles, except through the Higgs field and gravitationally (this is not the case if we assume more particles, with which the sterile neutrino can interact, like in the heavy scalar decay picture). And this is where we get the name sterile neutrinos. And these sterile neutrinos, as was noted, are assumed to be massive, many times more massive than the other neutrinos and maybe other particles as well.

If we briefly take a look at dark matter particles, we note that they are massive weakly interacting particles, i.e. they interact through gravity and nothing else that we know of. And as we know, sterile neutrinos are weakly interacting and are massive. Thus the GNM sterile neutrinos could be dark matter particles.

2.2 Boltzmann and Einstein equations in the early universe

But before we can go and start working on the sterile neutrinos and their possible connection to dark matter, one must understand the background in which we are working. As one knows, the particles we know today were all created during the early universe period, right after the Big Bang, via a process $2\gamma \to particle + antiparticle$. This includes dark matter particles. So if we want to talk about sterile neutrinos being a possible candidate for dark matter particles, we need to talk about this period.

However, there were many particles created in the early universe. Too many to write equations of motion for each and everyone of them. Because of this, we use the methods from statistical physics and our parameters usually are the distribution functions f and temperature T among other things like mass of the particles. Thus, we require the Boltzmann equation for these distribution functions, since it allows us to model their evolution and in turn how particles are distributed.

But particles have mass and that mass generates gravitational fields. And particles are distributed in various ways, so the generated gravitational fields will be different throught all of spacetime. To account for this, we use a perturbed metric

$$\begin{aligned} g_{00} &= -1 - 2\Psi \\ g_{0i} &= 0 \\ g_{ij} &= a^2\delta_{ij}\left(1 + 2\Phi\right), \end{aligned} \tag{2}$$

where a is the scale factor; Ψ corresponds to the perturbation to the spatial curvature and Φ correspons to the Newtonian potential. And these potentials also change over time, so we will have to use the Einstein's equations for them aswell. This metric is a perturbation of the Friedmann-Robertson-Walker (FRW) metric, which is just

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix}. \tag{3}$$

This metric accounts for the simple flat spacetime expansion. An additional reason, why we use the perturbed metric is the fact, that we assume the universe not to be thermally at equilibrium.

In the following sections, we will show the basics for early universe calculations, by calculating the densities of various particles, including cold dark matter (not sterile neutrinos) and how their distribution functions affect the perturbations of the metric. These calculations will be taken from the Dodelson's book on cosmology *Modern Cosmology* [2]. The main bulk of these sections is to give a background of understanding for the later calculations, although most of what is written here, will not be used in the main calculations of this work.

2.2.1 Boltzmann equations

We turn to calculate the densities of particles in the early universe. Specifically we will be calculating the densities of photons, cold dark matter and baryons (electrons, protons and so on; this might seem contradictory as electrons are not baryons, but it is just common to lump the two groups together and name them baryons). To do that, we will use the Boltzman equation. It can be shown schematically as

$$\frac{df}{dt} = C[f], \tag{4}$$

where f, is the density distribution function, which depends on coordinates x and momenta P[1]. The term $C\,[f]$ corresponds to all the collision terms.

For photons

Colisionless Boltzmann equation

To begin the calculations we must first calculate the collisionless Boltzman equation. Then we can calculate the collision terms and combine the two parts in to one simple equation.

So let's begin by defining the momentum vector[2]

$$P^{\mu} = \frac{dx^{\mu}}{d\lambda}, \tag{5}$$

where λ parametrizes the particle's path. For a massless photon, the scalar product of this momentum is

$$P^2 = g_{\mu\nu} P^{\mu} P^{\nu} = 0, \tag{6}$$

where $g_{\mu\nu}$ is the perturbed metric (2). We can expand this scalar product

$$P^2 = -\left(1 + 2\Psi\right)\left(P^0\right)^2 + p^2 = 0, \tag{7}$$

where $p^2 = g_{ij} P^i P^j$. Thus using this equation, we can write down P^0

$$P^0 = \frac{p}{\sqrt{1 + 2\Psi}} = p\left(1 - \Psi\right). \tag{8}$$

In the last equality we used the first-order perturbation theory.

Eq. (8) allows us to eliminate P^0 in favor of p. Having that, we can write down the collisionless Boltzman equation, where we expand the time derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \cdot \frac{dx^i}{dt} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i} \cdot \frac{d\hat{p}^i}{dt}, \tag{9}$$

where $\hat{p}^i$ is the unit vector and it stands for the direction vector of momentum. The last term in eq.

[1] Here x, P stands for x^{μ}, P^{μ}, all spatial coordinates, the time coordinate, the energy and the momentum.

[2] Note, that here it is not the derivative of a coordinate to gives us velocity, but a derivative of the parametrized path to gives us momentum of massless photons (if we had mass the momentum would be zero!).

(9) is a second order term. To see that recall that the Bose-Einstein distribution does not depend on the direction of momentum. Thus the only way that $\frac{\partial f}{\partial \hat{p}^i}$ is non-zero is if it's first order. But so is the term that it multiplies, because it depends on the small potentiantls from the perturbed metric. Thus the whole term is of second-order which means we can neglect it.

Let's focus on the second term on the right-hand side of the eq.(9). We can rewrite it, using our definition of momentum (5)

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{P^i}{P^0}. \tag{10}$$

Since P^i is proportional to the unit vector $\hat{p}^i$, we can write the proportionality using a constant C

$$P^i = \hat{p}^i C. \tag{11}$$

Now using the scalar product of the spatial momentum $p^2 = g_{ij}\hat{p}^i\hat{p}^j C^2$,we can write down that the constant $C = \frac{p(1-\Phi)}{a}$ and with it we can rewrite eq.(10) using eq.(11)

$$\frac{dx^i}{dt} = \frac{\hat{p}^i}{a}\left(1 + \Psi - \Phi\right). \tag{12}$$

Note, that in an overdense region, where $\Psi < 0$ and $\Phi > 0$,the terms in parentheses is less than one. This means that the photon slows down when traveling through an overdense region. However, in eq.(9) the fraction that multiplies (12) is first-order term. That means that we can neglect the gravitational potentials and the second term would just be

$$\frac{\hat{p}^i}{a}\cdot\frac{\partial f}{\partial x^i}. \tag{13}$$

The last term in eq. (9) is a bit more difficult. For it, we need to look at the time component of the geodesic equation

$$\frac{dP^0}{d\lambda} = -\Gamma^0{}_{\alpha\beta}P^\alpha P^\beta. \tag{14}$$

After making quite a handful of manipulations[3] we arrive at the final result

$$\frac{1}{p}\frac{dp}{dt} = -H - \frac{\partial\Phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}. \tag{15}$$

With this result we can now rewrite eq.(9)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\cdot\frac{\partial f}{\partial x^i} + p\frac{\partial f}{\partial p}\left(-H - \frac{\partial\Phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right). \tag{16}$$

Zero and first order equations for the Bose-Einstein distribution

This section will be short, because the only thing that is necessary is the definitions and some results.

[3] Here I refer the reader the book by Dodelson [2], where these calculations are made and their results are taken here to be true.

To begin, let us define the Bose-Einstein distribution that depends on the unit vector $\hat{p}^i$

$$f\left(x,p,\hat{p}\right)=\left(\exp\left\{\frac{p}{T\left(t\right)\left[1+\Theta\left(x,\hat{p}\right)\right]}\right\}-1\right)^{-1}. \tag{17}$$

Here, the temperature T depends only on time and is a zero-order term. The term Θ is a perturbation to the distribution, which describes the inhomogeneities and anisotropies in the photon distribution. This perturbation is small, so if we expand around it, we can write

$$f=f^{(0)}-p\frac{\partial f^{(0)}}{\partial p}\Theta, \tag{18}$$

where $f^{(0)}$ is the zero-order distribution

$$f^{(0)}=\left(\exp\left\{\frac{p}{T}\right\}-1\right)^{-1}. \tag{19}$$

For the zero-order equation, one can insert the zero-order distribution (19) into the Boltzman equation (16) and would get the result

$$\left(-\frac{dT/dt}{T}-\frac{da/dt}{a}\right)\frac{\partial f^{(0)}}{\partial p}=0. \tag{20}$$

The first-order equation now requires us to insert the full expansion (18) into the Boltzman equation (16). The result, one would get when doing so is

$$\frac{df}{dt}=-p\frac{\partial f^{(0)}}{\partial p}\left(\frac{\partial\Theta}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}+\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right). \tag{21}$$

Collision terms

Now that we have a collisionless Boltzmann equation for photons and having applied it to the Bose-Einstein distribution, we must consider more of reality. In the cosmic soup, there aren't just photons in it. There are also other particles. And photons can scatter from them, specifically from electrons in what is called the Compton Scattering. This would show in our Boltzman equation as collision terms.

We consider a scattering process where

$$e^-\left(q\right)+\gamma\left(p\right)\leftrightarrow e^-\left(q'\right)+\gamma\left(p'\right), \tag{22}$$

where each particle's momentum is indicated explicitly. The collision terms can be written explicitly for this process as

$$C\left[f\right]=\frac{1}{p}\int\frac{d^{3}q}{\left(2\pi\right)^{3}2E_{e}\left(q\right)}\int\frac{d^{3}q'}{\left(2\pi\right)^{3}2E_{e}\left(q'\right)}\int\frac{d^{3}p'}{\left(2\pi\right)^{3}2E_{e}\left(p'\right)}\left|M\right|^{2}\left(2\pi\right)^{4}$$
$$\times\delta^{3}\left(\overrightarrow{p}+\overrightarrow{q}-\overrightarrow{p}'-\overrightarrow{q}'\right)\delta\left(E\left(p\right)+E_{e}\left(q\right)-E\left(p'\right)-E_{e}\left(q'\right)\right)$$
$$\times\left(f_{e}\left(\overrightarrow{q}'\right)f\left(\overrightarrow{p}'\right)-f_{e}\left(\overrightarrow{q}\right)f\left(\overrightarrow{p}\right)\right).\tag{23}$$

As we can see, this equation is quite messy. The rewritten eq. (23), after many substitutions and algebraic actions [2] is

$$C\left[f\right]=-p\frac{\partial f^{(0)}}{\partial p}n_{e}\sigma_{T}\left(\Theta_{0}-\Theta+\hat{p}\cdot\overrightarrow{v_{b}}\right),\tag{24}$$

where n_e is the electron density, σ_T is the Thomsom cross–section and $v_b$is the baryonic velocity. The other parameter Θ_0 is

$$\Theta_{0}\left(x\right)=\frac{1}{4\pi}\int d\Omega\Theta\left(\hat{p},x\right),\tag{25}$$

which in simple words describes the deviation of the monopole at a given point in space form its average in all space.

Having the collision terms, we can write down the full Boltzmann equations for photons. To do so, we equate the two equations (16) and (24)

$$\frac{\partial\Theta}{\partial t}+\frac{\hat{p}^{i}}{a}\frac{\partial\Theta}{\partial x^{i}}+\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^{i}}{a}\frac{\partial\Psi}{\partial x^{i}}=n_{e}\sigma_{T}\left(\Theta_{0}-\Theta+\hat{p}\cdot\overrightarrow{v_{b}}\right),\tag{26}$$

which can be rewritten in terms of conformal time $\eta=\int_0^t\frac{dt'}{a(t')}$

$$\dot{\Theta}+\hat{p}^{i}\frac{\partial\Theta}{\partial x^{i}}+\dot{\Phi}+\hat{p}^{i}\frac{\partial\Psi}{\partial x^{i}}=n_{e}\sigma_{T}a\left(\Theta_{0}-\Theta+\hat{p}\cdot\overrightarrow{v_{b}}\right),\tag{27}$$

where the dots represent the derivatives with respect to conformal time.

Let's introduce more substitutions. Firstly, we introduce the Fourier transform, which can be written simply as

$$\Theta\left(\overrightarrow{x}\right)=\int\frac{d^{3}k}{\left(2\pi\right)^{3}}e^{i\overrightarrow{k}\cdot\overrightarrow{x}}\tilde{\Theta}\left(\overrightarrow{k}\right).\tag{28}$$

Let's also introduce the angle between the wavenumber $\overrightarrow{k}$ and the photon direction $\hat{p}$

$$\mu=\frac{\overrightarrow{k}\cdot\hat{p}}{\sqrt{k^{i}k^{i}}}.\tag{29}$$

And now let's define the optical depth

$$\tau\left(\eta\right)=\int_{\eta}^{\eta_{0}}d\eta'n_{e}\sigma_{T}a.\tag{30}$$

so that

$$\dot{\tau}=-n_{e}\sigma_{T}a.\tag{31}$$

With these definitions we have the Boltzmann equation

$$\dot{\tilde{\Theta}} + \mathrm{i}k\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + \mathrm{i}k\mu\tilde{\Psi} = -\dot{\tau}\left(\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b\right). \tag{32}$$

For cold dark matter

Having finished working with photons we can take a look at the most domimant matter - dark matter. First, let's note that at the temperatures of our desired study, dark matter generation is over and cannot begin, since the energy needed is lower than the mass of the cold dark matter. That is why we have cold dark matter.

Secondly, dark matter behaves like a fluid and so can be completely described by the energy-momentum tensor. Yet we will go the route of Boltzmann equation. This will help us when we have to work with Bolztmann equations for baryons.

Thirdly, we must note, that unlike photons or neutrinos, dark matter is nonrelativistic and it does not interact with any other constituents in the universe (except gravitationally but that effect appears in Einstein's equations), so there will be no collision terms in the Boltzmann equation. This is not exactly true, since we only know that dark matter does not interact electromagnetically and interacts with other particles via gravity, so we cannot assume any other interactions. However, since we don't know of any other possible interactions, we will go the simplest route and assume no collision terms.

With that out of the way, we can begin working on the Bolztmann equation. First, we rewrite the four-momentum scalar product for particles with mass

$$g_{\mu\nu}P^\mu P^\nu = -m^2, \tag{33}$$

where m is the mass of the dark matter particle. Also, we can define the energy

$$E = \sqrt{p^2 + m^2}. \tag{34}$$

We will use energy as one of the variables in the Boltzmann equation. So now we can write down the four-momentum of a massive particle

$$P^\mu = \left(E\left(1 - \Psi\right), p\hat{p}^i \frac{1 - \Phi}{a}\right). \tag{35}$$

Having all of these components we can now expand the total time derivative (4) for the dark matter distribution function f_{dm}

$$\frac{df_{dm}}{dt} = \frac{\partial f_{dm}}{\partial t} + \frac{\partial f_{dm}}{\partial x^i} \cdot \frac{dx^i}{dt} + \frac{\partial f_{dm}}{\partial E} \cdot \frac{dE}{dt} + \frac{\partial f_{dm}}{\partial \hat{p}^i} \cdot \frac{d\hat{p}^i}{dt}. \tag{36}$$

The equation resembles that of (9) just with one different variable. As it was before, the last term vanishes since it is a second-order term. If we work through the algebra, which is much the same as it was in the photon case our Boltzman equation (36) changes to

$$\frac{\partial f_{dm}}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f_{dm}}{\partial x^i} - \frac{\partial f_{dm}}{\partial E}\left(\frac{da/dt}{a}\frac{p^2}{E} + \frac{p^2}{E}\frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i p}{a}\frac{\partial \Psi}{\partial x^i}\right) = 0. \tag{37}$$

In the massless case this equation reduces to eq. (16).

When we dealt with photons we used the knowledge of their distribution function and perturbed it. However, for cold dark matter we don't need to know it. We can neglect the thermal motion of it, since it is nonrelativistic, thus the effects would be negligable. So to proceed further we will take moments of eq. (37).

We start by multiplying the eq. (37) with phase space volume $\frac{d^3p}{(2\pi)^3}$ and integrate it. We now need to introduce some definitions. First of which is the dark matter density

$$n_{dm} = \int \frac{d^3p}{(2\pi)^3} f_{dm}. \tag{38}$$

Another is the velocity

$$v_i = \frac{1}{n_{dm}} \int \frac{d^3p}{(2\pi)^3} f_{dm} \frac{p\hat{p}^i}{E}. \tag{39}$$

Then we calculate the integral

$$\begin{aligned} \int \frac{d^3p}{(2\pi)^3} p \frac{\partial f_{dm}}{\partial p} &= \frac{4\pi}{(2\pi)^3} \int_0^\infty dp p^3 \frac{\partial f_{dm}}{\partial p} \\ &= -3n_{dm}. \end{aligned}$$

Using all these definitions and the fact that $\frac{dE}{dp} = \frac{p}{E}$, and putting them in the eq. (37) we get the zeroth moment of the Bolztmann equation

$$\frac{\partial n_{dm}}{\partial t} + \frac{1}{a}\frac{\partial \left(n_{dm} v^i\right)}{\partial x^i} + 3\left(\frac{da/dt}{a} + \frac{\partial \Phi}{\partial t}\right) n_{dm} = 0. \tag{40}$$

Now let us write down the zero-order and first order terms. Since velocity and the metric perturbations are first-order, then we get from eq. (40) only the zero-order terms

$$\frac{\partial n_{dm}^{(0)}}{\partial t} + 3\frac{da/dt}{a} n_{dm}^{(0)} = 0, \tag{41}$$

where $n_{dm}^{(0)}$ is the zero-order, homogeneous part of the dark matter density. However, the full dark matter density is

$$n_{dm} = n_{dm}^{(0)} \left(1 + \delta\left(x\right)\right), \tag{42}$$

where $n_{dm}^{(0)} \delta\left(x\right)$ is a first-order term and δ is the fractional overdensity $\frac{\delta\rho}{\rho}$. If we now introduce this density to eq. (40), we will get the first-order equation

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0. \tag{43}$$

However this is not a complete set of functions because we have another variable that we did not calculate. That is the velocity v^i. We have it's definition from eq. (39), but we need to know how it evolves over time. For this, we must return to the eq. (37) and take it's first moment. To do this, we multiply the eq. (37) by $\frac{d^3 p \hat{p}^j}{(2\pi)^3} \frac{p}{E}$ and then integrate it. We get the first moment equation to be

$$\frac{\partial \left(n_{dm} v^j\right)}{\partial t} + 4\frac{da/dt}{a} n_{dm} v^j + \frac{n_{dm}}{a} \frac{\partial \Psi}{\partial x^j} = 0. \tag{44}$$

From here we can take the first-order terms and do some differentiation to get the first-order equation for velocity

$$\frac{\partial v^j}{\partial t} + \frac{da/dt}{a} v^j + \frac{1}{a} \frac{\partial \Psi}{\partial x^j} = 0. \tag{45}$$

Eqs. (43) and (45) complete the set for the first-order equations that we need. We can also rewrite them in terms of conformal time and Fourier transforms. We start with the Fourier transform of the velocity

$$\tilde{v}^i = \int_{-\infty}^{\infty} v^i e^{\mathrm{i}\omega k^i} dk_i. \tag{46}$$

Now, if we assume that the velocity is irrotational [2], we can assume that every velocity component is a projection of the velocity vector onto the wave-direction vector $\overrightarrow{k}$, then we have the identity $\tilde{v} = \tilde{v}^i \frac{k}{k^i}$, where $k = \sqrt{k^i k^i}$. And if we not put everything together and switch to conformal time, we have the equations

$$\begin{aligned} \dot{\tilde{\delta}} + \mathrm{i}k\tilde{v} + 3\dot{\tilde{\Phi}} &= 0 \\ \dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} + \mathrm{i}k\tilde{\Psi} &= 0. \end{aligned} \tag{47}$$

For baryons

Now that we have written the Boltzmann equation for both photons and cold dark matter, we can move on to baryons. Unlike cold dark matter, baryons will have collision terms. Electrons and protons are coupled by the Coulomb scattering. Also, electrons are coupled to photons by the Compton scattering. However, the Coulomb scattering rate is much larger than the expansion rate at all epochs of our interest, that it forces the electron and proton overdensities to have the same value

$$\frac{\rho_e - \rho_e^{(0)}}{\rho_e^{(0)}} = \frac{\rho_p - \rho_p^{(0)}}{\rho_p^{(0)}} = \delta_b. \tag{48}$$

On a similar note, their velocity also become the same

$$\overrightarrow{v}_e = \overrightarrow{v}_p = \overrightarrow{v}_b. \tag{49}$$

Now we can write down the Boltzmann equations for both electrons and protons

$$\begin{aligned} \frac{df_e}{dt} &= C_{Couloumb} + C_{Compton} \\ \frac{df_p}{dt} &= C_{Coulomb}. \end{aligned} \tag{50}$$

Since we are dealing with particles with mass, we can take the same route we took with cold dark matter. So for that, let's take the electron Botlzmann equation and multiply it by the phase element $\frac{d^3p}{(2\pi)^3}$ and integrate it. What we get is the identical equation to the eq. (40)

$$\frac{\partial n_e}{\partial t} + \frac{1}{a}\frac{\partial\left(n_e v_b^i\right)}{\partial x^i} + 3\left(\frac{da/dt}{a} + \frac{\partial\Phi}{\partial t}\right) n_e = 0, \tag{51}$$

where the collision terms vanish. That happens because when we multiply an unintegrated collision term by a conserved quantity and then integrate, we get zero [2]. Now if we perturb the eq. (51) and switch to the familiar Fourier space and conformal time we get the equation identical to that of dark matter

$$\dot{\tilde{\delta}}_b + \mathrm{i}k\tilde{v}_b + 3\dot{\tilde{\Phi}} = 0. \tag{52}$$

The second equation for baryons is a little trickier. Firstly, we need to take the first moments of both the Boltzmann equations (50) for protons and electrons and add them together. We do this as we did for the cold dark matter case but here we will multiply each equation by it's momentum and then by the mass m. So the results from the dark matter will carry over here. Except for the collision terms. Since the proton mass m_p is so much more bigger than the electron mass in the sum, it will only dominate.

After doing algebra calculations, we get the first moment equation

$$m_p\frac{\partial\left(n_b v_b^j\right)}{\partial t} + 4\frac{da/dt}{a} m_p n_b v_b^j + \frac{n_b m_p}{a}\frac{\partial\Psi}{\partial x^j} = C, \tag{53}$$

where C is the sum of all the collision terms. We can make the Coloumb scatter terms vanish by using the law of momentum conservation. Then the only term that stays is the integrated Compton scattering term. But we already have the Compton scattering term from eq. (24). All we need to do is switch to Fourier space, multiply it by $\frac{p\mu}{\rho_b}$ and integrate over all momentum. Of course, that is not such a simple task, unless we introduce another definition

$$\Theta_1 = \mathrm{i}\int_{-1}^{1}\frac{d\mu}{2}\mu\Theta\left(\mu\right). \tag{54}$$

We now can write down the second equation for baryons

$$\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b + \mathrm{i}k\tilde{\Psi} = \dot{\tau}\frac{4\rho_\gamma}{3\rho_b}\left(3\mathrm{i}\tilde{\Theta}_1 + \tilde{v}_b\right). \tag{55}$$

You may be wondering why is there a factor of ρ_b. That simply arises from the fact, that moving electrons is difficult because they are tightly coupled to protons. If the proton was infinitely heavy and so $\rho_b \to \infty$ we would get no effect from Compton scattering.

2.2.2 Einstein equations

Now that we know how the perturbations in the metric affect particle distribution functions, we must consider how do these distribution functions affect our perturbations. Because as we all know matter does affect gravity. So these particle distributions will changes how the perturbtions Ψ and Φ evolve over time.

To do that, we must take a look at Einstein equations. For those, who are not that well informed on general relativity, I again refer you to the Appendix A. For the calculations we will need to take into account how the Christoffel symbols and especially Ricci tensor look with the perturbed FRW metric.

Christoffel symbols and the Ricci tensor

To calculate the Christofel symbols one must look at how they are defined in eq.(188). The Christoffel symbols will be shown as they are in Fourier space. That is done for more simplicity and ease when writing the full Einstein's equations.

Let us start by simply writing the time components

$$\Gamma^0{}_{00} = \frac{\dot{\Psi}}{a}, \tag{56}$$

where we have only the time components of the Christoffel symbol. It is easier to go through this route where we will now add atleast one spatial component, because there are so many Christoffel symbols and their components. So it is much easier to see how they change, when we add only spatial components.

The next Christoffel symbols are

$$\begin{aligned} \Gamma^0{}_{0i} = \Gamma^0{}_{i0} &= \tfrac{\partial \Psi}{\partial x^i} \longrightarrow \mathrm{i}k_i \Psi \\ \Gamma^0{}_{ij} &= \delta_{ij} a^2 \left(H + 2H\left[\Phi - \Psi\right] + \tfrac{\dot{\Phi}}{a} \right) \\ \Gamma^i{}_{00} &= \tfrac{\mathrm{i}k^i}{a^2}\Psi \\ \Gamma^i{}_{j0} = \Gamma^i{}_{0j} &= \delta^i{}_j \left(H + \tfrac{\dot{\Phi}}{a} \right) \\ \Gamma^i{}_{jk} &= \mathrm{i}\Phi \left(\delta_{ij}k_k + \delta_{ik}k_j - \delta_{jk}k_i \right). \end{aligned} \tag{57}$$

Since we now have our Christoffel symbols, we can move on to the Ricci tensor and the Ricci scalar. However, these quantities require quite a bit of calculation, since they have a lot of terms. Instead of writing their full calculations, just like with Christoffel symbols we will just write down the final results and refer to the book [2].

For the Ricci tensor we have the time components

$$R_{00} = -3\frac{d^2a/dt^2}{a} - \frac{k^2}{a^2}\Psi - 3\frac{\partial}{\partial t}\left(\frac{\dot{\Phi}}{a}\right) + 3H\left(\frac{\dot{\Psi}}{a} - 2\frac{\dot{\Phi}}{a}\right), \tag{58}$$

while the space-space part[4] is

$$R_{ij} = \delta_{ij}\left(\left[2a^2H^2 + a\frac{d^2a}{dt^2}\right][1+2\Phi-2\Psi] + a^2H\left[6\frac{\dot{\Phi}}{a} - \frac{\dot{\Psi}}{a}\right] + a^2\frac{\partial}{\partial t}\left(\frac{\dot{\Phi}}{a}\right) + k^2\Phi\right)$$
$$+ k_ik_j\left(\Phi+\Psi\right). \tag{59}$$

Having the Ricci tensor we can write down the Ricci scalar

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} \\ &= (-1+2\Psi)\left(-3\frac{d^2a/dt^2}{a} - \frac{k^2}{a^2}\Psi - 3\frac{\partial}{\partial t}\left(\frac{\dot{\Phi}}{a}\right) + 3H\left(\frac{\dot{\Psi}}{a} - 2\frac{\dot{\Phi}}{a}\right)\right) \\ &+ 3\left(\frac{1-2\Phi}{a^2}\right)(\left(2a^2H^2 + a\frac{d^2a}{dt^2}\right)(1+2\Phi-2\Psi) \\ &+ a^2H\left(6\frac{\dot{\Phi}}{a} - \frac{\dot{\Psi}}{a}\right) + a^2\frac{\partial}{\partial t}\left(\frac{\dot{\Phi}}{a}\right) + k^2\Phi + \frac{k^2\left(\Phi+\Psi\right)}{3}), \end{aligned} \tag{60}$$

where the first-order part is

$$\begin{aligned} \delta R = &-12\Psi\left(H^2 + \frac{d^2a/dt^2}{a}\right) + \frac{2k^2}{a^2}\Psi + 6\frac{\partial}{\partial t}\left(\frac{\dot{\Phi}}{a}\right) \\ &- 6H\left(\frac{\dot{\Psi}}{a} - 4\frac{\dot{\Phi}}{a}\right) + 4\frac{k^2\Phi}{a^2}, \end{aligned} \tag{61}$$

where δR now denotes first-order terms of the Ricci scalar R.

Two components of the Einstein equations

Now that we have the Ricci scalar and the Christoffel symbols, we can finally calculate evolutions for the metric perturbations. To do that, we take a look at the Einstein equations

$$G^{\mu}{}_{\nu} = 8\pi G T^{\mu}{}_{\nu}. \tag{62}$$

Since the Einstein equations have ten components that would seem a daunting task. But we need only two components. The remaining eight would either be zero at first-order or be redundant.

To start calculating the first component we take the time-time component of the Einstein equations. So we need to evaluate the Eintsein tensor and the stress-energy tensor. Let us start with the Einstein tensor

$$G^{0}{}_{0} = g^{00}\left(R_{00} - \frac{1}{2}g_{00}R\right) = \left(-1+2\Psi\right)R_{00} - \frac{R}{2}. \tag{63}$$

Since we have both the time-time component of the Ricci tensor (59) and the perturbed Ricci scalar

[4] We do not take space-time or time-space parts because in the later calculations we will only need space-space and time-time components of the Ricci tensor.

(61) we can write the first-order part of the time-time component of the Einstein tensor (63)

$$\delta G^0{}_0 = -6H\frac{\dot{\Phi}}{a} + 6\Psi H^2 - 2\frac{k^2\Phi}{a^2}. \tag{64}$$

Now we need to calculate the time-time component of the stress-energy tensor. Recalling, that the time-time component $T^0{}_0$ of the stress-energy tensor is the energy density. So the stress-energy tensor for our purposes would be an integral over the distribution functions for each species

$$T^0{}_0 = -\sum_{all\ species\ i} g_i \int \frac{d^3p}{(2\pi)^3} E_i(p) f_i, \tag{65}$$

where g_i is the spin degeneracy of the species.

To calculate the first-order part of the stress-energy tensor, we must calculate the first-order part of the distribution functions for photons, neutrinos, dark matter and baryons. For dark matter and baryons we have it easy, since we already defined the right hand side as $-\rho_i(1+\delta_i)$,where i stands for dark matter or baryons. For photons however we need to use the perturbed distribution function (18), which after some calulation we get the stress-energy tensor part for photons

$$T^0{}_0 = -\rho_\gamma(1+4\Theta_0). \tag{66}$$

Neutrinos on the other hand are very easy. We take them to be massless and so the first-order contribution is identical in form to photons

$$T^0{}_0 = -\rho_\nu(1+4\aleph_0), \tag{67}$$

where $\aleph_0$ is similar to Θ_0 but for neutrinos. We take them to have the similar forms because we approximate that neutrinos are massless and so we can use the same Bose-Einstein distribution function, with the perturbation now changed to $\aleph_0$ to distinguish them from photons.

Combining both the first-order terms of the Einstein tensor and stress-energy tensor, we get the first equation

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi G a^2\left(\rho_{dm}\delta_{dm} + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\aleph_0\right) \tag{68}$$

To obtain the second evolution equation we must focus on the spatial part of the Einstein tensor

$$G^i{}_j = g^{ik}\left(R_{kj} - \frac{g_{kj}}{2}R\right) = \frac{\delta^{ik}(1-2\Phi)}{a^2}R_{kj} - \delta^i_j R. \tag{69}$$

If we now insert the calculated Ricci tensor and scalar, we well have

$$G^i{}_j = A\delta^i_j + \frac{k^i k_j(\Phi+\Psi)}{a^2}, \tag{70}$$

where A stands for more than ten terms. In order to not write them all out, we can make them vanish. For this we need to calculate the longitudinal, traceless part of the Einstein tensor. To do this, we multiply eq.(70) with $\hat{k}_i\hat{k}^j - \frac{1}{3}\delta_i^j$, which is a projection operator. With it all the terms with δ_{ij}vanish,, which leaves us with

$$\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta_i^j\right) G^i{}_j = \frac{2}{3a^2}k^2\left(\Phi + \Psi\right). \tag{71}$$

We must calculate the longitudinal, traceless part for the stress-energy tensor as well. For this we must consider that we are calculating the terms that come with pressure. Yet we know that baryons and dark matter are pressureless. So we would need to calculate the stress-energy tensor for neutrinos and photons

$$\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta_i^j\right) T^i{}_j = \sum_{all\ species\ i} g_i \int \frac{d^3p}{(2\pi)^3}\frac{p^2\mu^2 - \frac{1}{3}p^2}{E_i(p)} f_i. \tag{72}$$

After some calculations the photon part is

$$\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta_i^j\right) T^i{}_j = -\frac{8\rho^{(0)}\Theta_2}{3}, \tag{73}$$

which in form is similar for neutrinos as well. $\Theta_2$is quadrupole

$$\Theta_2 = \int_{-1}^{1}\frac{d\mu}{2}\frac{2P(\mu)}{3}\Theta(\mu), \tag{74}$$

where $P(\mu)$ is a Legendre polynomial, which is equal to $\frac{2}{3}P(\mu) = \mu^2 - \frac{1}{3}$.

If we now equate the Einstein and stress-energy tensors longitudinal, traceless parts, we get the second equation of evolution

$$k^2\left(\Phi + \Psi\right) = -32\pi G a^2\left(\rho_\gamma\Theta_2 + \rho_\nu\aleph_2\right). \tag{75}$$

With the equations (68) and (75) we can see what effect the particle densities have on the perturbed metric and how it evolves. However, when the metric changes, the distributions shift. So we have a system of density equations from Boltzmann equations and Einstein's equations for the perturbed metric. Using these, one can model how the universe evolves in time, however not analitically. That is because all of these equations are not linearly independant from each other. Thus, it requires to solve them all at the same time, which can only be done numerically.

2.3 Dodelson-Widrow model

Now that we have an understanding of how the early universe evolves, we can start taking a look at sterile neutrinos. To know if neutrinos can be a candidate for dark matter, we first must know how to calculate the density of sterile neutrinos. Sterile neutrinos can be produced via oscillations $\nu_L \rightarrow \nu_R$ [4]. As was mentioned previously, the high temperatures of the early universe allowed for the production of particles via the process $2\gamma \rightarrow particle + antiparticle$. This would give us only the left-handed neutrinos. As was mentioned, the right-handed neutrinos would only be produced through the neutrino oscillations. But their production, as with other particles, stops when the temperature of the universe reaches their mass. One can see that, because of the energy-mass relation $E^2 = m^2 + \overrightarrow{p}^2$. Once the temperature reaches the mass of the particle, only particle with zero momentum can be produced, and below this temperature no particles (of that mass) could be produced. In this and later sections, we will assume the thermal equilibrium of the early universe.

The probability of observing the sterile or right-handed neutrino after a time t is $\sin^2 2\theta_M \sin^2 \frac{vt}{L}$, if we assume that we start with only monoenergetic left-handed neutrinos, where θ_M is the mixing angle in the medium, L is the oscillation length and v is the velocity of the neutrinos. Now, in vacuum, the mixing angle, assuming the seesaw mechanism is in play, i.e. $\mu \ll M$,is $\theta^2 = \frac{m_\nu}{M}$[5]. The oscillation length in vacuum $L = \frac{4E}{M^2 - \mu^2}$,where E is the energy of the neutrinos. In the early Universe, the observation time t is replaced by the interaction time for the left-handed neutrinos. That is because the observation time is a rather difficult variable to work with, since we don't even know how much time passes for temperature to change by one degree. So it is better to work with variables that are more easily calculable. And of course, we assume that the collision time is always much greater than the oscillation time, that $\sin^2 \frac{vt}{L}$ averages to $\frac{1}{2}$ [4]. Most of the calculations done below are based on the Dodelson-Widrow paper [4].

Since we are talking about collisions, we then begin with the Boltzmann equation

$$\frac{d}{dt} f = C[f]. \tag{76}$$

For the sterile neutrino production the Boltzmann equation is

$$\left(\frac{\partial}{\partial t} - HE\frac{\partial}{\partial E}\right) f_S\left(E, t\right) = \left(\frac{1}{2}\sin^2\left(2\theta_M\left(E, t\right)\right)\mathit{\Gamma}\left(E, t\right)\right) f_A\left(E, t\right), \tag{77}$$

where H is the Hubble parameter; $\mathit{\Gamma}$ is the collision rate

$$\mathit{\Gamma} \simeq \frac{7\pi}{24} G^2_{Fermi} T^4 E, \tag{78}$$

where G^2_{Fermi} is the four-Fermi interaction constant. This is a coupling constant when four fermions interact, without an internal line in the Feynman diagram (typically in a Feynaman diagram we would have four fermions interacting as external lines, with a mediating particle, e.g. W bozons, as an internal line. However, in the four-Fermi interaction, we take, that the energies are rather low -

[5]The exact mixing angle is $\sin^2\theta = \frac{m_\nu}{M + m_\nu}$.However, if we assume that $M \gg m_\nu$, then $\theta \ll 1$ and we get that $\sin\theta \approx \theta$; $M + m_\nu \approx M$; $\theta^2 = \frac{m_\nu}{M}$.

compared to the mass of the mediating particle. So using the the mass energy relation $E^2 = m^2 + \overrightarrow{p}^2$, one can see that the mediating particle's mass is much larger than it's momentum $m \gg |\overrightarrow{p}|$. Because of this, we can neglect the momentum and treat the interaction as point-like, since we treat the mediating particle as not propagating. This is shown in Fig.1).

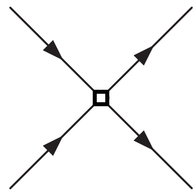

Fig. 1 Four-Fermi interaction. This is a point interaction with no mediating particle. The lines with arrows represent the interacting particles - arrows to the right are fermions and arrows to the left are antifermions.

The mixing angle is now the mixing angle in the medium and not in vacuum, since we are talking about the early universe (there are left-handed neutrinos that the sterile neutrinos interact with). Thus the mixing angle, which can be derived as shown in [11], is

$$\sin^2(2\theta_M) = \frac{\mu^2}{\mu^2 + [(c\Gamma E/M) + (M/2)]^2}, \tag{79}$$

where $c \simeq 4sin^2(2\theta_W)/15\alpha \simeq 26$; $sin^2(2\theta_W)$ is the weak mixing angle, which comes from the neutrino-lepton scatterings and is part of the left-handed thermal potential [11]; α is the fine structure constant $\alpha \approx 137^{-1}$.

f_A and f_S in eq. (77) are the active (left-handed) neutrino and sterile neutrino density distribution functions. Eq. The mixing angle is different, since $\theta = \frac{\mu}{M}$ is only valid for neutrinos in a vacuum.

For the active neutrinos, since they are fermions, their distribution function is

$$f_A = \frac{1}{e^{E/T} + 1} \approx \frac{1}{e^{p/T} + 1}; \tag{80}$$

here we have taken $E \approx p$,since $E = \sqrt{\overrightarrow{p}^2 + m^2}$and because the temperature T of the universe is so big compared to the mass of the neutrinos, then $\overrightarrow{p}^2 \gg m^2$. We assume that the active neutrino distribution density is as it is since we assume that there is a thermal equilibrium.

Because we can calculate the sterile neutrino density distribution f_S, from eq. (77), we can calculate the density of the sterile neutrinos. To do that, we must integrate the sterile neutrino density over all the momenta to get the sterile neutrino number density

$$n_{S,A} = 2\int \frac{d^3p}{(2\pi)^3} f_{S,A}. \tag{81}$$

And this equation holds for both sterile and active neutrinos. So to get the equation for the sterile neutrino number density evolution, we integrate the Boltzmann equation (77)

$$2\int \frac{d^3p}{(2\pi)^3}\left(\frac{\partial}{\partial t} - HE\frac{\partial}{\partial E}\right) f_S(E,t) = \int \frac{d^3p}{(2\pi)^3}\left(\sin^2(2\theta_M(E,t))\,\Gamma(E,t)\right) f_A(E,t)\,. \tag{82}$$

Since the Boltzmann equation is in it's general form as shown in eq. (76), that means in our case

$$\frac{d}{dt} = \frac{\partial}{\partial t} - HE\frac{\partial}{\partial E}. \tag{83}$$

Thus, we can switch the time derivative and the integration, to get

$$\frac{dn_S}{dt} = \int \frac{d^3p}{(2\pi)^3}\left(\sin^2(2\theta_M(E,t))\,\Gamma(E,t)\right) f_A(E,t)\,. \tag{84}$$

If we want to compare the active and sterile neutrino number densities, then we can define the ratio

$$r = \frac{n_S}{n_A}. \tag{85}$$

Now, we can take the derivative of the ratio

$$\begin{aligned}\frac{dr}{dt} &= \frac{1}{n_A}\frac{dn_S}{dt} - \frac{n_s}{n_A^2}\frac{dn_A}{dt}\\ &= \frac{1}{n_A}\frac{dn_S}{dt} - r\frac{d\ln n_A}{dt}.\end{aligned}$$

We already know what the first term is in this equation from eq. (84). The second term is the evolution of the active neutrino number density. Sterile neutrino production happens, when active neutrinos oscillate to sterile neutrinos. Yet the active neutrinos are continuously produced in the thermal early universe bath. So we can assume that the active neutrino number density doesn't change, i.e. $\frac{d\ln n_A}{dt} = 0$. The production eventually will stop. However, we assume that the sterile neutrino mass is much larger than the active neutrino mass $M \gg \mu$. Thus for our purposes, we don't need to look to the point when the active neutrino production ceases, since by that point the sterile neutrinos will have been not produced for a significant portion of time (that time being the time it takes to get from temperature $T = M$ to temperature $T = \mu$). Thus our assumption holds.

Another thing we can do, is replace the time variable t with the Robertson-Walker scale factor a. To do that, we use the differentiation variable change

$$\frac{dr}{dt} = \frac{dr}{d\ln a}\frac{d\ln a}{dt}. \tag{86}$$

We use the logarthmic scale factor for a reason. It comes from the definition of the Hubble parameter

$$H = \frac{\dot{a}}{a}, \tag{87}$$

where $\dot{a} = \frac{da}{dt}$. Thus, we can replace the derivative

$$\frac{1}{a}\frac{da}{dt} = \frac{d\ln a}{dt} = H. \tag{88}$$

And from this we get

$$\frac{dr}{d\ln a} = \frac{1}{H \cdot n_A}\frac{dn_S}{dt}. \tag{89}$$

We don't replace the sterile neutrino number density time derivative, since we already know what it is from eq. (84). Introducing the new variable $\gamma = \frac{1}{n_A}\frac{dn_S}{dt}$,we finally have the evolution of the sterile neutrino and active neutrino number densities

$$\frac{dr}{d\ln a} = \frac{\gamma}{H}. \tag{90}$$

But in this work, we don't want to calculate the ratio between active and sterile neutrinos. No, we wish to see, if sterile neutrinos are dark matter particles, what kind of mass would they have. For this purpose, we need to calculate the density parameter of sterile neutrinos and see, when it is within the dark matter density parameter bounds, i.e. $0.25 \leq \Omega_S \leq 0.26$. To calculate the sterile neutrino density parameter, we need to calculate the density of sterile neutrinos and the critical density as per the definition of the density parameter

$$\Omega_T = \frac{\rho_{TS}}{\rho_{crit}}, \tag{91}$$

where ρ_{TS} is the sterile neutrino density from thermal production and ρ_{crit} is the critical density

$$\rho_{crit} = \frac{3H^2}{8\pi G}. \tag{92}$$

H is the Hubble parameter and G is the Newton's gravitational constant. One must recognize that it's not the Hubble constant H_0 being used here so the critical density will depend on the temperature of the universe (see appendix C.1).

The sterile neutrino density is fairly easy to calculate as it is

$$\rho_{TS} = n_{TS} \cdot M. \tag{93}$$

Thus we need to calculate the sterile neutrino number density n_{TS} whose evolution equation is given by (84). We can write down the precise form of the number density, derived in the appendix C.1

$$n_{TS} = -\int_{\infty}^{M} \frac{7\mu^2 G_{Fermi}^2 T m_P}{96\pi^2\sqrt{(0.1g_*)}} dT \int_0^{\infty} \frac{p^3 dp}{\left(\mu^2 + \left[\frac{c\Gamma p}{M} + \frac{M}{2}\right]^2\right)\left(e^{\frac{p}{T}} + 1\right)}. \tag{94}$$

One can see, that for small and big momenta p the number density is rather negligable. As for high temperatures, one would expect the production to be the highest at these temperatures, but that is not the case, since we have an additional T^{-8} in the denominator because of the collision rate Γ^2

term. Thus we expect that the production actually should not start at the highest temperatures, but rather have a peak in between the maximum temperature of the universe and the sterile neutrino mass M. This is because at low temperatures the exponent begins to dominate our expression and thus the production again minimizes.

2.4 Sterile neutrino production from scalar decay

Of course, sterile neutrinos can be produced via other methods, not just through oscillations from active neutrinos. One of these methods is the heavy scalar decay. As such, we assume that there is a very heavy scalar - heavier than the sterile neutrino in such a way, that $\frac{M}{m_\phi} \ll 1$, where m_ϕ is the mass of the scalar. We can write the relevant Lagrangian as

$$\mathcal{L}_{rel} = -\frac{1}{2}\sigma\phi\bar{N}N - \frac{1}{2}m_\phi^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{95}$$

where σ is the Yukawa coupling for scalar-sterile neutrino interaction and the λ is the selfcoupling of the scalar field. We could have more terms, but for now these are the most relevant ones. In this section, all of the calculations and results are referenced from Marco Drewes' paper *Sterile neutrino Dark Matter production from scalar decay in a thermal bath* [5].

2.4.1 Correlations functions, spectral density and the loss/gain rates

To begin with, we need to talk about the correlation functions and the S-matrix.In non-equilibrium systems at large density, like the one we might have in the early universe, the usual methods for calculating S-matrix elements that we know from QFT cannot be used since there are no well-defined notions of asymptotic states, and the properties of quasiparticles in a medium may differ sgnificantly from those of particles in a vacuum. Yet, one can use the correlation functions of the fields to calculate observables, without needing to look at the asymptotic states or free particles. There are two independent two point functions for each field. For a real scalar field ϕ these are often chosen to be the connected Wightman functions

$$\Delta^{>}(x_1, x_2) = \langle \phi(x_1)\,\phi(x_2)\rangle_c\,,\ \Delta^{<}(x_1, x_2) = \langle \phi(x_2)\,\phi(x_1)\rangle_c\,, \tag{96}$$

where

$$\langle ... \rangle = \mathrm{Tr}\,(\hat{\rho} ...) \tag{97}$$

is the quantum statistical average and $\hat{\rho}$ is the density operator of the system. It is defined as

$$\hat{\rho} = \sum_n p_n \left|\phi_n\right\rangle \left\langle \phi_n\right|, \tag{98}$$

where p_n is not the momentum, but the probability of the system being in the state$|\phi_n\rangle$. The sum can be infinite.

From this we can have the linear combinations

$$\begin{aligned} \Delta^{-}(x_1, x_2) &= \mathrm{i}\left(\Delta^{>}(x_1, x_2) - \Delta^{<}(x_1, x_2)\right) \\ \Delta^{+}(x_1, x_2) &= \tfrac{1}{2}\left(\Delta^{>}(x_1, x_2) + \Delta^{<}(x_1, x_2)\right). \end{aligned} \tag{99}$$

Using these relations, we can define the scalar spectral density ρ_ϕ

$$\rho_\phi(q) = -\mathrm{i}\Delta^-(q) = -\mathrm{i}\int \frac{d^4x}{(2\pi)^4}\Delta^-(x)\, e^{\mathrm{i}qx}, \tag{100}$$

not to be confused with the density operator $\hat{\rho}$, since the spectral density ρ_ϕ is a quantity not an operator. Assuming that the other particle fields are in thermal equilibrium, when the sterile neutrinos N are produced (the scalar ϕ may or may not be in equilibrium, depending on whether N production during freeze-in is relevant) the correlation functions only depend on the relative coordinate $x_1 - x_2$, and they are related by the Kubo-Martin-Schwinger (KMS) relations

$$\Delta^<(q) = e^{-q_0/T}\Delta^>(q), \tag{101}$$

where $\Delta^{\lessgtr}(q)$ are the Fourier transforms of the $\Delta^{\lessgtr}(x)$. q_0 is essentially energy.

Now we have two main properties of our calculations - the spectral density ρ_ϕ and the correlation functions in eq. (96). With that, we can start thinking of the scalar decay. And for that we need to calculate the spectral density.

For a scalar ϕ that couples to a plasma in equilibrium, $\rho_\phi(q)$, which depends on the four-vector q, at leading order can be expressed as

$$\rho_\phi(q) = \frac{-2\mathrm{Im}\Pi_\phi^R(q) + 2q_0\epsilon}{\left(q_0^2 - m_\phi^2 - \boldsymbol{q}^2 - \mathrm{Re}\Pi_\phi^R(q)\right)^2 + \left(\mathrm{Im}\Pi_\phi^R(q) + q_0\epsilon\right)^2} \tag{102}$$

(even if ϕ is not in equilibrium), where this arrives from the definitions of the correlation functions (see eq. (100)). $\Pi_\phi^R(q)$ is the retarted self-energy, which also depends on T. The retarded self-energy can be calculated from the correction at one loop level. If $\hat{\Omega}_{\boldsymbol{q}}$ is the pole of $\rho_\phi(q)$ we also have

$$\Omega_{\phi\boldsymbol{q}} = \mathrm{Re}\hat{\Omega}_{\boldsymbol{q}},\ \Gamma_{\phi\boldsymbol{q}} = 2\mathrm{Im}\hat{\Omega}_{\boldsymbol{q}}, \tag{103}$$

where $\Omega_{\phi\boldsymbol{q}}$ is the mass shell for the scalar particles.

In weakly coupled theories one observes the hierarchy

$$\Gamma_{\phi\boldsymbol{q}} \ll \Omega_{\phi\boldsymbol{q}} \tag{104}$$

and can make the Breit-Wigner approximation

$$\rho_\phi^{BW}(q) = 2\mathcal{Z}\frac{q_0\Gamma_{\phi\boldsymbol{q}}}{\left(q_0^2 - \Omega_{\phi\boldsymbol{q}}^2\right)^2 + \left(q_0\Gamma_{\phi\boldsymbol{q}}\right)^2} + \rho_\phi^{cont}(q), \tag{105}$$

where the residue is

$$\mathcal{Z} = \left(1 - \frac{1}{2\Omega_{\phi\boldsymbol{q}}}\frac{\partial\mathrm{Re}\Pi_\phi^{\mathrm{R}}(q)}{\partial q_0}\right)^{-1}_{q_0=\Omega_{\phi\boldsymbol{q}}}. \tag{106}$$

Then the dispersion relations can be obtained by solving the equation

$$\Omega_{\phi\boldsymbol{q}}^2 - m_\phi^2 - \boldsymbol{q}^2 - \mathrm{Re}\Pi_\phi^R(q)\,|_{q_0=\Omega_{\phi\boldsymbol{q}}} = 0. \tag{107}$$

And so the damping rate is given by

$$\Gamma_{\phi\boldsymbol{q}} = -\mathcal{Z}\frac{\mathrm{Im}\Pi_\phi^R(q)}{q_0}|_{q_0=\Omega_{\phi\boldsymbol{q}}}. \tag{108}$$

Analogous to $\Delta^{\lessgtr}$, one can introduce self-energies $\Pi_\phi^{\lessgtr}$ and define

$$\Pi_\phi^-(q) = \Pi_\phi^> - \Pi_\phi^< = 2\mathrm{iIm}\Pi_\phi^R(q). \tag{109}$$

With this we can define

$$\Gamma_{\phi\boldsymbol{q}} = \mathcal{Z}\frac{\mathrm{i}\Pi_\phi^-(q)}{q_0}|_{q_0=\Omega_{\phi\boldsymbol{q}}} = \Gamma_{\phi\boldsymbol{q}}^> - \Gamma_{\phi\boldsymbol{q}}^<, \tag{110}$$

which are gain and loss rates for ϕ-particles. With the KMS-relation $\Pi_\phi^< = e^{-q_0/T}\Pi_\phi^>$ it is easy to see that the detailed balanced relation

$$\frac{\Gamma_{\phi\boldsymbol{q}}^<}{\Gamma_{\phi\boldsymbol{q}}^>} = e^{-\Omega_{\phi\boldsymbol{q}}/T} \tag{111}$$

holds and we have the damping rate

$$\Gamma_{\phi\boldsymbol{q}}^< = f_B\left(\Omega_{\phi\boldsymbol{q}}\right)\Gamma_{\phi\boldsymbol{q}}, \tag{112}$$

where $f_B\left(\Omega_{\phi\boldsymbol{q}}\right)$ is the Bose-Einstein distribution.

2.4.2 The general Boltzmann equations for the scalar decay

With the retarded self-energies from eq. (109) and the loss/gain rates from eqs. (110) and (111) we can start calculating the N-particle number density. Firstly, we must assume that the coupling σ in the Lagrangian (95) is very small, i.e. $\sigma \ll 1$. Then, we can look at the Boltzmann equation for the scalar field ϕ

$$\partial_t f_{\phi\boldsymbol{q}} = -\Gamma_{\phi\boldsymbol{q}}\left(f_{\phi\boldsymbol{q}} - \bar{f}_{\phi\boldsymbol{q}}\right), \tag{113}$$

where $f_{\phi\boldsymbol{q}} = f_\phi\left(\Omega_{\phi\boldsymbol{q}}\right)$; and

$$\bar{f}_{\phi\boldsymbol{q}} = \left(\frac{\Gamma_{\phi\boldsymbol{q}}^>}{\Gamma_{\phi\boldsymbol{q}}^<} - 1\right)^{-1}. \tag{114}$$

Here we can define an expansion

$$\Gamma_{\phi\boldsymbol{q}} = \tilde{\Gamma}_{\phi\boldsymbol{q}} + \tilde{\Gamma}_0, \tag{115}$$

where $\Gamma_{\phi\boldsymbol{q}}$ is the total thermal ϕ-width and $\tilde{\Gamma}_0 = \Gamma_{\phi\boldsymbol{q}}|_{\sigma=0}$ is $\Gamma_{\phi\boldsymbol{q}}$ at zeroth order in σ. We are able to do this expansion only because of the small coupling σ. So our Boltzmann equation turns to

$$\partial_t f_{\phi\boldsymbol{q}} = -\left[\tilde{\Gamma}_{\phi\boldsymbol{q}} + \tilde{\Gamma}_0\right]\left(f_{\phi\boldsymbol{q}} - \bar{f}_{\phi\boldsymbol{q}}\right). \tag{116}$$

In the case of $\lambda \gg \sigma$, we would have $\tilde{\Gamma}_{\phi\boldsymbol{q}} \ll \tilde{\Gamma}_0$.

As one can see, the main issue in calculating the scalar density distribution is finding the damping rate and subsequent loss and gain rates. Since they can be calculated using the retarded self-energy $\Pi^R_\phi(q)$, we can calculate their expressions for the one-loop level

$$\begin{aligned}\tilde{\Pi}^<_\phi(q) &= \frac{\mathrm{i}\sigma^2}{2}\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\left(S^<_N(p)\,S^>_N(p-q)\right)\\ &= -\frac{\mathrm{i}\sigma^2}{2}\int\frac{d^4p}{(2\pi)^4}\left(1-f_N(p_0-q_0)\right)f_N(p_0)\,\mathrm{tr}\left(\rho_N(p)\,\rho_N(p-q)\right)\end{aligned} \tag{117}$$

$$\begin{aligned}\tilde{\Pi}^>_\phi(q) &= \frac{\mathrm{i}\sigma^2}{2}\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\left(S^>_N(p)\,S^<_N(p-q)\right)\\ &= -\frac{\mathrm{i}\sigma^2}{2}\int\frac{d^4p}{(2\pi)^4}\left(1-f_N(p_0)\right)f_N(p_0-q_0)\,\mathrm{tr}\left(\rho_N(p)\,\rho_N(p-q)\right)\end{aligned} \tag{118}$$

$$\tilde{\Pi}^-_\phi(q) = \frac{\mathrm{i}\sigma^2}{2}\int\frac{d^4p}{(2\pi)^4}\left(f_N(p_0)-f_N(p_0-q_0)\right)\mathrm{tr}\left(\rho_N(p)\,\rho_N(p-q)\right). \tag{119}$$

In these retarted self-energy relations we use the correlaton functions $S^<_N$ and $S^>_N$, which are the correlation functions for the fermions N. These correlation functions are the same in form as for the scalar fields (96), you just replace the scalar fields with the fermion fields.

And here comes our second approximation because of the small coupling assumption we can approximate the spectral density with the free spectral density

$$\rho_N(p) \approx 2\pi\mathrm{sign}(p_0)\left(\not{p} + m_N\right)\delta\left(p^2 + m_N^2\right). \tag{120}$$

The masses and self-energies use the N indices since the scalar ϕ decays into the N field and so the loop is made of internal N fields.

Before we calculate everything two more things must be mentioned. Firstly, the mass of the scalar m_ϕ is bigger than our sterile neutrino mass as mentioned earlier, i.e. $m_N \ll m_\phi$. Another thing is that since the scalar field in general can have self-interactions as shown in the relevant Lagrangian (95), thus our ϕ-quasiparticle pole $\Omega_{\phi\boldsymbol{q}} \neq \omega_{\phi\boldsymbol{q}} = \sqrt{m_\phi^2 + \boldsymbol{q}^2}$. The explanation for this is that the effective mass of the quasiparticle changes with temperature. Thus we have

$$\Omega^2_{\phi\boldsymbol{q}} \approx \boldsymbol{q}^2 + M_\phi^2, \tag{121}$$

where we have our thermal mass

$$M_\phi^2 = m_\phi^2 + \frac{\lambda}{24}T^2. \tag{122}$$

And we also assume the vacuum decay rate to be [12]

$$\tilde{\Gamma}_0 = \frac{\sigma^2}{16\pi} m_\phi. \tag{123}$$

Using all of the approximations and eqs.(110), (111), (117), (118), (119) and (120) we can now show that the rates are

$$\tilde{\Gamma}^{>}_{\phi \boldsymbol{q}} = \frac{\sigma^2}{16\pi} \frac{M_\phi}{\Omega_{\phi \boldsymbol{q}}} \frac{M_\phi}{\mathrm{q}} \int_{(\Omega_{\phi \boldsymbol{q}} - \mathrm{q})/2}^{(\Omega_{\phi \boldsymbol{q}} + \mathrm{q})/2} d\mathrm{p} \left(1 - f_N(\mathrm{p})\right) \left(1 - f_N\left(\Omega_{\phi \boldsymbol{q}} - \mathrm{p}\right)\right) \tag{124}$$

$$\tilde{\Gamma}^{<}_{\phi \boldsymbol{q}} = \frac{\sigma^2}{16\pi} \frac{M_\phi}{\Omega_{\phi \boldsymbol{q}}} \frac{M_\phi}{\mathrm{q}} \int_{(\Omega_{\phi \boldsymbol{q}} - \mathrm{q})/2}^{(\Omega_{\phi \boldsymbol{q}} + \mathrm{q})/2} d\mathrm{p} f_N(\mathrm{p}) f_N\left(\Omega_{\phi \boldsymbol{q}} - \mathrm{p}\right) \tag{125}$$

and

$$\tilde{\Gamma}_{\phi \boldsymbol{q}} = \tilde{\Gamma}_0 \frac{M_\phi}{\Omega_{\phi \boldsymbol{q}}} \frac{M_\phi}{m_\phi \mathrm{q}} \int_{(\Omega_{\phi \boldsymbol{q}} - \mathrm{q})/2}^{(\Omega_{\phi \boldsymbol{q}} + \mathrm{q})/2} d\mathrm{p} \left(1 - f_N(\mathrm{p}) - f_N\left(\Omega_{\phi \boldsymbol{q}} - \mathrm{p}\right)\right), \tag{126}$$

where we integrate over the neutrino's momentum's absolute value $\mathrm{p} = |\boldsymbol{p}|$ and $\mathrm{q} = |\boldsymbol{q}|$ is the absolute value of the scalar's momentum.

Now we can finally calculate the sterile neutrino distribution function or the sterile neutrino number density. And this number density is the same as the Boltzmann equation for the scalar ϕ with a different sign

$$\partial_t n_N = 2 \int \frac{d^3 \boldsymbol{q}}{(2\pi)^3} \tilde{\Gamma}_{\phi \boldsymbol{q}} \left(f_{\phi \boldsymbol{q}} - \bar{f}_{\phi \boldsymbol{q}}\right). \tag{127}$$

There is no $\tilde{\Gamma}^{(0)}_{\phi \boldsymbol{q}}$ term since we have no thermal corrections for the N field, i.e.

$$\Omega^2_{N \boldsymbol{q}} \approx m_N^2 + \boldsymbol{q}^2 = \omega^2_{N \boldsymbol{q}}. \tag{128}$$

If the main contribution to $\tilde{\Gamma}_{\phi \boldsymbol{q}}$ comes from $1 \longrightarrow 2$ decays and their inverse, one can express

$$\tilde{\Gamma}_{\phi \boldsymbol{q}} = \int \frac{d^3 \boldsymbol{p}}{(2\pi)^3} \frac{\sigma^2 \pi}{2\Omega_{\phi \boldsymbol{q}} |\boldsymbol{p} - \boldsymbol{q}|} \left(\Omega_{\phi \boldsymbol{q}} - \mathrm{qx}\right) \left(1 - f_N(p) - f_N(|\boldsymbol{p} - \boldsymbol{q}|)\right) \delta\left(\Omega_{\phi \boldsymbol{q}} - \Omega_{N \boldsymbol{p}} - \Omega_{N \boldsymbol{p} - \boldsymbol{q}}\right), \tag{129}$$

where qx is a four-vector scalar product of the scalar particle's momentum $\mathrm{q} = |\boldsymbol{q}|$ and $\mathrm{x} = \frac{\boldsymbol{pq}}{\mathrm{pq}}$ - it's cosine of the angle between the spatial vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ [6].

Inserting all of what we have into our expression for the sterile neutrino Boltzmann equation,we get that it is

$$\partial_t f_{N \boldsymbol{p}} = 2 \frac{\tilde{\Gamma}_0}{m_\phi} \frac{M_\phi^2}{\boldsymbol{p}^2} \int_{M_\phi^2/(4p)+p}^{\infty} d\Omega_{\phi \boldsymbol{q}} \left(1 - f_N(p) - f_N\left(\Omega_{\phi \boldsymbol{q}} - p\right)\right) \left(f_{\phi \boldsymbol{q}} - \bar{f_{\phi \boldsymbol{q}}}\right). \tag{130}$$

If we assume that when the sterile neutrino production occurs, the scalar ϕ is in thermal equilibrium

[6]Note, one might think that this is a weird mixing between four-vectors and three-vectors. However, it is not, it's actaully just a scalar product of three-vectors $\boldsymbol{q}$ and $\boldsymbol{p}$ divided by the absolute values of $\boldsymbol{p}$ and $\boldsymbol{q}$.

[13], then we can simplify our Boltzmann equation to

$$\frac{df_N\left(p\right)}{dt} = \frac{2m_\phi \tilde{\Gamma}_0}{p^2} \int_{p+m_\phi^2/(4p)}^{\infty} f_{\phi q} d\Omega_{\phi q}. \tag{131}$$

2.4.3 A specific case of scalar decay

Now we know, that if we want to calculate the sterile neutrino density from the scalar decay, we must calculate the equations (116) and (131). Now, the sterile neutrino density distribution equation (131) would be easy to calculate if not for the scalar Boltzmann equation (116). We can see that from the fact, that it uses the dampening rate, which in turn has our sterile neutrino density distribution as a function in the dampening rate's definition (see eq. (126)). Thus we have a problem of coupled equations which is not so easilly solved numerically, since we have to solve all the equations at the same time. And the analytical solution most likely doesn't exist.

But what if we can make it simpler. For instance, if we say, that the self-interaction coupling λ from the Lagrangian (95) is much bigger than the scalar's ϕ coupling to the sterile neutrino fields σ, i.e. $\lambda \gg \sigma$,then our Boltzmann equation for the scalar ϕ changes from eq. (116) to

$$\frac{df_{\phi \boldsymbol{q}}}{dt} = -\tilde{\Gamma}_0 \left(f_{\phi \boldsymbol{q}} - \bar{f}_{\phi \boldsymbol{q}}\right). \tag{132}$$

This is the case, because if $\lambda \gg \sigma$, then we have $\tilde{\Gamma}_{\phi \boldsymbol{q}} \ll \tilde{\Gamma}_0$ as was mentioned in the previous section. So we can neglect the $\tilde{\Gamma}_{\phi \boldsymbol{q}}$ term.

But we can make this even simpler to calculate. Firstly, we will be working in the scalar ϕ equilibrium case, i.e. eq. (131) applies. Secondly, we will assume that degrees of freedom are constant. This may not be apparent why we should do it like this, but in the next sections, this will be explained.

Now we can begin with the main argument for our simplification. Let us assume, that the temperature of the early universe is bigger than the mass of the scalar, i.e. $T > m_\phi$. If we assume that the scalar particles ϕ are produced via the process $2\gamma \longrightarrow particle + antiparticle$, one can assume that the distribution function of the scalar ϕ would be the Bose-Einstein function. This just comes from the scalar particle definition. So in this period we assume that the scalar particles are thermally produced. But some of these particles decay into sterile neutrinos. And yet the number of the scalar particles does not change, since we assume that even if the scalar particles decay as many or maybe even more particles are produced. This means, that in eq. (131) we can insert the Bose-Einstein distribution instead of the scalar density distribution $f_{\phi \boldsymbol{q}}$

$$\frac{df_N}{dt} = \frac{2m_\phi \tilde{\Gamma}_0}{p^2} \int_{p+m_\phi^2/(4p)}^{\infty} e^{-\Omega_{\phi \boldsymbol{q}}/T} d\Omega_{\phi \boldsymbol{q}}, \tag{133}$$

where we have assumed that

$$f_{\phi \boldsymbol{q}} = \frac{1}{e^{\Omega_{\phi \boldsymbol{q}}/T} - 1} \approx e^{-\Omega_{\phi \boldsymbol{q}}/T}. \tag{134}$$

We can make this assumption since the temperatures T are so big compared to $T = 0$ K.

Having our simplified Boltzmann equation (133) we can start calculating. We begin with evaluating the first integral, which is rather easy to integrate

$$\int_{p+m_\phi^2/(4p)}^{\infty} e^{-\Omega_{\phi\boldsymbol{q}}/T} d\Omega_{\phi\boldsymbol{q}} = -Te^{-\Omega_{\phi\boldsymbol{q}}/T}|^{\infty}_{p+\frac{m_\phi^2}{4p}}. = Te^{-\frac{p+m_\phi^2/(4p)}{T}}. \tag{135}$$

And thus the Boltzmann equation becomes

$$\frac{df_N}{dt} = \frac{2m_\phi \tilde{\Gamma}_0}{p^2} Te^{-\frac{p+m_\phi^2/(4p)}{T}}. \tag{136}$$

Now, one has to integrate this equation over temperature and momentum. To do this, we need to replace the time variable with temperature as was done before in the thermal sterile neutrino production (see appendix C.1). After one does this, one must realise that we cannot integrate over the whole temperature range, since the scalar particle distribution function is of the form we use here, up until scalar particles are made, so until $T = m_\phi$. Thus, the integral becomes

$$f_N = -\frac{2m_\phi m_p \tilde{\Gamma}_0}{p^2 \pi} \int_{\infty}^{m_\phi} \sqrt{\frac{90}{g_*}} \frac{1}{T^2} e^{-\frac{p+m_\phi^2/(4p)}{T}} dT. \tag{137}$$

The number density is the integration over the whole momentum space. Since we do not care about the direction of the momentum, we have

$$d^3p = 4\pi p^2 dp. \tag{138}$$

Thus the number density is

$$n_{ES} = -\int_{\infty}^{m_\phi} \int_0^{\infty} \frac{2m_\phi m_p \tilde{\Gamma}_0}{\pi^3 T^2} \sqrt{\frac{90}{g_*}} e^{-\frac{p+m_\phi^2/(4p)}{T}} dp dT. \tag{139}$$

From this, we can turn to the second period of sterile neutino production - when the temperature T is smaller than the mass of the scalar ϕ, i.e. $T < m_\phi$. For this, we look back at the scalar Boltzmann equation (132). We see that the second term in this equation is positive, which means that particles are produced. Yet, in the second time period, the production of scalar particle ϕ is impossible, due to the temperature not being able to produce the particles. Thus we can neglect the second term and have our simplified Boltzmann equation

$$\frac{df_{\phi\boldsymbol{q}}}{dt} = -\tilde{\Gamma}_0 f_{\phi\boldsymbol{q}}. \tag{140}$$

Replacing the time variable with temperature, we have

$$\frac{df_{\phi\boldsymbol{q}}}{dT} = \sqrt{\frac{90}{g_*}} \frac{m_p \tilde{\Gamma}_0}{\pi T^3} f_{\phi\boldsymbol{q}.}. \tag{141}$$

This is easy to integrate, but we have to remember that the function that we integrate should be a function of two variables - the temperature T and the scalar's momentum $\boldsymbol{q}^2$. So we integrate the Boltzmann equation

$$\ln f_{\phi\boldsymbol{q}} = -\frac{C_1\tilde{\Gamma}_0}{2T^2} + C_2\left(\boldsymbol{q}^2\right), \tag{142}$$

where $C_1 = \sqrt{\frac{90}{g_*}}\frac{m_P}{\pi}$. We then exponantiate this equation

$$f_{\phi\boldsymbol{q}} = e^{-\frac{C_1\tilde{\Gamma}_0}{2T^2}+C_2\left(\boldsymbol{q}^2\right)}. \tag{143}$$

To find the missing function C_2, we must use the initial condition. This condintion is rather simple - at temperature $T = m_\phi$, the distribution functions of the scalar particles at both temperature ranges $T > m_\phi$ and $T < m_\phi$ must be the same. Thus, we have

$$f_{\phi\boldsymbol{q}}\left(T = m_\phi\right) = e^{-\frac{\Omega_{\phi\boldsymbol{q}}|_{T=m_\phi}}{m_\phi}} = e^{-\frac{C_1\tilde{\Gamma}_0}{2m_\phi^2}+C_2\left(\boldsymbol{q}^2\right)} \tag{144}$$

or

$$-\frac{\Omega_{\phi\boldsymbol{q}}|_{T=m_\phi}}{m_\phi} = -\frac{C_1\tilde{\Gamma}_0}{2m_\phi^2} + C_2\left(\boldsymbol{q}^2\right). \tag{145}$$

So our missing function is

$$C_2\left(\boldsymbol{q}^2\right) = \frac{C_1\tilde{\Gamma}_0}{2m_\phi^2} - \frac{\Omega_{\phi\boldsymbol{q}}|_{T=m_\phi}}{m_\phi}. \tag{146}$$

And the scalar distribution function is

$$f_{\phi\boldsymbol{q}} = e^{-\frac{C_1\tilde{\Gamma}_0}{2T^2}+\frac{C_1\tilde{\Gamma}_0}{2m_\phi^2}-\frac{\Omega_{\phi\boldsymbol{q}}|_{T=m_\phi}}{m_\phi}}. \tag{147}$$

From here, we can start calculating the neutrino number density. But before that, there are several more steps we must make. Firstly, what is the mass shell $\Omega_{\phi\boldsymbol{q}}$ at temperature $T = m_\phi$? Well, if we look at it's fomula in eqs. (121) and (122), we can see that it must be

$$\Omega_{\phi\boldsymbol{q}}|_{T=m_\phi} = \sqrt{\boldsymbol{q}^2 + m_\phi^2\left(1 + \frac{\lambda}{24}\right)}. \tag{148}$$

Now $1 + \frac{\lambda}{24}$ is rather a small number even if we take for the coupling to be $\lambda = 0.1$ compared to the mass of the scalar. Thus, we can approximate

$$\Omega_{\phi\boldsymbol{q}}|_{T=m_\phi} = \sqrt{\boldsymbol{q}^2 + m_\phi^2\left(1 + \frac{\lambda}{24}\right)} \approx \sqrt{\boldsymbol{q}^2 + m_\phi^2}. \tag{149}$$

Secondly, we need to take a look at the integration variable that is the mass shell in eq. (131). We know, from eqs. (121) and (122) that it should be

$$\Omega_{\phi \boldsymbol{q}} = \sqrt{\boldsymbol{q}^2 + m_\phi^2 + \frac{\lambda}{24}T^2} \tag{150}$$

or

$$\Omega_{\phi \boldsymbol{q}} = \sqrt{\boldsymbol{q}^2 + m_\phi^2}\sqrt{1 + \frac{\frac{\lambda}{24}T^2}{\boldsymbol{q}^2 + m_\phi^2}}. \tag{151}$$

If we now take a look at the second term in the second square root, i.e. $\frac{\frac{\lambda}{24}T^2}{\boldsymbol{q}^2+m_\phi^2}$ one might notice that it too is quite a small number. This is because the temperatures we are integrating all of the distribution function are $T \leq m_\phi$, which in turn means, that the effective mass M_ϕ is $M_\phi \approx m_\phi$ as is noted in the ref. [13]. That means if take for $\boldsymbol{q}^2 = 0$ we have $\sqrt{1 + \frac{\lambda}{24}}$ at the highest possible temperature, i.e. $T = m_\phi$, we would have

$$\sqrt{1 + \frac{\lambda}{24}} \approx 1 + \frac{1}{2}\frac{\lambda}{24}. \tag{152}$$

Taking $\lambda = 0.1$, we get

$$1 + \frac{1}{2}\frac{0.1}{24} \approx 1.002 \tag{153}$$

which is just a number that is very close to 1. And that's more true the smaller the temperature is. Thus our variable becomes

$$\Omega_{\phi \boldsymbol{q}} \approx \sqrt{\boldsymbol{q}^2 + m_\phi^2}. \tag{154}$$

Now, inserting the scalar distribution function into the Boltzmann equation, we have

$$\frac{df_N}{dt} = \frac{2m_\phi \tilde{\Gamma}_0}{p^2} e^{-\frac{C_1\tilde{\Gamma}_0}{2T^2} + \frac{C_1\tilde{\Gamma}_0}{2m_\phi^2}} \int_{p+m_\phi^2/(4p)}^{\infty} e^{-y/m_\phi} dy, \tag{155}$$

where $y = \sqrt{\boldsymbol{q}^2 + m_\phi^2}$. The neutrino distribution function is then

$$f_N = -\frac{2m_\phi^2 m_p \tilde{\Gamma}_0}{p^2 \pi}\sqrt{\frac{90}{g_*}} e^{\frac{C_1\tilde{\Gamma}_0}{2m_\phi^2} - \frac{p+m_\phi^2/(4p)}{m_\phi}} \int_{m_\phi}^{2M} \frac{e^{-\frac{C_1\tilde{\Gamma}_0}{2T^2}}}{T^3} dT. \tag{156}$$

Here, one might ask why do we integrate from the scalar's mass m_ϕ to the double of the sterile neutrino mass $2M$. Firstly, since we split the sterile neutrino period into two, with one period ending at m_ϕ, when the scalar particle production ends, it only seems natural, for the second period to begin, where the first ended. Secondly, we have previously assumed that one scalar particle can produce two sterile neutrinos. Thus the lowest temperature we can reach before the sterile neutrino production is over is when the scalar particle can produce only two sterile neutrinos, both of which have zero momentum. That temperature is $T = 2M$. Then the number density is

$$n_{LS} = -\frac{16m_\phi^2 m_p \tilde{\Gamma}_0}{(2\pi)^3}\sqrt{\frac{90}{g_*}}e^{\frac{C_1\tilde{\Gamma}_0}{2m_\phi^2}} \int_0^\infty e^{-\frac{p+m_\phi^2/(4p)}{m_\phi}} dp \int_{m_\phi}^{2M} \frac{e^{-\frac{C_1\tilde{\Gamma}_0}{2T^2}}}{T^3} dT \tag{157}$$

Thus our full number density would be the sum of the early time period - the period when the scalar particles were produced - and the late time period, when no scalar particles were produced

$$n_N = n_{ES} + n_{LS}. \tag{158}$$

3 Calculations and results

Now that we have a few methods of calculating the sterile neutrino density parameter, we can try and find what kind of mass would the sterile neutrino need to have for it to be considered a dark matter particle candidate. This will be done in two ways. First, we will look into the thermal production of sterile neutrinos. Then, we shall work with the scalar decay production method and in the end, combining it with the thermal production. In our calculations, we assume that the sterile neutrinos are stable and do not decay further.

3.1 Thermal production

In this section, we shall show the main results of the thermal production. To begin calculations we start with eq. (94)

$$n_{TS} = -\int_{\infty}^{M} \frac{7\mu^2 G_{Fermi}^2 T m_P}{96\pi^2\sqrt{(0.1g_*)}} dT \int_0^{\infty} \frac{p^3 dp}{\left(\mu^2 + \left[\frac{c\Gamma p}{M} + \frac{M}{2}\right]^2\right)\left(e^{\frac{p}{T}} + 1\right)}. \tag{159}$$

Next, we need the density, which is rather easy

$$\rho_{TS} = n_{TS} \cdot M. \tag{160}$$

The density parameter is density over critical density

$$\Omega_{TS} = \frac{\rho_{TS}}{\rho_{crit}}. \tag{161}$$

Now the critical density is defined as

$$\rho_{crit} = \frac{3H^2}{8\pi G}. \tag{162}$$

As was mentioned in the previous sections, the Hubble parameter H is not the Hubble constant H_0. That is because, we frankly just don't know how the sterile neutrino density will evolve to the present day. However, the density parameter must stay constant, which comes from the fact, that we assume that neutrinos are stable. Thus, we have to calculate the critical density at the time when the sterile neutrino production is finished.

To do this, we need to look at both the definitions of the Hubble parameter and the gravitational constant G. For the Hubble parameter case, it is fairly easy, as we can look to the appendix C.1, specifically eq. (216) and just use it

$$H^2 = \frac{\pi^2}{90} g_* \frac{T^4}{m_P^2}. \tag{163}$$

For the gravitational constant we have the definition of the Planck mass [1]

$$m_P = \frac{1}{\sqrt{8\pi G}}. \tag{164}$$

Thus, we can insert the definitions into the critical density

$$\rho_{crit} = 3 \cdot \frac{\pi^2}{90} g_* T^4 = \frac{\pi^2 g_* T^4}{30}. \tag{165}$$

Now, if we are looking at the time, when the production ends, that means $T = M$. And for the degrees of freedom, let us assume they are $g_* = 100$. This will be apparent later. However, now we have not only the expression of the critical density

$$\rho_{crit} = \frac{10\pi^2 M^4}{3}, \tag{166}$$

but also the density parameter

$$\Omega_{TS} = \frac{3n_{TS}}{10\pi^2 M^3}. \tag{167}$$

If one looks at eq. (94), one will see, that this equation probably does not have an anallitical solution. Thus, we need to use a computer program. In this case, we used the program MATLAB, it's code is shown in appendix C.2. The results we got are shown in Fig. 2

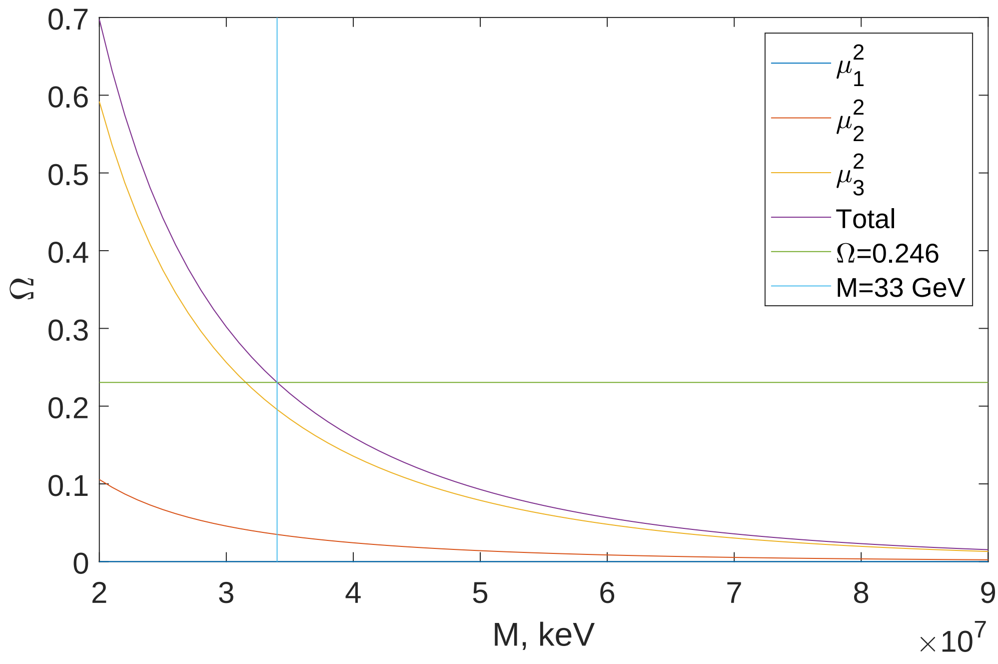

Fig. 2 The dependence of the sterile neutrino density parameter on the mass of the sterile neutrino. The three bottom lines are each active neutrinos contribution to the total density parameter of the sterile neutrino (in purple). The density parameter is $\Omega = 0.246$,when the sterile neutrino mass is 33 GeV.

From Fig. 2 we can see how the sterile neutrino density parameter denpends on the mass of the sterile neutrino. The lowest three lines in Fig. 2 are the separate additions to the density parameter from each active neutrino. We can see, that the lower the mass of the active neutrino the less it is able to produce sterile neutrinos, which one could see from the eq. (79). And from this graph one can also understand why we chose the degrees of freedom g_* to be $g_* = 100$. That is because, we do not cross the threshold in temperature for the degrees of freedom to change from this value, as one can see here

$$g_* \approx \begin{cases} 100, \ T > 300 \text{ MeV} \\ 10, \ 300 \text{ MeV} > T > 1 \text{ MeV} \\ 3, \ T < 1 \text{ MeV} \end{cases} . \tag{168}$$

There is of course, the more general form of the mixing angle, for sterile neutrino thermal production [13]

$$\sin^2(2\theta_M) = \frac{\sin^2(2\theta)}{\sin^2(2\theta) + \left[\cos(2\theta) - \frac{2p}{M^2}(V_D + V_T)\right]^2}, \tag{169}$$

where $\sin^2(2\theta)$ is the mixing angle in the vacuum ($\sin^2(2\theta) = 4\sin^2(\theta)\cos^2(\theta)$); V_D and V_T are finite density and finite temperature matter potentials. V_D dominates in highly matter-antimatter asymmetric environments, like stars; in our case it is negligable. The finite temperature matter potential is given as

$$V_T = -G_{eff}^2 T^4 p, \tag{170}$$

where $G_{eff}^2 = CG_{Fermi}^2$ and C is just some number; the finite temperature matter potential in this approximation is simply a temperature dependant shift in the effective in-medium mass.

We can see, that the eq. (170) resembles one of the terms in eq.(79), specifically, the right-most term in the denominator. If we now reshuffled some terms in eq. (169), like in appendix C.3, we can see that our eq. (169) becomes

$$\sin^2(2\theta_M) = \frac{\mu^2}{\mu^2 + \left(\frac{M}{2} - \frac{p}{M}V_T\right)^2}. \tag{171}$$

And from this equation and the definition of the finite temperature potential (170), we get the coefficient to be

$$G_{eff}^2 = \frac{7\pi}{24} G_{Fermi}^2 c. \tag{172}$$

3.2 The Scalar Decay

The scalar decay case follows a similar path to the thermal production, albeit a bit different. That's because instead of one parameter, the mass of the sterile neutrinos, we have three: the mass of the sterile neutrino M, the mass of the scalar m_ϕ, and the Yukawa coupling between the scalar and sterile neutrino field σ.

In this case, we don't need to calculate the density parameter all over again, we can just take it from the thermal production section, however, there is a problem with this aproach. You may notice in eqs. (139) and (157), that we finish our integrations over temperature at different times - mainly at $T = m_\phi$ and at $T = 2M$. This means, our temperature in the definition of the critical density is different. And so our density parameters are different as well

$$\Omega_{ES} = \frac{3n_{ES}M}{10\pi^2 m_\phi^4}, \ \Omega_{LS} = \frac{3n_{LS}}{20\pi^2 (2M)^3}. \tag{173}$$

Equations (139) and (157) as was the case with thermal production most likely do not have analytical solutions. So we turn to the MATLAB program (see appendix C.4 for the program used in this work). Using a mass of $7.5 \cdot 10^7$ GeV for the mass of the scalar m_ϕ, we were able to get the graph shown in Fig. 3. The choice of this massive scalar mass was done because later we would want to combine both the scalar decay and thermal production methods. Then it would be wise to have density parameters from both methods in the same range of possible sterile neutrinos masses. Thus, to have sensible results in this range, one has to choose this kind of massive scalar mass.

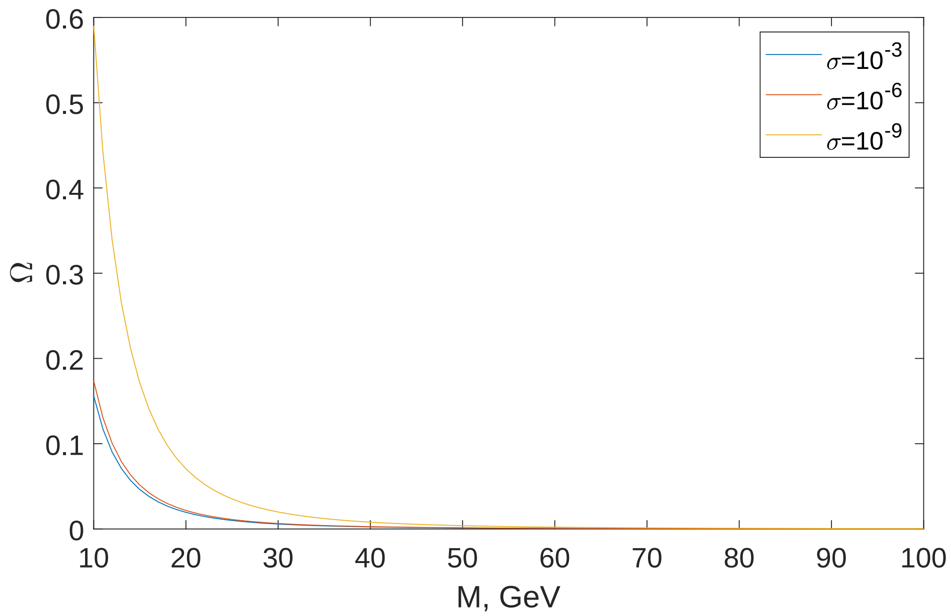

Fig. 3 Sterile neutrino number density with respect to the sterile neutrino mass, when the production method is solely the heavy scalar decay.

In Fig. 3 one can see, that we chose several couplings. The first two couplings give us a fairly similar picture and a mass of sterile neutrinos close to 10 GeV or lower. However, as the coupling lowers, we see that the mass of sterile neutrinos grows as well. This will be elaborated later.

For now, let us ask a simple question - could sterile neutrinos be produced solely because of scalar decay. The answer is rather simple - yes. But would that be realistic is another question. Since, as we know, the early universe's temperature was so high, that many heavy particles were produced. Not only that, in the previous section we discussed how from oscillations between active and sterile neutrinos, we could produce sterile neutrinos. And that required very high temperatures. The ones, we assume to be present when the scalar decay would take place. So, we can assume, that the scalar production in our universe would not be the only source for the sterile neutrino production. We must add thermal production to it as well.

That is rather simple, since the whole sterile neutrino density parameter is

$$\Omega_S = \Omega_{TS} + \Omega_{ES} + \Omega_{LS}. \tag{174}$$

That was done and shown in Fig. 4. In it, we chose the coupling to be $\sigma = 10^{-6}$ and the mass of the scalar $m_\phi = 7.5 \cdot 10^7$ GeV. We can see, that with these parameters, the scalar decay is actually negligable and the mass of the sterile neutrinos is still around 33 GeV.

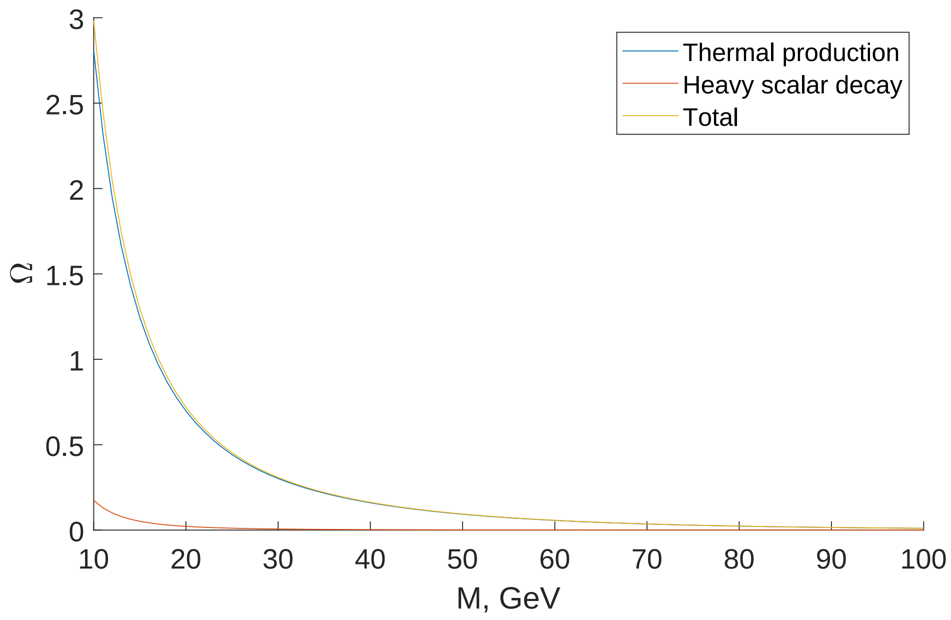

Fig. 4 Sterile neutrino number density with respect to the sterile neutrino mass, when we have two sterile neutrino production methods.

However, that is why it was said, that the scalar decay is rather different from the thermal production. For instance, let us say, that the coupling is $\sigma = 10^{-6}$ and is constant. What would happen to the mass of the sterile neutrinos if the mass of the scalar changed? The answer to that question can be found in Fig. 5. In this graph, we chose many different masses for the scalar particles, calclulated the sterile neutrino density parameter's dependence on the mass of the sterile neutrinos and chose those mass which gave the density parameter in the range $0.24 \leq \Omega_S \leq 0.265$. One may note, that this range is rather different from the one given in the previous sections, but that is because that accuracy of the calculations fell short due to the chosen sterile neutrino mass range, i.e. the chosen difference between two neighbouring neutrino masses was rather big and thus could give us density parameters that were close to the previously mentioned range of $0.25 \leq \Omega_S \leq 0.26$,but would not make it and so would not give us any results.

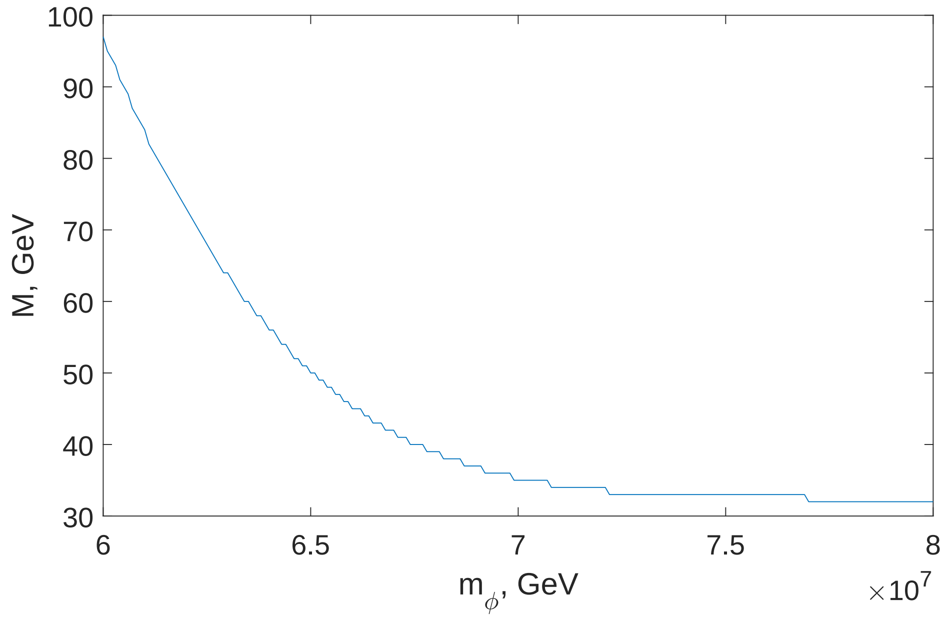

Fig. 5 Sterile neutrino mass, with respect to the mass of the scalar particle. The process of calculating the sterile neutrino density parameter's dependence was done for every scalar mass value. The sterile neutrino mass, that gave us a density parameter, which fulfilled the condintion $0.24 \leq \Omega_S \leq 0.265$ was chosen and plotted in this graph.

In Fig. 5 we can see that the mass of the sterile neutrinos goes down and approaches the thermal production mark of 33 GeV as the mass of the scalar grows. That is because of one simple factor - the integration period. Remember, in eq. (139) the temperature integral stops at the mass of the scalar. Thus, the heavier the mass of the scalar ϕ, the less scalar particles we have, which in turn gives us less sterile neutrinos. Because of that, the sterile neutrino production becomes dominated by the thermal production.

The other thing we can do is say that the mass of the scalar is $m_\phi = 6.6 \cdot 10^7$ GeV and is constant, but the coupling can now be varied. The graph is shown in Fig. 6. The reason why we chose the scalar mass to be as it is, was to introduce a significant presence of scalar decay, but not too significant, so the scalar decay method wouldn't dominate. I will refer you to Fig. 5. In that graph, we see, that the previous choice of scalar mass to be $m_\phi = 7.5 \cdot 10^7$ GeV, meant that the thermal decay would dominate the sterile neutrino production process. If the mass of the scalar would be any more smaller, the scalar decay would dominate the production.

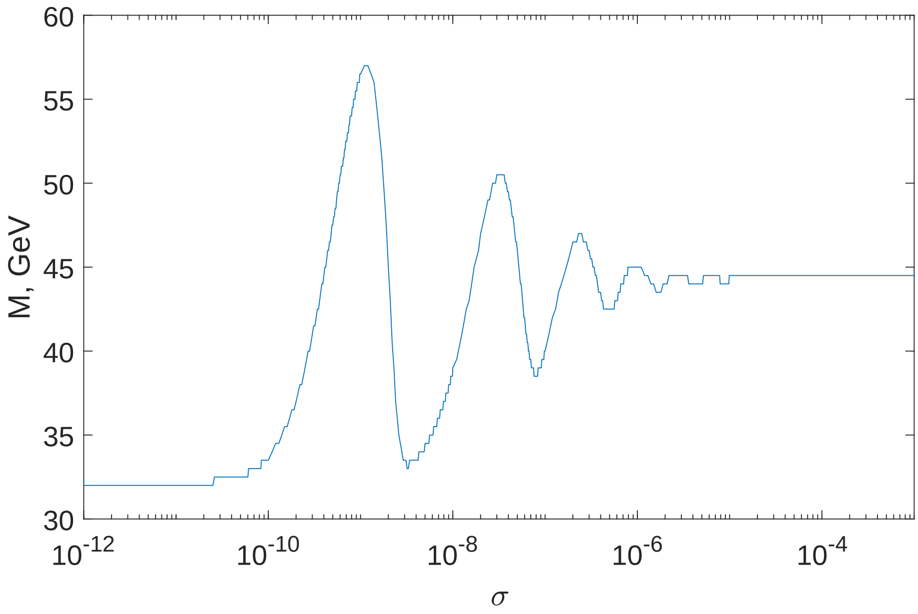

Fig. 6 Sterile neutrino mass with respect to the Yukawa coupling between the sterile neutrino and the scalar fields. The process of calculating the sterile neutrino density parameter's dependence was done for every Yukawa coupling's value. The sterile neutrino mass, that gave us a density parameter, which fulfilled the condintion $0.25 \leq \Omega_S \leq 0.26$ was chosen and plotted in this graph.

To understand Fig. 6, one must know several things. Firstly, there were over eight hundred different coupling values. However, we wanted to have the same amount of couplings between two neighbouring powers of 10. For example, there are the same amount of coupling values between the values $10^{-9} \leq \sigma \leq 10^{-8}$ and $10^{-4} \leq \sigma \leq 10^{-3}$. But the problem was, that the intervals could not have the same step, otherwise it would fit for one range, but be too small for the other. So, a generation program for these couplings was written. Yet it was not without it's problems, as it introduced artifacts that slightly shifted the couplings.

Second thing to note is that while for the previous graphs we were content with a simpler and a smaller mass range, here that did not work. To understand this, the potential sterile neutrino masses, for Figs. 3, 4 and 5 were from 10 GeV to 100 GeV, each differing from the neighbouring masses by 1 GeV. However, for attempts to actually plot the graph in Fig. 6, that was not enough. Thus the difference between neighbouring mass values was changed from 1 GeV to 0.5 GeV. This seems like a small change, but that doubled the amount of iterations we were working with and added a bit more accuracy to our calculations. And due to this newfound accuracy, the density parameter range, i.e. the range the density parameter must fall into, for us to pick a mass and plot it in Fig. 6, was changed to $0.25 \leq \Omega_S \leq 0.265$, to more accurately reflect our own parameters. And even then, the graph in Fig. 6 still looks like it can be smoothened. That was not done due to time constraints, as this graph took too much of time to make.

With this in mind, we can look at the Fig. 6. Firstly, one might notice that in Fig. 6's left side we have a constant sterile neutrino mass as the Yukawa coupling is very small. With the added accuracy, the sterile neutrino mass is 32.5 GeV in that region. This is important, because in the very weak coupling scenario, we expect the scalar decay to be negligable, as one can see from the left side in Fig. 6. That means, that the sterile neutrino mass solely comes from the thermal production. Thus the mass of the sterile neutrino just from thermal production must be in the range $32 \leq M \leq 33$ GeV for the neutrino to be considered a candidate for dark matter particles.

As the coupling starts growing in Fig. 6 we can see a peculiar thing happen - the mass to coupling graph begin to resemble a graph of a dampened oscillator. The biggest point of these oscillations almost reaches 70 GeV, while the lowest point tries to approach the thermal mass limit. Now, why do these oscillations occur? If we look into eq. (157) as it is more dependant on the Yukawa coupling than eq. (139) we see that the coupling appears in all sorts of places like the exponents and just as a second power (remember the $\tilde{\Gamma}_0$ definition from eq. (123)). And we see that the sterile neutrino mass appears as the temperature variable atleast in M^{-3} power. That means, to keep eq. (157) somewhat constant, the mass must shift dramatically in response to the change in the coupling. Why it does so in an oscillatory manner, is unknown and a topic for future investigations.

But, when the coupling becomes very big in comparison with it's lowest value, the oscillations stop and we get a constant value at around $M = 51$ GeV. In this region scalar decay dominates as one can see that the mass almost matches that of the mass in Fig. 5. How to explain this? Well, as the coupling is relatively big, that means that more and more scalar particles decay into sterile neutrinos as opposed to when it was very low, when scalar particles would decay very rarely into sterile neutrinos. Thus we have a process when scalar particles are created they relatively instantly decay into sterile neutrino, scalar particles are produced and decay again so on and so forth. This tells us that the sterile neutrino production not only is dominated by scalar decay but by the early scalar decay, i.e. eq. (139). That is because of the relatively fast decay, once the threshold of $T = m_\phi$ is passed, we don't really have any scalar particles to decay into sterile neutrinos.

4 Conclusions

In this work, we set out to see if the sterile neutrino particle could be a possible candidate for dark matter and under what circumstances that could be. We began with a short introduction to sterile neutrinos and the main reason for a need of them. For this we shortly introduced the Standard Model of particle physics (SM) and the Grimus-Neufeld model (GNM).

From this, we took a look at the basics of cosmology, starting with the Boltzmann equation. We used it to calculate how the different types of particles distribute in the early universe when we have small perturbances in the form of thermal anisotropies and the perturbed metric in eq. (2). Using these equations, we arrived at the equations of motion for the perturbations. The second step in the understanding of cosmological basics was to take a look at how the perturbations of the metric themselves evolve in response to the changing densities of particles because of the Boltzmann equation. When that was done, a conclusion was reached that solving all of the equations was not an easy task and requires to solve them numerically.

Having the cosmological background, we went over the Dodelson-Widrow model [4] in regards of sterile neutrino production via oscillations from active neutrinos. This required the introduction of the Seesaw mechanism and how it changes when going from a vacuum to a medium. From this, we calculated what the number density of sterile neutrinos should be in eq. (94).

We also introduced a second method of sterile neutrino production via heavy scalar decay [5]. We introduced how one could calculate the dampening and the loss/gain rates from the retarded self-energies, which come from the one loop correction to the heavy scalar propogator. After this, we arrived at a concrete Boltzmann equation for sterile neutrinos in eq. (131) when the heavy scalar and the universe are in thermal equilibrium. After introducing assumptions about the scalar decay process we calculated the sterile neutrino number densities, shown in eqs. (139) and (157). The assumptions were: the possibility of splitting the sterile neutrino production period into two periods: the first one being the period when scalar particles were produced thermally and decayed into sterile neutrinos, while the second one being the period when we can't produce scalar particles but have leftovers from the first period and they decay into sterile neutrinos; the second assumption was that the Yukawa coupling between the scalar and sterile neutrino fields σ was very small compared to the scalar self-coupling λ, i.e. $\sigma \ll \lambda$.

Using the calculated results from the previous sections, more specifically the Dodelson-Widrow model and the heavy scalar decay under our imposed assumptions, we used the program MATLAB to calculate the sterile neutrino density parameter. In the first case for thermal production, the graph we got is shown in Fig. 2. From it we can determine that for sterile neutrinos to have a density parameter $\Omega_{TS} = 0.246$ the mass of the sterile neutrinos has to be $M = 33$ GeV. We also determined what the finite temperature potential looks like in our case.

We then calculated the density parameter's dependence on the sterile neutrino mass when the sole production method is heavy scalar decay, which is shown in Fig. 3, when the heavy scalar mass is $m_\phi = 7.5 \cdot 10^7$ GeV. However, seeing as in our universe we expect sterile neutrinos to oscillate, scalar decay could not be the sole production method and must be combined with thermal production, we got the graph as shown in Fig. 4, where the mass of the scalar is the same as in Fig.

3 and the Yukawa coupling is $\sigma = 10^{-6}$. In this graph, we can see that under these circumstances the thermal production dominates and the mass of sterile neutrinos, in the case of Fig. 4 is around $M = 33$ GeV.

However, while the thermal production has one parameter - the sterile neutrino mass M - the heavy scalar decay method has three - the Yukawa coupling σ, the heavy scalar mass m_ϕ and the sterile neutrino mass M. Thus we chose to explore how does the sterile neutrino mass depend on the other two parameters when we have both heavy scalar decay and thermal production methods. That was done to see more accuratelly the bounds at which the thermal production dominates or the heavy scalar decay does.

We began with varying the heavy scalar mass m_ϕ as show in Fig. 5, while the Yukawa coupling was $\sigma = 10^{-6}$. One can see from this graph, that as the heavy scalar mass decreases, for the sterile neutrino to be a candidate for dark matter particles, the neutrino's mass must increase. That is because the lower the scalar mass, the longer we can produce the scalar particles as seen in eq. (139) and thus we can create more sterile neutrinos for longer via the scalar decay. While if the scalar mass increases, the sterile neutrino mass approaches the thermal production limit, as one would expect.

From there, we chose to vary the Yukawa coupling, while the scalar mass was $m_\phi = 6.6 \cdot 10^7$ GeV. This mass was chosen because as seen in Fig. 5 it would allow for less thermal production domination, unlike the mass we used earlier. The resulting graph is shown in Fig. 6, where on the left-most side we can see, that when the Yukawa coupling is very small, the scalar decay is negligible and gives a more accurate mass for the sterile neutrino, that being $M = 32.5$ GeV. Using this graph, one can argue that with enough accuracy, the sterile neutrino mass falls in the range $32 \leq M \leq 33$ GeV. However, on ther right-most side, when the Yukawa coupling is the largest, we have heavy scalar decay domination, and the sterile neutrino mass is around $M = 51$ GeV. This comes from the fact, that if the coupling is big enough, when a scalar particle is produced it then decays relatively fast to sterile neutrinos. This process repeats until we reach the threshold of $T = m_\phi$ at which point the scalar particle production and in turn sterile neutrino production stops since all the scalar particles would be decayed. This implies that while the Yukawa coupling is relatively big, only the early scalar decay, i.e. eq. (139) is important, as it dominates over other processes.

Yet there are problems. First of all is the accuracy of the calculations. The potential sterile neutrino mass range was indeed small, as the difference between two neighbouring mass values was only 1 GeV. In the case of Fig. 6 it was somewhat increased, by changing the difference to be 0.5 GeV, yet it still gives us not so smooth graphs and somewhat inaccurate masses.

The other problem lies with the models themselves. Specifically, the heavy scalar decay. While the thermal production model or the Dodelson-Widrow model has it's own problems like the unspecified masses of active neutrinos, since we only know their mass difference or the assumption of total thermal equilibriun (this is present in heavy scalar decay as well), the heavy scalar decay has it's own set of problems. First of all, to make it simpler, assumptions were made that may have stretched the limits of possible reality. One of these is the splitting of the scalar to sterile neutrino production period into two. Whilst the second is rather heavilly reliant on the Yukawa coupling between the two fields, the first period has only one term with it. And this discrepency might introduce differences if we just took eq. (113), where if we were to remove the decay part (the $f_{\phi q}$ term on the right-hand side) we would have a square of the coupling after integrating. Thus, we would have had two terms with the coupling in the eq. (139).

The problem that is present in both models is the issue of the sterile neutrino mass. The sterile neutrinos are quite heavy. So heavy in fact that they dwarf the mass of the proton which is usually around 938 MeV. And in the papers such as [4, 5, 13] they mainly deal with sterile neutrinos in the keV range. So where does this quite big difference come from. The main factor might be the range of integration. We chose, that for the temperatures, the integration would begin from $T \to \infty$. It is a valid assumption, since we assume that the temperature in the early universe, right after the Big Bang, was very huge, nearly infinite. We would technically assume the Planck temperature, but as it is so big compared to the masses we were working with, we can just choose infinity as our starting point instead and make calculations easier. And in the previously mentioned papers, they usually don't even begin integrating at infinity, which could explain this difference.

However, having said all that, we could ask, which model is better. In the case of less parameters, the Dodelson-Widrow model is the superior one, since we really only have one parameter - the sterile neutrino mass M. While the scalar decay method suffers from too many parameters. But, there is a valid reason to use the scalar decay method. Firstly, if supersymmetry (SUSY) were to be true, we would have additional, heavier particles, some of which were scalars, which decayed. Secondly, we know that in the early universe there was a period of inflation [1]. And to model this period we use a slow rolling potential for a scalar field which, when it reaches it's minimum, decays. So we might have good reason to assume, that there were heavy scalar decay. And so, it might actually be better to use both the Dodelson-Widrow model with the heavy scalar decay.

A Relativity

In this short appendix, I will go over the basics of Einstein's Special and General Relativity theories. They will not be presented in depth. Only the main ideas will be gone over.

Special Relativity

To begin the discussion of Special Relativity (SR) let us first consider it's main postulates:

- There are no experiments to measure an absolute velocity and all the laws of physics behave in the same way regardless of the velocity;
- The Speed of Light is the same in all inertial systems and nothing can move faster than it.

These postulates allowed Einstein to construct his SR. But for our short discussion it is enough to know them.

With that, we can move on to some definitions and rules. But before that, we must state that the speed of light is $c = 1$. Why do we do this? Because it makes equations pretty. Now we can define a vector A^{μ}. The index μ goes over from 0 to 3, 0 representing the time coordinate (all greek letters used for indices start from 0, while all latin letter indices start from 1). So A^0 would be the timelike part of the vector. The rest would be spacelike. Simply put, the spacelike like part of the vector is the same in three dimensional space as it is in four dimensional space.

To calculate the scalar product of a vector we must introduce index lowering or raising. Index lowering is just writing the index lower like A_{μ}, rather then like A^{μ} and vice verca. But that is not done simply but just writing the index lower than it was. For this procedure we must introduce the metric. A metric is a tensor that defines how we multiply vectors. As anybody, who has studied vector calculus can remember, the scalar product multiplies in such a way because of the basis vectors $\vec{e}_{\mu}$. So when we multiply a vector with itself, we write

$$\vec{A} \cdot \vec{A} = \vec{e}_{\alpha} \vec{e}_{\beta} A^{\alpha} A^{\beta}. \tag{175}$$

The multiplication $\vec{e}_{\alpha} \vec{e}_{\beta}$ is the metric $g_{\alpha\beta}$. With it, we can lower or raise the indices of any vector or tensor (a note, a vector has one index, but a tensor has two or more). This lower(raising) can be written as such

$$A_{\nu} = g_{\mu\nu} A^{\mu} = g_{0\nu} A^0 + g_{1\nu} A^1 + g_{2\nu} A^2 + g_{3\nu} A^3. \tag{176}$$

This is where we introduce Einstein's summation over indices, which just states that when we have a multiplication like eq. (176), we must sum over all the matching indice so that the left side and the right side of the equation has the same indices that are not summed over. The summed over indices are sometimes called dummy indices.

With this we can move on to Minkowski space and SR in general. For that we must define the spacetime interval. This interval is invariant under Lorentz transformations and is a four dimensional distance between events. All inertial observers measure it to be the same between two events. It is defined as

$$ds^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta. \tag{177}$$

In Minkowski space the metric[7] is

$$g_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{178}$$

so that the spacetime interval (177) becomes

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}\overrightarrow{x}^2, \tag{179}$$

where $\mathrm{d}\overrightarrow{x}^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ in Cartesian coordinates.

Now one could write out explicitly Lorentz transformations and velocity summation and many more things, but for the purposes of this paper, we only need to define a few more things[8]. More concretely we need to define the four-momentum and the energy-momentum tensor.

Firstly, let's begin with the four-momentum. It is defined as

$$\overrightarrow{p} \rightarrow (E, p^x, p^y, p^z) \tag{180}$$

in Cartesian coordinates, where E is the energy of the object. The scalar product for the four-momentum is

$$\overrightarrow{p} \cdot \overrightarrow{p} = -E^2 + \boldsymbol{p}^2, \tag{181}$$

where $\boldsymbol{p}^2$ is the scalar product of momentum in three dimensional space. However if we take a look from the object's perspective, the momentum changes. We know that momentum is defined as $p^\mu = mv^\mu$,where m is the mass of the object and v^μ is it's four-velocity. But from the object's perspective, it's four-velocity is different. It is the basis velocity u^μ which has in the object's perspective only one component $u^0 = 1$, which just means that the object is moving in time. So from the object's perspective $\overrightarrow{p} \cdot \overrightarrow{p} = m^2 \overrightarrow{u} \cdot \overrightarrow{u} = -m^2$. But this product must be the same in all inertial systems, so we have

$$-E^2 + \boldsymbol{p}^2 = -m^2 \tag{182}$$

or

$$E^2 = \boldsymbol{p}^2 + m^2, \tag{183}$$

which is a well known identity.

From this, we can turn our attentions to the energy-momentum tensor $T^{\alpha\beta}$. Before we write it explicitly, we must talk about it's components since the tensor changes from system to system:

[7]It depends on how you define it, but most cosmologist take the metric to be -,+,+,+, while most particle phisicists take it to be +,-,-,-

[8]For those who want to study SR more indepth, I suggest reading Bernard Schutz's "A first Course in General Relativity" [9].

- T^{00} – is always the energy density ρ;
- T^{0i} – is the energy flux through $x^i = const.$ surface;
- T^{i0} – is the momentum density components;
- T^{ij} – is the stress tensor (this is why this tensor is also referred to stress energy tensor).

The calculations are more than we need to write out the tensor. For ideal fluids the tensor is

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{184}$$

This tensor has also a very useful identity which is the energy-momentum tensor's conservation law

$$\partial_\beta T^{\alpha\beta} = 0. \tag{185}$$

General Relativity

General Relativity (GR) takes SR a step further. To understand this statement we must understand the tenents of GR:

- You can define a system only locally, not globally;
- No experiment can differentiate the sytem between that of moving with accelaration and that of being in a gravitational field;
- Einsten's equivalence principle - the free-falling ovserve in the gravitational field sees his local surrounding as if he had been in Minkowski space.

In a gravitational field, things move differently than in Minkowski space, since gravity affects motion. However, the equations for systems in gravitational fields are of similar form to the equations in SR.

For instance, let's take eq.(185). Here we see the law written down in flat spacetime. But in curved spacetime and so in a gravitational field, one can write the same law in a very familiar form

$$\nabla_\beta T^{\alpha\beta} = 0, \tag{186}$$

where ∇_β is the covariant derivative. The form does resemble that of eq. (185) but it has a different derivative. In fact, you can rewrite any law by replacing derivatives with covariant derivatives and you will have laws that have been affected by a gravitational field.

But what is a covariant derivative, how does it differ from a normal one? Before one can answer that, one must understand a few things about curvature and it's relation to gravity. To do that, we must firstly discuss the manifold - what is it? To put it simply, a manifold is the curvature of spacetime taking form. But there is a special feature of a manifold - on small enough scales the manifold resembles Minkowski space. Or in other words on small scales the manifold resembles Minkowski flat space. One can imagine this with our own experience of Earth. When we are walking in the streets or parks, we see that the Earth appears flat, disregarding any hills and pits; however, from space we can see that the Earth is definetaly not flat, but a sphere (more like of a spherical shape, because of the Earth's spinning it bulges out). The same goes for a manifold - on large scales you apply GR, but on small scales one can apply only SR and not be wrong.

Having said that, one might ask how do you write down this curvature mathematically and relate it's features. Well we already know that way, because we already used it previously - the metric. In Minkowski space we have the very simple metric (178), but in curved spacetime the metric is not so simple and can have many terms, even crossterms (the terms that are not diagonal in the matrix), depending on many variables. And with that the metric dictates how curved is spacetime.

But how does this relate to gravity? For that we must come back to the covariant derivatives. The covariant derivative for a vector V^μ can be written explicitly as

$$\nabla_\beta V^\mu = \partial_\beta V^\mu + V^\sigma \Gamma^\mu_{\sigma\beta}, \tag{187}$$

where $\Gamma^\mu_{\sigma\beta}$ is the Christoffel symbol, which can be written out explicitly as

$$\Gamma^\mu_{\sigma\beta} = \frac{1}{2} g^{\alpha\mu} \left(\partial_\beta g_{\alpha\sigma} + \partial_\sigma g_{\alpha\beta} - \partial_\alpha g_{\sigma\beta} \right). \tag{188}$$

Now we can see how the metric and in turn the curvature of spacetime influence how laws behave in a curved spacetime[9]. We have to note some behaviour of the covariant derivative. For instance, if the vector's index is lowered, then the term in eq.(187) with the Christoffel symbol changes sign

$$\nabla_\beta V_\mu = \partial_\beta V_\mu - \Gamma^\sigma_{\mu\beta} V_\sigma. \tag{189}$$

Also, we get as many Christoffel symbols (with different signs with different types for the types of indices) as there are indices on the differentiable quantity, such as

$$\nabla_\beta B^\mu_\nu = \partial_\beta B^\mu_\nu + B^\alpha_\nu \Gamma^\mu_{\alpha\beta} - \Gamma^\alpha_{\nu\beta} B^\mu_\alpha. \tag{190}$$

With these definitions, we can move on to the real question - how does curvature affect gravity. For that we must ask ourselves what is the shortest path or a geodesic in the curved spacetime. In Minkowski space we already know the geodesics between to points to be a straight line. But in curved spacetime that is different. To calculate the geodesics, we use the geodesic equation

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\tau} \frac{dx^\beta}{d\lambda} \frac{dx^\tau}{d\lambda} = 0, \tag{191}$$

[9]For more reading on how these and other relations can be derived and for more indepth look on GR, I refer you to Kip Thorne's, Charles Misner's and John Wheeler's book "Gravitation" [14]

where λ is the scalar parameter of motion, which can be proper time. If we solved this equation for say spherical curvature we would get geodesics that are orbits or infalling trajectories. With the geodesic equation we can safely say that gravity is a misnomer. Objects fall because that is the shortest distance, a path of least resistance. There is no force of gravity. Gravity is the curvature of spacetime.

Of course, there is one other question - how do objects then interact with gravity, i.e. how do objects curve spacetime. For that we need only a few definitions. Firstly, we define the Riemann tensor

$$R^{\alpha}_{\ \mu\lambda\sigma} = \partial_\lambda \Gamma^{\alpha}_{\mu\sigma} - \partial_\sigma \Gamma^{\alpha}_{\mu\lambda} + \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\nu}_{\mu\sigma} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\nu}_{\mu\lambda}. \tag{192}$$

A neet feature of the Riemann tensor is that if atleast one component out of twenty is not zero, then the spacetime is curved. Using the Riemann tensor we can define the Ricci tensor

$$R_{\beta\nu} = R^{\alpha}_{\ \beta\alpha\nu} \tag{193}$$

and using it we can define the scalar curvature

$$R = R^{\nu}_{\nu} = g^{\beta\nu} R_{\beta\nu}. \tag{194}$$

Do not confuse the scalar curvature with the radius. Although one can apply the scalar curvature to a sphere and get that it equals it's radius, that doesn't mean anything for other shapes. Now using eq.(193) and eq. (194) we can combine them to give us the Einstein tensor

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R, \tag{195}$$

which has a peculiar identity

$$\nabla_\beta G^{\alpha\beta} = 0. \tag{196}$$

So using identities (186) and (196) we can write the Einstein equations for gravity

$$G^{\alpha\beta} = 8\pi G T^{\alpha\beta}, \tag{197}$$

where G is the gravitational constant. We can see two sides to this equation - the left side denotes curvature, while the right side denotes energy density. So curvature "tells" how the energy density must move, but the energy density "tells" how spacetime curves.

And just for show, we can write down the spacetime interval for the metric of a spherically symmetric spacetime in spherical coordinates

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2, \tag{198}$$

where $d\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ and M the mass of the object that creates the curvature. Anybody, who is attentive, can see that this metric has singularities at $r = 0$ and $r = 2GM$, now know as a black holes singularity and the Event Horizon, respectively. To anybody who doesn't know, this is the Schwarzchild metric also known as the metric for an eternal black hole.[10]

[10]For anybody who wants a deeper exploration of black holes, I suggest reading Tai Chow's book "Gravity, Black Holes, and the very Early Universe" [15]

B Quantum Field Theory

In this appendix I would like to acquaint you with the theory of quantum fields. However, this theory is long and requires years of study and cannot easily be simplified to a few pages of text and formulas. Because of that reason I will ommit some sections like renormalization or symmetry groups and instead focus on a small number of basics of quantum field theory (QFT).

The explanation here will differ from that of books about QFT like that of Aitchison [6]. In those books, typically one gets acquainted with special relativity, then with classical field theory and only then does the quantization happen. It is a very important process for those, who wish to study QFT, but here that process will be abandoned for a simpler explanation of QFT where we will not do the tedious steps to get to it and just simply start from it, laying down the basics. But I will recommend, for those, who wish to study QFT indepth to read books like the ones by Aitchison [6, 7] or Schwartz [8].

Quantum fields

Before we begin our talk about quantum fields, we need to lay down some ground rules. Firstly, when we are working with quantum fields, we must also work with special relativity. Thus, we are working with four dimension of spacetime and we are using natural units

$$\hbar = c = 1, \tag{199}$$

where $\hbar$ is the reduced Planck's constant and c is the speed of light. Another thing we must understand is what a field is. A field is simply a function $\psi(x^\mu) = \psi(t, \boldsymbol{x})$ that has a value in spacetime point x^μ. This function or field extends all over spacetime, but has non-zero values only in certain spacetime points (some quantum fields have a non-zero vacuum value, meaning that in all spacetime points it has some fixed non-zero value, but that does not mean, that the field cannot acquire higher values).

These values, that a field acquires are interesting in a way, because the quantum fields oscillate about it's vacuum value and it's maximums are the acquired value, which we can interpret as a particle. Due to the Heisenberg's uncertainty principle, the field always oscillates. That is why, the quantum fields are expanded in a Fourier series

$$\hat{\phi} = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3 \sqrt{2\omega}} \left(\hat{a}(k)\, e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\omega t} + \hat{a}^\dagger(k)\, e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} + \mathrm{i}\omega t} \right), \tag{200}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators, respectively, and $\boldsymbol{k}^2 = \omega^2 - m^2$, where m is the mass of the particle,that the field describes. The value, the field acquires, can be interpreted to create the particle of momentum k and frequency ω.

Different types of fields

Although we have the Fourier expansion of a quantum field in eq. (200) it only applies to one type of field - the scalar field. Besides the scalar field we have other types of fields and their formulations to consider. We will begin with the fields and then move onto formulations.

Scalar fields

Since we started with scalar fields, we can continue to add some smalls things about them to make their description complete. Firstly, we must note that the scalar fields can be complex. That is, they can be constructed of two real fields in a complex formulation like

$$\hat{\phi} = \frac{1}{\sqrt{2}} \left(\phi_1 + \mathrm{i}\phi_2\right), \tag{201}$$

$$\hat{\phi}^\dagger = \frac{1}{\sqrt{2}} \left(\phi_1 - \mathrm{i}\phi_2\right). \tag{202}$$

Secondly, we also must note that the scalar fields have spin-0. That allows us to construct any order of their interaction terms with themselves. With that, there isn't much left to talk about the scalar fields, except that scalar fields satisfy the Klein-Gordon equation

$$\left(\Box - m^2\right) \hat{\phi} = 0, \tag{203}$$

where $\Box = -\partial_t^2 + \nabla^2$ is the d'Alambert operator. A typical example of a scalar field is the Higgs field.

Spinor fields

Spinor fields are a bit different from scalar fields. Firstly, spinor fields $\hat{\psi}_\alpha$ have a spinor index α, which tells us, that spinors have several components - four to be exact (two for a particle, two for an antiparticle). Spinors also have half integer spin - $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ so their interaction terms are constrained, meaning that we cannot construct any kind of term with spinor fields, but more on that later.

Another important thing about spinor fields is that they satisfy the Dirac equation (it is in a way backwards, since spinors were „discovered" by solving the Dirac equation)

$$\left(\mathrm{i}\gamma^\mu \partial_\mu - m\right) \hat{\psi} = 0, \tag{204}$$

where γ^μ is the gamma matrix, which is constructed from Pauli matrices (gamma matrices are 4×4) and how it is constructed depends on our formalism. Gamma matrices are very important, because they allow us to construct interaction terms and work with spin. A typical example of a spinor field is a fermion.

Vector fields

The final type of fields are vector fields $\hat{A}^{\mu}$. They are integer spin fields and like spinor fields, they have more than one component (again, four to be exact). The Maxwell or gauge vector fields must satisfy the equation below, which is imposed by the Lorentz condition

$$\Box \hat{A}^{\mu} = 0. \tag{205}$$

Vector fields can be massive, meaning that they can have mass, but they can be massless as well. A typical example of a gauge vector field is the photon and all the other force carriers.

Dirac, Weyl and Majorana fermions

We have talked about spinors and when talking about the gamma matrices, we mentioned that their form depends on the formalism we use. This is where we talk about formalisms. However, we won't get into to much detail, because each on their own formalisms are quite extensive and we don't really need to talk about them much.

We begin with what we already know - Dirac fermions. These types of fermions are rather simple - their fields satisfy the Dirac equation (eq. (204)). The Dirac fermion fields are complex. But there was one feature we haven't really talked about. It was mentioned that spinors have four components - two for a particle and two for an antiparticle. For now, let's stick with just the particle side. We know, that Dirac fermions are massive. However, each component of the spinor is not massive in the SM. In fact, they are massless. The mass only comes from the interaction with the Higgs field, which in turn requires that two different components of different chirality to be in the interaction term with the Higgs field. In general, the two different components can have mass.

But what is chirality. To put it simply, chirality is the eigenvalue of the gamma matrix γ^5. It gives us the „handedness" of the field. This matrix is constructed as the product of all the gamma matrices

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3. \tag{206}$$

We have two possible states of „handedness" - left and right.

What does this have to do with spinors? Only that, the spinor has two components, that have also two components (making four). The first two components have opposite chirality - one is lefthanded and the other is righthanded. These either lefthanded or righthanded components of fermions are known as Weyl fermions. Physical Weyl fermions would be massless, so if we want to describe real fermions, we need two Weyl fermions with opposite chirality. That is the problem with Weyl fermions. They usually form an incomplete picture of the real particles that we observe. To this day, there have been no known detections of particles that are Weyl fermions, although for a brief period in time, people thought that neutrinos were Weyl, before they were shown to have a mass.

Lastly, we need to talk about another formalism of fermions. We mentioned that Dirac fermion fields are complex, they give complex solutions. However, if we define the gamma matrices in another way, we can have solutions that are real. Those are Majorana fermions. Their solutions must always be real. And that gives us the result that Majorana fermions are their own antiparticles.

There are ways to go between the Dirac and Majorana formalism, however, we don't discuss them here, but one can be referred to Pal's work [16]. To this day, as with Weyl fermions, there have been no observations of Majorana fermions as free particles. But it is hypothesised that neutrinos can be Majorana fermions and that there might be a heavy Majorana neutrino in addition to the known neutrinos. This is one part of the Grimus-Neufeld model.

Interaction terms and Lagrangians

Having talked about all the kinds of fields that there can be, we can now move onto the bulk of QFT - how it is written down. To do that we use Lagrangian densities or Lagrangians[11]. For that, we lay the ground rules for the Lagrangians. Firstly, Lagrangians are real scalars. This means, we can have complex Lagrangian terms but there must also be a complex conjugate of that term so that the total Lagrangian is real; and we cannot have any terms that are just vectors or even tensors - they can be constructed of such things, but in the end, the indices must be summed over to give us a scalar.

Secondly, the Lagrangian must have spin zero and be Lorentz invariant. This gives us terms that cannot be left in a non-zero spin - fields in the constructed terms can have any kind of spin, but in the end it must sum to zero. And the Lorentz invariance comes from special relativity. The Lagrangian must be the same for all observers. The fields may be different for different observers, but the Lagrangian remains the same.

With that in mind, we can move onto the actual terms. For different kinds of fields, we have different kinds of terms and how we write them down. Let's begin with the standard scalar field. For a real scalar field we have the Lagrangian

$$\hat{\mathcal{L}} = \frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} - \frac{1}{2}m^2\hat{\phi}^2. \tag{207}$$

Since scalar fields are spin-0 then we can come up with a multitude of terms with any amount of scalar fields in them, although, these interaction terms grow less probable the higher the amount of fields grows.

Spinor fields are more complex for two reasons. Firstly, we have non-zero spin fields, which must be accounted for. Secondly, in typical spinor Lagrangians interactions with other kinds of fields start to occur (they were possible with scalar fields, but due to the historical significance of quantum electrodynamics (QED), I wanted to move the discussion about those interactions here) and we need to incorporate them aswell.

Let us begin with a single electron interacting with a vector field, let's say a photon. To account for the first problem, we just write down the simple Dirac Lagrangian

$$\hat{\mathcal{L}} = \bar{\hat{\psi}}\left(\mathrm{i}\gamma^\mu\partial_\mu - m\right)\hat{\psi}, \tag{208}$$

where $\bar{\hat{\psi}} = \hat{\psi}^\dagger\gamma^0$. This only includes our electron, we need to include the photon and it's interaction

[11] Physicists, who work with QFT typically use the word Lagrangian even though they are working with Lagrangian densities. There will be no exceptions here in regards of the words used ofr definitions.

with the electron. To do that, we change our derivative a little bit

$$\partial_\mu \to D_\mu = \partial_\mu + \mathrm{i}g\hat{A}_\mu, \tag{209}$$

where g is the coupling constant (typically for an electron interacting with a photon field we would have the electric charge e or q) and $\hat{A}_\mu$ is our photon field. Switching this derivative with what we had and introducing the electromagnetic field or photon propagation term $\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ our Lagrangian changes to

$$\hat{\mathcal{L}} = \bar{\hat{\psi}}\left(\mathrm{i}\gamma^\mu D_\mu - m\right)\hat{\psi} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{210}$$

Using these simple rules, one can construct any kind of a Lagrangian with any kind of terms. However, we want a Lagrangian that describes the particles and interaction that we observe. Thus some limitations must apply, like limiting the power of the possible terms. One such Lagrangian is commonly known as the Stadard Model and it is the most accurate model we have

$$\begin{aligned}\hat{\mathcal{L}}_{\mathrm{ESM}} = \sum_\psi \bar{\hat{\psi}}\mathrm{i}\gamma^\mu D_\mu\hat{\psi} - \frac{1}{4}\hat{W}_a^{\mu\nu}\hat{W}^a_{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}G^a_{\mu\nu}G_a^{\mu\nu} \\ + \left(D_\mu\hat{H}\right)^\dagger\left(D^\mu\hat{H}\right) - V\left(\hat{H}\right) + \sum_\psi\left(A_\psi^2\bar{\hat{\psi}}\hat{H}\hat{\psi} + h.c.\right),\end{aligned} \tag{211}$$

where $\hat{W}_a^{\mu\nu}$is the field strength tensor of the weak interaction and $B^{\mu\nu}$ is the field strength tensor of the U(1) gauge field; $G^a_{\mu\nu}$ is the gluon field strength tensor;$\hat{H}$ is the scalar Higgs field, which is a complex doublet; $V\left(\hat{H}\right)$ is the potential of the Higgs field; $\hat{\psi}$ is every fermion and quark we know. The last term is the Higgs field's interaction with the fermion fields, i.e. the terms that give the masses of particles.

Of course, this is not the complete description. We would also need to talk about symmetries and how they can be broken. Also, we would need to describe the different interactions possible and how the fields couple to the Higgs field to give the particles mass. However, that discussion would not fit in a short appendix. For that reason, these topics and many others, would not be described here, however if one wishes to study more on the topic of QFT, I refer you to the books by Aitchison and Schwartz [6–8].

C Calculating the sterile neutrino number density

C.1 Writing down the number density

To get the number density, we start with the result we had from the Boltzmann equation (84)

$$\frac{dn_S}{dt} = 2\int \frac{d^3p}{(2\pi)^3} \sin^2(2\theta_M)\,\Gamma f_A, \tag{212}$$

where the functions in front of the integral are given by eqs. (79), (78) and (80). Here it is assumed that $E \approx p$. We start from this step, since we only want the number density of the sterile neutrinos. That is because, by multiplying it with the sterile neutrino mass, we get the sterile neutrino density, which can be used to compare to the dark matter density. Another reason we start from this point and not from eq. (90) is because we don't need to compare the sterile neutrino density to active neutrino density. And so we can just calculate the sterile neutrino density because we already know how the functions underneath the integral depend on momentum and temperature (more on this later).

Having said that, we can then start our work with the integral. For starters, it doesn't really matter which direction does our momentum point to. We can always choose a coordinate system in such a way, that we do not look at the direction. So, we can choose to go from the Cartesian momentum coordinate space to the polar momentum coordinate space, giving us

$$d^3p = 4\pi p^2 dp. \tag{213}$$

Having done that, we can just put in our functions

$$\begin{aligned}\int_0^\infty \frac{d^3p}{(2\pi)^3} \sin^2(2\theta_M)\,\Gamma f_A &= 4\pi \int_0^\infty \frac{p^2dp}{(2\pi)^3} \frac{\mu^2}{\mu^2 + \left(\frac{c\Gamma p}{M} + \frac{M}{2}\right)^2} \frac{7\pi}{24} G_{Fermi}^2 T^4 p \frac{1}{e^{\frac{p}{T}+1}} \\ &= 4\pi \frac{7\pi G_{Fermi}^2 T^4 \mu^2}{24(2\pi)^3} \int_0^\infty \frac{p^3dp}{\left(\mu^2 + \left[\frac{c\Gamma p}{M} + \frac{M}{2}\right]^2\right)\left(e^{\frac{p}{T}} + 1\right)}.\end{aligned} \tag{214}$$

Our main variables in this integral are the sterile neutrino mass M, the temperature T, momentum p and the active neutrino mass, which is expressed via the Dirac mass $\mu^2 = m_\nu M$.

Now that we have our momentum integral, we need to take a look at our differentiation with respect to time variable t. Now, one could say, that we don't really need to do anything here. However, we know that as the universe expanded and thus got older, i.e. time passed, the universe cooled. That means the temperature depends on time in some way. And looking at our integral in eq. (214), we can see that only the dependance on temperature will survive the integration. And one can also see that there is no explicit time dependance. So we need to change our differentiation variable from t to T.

Another thing to note, that technically we have a differential equation in eq. (212). Now, for solving this equation analytically, nothing really changes, since we would still have to integrate it.

However, some computer programs can take differential equations and solve them numerically. And they also can integrate at the same time. In our case, we don't really know our sterile neutrino density's n_S initial conditions. One could make some assumptions and in turn have the initial conditions, but in this work that will not be done and we will only use integration.

To use integration, we need to know how to change our integration variables. We don't really know how temperature depends on time, but we do know how the scale factor a depends on time in the radiation dominated era [1]

$$a \sim t^{1/2}. \tag{215}$$

And we also know how temperature is related to the Hubble parameter [1]

$$H^2 = \frac{\pi^2}{90} g_* \frac{T^4}{m_P^2}, \tag{216}$$

where g_* is the degrees of freedom, which depend on the temperature

$$g_* \approx \begin{cases} 100, \; T > 300 \text{ MeV} \\ 10, \; 300 \text{ MeV} > T > 1 \text{ MeV} \\ 3, \; T < 1 \text{ MeV} \end{cases}; \tag{217}$$

and m_P is the Planck mass. Lastly, there is the usual definition of the Hubble parameter

$$H = \frac{\dot{a}}{a} = \frac{\frac{da}{dt}}{a}. \tag{218}$$

Thus, we can take the scale factor to be

$$a = Ct^{1/2}, \tag{219}$$

where C is just some constant. Then we will have the Hubble parameter

$$H = \frac{da}{dt}\frac{1}{a} = \frac{1}{2}Ct^{-1/2} \cdot C^{-1}t^{-1/2} = \frac{1}{2t}. \tag{220}$$

From this, we can equate the right hand sides of eqs. (216) and (220), to get

$$4t^2 = \frac{90 m_P^2}{\pi^2 g_* T^4}. \tag{221}$$

We can take the square root of this equation to get

$$2t = \frac{\sqrt{90} m_P}{\pi \sqrt{g_*} T^2}. \tag{222}$$

Now, all we have to do is take the differential of this equation and we finally get

$$dt = -\frac{\sqrt{90}m_P}{\pi\sqrt{g_*}T^3}dT. \tag{223}$$

Thus our sterile neutrino number density equation becomes

$$\begin{aligned} n_S &= 4\pi \int dt \frac{7\pi G_{Fermi}^2 T^4 \mu^2}{24\,(2\pi)^3\,\pi} \int_0^\infty \frac{p^3 dp}{\left(\mu^2 + \left[\frac{c\Gamma p}{M} + \frac{M}{2}\right]^2\right)\left(e^{\frac{p}{T}} + 1\right)} = \\ &= -\int_\infty^M \frac{7\mu^2 G_{Fermi}^2 T m_P \sqrt{10}}{16\pi^2\sqrt{g_*}} dT \int_0^\infty \frac{p^3 dp}{\left(\mu^2 + \left[\frac{c\Gamma p}{M} + \frac{M}{2}\right]^2\right)\left(e^{\frac{p}{T}} + 1\right)}. \end{aligned} \tag{224}$$

To get the density parameter Ω_S we need the density of sterile neutrinos ρ_S. It is very simple

$$\rho_S = n_S M. \tag{225}$$

Having the density, we can just plug it in in the density parameter

$$\Omega_S = \frac{\rho_S}{\rho_{crit}} = \frac{n_S M}{\rho_{crit}}, \tag{226}$$

where $\rho_{crit} = \frac{3H^2}{8\pi G}$ is the critical density.

C.2 Using MATLAB to calculate the number density integral

In this section we show, how to use the mathematical program MATLAB to calculate the number density. Before we do that, we need to consider our double integral's (224) boundaries. The momentum integral boundaries stay the same - from 0 to ∞. However, the temperature integral is different. As was noted, the production of particles cannot happen, if the temperature is lower than the particle's mass. So, our sterile neutrino density integral (224) integrates only to the mass of the sterile neutrino. And remembering how the degrees of freedom change, depending on the tempereature (eq. (217)), our integrals splits. If the sterile neutrino mass is $M < 1$ MeV,then we will have three integrals; if the sterile neutrino mass is 1 MeV $< M <$ 300 MeV, we will have two integrals and so on. The integration will end at the neutrino's mass in all of the cases. It will begin at ∞ (technically it should begin at Planck temperature $T_{Planck} = 1.416784(16) \times 10^{32}$K, which as you can imagine is very huge. And since it dwarfs the masses we are working with, we can take replace it with ∞ since it will have the same result).

Now that we have sorted out the issues with the integral, we need to calculate it. For that, we use the mathematical program MATLAB. This program allows the user to work with data and mathematical functions, but for our purposes what is most important is that it allows for us to numerically calculate integrals. A thing to note, is that MATLAB also allows to calculate differential equations numerically. And even though the integral (224) is given originally as differential equation eq. (212), we do not use differentiation methods, because we don't know the exact initial conditions, since even the momentum integral needs to be done numerically. But in theory, one could be able to use differentiation methods to calculate the number density.

To begin with, we need to define all of our constants. Those definitions are shown in the figure below

```
close all
clear all
mp=10^18; %%GeV
delta_m_21=7.53*10^(-5); %%+-0.18 eV^2
delta_m_31=2.44*10^(-3); %%+-0.06 eV^2
m_1=0;
m_2=delta_m_21; %%these are squares
m_3=delta_m_31-delta_m_21; %%these are squares
g_star=[3;10;100];
c=26;
M=[0.02:0.001:0.09]*10^9; %%keV
G_Fermi=1.166787*10^(-5); %%GeV^-2
%%for keV
m_1n=m_1*10^(-6);
m_2n=m_2*10^(-6);
m_3n=m_3*10^(-6);
mpn=mp*10^6;
G_Fermin=G_Fermi*10^(-12);
mn=[m_1n;m_2n;m_3n;];
```

Fig. 7 The first part of the code for the MATLAB program used to calculate the sterile neutrino number density, when we have the thermal production mechanism. Here are defined the main constants and parameters that we use. For simplicity, they are first written in the form of their most known units of measurement and then written in terms of keV units.

Here we have m_1, m_2 and m_3 as the squared active neutrino masses, where we have chosen that one is so small it is basically 0. Of course, none of these constants are given in the same units, since they differ so much from each other. And since we are working with keV scale, because of sterile neutrinos, we choose that every constant, whose units are not of order keV, to be so.

The next part is to write down the integral, which is shown in the figure below

```
sum=0;
for i=1:length(mn)
for j=1:length(M)
%%functions
mu=sqrt(mn(i)).*M(j);
konsta=-7*G_Fermin.^2*mu*mpn*sqrt(10)/(16*pi^2);
Gamma=@(p,T) 7.*pi./24.*(G_Fermin.^2).*(T.^4).*p;
firstp=@(p,T) (p.^3)./(mu+(c.*Gamma(p,T).*p./M(j)+M(j)./2).^2);
secondp=@(p,T) 1./(exp(p./T)+1);
firstT=@(T) T;
theint=@(p,T,g_star) konsta.*firstT(T).*firstp(p,T)
.*secondp(p,T)./sqrt(g_star);
%%integrals
if M(j)>300*10^3
res1(i,j)=integral2(@(p,T)theint(p,T,g_star(3)),0, Inf, Inf, M(j));
end
if M(j)<=300*10^3
if M(j)>10^3
res1(i,j)=integral2(@(p,T)theint(p,T,g_star(3)),0, Inf, Inf, 300*10^6)
+integral2(@(p,T)theint(p,T,g_star(2)),0, Inf, 300*10^6, M(j));
end
end
if M(j)<=10^3
res1(i,j)=integral2(@(p,T)theint(p,T,g_star(3)),0, Inf, Inf, 300*10^6)
+integral2(@(p,T)theint(p,T,g_star(2)),0, Inf, 300*10^6, 10^6)
+integral2(@(p,T)theint(p,T,g_star(1)),0, Inf, 10^6, M(j));
end
rho(i,j)=res1(i,j)*M(j);
rho_crit=((pi.^2).*10./3).*(M(j).^4);
omega(i,j)=rho(i,j)./rho_crit;
end
plot(M,omega(i,:));
sum=sum+omega(i,:);
end
plot(M,sum);
```

Fig. 8 The second part of the code for the MATLAB program used to calculate the sterile neutrino number density, when we have the thermal production mechanism. Here, we define the functions, that are in the main eq. (94). Below them we calculate the integrals at the appropriate periods of temperature, to account for the shift in the degrees of freedom (see eq. (217)).

Here, we write out the integral functions explicitly in the functions of $Gamma$, $firstp$, $secondp$, $firstT$ so that our integral function $theint$ does not get cluttered and looks nicer. Then, we can see the separations because of the degrees of freedom in the if functions, which look at the sterile neutrino's mass and determine how many integrals will we have. After that, we just calculate the integral, using the $integral2$ function, which is a MATLAB function and allows us to calculate the dual integral. This function has several modes, but changing them might not allow for infinite boundaries to be used, so it was left unchosen, because the $integral2$ function will automatically choose the mode, that allows for such boundaries.

We calculate this double integral for all active neutrino masses mn and for all the possible sterile neutrino masses $M(j)$ and the sterile neutrino densities are then calculated and shown in Fig. 2.

C.3 Calculating the coefficient for the finite temperature matter potential

We have to begin with a few equations and notations. Firstly, we begin with the seesaw mechanism, mainly

$$\sin^2(\theta) \approx \frac{m_\nu}{M} \tag{227}$$

and

$$m_\nu = \frac{\mu^2}{M}, \tag{228}$$

where μ again is the Dirac mass and we have used the fact that $M \gg m_\nu$. Then, we rewrite the equation (169)

$$\sin^2(2\theta_M) = \frac{\sin^2(2\theta)}{\sin^2(2\theta) + \left[\cos(2\theta) - \frac{2p}{M^2}V_T\right]^2}. \tag{229}$$

Note here that instead of the previously mentioned sine, we have a double angle $\sin^2(2\theta)$. So we begin with writing out the double angle sine and cosine

$$\sin^2(2\theta) = 4\sin^2(\theta)\cos^2(\theta), \tag{230}$$

$$\cos(2\theta) = 1 - 2\sin^2(\theta). \tag{231}$$

Here, we can use the facts, that $M \gg m_\nu$ and $\cos^2(\theta) = 1 - \sin^2(\theta) \approx 1$. Thus our double angle sine and cosine change to

$$\sin^2(2\theta) = 4\frac{m_\nu}{M} \tag{232}$$

and

$$\cos(2\theta) = 1. \tag{233}$$

With this, we can rewrite our sine in eq. (229)

$$\sin^2(2\theta_M) = \frac{4\frac{m_\nu}{M}}{4\frac{m_\nu}{M} + \left[1 - \frac{2p}{M^2}V_T\right]^2}. \tag{234}$$

Now, we can divide by $\frac{4}{M}$,to get

$$\sin^2(2\theta_M) = \frac{m_\nu}{m_\nu + \frac{M}{4}\left[1 - \frac{2p}{M^2}V_T\right]^2}. \tag{235}$$

From here we use the eq. (228) and we get

$$\sin^2(2\theta_M) = \frac{\frac{\mu^2}{M}}{\frac{\mu^2}{M} + \frac{M}{4}\left[1 - \frac{2p}{M^2}V_T\right]^2}. \tag{236}$$

Then, we again divide by $\frac{1}{M}$ and the result is

$$\sin^2(2\theta_M) = \frac{\mu^2}{\mu^2 + \frac{M^2}{4}\left[1 - \frac{2p}{M^2}V_T\right]^2}. \tag{237}$$

To make this result a bit more like (79), we „put in“ the parameter $\frac{M^2}{4}$ under the brackets, to get the result shown in eq. (171)

$$\sin^2(2\theta_M) = \frac{\mu^2}{\mu^2 + \left(\frac{M}{2} - \frac{p}{M}V_T\right)^2}. \tag{238}$$

C.4 Using the MATLAB program for heavy scalar decay

To begin with, the program used for the integration of the heavy scalar decay was quite short and is shown in the figure below.

```
close all
clear all
coupling=[10^(-3)];
m_phi=6.9*10^7; %%GeV
m_p=10^18; %%GeV
M=[10:1:100];
g_star=100;
lambda=0.1;
for k=1:length(coupling)
        Gamma=(coupling(k).^2)*m_phi./(16.*pi);
        C_1=(m_p/pi ).*sqrt(90/g_star);
        %%functions
        temp=@(T) 1./(T.^2);
        expon=@(p,T) exp(-(p+(m_phi.^2)./(4.*p))/T);
        C_2=-2.*C_1.*Gamma./(pi^2);
        int=@(p,T) temp(T).*expon(p,T);
        C_3=-2.*(m_phi.^2).*C_1.*Gamma./(pi.^2);
        C_4=exp(C_1.*Gamma./(2.*(m_phi.^2)));
        C_5=C_4.*C_3;
        expon_2=@(T) exp(-C_1.*Gamma./(2.*(T.^2)));
        temp_2=@(T) 1./(T.^3);
        expon_3=@(p) exp(-(p+(m_phi.^2)./(4.*p))./m_phi);
        int_2=@(p,T) expon_3(p).*temp_2(T).*expon_2(T);
        %%integrals
        for i=1:length(M)
                n_1=C_2.*integral2(int,0,Inf,Inf,m_phi);
                omega_1(i,k)=3*n_1./pi^2./10./(m_phi.^3);
                n_2=C_5.*integral2(int_2,0,Inf,m_phi, 2.*M(i));
                omega_2(i,k)=3*n_2./pi^2./10./((2*M(i)).^3);
                omega(i,k)=omega_1(i,k)+omega_2(i,k);
        end
        plot(M,omega)
end
```

Fig. 9 The code of the MATLAB program used to calculate the sterile neutrino density parameter just from the scalar decay process. Before the for function we have the parameters, then in the for function we define the function used in the calculations. At the second for we calculate the integrals from eqs. (139) and (157) and in turn the density parameter.

At the top - before the for function we have the definitions and parameters we use for the program. In the for function we have a constant $\tilde{\Gamma}_0$ that we see in eq. (123), but here it changes when the coupling changes.

Below we have the functions or the cut up parts of both integrals in eqs. (139) and (157). We do this for the sake of keeping track of smaller parts so as to see mistakes easier. Lastly, we have the integrals, which use the same $integral2$ function of MATLAB from the thermal production (see appendix C.2). Except here we have to calculate everything a bit differently. Firstly, eq. (139) holds only till the temperature reaches the mass of the scalar, i.e. $T = m_{\phi.}$. When that is done, the first process, a.k.a. early scalar decay in eq. (139) stops and the second process, the late scalar decay in eq.(157) begins. Now the integration does not become harder. The program will integrate up to a point we ask it to. However, there might come a question is this okay. Do we have to calculate additional things for these density parameters? As it turns out - no. We just have to correctly calculate the critical densities as shown in eq. (173). That is done in functions $omega\_1$ and $omega\_2$,where we divide by the appropriate masses or the temperature thresholds.

References

[1] S. M. Carroll, *Spacetime and Geometry*. Cambridge University Press, 7 2019.

[2] S. Dodelson, *Modern Cosmology*. Amsterdam: Academic Press, 2003.

[3] Y. Fukuda *et al.*, Measurements of the solar neutrino flux from Super-Kamiokande's first 300 days, *Phys. Rev. Lett.*, 1998, **81**, 1158–1162. [Erratum: Phys.Rev.Lett. 81, 4279 (1998)].

[4] S. Dodelson and L. M. Widrow, Sterile neutrinos as dark matter, *Physical Review Letters*, Jan 1994, **72**, 17–20.

[5] M. Drewes and J. U. Kang, Sterile neutrino dark matter production from scalar decay in a thermal bath, *Journal of High Energy Physics*, May 2016, **2016**.

[6] I. J. R. Aitchison and A. J. G. Hey, *Gauge theories in particle physics: A practical introduction. Vol. 1: From relativistic quantum mechanics to QED*. Bristol, UK: CRC Press, 2012.

[7] I. J. R. Aitchison and A. J. G. Hey, *Gauge theories in particle physics: A practical introduction. Vol. 2: Non-Abelian gauge theories: QCD and the electroweak theory*. Bristol, UK: CRC Press, 2012.

[8] M. D. Schwartz, Quantum Field Theory and the Standard Model, 2013.

[9] B. Schutz, *A First Course in General Relativity*. Cambridge University Press, 2 ed., 2009.

[10] G. Senjanović, Neutrino mass: From LHC to grand unification, *Nuovo Cimento Rivista Serie*, Feb. 2011, **34**, 1–68.

[11] J. M. Cline, Constraints on almost Dirac neutrinos from neutrino - anti-neutrino oscillations, *Phys. Rev. Lett.*, 1992, **68**, 3137–3140.

[12] M. Drewes and J. U. Kang, Sterile neutrino Dark Matter production from scalar decay in a thermal bath, *JHEP*, 2016, **05**, 051.

[13] A. Boyarsky, M. Drewes, T. Lasserre, S. Mertens, and O. Ruchayskiy, Sterile neutrino dark matter, *Progress in Particle and Nuclear Physics*, Jan 2019, **104**, 1–45.

[14] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation / Charles W. Misner, Kip S. Thorne, John Archibald Wheeler*. W. H. Freeman San Francisco, 1973.

[15] T. Chow, *Gravity, Black Holes, and the Very Early Universe: An Introduction to General Relativity and Cosmology*. 01 2008.

[16] P. B. Pal, Dirac, Majorana and Weyl fermions, *Am. J. Phys.*, 2011, **79**, 485–498.

Fizikinės evoliucijos ankstyvoje Visatoje skaičiavimų metodai

Aurimas Vitkus

Santrauka

Dalelių fizikos modeliai, kaip standartinis modelis, neturi jokio paaiškinimo iš kokių dalelių sudaryta tamsioji materija. Taip pat, standartiniame modelyje, neutrinai neturi masės, kas yra paneigta šiuolaikinių eksperimentų. Grimus-Neufeld modelis, standartinio modelio papildymas, kuriame yra pridėdamas vienas masyvus Majorana neutrinas ir antras Higso dubletas, išsprendžia neutrinų masės problemą. Taip pat suteikia galima teigti, kad pridėtas sterilus neutrinas gali būti kandidatas tamsiajai materijai.

Šiame darbe yra aptariami sterilių neutrinų produkcijos metodai ankstyvoje Visatoje ir išvedamos pagrindinės sterilių neutrinų tankio formulės. Remiantis jomis, yra apskaičiuojama sterilių neutrinų tankio priklausomybė nuo jų masės iš jos randama sterilaus neutrino masė, su kuria tankio parametras atitinką tamsiosios materijos tankio parametrą.

Darbo pradžioje yra aptariami kosmologijos pagrindai, kurie yra reikalingi suprasti sterilių neutrinų produkcijos metodus. Šie pagrindai yra Boltzmann'o ir Einšteino lygtys. Jomis pasinaudojus išvedamos išraiškos, kurios apibūdina visų dalelių tankių evoliuciją ankstyvoje Visatoje. Po to yra pereinama prie Dodelson-Widrow modelio arba terminės produkcijos. Jo veikimo principas yra aktyvių neutrinų osciliavimas į sterilius neutrinus. Pasinaudojus šiuo modeliu, yra išvedama sterilių neutrinų skaičiaus tankio formulė bei sterilių ir aktyvių neutrinų santykio evoliucija.

Po to yra aptariamas antrasis sterilių neutrinų gamybos metodas - sunkaus skaliario skilimo. Apibrėžiami papildomi reikalingi dydžiai bei parametrai: skaliaro masė, sąveikos konstanta tarp skaliaro ir neutrino laukų. Išvedamos dvi sterilių neutrinų skaičiaus tankio formulės dviems atskiriems periodams: kai skaliarai buvo termiškai gaminami ir skilo į sterilius neutrinus ir kai skaliarų produkcija sustoja ir likę skaliarai skilinėjo.

Pasinaudojus MATLAB programa ir sterilių neutrinų skaičiaus tankio formulėmis buvo apskaičiuoti tankio parametrų vertės, kurios buvo palygintos su tamsiosios materijos tankio parametro verte. Terminės produkcijos atveju, sterilių neutrinų tankio parametras yra $\Omega_{TS} = 0.246$, kai sterilių neutrinų masė yra $M = 33$ GeV. Sunkaus skaliaro atveju, pasirinkus skaliaro masę $m_{\phi} = 7.5 \cdot 10^{7}$ GeV ir sąveikos konstantą $\sigma = 10^{-6}$, sterilių neutrinų masė turi būti apie $M \approx 10$ GeV. Apjungus šiuos modelius buvo pastebėta, kad kuo mažesnė skaliaro masė, tuo labiau dominuoja skaliaro skilimo metodas ir sterilių neutrinų masė turi didėti, kad jų tankio parametras atitiktų tamsiosios materijos tankio parametrą. Taip pat, buvo pastebėta, kad su ypač maža sąveikos konstanta, sterilių neutrinų produkcijoje dominuoja terminė produkcija. Atvirkščiu atveju, kai sąveikos konstanta yra labai didelė, dominuoja skaliaro skilimo metodas. Tarpiniu atveju atsiranda svyravimai sterilių neutrinų masės priklausomybėje nuo sąveikos konstantos. Šių svyravimų atsiradimas nėra paaiškintas, bet yra manau, kad jie atsiranda dėl antrojo skaliaro skilimo etapo ir terminės produkcijos konkuravimo.

Pabaigoje yra aptariami modelių skirtuma:parametrų kiekis; terminės produkcijos metodo pranašumas parametrų kiekio atžvilgiu prieš sunkaus skaliaro skilimo metodą. Taip pat yra aptariamos

metodų problemos: neutrinų masė, kuri skiriasi nuo straipsnių, kuriais buvo remtasi šiame darbe; bei skaičiavimų tikslumas. Pirmoji problema yra paaiškinama temperatūros integravimo ruožu; šiame darbe temperatūra pradedama integruoti nuo begalybės, o remtuose straipsniuose nuo 200 MeV. Antroji problema yra paaiškinama potencialios sterilaus neutrinų masės ruožo per didelio žingsnio pasirinkimu.