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NONHOMOGENEOUS BOUNDARY VALUE PROBLEM FOR THE STATIONARY NAVIER–STOKES SYSTEM IN DOMAINS WITH NONCOMPACT BOUNDARIES

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VILNIAUS UNIVERSITETAS

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STACIONARI NAVJĖ–STOKSO SISTEMA SU NEHOMOGENINE KRAŠTINE SĄLYGA SRITYSE SU NEKOMPAKTIŠKAIS KRAŠTAIS

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Contents

Chapter 1

Introduction

The Navier–Stokes equations are a mathematical model aimed to describe the motion of an incompressible viscous fluid, for instance, water, glycerine, oil, etc. We consider the stationary nonhomogeneous boundary value problem for the Navier-Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p &= 0 \quad \text{in} \quad \Omega, \\ \text{div}\ \mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{a} \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

Here $\mathbf{u} = \mathbf{u}(x) = \big(u_1(x), u_2(x), u_3(x)\big)$ and $p = p(x)$ are the unknown velocity field and the pressure of the fluid, while $\mathbf{a}(x) = \big(a_1(x), a_2(x), a_3(x)\big)$ is given boundary value; $\nu > 0$ is the constant coefficient of viscosity.

In the thesis problem (1.1) is studied in domains Ω with noncompact multiply connected boundaries $\partial\Omega$. We consider domains with two types outlets to infinity: paraboloidal and layer type. Both of them can expand at infinity. We prove the existence of solutions to problems (1.1) in the case when the boundary value $\mathbf{a}$ has nonzero fluxes over the connected components of the boundary $\partial\Omega$.

Actuality and literature review

The Navier–Stokes equations are of great importance in the mathematical hydrodynamics. The solvability of the boundary and initial-boundary value problems for the Navier–Stokes equations has been studied in many papers and monographs (see, for example, [13], [31], [68]). Indeed, such problems are important from both points of view: applications and theoretically. Concerning the applications, one could image, for example, a flow of oil in a system of pipelines, blood flow, etc. The rigorous mathematical analysis of Navier–Stokes equations started at the beginning of the XX century in works of the famous French mathematician prof. J.

Leray. He had formulated several problems (called Leray's problem) which remain open until nowadays.

Bounded domain

In bounded domains Ω with multiply connected boundaries $\partial\Omega$ consisting of N disjoint components Γ_j problem (1.1) was studied first by prof. J. Leray in his

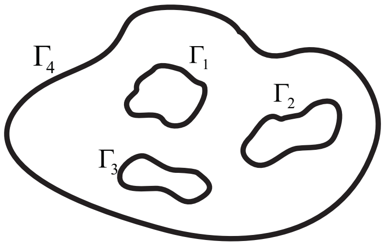

Figure 1.1: Domain Ω.

celebrated paper [36] published in 1933, and thereafter by many mathematicians (see [1], [2], [4], [5], [7]–[14], [23], [24], [27]–[30], [41], [53]–[58], [65] [67], [69], etc.). Continuity equation (1.1_2) implies in the case of a bounded domain Ω the necessary compatibility condition for the boundary value $\mathbf{a}$:

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=1}^{N}\int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = 0, \tag{1.2}$$

where $\mathbf{n}$ is a unit vector of the outward (with respect to Ω) normal to $\partial\Omega$. However, for a long time the existence of a weak solution $\mathbf{u}\in W^{1,2}(\Omega)$ to problem (1.1) was proved only either under the condition of zero fluxes

$$\mathcal{F}_j = \int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = 0, \qquad j = 1, 2, \dots, N, \tag{1.3}$$

(see, for example, [30], [31], [36], [69]), or assuming the fluxes $\mathcal{F}_j$ to be sufficiently small (see, for example, [2], [9], [10], [13], [14], [29]), or under certain symmetry assumptions on the domain Ω and the boundary value $\mathbf{a}$ (see, for example, [1], [11], [12], [28], [41], [53], [54], [58]). Recently problem (1.1) was solved for arbitrary large flux $\mathcal{F}$ having "correct" sign in a plane domain with two connected components of the boundary (the fluid flows in through the outer component, and flows out through the inner component) [27]. Condition (1.3) requires the fluxes $\mathcal{F}_j$ of the boundary value $\mathbf{a}$ to be zero separately on each connected component Γ_j of the boundary $\partial\Omega$, while the compatibility condition (1.2) means only that the total flux is zero. Obviously, condition (1.3) is stronger than (1.2), and (1.3) does not allow the presence of sinks and sources. In [36] J. Leray formulated a question whether problem (1.1) is solvable only under the necessary compatibility condition (1.2). In general case this *Leray's problem* still remains open despite of efforts of many mathematicians (see the above references and, in particular, the review papers [53], [54]).

Homogeneous boundary value problem in unbounded domain

In domains with noncompact boundaries problem (1.1) with homogeneous boundary conditions (below we denote it by (1.1_0)) was exhaustively studied during the last 35 years. After the work of J. Heywood [22] such problems have got a great progress in a row of papers. Problem (1.1_0) was investigated in a wide class of domains Ω having "outlets to infinity", and it was found that for the correct formulation of (1.1_0) it is necessary to prescribe additional conditions: for example, fluxes over the cross sections of outlets to infinity (see [22], [25], [32], [33], [59], [60]). However, any solenoidal vector field $\mathbf{v}(x)$ with the finite Dirichlet integral $\int\limits_{\Omega} |\nabla \mathbf{v}|^2 dx < \infty$ has necessary zero fluxes over the cross sections of "narrow" (for example, cilindrical) outlets (see [59]). Therefore, the usual energy estimates method becomes insufficient in this case, and the Navier–Stokes problem (1.1_0) with additionally prescribed fluxes in "narrow" outlets has to be studied in a class of functions having infinite Dirichlet integrals. The basic results concerning such problems were obtained by O.A. Ladyzhenskaya and V.A. Solonnikov [34], [61]–[64]. In [34] the special technique of integral estimates (so–called "techniques of Saint–Venant's principle") is developed and the existence of solutions having prescribed fluxes is proved. The solutions have either finite or infinite Dirichlet integral over the outlets to infinity depending on geometrical properties of them. Note that all these results are obtained without any restrictions on data assuming only that the total flux is equal to zero. There are also many papers (see, for example, [43], [49], [48]) devoted to the investigation of related questions, such as regularity, asymptotic behavior and uniqueness of solutions to the steady Navier–Stokes problem in domains with noncompact boundaries.

Nonhomogeneous boundary value problem in unbounded domain

However, not much is known about the nonhomogeneous boundary value problem (1.1) in domain with noncompact boundaries. To the best of our knowledge the first time problem (1.1) with nonhomogeneous boundary condition was solved without prescribing a "smallness condition" in 1999 by S.A. Nazarov and K.

Pileckas [42]. In [42] problem (1.1) was studied in an infinite layer $\mathbb{L} = \{x \in \mathbb{R}^3 : 0 < x_3 < 1\}$ under the assumption that on the bottom $S_0 = \{x \in \mathbb{R}^3 : x_3 = 0\}$ there is a compactly supported sink or source of an arbitrary intensity:

$$\operatorname{supp} \mathbf{a} \subset B_{R_0} = \{x : \sqrt{x_1^2 + x_2^2} < R_0, x_3 = 0\}, \quad \int\limits_{B_{R_0}} a_3(x_1, x_2) dx_1 dx_2 = \mathcal{F},$$

where the flux $\mathcal{F}$ is arbitrary large. In [42] the existence of at least one weak solution is proved. Moreover, in [46], [52] the asymptotic behavior of this solution is investigated. Notice that the constructed solution has an infinite Dirichlet integral.

Next, we should mention the series of papers by H. Morimoto, H. Fujita and H. Morimoto [37]–[40], where problem (1.1) is studied in symmetric two-dimensional multiply connected domains Ω with channel-like outlets to infinity containing a

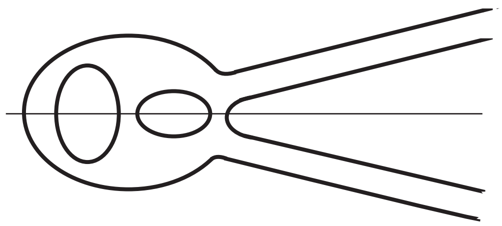

Figure 1.2: Domain Ω.

finite number of "holes" (so called, $\mathbb{Y}$-shaped, $\mathbb{V}$-shaped, $\mathbb{I}$-shaped, cross-shaped channels, or semi-infinite channels). Assuming that the boundary value $\mathbf{a}$ is zero on the "outer" boundary and that $\mathbf{a}$ satisfies symmetry assumptions on bounded connected components of $\partial\Omega$ (boundaries of the "holes"), it is proved in [37]–[40] that problem (1.1) admits at least one solution which tends in every channel to the corresponding Poiseuille flow. These results are obtained under assumptions that the total flux is zero (i.e., the sum of fluxes of the Poiseuille flows and of the fluxes of the boundary value $\mathbf{a}$ over the holes is equal to zero), the fluxes of Poiseuille flows are assumed to be "small", while the fluxes of boundary value $\mathbf{a}$ can be arbitrary large.

Recently, J. Neustupa [44], [45] has studied problem (1.1) in unbounded domains Ω with multiply connected boundaries. He supposed that the fluxes of $\mathbf{a}$ over bounded components of the boundary are "small", but he did not impose any conditions on fluxes over infinite components of the boundary (of course, the total flux is equal to zero). Assuming that the boundary value $\mathbf{a}$ admits a solenoidal extension $\mathbf{A}$ with $\mathbf{A} \in L^3(\Omega)$, $\nabla\mathbf{A} \in L^2(\Omega)$, J. Neustupa proved the existence of at least one solution to (1.1). Note that the existence of the solenoidal extension $\mathbf{A}$ with above properties and also the method used in [45] (a priori estimates of the solution are obtained using a contradiction argument) impose restriction on the domain Ω. The solutions found in [45] have finite Dirichlet integrals and, therefore, the domain Ω should expand at infinity sufficiently rapidly, in order to have enough place to transfer a flux of the fluid from a bounded part of $\partial\Omega$ to infinity.

The advanced theory of the Stokes and the stationary Navier–Stokes equations in exterior domains is presented in the books of G.P. Galdi ([13]). The solvability of the Navier–Stokes equations and asymptotics of the solutions in exterior domains was also studied in many papers [6], [15], [16], [17], [18], [19] and [20].

In the thesis we study the stationary Navier–Stokes equations with nonhomogeneous boundary conditions in domains Ω which may have two types of outlets to infinity: paraboloidal and layer type. The boundary $\partial\Omega$ is multiply connected and consists of connected noncompact components, forming the outer boundary, and connected compact components, forming the inner boundary. We suppose that the fluxes over the components of the inner boundary are sufficiently small, while we do not impose any restrictions on fluxes over the infinite components of the outer boundary. Note that the total flux of sources and sinks is equal to zero. Depending on the geometry of outlets to infinity, the Dirichlet integral of the solution may be either finite or infinite.

Aims and problems

The main aim of the dissertation is the analysis of the stationary Navier–Stokes equations with nonhomogeneous boundary conditions in unbounded domains with noncompact multiply connected boundaries. To prove the existence of solutions of the boundary value problem for the Navier–Stokes equations we need to construct a suitable extension $\mathbf{A}$ of boundary value $\mathbf{a}$. Since the construction of the extension is complicated, we start with two examples. In the first case we consider nonhomogeneous value boundary problem for the stationary Navier–Stokes equations in the domain with one paraboloidal outlet to infinity, and in the second case - the domain consists of two connected layers. The next part of the thesis consists of the generalization of these examples. We study the following problems:

- the solvability of the nonhomogeneous boundary value problem for the stationary Navier– Stokes equations in the domain with one paraboloidal outlet to infinity,
- the solvability of the nonhomogeneous boundary value problem for the stationary Navier– Stokes equations in the domain consisting of two connected layer type outlets,

- the construction of special extensions of the boundary value in domains with finite number of paraboloidal outlets to infinity,
- the construction of special extensions of the boundary value in domains with finite number of paraboloidal and layer type outlets to infinity.

Methods

In the thesis we apply the methods of functional analysis, fix point theory, properties of Sobolev spaces, estimates of Saint Venant type and some special techniques developed by V.A. Solonnikov [61] and O.A. Ladyzhenskaya [30], [31]. The most important is the Hopf's cut–off functions techniques which is used to construct special vector fields satisfying Leray–Hopf inequalities.

Novelty

All results obtained in the thesis are new. The existence of solutions with infinite Dirichlet integral of the nonhomogeneous boundary value problem for the stationary Navier–Stokes equations in domains with noncompact multiply connected boundaries was not known.

Structure of the dissertation and main results

Dissertation consists of six chapters, conclusions and bibliography. The first chapter contains a short review about actuality and history of the problem. It describes also shortly the main results obtained in the dissertation.

The second chapter provides the reader with some preliminaries such as basic notations and known auxiliary results used in this thesis.

In the third chapter we formulate the problem in general case and give general schemes of the proof of its solvability.

In the fourth chapter we consider two examples, i.e., we study the problem in a domain with one paraboloidal outlet to infinity and in a domain consisting of two connected layers. For both examples we construct a suitable extension of the boundary value and prove the existence of a solution.

In the fifth chapter we generalize the first example and consider the problem in domains with finite number of paraboloidal outlets to infinity for the two and three dimensional cases. We construct a suitable extension of the boundary value and formulate the theorem about the existence of a solution (without proof, because the proof is the same as in the case of one paraboloidal outlet (Subsection 4.1.3)).

In the sixth chapter we consider the problem in domains with finite number of paraboloidal and layer type outlets to infinity. We construct an extension of the boundary value and formulate the existence theorem.

Dissemination of results

The results of this thesis were presented in the following conferences

- *Parabolic and Navier–Stokes equations 2012,* Bedlewo, Poland, September 2 - 8, 2012.
- *53th Conference of Lithuanian Mathematical Society,* Klaipėda, Lithuania, June 11 - 12, 2012.
- *52th Conference of Lithuanian Mathematical Society,* Vilnius, Lithuania, June 16 - 17, 2012.
- *Regularity Aspects of PDE,* Bedlewo, Poland, September 5 - 11, 2010.

Schools

- *International Summer School on Evolution Equations "EVEQ 2012",* Prague, Czech Republic, July 9 - 13, 2012.
- *12th School "Mathematical Theory in Fluid Mechanics",* Kacov, Czech Republic, May 27 - June 3, 2011.

Contributing talks were given at the seminars at the Department of Differential Equations and Numerical Mathematics in Vilnius University.

Publications

The results of this thesis are published in the following papers:

Published papers:

1. K. Kaulakytė, K. Pileckas, On the nonhomogeneous boundary value problem for the Navier–Stokes system in a class of unbounded domains, *Journal of Mathematical Fluid Mechanics,* **14**, No. 4 (2012), 693 - 716.
2. K. Kaulakytė, Stationary Navier–Stokes equations with nonhomogeneous boundary condition in a system of two connected layers, *Lithuanian Mathematical Journal (conference works),* **53**, (2012), 13-18.

3. K. Kaulakytė, Stationary Navier–Stokes problem with nonhomogeneous boundary condition in an unbounded domain, *Lithuanian Mathematical Journal (conference works),* **52**, (2011), 28-33.

Submitted papers:

1. K. Kaulakytė, K. Pileckas, On nonhomogeneous boundary value problem for Navier–Stokes system in domains with layer type outlets to infinity, *Topological Methods in Nonlinear Analysis,* (2012), 29 p.

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I would like to thank my supervisor, Professor Konstantinas Pileckas for his help and support.

Chapter 2

Notations and preliminary results

We use $c,\ C, c_j,\ j = 1, 2, ...,$ to denote constants whose numerical values or whose dependence on parameters in unessential to our considerations. In such case c may have different values in a single computation.

Let V be a Banach space. The norm of an element u in the function space V is denoted by $\|u\|_V$. Vector-valued functions are denoted by bold letters; spaces of scalar and vector-valued functions are not distinguished in notations. The vector-valued function $\mathbf{u} = (u_1, \dots, u_n)$ belongs to the space V, if $u_i \in V, i = 1, \dots, n$, and $\|\mathbf{u}\|_V = \sum\limits_{i=1}^{n} \|u_i\|_V$.

Let Ω be an arbitrary domain in $\mathbb{R}^n$. As usual, denote by $C^\infty(\Omega)$ the set of all infinitely differentiable functions defined on Ω and let $C_0^\infty(\Omega)$ be the subset of all functions from $C^\infty(\Omega)$ with compact support in Ω. For a given nonnegative integer k and $q > 1$, $L^q(\Omega)$ and $W^{k,q}(\Omega)$ indicate the usual Lebesgue and Sobolev spaces with the norms

$$\|u\|_{L^q(\Omega)}\| = \Big(\int\limits_{\Omega} |u(x)|^q\, dx\Big)^{1/q} \ \text{ and } \ \|u\|_{W^{k,q}(\Omega)} = \Big(\sum_{|\alpha|=0}^{k} |D^\alpha u(x)|^q\, dx\Big)^{1/q},$$

respectively. $W^{k-1/q,q}(\partial\Omega)$ is the trace space on $\partial\Omega$ of functions from $W^{k,q}(\Omega)$ with the norm

$$\|u\|_{W^{k-1/q,q}(\partial\Omega)} = \inf\{\|\widehat{u}\|_{W^{k,q}(\Omega)} : \widehat{u} = u \text{ on } \partial\Omega\}.$$

$\mathring{W}^{k,q}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of $W^{k,q}(\Omega)$; we write $u \in W^{k,q}_{loc}(\Omega)$ if $u \in W^{k,q}(\Omega')$ for any bounded subdomain Ω' with $\overline{\Omega'} \subset \Omega$.

Let $D(\Omega)$ be the Hilbert space of vector functions formed as the closure of $C_0^\infty(\Omega)$ in the Dirichlet norm $\|\mathbf{u}\|_{D(\Omega)} = \|\nabla\mathbf{u}\|_{L^2(\Omega)}$ generated by the scalar product

$$(\mathbf{u}, \mathbf{v}) = \int\limits_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx,$$

where $\nabla \mathbf{u} : \nabla \mathbf{v} = \sum\limits_{j=1}^{n} \nabla u_j \cdot \nabla v_j = \sum\limits_{j=1}^{n} \sum\limits_{k=1}^{n} \frac{\partial u_j}{\partial x_k} \frac{\partial v_j}{\partial x_k}$. Denote by $J_0^\infty(\Omega)$ the set of all solenoidal ($\operatorname{div} \mathbf{u} = 0$) vector fields $\mathbf{u}$ from $C_0^\infty(\Omega)$. By $\widehat{H}(\Omega)$ we indicate the subspace of $D(\Omega)$ consisting of solenoidal vector fields, and by $H(\Omega)$ – the space formed as the closure of $J_0^\infty(\Omega)$ in the Dirichlet norm. Obviously, $H(\Omega) \subset \widehat{H}(\Omega)$. In general, the spaces $\widehat{H}(\Omega)$ and $H(\Omega)$ do not coincide (see, for example, [22], [25], [32], [59], [61]). However, if Ω is a bounded domain with Lipschitz boundary, then $H(\Omega) = \widehat{H}(\Omega)$ (see [32]).

Lemma 2.1. *(Cauchy inequality with ε). For any $a,\ b \in \mathbb{R}$ the following inequality*

$$|ab| \leq \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2 \quad \forall \varepsilon > 0, \tag{2.1}$$

holds.

Lemma 2.2. *(Hölder inequality). Let $q > 1,\ \frac{1}{q} + \frac{1}{q'} = 1,\ f \in L^q(\Omega),\ g \in L^{q'}(\Omega)$. Then the following inequality*

$$\begin{aligned} \left| \int\limits_{\Omega} f(x) g(x) \, dx \right| &\leq \Big(\int\limits_{\Omega} |f(x)|^q \, dx \Big)^{1/q} \Big(\int\limits_{\Omega} |g(x)|^{q'} \, dx \Big)^{1/q'} \\ &= \|f\|_{L^q(\Omega)} \|g\|_{L^{q'}(\Omega)} \end{aligned} \tag{2.2}$$

holds. For $q = q' = 2$ Hölder inequality

$$\left| \int\limits_{\Omega} f(x) g(x) \, dx \right| \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)} \tag{2.3}$$

is called the Cauchy–Schwarz inequality.

Lemma 2.3. *(Poincaré inequality). Let Ω be bounded domain in $\mathbb{R}^n$. Then for any $u \in \mathring{W}^{1,2}(\Omega)$ the inequality*

$$\int\limits_{\Omega} |u(x)|^2 \, dx \leq c(\operatorname{diam}(\Omega))^2 \int\limits_{\Omega} |\nabla u(x)|^2 \, dx, \tag{2.4}$$

holds, where c is an absolute constant.

Lemma 2.4. *(Ladyzhenskaya inequality). Let $\Omega \in \mathbb{R}^3,\ u \in \mathring{W}^{1,2}(\Omega)$. Then the following inequality*

$$\|u\|^4_{L^4(\Omega)} \leq (4/3)^{3/2} \|u\|_{L^2(\Omega)} \|\nabla u\|^3_{L^2(\Omega)} \tag{2.5}$$

holds.

For the proofs of the last two lemmas see, for example, [3] and [31], respectively.

Lemma 2.5. *Let* Ω *be a bounded domain with Lipschitz boundary and* $f \in L^2(\Omega)$ *satisfies the condition*

$$\int\limits_\Omega f\,dx = 0.$$

Then problem

$$\begin{cases} \text{div } \boldsymbol{u} &= f, \ \in \Omega, \\ \boldsymbol{u} &= 0, \quad x \in \partial\Omega. \end{cases} \tag{2.6}$$

has a solution $\boldsymbol{u} \in \overset{\circ}{W}{}^{1,2}(\Omega)$ *and there holds the estimate*

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} \leq c\|\nabla f\|_{L^2(\Omega)} \tag{2.7}$$

with the constant c *independent of* $\boldsymbol{u}$ *and* f.

Lemma 2.6. *Let* Ω *be a bounded domain with Lipschitz boundary. If* $\boldsymbol{\varphi} \in W^{1/2,2}(\partial\Omega)$ *and* $\int\limits_{\partial\Omega} \boldsymbol{\varphi}\cdot\boldsymbol{n}\,dS = 0$, *then there exists solenoidal extension* $\boldsymbol{\Phi} \in W^{1,2}(\Omega)$ *of function* $\boldsymbol{\varphi}$*:*

$$\begin{cases} \text{div } \boldsymbol{\Phi} &= 0, \ x \in \Omega, \\ \boldsymbol{\Phi} &= \boldsymbol{\varphi}, \quad x \in \partial\Omega. \end{cases} \tag{2.8}$$

Morover, the estimate

$$\|\boldsymbol{\Phi}\|_{W^{1,2}(\Omega)} \leq c\|\boldsymbol{\varphi}\|_{W^{1/2,2}(\partial\Omega)} \tag{2.9}$$

holds.

For the proofs of the last two lemmas see, for example, [32].

Let us introduce bounded domains

$$\omega_R^{(1)} = \{x : |x'| < g(x_3), \ R < x_3 < R + \frac{g(R)}{2L}\}$$

and

$$\omega_R^{(2)} = \{x : x_3 < h(|x'|), \ R < |x'| < 2R\},$$

where the function g satisfies the Lipschitz condition

$$|g(t_1) - g(t_2)| = L|t_1 - t_2|, \ t_1, t_2 \geq 1, \quad g(t) \geq 1 \ \ \forall t,$$

and the function h possess the following properties

$$\mu_1 h(t) \leq \max_{t \leq t_1 \leq 2t} h(t_1) \leq \mu_2 h(t), \quad h(t) \geq 1 \ \ \forall t \geq 1,$$

$$|h(t_1) - h(t_2)| \le L(t)|t_1 - t_2|, \quad t_1, t_2 \in [t, 2t].$$

Here μ_1, μ_2 are certain positive constants and for $L(t)$ holds the inequality

$$\frac{L(t) \cdot t}{h(t)} \le const, \quad L(t) \le const \quad \forall t.$$

Lemma 2.7. *Let $u \in \overset{\circ}{W}{}^{1,2}(\omega_R^{(j)})$, $j = 1, 2$. Then the following inequalities*

$$\int\limits_{\omega_R^{(1)}} |u(x)|^2\,dx \le c\,g^2(R) \int\limits_{\omega_R^{(1)}} |\nabla u(x)|^2\,dx, \tag{2.10}$$

$$\int\limits_{\omega_R^{(2)}} |u(x)|^2\,dx \le c\,h^2(R) \int\limits_{\omega_R^{(2)}} |\nabla u(x)|^2\,dx, \tag{2.11}$$

hold, where the constant c is independent of u and R.

Lemma 2.8. *Let $u \in \overset{\circ}{W}{}^{1,2}(\omega_R^{(j)})$. Then the following inequalities*

$$\|u\|_{L^4(\omega_R^{(1)})} \le c\,g^{1/4}(R)\|\nabla u\|_{L^2(\omega_R^{(1)})}, \tag{2.12}$$

$$\|u\|_{L^4(\omega_R^{(2)})} \le c\,h^{1/4}(R)\|\nabla u\|_{L^2(\omega_R^{(2)})} \tag{2.13}$$

hold, where the constant c is independent of u and R.

The proof of this lemma follows directly from Ladyzenskaya inequality (2.5) and Poincaré inequalities (2.10), (2.11).

Lemma 2.9. *Let $f \in L^2(\omega_R^{(1)})$ and*

$$\int\limits_{\omega_R^{(1)}} f\,dx = 0.$$

Then problem (2.6) *admits a solution $\boldsymbol{u} \in \overset{\circ}{W}{}^{1,2}(\omega_R^{(j)})$ satisfying the estimate*

$$\|\nabla \boldsymbol{u}\|_{L^2(\omega_R^{(1)})} \le c\|\nabla f\|_{L^2(\omega_R^{(1)})} \tag{2.14}$$

with the constant c independent of $\boldsymbol{u}$, f and R.

Lemma 2.10. *Let $f \in L^2(\omega_R^{(2)})$ and*

$$\int\limits_{\omega_R^{(2)}} f\,dx = 0.$$

Then problem (2.6) admits a solution $\boldsymbol{u} \in \overset{\circ}{W}{}^{1,2}(\omega_R^{(2)})$ *satisfying the estimate*

$$\|\nabla \boldsymbol{u}\|_{L^2(\omega_R^{(2)})} \leq c\frac{R}{h(R)}\|\nabla f\|_{L^2(\omega_R^{(2)})} \tag{2.15}$$

with the constant c *independent of* $\boldsymbol{u}$, f *and* R.

For the proofs of the last two lemmas see in [32] and [47], respectively.

Lemma 2.11. *(Gauss-Ostrogradsky theorem). Let* $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, *be a bounded domain with Lipschitz boundary* $\partial\Omega$. *Then the following relation*

$$\int\limits_{\Omega} \operatorname{div} \boldsymbol{F} dS = \int\limits_{\partial\Omega} \boldsymbol{F} \cdot \boldsymbol{n} dS$$

holds.

Lemma 2.12. *(Stokes theorem). Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with Lipschitz boundary* $\partial\Omega$. *Then the following relation*

$$\int\limits_{\Omega} \operatorname{curl} \boldsymbol{F} dS = \int\limits_{\partial\Omega} \boldsymbol{F} \cdot dl$$

holds.

Lemma 2.13. *(Leray-Schauder theorem). Let* H *be a Hilbert space and* $\mathcal{A} : H \to H$ *be a nonlinear compact operator. If norms of all possible solutions of operator equation*

$$u^{(\lambda)} = \lambda \mathcal{A} u^{(\lambda)}, \quad \lambda \in [0, 1],$$

are bounded by the same constant c *independent of* λ, *i.e.,*

$$\|u^{(\lambda)}\|_{H(\Omega)} \leq c \quad \forall \lambda \in [0, 1],$$

then the operator equation

$$u = \mathcal{A}u$$

has at least one solution $u \in H$ *(see, for example, [31]).*

Lemma 2.14. *Let* Ω *be a bounded domain with Lipschitz boundary* $\partial\Omega$, $\mathcal{L} \subseteq \partial\Omega$ *and the function* $\boldsymbol{h} \in W^{1/2,2}(\partial\Omega)$ *satisfies the conditions* $\int\limits_{\mathcal{L}} \boldsymbol{h} \cdot \boldsymbol{n} dS = 0$, $\operatorname{supp} \boldsymbol{h} \subset \mathcal{L}$. *Then* $\boldsymbol{h}$ *can be extended inside* Ω *in the form*

$$\mathbf{b}_0^*(x, \varepsilon) = \operatorname{curl}(\chi(x, \varepsilon)\boldsymbol{E}(x)), \tag{2.16}$$

where $\boldsymbol{E} \in W_2^2(\Omega)$, $\operatorname{curl} \boldsymbol{E}|_{\partial\Omega} = \mathbf{h}$ *and* χ *is a Hopf's type cut-off function, i.e.,* χ

is smooth, $\chi(x,\varepsilon)=1$ *on* $\mathcal{L}$, $\operatorname{supp}\chi$ *is contained in a small neighborhood of* $\mathcal{L}$ *and*

$$|\nabla\chi(x,\varepsilon)| \le \frac{\varepsilon c}{\operatorname{dist}(x,\mathcal{L})}. \tag{2.17}$$

The constant c *is independent of* ε.

Lemma 2.15. *Let* Ω *be a bounded domain with Lipschitz boundary* $\partial\Omega$, $\mathcal{L}\subseteq\partial\Omega$, $u\in W^{1,2}(\Omega)$ *and* $u|_{\mathcal{L}}=0$. *Then the following estimate*

$$\int\limits_{\Omega}\frac{u^2\,dx}{\operatorname{dist}^2(x,\mathcal{L})} \le c\int\limits_{\Omega}|\nabla u|^2\,dx \tag{2.18}$$

holds.

For proofs of the last two lemmas see [31].

Lemma 2.16. *Let non-negative numbers* y_k, $k=1,...,N$, *satisfy inequalities* $y_{k+1}\ge y_k$ *and*

$$y_k \le c_1(y_{k+1}-y_k)+c_2\kappa_k^{1/2}(y_{k+1}-y_k)^{3/2}+\frac{1}{2}Q_k, \tag{2.19}$$

where Q_k *satisfy*

$$\frac{1}{2}Q_k \ge c_1(Q_{k+1}-Q_k)+c_2\kappa_k^{1/2}(Q_{k+1}-Q_k)^{3/2}. \tag{2.20}$$

If $N<\infty$ *and* $y_N\le Q_N$, *then* $y_k\le Q_k \ \ \forall k<N$.

The proof of this lemma see in [61].

Let $\mathcal{M}$ be a closed set in $\mathbb{R}^n$. By $\Delta_{\mathcal{M}}(x)$ we denote the regularized distance from the point x to the set $\mathcal{M}$. Notice that $\Delta_{\mathcal{M}}(x)$ is infinitely differentiable function in $\mathbb{R}^n\setminus\mathcal{M}$ and the following inequalities

$$a_1 d_{\mathcal{M}}(x)\le\Delta_{\mathcal{M}}(x)\le a_2 d_{\mathcal{M}}(x),\quad |D^\alpha\Delta_{\mathcal{M}}(x)|\le a_3 d_{\mathcal{M}}^{1-|\alpha|}(x) \tag{2.21}$$

hold. Here $d_{\mathcal{M}}=dist(x,\mathcal{M})$ is the real distance from x to $\mathcal{M}$, the positive constants a_1,a_2 depend only on the dimension n, while a_3 depends on n and on the order of differentiation $|\alpha|$ (see [66]).

Let γ be an infinite [1] smooth simple curve. Define in $\mathbb{R}^3$ the vector field $\mathbf{b}(x)$,

[1]Defining an infinite curve we have in mind that the curve is infinite to both sides, so it is "closed" at infinity.

corresponding to γ, by the formula

$$\mathbf{b}(x) = \frac{1}{4\pi} \oint\limits_{\gamma} \frac{x-y}{|x-y|^3} \times d\mathbf{l}_y. \tag{2.22}$$

Vector field $\mathbf{b}(x)$ is a magnetic field generating, upon passage through γ, an electric flow of unit intensity.

If γ is an x_3 axis, then magnetic field $\mathbf{b}(x)$ can be rewritten in the form

$$\mathbf{b}(x) = \frac{1}{2\pi}\Big(-\frac{x_2}{|x'|^2}, \frac{x_1}{|x'|^2}, 0\Big). \tag{2.23}$$

Lemma 2.17. *The vector field* $\boldsymbol{b}$ *is solenoidal in* $\mathbb{R}^3 \setminus \gamma$, $\operatorname{curl} \boldsymbol{b} = 0$, *and the circulation of* $\boldsymbol{b}$ *along any closed contour, enveloping* γ, *is equal to* -1 *if the direction of integration along this contour and along* γ *are connected by the gimlet rule. If this contour does not envelop* γ, *then the circulation of* $\boldsymbol{b}$ *along it is equal to zero. At points whose distance from* γ *is not less than* d_0, *we have the estimate*

$$|D_x^\alpha \boldsymbol{b}(x)| \le \frac{c(\alpha, d_0)}{d_\gamma^{1+|\alpha|}(x)},$$

where $d_\gamma(x) = \operatorname{dist}(x, \gamma)$, d_0 *is sufficiently small positive number.*

For the proof see in [59].

Below we will use the following cut-off functions. Denote by $0 \le \Psi \le 1$ and ϱ the smooth monotone functions such that

$$\Psi(t) = \begin{cases} 0, & t \le 0, \\ 1, & t \ge 1, \end{cases} \tag{2.24}$$

$$\varrho(\tau) = \begin{cases} \dfrac{a_1}{2} d_0, & \text{for } \ \tau \le \dfrac{a_2}{2} d_0, \\ \tau, & \text{for } \tau \le a_2 d_0, \end{cases} \tag{2.25}$$

where a_1, a_2 are constants from inequality (2.21).

Chapter 3

Formulation of the problem and the general scheme

Let $\Omega \subset \mathbb{R}^3$ be an unbounded domain which splits outside the ball $B_{R_0}(0) = \{x \in \mathbb{R}^3 : |x| < R_0\}$ into $J \geq 1$ noncompact disjoint components, i.e.,

$$\Omega = \Omega_0 \bigcup \Omega^{(1)} \bigcup \Omega^{(2)} \bigcup ... \bigcup \Omega^{(J)},$$

where $\Omega_0 = \Omega \bigcap B_{R_0}(0)$. The unbounded components $\Omega^{(j)},\ j = 1, ..., J$, are called "outlets" to infinity. We study two types of outlets to infinity: paraboloidal and layer type outlets. When the cross section of paraboloidal outlet is constant, we have a cylinder. Therefore, cylinders are included as well. We denote paraboloidal outlets by D_j and layer type outlets - by E_j.

The bounded domain Ω_0 has the form

$$G_0 \setminus \bigcup_{i=1}^{I} \overline{G}_i,$$

where G_0 and $G_i, i = 1, \dots, I,\ I \geq 0$, are bounded simply connected domains such that $\overline{G}_i \subset G_0,\ \overline{G}_{i_1} \bigcap \overline{G}_{i_2} = \emptyset$ for $i_1 \neq i_2$. Therefore, boundary of the domain Ω consists of inner and outer boundaries. Let us denote them by Γ and S, respectively. Both of these boundaries may consist of finite number of components, i.e., $\Gamma = \bigcup\limits_{i=1}^{I} \Gamma_i$, $\Gamma_i = \partial G_i$, and $S = \bigcup\limits_{m=1}^{M} S^{(m)}$. Components of inner boundary are bounded surfaces, and components of outer boundary are unbounded.

3.1 Formulation of problem

We consider the stationary Navier–Stokes system with nonhomogeneous boundary condition in a domain Ω with outlets to infinity

$$\begin{cases} -\nu\Delta\mathbf{u}+\big(\mathbf{u}\cdot\nabla\big)\mathbf{u}+\nabla p &= 0 \quad \text{in} \quad \Omega,\\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega,\\ \mathbf{u} &= \mathbf{a} \quad \text{on} \quad \partial\Omega,\\ \int\limits_{\sigma_j(R)} \mathbf{u}\cdot\mathbf{n}\,dS &= \mathcal{F}_j, \quad j=1,2,\dots,J,\ R\geq R_0, \end{cases} \tag{3.1}$$

where $\mathcal{F}_j$, $j=1,...,J$, are the prescribed fluxes of the velocity field over cross sections $\sigma_j(R)$ of the outlets $\Omega^{(j)}$, $\mathbf{n}$ is the unit vector of the normal to σ_j.

We assume that the boundary value $\mathbf{a}\in W^{1/2,2}(\partial\Omega)$ has a compact support:

$$\operatorname{supp}\mathbf{a}\subset\partial\Omega\bigcap B_{R_1}(0)=\big(\Gamma\bigcup S\big)\bigcap B_{R_1}(0),\ R_1\geq R_0. \tag{3.2}$$

Denote $\Lambda_m=\operatorname{supp}\mathbf{a}\bigcap S^{(m)}\subset S^{(m)}\bigcap B_{R_1}(0)$.

Integrating the divergence equation $\operatorname{div}\mathbf{u}=0$ over the domain $\Omega\cap B_R(0)$ with sufficiently large R we obtain the following necessary compatibility condition for the fluxes $\mathcal{F}_j$ and the boundary value $\mathbf{a}$:

$$\begin{aligned} 0&=\int\limits_{\Omega\cap B_R(0)}\operatorname{div}\mathbf{u}\,dx=\int\limits_{\partial(\Omega\cap B_R(0))}\mathbf{u}\cdot\mathbf{n}\,dS\\ &=\sum_{i=1}^{I}\int\limits_{\Gamma_i}\mathbf{a}\cdot\mathbf{n}\,dS+\sum_{m=1}^{M}\int\limits_{\Lambda_m}\mathbf{a}\cdot\mathbf{n}\,dS+\sum_{j=1}^{J}\int\limits_{\sigma_j(R)}\mathbf{u}\cdot\mathbf{n}\,dS. \end{aligned} \tag{3.3}$$

Denote by

$$\mathbb{F}_i^{(inn)}=\int\limits_{\Gamma_i}\mathbf{a}\cdot\mathbf{n}\,dS,\ i=1,...,I,\qquad \mathfrak{F}_m^{(out)}=\int\limits_{\Lambda_m}\mathbf{a}\cdot\mathbf{n}\,dS,\ m=1,...,M,$$

the fluxes of the boundary value $\mathbf{a}$ over connected components of the inner and the outer boundaries, respectively. Then condition (3.3) can be written as

$$\sum_{i=1}^{I}\mathbb{F}_i^{(inn)}+\sum_{m=1}^{M}\mathfrak{F}_m^{(out)}+\sum_{j=1}^{J}\mathcal{F}_k=0. \tag{3.4}$$

Condition (3.4) means that the total flux is equal to zero.

Assumption (3.2) on the boundary value $\mathbf{a}$ is made in order to insure that the fluxes $\mathfrak{F}_m^{(out)}$ of $\mathbf{a}$ over unbounded parts $S^{(m)}$ of the boundary $\partial\Omega$ have sense. In general, the question to which functional space belongs the trace of the solenoidal vector field having a finite Dirichlet integral in a domain with noncompact boundary is rather complicated and the answer essentially depends on the geometry of outlets to infinity (see [45], Example 3). This question is related to a problem of the correct functional setting for the divergence operator. It is well known that for domains Ω with outlets to infinity the problem

$$\begin{cases} \operatorname{div} \mathbf{u} &= g \qquad \text{in} \quad \Omega, \\ \mathbf{u} &= 0 \qquad \text{on} \quad \partial\Omega, \end{cases}$$

with arbitrary right–hand side $g \in L^2(\Omega)$ is not solvable in a class of functions $\mathbf{u}$ with finite Dirichlet integral, and the inverse of the divergence operator is well defined on some weighted subspace of $L^2(\Omega)$ (see [60] for domains with the "paraboloidal" outlets, and [42] for the infinite layer). The similar weighted subspaces of $W^{1/2,2}(\partial\Omega)$ will appear, if we try to describe traces of solenoidal vector field with finite Dirichlet integrals in noncompact domains (see, for example, [50]). However, even for elements of these complicated spaces it may not make sense to speak about the fluxes over infinite parts of the boundary (the integral $\int\limits_{S^{(m)}} \mathbf{a}\cdot\mathbf{n}\, dS$ could be infinite). Therefore, having in mind that we allow also solutions with infinite Dirichlet integrals, we suppose, for simplicity, that $\mathbf{a}$ has a compact support.

3.2 Solvability of problem (3.1); general scheme

Definition 3.1. *By a weak solution of problem* (3.1) *we understand a solenoidal vector field* $\mathbf{u} \in W^{1,2}_{loc}(\Omega)$ *satisfying the boundary condition* $\mathbf{u}\big|_{\partial\Omega} = \boldsymbol{a}$*, the flux conditions*

$$\int\limits_{\sigma_j(R)} \mathbf{u}\cdot\mathbf{n} dS = \mathcal{F}_j, \quad j = 1, 2, \ldots, J, \quad R \geq R_0, \tag{3.5}$$

and the integral identity

$$\nu \int\limits_{\Omega} \nabla \boldsymbol{u} : \nabla \boldsymbol{\eta}\, dx - \int\limits_{\Omega} (\boldsymbol{v}\cdot\nabla)\boldsymbol{\eta}\cdot\boldsymbol{v}\, dx = 0 \quad \forall\, \boldsymbol{\eta} \in J_0^{\infty}(\Omega). \tag{3.6}$$

Assume that the necessary compatibility condition (3.3) is valid. Let $\mathbf{A} \in W^{1,2}_{loc}(\Omega)$ be a solenoidal extension of the boundary value $\mathbf{a}$ satisfying flux condition (3.5):

$$\operatorname{div} \mathbf{A} = 0, \quad \mathbf{A}\big|_{\partial\Omega} = \mathbf{a}, \quad \int\limits_{\sigma_j(R)} \mathbf{A}\cdot\mathbf{n} dS = \mathcal{F}_j, \quad j = 1, 2, \ldots, J.$$

We reduce problem (3.6) to the problem with homogeneous boundary condition and zero fluxes. We substitute $\mathbf{u} = \mathbf{v} + \mathbf{A}$ into (3.6)

$$\begin{aligned} &\nu \int_\Omega \nabla \mathbf{v} : \nabla \boldsymbol{\eta}\, dx - \int_\Omega (\mathbf{v} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{v}\, dx - \int_\Omega (\mathbf{A} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{v}\, dx - \int_\Omega (\mathbf{v} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A}\, dx \\ &= \int_\Omega (\mathbf{A} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A}\, dx - \nu \int_\Omega \nabla \mathbf{A} : \nabla \boldsymbol{\eta}\, dx \quad \forall \boldsymbol{\eta} \in J_0^\infty(\Omega), \end{aligned} \tag{3.7}$$

and look for the new unknown velocity field $\mathbf{v} \in W^{1,2}_{loc}(\Omega)$ satisfying (3.7) and such that

$$\operatorname{div} \mathbf{v} = 0, \quad \mathbf{v}\big|_{\partial\Omega} = 0, \quad \int_{\sigma_j(R)} \mathbf{v} \cdot \mathbf{n} dS = 0, \quad j = 1, 2, \ldots, J.$$

The existence of $\mathbf{v}$ satisfying integral identity (3.7) could be proved following the general scheme proposed by V.A. Solonnikov [61] (see also [34], [62]–[64]). It is assumed in [61] that there is a sequence of bounded domains $\{\Omega_{(l)},\ l \geq 1\}$ such that $\Omega_{(l)} \subset \Omega_{(l+1)}$ and $\Omega_{(l)}$ exhausts Ω as $l \to \infty$. The first step is to prove the existence of a weak solution $\mathbf{v}^{(l)}$ for any bounded domain $\Omega_{(l)}$, i.e., to find in every $\Omega_{(l)}$ a vector field $\mathbf{v}^{(l)} \in H(\Omega_{(l)})$ satisfying the integral identity

$$\begin{aligned} &\nu \int_{\Omega_{(l)}} \nabla \mathbf{v}^{(l)} : \nabla \boldsymbol{\eta}\, dx - \int_{\Omega_{(l)}} (\mathbf{v}^{(l)} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{v}^{(l)}\, dx \\ &- \int_{\Omega_{(l)}} (\mathbf{A} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{v}^{(l)}\, dx - \int_{\Omega_{(l)}} (\mathbf{v}^{(l)} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A}\, dx \\ &= \int_{\Omega_{(l)}} (\mathbf{A} \cdot \nabla)\boldsymbol{\eta} \cdot \mathbf{A}\, dx - \nu \int_{\Omega_{(l)}} \nabla \mathbf{A} : \nabla \boldsymbol{\eta}\, dx \qquad \forall \boldsymbol{\eta} \in H(\Omega_{(l)}). \end{aligned} \tag{3.8}$$

Since $\Omega_{(l)}$ is a bounded domain, it is well known (see [31]) that integral identity (3.8) is equivalent to the operator equation in the space $H(\Omega_{(l)})$:

$$\mathbf{v}^{(l)} = \mathcal{A}\mathbf{v}^{(l)}$$

with the compact operator $\mathcal{A}$. Therefore, by Leray–Schauder Theorem (see Lemma 2.13), for the existence of the solution $\mathbf{v}^{(l)}$ it is enough to prove that all possible solutions $\mathbf{v}^{(l,\lambda)}$ of the operator equation

$$\mathbf{v}^{(l,\lambda)} = \lambda \mathcal{A} \mathbf{v}^{(l,\lambda)}, \quad \lambda \in [0, 1]$$

are bounded by the same constant independent of λ. To obtain an a priori estimate we have to construct a special extension $\mathbf{A}$ of the boundary value $\mathbf{a}$ satisfying for arbitrary solenoidal vector field $\mathbf{w} \in W^{1,2}_{loc}(\Omega)$, $\mathbf{w}\big|_{\partial\Omega} = 0$, the inequalities

$$\int\limits_{\Omega_{(k)}} |\mathbf{A}|^2 |\mathbf{w}|^2 \, dx \le \delta \int\limits_{\Omega_{(k)}} |\nabla \mathbf{w}|^2 \, dx,$$
$$\int\limits_{\Omega_{(k)} \setminus \Omega_{(k-1)}} |\mathbf{A}|^2 |\mathbf{w}|^2 \, dx \le \delta \int\limits_{\Omega_{(k)} \setminus \Omega_{(k-1)}} |\nabla \mathbf{w}|^2 \, dx. \tag{3.9}$$

Here ε is arbitrary small positive number, which we fix later in order to prove the estimates. Inequalities (3.9) are usually called Leray–Hopf inequalities. Using Cauchy–Schwarz (2.3) and Poincaré (2.4) inequalities, from (3.9) we derive that

$$\Big| \int\limits_{\Omega_{(k)}} (\mathbf{w} \cdot \nabla)\mathbf{w} \cdot \mathbf{A} \, dx \Big| \le \delta \int\limits_{\Omega_{(k)}} |\nabla \mathbf{w}|^2 \, dx,$$
$$\Big| \int\limits_{\Omega_{(k)} \setminus \Omega_{(k-1)}} (\mathbf{w} \cdot \nabla)\mathbf{w} \cdot \mathbf{A} \, dx \Big| \le \delta \int\limits_{\Omega_{(k)} \setminus \Omega_{(k-1)}} |\nabla \mathbf{w}|^2 \, dx. \tag{3.10}$$

Let $\mathbf{A}$ satisfies (3.10). Taking in (3.8) $\boldsymbol{\eta} = \mathbf{v}^{(l)}$ and choosing sufficiently small ε, we obtain

$$\nu \int\limits_{\Omega_{(l)}} |\nabla \mathbf{v}^{(l)}|^2 \, dx = -\nu \int\limits_{\Omega_{(l)}} \nabla \mathbf{A} : \nabla \mathbf{v}^{(l)} \, dx + \int\limits_{\Omega_{(l)}} \big(\mathbf{A} \cdot \nabla\big)\mathbf{v}^{(l)} \cdot \mathbf{A} \, dx$$

$$+ \int\limits_{\Omega_{(l)}} \big(\mathbf{v}^{(l)} \cdot \nabla\big)\mathbf{v}^{(l)} \cdot \mathbf{A} \, dx \le c\Big(\|\nabla \mathbf{A}\|^2_{L^2(\Omega_{(l)})} + \|\mathbf{A}\|^4_{L^4(\Omega_{(l)})}\Big) + \frac{\nu}{2} \int\limits_{\Omega_{(l)}} |\nabla \mathbf{v}^{(l)}|^2 \, dx.$$

Therefore,

$$\int\limits_{\Omega_{(l)}} |\nabla \mathbf{v}^{(l)}|^2 \, dx \le c_1 \Big(\|\nabla \mathbf{A}\|^2_{L^2(\Omega_{(l)})} + \|\mathbf{A}\|^4_{L^4(\Omega_{(l)})}\Big) \equiv c_1 \Phi(l),$$

where c_1 is independent of l. Thus, the existence of $\mathbf{v}^{(l)}$ follows from the Leray–Schauder Theorem.

Next, we have to control the Dirichlet integral of $\mathbf{v}^{(l)}$ over subdomain $\Omega_{(k)} \subset \Omega_{(l)}, k \le l$. If $\Phi(l)$ grows "not too fast" as $l \to \infty$, then using the second of inequalities (3.10) it can be proved that

$$\int\limits_{\Omega_{(k)}} |\nabla \mathbf{v}^{(l)}|^2 \, dx \le c_2 \Phi(k) \tag{3.11}$$

with c_2 independent of k and l.

Since for every bounded domain $\Omega_{(k)}$ the embedding $W^{1,2}(\Omega_{(k)}) \hookrightarrow L^4(\Omega_{(k)})$ is compact, estimate (3.11) guarantees the existence of a subsequence $\{\mathbf{v}^{(l_m)}\}$ which converges weakly in $\mathring{W}^{1,2}(\Omega_{(k)})$ and strongly in $L^4(\Omega_{(k)})$ for $\forall k > 0$ (such subsequence could be constructed by Cantor diagonal process). Taking in integral identity (3.8) an arbitrary test function $\boldsymbol{\eta}$ with compact support we can find k such that $\operatorname{supp}\boldsymbol{\eta} \subset \Omega_{(k)}$ and, hence $\boldsymbol{\eta} \in H(\Omega_{(k)})$. Extending $\boldsymbol{\eta}$ by zero into $\Omega \setminus \Omega_{(k)}$, and considering all integrals in (3.8) as integrals over Ω, we can pass in (3.8) to a limit as $l_m \to \infty$. As a result we get for the limit vector function $\mathbf{v}$ integral identity (3.7). Obviously, estimate (3.11) remains valid also for $\mathbf{v}$. Moreover, the obtained solution has the following property: if the vector field $\mathbf{A}$ has the finite Dirichlet integral over some outlet $\Omega^{(j)}$, then the weak solution $\mathbf{v}$ also has finite norm $\|\nabla \mathbf{v}\|_{L^2(\Omega^{(j)})}$. Finally, we mention that the weak solution is unique in a class of functions that grow at infinity not "too fast" (see [34], [61]). The above mentioned results were obtained in [34], [61] in the case of the homogeneous boundary conditions ($\mathbf{a}(x) = 0$).

In this thesis we reduce the nonhomogeneous boundary value problem to homogeneous one, by constructing a suitable extension of the boundary value $\mathbf{a}$, and then we apply the above mentioned methods to the reduced problem. Therefore, the main purpose of the thesis is to construct an appropriate extension of boundary value which gives the possibility to reduce the nonhomogeneous boundary conditions to the homogeneous ones. This extension is constructed as the sum

$$\mathbf{A} = \mathbf{B}^{(inn)} + \sum_{m=1}^{M} \mathbf{B}_m^{(out)} + \mathbf{B}^{(flux)},$$

where $\mathbf{B}^{(inn)}$ extends the boundary value $\mathbf{a}$ from the inner boundary Γ, $\mathbf{B}_m^{(out)}$ extend $\mathbf{a}$ from the connected component $S^{(m)}$ of the noncompact outer boundary S, and $\mathbf{B}^{(flux)}$ has zero boundary value over $\partial\Omega$ and removes the fluxes over the cross sections of outlets to infinity. The vector fields $\mathbf{B}_m^{(out)}$ and $\mathbf{B}^{(flux)}$ are constructed to satisfy Leray–Hopf's inequalities (3.10) which allow to obtain a priori estimates of the solution for arbitrary large fluxes $\mathfrak{F}_m^{(out)}$ and $\mathcal{F}_j$. The construction of the vector fields $\mathbf{B}_m^{(out)}$ and $\mathbf{B}^{(flux)}$ is based on methods proposed in [33], [59], [61]. Notice that the Leray–Hopf's inequality cannot be true, in general, for the vector field $\mathbf{B}^{(inn)}$. If the fluxes of the boundary value over connected components of the boundary do not vanish, in [67], [23], [4] there are constructed counterexamples showing that in bounded domains Leray–Hopf inequality can be false whatever the choice of the solenoidal extension is taken. Therefore, we have to suppose that the fluxes $\mathbb{F}_i^{(inn)}$ of $\mathbf{a}$ over the compact components Γ_i of the inner boundary Γ are "sufficiently small".

3.3 Construction of the extension; general scheme

The first step in the construction of the extension $\mathbf{A}$ is to reduce the problem (3.1) to the case of boundary value with zero fluxes and to "transport" the fluxes from bounded parts of $\partial\Omega$ to infinity, i.e., we have to construct solenoidal vector fields $\mathbf{b}_m^{(out)}, m = 1, \dots, M$, and $\mathbf{b}^{(inn)}$ satisfying the following conditions

$$
\begin{aligned}
(a)\ & \int\limits_{\Lambda_m} \mathbf{b}_m^{(out)} \cdot \mathbf{n}\, dS = - \int\limits_{\sigma_{j_*}(R)} \mathbf{b}_m^{(out)} \cdot \mathbf{n}\, dS = \mathfrak{F}_m^{(out)},\ m = 1, \dots, M, \\
& \int\limits_{\Gamma_i} \mathbf{b}^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}_i^{(inn)},\ i = 1, \dots, I, \\
& \int\limits_{\sigma_{j_*}(R)} \mathbf{b}^{(inn)} \cdot \mathbf{n}\, dS = - \sum_{i=1}^{I} \mathbb{F}_i^{(inn)} = -\mathbb{F}^{(inn)}, \quad j_* \in \{1, \dots, J\};
\end{aligned}
$$

$$
(b)\quad \operatorname{supp} \mathbf{b}_m^{(out)},\ \operatorname{supp} \mathbf{b}^{(inn)} \subset \Big(\Omega \bigcap B_{R_2}(0)\Big) \bigcup \Omega^{(j_*)} \text{ for some } R_2 > R_1.
$$

The vector fields $\mathbf{b}_m^{(out)}$ and $\mathbf{b}^{(inn)}$ "drain" the fluxes from the outer and inner boundaries to some outlet to infinity $\Omega^{(j_*)}$. Following the terminology of H. Fujita [11] we call this method by *virtual drain method* and vector functions $\mathbf{b}_m^{(out)}$ and $\mathbf{b}^{(inn)}$ by *virtual drain functions.*

By constructing virtual drain functions $\mathbf{b}_m^{(out)}$ and $\mathbf{b}^{(inn)}$ we have an arbitrariness in choosing the outlet $\Omega^{(j_*)}$ to which we "drain" the fluxes. We choose the "widest" outlet $\Omega^{(j_*)}$, in order to minimize the generated by the drain function dissipation of energy (Dirichlet integral). Choosing different virtual drain functions we may get, in general, different solutions of problem (3.1) (a solution is known to be unique only for small data). It is well known (see [31]) that a solenoidal vector field with a nonzero flux over cross sections of the outlets to infinity can have finite Dirichlet integral only if this outlet is sufficiently "wide". In order to explain this, consider the outlet D which has the form

$$
D = \{x : |x'| < g(x_3),\ x_3 > 1\},\ x' = (x_1, x_2),
$$

and function g satisfies the Lipschitz condition. Take a solenoidal vector field $\mathbf{u}$ with the finite Dirichlet integral over the paraboloidal outlet D and let $\mathcal{F}$ be the flux of $\mathbf{u}$ over the cross section $\sigma(t) = \{x \in \mathbb{R}^3 : |x'| < g(t),\ x_3 = t\}$. By Cauchy–Schwarz (2.3) and Poincaré (2.4) inequalities we have

$$
|\mathcal{F}|^2 = \Big| \int\limits_{\sigma(t)} u_3(x', t) dx' \Big|^2 \le \pi g^2(t) \int\limits_{\sigma(t)} |u_3(x', t)|^2 dx'
$$

$$
\le c g^4(t) \int\limits_{\sigma(t)} |\nabla' u_3(x', t)|^2 dx',
$$

Dividing both sides of the last inequality by $g^4(t)$ and integrating over t from 1 to R we obtain

$$
\begin{aligned}
|\mathcal{F}|^2 \int\limits_1^R g^{-4}(t)dt &\le c\int\limits_1^R \int\limits_{\sigma(t)} |\nabla' u_3(x',t)|^2 dx'dt \\
&\le c\int\limits_D |\nabla \mathbf{u}|^2 dx < \infty.
\end{aligned}
\tag{3.12}
$$

From (3.12) it follows that either $\mathcal{F} = 0$ or $\int_1^\infty g^{-4}(t)dt < \infty$. The latter condition means that the outlet to infinity D is sufficiently "wide" (grows at infinity sufficiently rapidly). For example, if D is a cylinder ($g(t) \equiv 1$), there are no solenoidal vector fields with the finite Dirichlet integral and nonzero flux (see, for example, [59]).

After problem (3.1) is reduced (with the help of virtual drain functions) to the case of a boundary value with zero fluxes over all connected components of the boundary $\partial\Omega$, we can extend this boundary value from each component Λ_m, Γ_i into Ω using the standard method (see, for example, [31]), and we get solenoidal extensions $\mathbf{b}_{0\,m}^{(out)}$ and $\mathbf{b}_0^{(inn)}$, where $\mathbf{b}_{0\,m}^{(out)}$ satisfies Leray–Hopf inequalities (3.10). Finally, as in [59], [61], we construct in Ω a solenoidal vector field $\mathbf{B}^{(flux)}$ satisfying zero boundary conditions and having given fluxes over the cross sections of all outlets. The vector field

$$
\begin{aligned}
\mathbf{A} &= \big(\mathbf{b}^{(inn)} + \mathbf{b}_0^{(inn)}\big) + \sum_{m=1}^{M} \big(\mathbf{b}_m^{(out)} + \mathbf{b}_{0\,m}^{(out)}\big) + \mathbf{B}^{(flux)} \\
&= \mathbf{B}^{(inn)} + \sum_{m=1}^{M} \mathbf{B}_m^{(out)} + \mathbf{B}^{(flux)}
\end{aligned}
$$

gives the desired extension of the boundary value $\mathbf{a}$, and it satisfies the flux conditions (3.5).

Chapter 4

Examples

4.1 Domain with one paraboloidal outlet to infinity

In this chapter we consider the following problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p &= 0 \quad \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{a} \quad \text{on} \quad \partial\Omega \end{cases} \tag{4.1}$$

in the domain $\Omega \subset \mathbb{R}^3$ with one paraboloidal outlet to infinity $D_1 = \{x \in \Omega : |x'| < g_1(x_3),\ x_3 > 1\}$, i.e., $\Omega = \Omega_0 \bigcup D_1$, where $\Omega_0 = G_0 \setminus \overline{G}_1$, $\overline{G}_1 \subset G_0$, G_0 and G_1 are bounded simply connected domains. Denote $\partial G_1 = \Gamma_1 = \Gamma$. The outer boundary S consist of one component $S^{(1)}$. Therefore, $\partial\Omega = \Gamma_1 \bigcup S^{(1)}$. By the assumption the boundary value $\mathbf{a}$ has a compact support (see (3.2)), therefore $\Lambda_1 = \operatorname{supp}\mathbf{a} \bigcap S^{(1)} = S^{(1)} \bigcap B_{R_0}(0)$.

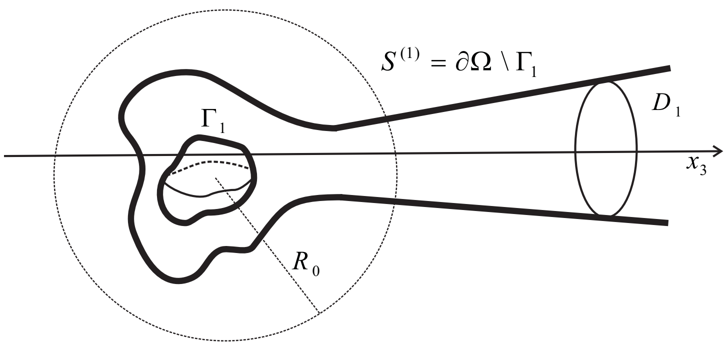

Figure 4.1: Domain Ω.

The function g_1 satisfies the Lipschitz condition

$$|g_1(t_1) - g_1(t_2)| \le L_1|t_1 - t_2|, \quad t_1, t_2 \ge 1, \quad g_1(t) \ge 1 \ \forall t. \tag{4.2}$$

Below we will use the following notations:

$$\Omega_{(l)} = \Omega_0 \cup D_1^{(l)}, \quad \omega_l = \Omega_{(l+1)} \setminus \Omega_{(l)},$$

where $D_1^{(l)} = \{x \in D_1 : x_3 < R_l\}$, $R_1 = 1$, $\ R_{l+1} = R_l + \dfrac{g_1(R_l)}{2L_1}$, $\ l \ge 1$.
Let

$$\mathbb{F}_1^{(inn)} = \int\limits_{\Gamma_1} \mathbf{a} \cdot \mathbf{n}\, dS, \quad \mathfrak{F}_1^{(out)} = \int\limits_{\Lambda_1} \mathbf{a} \cdot \mathbf{n}\, dS$$

be the fluxes of the boundary value $\mathbf{a}$ over the inner and outer boundaries, respectively. Since the total flux has to be equal to zero, condition (3.3) implies

$$\int\limits_{\sigma_1(R)} \mathbf{u} \cdot \mathbf{n}\, dS = -\big(\mathbb{F}_1^{(inn)} + \mathfrak{F}_1^{(out)}\big), \tag{4.3}$$

where $\sigma_1(R) = D_1 \bigcap \{x : x_3 = R\}$ is the cross section of the outlet D_1.

We construct a suitable extension $\mathbf{A} = \mathbf{B}_1^{(out)} + \mathbf{B}^{(inn)}$ and prove the existence of at least one solution to problem (4.1). Since the domain Ω has only one outlet to infinity, compatibility condition (3.4) has the form

$$\mathbb{F}_1^{(inn)} + \mathfrak{F}_1^{(out)} + \mathcal{F}_1 = 0. \tag{4.4}$$

Hence, in this case we cannot prescribe additionally the flux $\mathcal{F}_1$, i.e.,

$$\mathcal{F}_1 = -\big(\mathbb{F}_1^{(inn)} + \mathfrak{F}_1^{(out)}\big)$$

and the extension $\mathbf{B}^{(flux)}$ can be taken equal to zero.

4.1.1 Construction of the extension $\mathbf{B}^{(inn)}$

We start with the construction of the virtual drain function $\mathbf{b}^{(inn)}$. Let us first define in D_1 a solenoidal vector field $\mathbf{b}_1^{(inn)}$ such that

$$\mathbf{b}_1^{(inn)}(x)\big|_{\partial D_1 \cap \partial\Omega} = 0, \qquad \int\limits_{\sigma_1(R)} \mathbf{b}_1^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}_1^{(inn)}.$$

Let $\gamma_+ = \{x \in D_1 : |x'| = 0,\ x_3 > 1\}$. Define in D_1 the cut-off function

$$\zeta_1(x) = \Psi\Big(\ln\Big(\frac{\varrho(\delta(x))}{\Delta(x)}\Big)\Big), \tag{4.5}$$

where

$$\delta(x) = \Delta_{\gamma_+}(x), \quad \Delta(x) = \Delta_{\partial D_1 \cap \partial\Omega}(x),$$

functions Ψ and ϱ are given by formulas (2.24) and (2.25), respectively.

Lemma 4.1. *The function $\zeta_1(x)$ is equal to zero at those points of D_1 where $\varrho(\delta(x)) \le \Delta(x)$, while the $d_0/2$-neighborhood of the line γ_+ is contained in this set; $\zeta_1(x) = 1$ at those points of D_1 where $\Delta(x) \le e^{-1}\varrho(\delta(x))$. The following estimates*

$$\Big|\frac{\partial \zeta_1(x)}{\partial x_k}\Big| \le \frac{c}{\Delta(x)}, \qquad \Big|\frac{\partial^2 \zeta_1(x)}{\partial x_k \partial x_l}\Big| \le \frac{c}{\Delta^2(x)}$$

hold.

Proof. The proof of the lemma follows directly from the definition of the functions ζ_1, Ψ and ϱ, properties of the regularized distance (see estimates (2.21)) and the fact that $\mathrm{supp}\nabla\zeta_1$ is contained in the set where $\Delta(x) \le \varrho(\delta(x))$. □

Set

$$\mathbf{b}_1^{(inn)}(x) = -\mathbb{F}_1^{(inn)}\mathrm{curl}\big(\zeta_1(x)\mathbf{b}(x)\big) = -\mathbb{F}_1^{(inn)}\nabla\zeta_1(x) \times \mathbf{b}(x), \ x \in D_1,$$

where $\mathbf{b}(x)$ is the magnetic field defined by (2.23); properties of $\mathbf{b}(x)$ are given in Lemma 2.17.

Lemma 4.2. *The solenoidal vector field $\mathbf{b}_1^{(inn)}$ is infinitely differentiable, vanishes near the surface $\partial D_1 \cap \partial\Omega$ and the contour γ_+, the support of $\mathbf{b}_1^{(inn)}$ is contained in the set of points $x \in D_1$ satisfying the inequalities*

$$\varrho(\delta(x))e^{-1} \le \Delta(x) \le \varrho(\delta(x)). \tag{4.6}$$

Moreover,

$$\int\limits_{\sigma_1(R)} \mathbf{b}_1^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}_1^{(inn)},$$

and the following inequalities

$$|\mathbf{b}_1^{(inn)}(x)| \le \frac{c|\mathbb{F}_1^{(inn)}|}{d(x)}, \tag{4.7}$$

$$|\mathbf{b}_1^{(inn)}(x)| \le \frac{C|\mathbb{F}_1^{(inn)}|}{g_1^2(x_3)}, \quad |\nabla\mathbf{b}_1(x)| \le \frac{C|\mathbb{F}_1^{(inn)}|}{g_1^3(x_3)}, \tag{4.8}$$

hold. Here $d(x)=\mathrm{dist}(x, \partial D_1 \cap \partial\Omega)$.

Proof. Relation (4.6) follows from Lemma 4.1. Since $\zeta_1(x) = 1$ on $\partial D_1 \cap \partial\Omega$, by the Stokes Theorem (see Lemma 2.12) and properties of the magnetic field $\mathbf{b}$ we get

$$\int\limits_{\sigma_1(R)} \mathbf{b}_1^{(inn)} \cdot \mathbf{n}\, dS = -\mathbb{F}_1^{(inn)} \int\limits_{\sigma_1(R)} \operatorname{curl}\big(\zeta_1 \mathbf{b}\big) \cdot \mathbf{n}\, dS = -\mathbb{F}_1^{(inn)} \oint\limits_{\partial\sigma_1(R)} \mathbf{b} \cdot d\mathbf{l} = \mathbb{F}_1^{(inn)}.$$

From the definition of $\mathbf{b}_1^{(inn)}(x)$ and Lemma 4.1 follow the estimates

$$\begin{aligned} &|\mathbf{b}_1^{(inn)}(x)| \le |\mathbb{F}_1^{(inn)}||\nabla\zeta_1(x)||\mathbf{b}(x)| \le \frac{c|\mathbb{F}_1^{(inn)}|}{\Delta(x)\,|x'|}, \\ &|\nabla \mathbf{b}_1^{(inn)}(x)| \le |\mathbb{F}_1^{(inn)}|\Big(|\nabla(\nabla\zeta_1(x))||\mathbf{b}(x)| + |\nabla\zeta_1(x)||\nabla\mathbf{b}(x)|\Big) \\ &\le c|\mathbb{F}_1^{(inn)}|\Big(\frac{1}{\Delta^2(x)|x'|} + \frac{1}{\Delta(x)|x'|^2}\Big). \end{aligned} \tag{4.9}$$

It is easy to see that for points $x \in \operatorname{supp} \mathbf{b}_1^{(inn)}$ the inequalities

$$0 < c_1\delta(x) \le \Delta(x) \le c_2\delta(x), \qquad c_3 g_1(x_3) \le |x'| \le c_4 g_1(x_3) \tag{4.10}$$

hold. Since $\operatorname{dist}(x,\gamma_+) = |x'|$ for $x \in D_1$ and $\operatorname{dist}(x,\gamma_+) \ge d_0/2$ for $x \in \operatorname{supp} \mathbf{b}_1^{(inn)}$, estimates (4.7), (4.8) follow from (4.9), (4.10). □

Lemma 4.3. *For any vector field $\mathbf{w} \in W^{1,2}_{loc}(D_1)$ with $\mathbf{w}|_{\partial D_1 \cap \partial\Omega} = 0$ the following inequalities*

$$\begin{aligned} \int\limits_{D_1^{(k)}} |\mathbf{b}_1^{(inn)}|^2 |\mathbf{w}|^2\, dx &\le c|\mathbb{F}_1^{(inn)}|^2 \int\limits_{D_1^{(k)}} |\nabla\mathbf{w}|^2\, dx, \\ \int\limits_{D_1^{(k+1)}\setminus D_1^{(k)}} |\mathbf{b}_1^{(inn)}|^2 |\mathbf{w}|^2\, dx &\le c|\mathbb{F}_1^{(inn)}|^2 \int\limits_{D_1^{(k+1)}\setminus D_1^{(k)}} |\nabla\mathbf{w}|^2\, dx \end{aligned} \tag{4.11}$$

hold. The constant c is independent of k.

Proof. Estimates (4.11) follow from (4.7) and the well known inequality

$$\int\limits_{D_1^{(k+1)}\setminus D_1^{(k)}} \frac{|\mathbf{w}(x)|^2}{d^2(x)}\, dx \le c \int\limits_{D_1^{(k+1)}\setminus D_1^{(k)}} |\nabla\mathbf{w}(x)|^2\, dx$$

(see Lemma 2.15). □

Let $x^{(1)} \in G_1$, be a point lying inside the "hole" G_1. Denote $q_1(x) = q(x - x^{(1)})$, where

$$q(x) = \frac{1}{4\pi}\frac{1}{|x|}$$

is the fundamental solution of the Laplace operator in $\mathbb{R}^3$, and let

$$\mathbf{b}_\sharp^{(inn)}(x) = \mathbb{F}_1^{(inn)} \nabla q_1(x). \tag{4.12}$$

Obviously,

$$\operatorname{div} \mathbf{b}_\sharp^{(inn)} = \mathbb{F}_1^{(inn)} \operatorname{div} \nabla q_1(x) = \mathbb{F}_1^{(inn)} \Delta q_1(x) = 0, \ ^1$$

Moreover, we have

$$\int\limits_{\Gamma_1} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}_1^{(inn)}, \quad \int\limits_{\partial\Omega_0} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n}\, dS = -\mathbb{F}_1^{(inn)}.$$

Denote

$$\mathbf{h}_1 = \begin{cases} 0, \quad x \in \Gamma_1, \\ \mathbf{b}_1^{(inn)}|_{\partial\Omega_0 \cap \overline{D}_1} - \mathbf{b}_\sharp^{(inn)}|_{\partial\Omega_0 \cap \overline{D}_1}, \quad x \in \partial\Omega_0 \bigcap \overline{D}_1, \\ -\mathbf{b}_\sharp^{(inn)}\big|_{\partial\Omega_0 \setminus \overline{D}_1}, \quad x \in \partial\Omega_0 \setminus (\overline{D}_1 \bigcup \Gamma_1). \end{cases}$$

Then

$$\int\limits_{\partial\Omega_0} \mathbf{h}_1 \cdot \mathbf{n}\, dS = \int\limits_{\partial\Omega_0 \cap \overline{D}_1} \mathbf{b}_1^{(inn)} \cdot \mathbf{n}\, dS - \int\limits_{\partial\Omega_0} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}_1^{(inn)} - \mathbb{F}_1^{(inn)} = 0. \tag{4.13}$$

Because of (4.13) the function $\mathbf{h}_1$ can be extended inside the domain Ω_0 as a solenoidal vector field $\mathbf{b}_{01}^{(inn)} \in W^{1,2}(\Omega_0)$ and

$$\begin{aligned} &\|\mathbf{b}_{01}^{(inn)}\|_{W^{1,2}(\Omega_0)} \le c\|\mathbf{h}_1\|_{W^{1/2,2}(\partial\Omega_0)} \\ &\le c\Big(\|\mathbf{b}_\sharp^{(inn)}\|_{W^{1/2,2}(\partial\Omega_0)} + \|\mathbf{b}_1^{(inn)}\|_{W^{1/2,2}(\partial\Omega_0 \cap D_1)}\Big) \le c|\mathbb{F}_1^{(inn)}|, \end{aligned} \tag{4.14}$$

where the constant c depends only on the domain Ω_0 (see Lemma 2.6). Define

$$\mathbf{b}^{(inn)} = \begin{cases} \mathbf{b}_\sharp^{(inn)} + \mathbf{b}_{01}^{(inn)}, \quad x \in \Omega_0, \\ \mathbf{b}_1^{(inn)}, \quad x \in D_1. \end{cases}$$

The vector field $\mathbf{b}^{(inn)}$ "removes" the non-zero flux from the component Γ_1 and, as we have mentioned before, we call it a virtual drain function.

[1] The function $\mathbf{v} = \nabla q$ is not only solenoidal and harmonic; it satisfies together with the pressure $p = \frac{1}{2}|\nabla q|^2$ the system of the homogeneous Navier–Stokes equations

$$-\nu\Delta\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0.$$

Set

$$\mathbf{h}_0 = \begin{cases} \mathbf{a} - \mathbf{b}_\sharp^{(inn)}|_{\Gamma_1}, & x \in \Gamma_1, \\ 0, & x \in \partial\Omega_0 \setminus \Gamma_1. \end{cases}$$

Obviously,

$$\int_{\Gamma_1} \mathbf{h}_0 \cdot \mathbf{n}\, dS = \int_{\Gamma_1} \mathbf{a} \cdot \mathbf{n}\, dS - \int_{\Gamma_1} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}_1^{(inn)} - \mathbb{F}_1^{(inn)} = 0.$$

Therefore, the function $\mathbf{h}_0$ can be extended inside Ω in the form

$$\mathbf{b}_0^{(inn)}(x) = \operatorname{curl}(\chi(x))\mathbf{E}(x)),$$

where $\mathbf{E} \in W_2^2(\Omega_0)$, $\operatorname{curl}\mathbf{E}|_{\partial\Omega_0} = \mathbf{h}_0$ and χ is a Hopf's type smooth cut-off function and $\chi(x) = 1$ on Γ_1. Moreover, for any $\mathbf{w} \in W_{loc}^{1,2}(\Omega)$ with $\mathbf{w}|_{\partial\Omega} = 0$ the following estimate

$$\int_{\Omega_0} |\mathbf{b}_0^{(inn)}(x)|^2 |\mathbf{w}(x)|^2\, dx \le c|\mathbb{F}_1^{(inn)}|^2 \int_{\Omega_0} |\nabla \mathbf{w}(x)|^2\, dx \tag{4.15}$$

holds (see Lemma 2.14 with $\varepsilon = 1$ and Lemma 2.15).

Finally, we put

$$\mathbf{B}^{(inn)} = \mathbf{b}^{(inn)} + \mathbf{b}_0^{(inn)}. \tag{4.16}$$

Lemma 4.4. *The vector field* $\mathbf{B}^{(inn)}$ *is solenoidal,* $\mathbf{B}^{(inn)}|_{\Gamma_1} = \mathbf{a}|_{\Gamma_1}$, $\mathbf{B}^{(inn)}|_{S^{(1)}} = 0$, $\mathbf{B}^{(inn)} \in W_{loc}^{1,2}(\Omega)$. *For any solenoidal* $\mathbf{w} \in W_{loc}^{1,2}(\Omega)$ *with* $\mathbf{w}\big|_{\partial\Omega} = 0$ *the following estimates*

$$\begin{aligned} &\int_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{B}^{(inn)}|^2 |\mathbf{w}|^2\, dx \le c|\mathbb{F}_1^{(inn)}|^2 \int_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla \mathbf{w}|^2\, dx, \\ &\int_{\Omega_{(k)}} |\mathbf{B}^{(inn)}|^2 |\mathbf{w}|^2\, dx \le c|\mathbb{F}_1^{(inn)}|^2 \int_{\Omega_{(k)}} |\nabla \mathbf{w}|^2\, dx \end{aligned} \tag{4.17}$$

hold. Moreover,

$$\begin{aligned} &|\mathbf{B}^{(inn)}(x)| \le \frac{C|\mathbb{F}_1^{(inn)}|}{g_1^2(x_3)}, \quad |\nabla \mathbf{B}^{(inn)}(x)| \le \frac{C|\mathbb{F}_1^{(inn)}|}{g_1^3(x_3)}, \; x \in D_1, \\ &|\mathbf{B}^{(inn)}(x)| + |\nabla \mathbf{B}^{(inn)}(x)| \le C|\mathbb{F}_1^{(inn)}|, \quad x \in \Omega \setminus D_1. \end{aligned} \tag{4.18}$$

Proof. By the construction $\mathbf{B}^{(inn)}$ is solenoidal, $\mathbf{B}^{(inn)}|_{\Gamma_1} = \mathbf{a}$, $\mathbf{B}^{(inn)}|_{S^{(1)}} = 0$. From (4.11), (4.14), (4.15) and definition (4.12) of the vector field $\mathbf{b}_\sharp^{(inn)}(x)$ follow the estimates

$$\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{B}^{(inn)}|^2|\mathbf{w}|^2\,dx = \int\limits_{D_1^{(k)}\setminus D_1^{(k-1)}} |\mathbf{b}_1^{(inn)}|^2|\mathbf{w}|^2\,dx$$

$$\leq c|\mathbb{F}_1^{(inn)}|^2 \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla\mathbf{w}|^2\,dx,$$

$$\int\limits_{\Omega_{(k)}} |\mathbf{B}^{(inn)}|^2|\mathbf{w}|^2\,dx \leq c\Big(\int\limits_{\Omega_0} |\mathbf{b}_\sharp^{(inn)}|^2|\mathbf{w}|^2\,dx + \int\limits_{\Omega_0} |\mathbf{b}_{01}^{(inn)}|^2|\mathbf{w}|^2\,dx$$

$$+\int\limits_{\Omega_0} |\mathbf{b}_0^{(inn)}|^2|\mathbf{w}|^2\,dx\Big) + \int\limits_{D_1^{(k)}} |\mathbf{b}_1^{(inn)}|^2|\mathbf{w}|^2\,dx$$

$$\leq c\big(\|\mathbf{b}_\sharp^{(inn)}\|^2_{L^4(\Omega_0)} + \|\mathbf{b}_{01}^{(inn)}\|^2_{L^4(\Omega_0)}\big)\|\mathbf{w}\|^2_{L^4(\Omega_0)} + c|\mathbb{F}_1^{(inn)}|^2\int\limits_{\Omega_0} |\nabla\mathbf{w}|^2\,dx$$

$$+c|\mathbb{F}_1^{(inn)}|^2\int\limits_{D_1^{(k)}} |\nabla\mathbf{w}|^2\,dx \leq c|\mathbb{F}_1^{(inn)}|^2\int\limits_{\Omega_{(k)}} |\nabla\mathbf{w}|^2\,dx$$

for any solenoidal $\mathbf{w}\in W^{1,2}_{loc}(\Omega)$ with $\mathbf{w}\big|_{\partial\Omega}=0$. Finally, inequalities (4.18) follow from (4.16), (4.8). □

Remark 4.1. It is easy to see from the above construction that in the case $\mathbb{F}_1^{(inn)}=0$ we have $\mathbf{b}_1^{(inn)}=0$ and, hence, $\operatorname{supp}\mathbf{B}^{(inn)}\subset\overline{\Omega}_0$.

4.1.2 Construction of the extension $\mathbf{B}_1^{(out)}$

We start with the construction of virtual drain function $\mathbf{b}_1^{(out)}$. Take any point $x^{(1)}\in\Lambda_1\subset S^{(1)}$. Let γ_{11} be a smooth simple curve which intersects $\partial\Omega$ at the

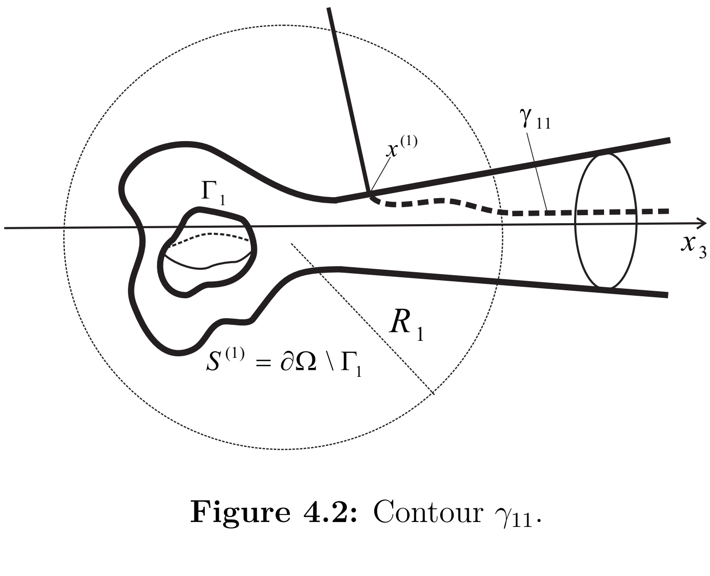

Figure 4.2: Contour γ_{11}.

point $x^{(1)}$ and

$$\gamma_{11} = \widehat{\gamma}_1 \bigcup \widehat{\gamma}_1^{(1)} \bigcup l_1,$$

where $\widehat{\gamma}_1$ is a semi-infinite line lying in D_1, $\widehat{\gamma}_1^{(1)} \subset (\Omega \cap B_{R_1}(0))$ is a finite simple curve connecting $\widehat{\gamma}_1$ and the point $x^{(1)}$, and $l_1 \subset \mathbb{R}^3 \setminus \Omega$ is a semi-infinite line starting at the point $x^{(1)}$.

In the domain Ω we introduce the virtual drain function

$$\mathbf{b}^{(out)}(x,\varepsilon) = \mathfrak{F}_1^{(out)} \mathrm{curl}(\zeta_{11}(x,\varepsilon) \cdot \mathbf{b}_1^{(1)}(x)) = \mathfrak{F}_1^{(out)} \nabla\zeta_{11}(x,\varepsilon) \times \mathbf{b}_1^{(1)}(x), \quad (4.19)$$

where

$$\zeta_{11}(x,\varepsilon) = \Psi\Big(\varepsilon \ln \frac{\varrho(\delta^{(11)}(x))}{\Delta_{\partial\Omega\setminus\Lambda_1}(x)}\Big), \quad (4.20)$$

$\delta^{(11)}(x) = \Delta_{\gamma_{11}}(x)$, $\mathbf{b}_1^{(1)}(x)$ is a magnetic field defined by formula (2.22) (the properties of $\mathbf{b}_1^{(1)}(x)$ are given in Lemma 2.17), functions Ψ and ϱ are defined by (2.24) and (2.25), respectively.

Lemma 4.5. *The vector field $\boldsymbol{b}_1^{(out)}$ is infinitely differentiable and solenoidal, $\boldsymbol{b}_1^{(out)}$ vanishes near the surface $\partial\Omega\setminus\Lambda_1$ and in a small neighborhood of the curve $\gamma_{11}\cap\overline{\Omega}$. The following estimates*

$$\begin{aligned} &|\boldsymbol{b}_1^{(out)}(x,\varepsilon)| \le \frac{c\varepsilon}{d_{\partial\Omega\setminus\Lambda_1}(x) d_{\gamma_{11}}(x)}, \\ &|\nabla \boldsymbol{b}_1^{(out)}(x,\varepsilon)| \le c\Big(\frac{1}{d^2_{\partial\Omega\setminus\Lambda_1}(x) d_{\gamma_{11}}(x)} + \frac{1}{d_{\partial\Omega\setminus\Lambda_1}(x)\cdot d^2_{\gamma_{11}}(x)}\Big), \end{aligned} \quad (4.21)$$

$$|\boldsymbol{b}_1^{(out)}(x,\varepsilon))| \le \frac{c(\varepsilon)|\mathfrak{F}_1^{(out)}|}{g_1^2(x_3)}, \ |\nabla \boldsymbol{b}_1^{(out)}(x,\varepsilon)| \le \frac{c(\varepsilon)|\mathfrak{F}_1^{(out)}|}{g_1^3(x_3)}, \ x \in D_1 \quad (4.22)$$

hold. Here $d_{\partial\Omega\setminus\Lambda_1}(x)$ and $d_{\gamma_{11}}(x)$ are distances from the point x to $\partial\Omega\setminus\Lambda_1$ and γ_{11}, respectively. The constant c in (4.21) *is independent of ε. Finally,*

$$\int\limits_{\Lambda_1} \boldsymbol{b}_1^{(out)} \cdot \boldsymbol{n}\, dS = \mathfrak{F}_1^{(out)}.$$

Proof. The first statement of the lemma follow from definitions (4.19), (4.20) of $\mathbf{b}_1^{(out)}(x,\varepsilon)$ and $\zeta_{11}(x,\varepsilon)$ and from the properties of the regularized distance (see estimates (2.21)). Estimates (4.21), (4.22) can be proved just is the same way as the analogous inequalities in [59], [61]. Since $\mathbf{b}_1^{(out)}(x,\varepsilon) = 0$ and $\zeta_{11}(x,\varepsilon) = 1$ on $\partial\Omega\setminus\Lambda_1$, the Ostrogradsky–Gauss and the Stokes formulas (see Lemma 2.11 and

Lemma 2.12, respectively) yield

$$\int\limits_{\Lambda_1} \mathbf{b}_1^{(out)} \cdot \mathbf{n}\, dS = - \int\limits_{\sigma_1(R)} \mathbf{b}_1^{(out)} \cdot \mathbf{n}\, dS = -\mathfrak{F}_1^{(out)} \int\limits_{\sigma_1(R)} \mathrm{curl}(\zeta_{11}\mathbf{b}_1^{(1)}) \cdot \mathbf{n}\, dS$$

$$= -\mathfrak{F}_1^{(out)} \int\limits_{\partial\sigma_1(R)} \zeta_{11}\mathbf{b}_1^{(1)} \cdot d\mathbf{l} = -\mathfrak{F}_1^{(out)} \int\limits_{\partial\sigma_1(R)} \mathbf{b}_1^{(1)} \cdot d\mathbf{l} = \mathfrak{F}_1^{(out)}.$$

□

Let $\mathbf{h}_1(x) = \mathbf{a}(x)|_{\Lambda_1} - \mathbf{b}_1^{(out)}(x,\varepsilon)|_{\Lambda_1}$. Then

$$\int\limits_{\Lambda_1} \mathbf{h}_1 \cdot \mathbf{n}\, dS = \int\limits_{\Lambda_1} \mathbf{a} \cdot \mathbf{n}\, dS - \int\limits_{\Lambda_1} \mathbf{b}_1^{(out)} \cdot \mathbf{n}\, dS = 0.$$

Therefore, $\mathbf{h}_1$ can be extended inside Ω in the form

$$\mathbf{b}_{01}^{(out)}(x,\varepsilon) = \mathrm{curl}\,(\chi_1(x,\varepsilon)\mathbf{E}_1(x)),$$

where $\mathbf{E}_1 \in W_2^2(\Omega_0)$, $\mathrm{curl}\,\mathbf{E}_1|_{\Lambda_1} = \mathbf{h}_1$ and χ_1 is a Hopf's type cut-off function such that $\chi_1(x,\varepsilon) = 1$ on Λ_1, $\mathrm{supp}\,\chi_1$ is contained in a small neighborhood of Λ_1, and

$$|\nabla\chi_1(x,\varepsilon)| \le \frac{\varepsilon c}{\mathrm{dist}(x,\Lambda_1)} \tag{4.23}$$

(see Lemma 2.14).

Define

$$\mathbf{B}_1^{(out)}(x,\varepsilon) = \mathbf{b}_1^{(out)}(x,\varepsilon) + \mathbf{b}_{0\,1}^{(out)}(x,\varepsilon).$$

Obviously,

$$\mathrm{div}\,\mathbf{B}_1^{(out)} = 0, \quad \mathbf{B}_1^{(out)}|_{\Lambda_1} = \mathbf{a}, \quad \mathbf{B}_1^{(out)}|_{\partial\Omega\setminus\Lambda_1} = 0.$$

Lemma 4.6. *The following estimates*

$$\begin{aligned} &\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{B}_1^{(out)}|^2|\mathbf{w}|^2\, dx \le \varepsilon c|\mathfrak{F}_1^{(out)}|^2 \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla\mathbf{w}|^2\, dx, \\ &\int\limits_{\Omega_{(k)}} |\mathbf{B}_1^{(out)}|^2|\mathbf{w}|^2\, dx \le \varepsilon c|\mathfrak{F}_1^{(out)}|^2 \int\limits_{\Omega_{(k)}} |\nabla\mathbf{w}|^2\, dx, \end{aligned} \tag{4.24}$$

hold for any solenoidal $\mathbf{w} \in W_{loc}^{1,2}(\Omega)$ *with* $\mathbf{w}\big|_{\partial\Omega} = 0$. *Moreover,*

$$|\mathbf{B}_1^{(out)}(x,\varepsilon)| \le \frac{C|\mathfrak{F}_1^{(out)}|}{g_1^2(x_3)}, \quad |\nabla\mathbf{B}_1^{(out)}(x)| \le \frac{C|\mathfrak{F}_1^{(out)}|}{g_1^3(x_3)}. \tag{4.25}$$

The constant c *in* (4.24) *does not depend on* ε *and* k, *while* $C = C(\varepsilon)$ *in* (4.25) *depends on* ε.

Proof. Inequality (4.21) yields (see Lemma 2.15)

$$\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{B}_1^{(out)}|^2|\mathbf{w}|^2dx \leq c\varepsilon|\mathfrak{F}_1^{(out)}|^2 \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} \frac{|\mathbf{w}|^2}{\mathrm{dist}^2(x,\partial\Omega)}dx$$

$$\leq c\varepsilon|\mathfrak{F}_1^{(out)}|^2 \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla\mathbf{w}|^2dx,$$

$$\int\limits_{\Omega_{(k)}} |\mathbf{B}_1^{(out)}|^2|\mathbf{w}|^2dx \leq c\varepsilon|\mathfrak{F}_1^{(out)}|^2 \int\limits_{\Omega_{(k)}} \frac{|\mathbf{w}|^2}{\mathrm{dist}^2(x,\partial\Omega)}dx \leq c\varepsilon|\mathfrak{F}_1^{(out)}|^2 \int\limits_{\Omega_{(k)}} |\nabla\mathbf{w}|^2dx.$$

Estimates (4.25) are the consequence of (4.22). □

4.1.3 Existence of a solution

We look for the solution $\mathbf{u}$ in the form

$$\mathbf{u}(x) = \mathbf{A}(x,\varepsilon) + \mathbf{v}(x), \tag{4.26}$$

where

$$\mathbf{A}(x,\varepsilon) = \mathbf{B}_1^{(out)}(x,\varepsilon) + \mathbf{B}^{(inn)}(x)$$

and as it follows from (4.17), (4.24) the following estimates

$$\begin{aligned}&\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{A}|^2|\mathbf{w}|^2\,dx \leq c(\varepsilon|\mathfrak{F}_1^{(out)}|^2 + |\mathbb{F}_1^{(inn)}|^2) \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla\mathbf{w}|^2\,dx,\\ &\int\limits_{\Omega_{(k)}} |\mathbf{A}|^2|\mathbf{w}|^2\,dx \leq c(\varepsilon|\mathfrak{F}_1^{(out)}|^2 + |\mathbb{F}_1^{(inn)}|^2) \int\limits_{\Omega_{(k)}} |\nabla\mathbf{w}|^2\,dx\end{aligned} \tag{4.27}$$

holds for any solenoidal $\mathbf{w} \in W^{1,2}_{loc}(\Omega)$ with $\mathbf{w}|_{\partial\Omega} = 0$.

Definition 4.1. *By a weak solution of problem* (4.1) *we understand a solenoidal vector field* $\mathbf{u}$ *which has representation* (4.26) *with the solenoidal vector field* $\boldsymbol{v} \in W^{1,2}_{loc}(\Omega)$, $\boldsymbol{v}|_{\partial\Omega} = 0$ *and satisfies the integral identity*

$$\begin{aligned}&\nu\int\limits_{\Omega} \nabla\boldsymbol{v} : \nabla\boldsymbol{\eta}\,dx - \int\limits_{\Omega}((\boldsymbol{A}+\boldsymbol{v})\cdot\nabla)\boldsymbol{\eta}\cdot\boldsymbol{v}\,dx - \int\limits_{\Omega}(\boldsymbol{v}\cdot\nabla)\boldsymbol{\eta}\cdot\boldsymbol{A}\,dx\\ &= -\nu\int\limits_{\Omega}\nabla\boldsymbol{A} : \nabla\boldsymbol{\eta}\,dx + \int\limits_{\Omega}(\boldsymbol{A}\cdot\nabla)\boldsymbol{\eta}\cdot\boldsymbol{A}\,dx \quad \forall\boldsymbol{\eta} \in J_0^{\infty}(\Omega).\end{aligned} \tag{4.28}$$

Theorem 4.1. *Assume that the boundary value* $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$ *has a compact support and that the flux* $\mathbb{F}_1^{(inn)}$ *is sufficiently small. Then there exists at least one* *weak solution* $\boldsymbol{u}$ *of problem* (4.1), (4.4) *and there holds the estimate*

$$\int\limits_{\Omega_{(l)}} |\nabla \boldsymbol{u}|^2\,dx \le c(data)\Big(1+\int\limits_1^{R_k}\frac{dx_3}{g_1^4(x_3)}\Big), \tag{4.29}$$

$$c(data)=c_0\Big(\|\boldsymbol{a}\|^2_{W^{1/2,2}(\partial\Omega)}+\|\boldsymbol{a}\|^4_{W^{1/2,2}(\partial\Omega)}\Big),$$

where c *is independent of* l.

Proof. In every bounded domain $\Omega_{(l)}$ there exists a vector field $\mathbf{v}^{(l)}\in H(\Omega_{(l)})$ satisfying the integral identity

$$\begin{gathered}\nu\int\limits_{\Omega_{(l)}}\nabla\mathbf{v}^{(l)}:\nabla\boldsymbol{\eta}\,dx-\int\limits_{\Omega_{(l)}}((\mathbf{A}+\mathbf{v}^{(l)})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{v}^{(l)}\,dx-\int\limits_{\Omega_{(l)}}(\mathbf{v}^{(l)}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\,dx\\ =-\nu\int\limits_{\Omega_{(l)}}\nabla\mathbf{A}:\nabla\boldsymbol{\eta}\,dx+\int\limits_{\Omega_{(l)}}(\mathbf{A}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\,dx\quad\forall\boldsymbol{\eta}\in H(\Omega_{(l)}).\end{gathered}\tag{4.30}$$

Indeed, this integral identity is equivalent to the operator equation

$$\mathbf{v}^{(l)}=\widehat{\mathcal{A}}\mathbf{v}^{(l)},\tag{4.31}$$

where $\widehat{\mathcal{A}}$ is a compact operator (see [31]). By Leray-Schauder Theorem (see Lemma 2.13) operator equation (4.31) admits at least one solution if norms of all possible solutions of the operator equation

$$\mathbf{v}^{(l,\lambda)}=\lambda\widehat{\mathcal{A}}\mathbf{v}^{(l,\lambda)},\quad\lambda\in[0,1],\tag{4.32}$$

are bounded by the same constant independent of λ.
Taking $\boldsymbol{\eta}=\mathbf{v}^{(l)2}$ in (4.30) and using Leray-Hopf (3.10) and Cauchy–Schwarz (2.3) inequalities, we obtain

$$\begin{gathered}\int\limits_{\Omega_{(l)}}|\nabla\mathbf{v}^{(l)}|^2\,dx=\frac{\lambda}{\nu}\int\limits_{\Omega_{(l)}}(\mathbf{v}^{(l)}\cdot\nabla)\mathbf{v}^{(l)}\cdot\mathbf{A}\,dx-\lambda\int\limits_{\Omega_{(l)}}\nabla\mathbf{A}:\nabla\mathbf{v}^{(l)}\,dx\\ +\frac{\lambda}{\nu}\int\limits_{\Omega_{(l)}}(\mathbf{A}\cdot\nabla)\mathbf{v}^{(l)}\cdot\mathbf{A}\,dx\le c(\varepsilon|\mathfrak{F}_1^{(out)}|^2+|\mathbb{F}_1^{(inn)}|^2)\int\limits_{\Omega_{(l)}}|\nabla\mathbf{v}^{(l)}|^2\,dx\\ +\Big(\int\limits_{\Omega_{(l)}}|\nabla\mathbf{A}|^2\,dx\Big)^{1/2}\cdot\Big(\int\limits_{\Omega_{(l)}}|\nabla\mathbf{v}^{(l)}|^2\,dx\Big)^{1/2}+\Big(\int\limits_{\Omega_{(l)}}|\mathbf{A}|^4\,dx\Big)^{1/2}\cdot\Big(\int\limits_{\Omega_{(l)}}|\nabla\mathbf{v}^{(l)}|^2\,dx\Big)^{1/2}.\end{gathered}\tag{4.33}$$

[2] For simplicity, we omit the index λ.

Using Cauchy inequality with ε (2.1), we recieve

$$(1 - c\varepsilon|\mathfrak{F}_1^{(out)}|^2 - c|\mathbb{F}_1^{(inn)}|^2)\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(l)})}$$

$$\le \frac{c}{\varepsilon}\bigg(\|\nabla\mathbf{A}\|^2_{L^2(\Omega_{(l)})} + \|\mathbf{A}\|^4_{L^4(\Omega_{(l)})}\bigg) + c\varepsilon\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(l)})}.$$

Taking sufficiently small ε and $\mathbb{F}_1^{(inn)}$, we have

$$\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(l)})} \le c\bigg(\|\nabla\mathbf{A}\|^2_{L^2(\Omega_{(l)})} + \|\mathbf{A}\|^4_{L^4(\Omega_{(l)})}\bigg). \tag{4.34}$$

Since

$$|\mathfrak{F}_1^{(out)}| + |\mathbb{F}_1^{(inn)}| \le c\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)},$$

$$\|\nabla\mathbf{A}\|^2_{L^2(\Omega_{(l)})} \le c\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}\big(1 + \int\limits_1^{R_l}\frac{dx_3}{g_1^4(x_3)}\big),$$

$$\|\mathbf{A}\|^4_{L^4(\Omega_{(l)})} \le c\|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)}\big(1 + \int\limits_1^{R_l}\frac{dx_3}{g_1^6(x_3)}\big) \le c\|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)}\big(1 + \int\limits_1^{R_l}\frac{dx_3}{g_1^4(x_3)}\big),$$

the inequality (4.34) yields

$$\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(l)})} \le c\big(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)}\big)\big(1 + \int\limits_1^{R_l}\frac{dx_3}{g_1^4(x_3)}\big). \tag{4.35}$$

Hence the existence of the solution $\mathbf{v}^{(l)}$ of operator equation (4.31) follows from Leray-Schauder Theorem.

Let us estimate the norm $\|\nabla\mathbf{v}^{(l)}\|_{L^2(\Omega_{(k)})}$ for $k < l$. We introduce the function

$$\mathbf{U}_k^{(l)}(x) = \begin{cases} \mathbf{v}^{(l)}(x), \quad x \in \Omega_{(k)}, \\ \theta_k(x)\mathbf{v}^{(l)}(x) + \widehat{\mathbf{v}}_k^{(l)}(x), \quad x \in \Omega_{(k+1)} \setminus \Omega_{(k)} = \omega_k, \\ 0, \quad \Omega \setminus \Omega_{(k+1)}, \end{cases} \tag{4.36}$$

where $\theta_k(x)$ is a smooth cut-off function with the following properties:

$$\theta_k(x) = \begin{cases} 1, \quad x \in \Omega_{(k)}, \\ 0, \quad x \in \Omega \setminus \Omega_{(k+1)}, \end{cases}$$

$$|\nabla\theta_k(x)| \le \frac{c}{g_1(R_k)}, \tag{4.37}$$

and $\widehat{\mathbf{v}}_k^{(l)}$ is the solution of the problem

$$\begin{aligned} \operatorname{div} \widehat{\mathbf{v}}_k^{(l)} &= -\nabla\theta_k \cdot \mathbf{v}^{(l)} \quad \text{in} \quad \omega_k, \\ \widehat{\mathbf{v}}_k^{(l)} &= 0 \quad \text{on} \quad \partial\omega_k. \end{aligned} \tag{4.38}$$

Since $\int\limits_{\omega_k} \nabla\theta_k \cdot \mathbf{v}^{(l)}\,dx = \int\limits_{\omega_k} \operatorname{div}(\theta\mathbf{v}^{(l)})\,dx = \int\limits_{\partial\omega_k} \theta\mathbf{v}^{(l)} \cdot \mathbf{n}\,dx = \int\limits_{\sigma_1(k+1)} v_3\,dx = 0$, the solution $\widehat{\mathbf{v}}_k^{(l)}$ of problem (4.38) exists and satisfies the estimate

$$\|\nabla\widehat{\mathbf{v}}_k^{(l)}\|_{L^2(\omega_k)} \leq c\|\nabla\theta_k \cdot \mathbf{v}^{(l)}\|_{L^2(\omega_k)}, \tag{4.39}$$

where c is independent of k (see Lemma 2.9). Using estimate (4.37) and Poincaré inequality (2.10), from (4.39) we derive the estimate

$$\|\nabla\widehat{\mathbf{v}}_k^{(l)}\|_{L^2(\omega_k)} \leq c\|\nabla\theta_k \cdot \mathbf{v}^{(l)}\|_{L^2(\omega_k)} \leq \frac{c}{g_1(R_k)}\|\mathbf{v}^{(l)}\|_{L^2(\omega_k)} \leq c\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)}. \tag{4.40}$$

Let us take in the integral identity (4.30) $\boldsymbol{\eta} = \mathbf{U}_k^{(l)}$. Since

$$\int\limits_{\Omega_{(k+1)}} ((\mathbf{v}^{(l)} + \mathbf{A}) \cdot \nabla)\mathbf{U}_k^{(l)} \cdot \mathbf{U}_k^{(l)}\,dx = 0,$$

we obtain

$$\begin{aligned} \nu \int\limits_{\Omega_{(k)}} |\nabla\mathbf{v}^{(l)}|^2\,dx &= \int\limits_{\omega_k} ((\mathbf{v}^{(l)}+\mathbf{A})\cdot\nabla)\mathbf{U}_k^{(l)}\cdot(\mathbf{v}^{(l)}-\mathbf{U}_k^{(l)})\,dx - \nu\int\limits_{\omega_k} \nabla\mathbf{v}^{(l)}:\nabla\mathbf{U}_k^{(l)}\,dx \\ &+ \int\limits_{\Omega_{(k+1)}} (\mathbf{v}^{(l)}\cdot\nabla)\mathbf{U}_k^{(l)}\cdot\mathbf{A}\,dx - \nu\int\limits_{\Omega_{(k+1)}} \nabla\mathbf{A}:\nabla\mathbf{U}_k^{(l)}\,dx + \int\limits_{\Omega_{(k+1)}} (\mathbf{A}\cdot\nabla)\mathbf{U}_k^{(l)}\cdot\mathbf{A}\,dx. \end{aligned} \tag{4.41}$$

Using inequalities (4.40), (2.12) and Puancaré inequality (2.10), we obtain

$$\|\mathbf{v}^{(l)}\|_{L^4(\omega_k)} \leq c\,g_1^{1/4}(R_k)\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)},$$

$$\|\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)}\|_{L^4(\omega_k)} \leq \|\mathbf{v}^{(l)}\|_{L^4(\omega_k)} + \|\widehat{\mathbf{v}}_k^{(l)}\|_{L^4(\omega_k)}$$

$$\leq c\,g_1^{1/4}(R_k)\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)} + c\,g_1^{1/4}(R_k)\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)} \leq c\,g_1^{1/4}(R_k)\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)},$$

$$\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\omega_k)} \leq \|\nabla(\theta\mathbf{v}^{(l)})\|_{L^2(\omega_k)} + \|\nabla\widehat{\mathbf{v}}_k^{(l)}\|_{L^2(\omega_k)}$$

$$\leq \frac{c}{g_1(R_k)}\|\mathbf{v}^{(l)}\|_{L^2(\omega_k)} + c\,\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)} + c\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)} \leq c\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)}.$$

Therefore, using the above estimates and inequalities (4.27) we can estimate the right hand side of (4.41) as follows

$$\Big|\int\limits_{\omega_k}((\mathbf{v}^{(l)}+\mathbf{A})\cdot\nabla)\mathbf{U}_k^{(l)}\cdot(\mathbf{v}^{(l)}-\mathbf{U}_k^{(l)})\,dx\Big|$$

$$\le \|\mathbf{v}^{(l)}\|_{L^4(\omega_k)}\,\|\mathbf{v}^{(l)}-\mathbf{U}_k^{(l)}\|_{L^4(\omega_k)}\,\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\omega_k)}$$

$$+\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\omega_k)}\big(\int\limits_{\omega_k}|\mathbf{A}|^2|\mathbf{v}^{(l)}-\mathbf{U}_k^{(l)}|^2\,dx\big)^{1/2}\le cg_1^{1/2}(R_k)\|\nabla\mathbf{v}^{(l)}\|^3_{L^2(\omega_k)}$$

$$+c(\sqrt{\varepsilon}|\mathfrak{F}_1^{(out)}|+|\mathbb{F}_1^{(inn)}|)\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)}\big(\int\limits_{\omega_k}|\nabla(\mathbf{v}^{(l)}-\mathbf{U}_k^{(l)})|^2\,dx\big)^{1/2}$$

$$\le cg_1^{1/2}(R_k)\|\nabla\mathbf{v}^{(l)}\|^3_{L^2(\omega_k)}+c(\sqrt{\varepsilon}|\mathfrak{F}_1^{(out)}|+|\mathbb{F}_1^{(inn)}|)\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)};$$

$$\nu\Big|\int\limits_{\omega_k}\nabla\mathbf{v}^{(l)}:\nabla\mathbf{U}_k^{(l)}\,dx\Big|\le\nu\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_k)}\,\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\omega_k)}\le\nu c\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)}$$

$$\Big|\int\limits_{\Omega_{(k+1)}}(\mathbf{v}^{(l)}\cdot\nabla)\mathbf{U}_k^{(l)}\cdot\mathbf{A}\,dx\Big|\le\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\Omega_{(k+1)})}\left(\int\limits_{\Omega_{(k+1)}}|\mathbf{v}^{(l)}|^2|\mathbf{A}|^2\,dx\right)^{1/2}$$

$$\le c(\sqrt{\varepsilon}|\mathfrak{F}_1^{(out)}|+|\mathbb{F}_1^{(inn)}|)\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(k+1)})}$$
$$\le c(\sqrt{\varepsilon}|\mathfrak{F}_1^{(out)}|+|\mathbb{F}_1^{(inn)}|)\Big(\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(k)})}+\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)}\Big);$$

$$\nu\Big|\int\limits_{\Omega_{(k+1)}}\nabla\mathbf{A}:\nabla\mathbf{U}_k^{(l)}\,dx\Big|+\Big|\int\limits_{\Omega_{(k+1)}}(\mathbf{A}\cdot\nabla)\mathbf{U}_k^{(l)}\cdot\mathbf{A}\,dx\Big|$$

$$\le c\Big(\|\nabla\mathbf{A}\|_{L^2(\Omega_{(k+1)})}+\|\mathbf{A}\|^2_{L^4(\Omega_{(k+1)})}\Big)\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\Omega_{(k+1)})}$$

$$\le\frac{c}{2\sqrt{\varepsilon}}\Big(\|\nabla\mathbf{A}\|_{L^2(\Omega_{(k+1)})}+\|\mathbf{A}\|^2_{L^4(\Omega_{(k+1)})}\Big)^2+\frac{c\sqrt{\varepsilon}}{2}\|\nabla\mathbf{U}_k^{(l)}\|^2_{L^2(\Omega_{(k+1)})}$$

$$\le\frac{c}{\sqrt{\varepsilon}}\Big(\|\nabla\mathbf{A}\|^2_{L^2(\Omega_{(k+1)})}+\|\mathbf{A}\|^4_{L^4(\Omega_{(k+1)})}\Big)+\frac{c\sqrt{\varepsilon}}{2}\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(k+1)})}$$

$$\le\frac{c}{\sqrt{\varepsilon}}\Big(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)}\Big)\Big(1+\int\limits_1^{R_l}\frac{dx_3}{g_1^4(x_3)}\Big)$$

$$+c\sqrt{\varepsilon}\Big(\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(k)})}+\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_{(k)})}\Big).$$

Therefore, from (4.41) it follows that

$$\int\limits_{\Omega_{(k)}}|\nabla\mathbf{v}^{(l)}|^2\,dx\le cg_1^{1/2}(R_k)\|\nabla\mathbf{v}^{(l)}\|^3_{L^2(\omega_k)}+c(\sqrt{\varepsilon}|\mathfrak{F}_1^{(out)}|+|\mathbb{F}_1^{(inn)}|)\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)}$$

$$+c(\sqrt{\varepsilon}|\mathfrak{F}_1^{(out)}|+|\mathbb{F}_1^{(inn)}|)\Big(\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(k)})}+\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)}\Big)$$

$$+\frac{c}{\sqrt{\varepsilon}}\bigg(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)}\bigg)\bigg(1+\int\limits_1^{R_l}\frac{dx_3}{g_1^4(x_3)}\bigg).$$

Taking $\mathbb{F}_1^{(inn)}$ and ε sufficiently small, we obtain

$$\int\limits_{\Omega_{(k)}}|\nabla\mathbf{v}^{(l)}|^2\,dx\le c_{**}g_1^{1/2}(R_k)\|\nabla\mathbf{v}^{(l)}\|^3_{L^2(\omega_k)}+c_*\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_k)}$$
$$+c_0\bigg(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)}\bigg)\bigg(1+\int\limits_1^{R_k}\frac{dx_3}{g_1^4(x_3)}\bigg).$$

Denote $y_k=\int\limits_{\Omega_{(k)}}|\nabla\mathbf{v}^{(l)}|^2\,dx$. Since $\int\limits_{\omega_k}=\int\limits_{\Omega_{(k+1)}}-\int\limits_{\Omega_{(k)}}$, we can rewrite the last inequality as

$$y_k\le c_*(y_{k+1}-y_k)+c_{**}g_1^{1/2}(R_k)(y_{k+1}-y_k)^{3/2}+Q_k, \tag{4.42}$$

where

$$Q_k=c(data)\bigg(1+\int\limits_1^{R_k}\frac{dx_3}{g_1^4(x_3)}\bigg),\ c(data)=c_0\bigg(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)}\bigg).$$

We have

$$c_*(Q_{k+1}-Q_k)+c_{**}g_1^{1/2}(R_k)(Q_{k+1}-Q_k)^{3/2}$$
$$=c_*c(data)\int\limits_{R_k}^{R_{k+1}}\frac{dt}{g_1^4(t)}+c_{**}g_1^{1/2}(R_k)\big(c(data)\int\limits_{R_k}^{R_{k+1}}\frac{dt}{g_1^4(t)}\big)^{3/2}.$$

If $\int\limits_1^{\infty}\frac{dt}{g_1^4(t)}<\infty$, then $\int\limits_{R_k}^{R_{k+1}}\frac{dt}{g_1^4(t)}\to 0$ as $k\to\infty$, and there holds the estimate

$$c_*c(data)\int\limits_{R_k}^{R_{k+1}}\frac{dt}{g_1^4(t)}+c_{**}g_1^{1/2}(R_k)\big(c(data)\int\limits_{R_k}^{R_{k+1}}\frac{dt}{g_1^4(t)}\big)^{3/2}\le\tilde{c}c(data)\bigg(1+\int\limits_1^{R_k}\frac{dx_3}{g_1^4(x_3)}\bigg)$$

for sufficiently large k. If $\int\limits_1^{\infty}\frac{dt}{g_1^4(t)}=\infty$, then [3]

$$c_*c(data)\int\limits_{R_k}^{R_{k+1}}\frac{dt}{g_1^4(t)}+c_{**}g_1^{1/2}(R_k)\big(c(data)\int\limits_{R_k}^{R_{k+1}}\frac{dt}{g_1^4(t)}\big)^{3/2}$$
$$\le\tilde{c}_*c(data)g_1^{-3}(R_k)+\tilde{c}_{**}(c(data))^{3/2}g_1^{-4}(R_k)$$
$$\le\tilde{c}_*c(data)+\tilde{c}_{**}(c(data))^{3/2}\le\tilde{c}c(data)\bigg(1+\int\limits_1^{R_k}\frac{dx_3}{g_1^4(x_3)}\bigg),$$

[3] We have used that $R_{k+1}-R_k=\frac{g_1(R_k)}{2L_1}$ and $\frac{1}{2}g_1(R_k)\le g_1(t)\le\frac{3}{2}g_1(R_k)$ for $t\in(R_k,R_{k+1})$ (see [59]).

because $\int\limits_1^{R_k} \dfrac{dt}{g_1^4(t)} \to \infty$ as $k \to \infty$.

Since Q_k satisfy condition (2.20), by Lemma 2.16 inequality (4.42) together with (4.35) implies the estimate

$$y_k = \int\limits_{\Omega_{(k)}} |\nabla \mathbf{v}^{(l)}|^2\, dx \le c(data)\left(1 + \int\limits_1^{R_k} \frac{dx_3}{g_1^4(x_3)}\right) \forall k \le l. \tag{4.43}$$

This estimate guarantees the existence of a subsection $\{\mathbf{v}^{(l_m)}\}$ which converges weakly in $\overset{\circ}{W}{}^{1,2}(\Omega_{(k)})$ and strongly in $L^4(\Omega_{(k)})$ for every k. Therefore, passing to a limit in the integral identity (4.30) we find that the limiting vector field $\mathbf{v}$ satisfies (4.28) for any $\boldsymbol{\eta} \in J_0^\infty(\Omega)$ (or for any $H(\Omega)$ with compact support). The sum $\mathbf{u} = \mathbf{A} + \mathbf{v}$ is a weak solution of the problem (4.1). The estimate (4.29) for $\mathbf{v}$ follows from (4.43). Since for $\mathbf{A}$ the analogous to (4.29) estimate is obvious, we obtain (4.29) for the sum $\mathbf{u} = \mathbf{A} + \mathbf{v}$. □

4.2 Domain with two connected layer type outlets to infinity

In this section the domain $\Omega \subset \mathbb{R}^3$ consists of two layer type outlets to infinity $E_1 = \{x \in \Omega : 1 < x_3 < h_1(|x'|)\}$ and $E_2 = \{x \in \Omega : -2 < x_3 < -1\}$, $\Omega_0 = \Omega \bigcap B_{R_0}(0)$. Let $\Omega_0 = G_0 \setminus G_1$, $\overline{G}_1 \subset G_0$, G_0 and G_1 are bounded simply connected domains. Let $\partial G_1 = \Gamma_1 = \Gamma$ be the inner boundary of $\partial\Omega$. The outer boundary of $\partial\Omega$ consists of three components, i.e., $S = S^{(1)} \cup S^{(2)} \cup S^{(3)}$:

$$S^{(1)} = \{x \in \partial E_1 : x_3 = h_1(x'|)\},$$

$$S^{(2)} = \{x \in \partial E_1 : x_3 = 1\} \bigcup \big(\partial H \setminus (\partial E_1 \bigcup \partial E_2)\big) \bigcup \{x \in \partial E_2 : x_3 = -1\},$$
$$S^{(3)} = \{x \in \partial E_2 : x_3 = -2\},$$

where $H \subset \Omega_0$ is a finite cylinder, containing the origin, which connects the layers E_1 and E_2. In the domain Ω we consider the following problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega, \\ \int\limits_{\sigma_j(R)} \mathbf{u}\cdot\mathbf{n}\, dS = \mathcal{F}_j, & R \ge R_0, \quad j = 1, 2, \end{cases} \tag{4.44}$$

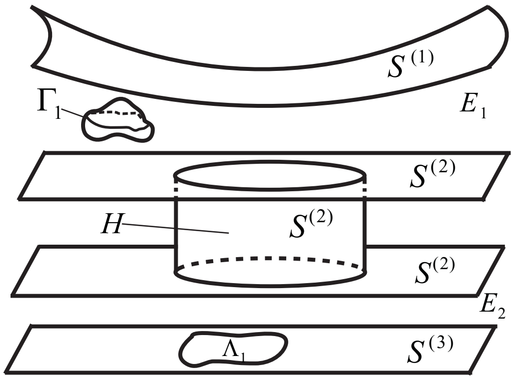

Figure 4.3: Domain Ω.

where $\sigma_j(R) = E_j \bigcap \{x : |x'| = R\}$ are the cross sections of layers E_j, $\mathcal{F}_j$ are prescribed fluxes over $\sigma_j(R)$, $j = 1, 2$.

Function $h_1(t)$ possess the following properties

$$\mu_1 h_1(t) \leq \max_{t \leq t_1 \leq 2t} h_1(t_1) \leq \mu_2 h_1(t), \quad h_1(t) \geq 1 \ \ \forall t \geq 1,$$

$$|h_1(t_1) - h_1(t_2)| \leq L_1(t)|t_1 - t_2|, \quad t_1, t_2 \in [t, 2t],$$

where μ_1, μ_2 are certain positive constants and for $L_1(t)$ holds the inequality

$$\frac{L_1(t) \cdot t}{h_1(t)} \leq const, \quad L_1(t) \leq const \ \ \forall t.$$

Suppose that the boundary value $\mathbf{a}$ has a compact support (see (3.2)) and $\Lambda_1 = \text{supp } \mathbf{a} \bigcap S \subset S^{(3)}$.

Below we use the following notations:

$$\Omega_{(l)} = \Omega_0 \cup E_1^{(l)} \cup E_2^{(l)}, \quad \omega_{jk} = \Omega_{(l+1)} \setminus \Omega_{(l)},$$

where $E_j^{(l)} = \{x \in E_j : |x'| < R_{jk}\}$, $R_{j1} = 1$, $R_{jl+1} = 2R_{jl}$, $l \geq 1$, $j = 1, 2$.
Let

$$\mathbb{F}_1^{(inn)} = \int_{\Gamma_1} \mathbf{a} \cdot \mathbf{n} \, dS, \quad \mathfrak{F}_1^{(out)} = \int_{\Lambda_1} \mathbf{a} \cdot \mathbf{n} \, dS$$

be the fluxes of the boundary value $\mathbf{a}$ over the inner and outer boundaries, respectively. Since the total flux has to be equal to zero, condition (3.3) implies

$$\mathcal{F}_1 + \mathcal{F}_2 + \mathbb{F}_1^{(inn)} + \mathfrak{F}_1^{(out)} = 0. \tag{4.45}$$

We construct the suitable extension $\mathbf{A} = \mathbf{B}_1^{(out)} + \mathbf{B}^{(inn)} + \mathbf{B}^{(flux)}$ and prove the existence of at least one solution to problem (4.44).

4.2.1 Construction of the extension $\mathbf{B}^{(inn)}$

Let us start with the construction of the "virtual drain" function $\mathbf{b}^{(inn)}$ which "drains" the flux $\mathbb{F}_1^{(inn)}$ from the inner boundary Γ to infinity. We choose the "widest" outlet E_1, in order to minimize the dissipation of energy generated by $\mathbf{b}^{(inn)}$. Let us first define in E_1 a solenoidal vector field $\mathbf{b}_1^{(inn)}$ such that

$$\mathbf{b}_1^{(inn)}(x)\big|_{\partial E_1\cap\partial\Omega}=0,\qquad \int\limits_{\sigma_1^{(E)}(R)}\mathbf{b}_1^{(inn)}\cdot\mathbf{n}\,dS=\mathbb{F}_1^{(inn)}.$$

Let $\gamma_1=\{x\in E_1:|x'|=0\}$. Define in E_1 a cut–off function

$$\zeta_1(x)=\Psi\Big(\ln\Big(\frac{\varrho(\delta(x))}{\Delta(x)}\Big)\Big),\tag{4.46}$$

where

$$\delta(x)=\Delta_{\gamma_1\cup S^{(1)}}(x),\quad \Delta(x)=\Delta_{\partial E_1\setminus S^{(1)}}(x),$$

functions Ψ and ϱ are defined by formulas (2.24) and (2.25), respectively.

Lemma 4.7. *The function $\zeta_1(x)$ is equal to zero at those points of $E_1\setminus\gamma_1$, where $\varrho(\delta(x))\le\Delta(x)$ and $\zeta_1(x)=1$ if $\Delta(x)\le e^{-1}\varrho(\delta(x))$. The following estimates*

$$\Big|\frac{\partial\zeta_1(x)}{\partial x_k}\Big|\le\frac{c}{\Delta(x)},\qquad\Big|\frac{\partial^2\zeta_1(x)}{\partial x_k\partial x_l}\Big|\le\frac{c}{\Delta^2(x)}\tag{4.47}$$

hold.

Proof. The proof of the lemma follows directly from the definition of the functions ζ_1,Ψ and ϱ, properties of the regularized distance (see estimates (2.21)) and the fact that supp$\nabla\zeta_1$ is contained in the set where $\Delta(x)\le\varrho(\delta(x))$. □

Define

$$\mathbf{b}_1^{(inn)}(x)=\mathbb{F}_1^{(inn)}\mathrm{curl}\,\big(\zeta_1(x)\mathbf{b}(x)\big)=\mathbb{F}_1^{(inn)}\nabla\zeta_1(x)\times\mathbf{b}(x),\ x\in E_1,$$

where $\mathbf{b}(x)$ is a magnetic field defined by (2.23) (the properties of $\mathbf{b}(x)$ are given in Lemma 2.17).

Lemma 4.8. *The solenoidal vector field $\mathbf{b}_1^{(inn)}$ is infinitely differentiable for $x\in E_1\setminus\{x:|x'|=0\}$, vanishes near the set $\partial E_1\bigcup\{x:|x'|=0\}$, and satisfies the conditions*

$$\int\limits_{\sigma_1(R)}\mathbf{b}_1^{(inn)}\cdot\mathbf{n}\,dS=\mathbb{F}_1^{(inn)},\quad \boldsymbol{n}=\Big(\frac{x_1}{|x'|},\frac{x_2}{|x'|},0\Big),\tag{4.48}$$

$$|\mathbf{b}_1^{(inn)}(x)|\le\frac{c|\mathbb{F}_1^{(inn)}|}{d(x)},\tag{4.49}$$

$$
\begin{aligned}
&|\mathbf{b}_1^{(inn)}(x)| \le \frac{C|\mathbb{F}_1^{(inn)}|}{h_1(|x'|)\,|x'|},\\
&|\nabla \mathbf{b}_1^{(inn)}(x)| \le \frac{C|\mathbb{F}_1^{(inn)}|}{h_1^2(|x'|)\,|x'|} + \frac{C|\mathbb{F}_1^{(inn)}|}{h_1(|x'|)\,|x'|^2},\ \ x \in E_1 \setminus \Omega_0.
\end{aligned} \tag{4.50}
$$

In (4.49) $d(x) = \mathrm{dist}(x, \partial E_1 \cap \partial\Omega \setminus S^{(1)})$.

Proof. Since $\big(\nabla\zeta_1 \times \mathbf{b}\big) \cdot \mathbf{n} = -\frac{1}{2\pi} \cdot \frac{\partial \zeta_1}{\partial x_3} \cdot \frac{1}{|x'|}$, we obtain

$$
\int\limits_{\sigma_1(R)} \mathbf{b}_1^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}_1^{(inn)} \int\limits_{\sigma_1(R)} \big(\nabla\zeta_1 \times \mathbf{b}\big) \cdot \mathbf{n}\, dS
$$

$$
\begin{aligned}
&= -\mathbb{F}_1^{(inn)} \frac{R}{2\pi} \int\limits_0^{2\pi} d\phi \int\limits_1^{h_1(R)} \frac{\partial \zeta_1}{\partial x_3} \cdot \frac{1}{R}\, d\,x_3 = -\mathbb{F}_1^{(inn)} \int\limits_1^{h_1(R)} \frac{\partial \zeta_1}{\partial x_3}\, d\,x_3\\
&= -\mathbb{F}_1^{(inn)} \big(\zeta_1(x_1, x_2, h_1(R)) - \zeta_1(x_1, x_2, 1)\big) = \mathbb{F}_1^{(inn)}.
\end{aligned}
$$

From the definition of $\mathbf{b}_1^{(inn)}(x)$ and estimates (4.47) it follows that

$$
\begin{aligned}
&|\mathbf{b}_1^{(inn)}(x)| \le |\mathbb{F}_1^{(inn)}||\nabla\zeta_1(x)||\mathbf{b}(x)| \le \frac{c|\mathbb{F}_1^{(inn)}|}{\Delta(x)\,|x'|},\\
&|\nabla \mathbf{b}_1^{(inn)}(x)| \le |\mathbb{F}_1^{(inn)}|\Big(|\nabla(\nabla\zeta_1(x))||\mathbf{b}(x)| + |\nabla\zeta_1(x)||\nabla\mathbf{b}(x)|\Big)\\
&\le c|\mathbb{F}_1^{(inn)}|\Big(\frac{1}{\Delta^2(x)|x'|} + \frac{1}{\Delta(x)|x'|^2}\Big).
\end{aligned} \tag{4.51}
$$

It is easy to see that for points $x \in \mathrm{supp}\,\mathbf{b}_1^{(inn)}$ the inequalities

$$
c_1 h_1(|x'|) \le \Delta(x) \le c_2 h_1(|x'|) \tag{4.52}
$$

hold with positive constants c_1 and c_2. Then estimates (4.49), (4.50) follow from (4.51), (4.52). □

Let us briefly describe the construction of the virtual drain function, which "removes" nonzero fluxes from the inner boundary Γ_1. Let $x^{(1)} \in G_1$, be the point lying inside the "hole" G_1. Denote $q_1(x) = q(x - x^{(1)})$, where

$$
q(x) = \frac{1}{4\pi} \frac{1}{|x|}
$$

is the fundamental solution of the Laplace operator in $\mathbb{R}^3$, and let

$$
\mathbf{b}_\sharp^{(inn)}(x) = \mathbb{F}_1^{(inn)} \nabla q_1(x).
$$

Then

$$\operatorname{div} \mathbf{b}_\sharp^{(inn)} = 0, \ \int\limits_{\Gamma_1} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n} \, dS = \mathbb{F}_1^{(inn)}, \ \int\limits_{\partial\Omega_0} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n} \, dS = -\mathbb{F}_1^{(inn)}$$

(remind that $\overline{G}_1 \subset \Omega_0$).

Set

$$\mathbf{h}_1 = \begin{cases} 0, \quad x \in \Gamma_1, \\ \mathbf{b}_1^{(inn)}|_{\partial\Omega_0 \cap \overline{E}_1} - \mathbf{b}_\sharp^{(inn)}|_{\partial\Omega_0 \cap \overline{E}_1}, \quad x \in \partial\Omega_0 \cap \overline{E}_1, \\ -\mathbf{b}_\sharp^{(inn)}\big|_{\partial\Omega_0 \setminus \overline{E}_1}, \quad x \in \partial\Omega_0 \setminus (\overline{E}_1 \cup \Gamma_1). \end{cases}$$

We have

$$\int\limits_{\partial\Omega_0} \mathbf{h}_1 \cdot \mathbf{n} \, dS = \int\limits_{\partial\Omega_0 \cap \overline{E}_1} \mathbf{b}_1^{(inn)} \cdot \mathbf{n} \, dS - \int\limits_{\partial\Omega_0} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n} \, dS = \mathbb{F}_1^{(inn)} - \mathbb{F}_1^{(inn)} = 0.$$

Therefore, the function $\mathbf{h}_1$ can be extended inside the domain Ω_0 as a solenoidal vector field $\mathbf{b}_{01}^{(inn)} \in W^{1,2}(\Omega_0)$ and

$$\begin{aligned} &\|\mathbf{b}_{01}^{(inn)}\|_{W^{1,2}(\Omega_0)} \le c\|\mathbf{h}_1\|_{W^{1/2,2}(\partial\Omega_0)} \\ &\le c\Big(\|\mathbf{b}_\sharp^{(inn)}\|_{W^{1/2,2}(\partial\Omega_0)} + \|\mathbf{b}_1^{(inn)}\|_{W^{1/2,2}(\partial\Omega_0 \cap \overline{E}_1)}\Big) \le c|\mathbb{F}_1^{(inn)}|, \end{aligned}$$

where the constant c depends only on the domain Ω_0 (see Lemma (2.6)).

As in Subsection 4.1.1, define the virtual drain function by the formula

$$\mathbf{b}^{(inn)} = \begin{cases} \mathbf{b}_\sharp^{(inn)} + \mathbf{b}_{01}^{(inn)}, \quad x \in \Omega_0, \\ \mathbf{b}_1^{(inn)}, \quad x \in E_1 \setminus \Omega_0, \\ 0, \quad x \in E_2 \setminus \Omega_0. \end{cases}$$

Set

$$\mathbf{h}_0 = \begin{cases} \mathbf{a} - \mathbf{b}_\sharp^{(inn)}|_{\Gamma_1}, \quad x \in \Gamma_1 \\ 0, \quad x \in \partial\Omega_0 \setminus \Gamma_1. \end{cases}$$

Then

$$\int\limits_{\Gamma_1} \mathbf{h}_0 \cdot \mathbf{n} \, dS = \int\limits_{\Gamma_1} \mathbf{a} \cdot \mathbf{n} \, dS - \int\limits_{\Gamma_1} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n} \, dS = \mathbb{F}_1^{(inn)} - \mathbb{F}_1^{(inn)} = 0,$$

and, therefore, the function $\mathbf{h}_0$ can be extended inside Ω_0 in the form

$$\mathbf{b}_0^{(inn)}(x) = \operatorname{curl}(\chi(x)\mathbf{E}(x)),$$

where $\mathbf{E} \in W_2^2(\Omega_0)$, $\operatorname{curl} \mathbf{E}|_{\partial\Omega_0} = \mathbf{h}_0$ and χ is a smooth cut-off function with $\chi(x) = 1$ on Γ_1. Moreover, for any $\mathbf{w} \in W^{1,2}_{loc}(\Omega)$ with $\mathbf{w}|_{\partial\Omega} = 0$ the following estimate

$$\int\limits_{\Omega_0} |\mathbf{b}_0^{(inn)}(x)|^2 |\mathbf{w}(x)|^2 \, dx \leq c|\mathbb{F}_1^{(inn)}|^2 \int\limits_{\Omega_0} |\nabla \mathbf{w}(x)|^2 \, dx$$

holds (see Lemma 2.14 with $\varepsilon = 1$ and Lemma 2.15). Finally, we put

$$\mathbf{B}^{(inn)} = \mathbf{b}^{(inn)} + \mathbf{b}_0^{(inn)}. \tag{4.53}$$

As in Subsection 4.1.1, the vector field $\mathbf{B}^{(inn)}$ satisfies the following lemma.

Lemma 4.9. *The vector field $\mathbf{B}^{(inn)}$ is solenoidal, $\mathbf{B}^{(inn)}|_{\Gamma_1} = \mathbf{a}|_{\Gamma_1}$, $\mathbf{B}^{(inn)}|_{S^{(m)}} = 0$, $m = 1, 2, 3$, $\mathbf{B}^{(inn)} \in W^{1,2}_{loc}(\Omega)$, $\mathbf{B}^{(inn)}(x) = 0$, $x \in E_2 \backslash \Omega_0$. For any $\mathbf{w} \in W^{1,2}_{loc}(\Omega)$ with $\mathbf{w}\big|_{\partial\Omega} = 0$ the following estimates*

$$\begin{aligned} &\int\limits_{\Omega_{(k)} \backslash \Omega_{(k-1)}} |\mathbf{B}^{(inn)}|^2 |\mathbf{w}|^2 \, dx \leq c|\mathbb{F}_1^{(inn)}|^2 \int\limits_{\Omega_{(k)} \backslash \Omega_{(k-1)}} |\nabla \mathbf{w}|^2 \, dx, \\ &\int\limits_{\Omega_{(k)}} |\mathbf{B}^{(inn)}|^2 |\mathbf{w}|^2 \, dx \leq c|\mathbb{F}_1^{(inn)}|^2 \int\limits_{\Omega_{(k)}} |\nabla \mathbf{w}|^2 \, dx, \end{aligned} \tag{4.54}$$

hold. Moreover,

$$\begin{aligned} &|\mathbf{B}^{(inn)}(x)| \leq \frac{C|\mathbb{F}_1^{(inn)}|}{h_1(|x'|)|x'|}, \; x \in E_1 \setminus \Omega_0, \\ &|\nabla \mathbf{B}^{(inn)}(x)| \leq \frac{C|\mathbb{F}_|^{(inn)}}{h_1^2(|x'|)|x'|} + \frac{C|\mathbb{F}_1^{(inn)}|}{h_1(|x'|)|x'|^2}, \; x \in E_1 \setminus \Omega_0, \\ &|\mathbf{B}^{(inn)}(x)| + |\nabla \mathbf{B}^{(inn)}(x)| \leq C|\mathbb{F}_1^{(inn)}|, \; x \in \Omega_0. \end{aligned} \tag{4.55}$$

4.2.2 Construction of the extension $\mathbf{B}_1^{(out)}$

We start with the construction of virtual drain function $\mathbf{b}_1^{(out)}$. Let $\gamma_{11} = \{x : |x'| = 0\}$ be an infinite line which intersects $\partial\Omega$ at the points $x^{(1)} \in \Lambda_1 \subset S^{(3)}$ and $x^{(0)} \in S^{(1)}$. Define

$$\mathbf{b}_1^{(out)}(x, \varepsilon) = -\mathfrak{F}_1^{(out)} \operatorname{curl}(\zeta_{11}(x, \varepsilon) \cdot \mathbf{b}_1^{(1)}(x)) = -\mathfrak{F}_1^{(out)} \nabla \zeta_{11}(x, \varepsilon) \times \mathbf{b}_1^{(1)}(x), \tag{4.56}$$

where

$$\zeta_{11}(x, \varepsilon) = \Psi\Big(\varepsilon \ln \frac{\delta^{(11)}(x)}{\Delta_{\partial\Omega \backslash \Lambda_1}(x)}\Big), \tag{4.57}$$

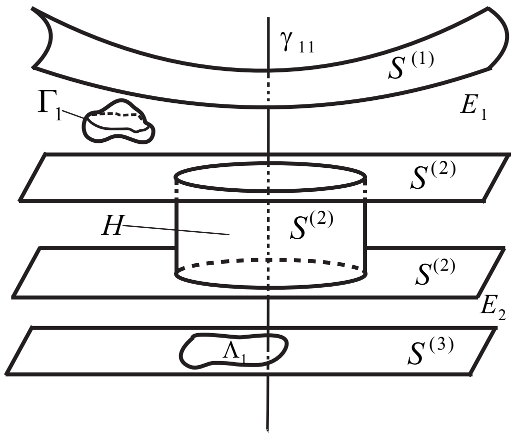

Figure 4.4: Contour γ_{11}.

$$\delta^{(11)} = \begin{cases} \rho_1(x)\Delta_{(\gamma_{11}\cup S^{(1)})\cap E_1}(x) + \rho_2(x)|x-x_0| \\ +(1-\rho_1(x)-\rho_2(x))\Delta_{\gamma_{11}\cup S^{(1)}}(x),\ x\in\partial\Omega\setminus(\gamma_{11}\cup S^{(1)}), x_0\in\gamma_{11}\cap\Omega_0, \\ 0,\ x\in\gamma_{11}\cup S^{(1)}, \end{cases}$$

$$\rho_j(x) = \begin{cases} 1,\ x\in E_j\setminus E_j^{(3)}, \\ 0,\ x\in(\partial\Omega\setminus E_j)\cup E_j^{(2)},\ j=1,2, \end{cases}$$

$\mathbf{b}_1^{(1)}(x)$ is the magnetic field (2.23) corresponding to the contour γ_{11} (the properties of $\mathbf{b}_1^{(1)}(x)$ are given in Lemma 2.17), the function Ψ is defined by formula (2.24).

Lemma 4.10. *The vector field $\boldsymbol{b}_1^{(out)}$ is infinitely differentiable and solenoidal, $\boldsymbol{b}_1^{(out)}$ vanishes near the surface $\partial\Omega\setminus\Lambda_1$, in a small neighborhood of the curve $\gamma_{11}\cap\overline{\Omega}$ and for $x\in E_2$ with $|x|\gg 1$. The following estimates*

$$\begin{aligned} &|\boldsymbol{b}_1^{(out)}(x,\varepsilon)| \le \frac{c\varepsilon}{d_{\partial\Omega\setminus(\Lambda_1\cup S^{(1)})}(x)|x'|}, \\ &|\nabla\boldsymbol{b}_1^{(out)}(x,\varepsilon)| \le c\Big(\frac{1}{d^2_{\partial\Omega\setminus(\Lambda_1\cup S^{(1)})}(x)|x'|} + \frac{1}{d_{\partial\Omega\setminus(\Lambda_1\cup S^{(1)})}(x)|x'|}\Big), \end{aligned} \tag{4.58}$$

$$\begin{gathered} |\boldsymbol{b}_1^{(out)}(x,\varepsilon))| \le \frac{c(\varepsilon)|\mathfrak{F}_1^{(out)}|}{h_1(|x'|)|x'|}, \\ |\nabla\boldsymbol{b}_1^{(out)}(x,\varepsilon)| \le \frac{c(\varepsilon)|\mathfrak{F}_1^{(out)}|}{h_1^2(|x'|)|x'|} + \frac{c(\varepsilon)|\mathfrak{F}_1^{(out)}|}{h_1(|x'|)|x'|^2},\ x\in E_1\setminus\Omega_0, \\ |\boldsymbol{b}_1^{(out)}(x,\varepsilon))| + |\nabla\boldsymbol{b}_1^{(out)}(x,\varepsilon)| \le c(\varepsilon)|\mathfrak{F}_1^{(out)}|,\ x\in\Omega\setminus E_1 \end{gathered} \tag{4.59}$$

hold. Here $d_{\partial\Omega\setminus(\Lambda_1\cup S^{(1)})}(x)$ is the distance from the point x to $\partial\Omega\setminus(\Lambda_1\cup S^{(1)})$. Finally,

$$\int\limits_{\Lambda_1} \boldsymbol{b}_1^{(out)}\cdot\boldsymbol{n}\,dS = \mathfrak{F}_1^{(out)}.$$

Proof. The first statement of the lemma follow from definitions (4.56), (4.57) of $\mathbf{b}_1^{(out)}(x,\varepsilon)$ and $\zeta_{11}(x,\varepsilon)$ and from the properties of the regularized distance (see estimates (2.21)). Ineqalities (4.58), (4.59) can be proved just is the same way as the analogous estimates in [59], [61]. Since $\mathbf{b}_1^{(out)}(x,\varepsilon)|_{\partial\Omega\setminus\Lambda_1}=0$, $\mathbf{b}_1^{(out)}(x,\varepsilon)=0$ for $x\in E_2$, $|x|\gg 1$ and $\zeta_{11}(x,\varepsilon)=1$ on $\partial\Omega\setminus(\Lambda_1\cup S^{(1)})$, and $\zeta_{11}(x,\varepsilon)=0$ on $S^{(1)}$, the Ostrogradsky–Gauss and the Stokes formulas (see Lemma 2.11 and Lemma 2.12, respectively) yield

$$\int\limits_{\Lambda_1}\mathbf{b}_1^{(out)}\cdot\mathbf{n}\,dS=-\int\limits_{\sigma_1(R)}\mathbf{b}_1^{(out)}\cdot\mathbf{n}\,dS=\mathfrak{F}_1^{(out)}\int\limits_{\sigma_1(R)}\big(\nabla\zeta_{11}\times\mathbf{b}_1^{(1)}\big)\cdot\mathbf{n}\,dS$$

$$=-\mathfrak{F}_1^{(out)}\frac{R}{2\pi}\int\limits_0^{2\pi}d\phi\int\limits_1^{h_1(R)}\frac{\partial\zeta_{11}}{\partial x_3}\cdot\frac{1}{R}\,d\,x_3=-\mathfrak{F}_1^{(out)}\int\limits_1^{h_1(R)}\frac{\partial\zeta_{11}}{\partial x_3}\,d\,x_3$$

$$=-\mathfrak{F}_1^{(out)}\big(\zeta_{11}(x_1,x_2,h_1(R))-\zeta_{11}(x_1,x_2,1)\big)=\mathfrak{F}_1^{(out)}.$$

□

Let $\mathbf{h}_1(x)=\mathbf{a}(x)|_{\Lambda_1}-\mathbf{b}_1^{(out)}(x,\varepsilon)|_{\Lambda_1}$. Then

$$\int\limits_{\Lambda_1}\mathbf{h}_1\cdot\mathbf{n}\,dS=\int\limits_{\Lambda_1}\mathbf{a}\cdot\mathbf{n}\,dS-\int\limits_{\Lambda_1}\mathbf{b}_1^{(out)}\cdot\mathbf{n}\,dS=0.$$

Therefore, $\mathbf{h}_1$ can be extended inside Ω in the form

$$\mathbf{b}_{01}^{(out)}(x,\varepsilon)=\operatorname{curl}\left(\chi_1(x,\varepsilon)\mathbf{E}_1(x)\right),$$

where $\mathbf{E}_1\in W_2^2(\Omega_0)$, $\operatorname{curl}\mathbf{E}_1|_{\Lambda_1}=\mathbf{h}_1$ and χ_1 is a Hopf's type cut-off function such that $\chi_1(x,\varepsilon)=1$ on Λ_1 (see Lemma 2.14).

Define

$$\mathbf{B}_1^{(out)}(x,\varepsilon)=\mathbf{b}_1^{(out)}(x,\varepsilon)+\mathbf{b}_{0\,1}^{(out)}(x,\varepsilon).$$

Obviously,

$$\operatorname{div}\mathbf{B}_1^{(out)}=0,\quad \mathbf{B}_1^{(out)}|_{\Lambda_1}=\mathbf{a},\quad \mathbf{B}_1^{(out)}|_{\partial\Omega\setminus\Lambda_1}=0,\quad \mathbf{B}_1^{(out)}=0, x\in E_2,\ |x|\gg 1.$$

Lemma 4.11. *For any solenoidal $\mathbf{w}\in W^{1,2}_{loc}(\Omega)$ with $\mathbf{w}\big|_{\partial\Omega}=0$ the following estimates*

$$\begin{gathered}\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}}|\mathbf{B}_1^{(out)}|^2|\mathbf{w}|^2\,dx\le\varepsilon c|\mathfrak{F}_1^{(out)}|^2\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}}|\nabla\mathbf{w}|^2\,dx,\\ \int\limits_{\Omega_{(k)}}|\mathbf{B}_1^{(out)}|^2|\mathbf{w}|^2\,dx\le\varepsilon c|\mathfrak{F}_1^{(out)}|^2\int\limits_{\Omega_{(k)}}|\nabla\mathbf{w}|^2\,dx,\end{gathered}\tag{4.60}$$

hold. Moreover,

$$|\mathbf{B}_1^{(out)}(x,\varepsilon)| \le \frac{C|\mathfrak{F}_1^{(out)}|}{h_1(|x'|)|x'|}, \quad x\in E_1\setminus\Omega_0,$$

$$|\nabla\mathbf{B}_1^{(out)}(x)| \le \frac{C|\mathfrak{F}_1^{(out)}|}{h_1^2(|x'|)|x'|} + \frac{C|\mathfrak{F}_1^{(out)}|}{h_1(|x'|)|x'|^2},\ x\in E_1\setminus\Omega_0, \tag{4.61}$$

$$|\mathbf{B}_1^{(out)}(x,\varepsilon)| + |\nabla\mathbf{B}_1^{(out)}(x)| \le C|\mathfrak{F}_1^{(out)}|,\ x\in E_2\setminus\Omega_0.$$

The constant c in (4.60) *does not depend on ε and k, while $C=C(\varepsilon)$ in* (4.61) *depends on ε.*

Proof. Inequality (4.58) yields

$$\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{B}_1^{(out)}|^2|\mathbf{w}|^2dx \le c\varepsilon|\mathfrak{F}_1^{(out)}|^2 \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} \frac{|\mathbf{w}|^2}{\mathrm{dist}^2(x,\partial\Omega)}dx,$$

$$\le c\varepsilon|\mathfrak{F}_1^{(out)}|^2 \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla\mathbf{w}|^2dx,$$

$$\int\limits_{\Omega_{(k)}} |\mathbf{B}_1^{(out)}|^2|\mathbf{w}|^2dx \le c\varepsilon|\mathfrak{F}_1^{(out)}|^2 \int\limits_{\Omega_{(k)}} \frac{|\mathbf{w}|^2}{\mathrm{dist}^2(x,\partial\Omega)}dx \le c\varepsilon|\mathfrak{F}_1^{(out)}| \int\limits_{\Omega_{(k)}} |\nabla\mathbf{w}|^2dx$$

(see Lemma 2.15 and [59] for details). Estimates (4.61) are the consequence of (4.59). □

4.2.3 Construction of the extension $\mathbf{B}^{(flux)}$

Now we need to compensate the fluxes over the cross sections of outlets to infinity, i.e., we have to construct a solenoidal vector field $\mathbf{B}^{(flux)}$ satisfying the flux conditions

$$\int\limits_{\sigma_1(R)} \mathbf{B}^{(flux)}\cdot\mathbf{n}\,dS = \mathcal{F}_1 + \mathbb{F}_1^{(inn)} + \mathfrak{F}_1^{(out)}, \quad \int\limits_{\sigma_2(R)} \mathbf{B}^{(flux)}\cdot\mathbf{n}\,dS = \mathcal{F}_2. \tag{4.62}$$

Denote

$$\mathbf{b}_{1,2}^{(flux)}(x,\varepsilon) = curl(\zeta_{1,2}(x,\varepsilon)\mathbf{b}^{(1,2)}(x)) = \nabla\zeta_{1,2}(x,\varepsilon)\times\mathbf{b}^{(1,2)}(x),$$

where $\mathbf{b}^{(1,2)}(x)$ is the magnetic field (2.23) corresponding to the contour γ_{11} (the properties of $\mathbf{b}^{(1,2)}(x)$ are given in Lemma 2.17),

$$\zeta_{1,2}(x,\varepsilon) = \Psi\left(\varepsilon\, ln\frac{\varrho(\Delta_{\gamma_{11}\cup S^{(2)}}(x))}{\Delta_{\partial\Omega\setminus S^{(2)}}(x)}\right),$$

the functions Ψ and ϱ are defined by formulas (2.24) and (2.25), respectively. The vector field $\mathbf{b}_{1,2}^{(flux)}$ has the following properties (see [59], [61], [47]).

Lemma 4.12. *The vector field* $\boldsymbol{b}_{1,2}^{(flux)}(x,\varepsilon)$ *is solenoidal,* $\boldsymbol{b}_{1,2}^{(flux)}|_{\partial\Omega}=0$ *and*

$$\int\limits_{\sigma_1(R)} \boldsymbol{b}_{1,2}^{(flux)}\cdot\mathbf{n}\,dS=-1,\quad \int\limits_{\sigma_2(R)} \boldsymbol{b}_{1,2}^{(flux)}\cdot\mathbf{n}\,dS=1.$$

For any solenoidal $\mathbf{w}\in W^{1,2}_{loc}(\Omega)$ *with* $\mathbf{w}\big|_{\partial\Omega}=0$ *the following estimates*

$$\begin{aligned}&\int\limits_{\Omega_{(m)}\setminus\Omega_{(m-1)}} |\boldsymbol{b}_{1,2}^{(flux)}|^2|\mathbf{w}|^2\,dx\le\varepsilon c\int\limits_{\Omega_{(m)}\setminus\Omega_{(m-1)}}|\nabla\mathbf{w}|^2\,dx,\\ &\int\limits_{\Omega_{(m)}}|\boldsymbol{b}_{1,2}^{(flux)}|^2|\mathbf{w}|^2\,dx\le\varepsilon c\int\limits_{\Omega_{(m)}}|\nabla\mathbf{w}|^2\,dx\end{aligned}\tag{4.63}$$

hold with the constant c *independent of* ε *and* m*. Moreover,*

$$\begin{aligned}&|\boldsymbol{b}_{1,2}^{(flux)}(x,\varepsilon)|\le\frac{C(\varepsilon)}{h_s(|x'|)|x'|},\\ &|\nabla\boldsymbol{b}_{1,2}^{(flux)}(x,\varepsilon))|\le\frac{C(\varepsilon)}{h_s^2(|x'|)|x'|}+\frac{C(\varepsilon)}{h_s(|x'|)|x'|^2},\quad x\in E_s\setminus\Omega_0,\ s=1,2,\\ &|\boldsymbol{b}_{1,2}^{(flux)}(x,\varepsilon)|+|\nabla\boldsymbol{b}_{1,2}^{(flux)}(x,\varepsilon))|\le C(\varepsilon),\ x\in\Omega_0.\end{aligned}\tag{4.64}$$

Vector field

$$\mathbf{B}^{(flux)}(x,\varepsilon)=-(\mathcal{F}_1+\mathbb{F}_1^{(inn)}+\mathfrak{F}_1^{(out)})\mathbf{b}_{1,2}^{(flux)}(x,\varepsilon)$$

satisfies the flux conditions (4.62).

4.2.4 Existence of a solution

We look for the solution $\mathbf{u}$ in the form

$$\mathbf{u}(x)=\mathbf{A}(x,\varepsilon)+\mathbf{v}(x),\tag{4.65}$$

where

$$\mathbf{A}(x,\varepsilon)=\mathbf{B}_1^{(out)}(x,\varepsilon)+\mathbf{B}^{(inn)}(x)+\mathbf{B}^{(flux)}(x,\varepsilon).$$

As it follows from (4.54), (4.60), (4.63), for any solenoidal $\mathbf{w} \in W^{1,2}_{loc}(\Omega)$ with $\mathbf{w}|_{\partial\Omega} = 0$ the following inequalities

$$\begin{aligned} &\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{A}|^2|\mathbf{w}|^2 dx \leq c(\varepsilon|\vec{\mathcal{F}}|^2+\varepsilon|\mathfrak{F}_1^{(out)}|^2+|\mathbb{F}_1^{(inn)}|^2) \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla\mathbf{w}|^2 dx, \\ &\int\limits_{\Omega_{(k)}} |\mathbf{A}|^2|\mathbf{w}|^2\,dx \leq c(\varepsilon|\vec{\mathcal{F}}|^2 + \varepsilon|\mathfrak{F}_1^{(out)}|^2 + |\mathbb{F}_1^{(inn)}|^2) \int\limits_{\Omega_{(k)}} |\nabla\mathbf{w}|^2\,dx, \end{aligned} \tag{4.66}$$

hold, where $|\vec{\mathcal{F}}| = \sqrt{\mathcal{F}_1^2 + \mathcal{F}_2^2}$, $\vec{\mathcal{F}} = (\mathcal{F}_1, \mathcal{F}_2)$.

Definition 4.2. *By a weak solution of problem* (4.44) *we understand a solenoidal vector field* $\mathbf{u}$ *which has the representation* (4.65) *with the solenoidal vector field* $\boldsymbol{v} \in W^{1,2}_{loc}(\Omega)$, $\boldsymbol{v}|_{\partial\Omega} = 0$, *satisfying the integral identity*

$$\begin{aligned} &\nu\int\limits_{\Omega} \nabla\boldsymbol{v} : \nabla\boldsymbol{\eta}\,dx - \int\limits_{\Omega}((\boldsymbol{A}+\boldsymbol{v})\cdot\nabla)\boldsymbol{\eta}\cdot\boldsymbol{v}\,dx - \int\limits_{\Omega}(\boldsymbol{v}\cdot\nabla)\boldsymbol{\eta}\cdot\boldsymbol{A}\,dx \\ &= -\nu\int\limits_{\Omega}\nabla\boldsymbol{A} : \nabla\boldsymbol{\eta}\,dx + \int\limits_{\Omega}(\boldsymbol{A}\cdot\nabla)\boldsymbol{\eta}\cdot\boldsymbol{A}\,dx \quad \forall\boldsymbol{\eta}\in J_0^\infty(\Omega). \end{aligned} \tag{4.67}$$

Theorem 4.2. *Assume that the boundary value* $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$ *has a compact support and that the flux* $\mathbb{F}_1^{(inn)}$ *is sufficiently small. Then there exists at least one weak solution* $\boldsymbol{u}$ *of problem* (4.44), (4.45) *and there holds the estimate*

$$\int\limits_{\Omega_{(l)}} |\nabla\boldsymbol{u}|^2\,dx \leq c(data)\left(1 + \int\limits_1^{2^k}\frac{dt}{t h_1^3(t)}\right), \tag{4.68}$$

where $c(data) = c_0\left(\|\boldsymbol{a}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\boldsymbol{a}\|^4_{W^{1/2,2}(\partial\Omega)} + |\vec{\mathcal{F}}|^2 + |\vec{\mathcal{F}}|^4\right)$.

Proof. In every bounded domain $\Omega_{(l)}$ there exists a vector field $\mathbf{v}^{(l)} \in H(\Omega_{(l)})$ satisfying the integral identity

$$\begin{aligned} &\nu\int\limits_{\Omega_{(l)}} \nabla\mathbf{v}^{(l)}:\nabla\boldsymbol{\eta}\,dx - \int\limits_{\Omega_{(l)}}((\mathbf{A}+\mathbf{v}^{(l)})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{v}^{(l)}\,dx - \int\limits_{\Omega_{(l)}}(\mathbf{v}^{(l)}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\,dx \\ &= -\nu\int\limits_{\Omega_{(l)}}\nabla\mathbf{A}:\nabla\boldsymbol{\eta}\,dx + \int\limits_{\Omega_{(l)}}(\mathbf{A}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{A}\,dx \quad \forall\boldsymbol{\eta}\in H(\Omega_{(l)}). \end{aligned} \tag{4.69}$$

Taking $\boldsymbol{\eta} = \mathbf{v}^{(l)}$ in (4.69) and using Leray–Hopf (3.10) and Cauchy–Schwarz (2.3) inequalities, we obtain

$$\nu \int\limits_{\Omega_{(l)}} |\nabla \mathbf{v}^{(l)}|^2\,dx$$

$$= \int\limits_{\Omega_{(l)}} (\mathbf{v}^{(l)}\cdot\nabla)\mathbf{v}^{(l)}\cdot\mathbf{A}\,dx - \nu\int\limits_{\Omega_{(l)}} \nabla\mathbf{A} : \nabla\mathbf{v}^{(l)}\,dx + \int\limits_{\Omega_{(l)}} (\mathbf{A}\cdot\nabla)\mathbf{v}^{(l)}\cdot\mathbf{A}\,dx$$

$$\le c(\varepsilon|\mathcal{F}_1|^2 + \varepsilon|\mathfrak{F}_1^{(out)}|^2 + |\mathbb{F}_1^{(inn)}|^2)\int\limits_{\Omega_{(l)}} |\nabla\mathbf{v}^{(l)}|^2\,dx \tag{4.70}$$

$$+\Big(\int\limits_{\Omega_{(l)}} |\nabla\mathbf{A}|^2\,dx\Big)^{1/2}\cdot\Big(\int\limits_{\Omega_{(l)}} |\nabla\mathbf{v}^{(l)}|^2\,dx\Big)^{1/2}$$

$$+\Big(\int\limits_{\Omega_{(l)}} |\mathbf{A}|^4\,dx\Big)^{1/2}\cdot\Big(\int\limits_{\Omega_{(l)}} |\nabla\mathbf{v}^{(l)}|^2\,dx\Big)^{1/2}.$$

Using Cauchy inequality with ε (2.1), we obtain

$$(\nu - c\varepsilon|\mathcal{F}_1|^2 - c\varepsilon|\mathfrak{F}_1^{(out)}|^2 - |\mathbb{F}_1^{(inn)}|^2)\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(l)})}$$

$$\le \frac{c}{\varepsilon}\bigg(\|\nabla\mathbf{A}\|^2_{L^2(\Omega_{(l)})} + \|\mathbf{A}\|^4_{L^4(\Omega_{(l)})}\bigg) + c\varepsilon\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(l)})}.$$

Taking sufficiently small $\mathbb{F}_1^{(inn)}$ and ε yields

$$\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(l)})} \le c\bigg(\|\nabla\mathbf{A}\|^2_{L^2(\Omega_{(l)})} + \|\mathbf{A}\|^4_{L^4(\Omega_{(l)})}\bigg). \tag{4.71}$$

Since

$$\|\nabla\mathbf{A}\|^2_{L^2(\Omega_{(l)})} + \|\mathbf{A}\|^4_{L^4(\Omega_{(l)})}$$
$$\le c\big(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)} + |\vec{\mathcal{F}}|^2 + |\vec{\mathcal{F}}|^4\big)\big(1 + \int\limits_1^{2^l}\frac{dt}{th_1^3(t)}\big),$$

from (4.71) it follows that

$$\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(l)})}$$
$$\le c\big(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)} + |\vec{\mathcal{F}}|^2 + |\vec{\mathcal{F}}|^4\big)\big(1 + \int\limits_1^{2^l}\frac{dt}{th_1^3(t)}\big). \tag{4.72}$$

Let us estimate the norm $\|\nabla \mathbf{v}^{(l)}\|_{L^2(\Omega_{(k)})}$ for $k < l$. Introduce the function

$$\mathbf{U}_k^{(l)}(x) = \begin{cases} \mathbf{v}^{(l)}(x), \quad x \in \Omega_{(k)}, \\ \theta_k(x)\mathbf{v}^{(l)}(x) + \sum\limits_{j=1}^{2} \widehat{\mathbf{v}}_{jk}^{(l)}(x), \quad x \in \bigcup\limits_{j=1}^{J} \omega_{jk}, \\ 0, \quad \Omega \setminus \Omega_{(k+1)}, \end{cases} \tag{4.73}$$

where $\theta_k(x)$ are smooth cut-off functions with the following properties:

$$\theta_k(x) = \begin{cases} 1, \quad x \in \Omega_{(k)}, \\ 0, \quad x \in \Omega \setminus \Omega_{(k+1)}, \end{cases}$$

$$|\nabla \theta_k(x)| \leq c\, 2^{-k}, \tag{4.74}$$

and $\widehat{\mathbf{v}}_{jk}^{(l)} \in \mathring{W}^{1,2}(\omega_{jk})$ are solutions of the problems

$$\begin{array}{rcl} \operatorname{div} \widehat{\mathbf{v}}_{jk}^{(l)} & = & -\nabla\theta_k \cdot \mathbf{v}^{(l)} \qquad \text{in} \quad \omega_{jk}, \\ \widehat{\mathbf{v}}_{jk}^{(l)} & = & 0 \qquad \text{on} \quad \partial\omega_{jk}. \end{array} \tag{4.75}$$

Since $\int\limits_{\omega_{jk}} \nabla\theta_k \cdot \mathbf{v}^{(l)} dx = \int\limits_{\omega_{jk}} \operatorname{div}(\theta_k \mathbf{v}^{(l)})\, dx = \int\limits_{\partial\omega_{jk}} \theta_k \mathbf{v}^{(l)} \cdot \mathbf{n}\, dx = \int\limits_{\sigma_j(k+1)} v_3\, dx = 0,$ there exists the solutions $\widehat{\mathbf{v}}_{jk}^{(l)}$ of problems (4.75) satisfying the estimates

$$\|\nabla \widehat{\mathbf{v}}_{jk}^{(l)}\|_{L^2(\omega_{jk})} \leq c(k)\|\nabla\theta_k \cdot \mathbf{v}^{(l)}\|_{L^2(\omega_{jk})}, \quad j = 1, 2, \tag{4.76}$$

where for $j = 1$ the constant $c(k)$ can be estimated by $\dfrac{c\, 2^k}{h_1(2^k)}$, and for $j = 2$ – by $c\, 2^k$, c is independent of k (see Lemma 2.5). Using (4.74) and Poincaré inequality (2.11), from (4.76) we derive

$$\begin{gathered} \|\nabla \widehat{\mathbf{v}}_{1k}^{(l)}\|_{L^2(\omega_{1k})} \leq \frac{c\, 2^k}{h_1(2^k)} \|\nabla\theta_k \cdot \mathbf{v}^{(l)}\|_{L^2(\omega_{1k})} \\ \leq \frac{c\, 2^k}{h_1(2^k)}\, 2^{-k} \|\mathbf{v}^{(l)}\|_{L^2(\omega_{1k})} \leq c\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{1k})}, \\ \|\nabla \widehat{\mathbf{v}}_{2k}^{(l)}\|_{L^2(\omega_{2k})} \leq c\, 2^k \|\nabla\theta_k \cdot \mathbf{v}^{(l)}\|_{L^2(\omega_{2k})} \leq c\|\mathbf{v}^{(l)}\|_{L^2(\omega_{2k})} \leq c\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{2k})}. \end{gathered} \tag{4.77}$$

Taking in integral identity (4.69) $\boldsymbol{\eta} = \mathbf{U}_k^{(l)}$ and using the fact that

$$\int\limits_{\Omega_{(k+1)}} ((\mathbf{v}^{(l)} + \mathbf{A}) \cdot \nabla)\mathbf{U}_k^{(l)} \cdot \mathbf{U}_k^{(l)}\, dx = 0,$$

we obtain

$$\nu \int\limits_{\Omega_{(k)}} |\nabla \mathbf{v}^{(l)}|^2\, dx = \int\limits_{\omega_{jk}} ((\mathbf{v}^{(l)} + \mathbf{A}) \cdot \nabla)\mathbf{U}_k^{(l)} \cdot (\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)})\, dx$$

$$-\nu \int\limits_{\omega_{jk}} \nabla \mathbf{v}^{(l)} : \nabla \mathbf{U}_k^{(l)}\, dx + \int\limits_{\Omega_{(k+1)}} (\mathbf{v}^{(l)} \cdot \nabla)\mathbf{U}_k^{(l)} \cdot \mathbf{A}\, dx \tag{4.78}$$

$$-\nu \int\limits_{\Omega_{(k+1)}} \nabla \mathbf{A} : \nabla \mathbf{U}_k^{(l)}\, dx + \int\limits_{\Omega_{(k+1)}} (\mathbf{A} \cdot \nabla)\mathbf{U}_k^{(l)} \cdot \mathbf{A}\, dx.$$

Using definition (4.73) of the function $\mathbf{U}_k^{(l)}$ and estimates (2.13), (4.77), we obtain

$$\|\mathbf{v}^{(l)}\|_{L^4(\omega_{1k})} \leq c\, h_1^{1/4}(2^k)\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{1k})},$$

$$\|\mathbf{v}^{(l)}\|_{L^4(\omega_{2k})} \leq c\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{2k})},$$

$$\|\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)}\|_{L^4(\omega_{1k})} \leq c\|\mathbf{v}^{(l)}\|_{L^4(\omega_{1k})} + c\|\widehat{\mathbf{v}}_{1k}^{(l)}\|_{L^4(\omega_{1k})}$$

$$\leq c\, h_1^{1/4}(2^k)\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{1k})} + c\, h_1^{1/4}(2^k)\|\nabla \widehat{\mathbf{v}}_{1k}^{(l)}\|_{L^2(\omega_{1k})} \leq c\, h_1^{1/4}(2^k)\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{1k})},$$

$$\|\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)}\|_{L^4(\omega_{2k})} \leq c\|\mathbf{v}^{(l)}\|_{L^4(\omega_{2k})} + c\|\widehat{\mathbf{v}}_{2k}^{(l)}\|_{L^4(\omega_{1k})} \leq c\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{2k})}.$$

Definition (4.73) of the function $\mathbf{U}_k^{(l)}$, estimate (4.74), inequalities (4.77) and Poincaré inequality (2.11) yield

$$\|\nabla \mathbf{U}_k^{(l)}\|_{L^2(\omega_{1k})} \leq c2^{-k}\|\mathbf{v}^{(l)}\|_{L^2(\omega_{1k})} + c\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{1k})} + \|\nabla \widehat{\mathbf{v}}_{1k}^{(l)}\|_{L^2(\omega_{1k})}$$

$$\leq c\, 2^{-k} h_1(2^k)\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{1k})} + c\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{1k})} \leq c\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{1k})},$$

$$\|\nabla \mathbf{U}_k^{(l)}\|_{L^2(\omega_{2k})} \leq c2^{-k}\|\mathbf{v}^{(l)}\|_{L^2(\omega_{2k})} + c\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{2k})} + \|\nabla \widehat{\mathbf{v}}_{1k}^{(l)}\|_{L^2(\omega_{2k})}$$

$$\leq c\, 2^{-k}\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{2k})} + c\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{2k})} \leq c\|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_{2k})},$$

Therefore, we can estimate the right hand side of (4.78) as follows

$$\Big| \int\limits_{\omega_{jk}} ((\mathbf{v}^{(l)} + \mathbf{A}) \cdot \nabla)\mathbf{U}_k^{(l)} \cdot (\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)})\, dx \Big|$$

$$\leq \|\mathbf{v}^{(l)}\|_{L^4(\omega_{jk})}\, \|\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)}\|_{L^4(\omega_{jk})}\, \|\nabla \mathbf{U}_k^{(l)}\|_{L^2(\omega_{jk})}$$

$$+\|\nabla \mathbf{U}_k^{(l)}\|_{L^2(\omega_{jk})}\Big(\int\limits_{\omega_{jk}} |\mathbf{A}|^2|\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)}|^2\, dx\Big)^{1/2}$$

$$\leq ch_1^{1/2}(2^k)\|\nabla \mathbf{v}^{(l)}\|^3_{L^2(\omega_{jk})}$$

$$+c(\sqrt{\varepsilon}|\mathfrak{F}_1^{(out)}|+\sqrt{\varepsilon}|\vec{\mathcal{F}}|+|\mathbb{F}_1^{(inn)}|)\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_{jk})}\big(\int\limits_{\omega_{jk}}|\nabla(\mathbf{v}^{(l)}-\mathbf{U}_k^{(l)})|^2\,dx\big)^{1/2}$$

$$\le ch_1^{1/2}(2^k)\|\nabla\mathbf{v}^{(l)}\|^3_{L^2(\omega_{jk})}+c(\sqrt{\varepsilon}|\mathfrak{F}_1^{(out)}|+\sqrt{\varepsilon}|\vec{\mathcal{F}}|+|\mathbb{F}_1^{(inn)}|)\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_{jk})};$$

$$\nu\Big|\int\limits_{\omega_{jk}}\nabla\mathbf{v}^{(l)}:\nabla\mathbf{U}_k^{(l)}\,dx\Big|\le\nu\|\nabla\mathbf{v}^{(l)}\|_{L^2(\omega_{jk})}\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\omega_{jk})}\le\nu c\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_{jk})};$$

$$\Big|\int\limits_{\Omega_{(k+1)}}(\mathbf{v}^{(l)}\cdot\nabla)\mathbf{U}_k^{(l)}\cdot\mathbf{A}\,dx\Big|\le\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\Omega_{(k+1)})}\bigg(\int\limits_{\Omega_{(k+1)}}|\mathbf{v}^{(l)}|^2|\mathbf{A}|^2\,dx\bigg)^{1/2}$$

$$\le c(\sqrt{\varepsilon}|\mathfrak{F}_1^{(out)}|+\sqrt{\varepsilon}|\vec{\mathcal{F}}|+|\mathbb{F}_1^{(inn)}|)\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(k+1)})}$$

$$\le c(\sqrt{\varepsilon}|\mathfrak{F}_1^{(out)}|+\sqrt{\varepsilon}|\vec{\mathcal{F}}|+|\mathbb{F}_1^{(inn)}|)\bigg(\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(k)})}+\sum_{j=1}^2\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_{jk})}\bigg);$$

$$\nu\Big|\int\limits_{\Omega_{(k+1)}}\nabla\mathbf{A}:\nabla\mathbf{U}_k^{(l)}\,dx\Big|+\Big|\int\limits_{\Omega_{(k+1)}}(\mathbf{A}\cdot\nabla)\mathbf{U}_k^{(l)}\cdot\mathbf{A}\,dx\Big|$$

$$\le c\bigg(\|\nabla\mathbf{A}\|_{L^2(\Omega_{(k+1)})}+\|\mathbf{A}\|^2_{L^4(\Omega_{(k+1)})}\bigg)\|\nabla\mathbf{U}_k^{(l)}\|_{L^2(\Omega_{(k+1)})}$$

$$\le\frac{c}{2\sqrt{\varepsilon}}\bigg(\|\nabla\mathbf{A}\|_{L^2(\Omega_{(k+1)})}+\|\mathbf{A}\|^2_{L^4(\Omega_{(k+1)})}\bigg)^2+\frac{c\sqrt{\varepsilon}}{2}\|\nabla\mathbf{U}_k^{(l)}\|^2_{L^2(\Omega_{(k+1)})}$$

$$\le\frac{c}{\sqrt{\varepsilon}}\bigg(\|\nabla\mathbf{A}\|^2_{L^2(\Omega_{(k+1)})}+\|\mathbf{A}\|^4_{L^4(\Omega_{(k+1)})}\bigg)+\frac{c\sqrt{\varepsilon}}{2}\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(k+1)})}$$

$$\le\frac{c}{\sqrt{\varepsilon}}\bigg(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)}+|\vec{\mathcal{F}}|^2+|\vec{\mathcal{F}}|^4\bigg)\bigg(1+\int\limits_1^{2^l}\frac{dt}{t\,h_1^3(t)}\bigg)$$

$$+c\sqrt{\varepsilon}\bigg(\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(k)})}+\sum_{j=1}^2\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_{(jk)})}\bigg).$$

Hence,

$$\nu\int\limits_{\Omega_{(k)}}|\nabla\mathbf{v}^{(l)}|^2\,dx\le c\,h_1^{1/2}(2^k)\sum_{j=1}^2\|\nabla\mathbf{v}^{(l)}\|^3_{L^2(\omega_{jk})}$$

$$+c(\sqrt{\varepsilon}|\mathfrak{F}_1^{(out)}|+\sqrt{\varepsilon}|\vec{\mathcal{F}}|+|\mathbb{F}_1^{(inn)}|)\sum_{j=1}^2\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_{jk})}$$

$$+\nu c\sum_{j=1}^2\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_{jk})}+c\varepsilon\bigg(\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\Omega_{(k)})}+\sum_{j=1}^2\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_{jk})}\bigg)$$

$$+\frac{c}{\varepsilon}\bigg(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)}+|\vec{\mathcal{F}}|^2+|\vec{\mathcal{F}}|^4\bigg)\bigg(1+\int\limits_1^{2^l}\frac{dt}{t\,h_1^3(t)}\bigg).$$

Taking ε and $\mathbb{F}_1^{(inn)}$ sufficiently small, we obtain

$$\nu\int\limits_{\Omega_{(k)}}|\nabla\mathbf{v}^{(l)}|^2\,dx\le c_{**}h_1^{1/2}(2^k)\sum_{j=1}^2\|\nabla\mathbf{v}^{(l)}\|^3_{L^2(\omega_{jk})}+c_*\sum_{j=1}^2\|\nabla\mathbf{v}^{(l)}\|^2_{L^2(\omega_{jk})}$$

$$+c_0\Big(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)}+|\vec{\mathcal{F}}|^2+|\vec{\mathcal{F}}|^4\Big)\Big(1+\int\limits_1^{2^k}\frac{dt}{t\,h_1^3(t)}\Big).$$

Denote $y_k=\int\limits_{\Omega_{(k)}}|\nabla\mathbf{v}^{(l)}|^2\,dx$. Since $\sum\limits_{j=1}^{2}\int\limits_{\omega_{jk}}=\int\limits_{\Omega_{(k+1)}}-\int\limits_{\Omega_{(k)}}$, we can rewrite the last inequality as

$$y_k\le c_*(y_{k+1}-y_k)+c_{**}h_1^{1/2}(2^k)(y_{k+1}-y_k)^{3/2}+Q_k, \tag{4.79}$$

where

$$Q_k=c(data)\Big(+\int\limits_1^{2^k}\frac{dt}{t\,h_1^3(t)}\Big),$$

$$c(data)=c_0\big(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}+\|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)}+|\vec{\mathcal{F}}|^2+|\vec{\mathcal{F}}|^4\big).$$

It is easy to see that

$$c_*(Q_{k+1}-Q_k)+c_{**}h_1^{1/2}(2^k)(Q_{k+1}-Q_k)^{3/2}$$

$$=c_*c(data)\int\limits_{2^k}^{2^{k+1}}\frac{dt}{th_1^3(t)}+c_{**}h_1^{1/2}(2^k)\big(c(data)\int\limits_{2^k}^{2^{k+1}}\frac{dt}{th_1^3(t)}\big)^{3/2}.$$

If $\int\limits_1^{\infty}\frac{dt}{th_1^3(t)}<\infty$, then $\int\limits_{2^k}^{2^{k+1}}\frac{dt}{th_1^3(t)}\to 0$ as $k\to\infty$, and for sufficiently large k there holds the estimate

$$c_*c(data)\int\limits_{2^k}^{2^{k+1}}\frac{dt}{th_1^3(t)}+c_{**}h_1^{1/2}(2^k)\big(c(data)\int\limits_{2^k}^{2^{k+1}}\frac{dt}{th_1^3(t)}\big)^{3/2}\le\tilde{c}c(data)\Big(1+\int\limits_1^{2^k}\frac{dt}{th_1^3(t)}\Big).$$

If $\int\limits_1^{\infty}\frac{dt}{th_1^3(t)}=\infty$, then[4]

$$c_*c(data)\int\limits_{2^k}^{2^{k+1}}\frac{dt}{th_1^3(t)}+c_{**}h_1^{1/2}(2^k)\big(c(data)\int\limits_{2^k}^{2^{k+1}}\frac{dt}{th_1^3(t)}\big)^{3/2}$$

$$\le\tilde{c}_*c(data)h_1^{-3}(2^k)+\tilde{c}_{**}(c(data))^{3/2}h_1^{-4}(2^k)\le\tilde{c}c(data)\Big(1+\int\limits_1^{2^k}\frac{dt}{th_1^3(t)}\Big),$$

because $\int\limits_1^{2^k}\frac{dt}{th_1^3(t)}\to\infty$ as $k\to\infty$. Since Q_k satisfy condition (2.20), by Lemma

[4]We have used that $\mu_1h_1(2^k)\le h_1(t)\le\mu_2h_1(2^k)\quad\forall t\in(2^k,2^{k+1})$, μ_1,μ_2 are positive constants.

2.16, inequality (4.79) together with (4.72) implies the estimate

$$y_k = \int\limits_{\Omega_{(k)}} |\nabla \mathbf{v}^{(l)}|^2 \, dx \le c(data)\left(1 + \int\limits_1^{2^k} \frac{dt}{t\, h_1^3(t)}\right) \forall k \le l. \tag{4.80}$$

This estimate guarantees the existence of a subsection $\{\mathbf{v}^{(l_m)}\}$ which converges weakly in $\overset{\circ}{W}{}^{1,2}(\Omega_{(k)})$ and strongly in $L^4(\Omega_{(k)})$ for every k. Therefore, passing to a limit in integral identity (4.69) we find that the limiting vector field $\mathbf{v}$ satisfies (4.67) for any $\boldsymbol{\eta} \in H(\Omega)$ with compact support. The sum $\mathbf{u} = \mathbf{A} + \mathbf{v}$ is a weak solution of problem (4.44). Estimate (4.68) for $\mathbf{v}$ follows from (4.80). Since for $\mathbf{A}$ the analogous to (4.68) estimate is obvious, we obtain (4.68) for the sum $\mathbf{u} = \mathbf{A} + \mathbf{v}$. □

Chapter 5

Domain with finite number of paraboloidal outlets

In this chapter we generalize the case of the domain Ω with one paraboloidal outlet to infinity (see Chapter 4, Section 4.1). Let domain $\Omega \subset \mathbb{R}^n, n = 2, 3$, be a domain with finite number of paraboloidal outlets to infinity $D_j,\ j = 1, ..., J$, i.e.,

$$\Omega = \Omega_0 \bigcup D_1 \bigcup D_2 \bigcup ... \bigcup D_J, \quad D_l \bigcap D_j = \emptyset,\ \ l \neq j,$$

where $\Omega_0 = \Omega \cap B_{R_0}(0)$.

We study the case of outlets to infinity D_j which in some cartesian coordinate systems $z^{(j)}$ [1] have the form

$$D_j = \left\{ z^{(j)} \in \mathbb{R}^n : z^{(j)\prime} g_j^{-1}(z_n^{(j)}) \in \sigma_j^{(0)},\ z_n^{(j)} > 1 \right\}, \quad j = 1, ..., J,$$

where $z^{(j)\prime} = (z_1^{(j)}, z_2^{(j)})$ if $n = 3$, $z^{(j)\prime} = z_1^{(j)}$ if $n = 2$, the functions $g_j(t) \geq 1$ satisfy the Lipschitz condition

$$|g_j(t_1) - g_j(t_2)| \leq L_j |t_1 - t_2| \ \ \forall t_1, t_2 \geq 1, \tag{5.1}$$

$\sigma_j^{(0)} \subset \mathbb{R}^{n-1}$ are bounded domains (in the two-dimensional case $\sigma_j^{(0)}$ are intervals). Without loss of generality we assume that σ_j contains the point $z^{(j)\prime} = 0$. In the three-dimensional case $\sigma_j^{(0)} \subset \mathbb{R}^2$ may be multiply connected. Then

$$\sigma_j^{(0)} = \sigma_{j0}^{(0)} \setminus \bigcup_{k=1}^{N_j} \overline{\sigma}_{jk}^{(0)},$$

where $\sigma_{jk}^{(0)}, k = 0, 1, \ldots, N_j$, are bounded simply connected plane domains such

[1] Notice that $z^{(j)}$ means the local coordinate system in the outlet D_j, while x is the global coordinate system in $\mathbb{R}^n$.

that $\overline{\sigma}_{jk}^{(0)} \subset \sigma_{j0}^{(0)}$, $k = 1, \ldots, N_j$, and $\overline{\sigma}_{jk}^{(0)} \cap \overline{\sigma}_{jl}^{(0)} = \emptyset$ for $k \neq l$, $k, l \neq 0$.

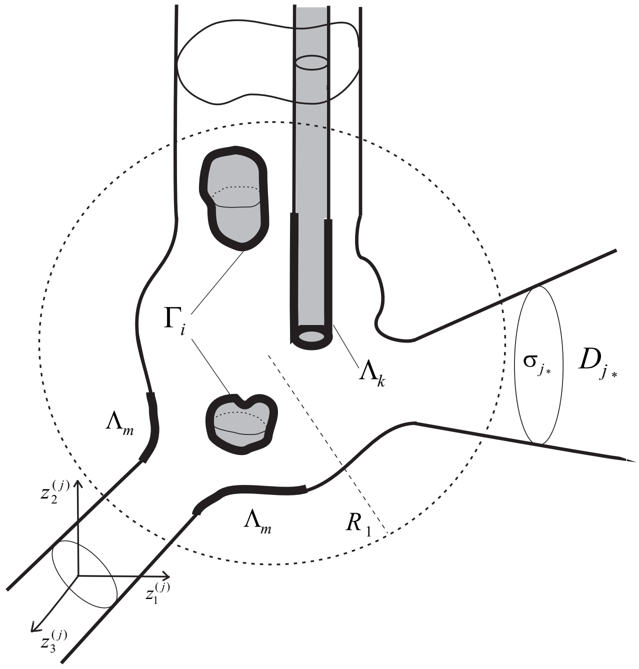

Figure 5.1: Domain Ω with paraboloidal outlets to infinity.

We assume that:

(i) The boundary $\partial\Omega$ is Lipschitz.

(ii) The bounded domain Ω_0 has the form

$$\Omega_0 = G_0 \setminus \bigcup_{i=1}^{I} \overline{G}_i,$$

where G_0 and $G_i, i = 1, \ldots, I, I \geq 0$, are bounded simply connected domains such that $\overline{G_i} \subset G_0$, $\overline{G}_{i_1} \bigcap \overline{G}_{i_2} = \emptyset$ for $i_1 \neq i_2$. Denote $\Gamma = \bigcup\limits_{i=1}^{I} \Gamma_i$, $\Gamma_i = \partial G_i$.

(iii) The outer boundary $S = \partial\Omega \setminus \Gamma$ consists of $M \geq 1$ disjoint unbounded components $S^{(m)}$. We denote by $\Lambda_m = \text{supp}\, \mathbf{a} \bigcap S \subset S^{(m)}$.

Below we will use the following notations:

$$R_{j1} = 1,\ R_{jl+1} = R_{jl} + \frac{g_j(R_{jl})}{2L_j},\ l \geq 1,\ j = 1, \ldots, J,$$

$$\Omega_{(l)} = \Omega_0 \bigcup D_1^{(l)} \bigcup \ldots \bigcup D_J^{(l)},\quad D_j^{(l)} = \left\{ z^{(j)} \in D_j :\ z_3^{(j)} < R_{jl} \right\},$$

$$\sigma_j(R) = D_j \bigcap \{ z^{(j)} : z_3^{(j)} = R \}.$$

In this chapter we consider the following problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p & = & 0 & \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} & = & 0 & \text{in} \quad \Omega, \\ \mathbf{u} & = & \mathbf{a} & \text{on} \quad \partial\Omega, \\ \int\limits_{\sigma_j(R)} \mathbf{u}\cdot\mathbf{n}\, dS & = & \mathcal{F}_j & \forall R \geq R_0, \ j = 1, 2, \ldots, J, \end{cases} \tag{5.2}$$

with the necessary compatibility condition

$$\sum_{i=1}^{I} \mathbb{F}_i^{(inn)} + \sum_{m=1}^{M} \mathfrak{F}_m^{(out)} + \sum_{j=1}^{J} \mathcal{F}_k = 0, \tag{5.3}$$

where $\sigma_j(R)$ is the cross section of the outlet D_j, $\mathbf{n}$ is the unit vector of the normal to σ_j. Remind that $\mathbb{F}_i^{(inn)}$ and $\mathfrak{F}_m^{(out)}$ are the fluxes of the boundary value $\mathbf{a}$ over the components of inner and outer boundaries, respectively.

Constructing virtual drain functions $\mathbf{b}_m^{(out)}$ and $\mathbf{b}^{(inn)}$ we choose the "widest" outlet, say D_{j_*}, in order to minimize the generated by the drain function dissipation of energy (Dirichlet integral).

5.1 Three dimensional case

5.1.1 Construction of the extension $\mathbf{B}^{(inn)}$

The idea of the construction of extension $\mathbf{B}^{(inn)}$ is the same as in Subsection 4.1.1. Therefore, we just formulate the main results and indicate the differences which appear. As in Subsection 4.1.1, we start with the construction of the virtual drain function $\mathbf{b}^{(inn)}$. We denote $\mathbb{F}^{(inn)} = \sum\limits_{i=1}^{I} \mathbb{F}_i^{(inn)}$ and first, we construct in D_{j_*} a solenoidal vector field $\mathbf{b}_{j_*}^{(inn)}$ such that

$$\mathbf{b}_1^{(inn)}(x)\big|_{\partial D_{j_*}\cap\partial\Omega} = 0, \qquad \int\limits_{\sigma_{j_*}(R)} \mathbf{b}_{j_*}^{(inn)}\cdot\mathbf{n}\; dS = \mathbb{F}^{(inn)}.$$

In order to simplify the notations, we omit the index j_* in notation of local coordinates $z^{(j_*)}$ and write just z. Let $\gamma_+ = \{z \in D_{j_*} : |z'| = 0, \ z_3 > 1\}$. Define in D_{j_*} a cut–off function

$$\zeta_{j_*}(x) = \Psi\Big(\ln\Big(\frac{\varrho(\delta(x))}{\Delta(x)}\Big)\Big), \tag{5.4}$$

where

$$\delta(x) = \Delta_{\gamma_+}(x), \quad \Delta(x) = \Delta_{\partial D_{j_*}\cap\partial\Omega}(x),$$

functions Ψ and ϱ are defined by formulas (2.24) and (2.25), respectively.

As in Chapter 4 (Subsection 4.1), we obtain the following properties of the function $\zeta_{j_*}(x)$.

Lemma 5.1. *The function $\zeta_{j_*}(x)$ is equal to zero at those points of D_{j_*}, where $\varrho(\delta(x)) \le \Delta(x)$, while the $d_0/2$-neighborhood of the line γ_+ is contained in this set; $\zeta_{j_*}(x) = 1$ at those points of D_{j_*} where $\Delta(x) \le e^{-1}\varrho(\delta(x))$. The following estimates*

$$\Big|\frac{\partial \zeta_{j_*}(x)}{\partial x_k}\Big| \le \frac{c}{\Delta(x)}, \qquad \Big|\frac{\partial^2 \zeta_{j_*}(x)}{\partial x_k \partial x_l}\Big| \le \frac{c}{\Delta^2(x)}$$

holds.

Define the vector field

$$\mathbf{b}_{j_*}^{(inn)}(x) = -\mathbb{F}^{(inn)}\mathrm{curl}\,\big(\zeta_{j_*}(x)\mathbf{b}(x)\big) = -\mathbb{F}^{(inn)}\nabla\zeta_{j_*}(x) \times \mathbf{b}(x),\ x \in D_{j_*},$$

where $\mathbf{b}(x)$ is a magnetic field defined by (2.23). The properties of $\mathbf{b}(x)$ are given in Lemma 2.17.

Lemma 5.2. *The solenoidal vector field $\mathbf{b}_{j_*}^{(inn)}$ is infinitely differentiable, vanishes near the surface $\partial D_{j_*} \cap \partial\Omega$ and the contour γ_+, and satisfies*

$$\int\limits_{\sigma_{j_*}(R)} \mathbf{b}_{j_*}^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}^{(inn)},$$

$$|\mathbf{b}_{j_*}^{(inn)}(x)| \le \frac{c|\vec{\mathbb{F}}^{(inn)}|}{d(x)}, \tag{5.5}$$

$$|\mathbf{b}_{j_*}^{(inn)}(x)| \le \frac{C|\vec{\mathbb{F}}_1^{(inn)}|}{g_{j_*}^2(z_3)}, \quad |\nabla \mathbf{b}_{j_*}(x)| \le \frac{C|\vec{\mathbb{F}}^{(inn)}|}{g_{j_*}^3(z_3)}, \tag{5.6}$$

where $d(x)=\mathrm{dist}(x, \partial D_{j_*}\cap\partial\Omega)$, $|\vec{\mathbb{F}}^{(inn)}| = \sqrt{(\mathbb{F}_i^{(inn)})^2}$, $\vec{\mathbb{F}}^{(inn)} = \big(\mathbb{F}_1^{(inn)}, \mathbb{F}_2^{(inn)}, \dots, \mathbb{F}_I^{(inn)}\big)$.

Proof. Since $\zeta_{j_*}(x) = 1$ on $\partial D_{j_*} \cap \partial\Omega$, by the Stokes Theorem (see Lemma 2.12) and properties of the magnetic field $\mathbf{b}$, we get

$$\int\limits_{\sigma_{j_*}(R)} \mathbf{b}_{j_*}^{(inn)} \cdot \mathbf{n}\; dS = -\mathbb{F}^{(inn)} \int\limits_{\sigma_{j_*}(R)} \mathrm{curl}\,\big(\zeta_{j_*}\mathbf{b}\big) \cdot \mathbf{n}\; dS = -\mathbb{F}^{(inn)} \oint\limits_{\partial\sigma_{j_*}(R)} \mathbf{b} \cdot d\mathbf{l}$$

$$= -\mathbb{F}^{(inn)}\Big(\oint\limits_{\partial\sigma_{j_*0}} \mathbf{b} \cdot d\mathbf{l} - \sum_{k=1}^{N_{j_*}} \oint\limits_{\partial\sigma_{j_*k}} \mathbf{b} \cdot d\mathbf{l}\Big) = \mathbb{F}^{(inn)}.$$

The rest of the proof is the same as in Lemma 4.2. □

Lemma 5.3. *For any vector field $\mathbf{w} \in W^{1,2}_{loc}(D_{j_*})$ with $\mathbf{w}|_{\partial D_{j_*}\cap\partial\Omega} = 0$ the following* *inequalities*

$$\int\limits_{D_{j_*}^{(k)}} |\mathbf{b}_{j_*}^{(inn)}|^2 |\mathbf{w}|^2\,dx \le c|\vec{\mathbb{F}}^{(inn)}|^2 \int\limits_{D_{j_*}^{(k)}} |\nabla \mathbf{w}|^2\,dx,$$
$$\int\limits_{D_{j_*}^{(k+1)}\setminus D_{j_*}^{(k)}} |\mathbf{b}_{j_*}^{(inn)}|^2 |\mathbf{w}|^2\,dx \le c|\vec{\mathbb{F}}^{(inn)}|^2 \int\limits_{D_{j_*}^{(k+1)}\setminus D_{j_*}^{(k)}} |\nabla \mathbf{w}|^2\,dx \tag{5.7}$$

hold with the constant c independent of k (see Lemma 2.15).

Let $x^{(i)} \in G_i$, be a point lying inside the "hole" G_i. Denote $q_i(x) = q(x - x^{(i)})$, where

$$q(x) = \frac{1}{4\pi}\frac{1}{|x|}$$

is the fundamental solution of the Laplace operator in $\mathbb{R}^3$, and let

$$\mathbf{b}_\sharp^{(inn)}(x) = \sum_{i=1}^{I} \mathbb{F}_i^{(inn)} \nabla q_i(x). \tag{5.8}$$

Then

$$\operatorname{div} \mathbf{b}_\sharp^{(inn)} = \sum_{i=1}^{I} \mathbb{F}_i^{(inn)} \operatorname{div} \nabla q_i(x) = \sum_{i=1}^{I} \mathbb{F}_i^{(inn)} \Delta q_i(x) = 0,$$
$$\int\limits_{\Gamma_i} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n}\; dS = \mathbb{F}_i^{(inn)}, \quad i = 1, ..., I,$$
$$\int\limits_{\partial\Omega_0} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n}\; dS = -\sum_{i=1}^{I} \mathbb{F}_i^{(inn)} = -\mathbb{F}^{(inn)}.$$

Denote

$$\mathbf{h}_1 = \begin{cases} 0, \quad x \in \Gamma_i, \quad i = 1, ..., I, \\ \mathbf{b}_{j_*}^{(inn)}|_{\partial\Omega_0 \cap D_{j_*}} - \mathbf{b}_\sharp^{(inn)}|_{\partial\Omega_0 \cap D_{j_*}}, \quad x \in \partial\Omega_0 \bigcap D_{j_*}, \\ -\mathbf{b}_\sharp^{(inn)}\big|_{\partial\Omega_0 \setminus \overline{D}_{j_*}}, \quad x \in \partial\Omega_0 \setminus (\overline{D}_{j_*} \bigcup \Gamma). \end{cases}$$

We have

$$\int\limits_{\partial\Omega_0} \mathbf{h}_1 \cdot \mathbf{n}\, dS = \int\limits_{\partial\Omega_0 \cap D_{j_*}} \mathbf{b}_1^{(inn)} \cdot \mathbf{n}\, dS - \int\limits_{\partial\Omega_0} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}^{(inn)} - \mathbb{F}^{(inn)} = 0. \tag{5.9}$$

Because of (5.9) the function $\mathbf{h}_1$ can be extended inside the domain Ω_0 as a solenoidal vector field $\mathbf{b}_{01}^{(inn)} \in W^{1,2}(\Omega_0)$ and

$$\|\mathbf{b}_{01}^{(inn)}\|_{W^{1,2}(\Omega_0)} \le c\|\mathbf{h}_1\|_{W^{1/2,2}(\partial\Omega_0)}$$
$$\le c\Big(\|\mathbf{b}_\sharp^{(inn)}\|_{W^{1/2,2}(\partial\Omega_0)} + \|\mathbf{b}_{j_*}^{(inn)}\|_{W^{1/2,2}(\partial\Omega_0 \cap D_{j_*})}\Big) \le c|\vec{\mathbb{F}}^{(inn)}|, \tag{5.10}$$

where the constant c depends only on the domain Ω_0 (see Lemma 2.6). Define

$$\mathbf{b}^{(inn)} = \begin{cases} \mathbf{b}_\sharp^{(inn)} + \mathbf{b}_{01}^{(inn)}, & x \in \Omega_0, \\ \mathbf{b}_{j_*}^{(inn)}, & x \in D_{j_*} \\ 0, & x \in D_j,\ j \neq j_*. \end{cases}$$

Vector field $\mathbf{b}^{(inn)}$ "removes" nonzero fluxes from the components Γ_i. Denote

$$\mathbf{h}_0 = \begin{cases} \mathbf{a} - \mathbf{b}_\sharp^{(inn)}|_{\Gamma_i}, & x \in \Gamma_i, \quad i = 1, ..., I, \\ 0, & x \in \partial\Omega_0 \setminus \Gamma. \end{cases}$$

Obviously,

$$\int\limits_{\Gamma_i} \mathbf{h}_0 \cdot \mathbf{n}\, dS = \int\limits_{\Gamma_i} \mathbf{a} \cdot \mathbf{n}\, dS - \int\limits_{\Gamma_i} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}_i^{(inn)} - \mathbb{F}_i^{(inn)} = 0.$$

Therefore, the function $\mathbf{h}_0$ can be extended inside Ω in the form

$$\mathbf{b}_0^{(inn)}(x) = \mathrm{curl}(\chi(x)\mathbf{E}(x)),$$

where $\mathbf{E} \in W_2^2(\Omega_0)$, $\mathrm{curl}\,\mathbf{E}|_{\partial\Omega} = \mathbf{h}_0$ and χ is a smooth cut-off function with $\chi(x) = 1$ on Γ (see Lemma 2.14 with $\varepsilon = 1$). Moreover, for any $\mathbf{w} \in W_{loc}^{1,2}(\Omega)$ with $\mathbf{w}|_{\partial\Omega} = 0$ the following estimate

$$\int\limits_{\Omega_0} |\mathbf{b}_0^{(inn)}(x)|^2 |\mathbf{w}(x)|^2\, dx \leq c|\vec{\mathbb{F}}^{(inn)}|^2 \int\limits_{\Omega_0} |\nabla \mathbf{w}(x)|^2\, dx \tag{5.11}$$

holds (see Lemma 2.15).

Finally, we put

$$\mathbf{B}^{(inn)} = \mathbf{b}^{(inn)} + \mathbf{b}_0^{(inn)}. \tag{5.12}$$

Lemma 5.4. *The vector field* $\mathbf{B}^{(inn)}$ *is solenoidal,* $\mathbf{B}^{(inn)}|_{\Gamma_i} = \mathbf{a}|_{\Gamma_i}$, $i = 1, ..., I$, $\mathbf{B}^{(inn)}|_{S^{(m)}} = 0$, $m = 1, ..., M$, $\mathbf{B}^{(inn)} \in W_{loc}^{1,2}(\Omega)$ *and* $\mathbf{B}^{(inn)} = 0$ *for* $x \in D_j$, $j \neq j_*$, $|x| \gg 1$. *For any solenoidal* $\mathbf{w} \in W_{loc}^{1,2}(\Omega)$ *with* $\mathbf{w}\big|_{\partial\Omega} = 0$ *the following estimates*

$$\begin{aligned} &\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{B}^{(inn)}|^2 |\mathbf{w}|^2\, dx \leq c|\vec{\mathbb{F}}^{(inn)}|^2 \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla \mathbf{w}|^2\, dx, \\ &\int\limits_{\Omega_{(k)}} |\mathbf{B}^{(inn)}|^2 |\mathbf{w}|^2\, dx \leq c|\vec{\mathbb{F}}^{(inn)}|^2 \int\limits_{\Omega_{(k)}} |\nabla \mathbf{w}|^2\, dx \end{aligned} \tag{5.13}$$

hold. Moreover,

$$|\mathbf{B}^{(inn)}(x)| \le \frac{C|\vec{\mathbb{F}}^{(inn)}|}{g_{j_*}^2(z_3)}, \quad |\nabla \mathbf{B}^{(inn)}(x)| \le \frac{C|\vec{\mathbb{F}}^{(inn)}|}{g_{j_*}^3(z_3)}, \ x \in D_{j_*},$$
$$|\mathbf{B}^{(inn)}(x)| + |\nabla \mathbf{B}^{(inn)}(x)| \le C|\vec{\mathbb{F}}^{(inn)}|, \ x \in \Omega \setminus D_{j_*}. \tag{5.14}$$

Proof is the same as in Chapter 4 (see Subsection 4.1.1, Lemma 4.4).

5.1.2 Construction of the extensions $\mathbf{B}_m^{(out)}$, $m = 1, ..., M$.

We start by the construction of the virtual drain functions $\mathbf{b}_m^{(out)}$, $m = 1, ..., M$. Take any point $x^{(m)} \in \Lambda_m \subset S^{(m)}$. Let γ_{mj_*} be a smooth simple curve which intersects $\partial\Omega$ at the point $x^{(m)}$ and has the form

$$\gamma_{mj_*} = \widehat{\gamma}_{j_*,+} \cup \widehat{\gamma}_{j_*}^{(m)} \bigcup l_m,$$

where $\widehat{\gamma}_{j_*}$ is a semi-infinite line lying in D_{j_*}, $\widehat{\gamma}_{j_*}^{(m)} \subset \Omega \bigcap B_{R_1}(0)$ is a finite simple

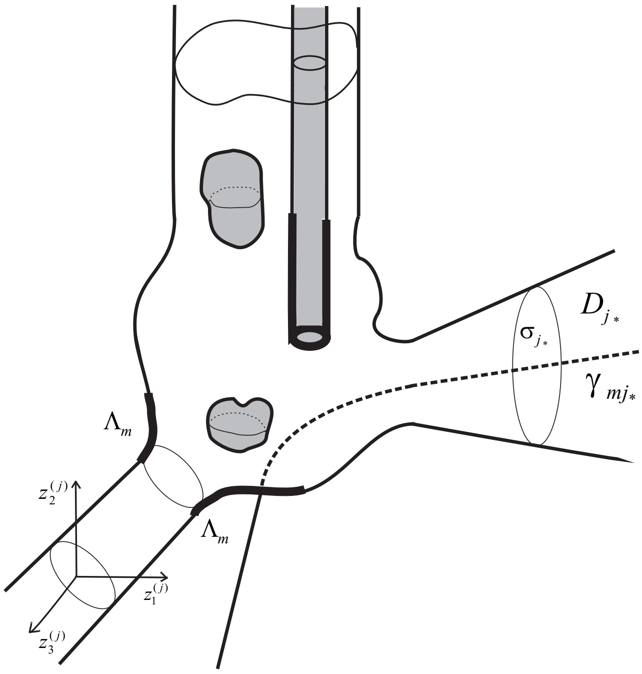

Figure 5.2: Contour γ_{mj_*}.

curve connecting $\widehat{\gamma}_{j_*}$ and the point $x^{(m)}$ and $l_m \subset \mathbb{R}^3 \setminus \Omega$ is a semi-infinite curve starting at the point $x^{(m)}$ and does not intersecting any D_j, $j = 1, ..., J$. Assume that the distance from γ_{mj_*} to $S^{(m)} \setminus \Lambda_m$ is not less than $d_0 > 0$, where d_0 is sufficiently small number, and that the direction of the curve γ_{mj_*} coincides with the direction of increase of the coordinate $z_3^{(j_*)}$.

In the domain Ω we introduce the virtual drain functions

$$\begin{aligned} \mathbf{b}_m^{(out)}(x,\varepsilon) &= \mathfrak{F}_m^{(out)}\mathrm{curl}(\zeta_{mj_*}(x,\varepsilon)\cdot \mathbf{b}_{j_*}^{(m)}(x)) \\ &= \mathfrak{F}_m^{(out)}\nabla\zeta_{mj_*}(x,\varepsilon)\times \mathbf{b}_{j_*}^{(m)}(x), \quad m=1,\dots,M, \end{aligned} \tag{5.15}$$

where $\mathbf{b}_{j_*}^{(m)}(x)$ is a magnetic field corresponding to the contour γ_{mj_*} (see formula (2.22) and Lemma 2.17),

$$\zeta_{mj_*}(x,\varepsilon) = \Psi\Big(\varepsilon \ln \frac{\delta^{(mj_*)}(x))}{\Delta_{\partial\Omega\setminus\Lambda_m}(x)}\Big), \tag{5.16}$$

Ψ is defined by formula (2.24),

$$\delta^{(mj_*)}(x) = \begin{cases} \rho_{j_*}(x)\Delta_{\widehat{\gamma}_{j_*}}(x) + \sum\limits_{j=1, j\neq j_*}^{J} \rho_j(x)|x-x_0^{(mj_*)}| + \\ +(1-\sum\limits_{j=1}^{J}\rho_j(x))\Delta_{\widehat{\gamma}_{j_*}^{(m)}}(x), \quad x\in\Omega\setminus\gamma_{mj_*}, \ x_0^{(mk)}\in\widehat{\gamma}_{j_*}^{(m)}, \\ 0, \qquad x\in\gamma_{mj_*}, \end{cases}$$

$$\rho_j(x) = \begin{cases} 1, & x\in D_j\setminus D_j^{(3)}, \\ 0, & x\in(\Omega\setminus D_j)\bigcup D_j^{(2)}. \end{cases}$$

Obviously, the function $\delta^{(mj_*)}(x)$ is continuous in $\overline{\Omega}$ and infinitely differentiable in $\Omega\setminus\gamma_{mj_*}$, $|\nabla\delta^{(mj_*)}(x)|$ is bounded. Moreover, it is easy to check that

$$\delta^{(mj_*)}(x) = \begin{cases} \Delta_{\widehat{\gamma}_{j_*}}(x), & x\in D_{j_*}\setminus D_{j_*}^{(3)}, \\ |x-x_0^{(mj_*)}|, & x\in D_j, \ |x|\gg 1, \ j\neq j_*, \\ \Delta_{\widehat{\gamma}_{j_*}^{(m)}}(x), & x\in\Omega_0. \end{cases}$$

Lemma 5.5. *The vector field $\boldsymbol{b}_m^{(out)}$ is infinitely differentiable and solenoidal, $\boldsymbol{b}_m^{(out)}$ vanishes near the surface $\partial\Omega\setminus\Lambda_m$, for $x\in D_j$, $j\neq j_*$, with $|x|\gg 1$, and in a small neighborhood of the curve $\gamma_{mj_*}\cap\overline{\Omega}$. The following estimates*

$$\begin{aligned} &|\boldsymbol{b}_m^{(out)}(x,\varepsilon)| \le \frac{c\varepsilon}{d_{\partial\Omega\setminus\Lambda_m}(x)d_{\gamma_{mj_*}}(x)}, \\ &|\nabla\boldsymbol{b}_m^{(out)}(x,\varepsilon)| \le c\Big(\frac{1}{d^2_{\partial\Omega\setminus\Lambda_m}(x)d_{\gamma_{mj_*}}(x)} + \frac{1}{d_{\partial\Omega\setminus\Lambda_m}(x)d^2_{\gamma_{mj_*}}(x)}\Big), \end{aligned} \tag{5.17}$$

$$|\boldsymbol{b}_m^{(out)}(x,\varepsilon))| \le \frac{c(\varepsilon)|\mathfrak{F}_m^{(out)}|}{g^2_{j_*}(z_3^{(j_*)})}, \ |\nabla\boldsymbol{b}_m^{(out)}(x,\varepsilon)| \le \frac{c(\varepsilon)|\mathfrak{F}_m^{(out)}|}{g^3_{j_*}(z_3^{(j_*)})}, \ x\in D_{j_*} \tag{5.18}$$

hold. Here $d_{\partial\Omega\setminus\Lambda_m}(x)$ *and* $d_{\gamma_{mj_*}}(x)$ *are distances from the point* x *to* $\partial\Omega\setminus\Lambda_m$ *and* γ_{mj_*}*, respectively. The constant* c *in* (5.17) *is independent of* ε*. Finally,*

$$\int\limits_{\Lambda_m} \boldsymbol{b}_m^{(out)}\cdot\boldsymbol{n}\,dS=\mathfrak{F}_m^{(out)}.$$

Proof. The first statement of the lemma follow from definitions (5.15), (5.16) of $\mathbf{b}_{dm}^{(out)}(x,\varepsilon)$ and $\zeta_{mj_*}(x,\varepsilon)$ and from the properties of the regularized distance (see estimates (2.21)). Since $\mathbf{b}_m^{(out)}(x,\varepsilon)=0$ and $\zeta_{mj_*}(x,\varepsilon)=1$ on $\partial\Omega\setminus\Lambda_m$, using the Ostrogradsky–Gauss and the Stokes theorems (see Lemma 2.11 and Lemma 2.12, respectively), we obtain

$$\int\limits_{\Lambda_m}\mathbf{b}_m^{(out)}\cdot\mathbf{n}\,dS=-\int\limits_{\sigma_{j_*}(R)}\mathbf{b}_m^{(out)}\cdot\mathbf{n}\,dS=-\mathfrak{F}_m^{(out)}\int\limits_{\sigma_{j_*}(R)}\operatorname{curl}(\zeta_{mj_*}\mathbf{b}_{j_*}^{(m)})\cdot\mathbf{n}\,dS$$

$$=-\mathfrak{F}_m^{(out)}\int\limits_{\partial\sigma_{j_*}(R)}\zeta_{mj_*}\mathbf{b}_{j_*}^{(m)}\cdot\mathbf{n}\,d\mathbf{l}=-\mathfrak{F}_m^{(out)}\int\limits_{\partial\sigma_{j_*}(R)}\mathbf{b}_{j_*}^{(m)}\cdot d\mathbf{l}=\mathfrak{F}_m^{(out)}.$$

□

Let $\mathbf{h}_m(x)=\mathbf{a}(x)|_{\Lambda_m}-\mathbf{b}_{dm}^{(out)}(x,\varepsilon)|_{\Lambda_m}$. Then

$$\int\limits_{\Lambda_m}\mathbf{h}_m\cdot\mathbf{n}\,dS=\int\limits_{\Lambda_m}\mathbf{a}\cdot\mathbf{n}\,dS-\int\limits_{\Lambda_m}\mathbf{b}_m^{(out)}\cdot\mathbf{n}\,dS=0.$$

Therefore, $\mathbf{h}_m$ can be extended inside Ω in the form

$$\mathbf{b}_{0\,m}^{(out)}(x,\varepsilon)=\operatorname{curl}\left(\chi_m(x,\varepsilon)\mathbf{E}_m(x)\right),$$

where $\mathbf{E}_m\in W_2^2(\Omega_0)$, $\operatorname{curl}\mathbf{E}_m|_{\Lambda_m}=\mathbf{h}_m$ and χ_m is a Hopf's type cut-off function such that $\chi_m(x,\varepsilon)=1$ on Λ_m (see Lemma 2.14). Define

$$\mathbf{B}_m^{(out)}(x,\varepsilon)=\mathbf{b}_m^{(out)}(x,\varepsilon)+\mathbf{b}_{0\,m}^{(out)}(x,\varepsilon).$$

Obviously, $\mathbf{B}_m^{(out)}$ has the following properties:

$$\operatorname{div}\mathbf{B}_m^{(out)}=0,\quad \mathbf{B}_m^{(out)}|_{\Lambda_m}=\mathbf{a},\quad \mathbf{B}_m^{(out)}|_{\partial\Omega\setminus\Lambda_m}=0,$$

$$\mathbf{B}_m^{(out)}=0,\ x\in\Omega\setminus D_{j_*},\ |x|\gg 1.$$

Lemma 5.6. *For any solenoidal* $\mathbf{w}\in W^{1,2}_{loc}(\Omega)$ *with* $\mathbf{w}\big|_{\partial\Omega}=0$ *the following esti-*

mates

$$\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{B}_m^{(out)}|^2|\mathbf{w}|^2\,dx \le \varepsilon c|\mathfrak{F}_m^{(out)}|^2 \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla\mathbf{w}|^2\,dx,$$

$$\int\limits_{\Omega_{(k)}} |\mathbf{B}_m^{(out)}|^2|\mathbf{w}|^2\,dx \le \varepsilon c|\mathfrak{F}_m^{(out)}|^2 \int\limits_{\Omega_{(k)}} |\nabla\mathbf{w}|^2\,dx, \tag{5.19}$$

hold. Moreover,

$$|\mathbf{B}_m^{(out)}(x,\varepsilon)| \le \frac{C(\varepsilon)|\mathfrak{F}_m^{(out)}|}{g_{j_*}^2(z_3^{(j_*)})}, \quad |\nabla\mathbf{B}_m^{(out)}(x,\varepsilon)| \le \frac{C(\varepsilon)|\mathfrak{F}_m^{(out)}|}{g_{j_*}^3(z_3^{(j_*)})},\ x\in D_{j_*},$$

$$|\mathbf{B}_m^{(out)}(x,\varepsilon)| + |\nabla\mathbf{B}_m^{(out)}(x,\varepsilon)| \le C(\varepsilon)|\mathfrak{F}_m^{(out)}|,\ x\in\Omega\setminus D_{j_*}. \tag{5.20}$$

Proof. The proof of this lemma is analogous to the proof of Lemma 4.24 in subsection 4.1.2. □

5.1.3 Construction of the extension $\mathbf{B}^{(flux)}$

Now we need to compensate the fluxes over the cross sections of outlets to infinity, i.e., we have to construct a solenoidal vector field $\mathbf{B}^{(flux)}$ satisfying the flux conditions

$$\int\limits_{\sigma_j(R)} \mathbf{B}^{(flux)}\cdot\mathbf{n}\,dS = \widehat{\mathcal{F}}_j, \quad j=1,\dots,J, \tag{5.21}$$

where

$$\widehat{\mathcal{F}}_j = \mathcal{F}_j, \quad j=1,\dots,J,\ j\ne j_*; \quad \widehat{\mathcal{F}}_{j_*} = \mathcal{F}_{j_*} + \mathbb{F}^{(inn)} + \mathfrak{F}^{(out)},$$

$\mathfrak{F}^{(out)} = \sum\limits_{m=1}^{M}\mathfrak{F}_m^{(out)}$. Note that in virtue of (3.3) the total flux is equal to zero:

$$\sum_{j=1}^{J}\widehat{\mathcal{F}}_j = 0.$$

Such flux carriers were constructed in [59], [61]. However, we briefly describe the construction. Let $\gamma^{(jk)}$ be an infinite smooth simple curve consisting of two semi-infinite lines $\widehat{\gamma}^{(j)}\subset D_j$, $\widehat{\gamma}^{(k)}\subset D_k$ and a finite curve $\widehat{\gamma}_j^{(k)}\subset\Omega\bigcap B_{R_1}(0)$ joining them. We assume that the distance from $\gamma^{(jk)}$ to $\partial\Omega$ is not less than $d_0>0$ and that the direction of the curve $\gamma^{(jk)}$ coincides with the direction of increase of the coordinate $z_3^{(j)}$.

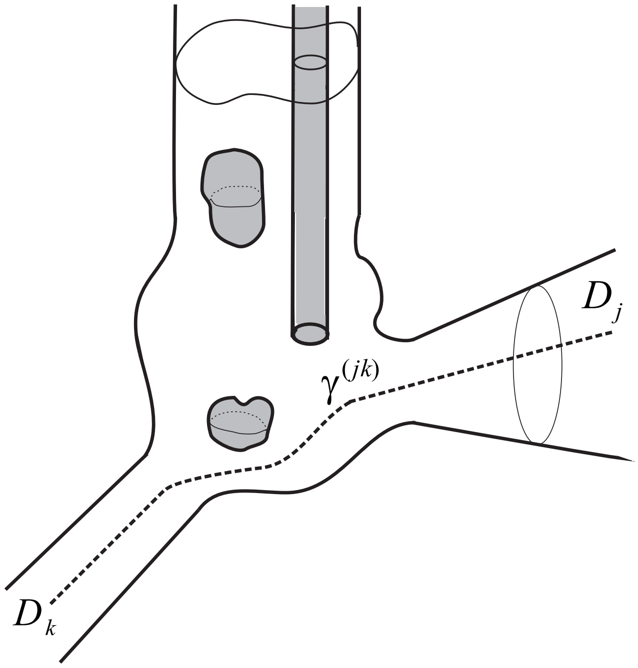

Figure 5.3: Contour $\gamma^{(jk)}$.

Denote

$$\mathbf{b}_{jk}^{(flux)}(x,\varepsilon) = \mathrm{curl}(\zeta_{jk}(x,\varepsilon)\mathbf{b}_j^{(k)}(x)) = \nabla\zeta_{jk}(x,\varepsilon)\times\mathbf{b}_j^{(k)}(x),$$

where $\mathbf{b}_j^{(k)}(x)$ is a magnetic field (2.22) corresponding to the contour $\gamma^{(jk)}$ (the properties of $\mathbf{b}_j^{(k)}(x)$ are given in Lemma 2.17),

$$\zeta_{jk}(x,\varepsilon) = \Psi\left(\varepsilon\ ln\frac{\delta^{(jk)}(x)}{\Delta_{\partial\Omega}(x)}\right),$$

Ψ is defined by formula (2.24),

$$\delta^{(jk)}(x) = \begin{cases} \rho_j(x)\Delta_{\widehat{\gamma}^{(j)}}(x) + \rho_k(x)\Delta_{\widehat{\gamma}^{(k)}}(x) + \sum\limits_{i\neq j,k}\rho_i(x)|x - x_0^{(jk)}| \\ +(1-\sum\limits_{i=1}^{J}\rho_i(x))\Delta_{\gamma^{(jk)}}(x),\ x_0^{(jk)}\in\widehat{\gamma}_j^{(k)},\ \text{for}\ \ x\in\Omega\setminus\gamma^{(jk)}, \\ 0,\ \ \text{for}\ \ x\in\gamma^{(jk)}, \end{cases}$$

$$\rho_i(x) = \begin{cases} 1, & x\in D_i\setminus D_i^{(3)}, \\ 0, & x\in(\Omega\setminus D_i)\bigcup D_i^{(2)}. \end{cases}$$

As before, the vector fields $\mathbf{b}_{jk}^{(flux)}$ have the following properties (see [59], [61]).

Lemma 5.7. *The vector field* $\boldsymbol{b}_{jk}^{(flux)}$ *is solenoidal,* $\boldsymbol{b}_{jk}^{(flux)}|_{\partial\Omega} = 0$*,* $\boldsymbol{b}_{jk}^{(flux)}(x,\varepsilon)$

$= 0$ *for* $x \in D_l$, $l \neq j, k$, $|x| \gg 1$, *and*

$$\int\limits_{\sigma_l(R)} \boldsymbol{b}_{jk}^{(flux)} \cdot \mathbf{n}\, dS = -\delta_{lj} + \delta_{lk}, \qquad l = 1, \ldots, J,$$

δ_{lj} *is the Kronecker's delta.*

For any solenoidal $\mathbf{w} \in W^{1,2}_{loc}(\Omega)$ *with* $\mathbf{w}\big|_{\partial\Omega} = 0$ *the following estimates*

$$\begin{aligned} &\int\limits_{\Omega_{(m)}\setminus\Omega_{(m-1)}} |\boldsymbol{b}_{jk}^{(flux)}|^2 |\mathbf{w}|^2\, dx \leq \varepsilon c \int\limits_{\Omega_{(m)}\setminus\Omega_{(m-1)}} |\nabla \mathbf{w}|^2\, dx, \\ &\int\limits_{\Omega_{(m)}} |\boldsymbol{b}_{jk}^{(flux)}|^2 |\mathbf{w}|^2\, dx \leq \varepsilon c \int\limits_{\Omega_{(m)}} |\nabla \mathbf{w}|^2\, dx \end{aligned} \tag{5.22}$$

hold with the constant c *independent of* ε *and* m*. Moreover,*

$$\begin{aligned} &|\boldsymbol{b}_{jk}^{(flux)}(x, \varepsilon)| \leq \frac{C}{g_s^2(z_3^{(s)})}, \quad |\nabla \boldsymbol{b}_{jk}^{(flux)}(x, \varepsilon))| \leq \frac{C}{g_s^3(z_3^{(s)})}, \quad x \in D_s, \\ &|\boldsymbol{b}_{jk}^{(flux)}(x, \varepsilon)| + |\nabla \boldsymbol{b}_{jk}^{(flux)}(x, \varepsilon))| \leq C, \ \ x \in \Omega \setminus (D_j \bigcup D_k), \end{aligned} \tag{5.23}$$

$s = j$ *or* $s = k$*.*

Let us take the vector fields $\mathbf{b}_{j,j+1}^{(flux)}$, $j = 1, ..., J-1$, and define

$$\mathbf{B}^{(flux)}(x, \varepsilon) = \sum_{j=1}^{J-1} \alpha_j \mathbf{b}_{j,j+1}^{(flux)}(x, \varepsilon), \quad \alpha_j = \sum_{l=i}^{j} \widehat{\mathcal{F}}_l, \quad j = 1, \ldots, J-1.$$

By the construction $\mathbf{B}^{(flux)}$ satisfies flux conditions (5.21).

5.1.4 Solvability of problem (5.2)

The vector field

$$\mathbf{A} = \mathbf{B}^{(inn)} + \sum_{m=1}^{M} \mathbf{B}_m^{(out)} + \mathbf{B}^{(flux)} \tag{5.24}$$

gives the desired extension of the boundary value $\mathbf{a}$. For all $\mathbf{w} \in W^{1,2}_{loc}(\Omega)$ with $\mathbf{w}\big|_{\partial\Omega} = 0$ from (5.13), (5.19), (5.22) follow the estimates

$$\begin{aligned} &\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{A}|^2 |\mathbf{w}|^2 dx \leq c(\varepsilon |\vec{\mathfrak{F}}^{(out)}|^2 + \varepsilon |\vec{\mathcal{F}}|^2 + |\vec{\mathbb{F}}^{(inn)}|^2) \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla \mathbf{w}|^2 dx, \\ &\int\limits_{\Omega_{(k)}} |\mathbf{A}|^2 |\mathbf{w}|^2\, dx \leq c(\varepsilon |\vec{\mathfrak{F}}^{(out)}|^2 + \varepsilon |\vec{\mathcal{F}}|^2 + |\vec{\mathbb{F}}^{(inn)}|^2) \int\limits_{\Omega_{(k)}} |\nabla \mathbf{w}|^2\, dx, \end{aligned} \tag{5.25}$$

where $|\vec{\mathcal{F}}|^2 = \sum\limits_{j=1}^{J} \mathcal{F}_j^2$, $\vec{\mathcal{F}} = (\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_J)$, $|\vec{\mathfrak{F}}^{(out)}|^2 = \sum\limits_{m=1}^{M} (\mathfrak{F}_m^{(out)})^2$, $\vec{\mathfrak{F}}_m^{(out)} = (\mathfrak{F}_1^{(out)}, \mathfrak{F}_2^{(out)}, ..., \mathfrak{F}_M^{(out)})$. The constant c is independent of ε and k. Moreover,

$$|\mathbf{A}(x)| \leq \frac{C(\varepsilon)}{g_j^2(z_3^{(j)})}, \ |\nabla \mathbf{A}(x)| \leq \frac{C(\varepsilon)}{g_j^3(z_3^{(j)})}, \quad x \in D_j, \quad j = 1, \dots, J. \tag{5.26}$$

From (5.26) follows the inequality

$$\int\limits_{\Omega_{(k)}} |\nabla \mathbf{A}(x)|^2 \, dx + \int\limits_{\Omega_{(k)}} |\mathbf{A}(x)|^4 \, dx \leq c(data)\Big(1 + \sum_{j=1}^{J} \int\limits_{1}^{R_{jk}} \frac{dz_3^{(j)}}{g_j^4(z_3^{(j)})}\Big),$$

$$c(data) = c_0\Big(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)} + |\vec{\mathcal{F}}|^2 + |\vec{\mathcal{F}}|^4\Big).$$

Since the proof of the existence of the weak solution to problem (5.2) is analogous to the considerations of Chapter 4 (see Subsection 4.1.3), we formulate the corresponding theorem without the proof.

Theorem 5.1. *Assume that the boundary value* $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ *has a compact support and that the fluxes* $\mathbb{F}_i^{(inn)}, i = 1, \dots, I$, *are sufficiently small. Let the compatibility condition* (5.3) *be satisfied. Then problem* (5.2) *admits at least one weak solution* $\mathbf{u} = \mathbf{v} + \mathbf{A} \in W^{1,2}_{loc}(\Omega)$, *where* $\mathbf{A}$ *is the vector field* (5.24) *and the following estimate*

$$\int\limits_{\Omega_{(k)}} |\nabla \mathbf{u}(x)|^2 \, dx \leq c(data)\Big(1 + \sum_{j=1}^{J} \int\limits_{1}^{R_{jk}} \frac{dz_3^{(j)}}{g_j^4(z_3^{(j)})}\Big),$$

holds.

In particular, if $\int\limits_1^\infty \frac{dt}{g_j^4(t)} < \infty, j = 1, \dots, J$, *then* $\int\limits_{D_j} |\nabla \mathbf{u}(x)|^2 \, dx < \infty$.

5.2 Two dimensional case

In this section we consider problem (5.2) in the two-dimensional domain $\Omega \subset \mathbb{R}^2$. Remind that for the two-dimensional Ω cross sections σ_j of outlets to infinity are open intervals, and we may assume, without loss of generality, that $\sigma_j^{(0)} = (-1, 1)$. As above, we look for $\mathbf{A}$ in the form

$$\mathbf{A} = \mathbf{B}^{(inn)} + \sum_{m=1}^{M} \mathbf{B}_m^{(out)} + \mathbf{B}^{(flux)}.$$

The idea of the construction is the same as in the three-dimensional case. Therefore, we only briefly indicate the differences arising for two-dimensional domains.

Constructing the virtual drain function $\mathbf{b}^{(inn)}$ we choose, as before, the "widest" outlet D_{j_*} and define in D_{j_*} the vector field $\mathbf{b}_{j_*}^{(inn)}$ by the formula

$$\mathbf{b}_{j_*}^{(inn)}(x) = -\mathbb{F}^{(inn)}\Big(\frac{\partial\tilde{\zeta}_{j_*}(x)}{\partial x_2},\ -\frac{\partial\tilde{\zeta}_{j_*}(x)}{\partial x_1}\Big),\quad x \in D_{j_*}^+.$$

Here $D_{j_*}^+ = \{z^{(j_*)} \in D_{j_*} : z_1^{(j_*)} > 0\}$,

$$\tilde{\zeta}_{j_*}(x) = \begin{cases} \zeta_{j_*}(x) = \Psi\Big(\ln\Big(\frac{\varrho(\delta(x))}{\Delta(x)}\Big)\Big), & x \in D_{j_*}^+, \\ 0, \quad x \in \mathbb{R}^2 \setminus D_{j_*}^+, \end{cases}$$

functions Ψ and ϱ are defined by formulas (2.24) and (2.25), respectively. Notice that $\zeta_{j_*}(x) = 0$ in a small neighborhood of the line $\gamma_+ = \{z^{(j_*)} : |z_1^{(j_*)}| = 0,\ z_2^{(j_*)} > 1\}$ and $\zeta_{j_*}(x) = 1$ near the boundary $\partial D_{j_*} \cap \partial\Omega$, and there holds the following estimates

$$\Big|\frac{\partial\zeta_{j_*}(x)}{\partial x_k}\Big| \le \frac{c}{\Delta(x)}, \qquad \Big|\frac{\partial^2\zeta_{j_*}(x)}{\partial x_k\partial x_l}\Big| \le \frac{c}{\Delta^2(x)}. \tag{5.27}$$

Lemma 5.8. *The solenoidal vector field $\mathbf{b}_{j_*}^{(inn)}(x)$ is infinitely differentiable, vanishes near the boundary $\partial D_{j_*} \cap \partial\Omega$ and in the neighborhood of the contour γ_+; the support of $\mathbf{b}_{j_*}^{(inn)}(x)$ is contained in the set of points $x \in D_{j_*}^+$ satisfying the inequalities*

$$\varrho\big(\delta(x)\big)e^{-1} \le \Delta(x) \le \varrho\big(\delta(x)\big). \tag{5.28}$$

The following estimates

$$|\mathbf{b}_{j_*}^{(inn)}(x)| \le \frac{c|\vec{\mathbb{F}}^{(inn)}|}{d(x)},\quad x \in D_{j_*}^+, \tag{5.29}$$

$$|\mathbf{b}_{j_*}^{(inn)}(x)| \le \frac{C|\vec{\mathbb{F}}^{(inn)}|}{g_{j_*}(z_2^{(j_*)})},\quad |\nabla\mathbf{b}_{j_*}(x)| \le \frac{C|\vec{\mathbb{F}}^{(inn)}|}{g_{j_*}^2(z_2^{(j_*)})} \tag{5.30}$$

hold. In (5.29) $d(x) = \mathrm{dist}(x, \partial D_{j_*} \cap \partial\Omega)$. *Finally,*

$$\int\limits_{\sigma_{j_*}(R)} \mathbf{b}_{j_*}^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}^{(inn)}.$$

Proof. Estimates (5.29) and (5.30) follow from the definition of $\mathbf{b}_{j_*}^{(inn)}(x)$, (5.27) and relations (5.28). By construction

$$\int_{\sigma_{j_*}(R)} \mathbf{b}_{j_*}^{(inn)} \cdot \mathbf{n}\, dS = \int_{-g_{j_*}(R)}^{g_{j_*}(R)} \mathbf{b}_{j_*}^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}^{(inn)} \int_{-g_{j_*}(R)}^{g_{j_*}(R)} \frac{\partial \tilde{\zeta}_{j_*}}{\partial z_1^{(j_*)}} dz_1^{(j_*)}$$

$$= \mathbb{F}^{(inn)} \Big(\tilde{\zeta}_{j_*}(g_{j_*}(R), R) - \tilde{\zeta}_{j_*}(-g_{j_*}(R), R) \Big) = \mathbb{F}^{(inn)}.$$

□

The rest of the construction of the virtual drain function $\mathbf{b}^{(inn)}$ and the extension $\mathbf{B}^{(inn)}$ is the same as in the three-dimensional case; we only mention that the fundamental solution of the Laplace operator in $\mathbb{R}^2$ has the form $q(x) = \dfrac{1}{2\pi} \ln |x|$ and that instead of (5.14) we have the following estimates

$$|\mathbf{B}^{(inn)}(x)| \le \frac{C|\vec{\mathbb{F}}^{(inn)}|}{g_{j_*}(z_2^{(j_*)})}, \quad |\nabla \mathbf{B}^{(inn)}(x)| \le \frac{C|\vec{\mathbb{F}}^{(inn)}|}{g_{j_*}^2(z_2^{(j_*)})}), \quad x \in D_{j_*}.$$

In order to construct the virtual drain functions $\mathbf{b}_m^{(out)}$, we take a point $x^{(m)} \in \Lambda_m \subset S_m$ and a smooth simple curve $\gamma_{mj_*} = \widehat{\gamma}_{j_*} \cup \widehat{\gamma}_{j_*}^{(m)}$, where $\widehat{\gamma}_{j_*} = \{z^{(j_*)} : z_1^{(j_*)} = 0, z_2^{(j_*)} > 1\}$ is a semi-infinite line lying in D_{j_*}, $\widehat{\gamma}_{j_*}^{(m)} \subset \Omega_0$ is a finite simple curve connecting $\widehat{\gamma}_{j_*}$ and the point $x^{(m)}$. Assuming that the distance from γ_{mj_*} to $S_m \setminus \Lambda_m$ is not less than $d_0 > 0$ and that the direction of the curve γ_{mj_*} coincide with the direction of increase of the coordinate $z_2^{(j_*)}$, we define, as in the three-dimensional case, a Hopf's cut-off function by the formula

$$\zeta_{mj_*}(x, \varepsilon) = \Psi\Big(\varepsilon \ln \frac{\delta^{(mj_*)}(x))}{\Delta_{\partial\Omega \setminus \Lambda_m}(x)} \Big), \tag{5.31}$$

Ψ is given by formula (2.24),

$$\delta^{(mj_*)}(x) = \begin{cases} \rho_{j_*}(x) \Delta_{\widehat{\gamma}_{j_*}}(x) + \sum\limits_{j=1, j \ne j_*}^{J} \rho_j(x) |x - x_0^{(mj_*)}| + \\ +(1 - \sum\limits_{j=1}^{J} \rho_j(x)) \Delta_{\widehat{\gamma}_{j_*}^{(m)}}(x), \quad x \in \Omega \setminus \gamma_{mj_*}, \; x_0^{(mk)} \in \widehat{\gamma}_{j_*}^{(m)}, \\ 0, \qquad x \in \gamma_{mj_*}, \end{cases}$$

$$\rho_j(x) = \begin{cases} 1, & x \in D_j \setminus D_j^{(3)}, \\ 0, & x \in (\Omega \setminus D_j) \bigcup D_j^{(2)}. \end{cases}$$

Further, defining $\tilde{\zeta}_{mj_*}(x, \varepsilon) = \zeta_{mj_*}(x, \varepsilon)$ on the right-hand side of curve γ_{mj_*} and

$\tilde{\zeta}_{mj_*}(x,\varepsilon)=0$ on the left-hand side of γ_{mj_*}, we introduce the vector field

$$\mathbf{b}_m^{(out)}(x,\varepsilon)=\mathfrak{F}_m^{(out)}\big(\frac{\partial\tilde{\zeta}_{mj_*}(x,\varepsilon)}{\partial x_2},\ -\frac{\partial\tilde{\zeta}_{mj_*}(x,\varepsilon)}{\partial x_1}\big).$$

Lemma 5.9. *The vector field $\boldsymbol{b}_m^{(out)}$ is infinitely differentiable and solenoidal, $\boldsymbol{b}_m^{(out)}$ vanishes near the boundary $\partial\Omega\setminus\Lambda_m$, for $x\in D_j,\ j\neq j_*$, with $|x|\gg 1$, and in a small neighborhood of the curve γ_{mj_*}. The following estimates*

$$|\boldsymbol{b}_m^{(out)}(x,\varepsilon)|\leq\frac{c\varepsilon}{d_{\partial\Omega\setminus\Lambda_m}(x)},\quad |\nabla\boldsymbol{b}_m^{(out)}(x,\varepsilon)|\leq\frac{c\varepsilon}{d^2_{\partial\Omega\setminus\Lambda_m}(x)},\tag{5.32}$$

$$|\boldsymbol{b}_m^{(out)}(x,\varepsilon))|\leq\frac{c(\varepsilon)|\mathfrak{F}_m^{(out)}|}{g_{j_*}(z_2^{(j_*)})},\quad |\nabla\boldsymbol{b}_m^{(out)}(x,\varepsilon)|\leq\frac{c(\varepsilon)|\mathfrak{F}_m^{(out)}|}{g^2_{j_*}(z_2^{(j_*)})},\quad x\in D_{j_*},\tag{5.33}$$

hold. The constant c in (5.32) *is independent of ε. Finally,*

$$\int\limits_{\Lambda_m}\boldsymbol{b}_m^{(out)}\cdot\boldsymbol{n}\,dS=\mathfrak{F}_m^{(out)}.$$

The extensions $\mathbf{B}_m^{(out)},\ m=1,...,M$, are constructed in the same way as in the three dimensional case and they have the same properties with the only difference that instead of (5.20) the following estimates

$$|\mathbf{B}_m^{(out)}(x,\varepsilon)|\leq\frac{C|\mathfrak{F}_m^{(out)}|}{g_{j_*}(z_2^{(j_*)})},\quad |\nabla\mathbf{B}_m^{(out)}(x,\varepsilon)|\leq\frac{C|\mathfrak{F}_m^{(out)}|}{g^2_{j_*}(z_2^{(j_*)})}\tag{5.34}$$

hold.

Finally, the solenoidal vector field $\mathbf{B}^{(flux)}$ satisfying the flux conditions

$$\int\limits_{\sigma_{j*}(R)}\mathbf{B}^{(flux)}\cdot\mathbf{n}\,dS=\widehat{\mathcal{F}}_j,\quad j=1,\ldots,J,$$

where $\widehat{\mathcal{F}}_j$ are defined analogously to in the three dimensional case, can be constructed as in [59], [61].

The vector field

$$\mathbf{A}=\mathbf{B}^{(inn)}+\sum_{m=1}^{M}\mathbf{B}_m^{(out)}+\mathbf{B}^{(flux)}\tag{5.35}$$

has all necessary properties that insure the validity of the following theorem.

Theorem 5.2. *Assume that the boundary value $\mathbf{a}\in W^{1/2,2}(\partial\Omega)$ has a compact support and that the fluxes $\mathbb{F}_i^{(inn)}, i=1,\ldots,I$, are sufficiently small. Let the compatibility condition* (5.3) *be satisfied. Then problem* (5.2) *admits at least one* *weak solution* $\mathbf{u} = \mathbf{v} + \mathbf{A} \in W^{1,2}_{loc}(\Omega)$*, where* $\mathbf{A}$ *is the vector field* (5.35)*. Moreover, the following estimate*

$$\int\limits_{\Omega_{(k)}} |\nabla \mathbf{u}(x)|^2 \, dx \le c(data)\Big(1 + \sum_{j=1}^{J} \int\limits_{1}^{R_{jk}} \frac{dz_2^{(j)}}{g_j^3(z_2^{(j)})}\Big),$$

$$c(data) = c_0\Big(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)} + |\vec{\mathcal{F}}|^2 + |\vec{\mathcal{F}}|^4\Big)$$

holds.

In particular, if $\int\limits_{1}^{\infty} \frac{dt}{g_j^3(t)} < \infty, j = 1, \ldots, J,$ *then* $\int\limits_{D_j} |\nabla \mathbf{u}(x)|^2 \, dx < \infty.$

Chapter 6

Domain with finite number of paraboloidal and layer type outlets to infinity

In this chapter we consider domain $\Omega \subset \mathbb{R}^3$ with two types of outlets to infinity: paraboloidal and layer type outlets. Layer type outlets we denote by $E_r,\ r = 1, ..., P$, and paraboloidal ones by $D_p, p = P+1, ..., J', J'+1, ..., J$. We assume that outlets $D_p, p = P+1, ..., J'$, are connected to the first layer E_1 and outlets D_p, $p = J'+1, ..., J$, are connected to the last layer E_P. Paraboloidal outlets with a constant cross section are cylinders. Notice that layer type outlets also can be expanding at infinity or not expanding. However, to expand can only the first and the last layer-type outlets (otherwise, they will intersect other outlets).

In some local coordinate systems $z^{(j)}$ outlets D_p and E_r have the forms:

$$D_p = \left\{ z^{(p)} : |z^{(p)\prime}| < g_p(z_3^{(p)}),\ z_3^{(p)} > 1 \right\},\ p = P+1, ..., J', J'+1, ..., J,$$

$$E_1 = \left\{ z^{(1)} : 0 < z_3^{(1)} < h_1(|z^{(1)\prime}|),\ |z^{(1)\prime}| > 1 \right\},$$

$$E_r = \left\{ z^{(r)} : 0 < z_3^{(r)} < 1,\ |z^{(r)\prime}| > 1 \right\},\ r = 2, ..., P-1,$$

$$E_P = \left\{ z^{(P)} : 0 < z_3^{(P)} < h_P(|z^{(P)\prime}|),\ |z^{(P)\prime}| > 1 \right\},$$

where the functions $g_p(t)$ satisfy Lipschitz conditions

$$|g_p(t_1) - g_p(t_2)| \le L_p |t_1 - t_2|, \quad t_1, t_2 \ge 1, \quad g_p(t) \ge 1\ \forall t,$$

and the functions $h_r(t)$, $r = 1$ and $r = P$, possess the following properties

$$\mu_1 h_r(t) \le \max_{t \le t_1 \le 2t} h_r(t_1) \le \mu_2 h_r(t), \quad h_r(t) \ge 1, \quad \forall t,$$

$$|h_r(t_1) - h_r(t_2)| \le L_r(t)|t_1 - t_2|, \quad t_1, t_2 \in [t, 2t]$$

where μ_1, μ_2 are certain positive constants and for $L_r(t)$ holds the inequality

$$\frac{L_r(t) \cdot t}{h_r(t)} \le const, \quad L_r(t) \le const \ \ \forall t.$$

Remark 6.1. Notice that functions $h_1,\ h_P$ and $g_p,\ p = P + 1, ..., J$, can be constants. In this case outlets to infinity E_1, E_P and $D_{P+1}, \dots, D_J$ do not expand at infinity. Instead of layers $E_r,\ r = 1, 2, ..., P$, one can take also outlets to infinity of the form $\{z^{(r)} : h_r^{(1)}(|z^{(r)\prime}|) < z_3^{(r)} < h_r^{(2)}(|z^{(r)\prime}|),\ |z^{(r)\prime}| > 1\}$, where

$$0 < h_r^{(0)} \le h_r^{(1)}(t) < h_r^{(2)}(t) \le h_r^{(*)} \ \ \forall t \ge 1,$$

$h_r^{(0)}$, $h_r^{(*)}$ are constants.

Layers E_r are connected to each other by bounded domains which we denote $H_l^{(j,j+1)} \subset \Omega_0,\ l = 1, ..., L$. Here the upper indices show which two layers are connected by $H_l^{(j,j+1)}$. Number of these connectors can be bigger than number of layers.

We assume that

(i) The boundary $\partial\Omega$ is Lipschitz.

(ii) The bounded domain Ω_0 has the form

$$\Omega_0 = G_0 \setminus \bigcup_{i=1}^{I} \overline{G}_i,$$

where G_0 and $G_i, i = 1, \dots, I,\ I \ge 0$, are bounded simply connected domains such that $\overline{G}_i \subset G_0,\ \overline{G}_{i_1} \cap \overline{G}_{i_2} = \emptyset$ for $i_1 \ne i_2$. Denote $\Gamma = \bigcup\limits_{i=1}^{I} \Gamma_i$, $\Gamma_i = \partial G_i$.

(iii) All connectors $H_l^{(j,j+1)}$ are contained in Ω_0: $H_l^{(j,j+1)} \subset \Omega_0$. By $\partial H_l^{(j,j+1)}$ we denote the "lateral" surface of $H_l^{(j,j+1)}$.

(iv) The outer boundary $S = \partial\Omega \setminus \Gamma$ consists of $P + 1$ disjoint unbounded connected components

$$S^{(0,1)} = \bigcup_{p=P+1}^{J'} \Upsilon_p \bigcup \mathfrak{X}^{(1)} \bigcup \{z^{(1)} \in \partial\Omega : z_3^{(1)} = h_1(|z^{(1)\prime}|),\ |z^{(1)\prime}| > 1\},$$

$$S^{(m,m+1)} = \{z^{(m)} \in \partial\Omega : z_3^{(m)} = 0,\ |z^{(m)\prime}| > 1\}\big) \bigcup \mathfrak{X}^{(m)} \bigcup$$
$$\big(\bigcup_{l=1}^{L} \partial H_l^{(m,m+1)} \setminus \Gamma\big) \bigcup \mathfrak{X}^{(m+1)}$$
$$\bigcup \{z^{(m+1)} \in \partial\Omega : z_3^{(m+1)} = 1,\ |z^{(m+1)\prime}| > 1\},\ L \ge 0, \quad m = 1, ..., P-1,$$

$$S^{(P,0)} = \bigcup_{p=J'+1}^{J} \Upsilon_p \bigcup \mathfrak{X}^{(P)} \bigcup \{z^{(P)} \in \partial\Omega : z_3^{(P)} = h_2(|z^{(P)\prime}|), |z^{(P)\prime}| > 1\},$$

where Υ_p is a lateral surface of D_p, $p = P+1, ..., J'$, $\mathfrak{X}^{(1)}$ is a union of surfaces connecting Υ_p, $p = P+1, ..., J'$, and $\{z^{(1)} \in \partial\Omega : z_3^{(1)} = h_1(|z^{(1)\prime}|),\ |z^{(1)\prime}| > 1\}$, $\mathfrak{X}^{(m)}$ is a union of surfaces connecting $\{z^{(m)} \in \partial\Omega : z_3^{(m)} = 0,\ |z^{(m)\prime}| > 1\}$ and $\big(\bigcup_{l=1}^{L} \partial H_l^{(m,m+1)} \setminus \Gamma\big)$, $\mathfrak{X}^{(m+1)}$ is a union of surfaces connecting $\big(\bigcup_{l=1}^{L} \partial H_l^{(m,m+1)} \setminus \Gamma\big)$ and $\{z^{(m+1)} \in \partial\Omega : z_3^{(m+1)} = 0,\ |z^{(m+1)\prime}| > 1\}$, $\mathfrak{X}^{(P)}$ is a union of surfaces connecting $\{z^{(P)} \in \partial\Omega : z_3^{(P)} = h_2(|z^{(P)\prime}|), |z^{(P)\prime}| > 1\}$ and Υ_p $p = J'+1, ..., J$.

In order to alleviate the notations, we omit the second index in $S^{(0,1)}$, $S^{(1,2)}$, ..., $S^{(P-1,P)}$, $S^{(P,0)}$ and set $S^{(0,1)} = S^{(0)}$, $S^{(1,2)} = S^{(1)}$, ..., $S^{(P-1,P)} = S^{(P-1)}$, $S^{(P,0)} = S^{(P)}$.

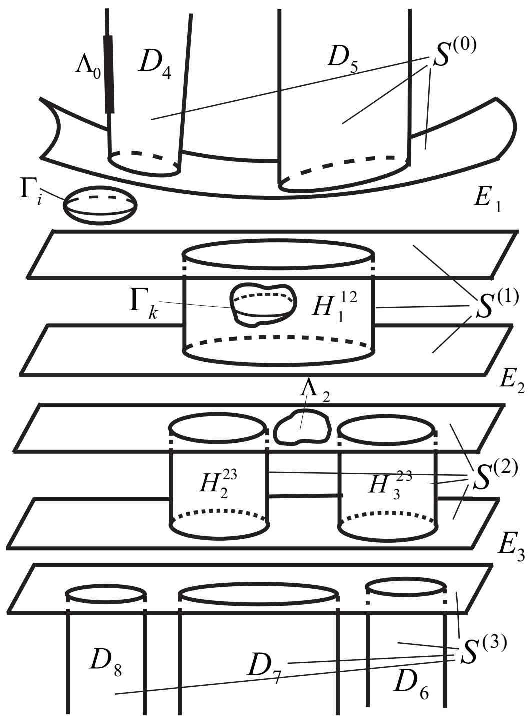

Figure 6.1: Domain Ω.

Below we shall use the following notations:

$$E_r^{(l)} = \left\{ z^{(r)} \in E_r :\ |z^{(r)\prime}| < R_{rl} \right\},$$

$$R_{r1} = 1,\ R_{rl+1} = 2R_{rl},\ l \geq 1,\ r = 1, \ldots, P,$$

$$D_p^{(l)} = \left\{ z^{(p)} \in D_p :\ z_3^{(p)} < R_{pl} \right\},$$

$$R_{p1} = 1,\ R_{pl+1} = R_{pl} + \frac{g_p(R_{pl})}{2L_p},\ l \geq 1,\ p = P+1, \ldots, J,$$

$$\Omega_{(l)} = \Omega_0 \cup E_1^{(l)} \cup \ldots \cup E_P^{(l)} \cup D_{P+1}^{(l)} \cup \ldots \cup D_J^{(l)},$$

$$\sigma_r^{(E)}(R) = E_r \cap \{z^{(r)} : |z^{(r)\prime}| = R\},\ \sigma_p^{(D)}(R) = D_p \cap \{z^{(p)} : z_3^{(p)} = R\},$$

i.e., $\sigma_p^{(D)}$ and $\sigma_r^{(E)} \subset \mathbb{R}^2$ are the cross sections of the outlets D_p and E_r, respectively.

Remark 6.2. Without loss of generality we assume that cross sections $\sigma_p^{(D)}$ are bounded simply connected domains. However, all results of this chapter remain valid in the case when the cross sections of paraboloidal outlets are bounded multiply connected domains (see Chapter 5).

Consider the following problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega, \\ \int\limits_{\sigma_r^{(E)}(R)} \mathbf{u}\cdot\mathbf{n}\, dS = \mathcal{F}_r^{(E)} & \forall R \geq 1,\ r = 1, 2, \ldots, P, \\ \int\limits_{\sigma_p^{(D)}(R)} \mathbf{u}\cdot\mathbf{n}\, dS = \mathcal{F}_p^{(D)} & \forall R \geq 1,\ p = P+1, \ldots, J, \end{cases} \tag{6.1}$$

where $\mathbf{n}$ is the unit vector of the normals to $\sigma_p^{(D)}$ and $\sigma_r^{(E)}$.

We suppose that the boundary value $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ has compact support

$$\operatorname{supp}\mathbf{a} \subset \partial\Omega_0. \tag{6.2}$$

Denote by Λ_m simply connected sets such that $\operatorname{supp}\mathbf{a} \cap S^{(m)} \subset \Lambda_m \subset S^{(m)}$, $m = 0, ..., P$.[1] Let

$$\mathbb{F}_i^{(inn)} = \int\limits_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\, dS,\ i = 1, ..., I, \quad \mathfrak{F}_m^{(out)} = \int\limits_{\Lambda_m} \mathbf{a}\cdot\mathbf{n}\, dS,\ m = 0, ..., P,$$

be the fluxes of the boundary value $\mathbf{a}$ over inner and outer boundaries, respectively. Then the necessary flux compatibility condition (3.3) can be written as

$$\sum_{i=1}^{I} \mathbb{F}_i^{(inn)} + \sum_{m=0}^{P} \mathfrak{F}_m^{(out)} + \sum_{r=1}^{P} \mathcal{F}_r^{(E)} + \sum_{p=P+1}^{J} \mathcal{F}_p^{(D)} = 0 \tag{6.3}$$

(the total flux is equal to zero).

[1]The sets $\operatorname{supp}\mathbf{a} \cap S^{(m)}$, $m = 0, ..., P$, are not necessary simply connected.

6.1 Construction of the extension $\mathbf{B}^{(inn)}$

Let us start with the construction of the "virtual drain" function $\mathbf{b}^{(inn)}$ which "drains" the the fluxes $\mathbb{F}_i^{(inn)}$ from the bounded parts Γ_i of the inner boundary Γ to infinity. Denote $\mathbb{F}^{(inn)} = \sum\limits_{i=1}^{I} \mathbb{F}_i^{(inn)}$, and, as usual, choose the "widest" outlet to infinity, in order to minimize the dissipation of energy generated by $\mathbf{b}^{(inn)}$. Suppose that such outlet is of the layer type, say E_1[2].

As in Chapter 4, Subsection 4.2.1 let us construct in E_1 a solenoidal vector field $\mathbf{b}_1^{(inn)}$ such that

$$\mathbf{b}_1^{(inn)}(x)\big|_{\partial E_1 \cap \partial\Omega} = 0, \qquad \int\limits_{\sigma_1^{(E)}(R)} \mathbf{b}_1^{(inn)} \cdot \mathbf{n}\, dS = \mathbb{F}^{(inn)}.$$

Introduce the infinite layer $\mathbb{L}_1 = \{y \in \mathbb{R}^3 : 0 < y_3 < h_1(|y'|),\ y' \in \mathbb{R}^2\}$ which for $|y'| > 1$ coincides with the outlet E_1. Let $\gamma_1 = \{y \in \mathbb{L} : |y'| = 0\}$. Define in $\mathbb{L}_1$ a cut–off function

$$\zeta_1(y) = \Psi\Big(\ln \Big(\frac{\varrho(\delta(y))}{\Delta(y)} \Big)\Big), \tag{6.4}$$

where

$$\delta(y) = \Delta_{\gamma_1 \cup \{y_3 = h_1(|y'|)\}}(y), \quad \Delta(y) = \Delta_{\partial\mathbb{L}_1 \setminus \{y_3 = h_1(|y'|)\}}(y),$$

functions Ψ and ϱ are defined by formulas (2.24) and (2.25), respectively.

Lemma 6.1. *The function $\zeta_1(y)$ is equal to zero at those points of $\mathbb{L}_1 \setminus \gamma_1$ where $\varrho(\delta(y)) \le \Delta(y)$ and $\zeta_1(y) = 1$ if $\Delta(y) \le e^{-1}\varrho(\delta(y))$. The following estimates*

$$\Big|\frac{\partial \zeta_1(y)}{\partial y_k}\Big| \le \frac{c}{\Delta(y)}, \qquad \Big|\frac{\partial^2 \zeta_1(y)}{\partial y_k \partial y_l}\Big| \le \frac{c}{\Delta^2(y)} \tag{6.5}$$

hold.

Set

$$\widehat{\mathbf{b}}_1^{(inn)}(y) = -\mathbb{F}^{(inn)} \mathrm{curl}\,\big(\zeta_1(y)\mathbf{b}_0(x)\big) = -\mathbb{F}^{(inn)} \nabla\zeta_1(y) \times \mathbf{b}_0(y),\ y \in \mathbb{L}_1,$$

where $\mathbf{b}_0(y)$ is a magnetic field defined by formula (2.23) with properties given in Lemma 2.17.

Lemma 6.2. *The solenoidal vector field $\widehat{\mathbf{b}}_1^{(inn)}$ is infinitely differentiable for $y \in$*

[2]To expand at infinity can only the first and the last layer type outlets E_1 and E_P. If E_1 and E_P are not growing at infinity, then we can "drain" the fluxes to any outlets E_r, however, for the unity of presentation, we choose in this case also the same outlet E_1. If we have to "drain" the fluxes from the bounded parts of $\partial\Omega$ to a paraboloidal outlet expanding at infinity, then extension $\mathbf{A}$ can be constructed combining the methods of this and previous chapters.

$\mathbb{L}_1 \setminus \{y : |y'| = 0\}$, *vanishes near the set* $\partial \mathbb{L}_1 \cup \{y : |y'| = 0\}$,

$$\int\limits_{\sigma_1^{(\mathbb{L}_1)}} \widehat{\mathbf{b}}_1^{(inn)} \cdot \mathbf{n} \, dS = \mathbb{F}^{(inn)}, \tag{6.6}$$

and

$$|\widehat{\mathbf{b}}_1^{(inn)}(y)| \le \frac{c|\vec{\mathbb{F}}^{(inn)}|}{d(y)}, \tag{6.7}$$

$$|\widehat{\mathbf{b}}_1^{(inn)}(y)| \le \frac{C|\vec{\mathbb{F}}^{(inn)}|}{h_1(|y'|)\,|y'|},$$

$$|\nabla \widehat{\mathbf{b}}_1^{(inn)}(y)| \le \frac{C|\vec{\mathbb{F}}^{(inn)}|}{h_1^2(|y'|)\,|y'|} + \frac{C|\vec{\mathbb{F}}^{(inn)}|}{h_1(|y'|)\,|y'|^2}, \; x \in E_1, \tag{6.8}$$

$$|\widehat{\mathbf{b}}_1^{(inn)}(y)| + |\nabla \widehat{\mathbf{b}}_1^{(inn)}(y)| \le C|\vec{\mathbb{F}}^{(inn)}|, \; x \in \Omega \setminus E_1.$$

In (6.7) $d(y) = \mathrm{dist}(y, \partial\mathbb{L}_1 \cap \partial\Omega \setminus \{y \in \partial\Omega : y_3 = h_1(|y'|)\})$, $|\vec{\mathbb{F}}^{(inn)}| = \sqrt{\sum\limits_{i=1}^{I} (\mathbb{F}_i^{(inn)})^2}$, $\vec{\mathbb{F}}^{(inn)} = (\mathbb{F}_1^{(inn)}, \mathbb{F}_2^{(inn)}, ..., \mathbb{F}_I^{(inn)})$.

Proof of this lemma is analogous to the proof of Lemma 4.8 (see Subsection 4.2.1).

Define

$$\mathbf{b}_1^{(inn)}(z^{(1)}) = \widehat{\mathbf{b}}_1^{(inn)}(z^{(1)})|_{E_1}. \tag{6.9}$$

Then, we can introduce the virtual drain function

$$\mathbf{b}^{(inn)} = \begin{cases} \mathbf{b}_\sharp^{(inn)} + \mathbf{b}_{01}^{(inn)}, & x \in \Omega_0, \\ \mathbf{b}_1^{(inn)}, & x \in E_1, \\ 0, & x \in \Omega \setminus (E_1 \cup \Omega_0), \end{cases}$$

where $\mathbf{b}_\sharp^{(inn)}$ and $\mathbf{b}_{01}^{(inn)}$ are constructed as in Chapter 4, Subsection 4.2.1.

Since

$$\mathbf{h}_0 = \begin{cases} \mathbf{a} - \mathbf{b}_\sharp^{(inn)}|_{\Gamma_i}, & x \in \Gamma_i, \; i = 1, \dots, I, \\ 0, & x \in \partial\Omega_0 \setminus \Gamma \end{cases}$$

is such that

$$\int\limits_{\Gamma_i} \mathbf{h}_0 \cdot \mathbf{n} \, dS = \int\limits_{\Gamma_i} \mathbf{a} \cdot \mathbf{n} \, dS - \int\limits_{\Gamma_i} \mathbf{b}_\sharp^{(inn)} \cdot \mathbf{n} \, dS = \mathbb{F}_i^{(inn)} - \mathbb{F}_i^{(inn)} = 0, \; i = 1, \dots, I,$$

we can extend the function $\mathbf{h}_0$ in the form

$$\mathbf{b}_0^{(inn)}(x) = \mathrm{curl}\,(\chi(x)\mathbf{E}(x)),$$

where $\mathbf{E} \in W_2^2(\Omega_0)$, $\operatorname{curl} \mathbf{E}|_{\partial\Omega_0} = \mathbf{h}_0$ and χ is a smooth cut-off function with $\chi(x) = 1$ on Γ (see Lemma 2.14).

Finally, we put

$$\mathbf{B}^{(inn)} = \mathbf{b}^{(inn)} + \mathbf{b}_0^{(inn)}. \tag{6.10}$$

Lemma 6.3. *The vector field* $\mathbf{B}^{(inn)}$ *is solenoidal,* $\mathbf{B}^{(inn)}|_{\Gamma_i} = \mathbf{a}|_{\Gamma_i}$, $i = 1, \dots, I$, $\mathbf{B}^{(inn)}|_{S^{(m)}} = 0$, $m = 0, \dots, P$, $\mathbf{B}^{(inn)} \in W_{loc}^{1,2}(\Omega)$, $\mathbf{B}^{(inn)}(x) = 0$, $x \in \Omega^{(j)}$, $j \neq 1, |x| \gg 1$. *For any* $\mathbf{w} \in W_{loc}^{1,2}(\Omega)$ *with* $\mathbf{w}\big|_{\partial\Omega} = 0$ *the following estimates*

$$\begin{gathered}
\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{B}^{(inn)}|^2 |\mathbf{w}|^2 \, dx \le c |\vec{\mathbb{F}}^{(inn)}|^2 \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla \mathbf{w}|^2 \, dx, \\
\int\limits_{\Omega_{(k)}} |\mathbf{B}^{(inn)}|^2 |\mathbf{w}|^2 \, dx \le c |\vec{\mathbb{F}}^{(inn)}|^2 \int\limits_{\Omega_{(k)}} |\nabla \mathbf{w}|^2 \, dx,
\end{gathered} \tag{6.11}$$

hold. Moreover,

$$\begin{gathered}
|\mathbf{B}^{(inn)}(x)| \le \frac{C|\vec{\mathbb{F}}^{(inn)}|}{h_1(|z^{(1)\prime}|)|z^{(1)\prime}|}, \ x \in E_1, \\
|\nabla \mathbf{B}^{(inn)}(x)| \le \frac{C|\vec{\mathbb{F}}^{(inn)}|}{h_1^2(|z^{(1)\prime}|)|z^{(1)\prime}|} + \frac{C|\vec{\mathbb{F}}^{(inn)}|}{h_1(|z^{(1)\prime}|)|z^{(1)\prime}|^2}, \ x \in E_1, \\
|\mathbf{B}^{(inn)}(x)| + |\nabla \mathbf{B}^{(inn)}(x)| \le C|\vec{\mathbb{F}}^{(inn)}|, \ x \in \Omega \setminus E_1.
\end{gathered} \tag{6.12}$$

6.2 Construction of the extension $\mathbf{B}_m^{(out)}$

In this section we construct the vector fields $\mathbf{B}_m^{(out)}$, $m = 0, \dots, P$, extending the boundary value $\mathbf{a}$ from the outer boundary S. We start with the construction of the "flux carriers" $\mathbf{b}_m^{(out)}$ which "drain" the fluxes $\mathfrak{F}_m^{(out)}$, $m = 0, ..., P$, from bounded parts of $S^{(m)}$ to infinity. The construction of $\mathbf{b}_m^{(out)}$, $m = 0, \dots, P$, depends on the location of Λ_m, $m = 0, ..., P$, i.e., there are some differences between the cases $m = 0$ and $m = 1, ..., P$. We submit the details of the construction for both cases in parallel.

1. Let $m = 1, ..., P$. Denote by γ_{m1} a smooth simple contours which intersect $\partial\Omega$ at the points $x^{(m)} \in \Lambda_m \subset S^{(m)}$ and $x^{(0)} \in S^{(0)} \cap \partial E_1$ and have the forms

$$\gamma_{m1} = \widehat{\gamma}_1 \cup l_1 \cup \widehat{\gamma}_1^{(m)} \cup l_m,$$

where $\widehat{\gamma}_1$ is a finite curve lying in E_1 and intersecting boundary $S^{(0)} \cap \partial E_1$ at the point $x^{(0)}$, $\widehat{\gamma}_1^{(m)} \subset \Omega \bigcap B_{R_1}(0)$ is a finite curve connecting $\widehat{\gamma}_1$ and the point $x^{(m)}$, l_0, $l_m \subset \mathbb{R}^3 \setminus \Omega$ are semi-infinite curves which begin at the points $x^{(0)}$ and $x^{(m)}$, respectively.

2. Let $m=0$. Denote by γ_{01} a smooth simple contour which intersects $\partial\Omega$ at the points $x^{(0)}\in\Lambda_0\subset S^{(0)}$ and $x^{(1)}\in S^{(1)}\cap\partial E_1$ and has the form:

$$\gamma_{01}=\widehat{\gamma}_1\cup l_1\cup\widehat{\gamma}_1^{(0)}\cup l_0,$$

where $\widehat{\gamma}_1\subset E_1$ is a finite curve intersecting the boundary $S^{(1)}\cap\partial E_1$ at $x^{(1)}$, $\widehat{\gamma}_1^{(0)}\subset\Omega\bigcap B_{R_1}(0)$ is a finite curve connecting $\widehat{\gamma}_1$ and $x^{(0)}$, $l_0,\ l_1\subset\mathbb{R}^3\setminus\Omega$ are semi-infinite curves which begin at $x^{(0)}$ and $x^{(1)}$, respectively.

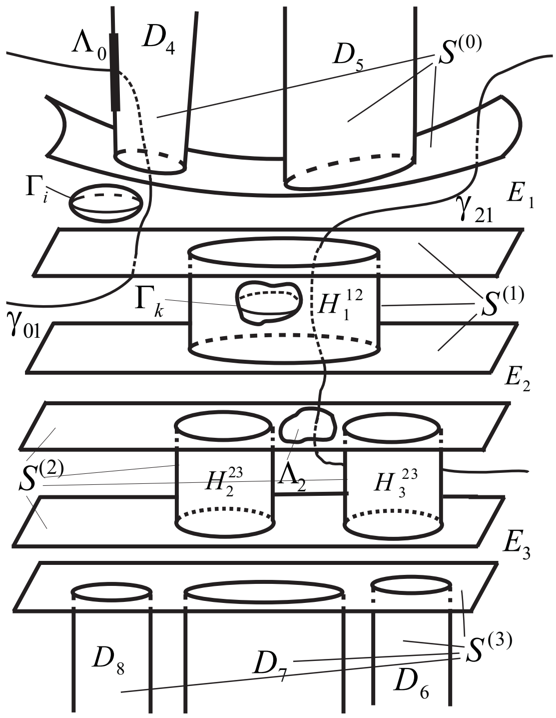

Figure 6.2: Contours γ_{01} and γ_{21}.

Assume that the direction of $\gamma_{m1}, m=0,...,P$, coincides with the direction of increase of coordinate $z_3^{(1)}$ and that the dist$(\gamma_{m1},\ S^{(m)}\setminus\Lambda_m)\geq d_0>0$, where d_0 is sufficiently small number, $m=0,...,P$.

In the domain Ω we introduce the virtual drain functions

$$\begin{aligned}\mathbf{b}_m^{(out)}(x,\varepsilon)&=\alpha_m\mathfrak{F}_m^{(out)}\mathrm{curl}(\zeta_{m1}(x,\varepsilon)\cdot\mathbf{b}_1^{(m)}(x))\\&=\alpha_m\mathfrak{F}_m^{(out)}\nabla\zeta_{m1}(x,\varepsilon)\times\mathbf{b}_1^{(m)}(x),\quad m=0,\dots,P,\end{aligned}\tag{6.13}$$

where $\mathbf{b}_1^{(m)}(x)$ is a magnetic field (2.22) corresponding to contour γ_{m1} (properties of $\mathbf{b}_1^{(m)}(x)$ are given in Lemma 2.17),

$$\alpha_m = \begin{cases} 1, & m = 1, \dots, P, \\ -1, & m = 0, \end{cases}$$

$$\zeta_{m1}(x,\varepsilon) = \Psi\Big(\varepsilon \ln \frac{\delta^{(m1)}(x)}{\Delta_{\partial\Omega\setminus(\Lambda_m\cup K)}(x)}\Big). \tag{6.14}$$

A function $\delta^{(m1)}(x)$ and a set K depend on the location of Λ_m. If $\Lambda_m \subset S^{(m)},\ m = 1, ..., P$, then $K = S^{(0)}$ and

$$\delta^{(m1)}(x) = \begin{cases} \rho_1(x)\Delta_{\widehat{\gamma}_1\cup(K\cap E_1)}(x) + \sum\limits_{\substack{j=2 \\ j\neq P+1,\dots,J'}}^{J} \rho_j(x)|x - x_0^{(m1)}| + \sum\limits_{j=P+1}^{J'} \rho_j(x)\delta_0 \\ +(1 - \sum\limits_{j=1}^{J} \rho_j(x))\Delta_{\gamma_{m1}\cup K}(x), \quad x \in \Omega \setminus (\gamma_{m1}\cup K),\ x_0^{(m1)} \in \widehat{\gamma}_1^{(m)}, \\ 0, \qquad x \in \gamma_{m1}\cup K. \end{cases}$$

If $\Lambda_0 \subset S^{(0)}$, then $K = S \setminus S^{(0)} = \bigcup\limits_{j=1}^{P} S^{(j)}$ and

$$\delta^{(01)}(x) = \begin{cases} \rho_1(x)\Delta_{\widehat{\gamma}_1\cup(K\cap E_1)}(x) + \sum\limits_{\substack{j=2 \\ j\neq P+1,\dots,J'}}^{J} \rho_j(x)\delta_0 + \sum\limits_{j=P+1}^{J'} \rho_j(x)|x - x_0^{(01)}| \\ +(1 - \sum\limits_{j=1}^{J} \rho_j(x))\Delta_{\gamma_{01}\cup K}(x), \quad x \in \Omega \setminus (\gamma_{01}\cup K),\ x_0^{(01)} \in \widehat{\gamma}_1^{(0)}, \\ 0, \qquad x \in \gamma_{01}\cup K, \end{cases}$$

where δ_0 is a sufficiently small positive constant,

$$\rho_j(x) = \begin{cases} 1,\ x \in D_j \setminus D_j^{(3)} \text{ or } x \in E_j \setminus E_j^{(3)}, \\ 0,\ x \in (\Omega \setminus D_j) \cup D_j^{(2)} \text{ or } x \in (\Omega \setminus E_j) \cup E_j^{(2)},\ j = 1, ..., J. \end{cases} \tag{6.15}$$

The function $\delta^{(m1)}(x)$ is continuous in the domain $\overline{\Omega}$ and infinitely differentiable in $\Omega \setminus (\gamma_{m1} \cup K)$, $|\nabla\delta^{(m1)}(x)|$ is bounded. Moreover, it is easy to check that

$$\delta^{(m1)}(x) = \begin{cases} \Delta_{\widehat{\gamma}_1\cup(K\cap E_1)}(x), & x \in E_1 \setminus E_1^{(3)}, \\ |x - x_0^{(m1)}|, & |x| \gg 1,\ x \in E_r, r = 2, ..., P, \\ & \quad \text{or } x \in D_p, p = J' + 1, ..., J, \\ \delta_0, & |x| \gg 1,\ x \in D_p,\ p = P + 1, ..., J', \\ \Delta_{\gamma_{m1}\cup K}(x), & x \in \Omega_0, \end{cases}$$

for $m=1,...,P$, and

$$\delta^{(01)}(x)=\begin{cases}\Delta_{\widehat{\gamma}_1\cup(K\cap E_1)}(x), & x\in E_1\setminus E_1^{(3)},\\ \delta_0,\ |x|\gg 1,\ x\in E_r,\ r=2,...,P,\ \text{or}\ x\in D_p,\ p=J'+1,...,J,\\ |x-x_0^{(01)}|,\quad |x|\gg 1,\ x\in D_p,\ p=P+1,...,J',\\ \Delta_{\gamma_{01}\cup K}(x), & x\in\Omega_0.\end{cases}$$

Lemma 6.4. *The vector field $\boldsymbol{b}_m^{(out)}$ is infinitely differentiable and solenoidal, $\boldsymbol{b}_m^{(out)}$ vanishes near the surface $\partial\Omega\setminus\Lambda_m$, in a small neighborhood of the curve $\gamma_{m1}\cap\overline{\Omega}$ and for $x\in\Omega\setminus E_1$ with $|x|\gg 1$. The following estimates*

$$|\boldsymbol{b}_m^{(out)}(x,\varepsilon)|\leq\frac{c\varepsilon}{d_{\partial\Omega\setminus(\Lambda_m\cup K)}(x)d_{\gamma_{m1}\cup K}(x)},\quad x\in\Omega,\tag{6.16}$$

$$\begin{aligned}&|\boldsymbol{b}_m^{(out)}(x,\varepsilon))|\leq\frac{c(\varepsilon)|\mathfrak{F}_m^{(out)}|}{h_1(|z^{(1)\prime}|)\,|z^{(1)\prime}|},\quad x\in E_1,\\ &|\nabla\boldsymbol{b}_m^{(out)}(x,\varepsilon)|\leq\frac{c(\varepsilon)|\mathfrak{F}_m^{(out)}|}{h_1^2(|z^{(1)\prime}|)\,|z^{(1)\prime}|}+\frac{c(\varepsilon)|\mathfrak{F}_m^{(out)}|}{h_1(|z^{(1)\prime}|)\,|z^{(1)\prime}|^2},\ x\in E_1,\end{aligned}\tag{6.17}$$

hold. The constant c in (6.16) *is independent of ε. Finally,*

$$\int\limits_{\Lambda_m}\boldsymbol{b}_m^{(out)}\cdot\boldsymbol{n}\,dS=\mathfrak{F}_m^{(out)}.$$

Proof. The first statement of the lemma follow from definitions (6.13), (6.14) of $\mathbf{b}_m^{(out)}$ and ζ_{m1} and from the properties of the regularized distance (see estimates (2.21)). Since $\mathbf{b}_m^{(out)}(x,\varepsilon)|_{\partial\Omega\setminus\Lambda_m}=0$, $\mathbf{b}_m^{(out)}(x,\varepsilon)=0$ for $x\in\Omega\setminus E_1$, $|x|\gg 1$, $\zeta_{m1}(x,\varepsilon)=1$ on $\partial\Omega\setminus(\Lambda_m\cup K)$ and $\zeta_{m1}(x,\varepsilon)=0$ on K, the Ostrogradsky–Gauss (see Lemma 2.11) and the Stokes (see Lemma 2.12) formulas yield

$$\begin{aligned}\int\limits_{\Lambda_m}\mathbf{b}_m^{(out)}\cdot\mathbf{n}\,dS&=-\int\limits_{\sigma_1^{(E)}(R)}\mathbf{b}_m^{(out)}\cdot\mathbf{n}\,dS\\ &=-\alpha_m\mathfrak{F}_m^{(out)}\int\limits_{\sigma_1^{(E)}(R)}\operatorname{curl}(\zeta_{m1}\mathbf{b}_1^{(m)})\cdot\mathbf{n}\,dS=-\alpha_m\mathfrak{F}_m^{(out)}\int\limits_{\partial\sigma_1^{(E)}(R)}\zeta_{m1}\mathbf{b}_1^{(m)}\cdot d\mathbf{l}\\ &=-\alpha_m\mathfrak{F}_m^{(out)}\Big(\int\limits_{l_0(R)}\zeta_{m1}\mathbf{b}_1^{(m)}\cdot d\mathbf{l}+\int\limits_{l_1(R)}\zeta_{m1}\mathbf{b}_1^{(m)}\cdot d\mathbf{l}\Big)\\ &=\begin{cases}-\mathfrak{F}_m^{(out)}\int\limits_{l_0(R)}\mathbf{b}_1^{(m)}\cdot d\mathbf{l},\quad m=1,...,P,\\ \mathfrak{F}_m^{(out)}\int\limits_{l_1(R)}\mathbf{b}_1^{(m)}\cdot d\mathbf{l},\quad m=0,\end{cases}\quad=\quad\mathfrak{F}_m^{(out)},\end{aligned}$$

where $l_0(R) = \partial\sigma_1^{(E)}(R) \cap S^{(0)}, \quad l_1(R) = \partial\sigma_1^{(E)}(R) \cap \big(\partial E_1 \setminus S^{(0)}\big)$.[3] $\square$

Let $\mathbf{h}_m(x) = \mathbf{a}(x)|_{\Lambda_m} - \mathbf{b}_m^{(out)}(x,\varepsilon)|_{\Lambda_m}$. Then

$$\int\limits_{\Lambda_m} \mathbf{h}_m \cdot \mathbf{n}\, dS = \int\limits_{\Lambda_m} \mathbf{a} \cdot \mathbf{n}\, dS - \int\limits_{\Lambda_m} \mathbf{b}_m^{(out)} \cdot \mathbf{n}\, dS = 0.$$

Therefore, $\mathbf{h}_m$ can be extended inside Ω in the form

$$\mathbf{b}_{0\,m}^{(out)}(x,\varepsilon) = \operatorname{curl}\left(\chi_m(x,\varepsilon)\mathbf{E}_m(x)\right),$$

where $\mathbf{E}_m \in W_2^2(\Omega_0)$, $\operatorname{curl}\mathbf{E}_m|_{\Lambda_m} = \mathbf{h}_m$ and χ_m is a Hopf's type cut-off function such that $\chi_m(x,\varepsilon) = 1$ on Λ_m (see Lemma 2.14).

Define

$$\mathbf{B}_m^{(out)}(x,\varepsilon) = \mathbf{b}_m^{(out)}(x,\varepsilon) + \mathbf{b}_{0\,m}^{(out)}(x,\varepsilon).$$

Obviously,

$$\operatorname{div}\mathbf{B}_m^{(out)} = 0, \quad \mathbf{B}_m^{(out)}|_{\Lambda_m} = \mathbf{a}, \quad \mathbf{B}_m^{(out)}|_{\partial\Omega\setminus\Lambda_m} = 0,$$

$$\mathbf{B}_m^{(out)} = 0, \; x \in \Omega \setminus E_1, \; |x| \gg 1.$$

Lemma 6.5. *For any solenoidal $\mathbf{w} \in W_{loc}^{1,2}(\Omega)$ with $\mathbf{w}\big|_{\partial\Omega} = 0$ the following estimates*

$$\begin{aligned}
&\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{B}_m^{(out)}|^2 |\mathbf{w}|^2\, dx \le \varepsilon c |\mathfrak{F}_m^{(out)}|^2 \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla\mathbf{w}|^2\, dx,\\
&\int\limits_{\Omega_{(k)}} |\mathbf{B}_m^{(out)}|^2 |\mathbf{w}|^2\, dx \le \varepsilon c |\mathfrak{F}_m^{(out)}| \int\limits_{\Omega_{(k)}} |\nabla\mathbf{w}|^2\, dx,
\end{aligned} \tag{6.18}$$

hold. The constant c does not depend on ε and k. Moreover,

$$\begin{aligned}
&|\mathbf{B}_m^{(out)}(x,\varepsilon)| \le \frac{C(\varepsilon)|\mathfrak{F}_m^{(out)}|}{h_1(|z^{(1)\prime}|)\,|z^{(1)\prime}|}, \quad x \in E_1,\\
&|\nabla\mathbf{B}_m^{(out)}(x,\varepsilon)| \le \frac{C(\varepsilon)|\mathfrak{F}_m^{(out)}|}{h_1^2(|z^{(1)\prime}|)\,|z^{(1)\prime}|} + \frac{C(\varepsilon)|\mathfrak{F}_m^{(out)}|}{h_1(|z^{(1)\prime}|)\,|z^{(1)\prime}|^2}, \quad x \in E_1,\\
&|\mathbf{B}_m^{(out)}(x,\varepsilon)| + |\nabla\mathbf{B}_m^{(out)}(x,\varepsilon)| \le C(\varepsilon)|\mathfrak{F}_m^{(out)}|, \; x \in \Omega \setminus E_1.
\end{aligned} \tag{6.19}$$

[3]Notice that the contours $l_0(R)$ and $l_1(R)$ have opposite direction.

Proof. Inequality (6.16) yields (see Lemma 2.15)

$$\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{B}_m^{(out)}|^2|\mathbf{w}|^2 dx \leq c\varepsilon|\mathfrak{F}_m^{(out)}|^2 \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} \frac{|\mathbf{w}|^2}{\text{dist}^2(x,\partial\Omega)}dx$$

$$\leq c\varepsilon|\mathfrak{F}_m^{(out)}|^2 \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla\mathbf{w}|^2 dx$$

$$\int\limits_{\Omega_{(k)}} |\mathbf{B}_m^{(out)}|^2|\mathbf{w}|^2 dx \leq c\varepsilon|\mathfrak{F}_m^{(out)}|^2 \int\limits_{\Omega_{(k)}} \frac{|\mathbf{w}|^2}{\text{dist}^2(x,\partial\Omega)}dx \leq c\varepsilon|\mathfrak{F}_m^{(out)}|^2 \int\limits_{\Omega_{(k)}} |\nabla\mathbf{w}|^2 dx.$$

Estimates (6.19) follow from (6.17). □

6.3 Construction of the extension $\mathbf{B}^{(flux)}$

Now, we need to compensate the fluxes over the cross sections of outlets to infinity, i.e., we have to construct a solenoidal vector field $\mathbf{B}^{(flux)}$ satisfying the flux conditions

$$\begin{aligned} &\int\limits_{\sigma_r^{(E)}(R)} \mathbf{B}^{(flux)}\cdot\mathbf{n}\,dS = \widehat{\mathcal{F}}_r^{(E)},\ r=1,\dots,P,\\ &\int\limits_{\sigma_p^{(D)}(R)} \mathbf{B}^{(flux)}\cdot\mathbf{n}\,dS = \mathcal{F}_p^{(D)},\ p=P+1,\dots,J, \end{aligned} \tag{6.20}$$

where

$$\widehat{\mathcal{F}}_r^{(E)} = \mathcal{F}_r^{(E)}, \quad r=2,\dots,P;\ \ \widehat{\mathcal{F}}_1^{(E)} = \mathcal{F}_1^{(E)} + \mathbb{F}^{(inn)} + \mathfrak{F}^{(out)},$$

$\mathfrak{F}^{(out)} = \sum\limits_{m=0}^{P} \mathfrak{F}_m^{(out)}$. Note that the total flux is equal to zero:

$$\sum_{r=1}^{P} \widehat{\mathcal{F}}_r^{(E)} + \sum_{p=P+1}^{J} \mathcal{F}_p^{(D)} = 0. \tag{6.21}$$

For the construction of the extension $\mathbf{B}^{(flux)}$, we use the vector field

$$\mathbf{b}^{(k,n)}(x) = \frac{1}{4\pi}\oint\limits_{\gamma^{(k,n)}} \frac{x-y}{|x-y|^3}\times d\mathbf{l}_y$$

which describes a magnetic field corresponding to the contour $\gamma^{(k,n)}$ (the properties of $\mathbf{b}^{(k,n)}$ are given in Lemma 2.17). However, introducing this contour we have to distinguish between three different cases.

1. Contour $\gamma^{(j,j+1)}$, $j = P+1,...,J'-1, J'+1,...,J-1$, goes through the two paraboloidal outlets. In this case $\gamma^{(j,j+1)}$ is an infinite smooth simple contour,

consisting of two semi-infinite lines $\widehat{\gamma}^{(j)} \subset D_j$, $\widehat{\gamma}^{(j+1)} \subset D_{j+1}$ and a finite curve $\widehat{\gamma}_j^{(j+1)} \subset \Omega \bigcap B_{R_1}(0)$ joining them. The direction of the curve $\gamma^{(j,j+1)}$ coincides with the direction of increase of the coordinate $z_3^{(j)}$.

2. Contour $\gamma^{(k,n)}$ goes through the two outlets, one of which is of paraboloidal type and another - of layer type. Actually, there are only two possibilities: either $k = J',\ n = 1$, or $k = P,\ n = J$. Define

$$\gamma^{(J',1)} = \widehat{\gamma}^{(J')} \cup \widehat{\gamma}^{(1)} \cup l_1 \cup \widehat{\gamma}_{J'}^{(1)},$$

$$\gamma^{(P,J)} = \widehat{\gamma}^{(P)} \cup \widehat{\gamma}^{(J)} \cup l_P \cup \widehat{\gamma}_P^{(J)},$$

where $\widehat{\gamma}^{(J')} \subset D_{J'}$, $\widehat{\gamma}^{(J)} \subset D_J$ are semi-infinite lines, $\widehat{\gamma}^{(1)} \subset E_1$, $\widehat{\gamma}^{(P)} \subset E_P$ are finite curves intersecting $\partial\Omega$ at the points $x^{(1)} \in S^{(1)} \cap \partial E_1$ and $x^{(P)} \in S^{(P-1)} \cap \partial E_P$, respectively, $\widehat{\gamma}_{J'}^{(1)}$, $\widehat{\gamma}_P^{(J)} \subset \Omega \bigcap B_{R_1}(0)$ are curves joining $\widehat{\gamma}^{(J')}$ with $\widehat{\gamma}^{(1)}$ and $\widehat{\gamma}^{(J)}$ with $\widehat{\gamma}^{(P)}$, and $l_1,\ l_P \subset \mathbb{R}^3 \setminus \Omega$ are semi-infinite curves which start at the points $x^{(1)}$ and $x^{(P)}$, respectively. The directions of the curves $\gamma^{(J',1)}$ and $\gamma^{(P,J)}$ coincide with the directions of the axis $z_3^{(1)}$ and $z_3^{(P)}$ in the layer type outlets to infinity.

3. Contour $\gamma^{(j,j+1)},\ j = 1, ..., P-1$, goes through the two layer type outlets. Then $\gamma^{(j,j+1)}$ is an infinite smooth simple contour which intersects $\partial\Omega$ at the points $x^{(j)} \in S^{(j-1)} \cap \partial E_j$ and $x^{(j+1)} \in S^{(j+1)} \cap \partial E_{j+1}$. Contour $\gamma^{(j,j+1)}$ consists of finite curves $\widehat{\gamma}^{(j)} \subset E_j$, $\widehat{\gamma}^{(j+1)} \subset E_{j+1}$ intersecting $\partial\Omega$ at the points $x^{(j)}$, $x^{(j+1)}$, respectively, a finite curve $\widehat{\gamma}_j^{(j+1)} \subset \Omega \bigcap B_{R_1}(0)$ joining them, and semi-infinite curves $l_j,\ l_{j+1} \subset \mathbb{R}^3 \setminus \Omega$ that begin at the points $x^{(j)}$, $x^{(j+1)}$:

$$\gamma^{(j,j+1)} = \widehat{\gamma}^{(j)} \cup l_j \cup \widehat{\gamma}^{(j+1)} \cup l_{j+1} \cup \widehat{\gamma}_j^{(j+1)}.$$

The direction of the curve $\gamma^{(j,j+1)}$ coincides with the direction of increase of the coordinate $z_3^{(j+1)}$. In all three cases we suppose that dist($\gamma^{(j,j+1)}$, $\partial\Omega$)$\geq d_0 > 0$, where d_0 is sufficiently small number.

Denote

$$\mathbf{b}_{k,n}^{(flux)}(x,\varepsilon) = \mathrm{curl}(\zeta_{k,n}(x,\varepsilon)\mathbf{b}^{(k,n)}(x)) = \nabla\zeta_{k,n}(x,\varepsilon) \times \mathbf{b}^{(k,n)}(x),$$

where

$$\zeta_{k,n}(x,\varepsilon) = \Psi\left(\varepsilon\ ln\frac{\delta^{(k,n)}(x)}{\Delta_{\partial\Omega\setminus K}(x)}\right).$$

In the last formula the set K depends on the location of the contour $\gamma^{(k,n)}$. In the first case, i.e., when $k = j, n = j+1$ and both outlets D_j and D_{j+1} are of paraboloidal type, we take $K = \emptyset$ and the construction of $\zeta_{j,j+1}(x,\varepsilon)$ is the same as in Chapter 5, Subsection 5.1.3. In the second case, when one of the outlets is

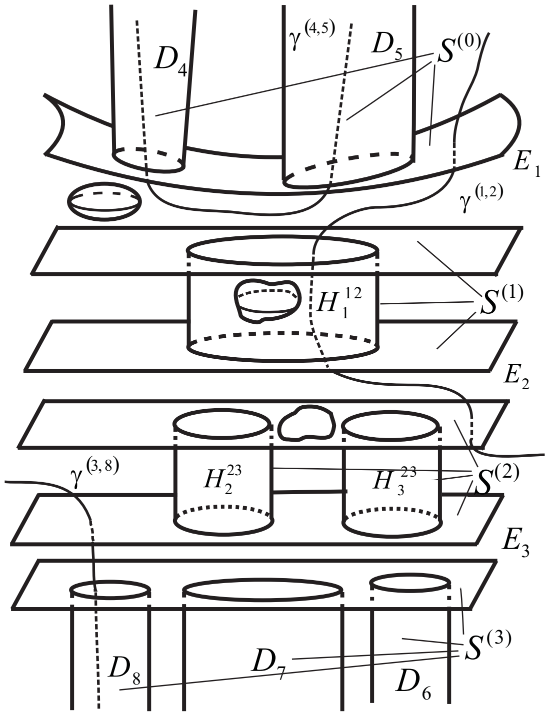

Figure 6.3: Contours $\gamma^{(1,2)}$, $\gamma^{(3,8)}$ and $\gamma^{(4,5)}$.

paraboloidal and another is of the layer type (i.e., we either have the pair $D_{J'}$ and E_1 or the pair D_J and E_P) we set $K = K^{(J',1)} = \bigcup_{j=1}^{P} S^{(j)}$,

$$\delta^{(J',1)}(x) = \begin{cases} \rho_{J'}(x)\Delta_{\widehat{\gamma}^{(J')}\cap D_{J'}} + \rho_1(x)\Delta_{(\widehat{\gamma}^{(1)}\cup K)\cap E_1}(x) + \sum\limits_{\substack{j=2,\\ j\neq J'+1,\dots,J}}^{J} \rho_j(x)\delta_0 \\ + \sum\limits_{j=P+1}^{J'-1} \rho_j(x)|x - x_0^{(J',1)}| + (1 - \sum\limits_{j=1}^{J} \rho_j(x))\Delta_{\gamma^{(J',1)}\cup K}(x), \\ x \in \Omega \setminus (\gamma^{(J',1)} \cup K),\ x_0^{(J',1)} \in \widehat{\gamma}_{J'}^{(1)}, \\ 0, \qquad x \in \gamma^{(J',1)} \cup K, \end{cases}$$

and $K = K^{(P,J)} = \bigcup_{j=0}^{P-1} S^{(j)}$,

$$\delta^{(P,J)}(x) = \begin{cases} \rho_J(x)\Delta_{\widehat{\gamma}^{(J)}\cap D_J} + \rho_P(x)\Delta_{(\widehat{\gamma}^{(P)}\cup K)\cap E_P}(x) + \sum\limits_{j=J'+1}^{J-1} \rho_j(x)|x - x_0^{(P,J)}| \\ + \sum\limits_{\substack{j=1,\\ j\neq P,J'+1,\dots,J}}^{J} \rho_j(x)\delta_0 + (1 - \sum\limits_{j=1}^{J} \rho_j(x))\Delta_{\gamma^{(P,J)}\cup K}(x), \\ x \in \Omega \setminus (\gamma^{(P,J)} \cup K),\ x_0^{(P,J)} \in \widehat{\gamma}_P^{(J)}, \\ 0, \qquad x \in \gamma^{(P,J)} \cup K, \end{cases}$$

respectively.

In the third case, when both outlets E_j and $E_{j+1}, j = 1, \dots, P-1$, are layers, $K = S \setminus S^{(j)}$ and

$$\delta^{(j,j+1)}(x) = \begin{cases} \rho_j(x)\Delta_{(\widehat{\gamma}^{(j)}\cup K)\cap E_j} + \rho_{j+1}(x)\Delta_{(\widehat{\gamma}^{(j+1)}\cup K)\cap E_{j+1}} + \sum\limits_{\substack{k=1\\k\neq j,j+1}}^{J} \rho_k(x)\delta_0 \\ +(1 - \sum\limits_{k=1}^{J} \rho_k(x))\Delta_{\gamma^{(j,j+1)}\cup K}(x), \quad x \in \Omega \setminus (\gamma^{(j,j+1)} \cup K), \\ 0, \qquad x \in \gamma^{(j,j+1)} \cup K. \end{cases}$$

The functions Ψ and ρ_j are defined by formulas (2.24) and (6.15), respectively. The vector fields $\mathbf{b}_{k,n}^{(flux)}$ have the following properties (see [59], [61], [47]).

Lemma 6.6. *The vector fields* $\boldsymbol{b}_{k,n}^{(flux)}(x,\varepsilon)$ *are solenoidal,* $\boldsymbol{b}_{k,n}^{(flux)}|_{\partial\Omega} = 0$, $\boldsymbol{b}_{k,n}^{(flux)}(x,\varepsilon) = 0$ *for* $x \in D_m, E_s, m, s \neq k, n$, $|x| \gg 1$, *and*

$$\int\limits_{\sigma_l(R)} \boldsymbol{b}_{k,n}^{(flux)} \cdot \mathbf{n}\, dS = -\delta_{lk} + \delta_{ln},$$

where $(k,n) = (j, j+1), j = 1, ..., P-1,\ P+1, ..., J'-1,\ J'+1, ..., J-1,\ (k,n) = (J', 1)$ *and* $(k,n) = (P, J)$, δ_{kn} *is the Kronecker's delta,* $\sigma_l(R)$ *stands either for* $\sigma_l^{(E)}(R)$ *or for* $\sigma_l^{(D)}(R)$.

For any solenoidal $\mathbf{w} \in W^{1,2}_{loc}(\Omega)$ *with* $\mathbf{w}\big|_{\partial\Omega} = 0$ *the following estimates*

$$\begin{aligned} \int\limits_{\Omega_{(m)}\setminus\Omega_{(m-1)}} |\boldsymbol{b}_{k,n}^{(flux)}|^2|\mathbf{w}|^2\, dx &\le \varepsilon c \int\limits_{\Omega_{(m)}\setminus\Omega_{(m-1)}} |\nabla\mathbf{w}|^2\, dx, \\ \int\limits_{\Omega_{(m)}} |\boldsymbol{b}_{k,n}^{(flux)}|^2|\mathbf{w}|^2\, dx &\le \varepsilon c \int\limits_{\Omega_{(m)}} |\nabla\mathbf{w}|^2\, dx \end{aligned} \tag{6.22}$$

hold with the constant c *independent of* ε *and* m. *Moreover,*

$$|\boldsymbol{b}_{k,n}^{(flux)}(x,\varepsilon)| \le \frac{C(\varepsilon)}{g_s^2(z_3^{(s)})}, \quad |\nabla \boldsymbol{b}_{k,n}^{(flux)}(x,\varepsilon))| \le \frac{C(\varepsilon)}{g_s^3(z_3^{(s)})}, \quad x \in D_s, \tag{6.23}$$

$$|\boldsymbol{b}_{k,n}^{(flux)}(x,\varepsilon)| \le \frac{C(\varepsilon)}{h_s(|z^{(s)\prime}|)|z^{(s)\prime}|},$$

$$|\nabla \boldsymbol{b}_{k,n}^{(flux)}(x,\varepsilon))| \le \frac{C(\varepsilon)}{h_s^2(|z^{(s)\prime}|)|z^{(s)\prime}|} + \frac{C(\varepsilon)}{h_s(|z^{(s)\prime}|)|z^{(s)\prime}|^2}, \quad x \in E_s, \tag{6.24}$$

$$|\boldsymbol{b}_{k,n}^{(flux)}(x,\varepsilon)| + |\nabla \boldsymbol{b}_{k,n}^{(flux)}(x,\varepsilon))| \le C(\varepsilon),\ x \in \Omega \setminus \Omega^{(k)} \cup \Omega^{(n)},$$

$s = k$ *or* $s = n$.

Let us take $\mathbf{b}^{(flux)}_{j,j+1}$, $j = 1, ..., J-1$, and define

$$\begin{aligned} \mathbf{B}^{(flux)}(x,\varepsilon) = \sum_{\substack{j=1\\ j\neq P,J'}}^{J-1} \beta_j \mathbf{b}^{(flux)}_{j,j+1}(x,\varepsilon) \\ +\beta_{J'}\mathbf{b}^{(flux)}_{J',1}(x,\varepsilon) + \beta_P \mathbf{b}^{(flux)}_{P,J}(x,\varepsilon). \end{aligned} \tag{6.25}$$

Calculating the fluxes of the vector field $\mathbf{B}^{(flux)}$ over the cross section $\sigma_j, j = 1, \dots, J$, and using (6.21), we find the coefficients β_j so that $\mathbf{B}^{(flux)}$ satisfies flux conditions (6.20).

6.4 Solvability of problem (6.1)

The vector field

$$\mathbf{A} = \mathbf{B}^{(inn)} + \sum_{m=0}^{P} \mathbf{B}^{(out)}_m + \mathbf{B}^{(flux)} \tag{6.26}$$

gives the desired extension of the boundary value $\mathbf{a}$. For all $\mathbf{w} \in W^{1,2}_{loc}(\Omega)$ with $\mathbf{w}\big|_{\partial\Omega} = 0$ from (6.11), (6.18), (6.22) follow the estimates

$$\begin{aligned} &\int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\mathbf{A}|^2|\mathbf{w}|^2 dx \\ &\leq c(\varepsilon|\vec{\mathfrak{F}}^{(out)}|^2 + \varepsilon|\vec{\mathcal{F}}^{(D)}|^2 + \varepsilon|\vec{\mathcal{F}}^{(E)}|^2 + |\mathbb{F}^{(inn)}|^2) \int\limits_{\Omega_{(k)}\setminus\Omega_{(k-1)}} |\nabla\mathbf{w}|^2 dx, \\ &\int\limits_{\Omega_{(k)}} |\mathbf{A}|^2|\mathbf{w}|^2\, dx \\ &\leq c(\varepsilon|\vec{\mathfrak{F}}^{(out)}|^2 + \varepsilon|\vec{\mathcal{F}}^{(D)}|^2 + \varepsilon|\vec{\mathcal{F}}^{(E)}|^2 + |\mathbb{F}^{(inn)}|^2) \int\limits_{\Omega_{(k)}} |\nabla\mathbf{w}|^2\, dx, \end{aligned} \tag{6.27}$$

where $|\vec{\mathcal{F}}^{(E)}|^2 = \sum\limits_{r=1}^{P} (\mathcal{F}^{(E)}_r)^2$, $\vec{\mathcal{F}}^{(E)} = (\mathcal{F}^{(E)}_1, \mathcal{F}^{(E)}_2, ..., \mathcal{F}^{(E)}_P)$, $|\vec{\mathcal{F}}^{(D)}|^2 = \sum\limits_{p=P+1}^{J} (\mathcal{F}^{(D)}_p)^2$, $\vec{\mathcal{F}}^{(D)} = (\mathcal{F}^{(D)}_{P+1}, \mathcal{F}^{(D)}_{P+2}, ..., \mathcal{F}^{(D)}_J)$, $|\vec{\mathfrak{F}}^{(out)}|^2 = \sum\limits_{m=0}^{P} (\mathfrak{F}^{(out)}_m)^2$,
$\vec{\mathfrak{F}}^{(out)} = (\mathfrak{F}^{(out)}_0, \mathfrak{F}^{(out)}_1, ..., \mathfrak{F}^{(out)}_P)$. The constant c in (6.27) is independent of ε and k. Moreover,

$$\begin{aligned} &|\mathbf{A}(x,\varepsilon)| \leq \frac{C(\varepsilon)}{g_p^2(z_3^{(p)})}, \ |\nabla\mathbf{A}(x,\varepsilon)| \leq \frac{C(\varepsilon)}{g_p^3(z_3^{(p)})}, \quad x \in D_p, \\ &|\mathbf{A}(x,\varepsilon)| \leq \frac{C(\varepsilon)}{h_r(|z^{(r)\prime}|)|z^{(r)\prime}|}, \\ &|\nabla\mathbf{A}(x,\varepsilon)| \leq \frac{C(\varepsilon)}{h_r^2(|z^{(r)\prime}|)|z^{(r)\prime}|} + \frac{C(\varepsilon)}{h_r(|z^{(r)\prime}|)|z^{(r)\prime}|^2}, \ x \in E_r, \\ &|\mathbf{A}(x,\varepsilon)| + |\nabla\mathbf{A}(x,\varepsilon)| \leq C(\varepsilon), \quad x \in \Omega_0, \end{aligned} \tag{6.28}$$

where $r = 1, \dots, P, \quad p = P+1, \dots, J$.

Estimates (6.28) yield the inequality

$$\int\limits_{\Omega_{(k)}} |\nabla \mathbf{A}(x,\varepsilon)|^2\,dx + \int\limits_{\Omega_{(k)}} |\mathbf{A}(x,\varepsilon)|^4\,dx$$
$$\leq c(data)\Big(1 + \sum_{r=1}^{P} \int\limits_1^{R_{rk}} \frac{d\tau}{\tau h_r^3(\tau)} + \sum_{p=P+1}^{J} \int\limits_1^{R_{pk}} \frac{dt}{g_p^4(t)}\Big),$$

where

$$c(data) = c_0\Big(\|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{a}\|^4_{W^{1/2,2}(\partial\Omega)} + |\vec{\mathcal{F}}^{(D)}|^2$$
$$+|\vec{\mathcal{F}}^{(E)}|^2 + |\vec{\mathcal{F}}^{(D)}|^4 + |\vec{\mathcal{F}}^{(E)}|^4\Big).$$

Leray–Hopf's inequalities (3.10) with given $\delta > 0$ follow from (6.27) if we take ε sufficiently small and assume that the fluxes $\mathbb{F}_i^{(inn)}, i = 1, \dots, I$, are sufficiently small. Therefore, the proof of the existence of the weak solution to problem (6.1), (6.3) is analogous to proofs given in Chapter 4, Subsections 4.1.3 and 4.2.4.

Theorem 6.1. *Assume that the boundary value* $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ *satisfies condition* (6.2) *and that the fluxes* $\mathbb{F}_i^{(inn)}, i = 1, \dots, I$, *are sufficiently small. Then problem* (6.1), (6.3) *admits at least one weak solution* $\mathbf{u} = \mathbf{v} + \mathbf{A} \in W^{1,2}_{loc}(\Omega)$, *where* $\mathbf{A}$ *is vector field* (6.26). *The following estimate*

$$\int\limits_{\Omega_{(k)}} |\nabla \mathbf{u}(x)|^2\,dx \leq c(data)\Big(1 + \sum_{r=1}^{P} \int\limits_1^{R_{rk}} \frac{d\tau}{\tau h_r^3(\tau)} + \sum_{p=P+1}^{J} \int\limits_1^{R_{pk}} \frac{dt}{g_p^4(t)}\Big),$$

holds. In particular, if either $\int\limits_1^{\infty} \frac{d\tau}{\tau\, h_r^3(\tau)} < \infty, \ r = 1, \dots, P,$ *or* $\int\limits_1^{\infty} \frac{dt}{g_p^4(t)} < \infty, \ p = P+1, \dots, J$, *then* $\int\limits_{\Omega^{(j)}} |\nabla \mathbf{u}(x)|^2\,dx < \infty$, *where* $\Omega^{(j)}$ *is either* E_j *or* D_j.

Remark 6.3. All obtained in the thesis results remain valid for the nonhomogeneous Navier–Stokes system if the external force $\mathbf{f}$ have an appropriate behavior at infinity.

Remark 6.4. The extension $\mathbf{A}$ of the boundary value $\mathbf{a}$, when we "drain" the fluxes from the bounded parts of $\partial\Omega$ to a paraboloidal outlet to infinity (in order to "minimize" the dissipation of the energy), can be constructed combining the methods of this and the previous chapters.

Conclusions

The main goal of the thesis was to construct a suitable extension of boundary value which gives the possibility to reduce the nonhomogeneous boundary conditions to the homogeneous ones. This extension is constructed in the form

$$\mathbf{A} = \mathbf{B}^{(inn)} + \sum_{m=1}^{M} \mathbf{B}_m^{(out)} + \mathbf{B}^{(flux)},$$

where $\mathbf{B}^{(inn)}$ extends the boundary value $\mathbf{a}$ from the inner boundary Γ, $\mathbf{B}_m^{(out)}$ extend $\mathbf{a}$ from the connected component $S^{(m)}$ of the noncompact outer boundary S, and $\mathbf{B}^{(flux)}$ has zero boundary value over $\partial\Omega$ and removes the fluxes over the cross sections of outlets to infinity. The vector fields $\mathbf{B}_m^{(out)}$ and $\mathbf{B}^{(flux)}$ are constructed to satisfy Leray–Hopf's inequalities which allows to obtain a priori estimates of the solution for arbitrary large fluxes $\mathfrak{F}_m^{(out)}$ and $\mathcal{F}_j$. However, Leray–Hopf's inequality cannot be true, in general, for the vector field $\mathbf{B}^{(inn)}$. If the fluxes of the boundary value over compact connected components of the boundary do not vanish, there are counterexamples (see [67], [4]) showing that in bounded domains Leray–Hopf inequality can be false whatever the choice of the solenoidal extension is taken. Therefore, we assume that the fluxes $\mathbb{F}_i^{(inn)}$ of $\mathbf{a}$ over the compact components Γ_i of the inner boundary Γ are "sufficiently small".

1. If fluxes $\mathbb{F}_i^{(inn)}$ are sufficiently small ($\mathfrak{F}_m^{(out)}$ and $\mathcal{F}_j$ are arbitrary), then problem (3.1) in domains with noncompact multiply connected boundaries admits at leats one solution, having either finite or infinite Dirichlet integral.

2. If fluxes $\mathbb{F}_i^{(inn)}$ are "large", then it is impossible to prove the existence of solution to problem (3.1) by using the extension method.

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