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NON-STATIONARY POISEUILLE TYPE SOLUTIONS FOR THE SECOND GRADE FLUID FLOW PROBLEM IN CYLINDRICAL DOMAINS

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VILNIAUS UNIVERSITETAS

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ANTROJO LAIPSNIO SKYSČIŲ TEKĖJIMO UŽDAVINIO NESTACIONARŪS PUAZEILIO TIPO SPRENDINIAI CILINDRINĖSE SRITYSE

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Contents

Introduction

In the thesis we study mathematical models of incompressible, homogeneous viscoelastic second grade fluid flows in certain two- and three-dimensional unbounded domains Π. More precisely, we assume that the domain Π is either the infinite channel $\Pi = \{x \in \mathbb{R}^2 : (x_1, x_2) \in \sigma \times \mathbb{R}\}$, where $\sigma = [-\frac{d}{2}, \frac{d}{2}]$ with $d > 0$ denoting the width of the channel, or the infinite cylinder $\Pi = \{x = (x', x_3) \in \mathbb{R}^3 : (x', x_3) \in \sigma \times \mathbb{R}\}$, where $x' = (x_1, x_2)$ and σ denotes the constant cross-section of the tube, i.e., σ is bounded, connected subset of $\mathbb{R}^2$ independent of x_3. We assume that the boundary $\partial\sigma$ is C^4-smooth. We are looking for a solution to the second grade fluid flow problem having a prescribed time dependent flux. Such problems are reduced to inverse problems with an unknown right-hand side corresponding to the pressure drop along the axis of the cylinder.

In the two dimensional case and in the case of axially symmetric cylinder we prove the global existence of a unique unidirectional flow and find the relationship between the flux and the gradient of the pressure.

In a general three-dimensional pipe (without axial symmetry) we prove the existence of a unique global solution for small data. We also prove that in the case of unidirectional data the velocity components perpendicular to the axis of the cylinder are secondary.

Time-periodic problem for the second grade fluid flow in a channel is also studied. For this problem we prove the existence of the unique solution and find the relationship between the flux and the gradient of the pressure.

Actuality and history of the problem

The mechanical behavior of fluids are specified by constitutive equations. For long time Navier-Stokes equation seemed to be most useful to describe non-Newtonian fluids motion. However, in many fields, such as food industry or bio-engineering, the fluids are mixtures and cannot be described by the Navier-Stokes equations. Examples of such combined fluids are gels, paints, oils, slurries, geological fluids, food products, blood, etc. These liquids have strong non-Newtonian characteristics. It is difficult to suggest a single model which would exhibit all properties of viscoelastic fluids. One of models to account for the rheological effects of viscoelastic fluid is the second grade fluid flow model. It belongs to the class of non-Newtonian Rivlin-Ericksen fluids of differential type (see [40], [86]). This model describes a large class of viscous fluids with polymer additives and viscoelastic liquids with "short memory". The equations governing the flow of the second grade fluid are one order higher than the Navier-Stokes equations.

The stress tensor for the second grade fluid is given by [74]

$$\mathbf{T} = p\mathbf{I} + \nu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2$$

where

$$\begin{aligned} \mathbf{A}_1 &= \nabla\mathbf{u} + (\nabla\mathbf{u})^T, \\ \mathbf{A}_2 &= \frac{d}{dt}\mathbf{A}_1 + \mathbf{A}_1\nabla\mathbf{u} + (\nabla\mathbf{u})^T\mathbf{A}_1, \end{aligned}$$

p and $\mathbf{u}$ are the pressure and the velocity field, ν is the coefficient of viscosity and α_1, α_2 are coefficients of the material (usually called normal stress moduli). Considerations on the stability of the rest state require ν and α_1 to be nonnegative ([18], [21], [32], [41]).

For fluids of the second grade, we express the divergence of the stress tensor $\mathbf{T}$ as follows

$$\nabla\cdot\mathbf{T} = -\nabla p + \nu\Delta\mathbf{u} + \alpha_1(\frac{\partial}{\partial t}\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\Delta\mathbf{u}) + \mathbf{N}_1(\mathbf{u}) + \mathbf{N}_2(\mathbf{u}) + \mathbf{N}_3(\mathbf{u}) \tag{0.1}$$

where

$$\begin{aligned} \mathbf{N}_1(\mathbf{u}) &= \alpha_1(\nabla\mathbf{u})^T : \nabla\mathbf{A}_1, \\ \mathbf{N}_2(\mathbf{u}) &= \alpha_1\nabla\cdot(\mathbf{A}_1\mathbf{W} - \mathbf{W}\mathbf{A}_1), \\ \mathbf{N}_3(\mathbf{u}) &= (\alpha_1 + \alpha_2)\nabla\cdot\mathbf{A}_1^2 \end{aligned}$$

with

$$\mathbf{W} = \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^T).$$

The flow of the fluid can be expressed by linear momentum

$$\rho\frac{\partial}{\partial t}\mathbf{u} - \nabla\cdot\mathbf{T} = \rho\mathbf{f}, \tag{0.2}$$

where $\mathbf{f}$ denotes the external force, and, for simplicity, the constant density of the fluid ρ is taken equal to 1. Substituting expression (0.1) into equation (0.2) and assuming that the motion is incompressible we obtain the following system of equations

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{u} - \alpha_1 \Delta \mathbf{u}) + \mathbf{u} \cdot \nabla(\mathbf{u} - \alpha_1 \Delta \mathbf{u}) - \nu \Delta \mathbf{u} + \nabla p - \mathbf{N}_1(\mathbf{u}) \\ \qquad -\mathbf{N}_2(\mathbf{u}) - \mathbf{N}_3(\mathbf{u}) = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0. \end{cases} \tag{0.3}$$

We consider the problem in the space-time region $\Pi^T = \Pi \times (0,T)$, where Π is sufficiently smooth, bounded two- or three-dimensional domain, and $T > 0$. We assume that the fluid adheres to the walls $\partial\Pi$ of the fixed container Π:

$$\mathbf{u}(x,t) = 0, \quad (x,t) \in \partial\Pi \times (0,T). \tag{0.4}$$

Finally, we append the initial condition

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad x \in \Pi. \tag{0.5}$$

Special cases of problem (0.3), (0.4), (0.5) have been considered by many authors. Oskolkov has investigated the case when all $\mathbf{N}_i = 0$ (see [57], [58], [59]). Using "vanishing viscosity" method he proved the existence and uniqueness of a global weak solution. In 1974, Dunn and Fosdick [21] showed that, in oder to make the fluid model consistent with thermodynamics, the material constants must be taken to satisfy

$$\nu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0.$$

Moreover, they showed that the rest state is asymptotically stable just when previous inequalities are strict. Instability results for $\alpha_1 < 0$ have been found in some special flow situations (see [18], [82]). In the thesis we recall the thermodynamical restriction $\alpha_1 + \alpha_2 = 0$ and take

$$\alpha_1 = -\alpha_2 := \alpha.$$

Therefore, we can rewrite problem (0.3), (0.4), (0.5) as

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{u} - \alpha \Delta \mathbf{u}) + \mathbf{u} \cdot \nabla(\mathbf{u} - \alpha \Delta \mathbf{u}) - \nu \Delta \mathbf{u} - \mathbf{N}_1(\mathbf{u}) - \mathbf{N}_2(\mathbf{u}) + \nabla p = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0, \\ \mathbf{u}|_{S^T} = 0, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x), \end{cases}$$

where $S^T = \partial\Pi \times (0,T)$.

However, in this work we refer another formulation of the problem obtained by observing that

$$\mathrm{curl}\Delta\mathbf{u}\times\mathbf{u}=\mathbf{u}\cdot\nabla\Delta\mathbf{u}+\nabla\cdot((\nabla\mathbf{u})^{\mathrm{T}}\mathbf{A}_1)-\nabla(\mathbf{u}\cdot\Delta\mathbf{u}+\frac{1}{4}|\mathbf{A}_1|^2).$$

Then the second grade fluid flow problem takes the form

$$\begin{cases} \frac{\partial}{\partial t}(\mathbf{u}-\alpha\Delta\mathbf{u})-\nu\Delta\mathbf{u}+\mathrm{curl}(\mathbf{u}-\alpha\Delta\mathbf{u})\times\mathbf{u}+\nabla\widetilde{p}=\mathbf{f}, \\ \mathrm{div}\mathbf{u}=0, \\ \mathbf{u}|_{S^T}=0, \quad \mathbf{u}(x,0)=\mathbf{u}_0(x) \end{cases} \tag{0.6}$$

with the modified pressure

$$\widetilde{p}=\alpha(\mathbf{u}\cdot\Delta\mathbf{u}+\frac{1}{4}A:A)-\frac{1}{2}(\mathbf{u}\cdot\mathbf{u})+p.$$

Let us remark, that in the two-dimensional case we understand curl and the vector product as usually in $\mathbb{R}^3$ assuming that vectors have zero third component and do not depend on x_3.

The first who have used equation (0.6) were Dunn and Fosdick [21]. Since then it has been the most usual of the equations considering the second grade fluids (see [6], [11], [13], [14], [20], [21], [33], [34], [39], [40], [60], [65], [66], [82], [86], and others). Cioranescu and Quazar in [15] (see also [13], [14]) have proved the existence of a unique global solution in two-dimensional bounded domains, and of a local solution in three-dimensional case.

In this thesis we are looking for the solution of system (0.6) in the cylinder $\Pi=\{x\in\mathbb{R}^n : x'\in\sigma\times\mathbb{R}\}$ which has the prescribed flux $F(t)$ over the cross-section σ[1]:

$$\int_\sigma u_n(x',x_n,t)dx'=F(t). \tag{0.7}$$

For the two-dimensional channel and the three-dimensional axially symmetric pipe we assume that the initial data and the external force have only the last component and are independent of the coordinate x_n. We look for an unidirectional (having just the last component) solution, which satisfies the flux condition. Such solution we call Poiseuille type solution. Different types of such exact solutions were computed numericaly in [4], [10], [16], [22], [31], [35], [43], [45], [51], [54], [70], [71], [77], [78], [79] etc.

[1] $\sigma=(-\frac{d}{2},\frac{d}{2})$, if $n=2$, and $\sigma=\{x'=(x_1,x_2)\in\mathbb{R}^2 : |x'|<1\}$, if $n=3$.

For $\alpha = 0$ the system (0.6) coincide with the Navier-Stokes system. For the non-stationary Navier-Stokes system Poiseuille type solutions with prescribed time dependent flux are found and investigated in [36], [37], [61]-[64]. The time periodic Navier-Stokes problem with given flux was considered in [9], [37].

Note that equations (0.6) describing the motion of fluids of second grade are of higher order than the Navier-Stokes equations. A significant difference between the Navier-Stokes system and equations (0.6) is that the nonlinear terms in Navier-Stokes equations are of lower order than the linear ones, while in (0.6) the nonlinear terms are of higher order. Therefore, in the three-dimensional pipe (without axial symmetry) the unidirectional solution for the second grade fluid flow problem is impossible, and the Poiseuille type solution has all three components. However, in this case the velocity components U_1, U_2 are secondary in comparison with the axial velocity U_3. The analogous results for the steady second grade fluid motion are obtained in [65], [66].

Notice that in the case of the three-dimensional pipe the most essential were the ideas proposed by Cioranescu, Quazar and Girault in [13]-[15]. As in these papers, we find the solutions by the Galerkin method using the special basis constructed in [13],[15]. Notice, that unlike [13]-[15], in oder to satisfy the flux condition, we have to solve the inverse problem with the unknown right-hand side corresponding to the pressure drop (gradient of the pressure).

Aims and problems

The aim of this thesis is to investigate the solvability and uniqueness of the solution to the second grade fluid flow problem with prescribed flux condition in the following cases:

- the initial boundary problem in the two-dimensional channel,
- the initial boundary value problem in the three-dimensional pipe with rotational symmetry,
- the time-periodic problem in the two-dimensional channel,
- the initial boundary value problem in the three-dimensional pipe.

Methods

In the thesis we apply the methods of functional analysis, properties of Sobolev spaces. To construct an approximate solution we use Galerkin method with the special bases.

Novelty

All results presented in this doctoral thesis are new. The Poiseuille type solution for the non-stationary second grade fluid flow problem with prescribed flux condition earlier were not studied. The existence results and estimates obtained for the solutions are new.

Structure of the thesis

The thesis consists of Introduction, 5 chapters, conclusions and bibliography. Introduction contains a short review about the history of the problem and describes shortly obtained results. In Chapter 1 we present basic notations and auxiliary propositions which are used in the work. The initial boundary value problem for the second grade fluid flow in the two-dimensional channel is studied in Chapter 2. In this chapter we construct the solution using the Galerkin method and prove that there exists the unique global solution. In Chapter 3 we obtain the analogous results in the three-dimensional cylinder with axial symmetry, assuming that the data are also axially symmetric. Chapter 4 is devoted to the study of the time periodic problem in the two dimensional channel. The last Chapter 5 deals with the second grade fluid flow problem in the three-dimensional pipe. For sufficiently small data we prove the existence of the unique solution having the prescribed flux. The existence is proved by the Galerkin method using the special basis constructed by Cioranescu and Quazar in [14], [15].

Approbation

The results of the thesis were presented at:

- International conference "Parabolic and Navier-Stokes Equations" (Będlewo, Poland, 2012, September 2 - 8),
- International conference "Regularity Aspects of Partial differential equations" (Będlewo, Poland, 2010. September 5 - 11),
- 51th conference of Lithuanian Mathematical Society (Šiauliai, 2010, June 17 - 18).
- 15th Master and PhD conference IVUS 2010 (Kaunas, 2010, May 13).
- International conference "Differential equations and their applications" (Panevėžys, 2009, September 10 - 12).
- 50th conference of Lithuanian Mathematical Society (Vilnius, 2009, June 18 - 19).
- Seminars on “Differential equations” at the department of Mathematics and Informatics of Vilnius university (Vilnius, 2009 - 2012).
- Seminars ”Differential equations and there applications” of the Institute of Mathematics and Informatics (Vilnius, 2007 - 2009).

Publications

The main results of the thesis are published in the following papers:

1. Kloviene N., Pileckas K., The second grade fluid flow problem in an infinite pipe, *Asymptotic Analysis*, (2012) (accepted for publication).

2. Kloviene N., Pileckas K., Nonstationary Poiseuille type solutions for the second grade fluid flow, *Lithuanian Mathematical Journal*, **52** (2), 155-171 (2012).

3. Kloviene N., On a third order initial boundary value problem in a plane domain, *Nonlinear Analysis: Modeling and Control*, **17** (3), 312-326 (2012).

4. Kloviene N., Poiseuille type solution of second grade fluid, *Proceedings of the 15th Master and PhD conference IVUS 2010*, 69-73 (2010).

5. Kloviene N., Periodic by time the second grade fluid flow in the infinite channel, *Lietuvos Matematikos Rinkinys, LMD darbai*, **50**, 30-35 (2009) (in lithuanian).

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Chapter 1

Preliminaries

1.1 Notation

i, j, k, l, n, p, r - natural numbers.

$\mathbb{R}^n$ - n-dimensional Euclidean space.

$x = (x_1, ..., x_n)$ - point in $\mathbb{R}^n$.

Ω - domain (open connected set) in $\mathbb{R}^n$.

$\partial\Omega$ - boundary of the domain Ω.

$\bar{\Omega}$ - closure of the domain Ω.

$\Omega^T = \Omega \times (0, T)$.

$|| \cdot ||_X$ - the norm in the Banach space X.

The vector-valued function $\mathbf{u} = (u_1, \ldots, u_n)$ belongs to the space V, if $u_i \in V, i = 1, \ldots, n$, and $\|\mathbf{u}\|_V = \big(\sum\limits_{i=1}^{n} \|u_i\|_V^2\big)^{1/2}$.

$\alpha = (\alpha_1, ..., \alpha_n), \alpha_i \geq 0,$ - multi-index, $|\alpha| = \alpha_1 + ... + \alpha_n$.

$D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$ - a differential operator of order $|\alpha|$, where $D^{\alpha_i} = \dfrac{\partial^{\alpha_i}}{\partial x_j}$.

C^k - the space, consisting of k-times continuously differentiable functions.

$C^\infty(\Omega)$ - the set of all infinitely differentiable functions defined on Ω.

$C_0^\infty(\Omega)$ - the subset of all functions from $C^\infty(\Omega)$ with compact support in Ω.

We say, that the boundary $\partial\Omega$ of the domain Ω is C^k, if for each point $x_0 \in \partial\Omega$ there exists $r > 0$ and a C^k - function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that in local coordinates we have $\partial\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_n = f(x_1, ..., x_{n-1})\}$, where $B(x_0, r)$ denotes a ball with the center in x_0 and the radial r.

$L_p(\Omega)$ - the Banach space, consisting of functions, whose p^{th} powers are integrable over Ω. The norm in $L_p(\Omega)$ is given by $||f||_{L_p(\Omega)} = \big(\int\limits_\Omega |f|^p dx\big)^{1/p}$.

$L_\infty(\Omega)$ - the Banach space of functions with the norm $\|f\|_{L_\infty(\Omega)} = \operatorname{ess\,sup}\limits_{x\in\Omega} |f(x)|$.

$W_p^l(\Omega)$ - the Sobolev space which consists of functions such that $D^\alpha f \in L_p(\Omega)$ for all week partial derivatives of order $|\alpha|$, $0 \leq |\alpha| \leq l$. The norm in $W_p^l(\Omega)$ is given by $||f||_{W_p^l(\Omega} = \big(\sum\limits_{|\alpha|=0}^{l} \int\limits_\Omega |D^\alpha f(x)|^p dx\big)^{1/p}$.

$\overset{\circ}{W}{}_p^l$ - the Sobolev space which is obtained as a closure of the set $C_0^\infty(\Omega)$ in the norm $\|f\|_{W_p^l(\Omega)}$.

$W_2^{2l,l}(\Omega^T)$ - the Hilbert space ($l > 0$), consisting of functions, whose derivatives $D_t^r D_x^\alpha f$ belong to $L_2(\Omega^T)$, $2r + |\alpha| \leq 2l$. The norm of $W_2^{2l,l}(\Omega)$ is
$||f||_{W_2^{2l,l}(\Omega)} = \big(\sum\limits_{j=0}^{2l} \sum\limits_{2r+|\alpha|=j} \int\limits_0^T \int_\Omega |D_t^r D_x^\alpha f(x,t)|^2 dxdt\big)^{1/2}$.

$W_2^{1,1}(\Omega^T)$ - the space with the norm

$$\|f\|_{W_2^{1,1}(\Omega^T)} = \Big(\int\limits_0^T (\|\frac{\partial f}{\partial t}(\cdot, t)\|^2_{L_2(\Omega)} + \|f(\cdot, t)\|^2_{W_2^1(\Omega)}) dt\Big)^{\frac{1}{2}}.$$

$W_2^{1,0}(\Omega)$ - the space with the norm $\|f\|_{W_2^{1,0}(\Omega)} = \big(\int\limits_0^T \|f(\cdot, t)\|^2_{W_2^1(\Omega)} dt\big)^{\frac{1}{2}}$.

$\overset{\circ}{W}{}_2^{1,0}(\Omega^T), \overset{\circ}{W}{}_2^{1,1}(\Omega^T)$ - subsets of spaces $W_2^{1,0}(\Omega^T), W_2^{1,1}(\Omega^T)$ consisting of functions such that $f(x,t)|_{\partial\Omega\times(0,T)} = 0$.

$W_2^{-1}(\Omega)$ - the dual space to $\overset{\circ}{W}{}_2^1(\Omega)$.

$\nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$ - gradient of the function f.

$\Delta f = \sum\limits_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$ - Laplacian operator of the function f.

$\mathbf{v} \cdot \mathbf{u} = \sum\limits_{i=1}^{n} v_i u_i$ - scalar product for functions $\mathbf{v}$ and $\mathbf{u}$.

$\mathbf{v} \times \mathbf{u} = (v_2u_3 - v_3u_2, v_3u_1 - v_1u_3, v_1u_2 - v_2u_1)$ if $\mathbf{v}$ and $\mathbf{u}$ are three-dimensional vectors, $\mathbf{v} \times \mathbf{u} = (0, 0, v_1u_2 - v_2u_1)$ if $\mathbf{v}$ and $\mathbf{u}$ are two-dimensional vectors.

$\nabla \cdot \mathbf{u} = \mathrm{div}\mathbf{u} = \sum\limits_{i=1}^{n} \frac{\partial u_i}{\partial x_i}$.

$\mathrm{curl}\mathbf{u} = \nabla \times \mathbf{u} = (\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2})$.

The operator curl in the two-dimensional case $\mathbf{u} = (u_1, u_2)$ we understand as usually in $\mathbb{R}^3$ assuming that vector-field has zero third component and the components are independent of x_3: $\mathrm{curl}\mathbf{u} = (0, 0, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2})$.

$\mathbf{A} : \mathbf{B} = \sum\limits_{ij=1}^{n} A_{ij}B_{ij}$ where A and B are $n \times n$ matrices with elements A_{ij}, B_{ij}.

We use letters $c,\ C, c_j,\ j = 1, 2, ...,$ to denote constants whose numerical values or whose dependence on parameters is unessential to our considerations. In such case c may have different values in a single computation.

1.2 Auxiliary results

Theorem 1.1. ***(Young Inequality with*** ε*) For all $a, b \in \mathbb{R}$ the following inequality*

$$|ab| \leq \frac{\varepsilon}{q}|a|^q + \frac{\varepsilon^{-q'/q}}{q'}|b|^{q'}, \qquad \forall \varepsilon > 0$$

holds, where $\frac{1}{q} + \frac{1}{q'} = 1$. If $q = 2$ we get, so called, ***Cauchy*** *inequality with ε:*

$$|ab| \leq \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2, \qquad \forall \varepsilon > 0.$$

Theorem 1.2. ***(Minkowski Inequality)*** *Assume $1 \leq p \leq \infty$ and $u, v \in L_p(\Omega)$. Then*

$$\|u + v\|_{L_p(\Omega)} \leq \|u\|_{L_p(\Omega)} + \|v\|_{L_p(\Omega)}.$$

Theorem 1.3. ***(Hölder Inequality)*** *(see [26]) Let $q > 1$, $f \in L_q(\Omega)$, $g \in L_{q'}(\Omega)$, where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, then*

$$|\int\limits_{\Omega} f(x)g(x)dx| \leq (\int\limits_{\Omega} |f(x)|^q dx)^{1/q} (\int\limits_{\Omega} |g(x)|^{q'} dx)^{1/q'} = ||f||_{L_q(\Omega)} ||g||_{L_{q'}(\Omega)}.$$

If $q = q' = 2$, the previous inequality is called ***Cauchy-Schwarz*** *inequality and takes the form*

$$|\int\limits_{\Omega} f(x)g(x)dx| \leq (\int\limits_{\Omega} |f(x)|^2 dx)^{1/2} (\int\limits_{\Omega} |g(x)|^2 dx)^{1/2} = ||f||_{L_2(\Omega)} ||g||_{L_2(\Omega)}.$$

Theorem 1.4. ***(Poincaré-Friedrichs Inequality)*** *(see [26]) Let $\Omega \subset \mathbb{R}^n$ is a bounded domain, then for all $u \in \overset{\circ}{W}{}_2^1(\Omega)$ the following inequality*

$$\int\limits_{\Omega} |u(x)|^2 d \leq \frac{1}{\lambda_1} \int\limits_{\Omega} |\nabla u(x)|^2 dx,$$

holds, where λ_1 is the smallest eigenvalue of the Dirichlet boundary problem for Laplace operator in Ω:

$$\begin{cases} \Delta u = \lambda u, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Theorem 1.5. ***(Sobolev Embedding)*** *(see [1]) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.*

- *If $l \geq 1$, $q \geq 1$,*

  $$n \geq ql, \quad r < \frac{qn}{n - ql},$$

  then the space $W_q^l(\Omega)$ is embedded into $L_r(\Omega)$ and

  $$||u||_{L_r(\Omega)} \leq c||u||_{W_q^l(\Omega)}.$$

- *If $n < ql$, then $W_q^l(\Omega)$ is embedded into $C^h(\bar{\Omega})$, where $h \leq (ql - n)/q$, and*

  $$||u||_{C^h(\bar{\Omega})} \leq c||u||_{W_q^l(\Omega)}.$$

Theorem 1.6. *(see [52]) If $u \in W_2^{2l,l}(\Omega^T)$, then $D_t^r D_x^\alpha u$ with $2r + |\alpha| < 2l - 1$ belong to* *the space* $W_2^{2l-2r-|\alpha|-1}(\Omega)$ *and there holds the inequality*

$$||D_t^r D_x^\alpha u(\cdot,t)||_{W_2^{2l-2r-|\alpha|-1}(\Omega)} \le c||u||_{W_2^{2l,l}(\Omega^T)}$$

with the constant c *independent of* $t \in [0,T]$.

Theorem 1.7. ***(Parseval Equality)*** *(see [88]) Let* $f \in L_2(0,2\pi)$, *then the following equality holds*

$$\frac{1}{\pi}\int\limits_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2),$$

where a_0, a_k, b_k *are the Fourier coefficients of the function f, e.i.* $a_0 = \int\limits_0^{2\pi} f(x)dx$, $a_k = \int\limits_0^{2\pi} f(x)\cos(kx)dx$, $\int\limits_0^{2\pi} f(x)\sin(kx)dx$.

Theorem 1.8. ***(Nečas Inequality)*** *(see [56]) Let* Ω *be a bounded domain. If* $u \in W_2^{-1}(\Omega)$, $\nabla u \in L_2(\Omega)$, *then* $u \in L_2(\Omega)$ *and the following inequality*

$$\|u\|_{L_2(\Omega)} \le c\Big(\|u\|_{W_2^{-1}(\Omega)} + \|\nabla u\|_{L_2(\Omega)}\Big)$$

holds.

Theorem 1.9. ***(Gronwall's inequality)*** *(see [26]) Let* η *be a nonnegative, absolutely continuous function on* $[0,T]$ *which satisfies for all* t *the differential inequality*

$$\eta'(t) \le \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ *and* $\psi(t)$ *are nonnegative, summable functions on* $[0,T]$. *Then*

$$\eta(t) \le exp\{\int\limits_0^t \phi(s)ds\}\Big(\eta(0) + \int\limits_0^t \psi(s)ds\Big)$$

for all $0 \le t \le T$.

In particular, if

$$\eta'(t) \le \phi(t)\eta(t) \quad on\ [0,T] \quad and\ \eta(0) = 0,$$

then

$$\eta(t) \equiv 0\ on\ [0,T].$$

Chapter 2

Channel flow

In this chapter we study the second grade fluid flow problem

$$\begin{cases} \frac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathrm{curl}(\mathbf{u} - \alpha\Delta\mathbf{u}) \times \mathbf{u} + \nabla\widetilde{p} = \mathbf{f}, \\ \mathrm{div}\mathbf{u} = 0, \\ \mathbf{u}|_{S^T} = 0, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x), \end{cases} \tag{2.1}$$

with the prescribed flux condition

$$\int_{-d/2}^{d/2} u_2(x_1, x_2, t)dx_1 = F(t). \tag{2.2}$$

in the channel $\Pi = \{x \in \mathbb{R}^2 : (x_1, x_2) \in \sigma \times \mathbb{R}\}$, where $\sigma = (-\frac{d}{2}, \frac{d}{2})$. We assume that the initial velocity $\mathbf{u}_0(x)$ and the external force $\mathbf{f}(x,t)$ are independent of the coordinate x_2 and have the forms

$$\mathbf{u}_0(x) = (0, u_0(x_1)), \qquad \mathbf{f}(x,t) = (0, f(x_1, t)).$$

Suppose that the following necessary compatibility condition

$$\int_{-d/2}^{d/2} u_0(x_1)dx_1 = F(0) \tag{2.3}$$

holds.

Deffinition. *The solution* $(\boldsymbol{u}(x,t), \widetilde{p}(x,t))$ *of problem* (2.1), (2.2) *such that*

$$\begin{aligned}&\boldsymbol{u}(x,t) = (0, U(x_1,t)),\\&\widetilde{p}(x,t) = -q(t)x_2 - \int\limits_{-d/2}^{x_1} U(y,t)\frac{\partial}{\partial y}\Big(U(y,t) - \alpha\frac{\partial^2 U(y,t)}{\partial y^2}\Big)dy + p_0(t),\end{aligned} \tag{2.4}$$

where $p_0(t)$ *is an arbitrary function, is called the Poiseuille type solution.*

Substituting expressions (2.4) into system (2.1), (2.2) we get, that the first equation in (2.1_1) and the divergence equation (2.1_3) are satisfied identically, while from (2.1_2) and the initial and boundary conditions we get the following inverse problem on $\sigma^T = \sigma \times (0,T)$:

$$\begin{cases}\dfrac{\partial}{\partial t}(U - \alpha\dfrac{\partial^2 U}{\partial x_1^2}) - \nu\dfrac{\partial^2 U}{\partial x_1^2} = q(t) + f,\\ U(-\dfrac{d}{2}, t) = U(\dfrac{d}{2}, t) = 0, \quad U(x_1, 0) = u_0(x_1),\\ \int\limits_{-d/2}^{d/2} U(x_1,t)dx_1 = F(t).\end{cases} \tag{2.5}$$

Notice that in (2.5) functions $u_0(x_1)$, $f(x_1,t)$ and $F(t)$ are given, while $U(x_1,t)$ and $q(t)$ are unknown and have to be found.

Denote by $\mathcal{M}(\sigma^T)$ the space of functions u such that $u \in \overset{\circ}{W}{}_2^{1,1}(\sigma^T)$ and $\dfrac{\partial^2 u}{\partial x_1 \partial t} \in L_2(\sigma^T)$.

Deffinition. *By a weak solution of problem* (2.5) *we understand a couple of functions* $(U,q) \in \mathcal{M}(\sigma^T) \times L_2(0,T)$ *satisfying for all* $t \in [0,T]$ *the integral identity*

$$\begin{aligned}&\int\limits_0^t\int\limits_\sigma \frac{\partial U(x_1,\tau)}{\partial \tau}\eta(x_1,\tau)dx_1d\tau + \alpha\int\limits_0^t\int\limits_\sigma \frac{\partial^2 U(x_1,\tau)}{\partial\tau\partial x_1}\frac{\partial\eta(x_1,\tau)}{\partial x_1}dx_1d\tau\\&+\nu\int\limits_0^t\int\limits_\sigma \frac{\partial U(x_1,\tau)}{\partial x_1}\frac{\partial\eta(x_1,\tau)}{\partial x_1}dx_1d\tau = \int\limits_0^t q(\tau)\int\limits_\sigma \eta(x_1,\tau)dx_1d\tau\\&+\int\limits_0^t\int\limits_\sigma f(x_1,\tau)\eta(x_1,\tau)dx_1d\tau \quad \forall \eta \in \overset{\circ}{W}{}_2^{1,0}(\sigma^T),\end{aligned} \tag{2.6}$$

the initial condition $U(x_1,0) = u_0(x_1)$ *and the flux condition* (2.5_3).

2.1 Construction of an approximate solution

We will argue similarly to the case of the Navier-Stokes equations (see [62]).

Let $\lambda_k = \dfrac{\nu\pi^2 k^2}{d^2}$ and $v_k(x_1) = \sqrt{\dfrac{2}{d}} \sin \dfrac{\pi k}{d}(x + \dfrac{d}{2})$ be eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\begin{cases} -\nu v_k''(x_1) = \lambda_k v_k(x_1), \\ v_k(-\frac{d}{2}) = v_k(\frac{d}{2}) = 0. \end{cases} \tag{2.7}$$

It is well known that the eigenfunctions v_k form a basis in $L_2(\sigma)$, $\overset{\circ}{W}{}_2^1(\sigma)$ and $W_2^2(\sigma)$; moreover,

$$\int\limits_{-d/2}^{d/2} v_k(x_1)v_l(x_1)dx_1 = \delta_{kl}, \quad \int\limits_{-d/2}^{d/2} v_k'(x_1)v_l'(x_1)dx_1 = 0, \quad k \neq l, \tag{2.8}$$

$$\nu \int\limits_{-d/2}^{d/2} |v_k'(x_1)|^2 dx_1 = \lambda_k, \tag{2.9}$$

where δ_{kl} - is the Kroneker delta.

The constant function $h(x_1) \equiv 1$ belongs to the space $L_2(\sigma)$ and therefore, it can be expressed as the Fourier series:

$$1 = \sum_{k=1}^{\infty} \beta_k v_k(x_1),$$

where $\beta_k = \int\limits_{-d/2}^{d/2} v_k(x_1)dx_1 = \dfrac{\sqrt{2d}}{\pi k}\big(1 + (-1)^{k+1}\big), k = 1, 2, ...,$ and $\sum\limits_{k=1}^{\infty} \beta_k^2 = d$. If $f \in L_2(\sigma^T)$, $u_0 \in \overset{\circ}{W}{}_2^1(\sigma)$, then

$$f(x_1, t) = \sum_{k=1}^{\infty} f_k(t)v_k(x_1), \quad u_0(x_1) = \sum_{k=1}^{\infty} a_k v_k(x_1)$$

with $f_k(t) = \int\limits_{-d/2}^{d/2} f_2(x_1, t)v_k(x_1)dx_1$, $a_k = \int\limits_{-d/2}^{d/2} u_0(x_1)v_k(x_1)dx_1$.

An approximate solutions $(U^{(N)}(x_1, t), q^{(N)}(t))$ of problem (2.5) are found as solutions of the following problems

$$\begin{cases} \dfrac{\partial}{\partial t}\big(U^{(N)}-\alpha\dfrac{\partial^2U^{(N)}}{\partial x_1^2}\big)-\nu\dfrac{\partial^2U^{(N)}}{\partial x_1^2} = q^{(N)}(t)\sum\limits_{k=1}^{N}\beta_k v_k(x_1)+\sum\limits_{k=1}^{N} f_k(t)v_k(x_1), \\ U^{(N)}(-\dfrac{d}{2},t) = U^{(N)}(\dfrac{d}{2},t) = 0, \quad U^{(N)}(x_1,0) = \sum\limits_{k=1}^{N} a_k v_k(x_1), \\ \int\limits_{-d/2}^{d/2} U^{(N)}(x_1,t)dx_1 = F(t). \end{cases} \tag{2.10}$$

We look for $U^{(N)}(x_1,t)$ in the form

$$U^{(N)}(x_1,t) = \sum_{k=1}^{N} w_k^{(N)}(t)v_k(x_1).$$

Substituting $U^{(N)}$ into equation (2.10) we easily find that

$$\begin{aligned} U^{(N)}(x_1,t) = \sum_{k=1}^{N}\Big(&\frac{\nu}{\nu+\alpha\lambda_k}\int\limits_0^t e^{\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}(\tau-t)}\big(\beta_k q^{(N)}(\tau)+f_k(\tau)\big)d\tau \\ &+e^{-\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}t}a_k\Big)v_k(x_1). \end{aligned} \tag{2.11}$$

Now, we find $q^{(N)}(t)$, in order to satisfy the flux condition (2.10_3). We have

$$\begin{aligned} \int\limits_{-d/2}^{d/2} U^{(N)}(x_1,t)dx_1 &= \sum_{k=1}^{N}\frac{\beta_k\nu}{\nu+\alpha\lambda_k}\int\limits_0^t e^{\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}(\tau-t)}q^{(N)}(\tau)d\tau\int\limits_{-d/2}^{d/2} v_k(x_1)dx_1 \\ &+\sum_{k=1}^{N}\Big(\frac{\nu}{\nu+\alpha\lambda_k}\int\limits_0^t e^{\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}(\tau-t)}f_k(\tau)d\tau+e^{-\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}t}a_k\Big)\int\limits_{-d/2}^{d/2} v_k(x_1)dx_1 = F(t). \end{aligned}$$

Thus, the function $q^{(N)}(t)$ is the solution of the Volterra integral equation of the first kind:

$$\begin{aligned} &\sum_{k=1}^{N}\frac{\beta_k^2\nu}{\nu+\alpha\lambda_k}\int\limits_0^t e^{\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}(\tau-t)}q^{(N)}(\tau)d\tau = F(t) \\ &-\sum_{k=1}^{N}\Big(\frac{\beta_k\nu}{\nu+\alpha\lambda_k}\int\limits_0^t e^{\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}(\tau-t)}f_k(\tau)d\tau+e^{-\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}t}a_k\beta_k\Big). \end{aligned} \tag{2.12}$$

Differentiating equation (2.12) we get the Volterra integral equation of the second kind

$$q^{(N)}(t)-\int\limits_0^t K^{(N)}(t,\tau)q^{(N)}(\tau)d\tau = \Phi^{(N)}(t), \tag{2.13}$$

where the kernel $K^{(N)}(t,\tau)$ is given by

$$K^{(N)}(t,\tau)=\frac{1}{\chi_N}\sum_{k=1}^{N}\frac{\beta_k^2\lambda_k\nu^2}{(\nu+\alpha\lambda_k)^2}e^{\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}(\tau-t)},$$
$$\chi_N=\sum_{k=1}^{N}\frac{\beta_k^2\nu}{\nu+\alpha\lambda_k}, \tag{2.14}$$

and

$$\Phi^{(N)}(t)=\frac{1}{\chi_N}\Big(F'(t)-\sum_{k=1}^{N}\frac{\beta_k\nu}{\nu+\alpha\lambda_k}\frac{d}{dt}\int_0^t e^{\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}(\tau-t)}f_k(\tau)d\tau+$$
$$+\sum_{k=1}^{N}\frac{\beta_k\lambda_k\nu}{\nu+\alpha\lambda_k}e^{-\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}t}a_k\Big). \tag{2.15}$$

It is well known that (2.13) admits a unique solution $q^{(N)}\in L_2(0,T)$ and

$$\|q^{(N)}\|_{L_2(0,T)}\le c_N\|\Phi^{(N)}\|_{L_2(0,T)}. \tag{2.16}$$

A priori we do not know if the constant c_N is independent of N. Below we will prove that c_N can be taken independently of N.

2.2 A priori estimates

Let us consider a "sufficiently smooth" solution $U(x_1,t)$ of the following initial boundary value problem

$$\begin{cases}\dfrac{\partial}{\partial t}(U-\alpha\dfrac{\partial^2U}{\partial x_1^2})-\nu\dfrac{\partial^2U}{\partial x_1^2}=M,\\ U(-\dfrac{d}{2},t)=U(\dfrac{d}{2},t)=0,\quad U(x_1,0)=\varphi(x_1).\end{cases} \tag{2.17}$$

Lemma 2.1. *Suppose that $M\in L_2(\sigma^T),\varphi\in \overset{\circ}{W}{}_2^1(\sigma)\cap W_2^2(\sigma)$, and let $U\in W_2^{2,1}(\sigma^T)$ be a solution of problem (2.17) such that $\dfrac{\partial U}{\partial t}\in W_2^2(\sigma^T)$, $\dfrac{\partial U}{\partial t}\Big|_{\partial\sigma}=0$. Then the following estimates*

$$\max_{t\in[0,T]}\|U(\cdot,t)\|^2_{W_2^1(\sigma)}+\|U\|^2_{W_2^{1,1}(\sigma^T)}+\Big\|\frac{\partial U}{\partial t}\Big\|^2_{W_2^1(\sigma^T)}$$
$$\le c\big(\|M\|^2_{L_2(\sigma^T)}+\|\varphi\|^2_{W_2^1(\sigma)}\big), \tag{2.18}$$

$$\max_{t\in[0,T]}\|U(\cdot,t)\|^2_{W_2^2(\sigma)}+\|U\|^2_{W_2^{2,1}(\sigma^T)}+\Big\|\frac{\partial U}{\partial t}\Big\|^2_{W_2^2(\sigma^T)}$$
$$\le c\big(\|M\|^2_{L_2(\sigma^T)}+\|\varphi\|^2_{W_2^2(\sigma)}\big) \tag{2.19}$$

hold.

Proof. Multiply equation (2.17) by $U(x_1,t)$, integrate by parts on the interval $(-\frac{d}{2},\frac{d}{2})$ and then integrate with respect to t. Using Cauchy, Cauchy-Schwarz and Poincaré-

Friedrichs inequalities we get

$$\begin{gathered}\int\limits_{-d/2}^{d/2}\big(|U(x_1,t)|^2+\alpha\Big|\frac{\partial U(x_1,t)}{\partial x_1}\Big|^2\big)dx_1+\nu\int\limits_0^t\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U(x_1,\tau)}{\partial x_1}\Big|^2dx_1d\tau\\ \le c\int\limits_0^t\int\limits_{-d/2}^{d/2}|M(x_1,\tau)|^2dx_1d\tau+\int\limits_{-d/2}^{d/2}\big(|\varphi(x_1)|^2+\alpha\Big|\frac{\partial\varphi(x_1)}{\partial x_1}\Big|^2\big)dx_1.\end{gathered}\tag{2.20}$$

Analogously, multiplying (2.17) by $\frac{\partial U(x_1,t)}{\partial t}$, integrating by parts on the interval $(-\frac{d}{2},\frac{d}{2})$ and then with respect to time yield the estimate

$$\begin{gathered}\int\limits_0^t\int\limits_{-d/2}^{d/2}\Big(\Big|\frac{\partial U(x_1,\tau)}{\partial\tau}\Big|^2+2\alpha\Big|\frac{\partial^2U(x_1,\tau)}{\partial x_1\partial\tau}\Big|^2\Big)dx_1d\tau+\nu\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U(x_1,t)}{\partial x_1}\Big|^2dx_1\\ \le 2\int\limits_0^t\int\limits_{-d/2}^{d/2}|M(x_1,\tau)|^2dx_1d\tau+\nu\int\limits_{-d/2}^{d/2}\Big|\frac{\partial\varphi(x_1)}{\partial x_1}\Big|^2dx_1.\end{gathered}\tag{2.21}$$

Finally, multiplying (2.17) by $-\frac{\partial^2U(x_1,t)}{\partial x_1^2}$ and by $-\frac{\partial^3U(x_1,t)}{\partial x_1^2\partial t}$ we derive

$$\begin{gathered}\int\limits_{-d/2}^{d/2}\Big(\Big|\frac{\partial U(x_1,t)}{\partial x_1}\Big|^2+\alpha\Big|\frac{\partial^2U(x_1,t)}{\partial x_1^2}\Big|^2\Big)dx_1+\nu\int\limits_0^t\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2U(x_1,\tau)}{\partial x_1^2}\Big|^2dx_1d\tau\\ \le c\int\limits_0^t\int\limits_{-d/2}^{d/2}|M(x_1,\tau)|^2dx_1d\tau+\int\limits_{-d/2}^{d/2}\Big(\Big|\frac{\partial\varphi(x_1)}{\partial x_1}\Big|^2+\alpha\Big|\frac{\partial^2\varphi(x_1)}{\partial x_1^2}\Big|^2\Big)dx_1\end{gathered}\tag{2.22}$$

and

$$\begin{gathered}\int\limits_0^t\int\limits_{-d/2}^{d/2}\Big(\Big|\frac{\partial^2U(x_1,\tau)}{\partial x_1\partial\tau}\Big|^2+\frac{\alpha}{2}\Big|\frac{\partial^3U(x_1,\tau)}{\partial x_1^2\partial\tau}\Big|^2\Big)dx_1d\tau+\frac{\nu}{2}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2U(x_1,t)}{\partial x_1^2}\Big|^2dx_1\\ \le c\int\limits_0^t\int\limits_{-d/2}^{d/2}|M(x_1,\tau)|^2dx_1d\tau+\frac{\nu}{2}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2\varphi(x_1)}{\partial x_1^2}\Big|^2dx_1.\end{gathered}\tag{2.23}$$

Estimate (2.18) follows from (2.20), (2.21), while estimate (2.19) - from (2.18), (2.22), (2.23). □

Consider now the approximate solution $U^{(N)}(x_1,t)$ of problem (2.5) constructed in Section 2.1. Since $U^{(N)}(x_1,t)$ is expressed as a finite sum (2.11), it satisfies assumptions of Lemma 2.1 and, therefore, considering $q^{(N)}(t)$ as a given right-hand side, we get the following

Lemma 2.2. *For the approximate solution $U^{(N)}(x_1,t)$ of problem* (2.5) *the following estimates*

$$\max_{t\in[0,T]}\|U^{(N)}(\cdot,t)\|^2_{W_2^1(\sigma)}+\|U^{(N)}\|^2_{W_2^{1,1}(\sigma^T)}+\Big\|\frac{\partial U^{(N)}}{\partial t}\Big\|^2_{W_2^1(\sigma^T)} \\ \le c\big(\|f^{(N)}\|^2_{L_2(\sigma^T)}+\|q^{(N)}h^{(N)}\|^2_{L_2(\sigma^T)}+\|u_0^{(N)}\|^2_{W_2^1(\sigma)}\big), \tag{2.24}$$

and

$$\max_{t\in[0,T]}\|U^{(N)}(\cdot,t)\|^2_{W_2^2(\sigma)}+\|U^{(N)}\|^2_{W_2^{2,1}(\sigma^T)}+\Big\|\frac{\partial U^{(N)}}{\partial t}\Big\|^2_{W_2^2(\sigma^T)} \\ \le c\big(\|f^{(N)}\|^2_{L_2(\sigma^T)}+\|q^{(N)}h^{(N)}\|^2_{L_2(\sigma^T)}+\|u_0^{(N)}\|^2_{W_2^2(\sigma)}\big) \tag{2.25}$$

hold. Here $f^{(N)}(x_1,t)=\sum\limits_{k=1}^{N}f_k(t)v_k(x_1)$, $h^{(N)}(x_1)=\sum\limits_{k=1}^{N}\beta_kv_k(x_1)$, $u_0^{(N)}(x_1)=\sum\limits_{k=1}^{N}a_kv_k(x_1)$ *and the constants in* (2.24) *and* (2.25) *do not depend on* N.

Next, we have to estimate the right-hand sides of inequalities (2.24), (2.25). Obviously,

$$\|f^{(N)}\|^2_{L_2(\sigma^T)}+\|u_0^{(N)}\|^2_{W_2^1(\sigma)}\le c\big(\|f\|^2_{L_2(\sigma^T)}+\|u_0\|^2_{W_2^1(\sigma)}\big),$$

and, if $u_0\in\overset{\circ}{W}{}_2^1(\sigma)\cap W_2^2(\sigma)$, then

$$\|u_0^{(N)}\|^2_{W_2^2(\sigma)}\le c\|u_0\|^2_{W_2^2(\sigma)}.$$

Constants in the last two inequalities are independent of N.

Let us consider the kernel $K^{(N)}(t,\tau)$ of the integral equation (2.13). Denote $\mathbb{Q}^T=(0,T)\times(0,T)$. Then

$$\begin{aligned}\chi_N\|K^{(N)}(t,\tau)\|_{L_2(\mathbb{Q}^T)}&\le\sum_{k=1}^{N}\Big\|\frac{\beta_k^2\lambda_k\nu^2}{(\nu+\alpha\lambda_k)^2}\,e^{\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}(\tau-t)}\Big\|_{L_2(\mathbb{Q}^T)}\\ &=\sum_{k=1}^{N}\Big(\int_0^T\int_0^T\frac{\beta_k^4\lambda_k^2\nu^4}{(\nu+\alpha\lambda_k)^4}\,e^{\frac{2\nu\lambda_k}{\nu+\alpha\lambda_k}(\tau-t)}dtd\tau\Big)^{1/2}\\ &=\sum_{k=1}^{N}\frac{\beta_k^2\nu}{2(\nu+\alpha\lambda_k)}\sqrt{e^{\frac{-2T\nu\lambda_k}{\nu+\alpha\lambda_k}}+e^{\frac{2T\nu\lambda_k}{\nu+\alpha\lambda_k}}-2}\\ &\le\sum_{k=1}^{N}\frac{\beta_k^2}{2}\sqrt{\frac{1+e^{\frac{4T\nu\lambda_k}{\nu+\alpha\lambda_k}}}{e^{\frac{2T\nu\lambda_k}{\nu+\alpha\lambda_k}}}}\le\sum_{k=1}^{N}\frac{\beta_k^2}{2}\sqrt{\frac{2e^{\frac{4T\nu\lambda_k}{\nu+\alpha\lambda_k}}}{e^{\frac{2T\nu\lambda_k}{\nu+\alpha\lambda_k}}}}\le\sum_{k=1}^{N}\frac{\beta_k^2}{\sqrt2}\,e^{\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}T}\\ &\le\sum_{k=1}^{N}\frac{\beta_k^2}{\sqrt2}\,e^{\frac{\nu}{\alpha}T}\le\frac{d}{\sqrt2}\,e^{\frac{\nu}{\alpha}T}.\end{aligned}\tag{2.26}$$

Since $\lim\limits_{N\to\infty}\chi_N = \sum\limits_{k=1}^{\infty}\dfrac{\beta_k^2\nu}{\nu+\alpha\lambda_k} = \chi_* < \infty$, the truncated kernels $K^{(N)}(t,\tau)$ converge in the norm of $L_2(\mathbb{Q}_T)$ to

$$K(t,\tau) = \chi_*^{-1}\sum_{k=1}^{\infty}\frac{\beta_k^2\lambda_k\nu^2}{(\nu+\alpha\lambda_k)^2}\, e^{\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}(\tau-t)}.$$

From (2.26) it follows that the constant c_N in inequality (2.16) could be chosen independent of N. Notice that for the Navier-Stokes equations the trancated kernels converge only in the norm of $L_1(\mathbb{Q}_T)$ (see [63], [36]).

The norm of the function $\Phi^{(N)}$ (see (2.15)) can be estimated as follows:

$$\begin{aligned}
&\chi_N\|\Phi^{(N)}\|_{L_2(0,T)} \le \|F'\|_{L_2(0,T)} + \sum_{k=1}^{N}\frac{\beta_k\nu}{\nu+\alpha\lambda_k}\|f_k\|_{L_2(0,T)}\\
&+\sum_{k=1}^{N}\frac{\beta_k\lambda_k\nu^2}{(\nu+\alpha\lambda_k)^2}\Big\|\int_0^t e^{\frac{\lambda_k\nu}{\nu+\alpha\lambda_k}(\tau-t)} f_k(\tau)d\tau\Big\|_{L_2(0,T)}\\
&+\sum_{k=1}^{N}\frac{\beta_k\lambda_k a_k\nu}{(\nu+\alpha\lambda_k)}\|e^{-\frac{\lambda_k\nu}{\nu+\alpha\lambda_k}t}\|_{L_2(0,T)}\\
&\le \|F'\|_{L_2(0,T)} + \Big(\sum_{k=1}^{N}\beta_k^2\Big)^{1/2}\Big(\sum_{k=1}^{N}\|f_k\|^2_{L_2(0,T)}\Big)^{1/2}\\
&+\frac{\nu}{\alpha}\Big(\sum_{k=1}^{N}\beta_k^2\Big)^{1/2}\Big(\sum_{k=1}^{N}\|f_k\|^2_{L_2(0,T)}\Big)^{1/2} + \sqrt{\frac{\nu}{\alpha}}\Big(\sum_{k=1}^{N}\beta_k^2\Big)^{1/2}\Big(\sum_{k=1}^{N}a_k^2\Big)^{1/2}\\
&\le c\big(\|F'\|_{L_2(0,T)} + \|f\|_{L_2(\sigma^T)} + \|u_0\|_{L_2(\sigma)}\big).
\end{aligned}\tag{2.27}$$

Now (2.16), (2.26), (2.27) yield

$$\|q^{(N)}\|_{L_2(0,T)} \le c\big(\|F'\|_{L_2(0,T)} + \|f\|_{L_2(\sigma^T)} + \|u_0\|_{L_2(\sigma)}\big).\tag{2.28}$$

Applying estimate (2.28) together with inequalities (2.24), (2.25) we prove the following

Lemma 2.3. *Suppose that $f\in L_2(\sigma^T), u_0\in \overset{\circ}{W}{}_2^1(\sigma), F\in W_2^1(0,T)$. Then for the approximate solution $\big(U^{(N)}(x_1,t), q^{(N)}(t)\big)$ of problem (2.5) the following estimate*

$$\begin{aligned}
&\max_{t\in[0,T]}\|U^{(N)}(\cdot,t)\|^2_{W_2^1(\sigma)} + \|U^{(N)}\|^2_{W_2^{1,1}(\sigma^T)} + \Big\|\frac{\partial U^{(N)}}{\partial t}\Big\|^2_{W_2^1(\sigma^T)}\\
&+\|q^{(N)}\|_{L_2(0,T)} \le c\big(\|f\|^2_{L_2(\sigma^T)} + \|F'\|^2_{L_2(0,T)} + \|u_0\|^2_{W_2^1(\sigma)}\big)
\end{aligned}\tag{2.29}$$

holds. If, in addition $u_0\in \overset{\circ}{W}{}_2^1(\sigma)\cap W_2^2(\sigma)$, then

$$
\begin{aligned}
&\max_{t\in[0,T]} \|U^{(N)}(\cdot,t)\|^2_{W_2^2(\sigma)} + \|U^{(N)}\|^2_{W_2^{2,1}(\sigma^T)} + \Big\|\frac{\partial U^{(N)}}{\partial t}\Big\|^2_{W_2^2(\sigma^T)} \\
&+\|q^{(N)}\|_{L_2(0,T)} \leq c\big(\|f\|^2_{L_2(\sigma^T)} + \|F'\|^2_{L_2(0,\,T)} + \|u_0\|^2_{W_2^2(\sigma)}\big).
\end{aligned} \tag{2.30}
$$

Let $q \in L_2(0,T)$ be a unique solution of the integral equation

$$
q(t) - \int\limits_0^t K(t,\tau)q(\tau)d\tau = \Phi(t), \tag{2.31}
$$

where

$$
\Phi(t) = \frac{1}{\chi_*}\big(F'(t) - \sum_{k=1}^{\infty} \frac{\beta_k\nu}{\nu+\alpha\lambda_k}\frac{d}{dt}\int\limits_0^t e^{\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}(\tau-t)} f_k(\tau)d\tau + \sum_{k=1}^{\infty}\frac{\beta_k\lambda_k\nu}{\nu+\alpha\lambda_k}e^{-\frac{\nu\lambda_k}{\nu+\alpha\lambda_k}t}a_k\big).
$$

Then

$$
\|q\|_{L_2(0,T)} \leq c\|\Phi\|_{L_2(0,T)} \leq c\big(\|F'\|_{L_2(0,T)} + \|f\|_{L_2(\sigma^T)} + \|u_0\|_{L_2(\sigma)}\big). \tag{2.32}
$$

Subtracting the integral equation (2.31) from (2.13) we get

$$
\begin{aligned}
&q^{(N)}(t) - q(t) - \int\limits_0^t K^{(N)}(t,\tau)(q^{(N)}(\tau) - q(\tau))d\tau \\
&= \int\limits_0^t (K^{(N)}(t,\tau) - K(t,\tau))q(\tau)d\tau + \Phi^{(N)}(t) - \Phi(t).
\end{aligned}
$$

Applying to the difference $(q^{(N)} - q)$ estimate (2.16) yields

$$
\begin{aligned}
&\|q^{(N)} - q\|_{L_2(0,T)} \\
&\leq c\big(\Big\|\int\limits_0^T (K^{(N)}(t,\tau) - K(t,\tau))q(\tau)d\tau\Big\|_{L_2(0,T)} + \|\Phi^{(N)} - \Phi\|_{L_2(0,T)}\big) \\
&\leq c\big(\|K^{(N)} - K\|_{L_2(\mathbb{Q}^T)}\|q\|_{L_2(0,T)} + \|\Phi^{(N)} - \Phi\|_{L_2(0,T)}\big).
\end{aligned} \tag{2.33}
$$

Obviously, $\Phi^{(N)} \to \Phi$ in $L_2(0,T)$. Therefore, we can pass in (2.33) to the limit as $N \to \infty$ and we get that

$$
\lim_{N\to\infty} \|q^{(N)} - q\|_{L_2(0,T)} = 0. \tag{2.34}
$$

Remark 2.1. Notice that the integral equation (2.28) gives the relation between the flux $F(t)$ and the pressure drop $q(t)$ (for the Navier-Stokes case see [36].

2.3 Decay of the approximate solution as $t \to \infty$

For simplicity we assume that $f(x_1,t) \equiv 0$. Let $\mathcal{W}^l_{2,\mu}(0,\infty), \mu > 0$, be the space of exponentially vanishing functions with the finite norm

$$\|F\|_{\mathcal{W}^l_{2,\mu}(0,\infty)} = \|\exp(\mu t)F(t)\|_{W^l_2(0,\infty)}.$$

Lemma 2.4. *Let $f(x_1,t) = 0$, $u_0 \in \overset{\circ}{W}{}^1_2(\sigma)$ and $F \in \mathcal{W}^1_{2,\mu}(0,\infty)$ with $\mu > 0$. Then for sufficiently large N the solution $\big(U^{(N)}(x_1,t), q^{(N)}(t)\big)$ of problem (2.10) satisfies the estimate*

$$\begin{aligned}
&\exp(\gamma^* t)\int\limits_{-d/2}^{d/2}\big(|U^{(N)}(x_1,t)|^2 + \Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2\big)dx_1\\
&+\int\limits_0^t\int\limits_{-d/2}^{d/2}\exp(\gamma^*\tau)\Big(\Big|\frac{\partial U^{(N)}(x_1,\tau)}{\partial \tau}\Big|^2 + \Big|\frac{\partial^2 U^{(N)}(x_1,\tau)}{\partial x_1\partial\tau}\Big|^2\Big)dx_1 d\tau\\
&+\int\limits_0^t \exp(\gamma^*\tau)|q^{(N)}(\tau)|^2 d\tau \le c\big(\|F\|^2_{\mathcal{W}^l_{2,\mu}(0,\infty)} + \|u_0\|^2_{W^1_2(\sigma)}\big),
\end{aligned}\tag{2.35}$$

where

$$\gamma_* = \begin{cases} \min\big(1,2\mu\big), & if \quad \min\big(1,2\mu\big) < \dfrac{\nu\pi^2}{d^2},\\ \dfrac{\nu\pi^2}{d^2} - \delta & if \quad \min\big(1,2\mu\big) \ge \dfrac{\nu\pi^2}{d^2}, \end{cases}$$

$\delta > 0$ is arbitrary small number. The constant c in (2.35) is independent of t.

Proof. Multiplying equations (2.10) by $U^{(N)}(x',t)$, integrating over $(-\frac{d}{2},\frac{d}{2})$, and using Cauchy inequality with ε we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{-d/2}^{d/2}\big(|U^{(N)}(x_1,t)|^2 + \alpha\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2\big)dx_1 + \nu\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1\\
&= q^{(N)}(t)\int\limits_{-d/2}^{d/2}U^{(N)}(x_1,t)dx_1 + q^{(N)}(t)\int\limits_{-d/2}^{d/2}(h^{(N)}(x_1)-1)U^{(N)}(x_1,t)dx_1\\
&\le \varepsilon|q^{(N)}(t)|^2 + \frac{1}{2\varepsilon}|F(t)|^2 + \frac{1}{2\varepsilon}\delta(N)\int\limits_{-d/2}^{d/2}|U^{(N)}(x_1,t)|^2 dx_1,
\end{aligned}\tag{2.36}$$

where $\delta(N) = \|h^{(N)} - 1\|^2_{L_2(\sigma)} \to 0$, $h^{(N)}(x_1) = \sum\limits_{k=1}^N \beta_k v_k(x_1)$.

Analogously, multiplying (2.10) by $\dfrac{\partial U^{(N)}(x',t)}{\partial t}$, we get the inequality (notice that by construction $\dfrac{\partial U^{(N)}(\pm\frac{d}{2},t)}{\partial t} = 0$):

$$
\begin{aligned}
&\int\limits_{-d/2}^{d/2} \Big(\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial t}\Big|^2 + \alpha\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1 \partial t}\Big|^2\Big) dx_1 \\
&+\frac{\nu}{2}\frac{d}{dt}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1 \le \varepsilon |q^{(N)}(t)|^2 + \frac{1}{2\varepsilon}\Big|\frac{d}{dt}F(t)\Big|^2 \\
&+\frac{1}{2\varepsilon}\delta(N)\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial t}\Big|^2 dx_1,
\end{aligned}
\tag{2.37}
$$

Let $v_0(x_1) = \frac{1}{2}\big(\frac{d^2}{4} - x_1^2\big)$, $\kappa_0 = \int\limits_{-d/2}^{d/2} v_0(x_1)dx_1 = \frac{d^3}{12}$. Then

$$
\begin{aligned}
&-\int\limits_{-d/2}^{d/2} \frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2} v_0(x_1)dx_1 = -\int\limits_{-d/2}^{d/2} U^{(N)}(x_1,t)\frac{\partial^2 v_0(x_1)}{\partial x_1^2}dx_1 \\
&= \int\limits_{-d/2}^{d/2} U^{(N)}(x_1,t)dx_1 = F(t).
\end{aligned}
$$

Therefore, multiplying (2.10) by $v_0(x_1)$ and integrating by parts we obtain

$$
\begin{aligned}
&\int\limits_{-d/2}^{d/2} \frac{\partial U^{(N)}(x_1,t)}{\partial t} v_0(x_1)dx_1 + \alpha\frac{d}{dt}F(t) + \nu F(t) = q^{(N)}(t)\kappa_0 \\
&+q^{(N)}(t)\int\limits_{-d/2}^{d/2}\big(h^{(N)}(x_1) - 1\big)v_0(x_1)dx_1.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|q^{(N)}(t)|^2\kappa_0^2 \le c\Big(|F(t)|^2 + \Big|\frac{d}{dt}F(t)\Big|^2 \\
&+\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial t}\Big|^2 dx_1 + |q^{(N)}(t)|^2\|h^{(N)} - 1\|^2_{L_2(\sigma)}\Big).
\end{aligned}
\tag{2.38}
$$

Taking in (2.38) N so that $\delta(N) \le \kappa_0^2/2$ yields the estimate

$$
|q^{(N)}(t)|^2 \le c\Big(|F(t)|^2 + \Big|\frac{d}{dt}F(t)\Big|^2 + \int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial t}\Big|^2 dx_1\Big). \tag{2.39}
$$

Now, inequalities (2.36), (2.37) and (2.39) give

$$
\frac{1}{2}\frac{d}{dt}\int\limits_{-d/2}^{d/2}\Big(|U^{(N)}(x_1,t)|^2 + (\alpha+\nu)\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2\Big)dx_1 + \nu\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1
$$

$$+\int\limits_{-d/2}^{d/2}\Big(\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial t}\Big|^2+\alpha\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1\partial t}\Big|^2\Big)dx_1\le c\big(|F(t)|^2+|\frac{d}{dt}F(t)|^2\big)$$
$$+c_1\varepsilon\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial t}\Big|^2dx_1+\frac{1}{2\varepsilon}\delta(N)\int\limits_{-d/2}^{d/2}\big|U^{(N)}(x_1,t)\big|^2dx_1$$
$$+\frac{1}{2\varepsilon}\delta(N)\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial t}\Big|^2dx_1.$$

Taking in the last inequality $\varepsilon=\dfrac{1}{4c_1}$ and N so large that $2c_1\delta(N)\le\dfrac{1}{4}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\limits_{-d/2}^{d/2}\big(|U^{(N)}(x_1,t)|^2+(\alpha+\nu)\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2\big)dx_1\\&+\nu\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2dx_1+\int\limits_{-d/2}^{d/2}\Big(\frac{1}{4}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial t}\Big|^2dx'\\&+\alpha\Big|\frac{\partial^2U^{(N)}(x_1,t)}{\partial x_1\partial t}\Big|^2\Big)dx_1\le c\big(|F(t)|^2+|\frac{d}{dt}F(t)|^2\big)\\&+2c_1\delta(N)\int\limits_{-d/2}^{d/2}\big|U^{(N)}(x_1,t)\big|^2dx_1.\end{aligned}\tag{2.40}$$

Because of the Poincaré-Friedrichs inequality

$$\int\limits_{-d/2}^{d/2}|U^{(N)}(x_1,t)|^2dx_1\le\frac{d^2}{\nu\pi^2}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2dx_1,$$

from (2.40) follows (for sufficiently large N) the estimate

$$\begin{aligned}&\frac{d}{dt}\int\limits_{-d/2}^{d/2}\big(|U^{(N)}(x_1,t)|^2+(\alpha+\nu)\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2\big)dx_1\\&+\gamma^*\int\limits_{-d/2}^{d/2}\big(|U^{(N)}(x_1,t)|^2+(\alpha+\nu)\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2\big)dx_1\\&+\int\limits_{-d/2}^{d/2}\Big(\frac{1}{2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial t}\Big|^2+\alpha\Big|\frac{\partial^2U^{(N)}(x_1,t)}{\partial x_1\partial t}\Big|^2\Big)dx_1\\&\le c\big(|F(t)|^2+|\frac{d}{dt}F(t)|^2\big),\end{aligned}\tag{2.41}$$

where γ^* is defined in the formulation of the lemma. Multiplying (2.41) by $\exp(\gamma^* t)$ and integrating with respect to t we receive the inequality

$$
\begin{aligned}
&\exp(\gamma^* t)\int\limits_{-d/2}^{d/2}\big(|U^{(N)}(x_1,t)|^2+(\alpha+\nu)\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2\big)dx_1\\
&+\int\limits_0^t\int\limits_{-d/2}^{d/2}\exp(\gamma^* t)\big(\frac{1}{2}\Big|\frac{\partial U^{(N)}(x_1,\tau)}{\partial \tau}\Big|^2+\alpha\Big|\frac{\partial^2 U^{(N)}(x_1,\tau)}{\partial x_1\partial\tau}\Big|^2\big)dx_1 d\tau\\
&\le c\int\limits_0^\infty \exp(\gamma^*\tau)\big(|F(t)|^2+|\frac{d}{dt}F(t)|^2\big)dt+c\|u_0\|^2_{W_2^1(\sigma)}\\
&\le c\big(\|F\|^2_{\mathcal{W}^l_{2,\mu}(0,\infty)}+\|u_0\|^2_{W_2^1(\sigma)}\big).
\end{aligned}
\tag{2.42}
$$

Finally, estimates (2.39) and (2.42) give

$$
\int\limits_0^t \exp(\gamma^*\tau)|q^{(N)}(\tau)|^2 d\tau \le c\big(\|F\|^2_{\mathcal{W}^l_{2,\mu}(0,\infty)}+\|u_0\|^2_{W_2^1(\sigma)}\big).
\tag{2.43}
$$

Estimate (2.35) follows from (2.42) and (2.43). □

2.4 Existence and uniqueness of the solution

Theorem 2.1. *Suppose that $f\in L_2(\sigma^T), u_0\in \mathring{W}_2^1(\sigma), F\in W_2^1(0,T)$. Then problem (2.5) admits a unique week solution $\big(U,q\big)\in\mathcal{M}(\sigma^T)\times L_2(0,T)$ and the following estimate*

$$
\begin{aligned}
&\max_{t\in[0,T]}\|U(\cdot,t)\|^2_{W_2^1(\sigma)}+\|U\|^2_{W_2^{1,1}(\sigma^T)}+\Big\|\frac{\partial U}{\partial t}\Big\|^2_{W_2^1(\sigma^T)}+\|q\|_{L_2(0,T)}\\
&\le c\big(\|F\|^2_{W_2^1(0,T)}+\|f\|^2_{L_2(\sigma^T)}+\|u_0\|^2_{W_2^1(\sigma)}\big)
\end{aligned}
\tag{2.44}
$$

holds.

If, in addition $u_0\in\mathring{W}_2^1(\sigma)\cap W_2^2(\sigma)$, then $U\in W_2^{2,1}(\sigma^T)$, $\dfrac{\partial U}{\partial t}\in W_2^2(\sigma^T)$, the equations (2.5) are satisfied almost everywhere in σ^T, and

$$
\begin{aligned}
&\max_{t\in[0,T]}\|U(\cdot,t)\|^2_{W_2^2(\sigma)}+\|U\|^2_{W_2^{2,1}(\sigma^T)}+\Big\|\frac{\partial U}{\partial t}\Big\|^2_{W_2^2(\sigma^T)}+\|q\|_{L_2(0,T)}\\
&\le c\big(\|F\|^2_{W_2^1(0,T)}+\|f\|^2_{L_2(\sigma^T)}+\|u_0\|^2_{W_2^2(\sigma)}\big).
\end{aligned}
\tag{2.45}
$$

Proof. Multiplying equations (2.10) by arbitrary $\eta\in\mathring{W}_2^{1,0}(\sigma^T)$ and integrating by parts we get the following integral identity

$$
\begin{aligned}
&\int\limits_{0}^{t}\int\limits_{-d/2}^{d/2}\frac{\partial U^{(N)}(x_1,\tau)}{\partial\tau}\eta(x_1,\tau)dx_1d\tau+\alpha\int\limits_{0}^{t}\int\limits_{-d/2}^{d/2}\frac{\partial^2U^{(N)}(x_1,\tau)}{\partial\tau\partial x_1}\frac{\partial\eta(x_1,\tau)}{\partial x_1}dx_1d\tau\\
&+\nu\int\limits_{0}^{t}\int\limits_{-d/2}^{d/2}\frac{\partial U^{(N)}(x_1,\tau)}{\partial x_1}\frac{\partial\eta(x_1,\tau)}{\partial x_1}dx_1d\tau\\
&=\int\limits_{0}^{t}q^{(N)}(\tau)\int\limits_{-d/2}^{d/2}h^{(N)}(x_1)\eta(x_1,\tau)dx_1d\tau\\
&+\int\limits_{0}^{t}\int\limits_{-d/2}^{d/2}f^{(N)}(x_1,\tau)\eta(x_1,\tau)dx_1d\tau\quad\forall\eta\in\overset{\circ}{W}{}_2^{1,0}(\sigma^T),\quad\forall t\in[0,T].
\end{aligned}
\tag{2.46}
$$

From estimates (2.29) and (2.34) it follows that there is a subsequence $\Big\{\big(U^{(N_l)},q^{(N_l)}\big)\Big\}$ such that

$$
\begin{aligned}
&U^{(N_l)}(\cdot,t)\rightharpoonup U(\cdot,t)\quad\text{in }\ W_2^1(\sigma)\ \ \forall t\in[0,T],\\
&U^{(N_l)}\rightharpoonup U,\ \ \frac{\partial U^{(N_l)}}{\partial x_1}\rightharpoonup\frac{\partial U}{\partial x_1},\ \ \frac{\partial U^{(N_l)}}{\partial t}\rightharpoonup\frac{\partial U}{\partial t},\ \ \frac{\partial^2U^{(N_l)}}{\partial x_1\partial t}\rightharpoonup\frac{\partial^2U}{\partial x_1\partial t}\ \text{ in }\ L_2(\sigma^T).\\
&q^{(N_l)}\to q\ \text{ in }\ L_2(0,T).
\end{aligned}
$$

Passing to a limit in (2.46) we get for U and q integral identity (2.6). Obviously, U satisfies the flux condition (2.5_3) and the initial condition. Moreover, from inequality (2.29) we get for $\big(U,q\big)$ estimate (2.44).

Let $u_0\in\overset{\circ}{W}{}_2^1(\sigma)\cap W_2^2(\sigma)$. Then, because of inequality (2.30), the subsequence $\Big\{\big(U^{(N_l)},q^{(N_l)}\big)\Big\}$ could be chosen so that in addition

$$
\begin{aligned}
&\frac{\partial^2U^{(N_l)}(\cdot,t)}{\partial x_1^2}\rightharpoonup\frac{\partial^2U(\cdot,t)}{\partial x_1^2}\quad\text{in }\ W_2^1(\sigma)\ \ \forall t\in[0,T],\\
&\frac{\partial^2U^{(N_l)}}{\partial x_1^2}\rightharpoonup\frac{\partial^2U}{\partial x_1^2},\ \ \frac{\partial^3U^{(N_l)}}{\partial x_1^2\partial t}\rightharpoonup\frac{\partial^3U}{\partial x_1^2\partial t}\ \text{ in }\ L_2(\sigma^T),
\end{aligned}
$$

and, obviously, for $\big(U,q\big)$ estimate (2.45) holds. Integrating by parts we get the identity

$$
\int\limits_{0}^{T}\int\limits_{-d/2}^{d/2}\Big(\frac{\partial}{\partial\tau}\Big(U-\alpha\frac{\partial^2U}{\partial x_1^2}\Big)-\nu\frac{\partial^2U}{\partial x_1^2}-q(t)-f\Big)\eta dx_1d\tau=0\ \ \forall\eta\in\overset{\circ}{W}{}_2^{1,0}(\sigma^T).
$$

Therefore, $\big(U,q\big)$ satisfy equations (2.5) almost everywhere in σ^T.

Let us prove the uniqueness. Let $\big(U^{[1]}(x_1,t),q^{[1]}(t)\big)$ and $\big(U^{[2]}(x_1,t),q^{[2]}(t)\big)$ be two weak solutions of problem (2.5). The difference $\big(V,s\big)=\big(U^{[1]}-U^{[2]},q^{[1]}-q^{[2]}\big)$ satisfies the integral identity

$$\int\limits_0^T \int\limits_{-d/2}^{d/2} \frac{\partial V(x_1,\tau)}{\partial \tau}\eta(x_1,\tau)dx_1d\tau + \alpha\int\limits_0^T \int\limits_{-d/2}^{d/2} \frac{\partial^2 V(x_1,\tau)}{\partial x_1\partial \tau}\frac{\partial \eta(x_1,\tau)}{\partial x_1}dx_1d\tau$$
$$+\nu\int\limits_0^T \int\limits_{-d/2}^{d/2}\frac{\partial V(x_1,\tau)}{\partial x_1}\cdot\frac{\partial \eta(x_1,\tau)}{\partial x_1}dx_1d\tau = \int\limits_0^T s(t)\int\limits_{-d/2}^{d/2}\eta(x_1,\tau)dx_1d\tau \quad \forall \eta \in \overset{\circ}{W}{}_2^{1,0}(\sigma^T).$$

Take $\eta(x_1,t) = V(x_1,t)$. Since $\int\limits_{-d/2}^{d/2} V(x_1,t)dx_1 = 0$ and $V(x_1,0)=0$, we obtain

$$\int\limits_{-d/2}^{d/2}|V(x_1,t)|^2dx_1 + \alpha\int\limits_{-d/2}^{d/2}\Big|\frac{\partial V(x_1,t)}{\partial x_1}\Big|^2dx_1 + 2\nu\int\limits_0^T\int\limits_{-d/2}^{d/2}\Big|\frac{\partial V(x_1,\tau)}{\partial x_1}\Big|^2dx_1d\tau = 0.$$

Therefore, $V(x_1,t)=0$. Taking in the identity (2.6) $\eta(x',t)$ such that $\int\limits_\sigma \eta(x',t)dx' = q(t)$ we get $\int\limits_0^t |q(\tau)|^2d\tau = 0$. Hence, $q(t)=0$. $\square$

From the estimate (2.35) for the approximate solution $\big(U^{(N)}, q^{(N)}\big)$ (see Lemma 2.4) follows the following result.

Theorem 2.2. *Let $f(x_1,t)=0$, $u_0 \in \overset{\circ}{W}{}_2^1(\sigma)$ and $F \in \mathcal{W}_{2,\mu}^1(0,\infty)$ with $\mu > 0$. Then the solution $\big(U,q\big)$ of problem (2.5) satisfies the estimate*

$$\begin{aligned}&\exp(\gamma^* t)\int\limits_{-d/2}^{d/2}\Big(|U(x_1,t)|^2 + \Big|\frac{\partial U(x_1,t)}{\partial x_1}\Big|^2\Big)dx_1\\ &+\int\limits_0^t\int\limits_{-d/2}^{d/2}\exp(\gamma^* t)\Big(\Big|\frac{\partial U(x_1,\tau)}{\partial \tau}\Big|^2 + \Big|\frac{\partial^2 U(x_1,\tau)}{\partial x_1\partial\tau}\Big|^2\Big)dx_1d\tau\\ &+\int\limits_0^t \exp(\gamma^*\tau)|q^{(N)}(\tau)|^2d\tau \le c\big(\|F\|^2_{\mathcal{W}^l_{2,\mu}(0,\infty)} + \|u_0\|^2_{W_2^1(\sigma)}\big),\end{aligned} \tag{2.47}$$

where

$$\gamma_* = \begin{cases} \min\big(1,2\mu\big), & \text{if } \min\big(1,2\mu\big) < \dfrac{\nu\pi^2}{d^2},\\ \dfrac{\nu\pi^2}{d^2} - \delta & \text{if } \min\big(1,2\mu\big) \ge \dfrac{\nu\pi^2}{d^2},\end{cases}$$

$\delta > 0$ is an arbitrary small number.

Chapter 3

Rotational pipe flow

Analogous results, as in Chapter 2 for the flow in channels, can be obtained in the three-dimensional case when the flow domain is an infinite pype $\Pi = \{x = (x', x_3) \in \mathbb{R}^3 : (x', x_3) \in \sigma \times \mathbb{R}\}$ with the circular cross-section $\sigma = \{x' : |x'| < 1\}$. Consider in the pipe Π the second grade fluid flow problem

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathrm{curl}(\mathbf{u} - \alpha\Delta\mathbf{u}) \times \mathbf{u} + \nabla\widetilde{p} = \mathbf{f}, \\ \mathrm{div}\mathbf{u} = 0, \\ \mathbf{u}|_{S^T} = 0, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x), \end{cases} \tag{3.1}$$

with additionally prescribed flux condition

$$\int\limits_{\sigma} u_3(x', x_3, t)dx' = F(t). \tag{3.2}$$

Suppose that the initial velocity $\mathbf{u}_0(x)$ and the right-hand side $\mathbf{f}(x,t)$ are axially symmetric, independent of the coordinate z and have the forms

$$\mathbf{u}_0(x) = (0, 0, u_{z0}(r)), \quad \mathbf{f}(x,t) = (0, 0, f_z(r,t)), \tag{3.3}$$

where (r, φ, z) are cylindrical coordinates in $\mathbb{R}^3$. Moreover, suppose that there holds the compatibility condition:

$$2\pi \int\limits_0^R r u_{z0}(r)dr = F(0). \tag{3.4}$$

We look for a axially symmetric solution of problem (3.1), (3.2) such that

$$
\begin{aligned}
&\mathbf{u}(r,\varphi,z,t) = (0,0,U(r,t)),\\
&\widetilde{p}(r,\varphi,z,t) = \widetilde{p}(r,z,t).
\end{aligned} \tag{3.5}
$$

Substituting expressions (3.5) into (3.1), (3.2) yields

$$
\begin{cases}
-2\alpha\dfrac{\partial U}{\partial r}\dfrac{\partial^2 U}{\partial r^2} - \alpha\dfrac{1}{r}\Big(\dfrac{\partial U}{\partial r}\Big)^2 + \dfrac{\partial \widetilde{p}}{\partial r} = 0,\\
\dfrac{\partial}{\partial t}\big(U - \alpha\big(\dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r}\big)\big) - \nu\big(\dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r}\big) + \dfrac{\partial \widetilde{p}}{\partial z} = f_z,\\
U(0,t) = U(R,t) = 0, \quad U(r,0) = u_{z0}(r),\\
2\pi\int\limits_0^R rU(r,t)dr = F(t).
\end{cases} \tag{3.6}
$$

From (3.6_1) it follows that

$$
\widetilde{p}(r,z,t) = -q(t)z + \alpha\int\limits_0^r \frac{1}{y}\frac{\partial}{\partial y}\Big(y\Big(\frac{\partial U(y,t)}{\partial y}\Big)^2\Big)dy + p_0(t), \tag{3.7}
$$

and we get for $\big(U,q\big)$ the inverse problem on the cross-section σ

$$
\begin{cases}
\dfrac{\partial}{\partial t}\big(U - \alpha\big(\dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r}\big)\big) - \nu\big(\dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r}\big) = q(t) + f_z,\\
U(0,t) = U(R,t) = 0, \quad U(r,0) = u_{z0}(r),\\
2\pi\int\limits_0^R rU(r,t)dr = F(t).
\end{cases} \tag{3.8}
$$

Deffinition. *The solution* $(\mathbf{u}(r,\varphi,z,t),\widetilde{p}(r,\varphi,z,t))$ *of problem* (3.1), (3.2) *having the form* (3.5), (3.7) *is called the Poiseuille type solution.*

Problem (3.8) could be studied just in the same way as the two-dimensional problem (2.5). The only difference is that as the basis in $L_2(\sigma)$ we take the eigenvalues $v_k \in \overset{\circ}{W}{}^1_2(\sigma)$ of the Laplace operator $\triangle' = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r}$:

$$
\begin{cases}
-\nu\triangle' v_k = \lambda_k v_k, \quad x' \in \sigma,\\
v_k|_{\partial\sigma} = 0.
\end{cases}
$$

Note, that $v_k(r) = J_0(\dfrac{\mu_k r}{R})/\big(\int\limits_0^R rJ_0^2(\dfrac{\mu_k r}{R})dr\big)^{\frac{1}{2}}$, $\lambda_k = (\dfrac{\mu_k}{k})^2$, where J_0 is the Bessel function and μ_k are the roots of the equation $J_0(\mu) = 0$ (see, for example, [5]).

Deffinition. *The pair* $\big(U, q\big) \in \mathcal{M}(\sigma^T) \times L_2(0, T)$ *is called the week solution of problem* (3.8) *if it satisfies the integral identity*

$$\int\limits_0^t\int\limits_0^R \frac{\partial U(r,\tau)}{\partial \tau}\eta(r,\tau)drd\tau + \alpha\int\limits_0^t\int\limits_0^R \frac{\partial^2 U(r,\tau)}{\partial\tau\partial r}\big(\frac{\partial \eta(r,\tau)}{\partial r} - \frac{1}{r}\eta(r,\tau)\big)drd\tau$$
$$+\nu\int\limits_0^t\int\limits_0^R \frac{\partial U(r,\tau)}{\partial r}\big(\frac{\partial \eta(r,\tau)}{\partial r} - \frac{1}{r}\eta(r,\tau)\big)drd\tau = \int\limits_0^t q(\tau)\int\limits_0^R \eta(r,\tau)drd\tau$$
$$+\int\limits_0^t\int\limits_0^R f_z(r,\tau)\eta(r,\tau)drd\tau \quad \forall \eta \in \overset{\circ}{W}{}_2^{1,0}(\sigma^T),$$

the initial condition $U(r, 0) = u_{z0}(r)$ *and the flux condition* (3.8_3).

There holds the following theorems.

Theorem 3.1. *Suppose that* $f_z \in L_2(\sigma^T), u_{z0} \in \overset{\circ}{W}{}_2^1(\sigma), F \in W_2^1(0, T)$. *Then problem* (3.8) *admits a unique week solution* $\big(U, q\big) \in \mathcal{M}(\sigma^T) \times L_2(0, T)$ *and the following estimate*

$$\max_{t\in[0,T]} \|U(\cdot,t)\|^2_{W_2^1(\sigma)} + \|U\|^2_{W_2^{1,1}(\sigma^T)} + \Big\|\frac{\partial U}{\partial t}\Big\|^2_{W_2^1(\sigma^T)} + \|q\|_{L_2(0,T)}$$
$$\le c\big(\|F\|^2_{W_2^1(0,T)} + \|f_z\|^2_{L_2(\sigma^T)} + \|u_{z0}\|^2_{W_2^1(\sigma)}\big)$$

holds.

If, in addition $u_{z0} \in \overset{\circ}{W}{}_2^1(\sigma) \cap W_2^2(\sigma)$, *then* $U \in W_2^{2,1}(\sigma^T)$, $\dfrac{\partial U}{\partial t} \in W_2^2(\sigma^T)$, *the equations* (3.8) *are satisfied almost everywhere in* σ^T, *and*

$$\max_{t\in[0,T]} \|U(\cdot,t)\|^2_{W_2^2(\sigma)} + \|U\|^2_{W_2^{2,1}(\sigma^T)} + \Big\|\frac{\partial U}{\partial t}\Big\|^2_{W_2^2(\sigma^T)} + \|q\|_{L_2(0,T)}$$
$$\le c\big(\|F\|^2_{W_2^1(0,T)} + \|f_z\|^2_{L_2(\sigma^T)} + \|u_{z0}\|^2_{W_2^2(\sigma)}\big).$$

Theorem 3.2. *Let* $f_z(r, t) = 0,\ u_{z0} \in \overset{\circ}{W}{}_2^1(\sigma)$ *and* $F \in \mathcal{W}^1_{2,\mu}(0, \infty)$ *with* $\mu > 0$. *Then the solution* $\big(U, q\big)$ *of problem* (3.8) *satisfies the estimate*

$$\exp(\gamma^* t)\int\limits_0^R \big(|U(r,t)|^2 + \Big|\frac{\partial U(r,t)}{\partial r}\Big|^2\big)\, rdr$$
$$+\int\limits_0^t\int\limits_0^R \exp(\gamma^*\tau)\big(\Big|\frac{\partial U(r,\tau)}{\partial \tau}\Big|^2 + \Big|\frac{\partial^2 U(r,\tau)}{\partial r\partial\tau}\Big|^2\big)\, rdrd\tau + \int\limits_0^t \exp(\gamma^*\tau)|q^{(N)}(\tau)|^2 d\tau$$
$$\le c\big(\|F\|^2_{\mathcal{W}^l_{2,\mu}(0,\infty)} + \|u_{z0}\|^2_{W_2^1(\sigma)}\big),$$

where

$$\gamma_* = \begin{cases} \min\big(1, 2\mu\big), & \text{if } \min\big(1, 2\mu\big) < \mu_1^2, \\ \mu_1^2 - \delta, & \text{if } \min\big(1, 2\mu\big) \geq \mu_1^2, \end{cases}$$

μ_1 *is the first positive root of the equation* $J_0(\mu) = 0$, $\delta > 0$ *is an arbitrary small number.*

The proofs of these theorems are exactly the same as for the two-dimensional case.

Remark 3.1. In Chapter 4 we study the three-dimensional problem for a non-symmetrical case (i.e., when the cross-section σ is an arbitrary bounded domain). In this case problem (3.1), (3.2) does not have the unidirectional solution and the velocity field has all three components.

Chapter 4

Time periodic channel flow

In the infinite channel $\Pi = \{x \in \mathbb{R}^2 : (x_1, x_2) \in (-\frac{d}{2}, \frac{d}{2}) \times \mathbb{R}\}$ we consider time-periodic (without loss of generality we assume that the period is equal to 2π) second grade fluid flow problem

$$\begin{cases} \frac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathrm{curl}(\mathbf{u} - \alpha\Delta\mathbf{u}) \times \mathbf{u} + \nabla\widetilde{p} = \mathbf{f}, \\ \mathrm{div}\mathbf{u} = 0, \\ \mathbf{u}|_{\partial\Pi\times(0,2\pi)} = 0, \quad \mathbf{u}(x,0) = \mathbf{u}(x,2\pi), \end{cases} \tag{4.1}$$

with the prescribed periodic flux

$$\int\limits_{-d/2}^{d/2} u_2(x,t) = F(t), \qquad F(0) = F(2\pi). \tag{4.2}$$

We look for the Poiseuille type solution $(\mathbf{u}(x,t), \widetilde{p}(x,t))$ of system (4.1), (4.2) in the form

$$\begin{aligned} &\mathbf{u}(x,t) = (0, U(x_1,t)), \\ &\widetilde{p}(x,t) = -q(t)x_2 - \int\limits_{-d/2}^{x_1} U(y,t)\frac{\partial}{\partial y}\Big(U(y,t) - \alpha\frac{\partial^2 U(y,t)}{\partial y^2}\Big)dy + p_0(t), \end{aligned} \tag{4.3}$$

where $p_0(t)$ is an arbitrary function, while $U(x_1,t)$ and $q(t)$ are time-periodic functions

$$U(x_1,0) = U(x_1,2\pi), \qquad q(0) = q(2\pi).$$

As in Chapter 2, the first equation in (4.1_1) and the divergence equation (4.1_3) are satisfied identically, while from (4.1_2) and the boundary conditions we receive the following inverse problem on the interval $(-\frac{d}{2}, \frac{d}{2})$:

$$\begin{cases} \frac{\partial}{\partial t}(U - \alpha \frac{\partial^2 U}{\partial x_1^2}) - \nu \frac{\partial^2 U}{\partial x_1^2} = q(t), \\ U(-\frac{d}{2}, t) = U(\frac{d}{2}, t) = 0, \quad U(x_1, 0) = U(x_1, 2\pi), \\ \int\limits_{-d/2}^{d/2} U(x_1, t) dx_1 = F(t). \end{cases} \tag{4.4}$$

4.1 Direct problem

First, we assume, that the function $q(t)$ is known and we study the following direct problem

$$\begin{cases} \frac{\partial}{\partial t}(U - \alpha \frac{\partial^2 U}{\partial x_1^2}) - \nu \frac{\partial^2 U}{\partial x_1^2} = q(t), \\ U(-\frac{d}{2}, t) = U(\frac{d}{2}, t) = 0, \quad U(x_1, 0) = U(x_1, 2\pi). \end{cases} \tag{4.5}$$

Theorem 4.1. *Let $q \in L_2(0, 2\pi)$ is 2π-periodic function. Then problem (4.5) admits the unique 2π-periodic solution $U \in W_2^{2,1}((-\frac{d}{2}, \frac{d}{2}) \times (0, 2\pi))$ and the following estimate*

$$\begin{aligned} &||U||_{W_2^{2,1}((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} + \left\| \frac{\partial U}{\partial x_1} \right\|_{W_2^{1,1}((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} \\ &+ \left\| \frac{\partial^3 U}{\partial t \partial x_1^2} \right\|_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} \leq c||q||_{L_2(0,2\pi)} \end{aligned} \tag{4.6}$$

holds.

Proof. Any 2π-periodic function from $L_2(0, 2\pi)$ can be expressed by a Fourier series, therefore,

$$q(t) = \frac{q_0^{(c)}}{2} + \sum_{n=1}^{\infty} (q_n^{(c)} \cos(nt) + q_n^{(s)} \sin(nt)). \tag{4.7}$$

We look for the approximate solution $U^{(N)}(x_1, t)$ in the form

$$U^{(N)}(x_1, t) = \frac{\varphi_0(x_1)}{2} + \sum_{n=1}^{N} (\varphi_n(x_1) \cos(nt) + \psi_n(x_1) \sin(nt)), \tag{4.8}$$

where coefficients $\varphi_n(x_1)$ and $\psi_n(x_1)$, $n = 0, 1, ..., N$, are found from the equations

$$\begin{cases} (\nu^2+\alpha^2n^2)\varphi_n'' - \alpha n^2\varphi_n - n\nu\psi_n = \alpha n q_n^{(s)} - \nu q_n^{(c)}, \quad x_1\in\sigma \\ (\nu^2+\alpha^2n^2)\psi_n'' - \alpha n^2\psi_n + n\nu\varphi_n = -\nu q_n^{(s)} - \alpha n q_n^{(c)}, \quad x_1\in\sigma \\ \varphi_n|_{\partial\sigma}=0, \quad \psi_n|_{\partial\sigma}=0, \end{cases} \tag{4.9}$$

$n=1,2,...,N$, $\varphi_0 = \dfrac{q_0^{(c)}}{8\nu}(d^2-4x_1^2)$.

Obviously, (4.9) is a strongly elliptic system. Let us show that it has only one solution. Let (φ_n,ψ_n) be the solution of the homogeneous system (4.9). Multiply the first equality of system (4.9) by $\psi_n(x_1)$, the second one by $\varphi_n(x_1)$ and then integrate them by parts on σ. Summing these equations yields

$$\nu\int\limits_\sigma |\varphi_n'(x_1)|^2dx_1 + \nu\int\limits_\sigma |\psi_n'(x_1)|^2dx_1 = 0.$$

Thus, $\varphi_n(x_1)=0,\ \psi_n(x_1)=0$ and from the theory of elliptic systems (see, for example, [17], [26]) we conclude, that for all $n\ge 1$ there exist just one solution $(\varphi_n,\psi_n)\in W_2^2(\sigma)\cap \dot W_2^1(\sigma)$ of the non-homogeneous problem (4.9).

Denote

$$q^{(N)}(t) = \frac{q_0^{(c)}}{2} + \sum_{n=1}^N (q_n^{(c)}\cos(nt) + q_n^{(s)}\sin(nt)). \tag{4.10}$$

It is easy to calculate that the approximate solution $(U^{(N)},q^{(N)})$ (see (4.8), (4.9)) is a solution of the following problem:

$$\begin{cases} \dfrac{\partial}{\partial t}(U^{(N)} - \alpha\dfrac{\partial^2U^{(N)}}{\partial x_1^2}) - \nu\dfrac{\partial^2U^{(N)}}{\partial x_1^2} = q^{(N)}(t), \\ U^{(N)}(-\dfrac{d}{2},t) = U^{(N)}(\dfrac{d}{2},t) = 0, \quad U^{(N)}(x_1,0) = U^{(N)}(x_1,2\pi). \end{cases} \tag{4.11}$$

Multiplying equality (4.11_1) by $U^{(N)}(x_1,t)$ and integrating by parts on $(-\frac{d}{2},\frac{d}{2})$ yield

$$\frac{1}{2}\frac{d}{dt}\int\limits_{-d/2}^{d/2}|U^{(N)}(x_1,t)|^2dx_1 + \frac{\alpha}{2}\frac{d}{dt}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2dx_1$$
$$+\nu\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2dx_1 = q^{(N)}(t)\int\limits_{-d/2}^{d/2}U^{(N)}(x_1,t)dx_1.$$

Integrate the latter identity with respect to t from 0 to 2π. Using the periodicity condition (4.11_2), Cauchy-Schwarz and Poincaré-Friedrichs inequalities we obtain

$$\nu\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1dt=\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2} q^{(N)}(t)U^{(N)}(x_1,t)dx_1dt$$
$$\le (\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}|q^{(N)}(t)|^2dx_1dt)^{1/2}(\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}|U^{(N)}(x_1,t)|^2dx_1dt)^{1/2}$$
$$\le c||q^{(N)}||_{L_2(0,2\pi)}(\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1dt)^{1/2}$$
$$\le c||q||_{L_2(0,2\pi)}(\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1dt)^{1/2}.$$

Thus,

$$\nu\Big\|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big\|_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))}\le c||q||_{L_2(0,2\pi)}. \tag{4.12}$$

Multiply now equality (4.11_1) by $\dfrac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}$ and integrate by parts on $(-\dfrac{d}{2},\dfrac{d}{2})$. Using (4.11_2) we get

$$\frac{1}{2}\frac{d}{dt}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1+\frac{\alpha}{2}\frac{d}{dt}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2 dx_1$$
$$+\nu\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2 dx_1=-q^{(N)}(t)\int\limits_{-d/2}^{d/2}\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}dx_1.$$

This equality yields

$$\nu\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2 dx_1dt=\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2} q^{(N)}(t)\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}dx_1dt$$
$$\le (\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}|q^{(N)}(t)|^2dx_1dt)^{1/2}(\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2dx_1dt)^{1/2}$$
$$\le c||q^{(N)}||_{L_2(0,2\pi)}(\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2 dx_1dt)^{1/2}$$
$$\le c||q||_{L_2(0,2\pi)}(\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2 dx_1dt)^{1/2}.$$

Thus, the following estimate

$$\nu\Big\|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big\|_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))}\le c||q||_{L_2(0,2\pi)} \tag{4.13}$$

holds.

The next step is to estimate $W_2^1(\sigma)$ - norm of $U^{(N)}(x_1,0)$. Let us multiply equation (4.11_1) by $tU^{(N)}(x_1,t)$ and integrate by parts on $(-\dfrac{d}{2},\dfrac{d}{2})$. Using the boundary condition (4.11_2) and the following identities

$$
\begin{aligned}
&2t\frac{\partial U^{(N)}(x_1,t)}{\partial t}U^{(N)}(x_1,t)=\frac{\partial}{\partial t}\big(t|U^{(N)}(x_1,t)|^2\big)-|U^{(N)}(x_1,t)|^2,\\
&t\frac{\partial}{\partial t}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2=\frac{\partial}{\partial t}\big(t\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2\big)-\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2
\end{aligned},
$$

we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{-d/2}^{d/2}t|U^{(N)}(x_1,t)|^2dx_1+\frac{\alpha}{2}\frac{d}{dt}\int\limits_{-d/2}^{d/2}t\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2dx_1\\
&+\nu t\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2dx_1=tq^{(N)}(t)\int\limits_{-d/2}^{d/2}U^{(N)}(x_1,t)dx_1\\
&+\frac{1}{2}\int\limits_{-d/2}^{d/2}|U^{(N)}(x_1,t)|^2dx_1+\frac{\alpha}{2}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2dx_1.
\end{aligned}
$$

Integrating the last equality by t from 0 to 2π and using Poincaré-Friedrichs and (4.12) inequalities we derive

$$
\begin{aligned}
&\pi\int\limits_{-d/2}^{d/2}|U^{(N)}(x_1,2\pi)|^2dx_1+\alpha\pi\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,2\pi)}{\partial x_1}\Big|^2dx_1\\
&+\nu\int\limits_0^{2\pi}t\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2dx_1dt=\int\limits_0^{2\pi}tq^{(N)}(t)\int\limits_{-d/2}^{d/2}U^{(N)}(x_1,t)dx_1dt\\
&+\frac{1}{2}\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}|U^{(N)}(x_1,t)|^2dx_1dt+\frac{\alpha}{2}\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2dx_1dt\\
&\le 2\pi c||q^{(N)}||_{L_2(0,2\pi)}\Big\|\frac{\partial U^{(N)}}{\partial x_1}\Big\|_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))}+c\Big\|\frac{\partial U^{(N)}}{\partial x_1}\Big\|^2_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))}\\
&+\frac{\alpha}{2}\Big\|\frac{\partial U^{(N)}}{\partial x_1}\Big\|^2_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))}\le c||q||^2_{L_2(0,2\pi)}.
\end{aligned}
\tag{4.14}
$$

Further, multiplying equation (4.11_1) by $t\dfrac{\partial U^{(N)}(x_1,t)}{\partial t}$, integrating by parts on $(-\dfrac{d}{2},\dfrac{d}{2})$ and over t from 0 to 2π yields

$$
\begin{aligned}
&\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}t\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial t}\Big|^2dx_1dt+\alpha\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}t\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial t\partial x_1}\Big|^2dx_1dt\\
&+\pi\nu\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,2\pi)}{\partial x_1}\Big|^2dx_1=\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}tq^{(N)}(t)\frac{\partial U^{(N)}(x_1,t)}{\partial t}dx_1dt
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\nu}{2}\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1 dt \le c||q^{(N)}||^2_{L_2(0,2\pi)} \\
&+\frac{1}{2}\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2} t\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial t}\Big|^2 dx_1 dt + c\Big\|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big\|^2_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} \\
&\le c||q||^2_{L_2(0,2\pi)} + \frac{1}{2}\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2} t\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial t}\Big|^2 dx_1 dt.
\end{aligned}
\tag{4.15}
$$

From (4.14) and (4.15) we obtain that

$$
||U^{(N)}(\cdot,2\pi)||^2_{W_2^1(-\frac{d}{2},\frac{d}{2})} \le c||q||^2_{L_2(0,2\pi)}.
$$

Multiplying equation (4.11_1) by $t\dfrac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}$, integrating by parts on $(-\dfrac{d}{2},\dfrac{d}{2})$, using the boundary condition (4.11_2) and the identity

$$
2t\frac{\partial^3 U^{(N)}(x_1,t)}{\partial t\partial x_1^2}\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2} = \frac{\partial}{\partial t}\Big(t\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2\Big) - \Big(\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big)^2,
$$

we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{-d/2}^{d/2} t\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1 + \frac{\alpha}{2}\frac{d}{dt}\int\limits_{-d/2}^{d/2} t\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2 dx_1 \\
&+\frac{\alpha}{2}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2 dx_1 + \nu t\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2 dx_1 \\
&= -tq^{(N)}(t)\int\limits_{-d/2}^{d/2}\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}dx_1 + \frac{1}{2}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1.
\end{aligned}
$$

From the last inequality it follows that

$$
\begin{aligned}
&\pi\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,2\pi)}{\partial x_1}\Big|^2 dx_1 + \alpha\pi\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2 U^{(N)}(x_1,2\pi)}{\partial x_1^2}\Big|^2 dx_1 \\
&+\frac{\alpha}{2}\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2 dx_1 dt + \nu\int\limits_0^{2\pi} t\int\limits_{-d/2}^{d/2}\Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2 dx_1 dt \\
&= -\int\limits_0^{2\pi} tq^{(N)}(t)\int\limits_{-d/2}^{d/2}\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}dx_1 dt + \frac{1}{2}\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1 dt \\
&\le 2\pi c||q^{(N)}||_{L_2(0,2\pi)}\Big\|\frac{\partial^2 U^{(N)}}{\partial x_1^2}\Big\|_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} \\
&+c\Big\|\frac{\partial U^{(N)}}{\partial x_1}\Big\|^2_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} \le c||q||^2_{L_2(0,2\pi)}.
\end{aligned}
\tag{4.16}
$$

Hence,

$$\Big\|\frac{\partial U^{(N)}(\cdot,2\pi)}{\partial x_1}\Big\|^2_{W_2^1(-\frac{d}{2},\frac{d}{2})} \le c||q||^2_{L_2(0,2\pi)}.$$

Because of the periodicity condition, the following two inequalities are valid

$$||U^{(N)}(\cdot,0)||^2_{W_2^1(-\frac{d}{2},\frac{d}{2})} \le c||q||^2_{L_2(0,2\pi)}, \tag{4.17}$$

$$\Big\|\frac{\partial^2 U^{(N)}(\cdot,0)}{\partial x_1^2}\Big\|^2_{L_2(-\frac{d}{2},\frac{d}{2})} \le c||q||^2_{L_2(0,2\pi)}. \tag{4.18}$$

Now, $U^{(N)}(x_1,t)$ can be interpretated as a solution of the initial-boundary value problem (4.11_1) with the initial condition $U^{(N)}(x_1,t)|_{t=0} = U^{(N)}(x_1,0)$:

$$\begin{cases} \dfrac{\partial}{\partial t}(U^{(N)} - \alpha\dfrac{\partial^2 U^{(N)}}{\partial x_1^2}) - \nu\dfrac{\partial^2 U^{(N)}}{\partial x_1^2} = q^{(N)}(t), \\ U^{(N)}(-\dfrac{d}{2},t) = U^{(N)}(\dfrac{d}{2},t) = 0, \quad U^{(N)}(x_1,t)|_{t=0} = U^{(N)}(x_1,0). \end{cases} \tag{4.19}$$

Multiply equation (4.19_1) by $U^{(N)}(x_1,t)$ and integrate by parts on $(-\frac{d}{2},\frac{d}{2})$ and by τ from 0 to t:

$$\begin{aligned} &\frac{1}{2}\int\limits_{-d/2}^{d/2}|U^{(N)}(x_1,t)|^2dx_1 + \frac{\alpha}{2}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2dx_1 + \nu\int\limits_0^t\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,\tau)}{\partial x_1}\Big|^2dx_1d\tau \\ &= \frac{1}{2}\int\limits_{-d/2}^{d/2}|U^{(N)}(x_1,0)|^2dx_1 + \frac{\alpha}{2}\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,0)}{\partial x_1}\Big|^2dx_1 + \int\limits_0^t\int\limits_{-d/2}^{d/2}q^{(N)}(\tau)U^{(N)}(x_1,\tau)dx_1d\tau \\ &\le \frac{1}{2}||U^{(N)}(x_1,0)||^2_{L_2(-\frac{d}{2},\frac{d}{2})} + \frac{\alpha}{2}\Big\|\frac{\partial U^{(N)}(x_1,0)}{\partial x_1}\Big\|^2_{L_2(-\frac{d}{2},\frac{d}{2})} \\ &+||q^{(N)}||_{L_2(0,2\pi)}(\int\limits_0^t\int\limits_{-d/2}^{d/2}|U^{(N)}(x_1,\tau)dx_1d\tau)^{1/2} \le c||q||^2_{L_2(0,2\pi)} + \frac{\nu}{2}\int\limits_0^t\int\limits_{-d/2}^{d/2}\Big|\frac{\partial U^{(N)}(x_1,\tau)}{\partial x_1}\Big|^2dx_1d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sup_{t\in[0,2\pi]}||U^{(N)}(\cdot,t)||^2_{L_2(-\frac{d}{2},\frac{d}{2})} + \alpha\sup_{t\in[0,2\pi]}\Big\|\frac{\partial U^{(N)}(\cdot,t)}{\partial x_1}\Big\|^2_{L_2(-\frac{d}{2},\frac{d}{2})} \\ &+\nu\Big\|\frac{\partial U^{(N)}}{\partial x_1}\Big\|^2_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} \le c||q||^2_{L_2(0,2\pi)}. \end{aligned} \tag{4.20}$$

In the same way, multiplying equation (4.19_1) by $\frac{\partial}{\partial t}U^{(N)}(x_1,t)$, integrating by parts on $(-\frac{d}{2},\frac{d}{2})$ and by τ from 0 to t, we obtain

$$\int_0^t \int_{-d/2}^{d/2} \Big|\frac{\partial U^{(N)}(x_1,\tau)}{\partial \tau}\Big|^2 dx_1 d\tau + \alpha \int_0^t \int_{-d/2}^{d/2} \Big|\frac{\partial^2 U^{(N)}(x_1,\tau)}{\partial \tau \partial x_1}\Big|^2 dx_1 d\tau$$

$$+\frac{\nu}{2} \int_{-d/2}^{d/2} \Big|\frac{\partial^2 U^{(N)}(x_1,\tau)}{\partial \tau \partial x_1}\Big|^2 dx_1 = \int_0^t \int_{-d/2}^{d/2} q^{(N)}(\tau) \frac{\partial U^{(N)}(x_1,\tau)}{\partial \tau} dx_1 d\tau$$

$$+\frac{\nu}{2} \int_{-d/2}^{d/2} \Big|\frac{\partial^2 U^{(N)}(x_1,\tau)}{\partial \tau \partial x_1}\Big|^2 dx_1 \le c||q||^2_{L_2(0,2\pi)} + \frac{1}{2} \int_0^t \int_{-d/2}^{d/2} \Big|\frac{\partial U^{(N)}(x_1,\tau)}{\partial \tau}\Big|^2 dx_1 d\tau.$$

Hence,

$$\begin{aligned} &\nu \sup_{t\in[0,2\pi]} \Big\|\frac{\partial U^{(N)}(\cdot,t)}{\partial x_1}\Big\|^2_{L_2(-\frac{d}{2},\frac{d}{2})} + \Big\|\frac{\partial U^{(N)}}{\partial t}\Big\|^2_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} \\ &+\alpha \Big\|\frac{\partial U^{(N)}}{\partial x_1}\Big\|^2_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} \le c||q||^2_{L_2(0,2\pi)}. \end{aligned} \tag{4.21}$$

Now, multiplying equation (4.19_1) by $\dfrac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}$, integrating by parts on $(-\dfrac{d}{2},\dfrac{d}{2})$ and by τ from 0 to t, using the boundary condition and the inequalities (4.17), (4.18) we derive

$$\frac{1}{2} \int_{-d/2}^{d/2} \Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1 + \frac{\alpha}{2} \int_{-d/2}^{d/2} \Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2 dx_1$$

$$+\nu \int_0^t \int_{-d/2}^{d/2} \Big|\frac{\partial^2 U^{(N)}(x_1,\tau)}{\partial x_1^2}\Big|^2 dx_1 d\tau = -\int_0^t q(\tau) \int_{-d/2}^{d/2} \frac{\partial^2 U^{(N)}(x_1,\tau)}{\partial x_1^2} dx_1 d\tau$$

$$+\frac{1}{2} \int_{-d/2}^{d/2} \Big|\frac{\partial U^{(N)}(x_1,0)}{\partial x_1}\Big|^2 dx_1 + \frac{\alpha}{2} \int_{-d/2}^{d/2} \Big|\frac{\partial^2 U^{(N)}(x_1,0)}{\partial x_1^2}\Big|^2 dx_1$$

$$\le c||q||^2_{L_2(0,2\pi)} + \frac{\nu}{2} \int_0^t \int_{-d/2}^{d/2} \Big|\frac{\partial^2 U^{(N)}(x_1,\tau)}{\partial x_1^2}\Big|^2 dx_1 d\tau.$$

Thus, the following inequality

$$\begin{aligned} &\sup_{t\in[0,2\pi]} \Big\|\frac{\partial U^{(N)}(\cdot,t)}{\partial x_1}\Big\|^2_{L_2(-\frac{d}{2},\frac{d}{2})} + \alpha \sup_{t\in[0,2\pi]} \Big\|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big\|^2_{L_2(-\frac{d}{2},\frac{d}{2}))} \\ &+\Big\|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big\|^2_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} \le c||q||^2_{L_2(0,2\pi)} \end{aligned} \tag{4.22}$$

holds.

Next, multiply equation (4.19_1) by $\dfrac{\partial^3 U^{(N)}(x_1,t)}{\partial \tau \partial x_1^2}$, integrate by parts on $(-\dfrac{d}{2},\dfrac{d}{2})$ and by τ from 0 to t. Using the boundary condition and inequalities (4.17), (4.18) we obtain

$$\int\limits_0^t \int\limits_{-d/2}^{d/2} \Big|\frac{\partial^2 U^{(N)}(x_1,\tau)}{\partial\tau\partial x_1}\Big|^2 dx_1 d\tau + \alpha \int\limits_0^t \int\limits_{-d/2}^{d/2} \Big|\frac{\partial^3 U^{(N)}(x_1,\tau)}{\partial\tau\partial x_1^2}\Big|^2 dx_1 d\tau$$
$$+\frac{\nu}{2}\int\limits_{-d/2}^{d/2} \Big|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial x_1^2}\Big|^2 dx_1 = -\int\limits_0^t q(\tau) \int\limits_{-d/2}^{d/2} \frac{\partial^3 U^{(N)}(x_1,\tau)}{\partial\tau\partial x_1^2} dx_1 d\tau$$
$$+\frac{\nu}{2}\int\limits_{-d/2}^{d/2} \Big|\frac{\partial^2 U^{(N)}(x_1,0)}{\partial x_1^2}\Big| dx_1 \le c||q||^2_{L_2(0,2\pi)} + \frac{\alpha}{2}\int\limits_0^t \int\limits_{-d/2}^{d/2} \Big|\frac{\partial^3 U^{(N)}(x_1,\tau)}{\partial\tau\partial x_1^2}\Big|^2 dx_1 d\tau.$$

From the latter inequality it follows that

$$\begin{aligned}&\Big\|\frac{\partial^2 U^{(N)}(x_1,t)}{\partial t\partial x_1}\Big\|^2_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} + \nu \sup_{t\in[0,2\pi]} \Big\|\frac{\partial^2 U^{(N)}(\cdot,t)}{\partial x_1^2}\Big\|^2_{L(-\frac{d}{2},\frac{d}{2})} \\ &+\alpha\Big\|\frac{\partial^3 U^{(N)}(x_1,t)}{\partial t\partial x_1^2}\Big\|^2_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} \le c||q||^2_{L_2(0,2\pi)}.\end{aligned} \tag{4.23}$$

From inequalities (4.12), (4.23) we conclude that the sequence $\{U^{(N)}\}$ is bounded in the space $W_2^{2,1}((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))$ and $\{\dfrac{\partial^3 U^{(N)}}{\partial t\partial^2 x_1}\}$ is bounded in the space $L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))$. Therefore, there exists subsequence $\{U^{(N_l)}\}$ such that

$$\begin{aligned}&U^{(N_l)} \rightharpoonup U \ \text{ in } \ W_2^{2,1}((-\tfrac{d}{2},\tfrac{d}{2})\times(0,2\pi)),\\ &\frac{\partial^3 U^{(N)}}{\partial t\partial^2 x_1} \rightharpoonup \frac{\partial^3 U}{\partial t\partial^2 x_1} \ \text{ in } \ L_2((-\tfrac{d}{2},\tfrac{d}{2})\times(0,2\pi)).\end{aligned}$$

Obviously, the limit function U is the week solution of problem (4.5). Moreover, the inequality (4.6) remains valid.

Let us prove that the solution is unique. Take $q(t)=0$. Repeating the proof of inequality (4.12) we get

$$\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2} \Big|\frac{\partial U^{(N)}(x_1,t)}{\partial x_1}\Big|^2 dx_1 dt = 0.$$

Hence, $U(x_1,t)=0.\square$

4.2 Relationship between the flux and the gradient of the pressure

The arguments of this section are similar to that of Galdi and Robertson in [37] for the Navier-Stokes equations. We show the relationship between the flux $F(t)$ and the gradient of the pressure $q(t)$. Consider the following problems

$$\begin{cases} \dfrac{\partial}{\partial t}(U_n^{(c)} - \alpha\dfrac{\partial^2 U_n^{(c)}}{\partial x_1^2}) - \nu\dfrac{\partial^2 U_n^{(c)}}{\partial x_1^2} = \cos(nt), \\ U_n^{(c)}(-\dfrac{d}{2},t) = U_n^{(c)}(\dfrac{d}{2},t) = 0, \quad U_n^{(c)}(x_1,0) = U_n^{(c)}(x_1,2\pi), \end{cases} \tag{4.24}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t}(U_n^{(s)} - \alpha\dfrac{\partial^2 U_n^{(s)}}{\partial x_1^2}) - \nu\dfrac{\partial^2 U_n^{(s)}}{\partial x_1^2} = \sin(nt), \\ U_n^{(s)}(-\dfrac{d}{2},t) = U_n^{(s)}(\dfrac{d}{2},t) = 0, \quad U_n^{(s)}(x_1,0) = U_n^{(s)}(x_1,2\pi). \end{cases} \tag{4.25}$$

The solutions of problems (4.24) and (4.25) are:

$$\begin{array}{l} U_n^{(c)}(x_1,t) = \varphi_n(x_1)\cos(nt) - \psi_n(x_1)\sin(nt) \\ U_n^{(s)}(x_1,t) = \psi_n(x_1)\cos(nt) + \varphi_n(x_1)\sin(nt), \end{array} \tag{4.26}$$

where pairs $(\varphi_n(x_1), \psi_n(x_1))$ are the solutions of the following systems

$$\begin{cases} -n\psi_n - \alpha n\psi_n'' - \nu\varphi_n'' = 1 \\ -n\varphi_n + \alpha n\varphi_n'' + \nu\psi_n'' = 0 \\ \varphi_n(-\dfrac{d}{2}) = \varphi_n(\dfrac{d}{2}) = 0 \\ \psi_n(-\dfrac{d}{2}) = \psi_n(\dfrac{d}{2}) = 0 \end{cases} \tag{4.27}$$

According to Theorem 4.1, problem (4.27) has the unique solution $(\varphi_n, \psi_n) \in W_2^2(-\frac{d}{2}, \frac{d}{2}) \cap \overset{\circ}{W}{}_2^1(-\frac{d}{2}, \frac{d}{2})$ for all $n \geq 0$.

Let us define

$$a_n = \int\limits_\sigma \varphi_n(x_1)dx_1, \quad b_n = -\int\limits_\sigma \psi_n(x_1)dx_1, \quad n = 0,1,2,\dots . \tag{4.28}$$

Lemma 4.1. *Let $(\varphi_n, \psi_n) \in W_2^2(-\frac{d}{2}, \frac{d}{2}) \cap \overset{\circ}{W}{}_2^1(-\frac{d}{2}, \frac{d}{2})$ be the solutions of systems (4.27).*

Then the inequalities

$$\nu||\varphi_n''||^2_{L_2(-\frac{d}{2},\frac{d}{2})} + \nu||\psi_n''||^2_{L_2(-\frac{d}{2},\frac{d}{2})} \leq \frac{d}{\nu}, \quad \forall n = 0, 1, 2, ... \tag{4.29}$$

hold. Moreover, the numbers a_n and b_n satisfies the properties

$$\begin{array}{ll} (a) & a_n > 0, \quad \forall n = 0, 1, 2, ...; \quad b_0 = 0, b_n > 0, \quad \forall n = 1, 2, ...; \\ (b) & a_n \leq \dfrac{d}{n}, \quad b_n \leq \dfrac{d}{n}, \quad \forall n = 1, 2, ...; \\ (c) & \lim\limits_{n\to\infty} (nb_n) = d. \end{array}$$

Proof. Multiplying (4.27_1) by $\varphi_n''(x_1)$, (4.27_2) by $\psi_n''(x_1)$, integrating on $(-\frac{d}{2}, \frac{d}{2})$ and summing the obtained equations, we get

$$\begin{aligned} &\int\limits_{-d/2}^{d/2} \big(n\psi_n(x_1)\varphi_n''(x_1) - n\varphi_n(x_1)\psi_n''(x_1)\big)dx_1 + \nu||\varphi_n''(x_1)||^2_{L_2(-\frac{d}{2},\frac{d}{2})} \\ &+\nu||\psi_n''(x_1)||^2_{L_2(-\frac{d}{2},\frac{d}{2})} = -\int\limits_{-d/2}^{d/2} \varphi_n''(x_1)dx_1. \end{aligned}$$

Integrating by parts yields

$$\begin{aligned} &\int\limits_{-d/2}^{d/2} \big(n\psi_n(x_1)\varphi_n''(x_1) - n\varphi_n(x_1)\psi_n''(x_1)\big)dx_1 \\ &= \int\limits_{-d/2}^{d/2} \big(-n\psi_n'(x_1)\varphi_n'(x_1) + n\varphi_n'(x_1)\psi_n'(x_1)\big)dx_1 = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\nu||\varphi_n''(x_1)||^2_{L_2(-\frac{d}{2},\frac{d}{2})} + \nu||\psi_n''(x_1)||^2_{L_2(-\frac{d}{2},\frac{d}{2})} = -\int\limits_{-d/2}^{d/2} \varphi_n''(x_1)dx_1 \\ &\leq \frac{\nu}{2}||\varphi_n''(x_1)||^2_{L_2(-\frac{d}{2},\frac{d}{2})} + \frac{d}{2\nu}. \end{aligned}$$

Obviously, from the last inequality follows (4.29).

Next, let us multiply (4.27_1) by $\psi_n(x_1)$, (4.27_2) by $\varphi_n(x_1)$ and integrate by parts on $(-\frac{d}{2}, \frac{d}{2})$:

$$-n||\psi_n||^2_{L_2(-\frac{d}{2},\frac{d}{2})} = \alpha \int\limits_{-d/2}^{d/2} |\psi_n'|^2 dx_1 - \nu \int\limits_{-d/2}^{d/2} \psi_n'\varphi_n' dx_1 - b_n,$$

$$-n||\varphi_n||^2_{L_2(-\frac{d}{2},\frac{d}{2})} = \alpha \int\limits_{-d/2}^{d/2} |\varphi'_n|^2 dx_1 + \nu \int\limits_{-d/2}^{d/2} \psi'_n \varphi'_n dx_1.$$

Summing these equations we get

$$b_n = n\big(||\varphi_n||^2_{L_2(-\frac{d}{2},\frac{d}{2})} + ||\psi_n||^2_{L_2(-\frac{d}{2},\frac{d}{2})}\big) + \alpha\big(||\psi'_n||^2_{L_2(-\frac{d}{2},\frac{d}{2})} + ||\varphi'_n||^2_{L_2(-\frac{d}{2},\frac{d}{2})}\big). \qquad (4.30)$$

If (4.27_1) is multiplied by $\varphi_n(x_1)$ and (4.27_2) by $\psi_n(x_1)$, then analogously as for b_n we obtain

$$a_n = \nu\big(||\psi'_n||^2_{L_2(-\frac{d}{2},\frac{d}{2})} + ||\varphi'_n||^2_{L_2(-\frac{d}{2},\frac{d}{2})}\big). \qquad (4.31)$$

Since the solution of problem (4.27) can not be identically to zero, from (4.30) and (4.31) follows the property (a).

Using Cauchy-Schwarz inequality from (4.28) we get

$$a_n \le \sqrt{d}||\varphi_n||_{L_2(-\frac{d}{2},\frac{d}{2})}, \quad b_n \le \sqrt{d}||\psi_n||_{L_2(-\frac{d}{2},\frac{d}{2})}, \quad n = 0, 1, 2, \dots . \qquad (4.32)$$

The property (b) follows from (4.30) and (4.32).

According to (4.30) and (4.32) we get

$$n||\psi_n||_{L_2(-\frac{d}{2},\frac{d}{2})} \le \sqrt{d}, \quad n = 1, 2, \dots . \qquad (4.33)$$

From (4.31) and (4.32) we obtain

$$\lim_{n\to\infty} ||\varphi'_n||_{L_2(-\frac{d}{2},\frac{d}{2})} = 0, \quad \lim_{n\to\infty} ||\psi'_n||_{L_2(-\frac{d}{2},\frac{d}{2})} = 0. \qquad (4.34)$$

Let us multiply (4.27_1) by $\chi \in C_0^\infty(-\frac{d}{2},\frac{d}{2})$ and integrate over $(-\dfrac{d}{2},\dfrac{d}{2})$:

$$-n \int\limits_{-d/2}^{d/2} \psi_n(x_1)\chi(x_1)dx_1 = \alpha n \int\limits_{-d/2}^{d/2} \psi'_n(x_1)\chi'(x_1)dx_1$$
$$+\nu \int\limits_{-d/2}^{d/2} \varphi'_n(x_1)\chi'(x_1)dx_1 + \int\limits_{-d/2}^{d/2} \chi(x_1)dx_1.$$

In the last equality passing to the limit and using (4.34) yields

$$\lim_{n\to\infty}\big(-n\int\limits_{-d/2}^{d/2}\psi_n(x_1)\chi(x_1)dx_1\big)=\int\limits_{-d/2}^{d/2}\chi(x_1)dx_1. \tag{4.35}$$

The set $C_0^\infty(-\frac{d}{2},\frac{d}{2})$ is dense in the space $L_2(-\frac{d}{2},\frac{d}{2})$. Hence, (4.35) remains valid for all $\chi\in L_2(-\frac{d}{2},\frac{d}{2})$. Taking $\chi(x_1)\equiv 1$ we obtain

$$\lim_{n\to\infty}(nb_b)=\lim_{n\to\infty}\big(-n\int\limits_{-d/2}^{d/2}\psi_n(x_1)dx_1\big)=\int\limits_{-d/2}^{d/2}dx_1=d.$$

□

The flux $F(t)=\int\limits_{-d/2}^{d/2}U(x_1,t)dx_1$ is a periodic function and $F\in L_2(0,2\pi)$. Therefore, it can be expressed by the Fourier series

$$F(t)=\frac{F_0^{(c)}}{2}+\sum_{n=1}^{\infty}\big(F_n^{(c)}\cos(nt)+F_n^{(s)}\sin(nt)\big). \tag{4.36}$$

Lemma 4.2. *Fourier coefficients* $(q_n^{(c)},q_n^{(s)})$ *and* $(F_n^{(c)},F_n^{(s)})$ *of the functions* $q(t)$ *and* $F(t)$*, are related by equalities*

$$F_n^{(c)}=a_nq_n^{(c)}-b_nq_n^{(s)},\quad F_n^{(s)}=b_nq_n^{(c)}+a_nq_n^{(s)},\quad n=0,1,2,..., \tag{4.37}$$

or, equivalently,

$$q_n^{(c)}=\frac{a_nF_n^{(c)}+b_n^{(s)}}{a_n^2+b_n^2},\quad q_n^{(s)}=\frac{a_nF_n^{(s)}-b_nF_n^{(c)}}{a_n^2+b_n^2}. \tag{4.38}$$

Proof. Let us define the functions

$$\widehat{U}_n^{(c)}(x_1,t)=U_n^{(c)}(x_1,-t),\quad \widehat{U}_n^{(s)}(x_1,t)=U_n^{(s)}(x_1,-t),\quad t\in[0,2\pi] \tag{4.39}$$

which are the solutions of the following problems

$$\begin{cases}\dfrac{\partial}{\partial t}(\widehat{U}_n^{(c)}+\alpha\dfrac{\partial^2\widehat{U}_n^{(c)}}{\partial x_1^2})+\nu\dfrac{\partial^2\widehat{U}_n^{(c)}}{\partial x_1^2}=-\cos(nt),\\ \widehat{U}_n^{(c)}(-\dfrac{d}{2},t)=\widehat{U}_n^{(c)}(\dfrac{d}{2},t)=0,\quad \widehat{U}_n^{(c)}(x_1,0)=\widehat{U}_n^{(c)}(x_1,2\pi),\end{cases} \tag{4.40}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t}(\widehat{U}_n^{(s)} + \alpha\dfrac{\partial^2 \widehat{U}_n^{(s)}}{\partial x_1^2}) + \nu\dfrac{\partial^2 \widehat{U}_n^{(s)}}{\partial x_1^2} = -\sin(nt), \\ \widehat{U}_n^{(s)}(-\dfrac{d}{2}, t) = \widehat{U}_n^{(s)}(\dfrac{d}{2}, t) = 0, \quad \widehat{U}_n^{(s)}(x_1, 0) = \widehat{U}_n^{(s)}(x_1, 2\pi). \end{cases} \tag{4.41}$$

It is obvious, that

$$\begin{aligned} U_n^{(c)}(x_1, t) &= \varphi_n(x_1)\cos(nt) + \psi_n(x_1)\sin(nt), \\ U_n^{(s)}(x_1, t) &= \psi_n(x_1)\cos(nt) - \varphi_n(x_1)\sin(nt). \end{aligned} \tag{4.42}$$

Multiplying equation (4.4_1) by $\widehat{U}_n^{(c)}(x_1, t)$ and integrating by parts on $(-\frac{d}{2}, \frac{d}{2})$ yields

$$\begin{aligned} &\int\limits_{-d/2}^{d/2} \frac{\partial U(x_1,t)}{\partial t}\widehat{U}_n^{(c)}(x_1,t)dx_1 - \alpha \int\limits_{-d/2}^{d/2} \frac{\partial U(x_1,t)}{\partial t}\frac{\partial^2 \widehat{U}_n^{(c)}(x_1,t)}{\partial x_1^2}dx_1 \\ &-\nu \int\limits_{-d/2}^{d/2} U(x_1,t)\frac{\partial^2 \widehat{U}_n^{(c)}(x_1,t)}{\partial x_1^2}dx_1 = \int\limits_{-d/2}^{d/2} q(t)\widehat{U}_n^{(c)}(x_1,t)dx_1. \end{aligned} \tag{4.43}$$

Using the equalities

$$\frac{\partial U(x_1,t)}{\partial t}\widehat{U}_n^{(c)}(x_1,t) = \frac{\partial}{\partial t}\big(U(x_1,t)\widehat{U}_n^{(c)}(x_1,t)\big) - U(x_1,t)\frac{\partial \widehat{U}_n^{(c)}(x_1,t)}{\partial t},$$

$$\frac{\partial U(x_1,t)}{\partial t}\frac{\partial^2 \widehat{U}_n^{(c)}(x_1,t)}{\partial x_1^2} = \frac{\partial}{\partial t}\big(U(x_1,t)\frac{\partial^2 \widehat{U}_n^{(c)}(x_1,t)}{\partial x_1^2}\big) - U(x_1,t)\frac{\partial^3 \widehat{U}_n^{(c)}(x_1,t)}{\partial t \partial x_1^2},$$

we rewrite (4.43) in the form

$$\begin{aligned} &\frac{d}{dt}\int\limits_{-d/2}^{d/2} U(x_1,t)\widehat{U}_n^{(c)}(x_1,t)dx_1 - \alpha\frac{d}{dt}\int\limits_{-d/2}^{d/2} U(x_1,t)\frac{\partial^2 \widehat{U}_n^{(c)}(x_1,t)}{\partial x_1^2}dx_1 \\ &- \int\limits_{-d/2}^{d/2} U(x_1,t)\big(\widehat{U}_n^{(c)}(x_1,t) + \alpha\frac{\partial^2 \widehat{U}_{nt}^{(c)}(x_1,t)}{\partial x_1^2} + \nu\frac{\partial^2 \widehat{U}_n^{(c)}(x_1,t)}{\partial x_1^2}\big)dx_1 \\ &= \int\limits_{-d/2}^{d/2} q(t)\widehat{U}_n^{(c)}(x_1,t)dx_1. \end{aligned}$$

Integrating the last equality by t from 0 to 2π, using periodicity condition, the definition of the functions $\widehat{U}_n^{(c)}$ and (4.28), we obtain

$$
\begin{aligned}
&-\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2} U(x_1,t)\big(\widehat{U}_n^{(c)}(x_1,t)+\alpha\frac{\partial^2\widehat{U}_{nt}^{(c)}(x_1,t)}{\partial x_1^2}+\nu\frac{\partial^2\widehat{U}_n^{(c)}(x_1,t)}{\partial x_1^2}\big)dx_1dt\\
&=\int\limits_0^{2\pi} q(t)\big(\int\limits_{-d/2}^{d/2}\widehat{U}_n^{(c)}(x_1,t)dx_1\big)dt=\int\limits_0^{2\pi} q(t)(a_n\cos(nt)-b_n\sin(nt))dt.
\end{aligned}
$$

It follows from (4.40_1) that

$$
\begin{aligned}
&-\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2} U(x_1,t)\big(\widehat{U}_n^{(c)}(x_1,t)+\alpha\frac{\partial^2\widehat{U}_{nt}^{(c)}(x_1,t)}{\partial x_1^2}+\nu\frac{\partial^2\widehat{U}_n^{(c)}(x_1,t)}{\partial x_1^2}\big)dx_1dt\\
&=\int\limits_0^{2\pi}\int\limits_{-d/2}^{d/2} U(x_1,t)\cos(nt)dx_1dt=\int\limits_0^{2\pi} F(t)\cos(nt)dt.
\end{aligned}
$$

Therefore, the following formula

$$
\int\limits_0^{2\pi} F(t)\cos(nt)dt=\int\limits_0^{2\pi} q(t)(a_n\cos(nt)-b_n\sin(nt))dt \tag{4.44}
$$

holds. Substituting series (4.10) and (4.36) into (4.41) we obtain the first formula in (4.37). The second one can be proved in the same way, taking $\widehat{U}_n^{(s)}(x_1,t)$ instead of $\widehat{U}_n^{(c)}(x_1,t)$.

Since $a_n^2+b_n^2>0$ from (4.37) follows (4.38).□

Lemma 4.3. *Let* $(F_n^{(c)},F_n^{(s)})$ *and* $(q_n^{(c)},q_n^{(s)})$, $n=0,1,2,...$, *satisfy the relations* (4.37) *or* (4.38). *If the Fourier series* (4.7) *converges in the space* $L_2(0,2\pi)$ *to some* $q\in L_2(0,2\pi)$, *then* (4.36) *converges to* $F\in L_2(0,2\pi)$. *Moreover, the following estimate is valid*

$$
||F||_{L_2(0,2\pi)}\le c_1||q||_{L_2(0,2\pi)}. \tag{4.45}
$$

Opposite, if the Fourier series (4.36) *converges to* $F\in W_2^1(0,2\pi)$, *then* (4.7) *converges to* $q\in L_2(0,2\pi)$ *and*

$$
||q||_{L_2(0,2\pi)}\le c_2||F||_{W_2^1(0,2\pi)}. \tag{4.46}
$$

Constants c_1 *and* c_2 *depends only on* d.

Proof. In addition to (4.37) and Lemma 4.1 (a), we have the relations

$$
F_0^{(c)}=a_0q_0^{(c)},\quad |F_n^{(c)}|^2+|F_n^{(s)}|^2=(a_n^2+b_n^2)(|q_n^{(c)}|^2+|q_n^{(s)}|^2),\quad n\ge 1. \tag{4.47}
$$

Using Lemma 4.1 (b) we get

$$\frac{|F_0^{(c)}|^2}{2} + \sum_{n=1}^{\infty} \left(|F_n^{(c)}|^2 + |F_n^{(s)}|^2\right) \leq a_0^2 \frac{|q_0^{(c)}|}{2} + 2d^2 \sum_{n=1}^{\infty} \left(|q_n^{(c)}|^2 + |q_n^{(s)}|^2\right).$$

If $q \in L_2(0,2\pi)$, then series (4.36) converges in $L_2(0,2\pi)$ and inequality (4.45) follows from Parseval equality.

From (4.47) it follows that

$$|q_n^{(c)}|^2 + |q_n^{(s)}|^2 \leq \frac{1}{b_n^2}\left(|F_n^{(c)}|^2 + |F_n^{(s)}|^2\right), \quad n \geq 1. \tag{4.48}$$

By Lemma 4.1 (c) we obtain, that there exists a number N_0 such that

$$b_n \geq \frac{d}{2n}, \quad \forall n \geq N_0.$$

Let us take $b_* = min\{b_1, ..., b_{N_0}\}$. Using Lemma 4.1 (a) we get the inequality $b_* > 0$. From Lemma 4.1 (b) and inequality (4.48) it follows that

$$\begin{aligned}&\frac{|q_0^{(c)}|^2}{2} + \sum_{n=1}^{\infty} \left(|q_n^{(c)}|^2 + |q_n^{(s)}|^2\right) \leq \frac{1}{a_0} \frac{|F_0^{(c)}|}{2} + \frac{1}{b_*^2} \sum_{n=1}^{\infty} \left(|F_n^{(c)}|^2 + |F_n^{(s)}|^2\right) \\ &+ \frac{4}{d^2} \sum_{n=1}^{\infty} n^2 \left(|F_n^{(c)}|^2 + |F_n^{(s)}|^2\right).\end{aligned}$$

If $F \in W_2^1(0,2\pi)$, then series (4.7) converges in $L_2(0,2\pi)$. From Parseval equality we get (4.46).□

4.3 Inverse problem

Theorem 4.2. *Let $F \in W_2^1(0,2\pi)$ be 2π-periodic function. There exists the unique 2π-periodic solution $(U(x_1,t), q(t)) \in W_2^{2,1}((-\frac{d}{2},\frac{d}{2}) \times (0,2\pi)) \times L_2(0,2\pi)$ of problem (4.4). Moreover, the following estimate*

$$\begin{aligned}&||U||_{W_2^{2,1}((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} + \left\|\frac{\partial U}{\partial x_1}\right\|_{W_2^{1,1}((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} \\ &+ \left\|\frac{\partial^3 U}{\partial t \partial x_1^2}\right\|_{L_2((-\frac{d}{2},\frac{d}{2})\times(0,2\pi))} \leq c||F||_{W_2^1(0,2\pi)}\end{aligned} \tag{4.49}$$

holds.

Proof. Let $F_n^{(c)}$ and $F_n^{(s)}$ be the coefficients of function $F(t)$. The function $q(t)$ is determinated by formula (4.7) with coefficients $q_n^{(c)}$ and $q_n^{(s)}$ defined by (4.38). In virtue of Lemma 4.3, series (4.7) converges in $L_2(0, 2\pi)$ and the limit function $q \in L_2(0, 2\pi)$. Moreover, inequality (4.46) is valid. From Theorem 4.1 it follows that problem (4.4) has the unique solution $U \in W_2^{2,1}((-\frac{d}{2}, \frac{d}{2}) \times (0, 2\pi))$ and estimate (4.6) is valid. Obviously, $(U(x_1, t), q(t))$ is the solution of problem (4.4) and from (4.6), (4.46) follows (4.49).

Let $F(t) = 0$. Multiply (4.4_1) by $U(x_1, t)$, integrate by parts on $(-\dfrac{d}{2}, \dfrac{d}{2})$ and by t from 0 to 2π:

$$\int\limits_0^{2\pi} \int\limits_{-d/2}^{d/2} \Big|\frac{\partial U(x_1, t)}{\partial x_1}\Big|^2 dx_1 dt = \int\limits_0^{2\pi} q(t) \int\limits_{-d/2}^{d/2} U(x_1, t) dx_1 dt = \int\limits_0^{2\pi} q(t) F(t) dt = 0.$$

Therefore, $U(x_1, t) = 0$. From (4.4_1) it follows that $q(t) = 0$.□

Remark 4.1. *The solution of problem* (4.4) *can be expressed by the following series*

$$\begin{aligned} U(x_1, t) = \frac{q_0^{(c)}}{2} \varphi_0(x_1) + \sum_{n=1}^{\infty} \big(\big(q_n^{(c)} \varphi_n(x_1) + q_n^{(s)} \psi_n(x_1)\big) \cos(nt) \\ + \big(q_n^{(s)} \varphi_n(x_1) - q_n^{(c)} \psi_n(x_1)\big) \sin(nt)\big), \end{aligned}$$

where functions $\varphi_n(x_1)$, $\psi_n(x_1)$, $n = 0, 1, 2, ...$, are the solutions of problems (4.27) *and the coeficients $q_n^{(c)}$, $q_n^{(s)}$ are expressed by formulas* (4.38).

Chapter 5

Flow in a pipe with arbitrary cross section

In this chapter we study the flow of the incompressible non-Newtonian second grade fluid flow problem

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + \mathrm{curl}(\mathbf{u} - \alpha\Delta\mathbf{u}) \times \mathbf{u} + \nabla\widetilde{p} = \mathbf{f}, \\ \mathrm{div}\mathbf{u} = 0, \\ \mathbf{u}|_{S^T} = 0, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x), \end{cases} \tag{5.1}$$

with additionally prescribed flux condition

$$\int\limits_{\sigma} u_3(x', x_3, t)dx' = F(t) \tag{5.2}$$

in the three-dimensional pipe $\Pi = \{x = (x', x_3) \in \mathbb{R}^3 : x' \in \sigma \subset \mathbb{R}^2, x_3 \in \mathbb{R}\}$, where the cross-section σ is an arbitrary bounded plane domain with the boundary $\partial\sigma$ of class C^4.

First, let us assume (as in the previous chapters) that the initial velocity $\mathbf{u}_0(x)$ and the exterior force $\mathbf{f}(x,t)$ are independent of the coordinate x_3 and have the forms

$$\mathbf{u}_0(x) = (0, 0, u_0(x')), \quad \mathbf{f}(x,t) = (0, 0, f(x', t)). \tag{5.3}$$

Let us try to find the unidirectional solution

$$\mathbf{u}(x,t) = (0, 0, U(x', t)). \tag{5.4}$$

Substituting expression (5.4) into system (5.1) we get

$$\begin{cases} -U\dfrac{\partial}{\partial x_1}(U-\alpha\Delta U)+\dfrac{\partial p}{\partial x_1}=0, \\ -U\dfrac{\partial}{\partial x_2}(U-\alpha\Delta U)+\dfrac{\partial p}{\partial x_2}=0, \\ \dfrac{\partial}{\partial t}(U-\alpha\Delta U)-\nu\Delta U+\dfrac{\partial p}{\partial x_3}=f, \\ U|_{\partial\sigma}=0, \quad U(x',0)=u_0. \end{cases}$$

The vector $(-U\frac{\partial}{\partial x_1}(U-\alpha\Delta U), -U\frac{\partial}{\partial x_2}(U-\alpha\Delta U))^T$ cannot be expressed as a gradient of some function. Therefore, we can not look for a solution of problem (5.1)-(5.2) in form (5.4), and even in the case of unidirectional data (having the form (5.3)) the velocity field has all three components.

In this chapter we consider a little bit more general case when data have all three components. Let us suppose that $\mathbf{f}$ and $\mathbf{u}_0$ do not depend on the coordinate x_3 and have the forms

$$\mathbf{u}_0(x)=(u_{01}(x'),u_{02}(x'),v_0(x')), \qquad \mathbf{f}(x,t)=(f_1(x',t),f_2(x',t),f_3(x',t)),$$

and look for the *Poiseuille* type solution $(\mathbf{u}(x,t),p(x,t))$ of system (5.1) having the following form

$$\begin{aligned} &\mathbf{u}(x,t)=(U_1(x',t),U_2(x',t),U_3(x',t)), \\ &p(x,t)=\widetilde{p}(x',t)-q(t)x_3+p_0(t), \end{aligned} \tag{5.5}$$

where $p_0(t)$ is an arbitrary function. Moreover, we look for the solution having the prescribed flux

$$\int\limits_\sigma U_3(x',t)\,dx'=F(t). \tag{5.6}$$

Obviously, in this case the initial velocity has to satisfy the necessary compatibility condition

$$\int\limits_\sigma v_0(x')dx'=F(0). \tag{5.7}$$

Substituting expressions (5.5) into equations (5.1), (5.2) we obtain the following problem on $\sigma^T=\sigma\times(0,T)$:

$$
\begin{cases}
\dfrac{\partial}{\partial t}(\mathbf{U}' - \alpha\triangle\mathbf{U}') - \nu\triangle\mathbf{U}' + \operatorname{curl}(\mathbf{U}' - \alpha\triangle\mathbf{U}') \times \mathbf{U}' \\
\quad -U_3(\nabla(U_3 - \alpha\triangle U_3)) + \nabla\widetilde{p} = \mathbf{f}', \\
\operatorname{div}\mathbf{U}' = 0, \\
\mathbf{U}'|_{\partial\sigma} = 0, \quad \mathbf{U}'(x',0) = \mathbf{u}_0'(x'), \\
\\
\dfrac{\partial}{\partial t}(U_3 - \alpha\triangle U_3) - \nu\triangle U_3 + (\mathbf{U}'\cdot\nabla)(U_3 - \alpha\triangle U_3) = f_3 + q(t), \\
U_3|_{\partial\sigma} = 0, \quad U_3(x',0) = v_0(x'), \\
\int\limits_{\sigma} U_3(x',t)dx' = F(t),
\end{cases}
\tag{5.8}
$$

where $\mathbf{f}'(x',t) = (f_1(x',t), f_2(x',t)), \mathbf{U}'(x',t) = (U_1(x',t), U_2(x',t))$. Notice that in (5.8) functions $\mathbf{u}_0(x')$, $\mathbf{f}(x',t)$ and $F(t)$ are given, while $\mathbf{U}(x',t)$, $\widetilde{p}(x',t)$ and $q(t)$ are unknown and have to be found.

Let us remind that for the two-dimensional vector-fields we understand the operator curl as the usual operator curl for the three-dimensional vectors with the third component equal to zero.

5.1 Function spaces and auxiliary results

In this section we define function spaces, which are used in the following calculus. Let $\sigma \subset \mathbb{R}^2$ be a bounded domain. Define the function space

$$
\mathring{\mathcal{X}}(\sigma) = \{u : u \in \mathring{W}_2^1(\sigma), \nabla(u - \alpha\Delta u) \in L_2(\sigma)\}
$$

with the norm

$$
\|u\|^2_{\mathring{\mathcal{X}}(\sigma)} = \|u\|^2_{W_2^1(\sigma)} + \|\nabla(u - \alpha\Delta u)\|^2_{L_2(\sigma)}.
$$

In the case $\partial\sigma \in C^3$ the space $\mathring{\mathcal{X}}(\sigma)$ is equivalent to $W_2^3(\sigma) \cap \mathring{W}_2^1(\sigma)$. Indeed, if $u \in W_2^3(\sigma) \cap \mathring{W}_2^1(\sigma)$ then, obviously, $u \in \mathring{\mathcal{X}}(\sigma)$ and $\|u\|_{\mathring{\mathcal{X}}(\sigma)} \le c\|u\|_{W_2^3(\sigma)}$. On the other hand, applying the Nečas inequality to Δu we get

$$
\begin{aligned}
&\|\Delta u\|_{L_2(\sigma)} \le c\Big(\|\Delta u\|_{W_2^{-1}(\sigma)} + \|\nabla\Delta u\|_{L_2(\sigma)}\Big) \\
&\le c\Big(\|\nabla u\|_{L_2(\sigma)} + \|\nabla\Delta u\|_{L_2(\sigma)}\Big).
\end{aligned}
\tag{5.9}
$$

Therefore, considering u as a solution of the Poisson equation

$$\begin{cases} -\Delta u = -\Delta u, \\ u|_{\partial\sigma} = 0, \end{cases}$$

we obtain the estimate

$$\begin{aligned} \|u\|_{W_2^3(\sigma)} &\leq c\Big(\|\Delta u\|_{L_2(\sigma)} + \|\nabla\Delta u\|_{L_2(\sigma)}\Big) \\ &\leq c\Big(\|\nabla u\|_{L_2(\sigma)} + \|\nabla\Delta u\|_{L_2(\sigma)}\Big) \leq c\|u\|_{\mathring{\mathcal{X}}(\sigma)}. \end{aligned} \tag{5.10}$$

Denote

$$\mathring{\mathcal{Y}}(\sigma) = \{\mathbf{u} : \mathbf{u} \in \mathring{W}_2^1(\sigma), \operatorname{curl}(\mathbf{u} - \alpha\Delta\mathbf{u}) \in L_2(\sigma), \operatorname{div}\mathbf{u} = 0\}.$$

The norm in $\mathring{\mathcal{Y}}(\sigma)$ is defined by

$$\|\mathbf{u}\|^2_{\mathring{\mathcal{Y}}(\sigma)} = \|\mathbf{u}\|^2_{W_2^1(\sigma)} + \|\operatorname{curl}(\mathbf{u} - \alpha\Delta\mathbf{u})\|^2_{L_2(\sigma)}.$$

In the case $\partial\sigma \in C^3$ the space $\mathring{\mathcal{Y}}(\sigma)$ is equivalent to $W_2^3(\sigma) \cap \mathring{W}_2^1(\sigma)$ and

$$\|\mathbf{u}\|_{W_2^3(\sigma)} \leq c\|\mathbf{u}\|_{\mathring{\mathcal{Y}}(\sigma)}. \tag{5.11}$$

The proof of these facts is given in [15].

Let

$$\mathring{\mathcal{W}}(\sigma^T) = \{u : D_x^\alpha u \in L_2(\sigma^T), |\alpha| \leq 3, \frac{\partial}{\partial t}u \in L_2(\sigma^T), \nabla\frac{\partial}{\partial t}u \in L_2(\sigma^T), u|_{\partial\sigma} = 0\}$$

with the norm

$$\|u\|^2_{\mathring{\mathcal{W}}(\sigma^T)} = \sum_{|\alpha|\leq 3}\|D_x^\alpha u\|^2_{L_2(\sigma^T)} + \|\frac{\partial}{\partial t}u\|^2_{L_2(\sigma^T)} + \|\nabla\frac{\partial}{\partial t}u\|^2_{L_2(\sigma^T)}$$

and

$$\mathring{\mathcal{V}}(\sigma^T) = \{\mathbf{u} : \mathbf{u} \in \mathring{\mathcal{W}}(\sigma^T),\ \operatorname{div}\mathbf{u} = 0\}.$$

Below, by constructing Galerkin approximations for the solution of problem (5.8), we use, following the ideas from [14] - [15], the special basis in the space $\mathring{\mathcal{Y}}(\sigma)$. Let $\{\mu_k\}$ and

$\{\mathbf{w}_k\}_{k\geq 1} \subset \overset{\circ}{\mathcal{Y}}(\sigma)$ be eigenvalues and eigenfunctions[1] of the following problem

$$\begin{aligned}&\int_\sigma \nabla(\mathbf{w}_k(x') - \alpha\Delta\mathbf{w}_k(x'))\cdot\nabla(\boldsymbol{\rho}(x') - \alpha\Delta\boldsymbol{\rho}(x'))dx'\\&= (\mu_k - 1)\int_\sigma(\mathbf{w}_k\rho(x') + \alpha\nabla\mathbf{w}_k(x')\cdot\nabla\boldsymbol{\rho}(x'))dx' \quad \forall\, \boldsymbol{\rho}\in\overset{\circ}{\mathcal{Y}}(\sigma).\end{aligned} \tag{5.12}$$

Theorem 5.1. (see [13], Lemma 4.1) *Let $\sigma \subset \mathbb{R}^2$ be a bounded simply connected domain with the boundary $\partial\sigma \in C^4$. Then*

- (5.12) *defines a countable set of eigenvalues $\mu_k > 1$, $k = 1, 2, \ldots$; the corresponding eigenfunctions $\{\mathbf{w}_k\}_{k\geq 1}$ constitute basiss in the spaces $\overset{\circ}{\mathcal{Y}}(\sigma)$, $\overset{\circ}{W}{}_2^1(\sigma)$ and $L_2(\sigma)$.*
- *The eigenfunctions $\mathbf{w}_k$ of* (5.12) *belong to the space $W_2^4(\sigma)$.*

Analogous basis can be constructed in the space $\overset{\circ}{\mathcal{X}}(\sigma)$. Let $\{\lambda_k\}$ and $\{w_k\}_{k\geq 1} \subset \overset{\circ}{\mathcal{X}}(\sigma)$ be eigenvalues and eigenfunctions of the problem

$$\begin{aligned}&\int_\sigma \nabla(w_k(x') - \alpha\Delta w_k(x'))\cdot\nabla(\rho(x') - \alpha\Delta\rho(x'))dx'\\&= (\lambda_k - 1)\int_\sigma(w_k\rho(x') + \alpha\nabla w_k(x')\cdot\nabla\rho(x'))dx' \quad \forall\, \rho\in\overset{\circ}{\mathcal{X}}(\sigma).\end{aligned} \tag{5.13}$$

Theorem 5.2. *Let $\sigma \subset \mathbb{R}^2$ be a bounded simply connected domain with the boundary $\partial\sigma \in C^4$. Then*

- (5.13) *defines a countable set of eigenvalues $\lambda_k > 1$, $k = 1, 2, \ldots$; the corresponding eigenfunctions $\{w_k\}_{k\geq 1}$ constitute basiss in $\overset{\circ}{\mathcal{X}}(\sigma)$, $\overset{\circ}{W}{}_2^1(\sigma)$ and $L_2(\sigma)$.*
- *The eigenfunctions w_k can be orthonormalized:*

$$\int_\sigma(w_k w_l + \alpha\nabla w_k\cdot\nabla w_l)dx' = \begin{cases} 0, & k\neq l,\\ 1, & k = l.\end{cases} \tag{5.14}$$

  Then

$$\int_\sigma\nabla(w_k - \alpha\Delta w_k)\cdot\nabla(w_l - \alpha\Delta w_l)dx' = \begin{cases} 0, & k\neq l,\\ \lambda_k - 1, & k = l.\end{cases} \tag{5.15}$$

- *The eigenfunctions w_k of* (5.13) *belong to the space $W_2^4(\sigma)$.*

[1] i.e., $\mathbf{w}_k$ are nontrivial solutions of (5.12).

Proof. The first property can be proved by standard arguments (see, for example, [17] Chapter 9.2). The second property follows direct from the integral identity (5.13). To prove the third property, we denote $\rho - \alpha\Delta\rho = \varphi$ and rewrite identity (5.13) in the form

$$\int\limits_\sigma \nabla(w_k(x') - \alpha\Delta w_k(x')) \cdot \nabla\varphi dx' = (\lambda_k - 1)\int\limits_\sigma w_k\varphi dx'.$$

Obviously, if $\rho \in \mathring{\mathcal{X}}(\sigma)$, then $\varphi \in W_2^1(\sigma)$. On the other hand, for any $\varphi \in W_2^1(\sigma)$ there exists a unique $\rho \in \mathring{\mathcal{X}}(\sigma)$ such that $\rho - \alpha\Delta\rho = \varphi$. Therefore, $w_k - \alpha\Delta w_k \in W_2^1(\sigma)$ can be interpreted as a weak solution to the following Neumann problem:

$$\begin{cases} -\Delta(w_k - \alpha\Delta w_k) = (\lambda_k - 1)w_k, \\ \dfrac{\partial(w_k - \alpha\Delta w_k)}{\partial n}\Big|_{\partial\sigma} = 0. \end{cases}$$

Since $w_k \in L_2(\sigma)$, we conclude that $w_k - \alpha\Delta w_k \in W_2^2(\sigma)$. Consider now w_k as a solution of the Dirichlet problem:

$$\begin{cases} w_k - \alpha\Delta w_k = w_k - \alpha\Delta w_k, \\ w_k|_{\partial\sigma} = 0. \end{cases}$$

Since $\sigma \in C^4$, it follows (see, for example, [26]) that $w_k \in W_2^4(\sigma) \cap \mathring{W}_2^1(\sigma)$. $\square$

5.2 Direct problem in a plane domain

For small data problem (5.8) can be solved by iterations, dividing it into two problems:

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{U}' - \alpha\Delta\mathbf{U}') - \nu\Delta\mathbf{U}' + curl(\mathbf{U}' - \alpha\Delta\mathbf{U}') \times \mathbf{U}' + \nabla\widetilde{p} \\ \qquad = U_3(\nabla(U_3 - \alpha\Delta U_3)) + \mathbf{f}', \\ \mathrm{div}_{x'}\mathbf{U}' = 0, \\ \mathbf{U}'|_{\partial\sigma} = 0, \quad \mathbf{U}'(x', 0) = \mathbf{u}_0'(x'), \end{cases} \tag{5.16}$$

with given U_3 and

$$\begin{cases} \dfrac{\partial}{\partial t}(U_3 - \alpha\Delta U_3) - \nu\Delta U_3 + (\mathbf{U}' \cdot \nabla)(U_3 - \alpha\Delta U_3) = f + q(t), \\ U_3|_{\partial\sigma} = 0, \quad U_3(x', 0) = v_0(x'), \\ \int\limits_\sigma U_3(x', t)dx' = F(t), \end{cases} \tag{5.17}$$

with given $\mathbf{U}'$. Problem (5.16) with the given right-hand side is the standard initial boundary value problem describing the motion of the second grade fluid in a bounded plane domain σ. Such two- and three-dimensional problems have been studied by several authors (see [13]-[15], [41]-[39], [60], [86], etc.). Problem (5.17) is an inverse problem (the function q in the right-hand side is unknown).

By studying the inverse problem (5.17) it is convenient to reduce it to the case of zero right-hand side f and the homogeneous initial condition. This could be done by subtracting the solution v of the direct problem

$$\begin{cases} \dfrac{\partial}{\partial t}(v-\alpha\Delta v)-\nu\Delta v+(\mathbf{U}'\cdot\nabla)(v-\alpha\Delta v)=f, \\ v|_{\partial\sigma}=0, \quad v(x',0)=v_0(x'). \end{cases} \tag{5.18}$$

5.2.1 Construction of approximate solutions

In this section we prove the existence of the unique solution to the problem

$$\begin{cases} \dfrac{\partial}{\partial t}(v-\alpha\Delta v)-\nu\Delta v+(\mathbf{W}\cdot\nabla)(v-\alpha\Delta v)=f, \\ v|_{\partial\sigma}=0, \quad v(x',0)=v_0(x'), \end{cases} \tag{5.19}$$

assuming, that the function $\mathbf{W}\in\mathring{\mathcal{V}}(\sigma^T)$ is given and satisfies the following condition

$$\sup_{t\in[0,T]}\|\mathbf{W}\|_{\mathring{X}(\sigma)}+\|\mathbf{W}\|_{\mathring{\mathcal{W}}(\sigma^T)}\le\delta_0, \tag{5.20}$$

where δ_0 is a sufficiently small constant.

Deffinition. *The function $v\in\mathring{\mathcal{W}}(\sigma^T)$ is called a weak solution of problem* (5.19) *if it satisfies for all $t\in[0,T]$ the integral identity*

$$\begin{aligned} &\int_0^t\int_\sigma(\partial_\tau v\,\eta+\alpha\nabla\partial_\tau v\cdot\nabla\eta)\,dx'd\tau+\nu\int_0^t\int_\sigma\nabla v\cdot\nabla\eta dx'd\tau \\ &=\int_0^t\int_\sigma f\eta\,dx'd\tau+\int_0^t\int_\sigma(\boldsymbol{W}\cdot\nabla)\eta\,(v-\alpha\Delta v)\,dx'd\tau\ \ \forall\,\eta\in\mathring{W}_2^{1,0}(\sigma^T) \end{aligned} \tag{5.21}$$

and the initial condition $v(x',0)=v_0(x')$.

Let $f \in L_2(\sigma^T)$, $v_0 \in \overset{\circ}{W}{}_2^1(\sigma)$. Then we can express them by the Fourier series

$$f(x',t) = \sum_{k=1}^{\infty} f_k(t)w_k(x'), \quad v_0(x') = \sum_{k=1}^{\infty} a_k w_k(x'),$$

where $f_k(t) = \int_\sigma f_3(x',t)w_k(x')dx'$, $a_k = \int_\sigma v_0(x')w_k(x')dx'$ and $\{w_k\} \subset \overset{\circ}{\mathcal{X}}(\sigma)$ are the eigenfunctions of problem (5.13).

We look for the approximate solutions $v^{(N)}(x',t)$ in the form

$$v^{(N)}(x',t) = \sum_{k=1}^{N} y_k^{(N)}(t)w_k(x'),$$

where the coefficients $y_k^{(N)}(t)$ are found from the integral equalities

$$\begin{aligned}&\int_\sigma \Big(\frac{\partial}{\partial t}v^{(N)}w_k + \alpha\nabla\frac{\partial}{\partial t}v^{(N)}\cdot\nabla w_k\Big)dx' + \nu\int_\sigma \nabla v^{(N)}\cdot\nabla w_k dx' \\ &= \int_\sigma f^{(N)}w_k dx' + \int_\sigma \mathbf{W}\cdot\nabla w_k(v^{(N)} - \alpha\Delta v^{(N)})dx', \quad k = 1,...,N,\end{aligned} \tag{5.22}$$

and the initial condition $v^{(N)}(x',0) = v_0^{(N)}(x')$, where

$$f^{(N)}(x',t) = \sum_{k=1}^{N} f_k(t)w_k(x'), \quad v_0^{(N)}(x') = \sum_{k=1}^{N} a_k w_k(x').$$

Since the eigenfunctions w_k are smooth ($w_k \in W_2^4(\sigma)$), the approximations $v^{(N)}(x',t)$ are solutions to the following initial boundary value problems

$$\begin{cases} \dfrac{\partial}{\partial t}(v^{(N)} - \alpha\Delta v^{(N)}) - \nu\Delta v^{(N)} + (\mathbf{W}\cdot\nabla)(v^{(N)} - \alpha\Delta v^{(N)}) = f^{(N)}, \\ v^{(N)}|_{\partial\sigma} = 0, \quad v^{(N)}(x',0) = v_0^{(N)}(x'). \end{cases} \tag{5.23}$$

Multiplying (5.22) by $w_j(x')$ and using (5.14) we derive the Cauchy problem for the system of linear ordinary differential equations

$$\begin{cases} y_k^{(N)\prime}(t) + \sum_{j=1}^{N}\Big(\dfrac{\nu}{\alpha} + m_{kj}(t)\Big)y_k^{(N)}(t) = f_k(t), \ k = 1,...,N, \\ y_k^{(N)}(0) = a_k, \qquad k = 1,...,N, \end{cases}$$

where $m_{kj}(t) = -\int_\sigma \big(\frac{\nu}{\alpha}w_k w_j - (\mathbf{W}\cdot\nabla)w_j(w_k - \alpha\Delta w_k)\big)\,dx'$.

The last system can be rewritten in the vector form:

$$\begin{cases} \mathbf{Y}^{(N)\prime}(t) + (\mathbb{J}^{(N)} + \mathbb{A}^{(N)}(t))\mathbf{Y}^{(N)}(t) = \mathbf{f}(t), \\ \mathbf{Y}^{(N)}(0) = \mathbf{a}, \end{cases} \tag{5.24}$$

where

$$\mathbf{Y}^{(N)}(t) = \begin{pmatrix} y_1^{(N)}(t) \\ \dots \\ y_N^{(N)}(t) \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f_1 \\ \dots \\ f_N \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} a_1 \\ \dots \\ a_N \end{pmatrix},$$

$\mathbb{J}^{(N)} = \mathrm{diag}(\frac{\nu}{\alpha}, ..., \frac{\nu}{\alpha})$ - diagonal matrix, $\mathbb{A}^{(N)}(t)$ is $(N \times N)$ - matrix with elements $m_{kj}(t)$.

Lemma 5.1. *Let* $f \in L_2(\sigma^T)$, $v_0 \in \overset{\circ}{W}{}_2^1(\sigma)$. *Suppose that* $\boldsymbol{W} \in \overset{\circ}{\mathcal{V}}(\sigma^T)$ *is given and satisfies* (5.20). *Then there exist a unique solution* $Y^{(N)} \in W_2^1(0,T)$ *of system* (5.24).

Proof. Let us prove that elements $m_{kj}(t)$ of the matrix $\mathbb{A}^{(N)}(t)$ are bounded. We have

$$\begin{aligned} &|m_{kj}(t)| = \Big| \int_\sigma -\frac{\nu}{\alpha} w_k(x')w_j(x') - (\mathbf{W}(x',t)\cdot\nabla)w_j(x')(w_k(x') - \alpha\Delta w_k(x'))dx' \Big| \\ &\le \frac{\nu}{\alpha} + \Big| \int_\sigma (\mathbf{W}(x',t)\cdot\nabla)(w_k(x') - \alpha\Delta w_k(x'))w_j(x')dx' \Big| \\ &\le \frac{\nu}{\alpha} + \|\mathbf{W}(\cdot,t)\|_{L_4(\sigma)} \|\nabla(w_k - \alpha\Delta w_k)\|_{L_2(\sigma)} \|w_j\|_{L_4(\sigma)} \\ &\le \frac{\nu}{\alpha} + c\|\mathbf{W}(\cdot,t)\|^{1/2}_{L_2(\sigma)} \|\nabla\mathbf{W}(\cdot,t)\|^{1/2}_{L_2(\sigma)} \|\nabla(w_k - \alpha\Delta w_k)\|_{L_2(\sigma)} \|\nabla w_j\|_{L_2(\sigma)} \\ &\le \frac{\nu}{\alpha} + c\Big(\frac{\lambda_k - 1}{\alpha}\Big)^{1/2} \sup_{t\in[0,T]} \|\mathbf{U}'(\cdot,t)\|^{1/2}_{L_2(\sigma)} \sup_{t\in[0,T]} \|\nabla\mathbf{W}(\cdot,t)\|^{1/2}_{L_2(\sigma)} \le \frac{\nu}{\alpha} + c\delta_0. \end{aligned}$$

Here we have used equalities (5.14), (5.15), the well known inequality

$$\|u\|^4_{L_4(\sigma)} \le c\|u\|^2_{L_2(\sigma)} \|\nabla u\|^2_{L_2(\sigma)} \le c\|\nabla u\|^4_{L_2(\sigma)}$$

which holds for any function $u \in \overset{\circ}{W}{}_2^1(\sigma)$ (see, for example, [52]) and the condition (5.20). Thus, all elements of the matrix $\mathbb{A}^{(N)}(t)$ are bounded functions and, therefore, the existence of the unique solution to problem (5.24) follows from standard results for linear ordinary differential equations (see, for example, [83]). □

5.2.2 A priori estimates

Lemma 5.2. *Let* $\boldsymbol{W} \in \overset{\circ}{\mathcal{V}}(\sigma^T)$ *satisfies condition* (5.20) *with sufficiently small* δ_0 *(*δ_0 *is subject to inequalities* (5.35), (5.37), (5.40) *below). Suppose that* $\partial\sigma \in C^4$, $f \in W_2^1(\sigma)$

and $v_0 \in W_2^3(\sigma) \cap \mathring{W}_2^1(\sigma)$*. Then for the approximate solution* $v^{(N)}$ *the following estimate*

$$\sup_{t\in[0,T]} \|v^{(N)}\|_{\mathring{\mathcal{X}}(\sigma)} + \|v^{(N)}\|_{\mathring{\mathcal{W}}(\sigma^T)} \le c\Big(\|f^{(N)}\|_{W_2^1(\sigma^T)} + \|v_0^{(N)}\|_{\mathring{\mathcal{X}}(\sigma)}\Big) \tag{5.25}$$

holds. Here c *does not depend on* N*.*

Proof. Let us multiply equalities (5.22) by $y_k^{(N)}(t)$ and sum them by k from 1 to N:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\sigma (|v^{(N)}|^2 + \alpha|\nabla v^{(N)}|^2)dx' + \nu\int_\sigma |\nabla v^{(N)}|^2 dx' \\
&= \int_\sigma f^{(N)} v^{(N)} dx' - \alpha\int_\sigma (\mathbf{W}\cdot\nabla)\Delta v^{(N)} v^{(N)} dx' \\
&\le \frac{1}{2\varepsilon}\int_\sigma |f^{(N)}|^2 dx' + c\varepsilon\int_\sigma |\nabla v^{(N)}|^2 dx' + \alpha \sup_{x'\in\sigma}|\mathbf{W}|\|v^{(N)}\|_{L_2(\sigma)}\|\nabla\Delta v^{(N)}\|_{L_2(\sigma)} \\
&\le \frac{1}{2\varepsilon}\int_\sigma |f^{(N)}|^2 dx' + \frac{c\delta_0^2}{\varepsilon}\|\nabla\Delta v^{(N)}\|^2_{L_2(\sigma)} + c\varepsilon\int_\sigma |\nabla v^{(N)}|^2 dx'.
\end{aligned}$$

Here we have applied Cauchy inequality with ε, Poincaré-Friedrichs inequality and the inequality

$$\begin{aligned}
&\sup_{x'\in\sigma}|\mathbf{W}| + \sup_{x'\in\sigma}|\nabla\mathbf{W}| \le c\|\mathbf{W}\|_{W_2^3(\sigma)} \\
&\le c\Big(\|\mathbf{W}\|_{W_2^1(\sigma)} + \|curl(\mathbf{W} - \alpha\Delta\mathbf{W})\|_{L_2(\sigma)}\Big) \le c\delta_0,
\end{aligned} \tag{5.26}$$

which follows from the Sobolev Embedding Theorem, (5.9) and (5.13). Taking $\varepsilon = \dfrac{\nu}{2c}$ yields

$$\begin{aligned}
&\frac{d}{dt}\int_\sigma (|v^{(N)}|^2 + \alpha|\nabla v^{(N)}|^2)dx' + \nu\int_\sigma |\nabla v^{(N)}|^2 dx' \\
&\le c\delta_0^2\|\nabla\Delta v^{(N)}\|^2_{L_2(\sigma)} + c\int_\sigma |f^{(N)}|^2 dx'.
\end{aligned} \tag{5.27}$$

Denote

$$\Phi^{(N)}(x',t) = (\mathbf{W}\cdot\nabla)(v^{(N)} - \alpha\Delta v^{(N)}) - \nu\Delta v^{(N)} - f^{(N)}. \tag{5.28}$$

Since the eigenfunctions $w_k \in W_2^4(\sigma)$, it follows that $\Phi^{(N)} \in W_2^1(\sigma)$.

Let us rewrite equalities (5.22) in the form:

$$\int_\sigma \Big(\frac{\partial}{\partial t}v^{(N)} w_k + \alpha\nabla\frac{\partial}{\partial t}v^{(N)}\cdot\nabla w_k\Big)dx' + \int_\sigma \Phi^{(N)} w_k dx' = 0. \tag{5.29}$$

Let $W^{(N)}(\cdot,t) \in \overset{\circ}{W}{}^1_2(\sigma) \cap W^3_2(\sigma)$ be the solution of the following problem

$$\begin{cases} -\alpha\Delta W^{(N)} + W^{(N)} = \Phi^{(N)}, \\ W^{(N)}|_{\partial\sigma} = 0. \end{cases} \tag{5.30}$$

Then

$$\int_\sigma (\alpha\nabla W^{(N)} \cdot \nabla\eta + W^{(N)}\eta)dx' = \int_\sigma \Phi^{(N)}\eta dx' \quad \forall\eta \in \overset{\circ}{W}{}^1_2(\sigma). \tag{5.31}$$

Taking in (5.31) $\eta = w_k$ we obtain from (5.29) the relations

$$\int_\sigma (\frac{\partial}{\partial t}v^{(N)}w_k + \alpha\nabla\frac{\partial}{\partial t}v^{(N)} \cdot \nabla w_k)dx' + \int_\sigma (W^{(N)}w_k + \alpha\nabla W^{(N)} \cdot \nabla w_k)dx' = 0.$$

By the definition of the eigenfunctions w_k (see (5.13)) we can rewrite the last equalities as follows

$$\begin{aligned} &\frac{1}{\lambda_k}\int_\sigma \big(\frac{\partial}{\partial t}v^{(N)}w_k + \alpha\nabla\frac{\partial}{\partial t}v^{(N)} \cdot \nabla w_k + \nabla(\frac{\partial}{\partial t}v^{(N)} - \alpha\Delta\frac{\partial}{\partial t}v^{(N)}) \cdot \nabla(w_k - \alpha\Delta w_k)\big)dx' \\ &+\frac{1}{\lambda_k}\int_\sigma \big(W^{(N)}w_k + \alpha\nabla W^{(N)} \cdot \nabla w_k + \nabla(W^{(N)} - \alpha\Delta W^{(N)}) \cdot \nabla(w_k - \alpha\Delta w_k)\big)dx' = 0. \end{aligned}$$

Multiplying these relations by $\lambda_k y_k^{(N)}(t)$ and summing from 1 to N yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\sigma \big(|v^{(N)}|^2 + \alpha|\nabla v^{(N)}|^2 + |\nabla(v^{(N)} - \alpha\Delta v^{(N)})|^2\big)dx' \\ &+\int_\sigma \big(W^{(N)}v^{(N)} + \alpha\nabla W^{(N)} \cdot \nabla v^{(N)}\big)dx' \\ &+\int_\sigma \nabla(W^{(N)} - \alpha\Delta W^{(N)}) \cdot \nabla(v^{(N)} - \alpha\Delta v^{(N)})dx' = 0. \end{aligned}$$

From (5.30), (5.31) it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\sigma \big(|v^{(N)}|^2 + \alpha|\nabla v^{(N)}|^2 + |\nabla(v^{(N)} - \alpha\Delta v^{(N)})|^2\big)dx' \\ &+\int_\sigma \Phi^{(N)}v^{(N)}dx' + \int_\sigma \nabla(v^{(N)} - \alpha\Delta v^{(N)}) \cdot \nabla\Phi^{(N)}dx' = 0. \end{aligned} \tag{5.32}$$

Substituting expression (5.28) of the function $\Phi^{(N)}$ into (5.32) gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\sigma \big(|v^{(N)}|^2 + \alpha|\nabla v^{(N)}|^2 + |\nabla(v^{(N)} - \alpha\Delta v^{(N)})|^2\big)dx' + \alpha\int_\sigma \mathbf{W} \cdot \nabla v^{(N)}\Delta v^{(N)}dx' \\ &+\nu\int_\sigma |\nabla v^{(N)}|^2dx' + \int_\sigma \nabla\big((\mathbf{W} \cdot \nabla)(v^{(N)} - \alpha\Delta v^{(N)})\big) \cdot \nabla(v^{(N)} - \alpha\Delta v^{(N)})dx' \end{aligned}$$

$$
\begin{aligned}
&+\frac{\nu}{\alpha}\int\limits_\sigma|\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2dx'-\frac{\nu}{\alpha}\int\limits_\sigma\nabla v^{(N)}\cdot\nabla(v^{(N)}-\alpha\Delta v^{(N)})dx'\\
&=\int\limits_\sigma f^{(N)}v^{(N)}dx'+\int\limits_\sigma\nabla f^{(N)}\cdot\nabla(v^{(N)}-\alpha\Delta v^{(N)})dx'.
\end{aligned}
\tag{5.33}
$$

The right-hand side of (5.33) contains the term with the fourth order derivative. However, this term can be estimated by the integral containing only derivatives up to the third order. The operations below are correct because the functions w_k belong to the space $W_2^4(\sigma)$. Denote for simplicity $v^{(N)}-\alpha\Delta v^{(N)}=u$. We have

$$
\begin{aligned}
&\int\limits_\sigma\nabla\big((\mathbf{W}\cdot\nabla)(v^{(N)}-\alpha\Delta v^{(N)})\big)\cdot\nabla(v^{(N)}-\alpha\Delta v^{(N)})dx'\\
&=\int\limits_\sigma\nabla\big((\mathbf{W}\cdot\nabla)u\big)\cdot\nabla u dx'=\int\limits_\sigma\{\nabla\mathbf{W}\cdot\nabla\}u\cdot\nabla u dx'\\
&+\int\limits_\sigma(\mathbf{W}\cdot\nabla)\nabla u\cdot\nabla u dx'=\int\limits_\sigma\{\nabla\mathbf{W}\cdot\nabla\}u\cdot\nabla u dx'\\
&+\frac12\int\limits_\sigma(\mathbf{W}\cdot\nabla)|\nabla u|^2dx'=\int\limits_\sigma\{\nabla\mathbf{W}\cdot\nabla\}u\cdot\nabla u dx'\\
&-\frac12\int\limits_\sigma\nabla\cdot\mathbf{W}|\nabla u|^2dx'=\int\limits_\sigma\{\nabla\mathbf{W}\cdot\nabla\}u\cdot\nabla u dx'\\
&\le c\int\limits_\sigma|\nabla\mathbf{W}||\nabla u|^2dx'\le c\sup_{x'\in\sigma}|\nabla\mathbf{W}|\int\limits_\sigma|\nabla u|^2dx'\\
&\le c\delta_0\int\limits_\sigma|\nabla u|^2dx'=c\delta_0\int\limits_\sigma|\nabla v^{(N)}-\alpha\nabla\Delta v^{(N)}|^2dx'.
\end{aligned}
\tag{5.34}
$$

Here we have used the notation $\{\nabla\mathbf{W}\cdot\nabla\}=\nabla(W_1\frac{\partial}{\partial x_1}+W_2\frac{\partial}{\partial x_2})$ and applied (5.26). Using (5.34) from (5.33) we get the estimate

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int\limits_\sigma\big(|v^{(N)}|^2+\alpha|\nabla v^{(N)}|^2+|\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2\big)dx'+\nu\int\limits_\sigma|\nabla v^{(N)}|^2dx'\\
&+\frac{\nu}{\alpha}\int\limits_\sigma|\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2dx'=\alpha\int\limits_\sigma\mathbf{W}\cdot\nabla\Delta v^{(N)}v^{(N)}dx'\\
&+\frac{\nu}{\alpha}\int\limits_\sigma\nabla v^{(N)}\cdot\nabla(v^{(N)}-\alpha\Delta v^{(N)})dx'+\int\limits_\sigma\nabla\big((\mathbf{W}\cdot\nabla)(v^{(N)}-\alpha\Delta v^{(N)})\big)\cdot\nabla(v^{(N)}-\alpha\Delta v^{(N)})dx'\\
&+\int\limits_\sigma f^{(N)}v^{(N)}dx'+\int\limits_\sigma\nabla f^{(N)}\cdot\nabla(v^{(N)}-\alpha\Delta v^{(N)})dx'\le\alpha\sup_{x'\in\sigma}|\mathbf{W}|\|\nabla\Delta v^{(N)}\|_{L_2(\sigma)}\|v^{(N)}\|_{L_2(\sigma)}\\
&+\frac{\nu}{\alpha}\|\nabla v^{(N)}\|_{L_2(\sigma)}\|\nabla(v^{(N)}-\alpha\Delta v^{(N)})\|_{L_2(\sigma)}+\frac{1}{2\varepsilon}\Big(\int\limits_\sigma|f^{(N)}|^2dx'+\int\limits_\sigma|\nabla f^{(N)}|^2dx'\Big)\\
&+(\frac{\varepsilon}{2}+c\delta_0)\int\limits_\sigma|\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2dx'+\frac{\varepsilon}{2}\int\limits_\sigma|\nabla v^{(N)}|^2dx'\\
&\le\frac{c\delta_0^2}{\varepsilon}\int\limits_\sigma|\nabla\Delta v^{(N)}|^2dx'+\varepsilon\int\limits_\sigma|\nabla v^{(N)}|dx'+\frac{c}{\varepsilon}\int\limits_\sigma|\nabla v^{(N)}|dx'\\
&+(\varepsilon+c\delta_0)\int\limits_\sigma|\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2dx'+\frac{1}{2\varepsilon}\int\limits_\sigma|f^{(N)}|^2dx'+\frac{1}{2\varepsilon}\int\limits_\sigma|\nabla f^{(N)}|^2dx'
\end{aligned}
$$

$$\leq \Big(\frac{c_1\delta_0^2}{\varepsilon}+\varepsilon+c_2\delta_0\Big)\int\limits_\sigma |\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2dx'+\varepsilon\int\limits_\sigma |\nabla v^{(N)}|dx'$$
$$+\frac{c(1+\delta_0^2)}{\varepsilon}\int\limits_\sigma |\nabla v^{(N)}|dx'+\frac{1}{2\varepsilon}\int\limits_\sigma |f^{(N)}|^2dx'+\frac{1}{2\varepsilon}\int\limits_\sigma |\nabla f^{(N)}|^2dx'.$$

Taking $\varepsilon=\min\{\frac{\nu}{2},\frac{\nu}{4\alpha}\}$ and assuming that δ_0 is sufficiently small, i.e.,

$$\frac{c_1\delta_0^2}{\varepsilon}+c_2\delta_0\leq\frac{\nu}{4\alpha}, \tag{5.35}$$

from the latter inequality we obtain

$$\begin{gathered}\frac{d}{dt}\int\limits_\sigma \big(|v^{(N)}|^2+\alpha|\nabla v^{(N)}|^2+|\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2\big)dx'\\ +\nu\int\limits_\sigma |\nabla v^{(N)}|^2dx'+\frac{\nu}{\alpha}\int\limits_\sigma |\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2dx'\\ \leq c\Big(\int\limits_\sigma(|f^{(N)}|^2+\int\limits_\sigma |\nabla f^{(N)}|^2)dx'+|\nabla v^{(N)}|^2dx'\Big).\end{gathered} \tag{5.36}$$

Integrating inequality (5.27) by t gives the estimates

$$\int\limits_\sigma \big(|v^{(N)}(x',t)|^2+\alpha|\nabla v^{(N)}(x',t)|^2\big)dx'+\nu\int\limits_0^t\int\limits_\sigma |\nabla v^{(N)}|^2dx'd\tau$$
$$\leq c\delta_0^2\int\limits_0^t\int\limits_\sigma |\nabla\Delta v^{(N)}|^2dx'd\tau+c\int\limits_0^t\int\limits_\sigma |f^{(N)}|^2dx'd\tau+\int\limits_\sigma \big(|v_0^{(N)}|^2+\alpha|\nabla v_0^{(N)}|^2\big)dx'$$
$$\leq c\delta_0^2\int\limits_0^t\int\limits_\sigma |\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2dx'd\tau+c_3\delta_0^2\int\limits_0^t\int\limits_\sigma |\nabla v^{(N)}|^2dx'd\tau$$
$$+c\int\limits_0^t\int\limits_\sigma |f^{(N)}|^2dx'd\tau+\int\limits_\sigma \big(|v_0^{(N)}|^2+\alpha|\nabla v_0^{(N)}|^2\big)dx'.$$

If

$$c_3\delta_0^2\leq\frac{\nu}{2}, \tag{5.37}$$

the last inequality yields

$$\begin{gathered}\int\limits_0^t\int\limits_\sigma |\nabla v^{(N)}|^2dx'd\tau\leq c\delta_0^2\int\limits_0^t\int\limits_\sigma |\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2dx'd\tau\\ +c\int\limits_0^t\int\limits_\sigma |f^{(N)}|^2dx'd\tau+\int\limits_\sigma \big(|v_0^{(N)}|^2+\alpha|\nabla v_0^{(N)}|^2\big)dx'.\end{gathered} \tag{5.38}$$

Integrating inequality (5.36) with respect to t and estimating the last term in right-hand side by (5.38) we obtain

$$\begin{aligned}
&\int\limits_\sigma \big(|v^{(N)}(x',t)|^2+\alpha|\nabla v^{(N)}(x',t)|^2+|\nabla(v^{(N)}(x',t)-\alpha\Delta v^{(N)}(x',t))|^2\big)dx'\\
&+\nu\int\limits_0^t\int\limits_\sigma |\nabla v^{(N)}|^2dx'd\tau+\int\limits_0^t\int\limits_\sigma |\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2dx'd\tau\\
&\le c\big(\int\limits_0^t\int\limits_\sigma |f^{(N)}|^2dx'd\tau+\int\limits_0^t\int\limits_\sigma |\nabla f^{(N)}|^2dx'd\tau\big)\\
&+c_4\delta_0^2\int\limits_0^t\int\limits_\sigma |\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2dx'd\tau\\
&+c\int\limits_\sigma \big(|v_0^{(N)}|^2+\alpha|\nabla v_0^{(N)}|^2+|\nabla(v_0^{(N)}-\alpha\Delta v_0^{(N)})|^2\big)dx'.
\end{aligned} \tag{5.39}$$

Assuming that

$$c_4\delta_0^2\le\frac{1}{2}, \tag{5.40}$$

from (5.39) follows the estimate

$$\begin{aligned}
&\int\limits_\sigma \big(|v^{(N)}(x',t)|^2+\alpha|\nabla v^{(N)}(x',t)|^2+|\nabla(v^{(N)}(x',t)-\alpha\Delta v^{(N)}(x',t))|^2\big)dx'\\
&+\int\limits_0^t\int\limits_\sigma \big(\nu|\nabla v^{(N)}|^2+|\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2\big)dx'd\tau\\
&\le c\int\limits_0^t\int\limits_\sigma \big(|f^{(N)}|^2+|\nabla f^{(N)}|^2\big)dx'd\tau\\
&+c\int\limits_\sigma \big(|v_0^{(N)}|^2+\alpha|\nabla v_0^{(N)}|^2+|\nabla(v_0^{(N)}-\alpha\Delta v_0^{(N)})|^2\big)dx'.
\end{aligned} \tag{5.41}$$

Let us multiply equalities (5.22) by $\dfrac{d}{dt}y_k^{(N)}(t)$ and sum them by k from 1 to N:

$$\begin{aligned}
&\int\limits_\sigma \big(|\frac{\partial}{\partial t}v^{(N)}|^2+\alpha|\nabla\frac{\partial}{\partial t}v^{(N)}|^2\big)dx'+\frac{\nu}{2}\frac{d}{dt}\int\limits_\sigma |\nabla v^{(N)}|^2dx'\\
&=\int\limits_\sigma (\mathbf{U}'\cdot\nabla)(v^{(N)}-\alpha\Delta v^{(N)})\frac{\partial}{\partial t}v^{(N)}dx'+\int\limits_\sigma f^{(N)}\frac{\partial}{\partial t}v^{(N)}dx'\\
&\le\frac{1}{2\varepsilon}\sup_{x'\in\sigma}|\mathbf{U}'|^2\int\limits_\sigma |\nabla(v^{(N)}-\alpha\Delta v^{(N)})|^2dx'+\varepsilon\int\limits_\sigma |\frac{\partial}{\partial t}v^{(N)}|^2dx'+\frac{1}{2\varepsilon}\int\limits_\sigma |f^{(N)}|^2dx'.
\end{aligned}$$

Taking $\varepsilon=\dfrac{1}{2}$, integrating with respect to t and applying (5.26), (5.41) we obtain

$$\nu\int\limits_{\sigma}|\nabla v^{(N)}(x',t)|^2dx'+\int\limits_0^t\int\limits_{\sigma}(|\partial_\tau v^{(N)}|^2+\alpha|\nabla\partial_\tau v^{(N)}|^2)dx'd\tau$$
$$\le c\int\limits_0^t\int\limits_{\sigma}\big(|f^{(N)}|^2dx'd\tau+|\nabla f^{(N)}|^2\big)dx'd\tau \tag{5.42}$$
$$+c\int\limits_{\sigma}\big(|v_0^{(N)}|^2+\alpha|\nabla v_0^{(N)}|^2+|\nabla(v_0^{(N)}-\alpha\Delta v_0^{(N)})|^2\big)dx'.$$

Estimate (5.25) follows from (5.41), (5.42) and the definitions of the norms. $\square$

5.2.3 Existence and uniqueness of the solution

Theorem 5.3. *Suppose that $\partial\sigma\in C^4$, $f\in W_2^1(\sigma^T), v_0\in W_2^3(\sigma)\cap \overset{\circ}{W}{}_2^1(\sigma)$ and $\boldsymbol{W}\in \overset{\circ}{\mathcal{V}}(\sigma^T)$ satisfies condition* (5.20) *with δ_0 subject to inequalities* (5.35), (5.37), (5.40). *Then problem* (5.19) *admits a unique week solution $v\in\overset{\circ}{\mathcal{W}}(\sigma^T)$ and the following estimate*

$$\sup_{t\in[0,T]}\|v\|_{\overset{\circ}{X}(\sigma)}+\|v\|_{\overset{\circ}{\mathcal{W}}(\sigma^T)}\le c\big(\|f\|_{W_2^1(\sigma^T)}+\|v_0\|_{\overset{\circ}{X}(\sigma)}\big) \tag{5.43}$$

holds.

Proof. Multiplying equations (5.23) by arbitrary function $\eta\in\overset{\circ}{W}{}_2^{1,0}(\sigma^T)$ and integrating by parts on σ and by τ from 0 to t we get the following integral identity

$$\int\limits_0^t\int\limits_{\sigma}(\frac{\partial}{\partial\tau}v^{(N)}\eta+\alpha\nabla\frac{\partial}{\partial\tau}v^{(N)}\cdot\nabla\eta)dx'd\tau+\nu\int\limits_0^t\int\limits_{\sigma}\nabla v^{(N)}\cdot\nabla\eta dx'd\tau$$
$$=\int\limits_0^t\int\limits_{\sigma}f^{(N)}\eta dx'd\tau+\int\limits_0^t\int\limits_{\sigma}\mathbf{W}\cdot\nabla\eta(v^{(N)}-\alpha\Delta v^{(N)})dx'd\tau\quad\forall t\in[0,T]. \tag{5.44}$$

From estimates (5.41), (5.42) it follows that there exists a subsequence $\{v^{(N_l)}\}$ such that

$$v^{(N_l)}(\cdot,t)\rightharpoonup v(\cdot,t)\quad\text{in }\ \overset{\circ}{X}(\sigma)\ \ \forall t\in[0,T],\quad v^{(N_l)}\rightharpoonup v\ \text{ in }\ \overset{\circ}{\mathcal{W}}(\sigma^T).$$

Passing in (5.44) to a limit as $N_l\to\infty$ we obtain for v integral identity (5.21). Obviously, v satisfies the initial condition. Moreover, from inequality (5.25) follows estimate (5.43).

Let us prove the uniqueness. Let $v^{[1]}$ and $v^{[2]}$ be two weak solutions of problem (5.19). The difference $V=v^{[1]}-v^{[2]}$ satisfies the integral identity

$$\int\limits_0^t\int\limits_{\sigma}(\frac{\partial}{\partial\tau}V\eta+\alpha\nabla\frac{\partial}{\partial\tau}V\cdot\nabla\eta)dx'd\tau+\nu\int\limits_0^t\int\limits_{\sigma}\nabla V\cdot\nabla\eta dx'd\tau$$
$$=\int\limits_0^t\int\limits_{\sigma}\mathbf{W}\cdot\nabla\eta(V-\alpha\Delta V)dx'd\tau\qquad\forall\,\eta\in\overset{\circ}{W}{}_2^{1,0}(\sigma^T).$$

Taking $\eta = V$ yields

$$\frac{1}{2}\int\limits_{\sigma}(|V|^2+\alpha|\nabla V|^2)dx'+\nu\int\limits_0^t\int\limits_{\sigma}|\nabla V|^2dx'd\tau=-\alpha\int\limits_0^t\int\limits_{\sigma}\mathbf{W}\cdot\nabla V\Delta V dx'd\tau.$$

Integrating by parts the right-hand side term we get

$$\begin{aligned}&-\alpha\int\limits_0^t\int\limits_{\sigma}\mathbf{W}\cdot\nabla V\Delta V dx'd\tau=\alpha\int\limits_0^t\int\limits_{\sigma}\big[-\frac{1}{2}\sum_{i=1}^2\nabla\cdot\mathbf{W}(\frac{\partial}{\partial x_i}V)^2\\&+\sum_{i=1}^2\sum_{j=1}^2\frac{\partial}{\partial x_i}W_j\frac{\partial}{\partial x_j}V\frac{\partial}{\partial x_i}V\big]dx'd\tau=\sum_{i=1}^2\sum_{j=1}^2\frac{\partial}{\partial x_i}W_j\frac{\partial}{\partial x_j}V\frac{\partial}{\partial x_i}Vdx'd\tau\\&\le c\int\limits_0^t\sup_{x'\in\sigma}|\nabla\mathbf{W}'|\int\limits_{\sigma}|\nabla V|^2dx'd\tau\le c\int\limits_0^t\|\mathbf{W}'\|_{W_2^3(\sigma)}\int\limits_{\sigma}|\nabla V|^2dx'd\tau.\end{aligned}$$

Therefore,

$$\int\limits_{\sigma}|\nabla V|^2dx'\le c\int\limits_0^t\|\mathbf{W}\|_{W_2^3(\sigma)}\int\limits_{\sigma}|\nabla V|^2dx'd\tau. \tag{5.45}$$

Define

$$u(s)=\exp(-\int\limits_0^s c||\mathbf{W}||_{W_2^3(\sigma)}d\tau)\int\limits_0^s c||\mathbf{W}||_{W_2^3(\sigma)}\int\limits_{\sigma}|\nabla V|^2dx'd\tau),\quad s\in(0,\tau).$$

Then

$$u'(s)=c||\mathbf{W}||_{W_2^3(\sigma)}\exp(-\int\limits_0^s c||\mathbf{W}||_{W_2^3(\sigma)}d\tau)\big(\int\limits_{\sigma}|\nabla V|^2dx'-\int\limits_0^s c||\mathbf{W}||_{W_2^3(\sigma)}\int\limits_{\sigma}|\nabla V|^2dx'd\tau\big).$$

From (5.45) it follows that

$$\int\limits_{\sigma}|\nabla V|^2dx'-\int\limits_0^t c||\mathbf{W}||_{W_2^3(\sigma)}\int\limits_{\sigma}|\nabla V|^2dx'd\tau\le 0,$$

and we get $u'(s)\le 0$. Integrating the last inequality from 0 to t and taking into account that $u(0)=0$ we get $u(s)\le 0$. Hence,

$$\int\limits_0^t||\mathbf{W}||_{W_2^3(\sigma)}\int\limits_{\sigma}|\nabla V|^2dx'd\tau\le 0.$$

Substituting this into (5.45) we get $\int\limits_{\sigma}|\nabla V|^2dx'\le 0$. Thus, $v^{[1]}=v^{[2]}$. □

5.3 Construction of an approximate solution to the inverse problem (5.8)

Let us come back to problem (5.8). In this section we construct an approximate solution and prove an a priori estimates of it.

Definition. *By a weak solution of problem* (5.8) *we understand the pair* $(\mathbf{U}, q) = (\mathbf{U}', U_3, q) \in \overset{\circ}{\mathcal{V}}(\sigma^T) \times \overset{\circ}{\mathcal{W}}(\sigma^T) \times L_2(0,T)$ *satisfying the initial condition* $\mathbf{U}(x', 0) = \mathbf{u}_0(x')$, *the flux condition*

$$\int\limits_{\sigma} U_3(x', t)dx' = F(t) \tag{5.46}$$

and the integral identity

$$\begin{aligned}
&\int\limits_0^t \int\limits_{\sigma} (\frac{\partial}{\partial \tau}\mathbf{U} \cdot \boldsymbol{\eta} + \alpha \nabla \frac{\partial}{\partial \tau}\mathbf{U} \cdot \nabla \boldsymbol{\eta})dx'd\tau + \nu \int\limits_0^t \int\limits_{\sigma} \nabla \mathbf{U} \cdot \nabla \boldsymbol{\eta} dx'd\tau \\
&+ \int\limits_0^t \int\limits_{\sigma} (\mathbf{U} \cdot \nabla)\mathbf{U} \cdot \boldsymbol{\eta} dx'd\tau + \alpha \int\limits_0^t \int\limits_{\sigma} (\mathbf{U} \cdot \nabla)\boldsymbol{\eta} \cdot \triangle \mathbf{U} dx'd\tau \\
&- \alpha \int\limits_0^t \int\limits_{\sigma} (\boldsymbol{\eta} \cdot \nabla)\mathbf{U} \cdot \triangle \mathbf{U} dx'd\tau = \int\limits_0^t \int\limits_{\sigma} q\eta_3 dx'd\tau + \int\limits_0^t \int\limits_{\sigma} \mathbf{f} \cdot \boldsymbol{\eta} dx'd\tau \\
&\forall \boldsymbol{\eta} = (\boldsymbol{\eta}', \eta_3) \in \overset{\circ}{W}{}_2^{1,0}(\sigma^T) \text{ with } \mathrm{div}\boldsymbol{\eta}' = 0, \ \forall \mathrm{t} \in [0, \mathrm{T}].
\end{aligned} \tag{5.47}$$

One can derive identity (5.47) multiplying equations (5.8_1) by $\boldsymbol{\eta}'$, equations (5.8_2) by η_3, summing the obtained equalities, integrating by parts on σ and then integrating by t. Notice that identity (5.47) is equivalent to the two following identities:

$$\begin{aligned}
&\int\limits_0^t \int\limits_{\sigma} (\frac{\partial}{\partial \tau}\mathbf{U}' \cdot \boldsymbol{\eta}' + \alpha \nabla \frac{\partial}{\partial \tau}\mathbf{U}' \cdot \nabla \boldsymbol{\eta}')dx'd\tau + \nu \int\limits_0^t \int\limits_{\sigma} \nabla \mathbf{U}' \cdot \nabla \boldsymbol{\eta}' dx'd\tau \\
&+ \int\limits_0^t \int\limits_{\sigma} (\mathbf{U}' \cdot \nabla)\mathbf{U}' \cdot \boldsymbol{\eta}' dx'd\tau + \alpha \int\limits_0^t \int\limits_{\sigma} (\mathbf{U}' \cdot \nabla)\boldsymbol{\eta}' \cdot \triangle \mathbf{U}' dx'd\tau \\
&- \alpha \int\limits_0^t \int\limits_{\sigma} (\boldsymbol{\eta}' \cdot \nabla)\mathbf{U}' \cdot \triangle \mathbf{U}' dx'd\tau + \int\limits_0^t \int\limits_{\sigma} \nabla U_3 \cdot \boldsymbol{\eta}'(U_3 - \alpha \triangle U_3)dx'd\tau \\
&= \int\limits_0^t \int\limits_{\sigma} \mathbf{f}' \cdot \boldsymbol{\eta}' dx'd\tau \quad \forall \boldsymbol{\eta}' \in \overset{\circ}{W}{}_2^{1,0}(\sigma^T) \ \text{ with } \ \mathrm{div}\, \boldsymbol{\eta}' = 0,
\end{aligned} \tag{5.48}$$

and

$$
\begin{aligned}
&\int\limits_0^t\int\limits_\sigma\Big(\frac{\partial}{\partial\tau}U_3\eta_3+\alpha\nabla\frac{\partial}{\partial\tau}U_3\cdot\nabla\eta_3\Big)dx'd\tau+\nu\int\limits_0^t\int\limits_\sigma\nabla U_3\cdot\nabla\eta_3 dx'd\tau\\
&+\int\limits_0^t\int\limits_\sigma \mathbf{U}'\cdot\nabla\eta_3(U_3-\alpha\triangle U_3)dx'd\tau\\
&=\int\limits_0^t\int\limits_\sigma q\eta_3dx'd\tau+\int\limits_0^t\int\limits_\sigma f_3\eta_3dx'd\tau\quad\forall\eta_3\in\overset{\circ}{W}{}_2^{1,0}(\sigma^T).
\end{aligned}
\tag{5.49}
$$

For the Galerkin approximations $\mathbf{U}^{(N)}(x',t)=(\mathbf{U}'^{(N)}(x',t),U_3^{(N)}(x',t))$ of the weak solution $\mathbf{U}(x',t)$ we look in the form

$$
\mathbf{U}'^{(N)}(x',t)=\sum_{k=1}^N c_k^{(N)}(t)\mathbf{w}_k(x'),\;\; U_3^{(N)}(x',t)=\sum_{k=1}^N b_k^{(N)}(t)w_k(x').
$$

Here the coefficient $c_1^{(1)}(t)$ is found from the equation

$$
\frac{d}{dt}c_1^{(1)}(t)+\nu c_1^{(1)}(t)\int\limits_\sigma|\nabla\mathbf{w}_1|^2dx'=f_1'(t),\quad c_1^{(1)}(0)=u_{01}',
\tag{5.50}
$$

while $c_k^{(N)}(t)$, $b_k^{(N)}(t)$, $k=1,...,N$, are found recurrently as the solutions of the Cauchy problems for the following systems of ordinary differential equations

$$
\left\{
\begin{aligned}
&\frac{d}{dt}b_k^{(N)}(t)+\nu\int\limits_\sigma\nabla U_3^{(N)}\cdot\nabla w_kdx'+\int\limits_\sigma(\mathbf{U}'^{(N)}\cdot\nabla)w_k(U_3^{(N)}-\alpha\Delta U_3^{(N)})dx'\\
&\quad=q^{(N)}(t)\int\limits_\sigma w_kdx'+f_{3k}(t),\\
&b_k^{(N)}(0)=v_{0k},\;\; k=1,\dots,N,\; N\ge1,
\end{aligned}
\right.
\tag{5.51}
$$

and

$$
\left\{
\begin{aligned}
&\frac{d}{dt}c_k^{(N)}(t)+\nu\int\limits_\sigma\nabla\mathbf{U}'^{(N)}\cdot\nabla\mathbf{w}_kdx'+\int\limits_\sigma(\mathbf{U}'^{(N)}\cdot\nabla)\mathbf{U}'^{(N)}\cdot\mathbf{w}_kdx'\\
&+\alpha\int\limits_\sigma(\mathbf{U}'^{(N)}\cdot\nabla)\mathbf{w}_k\cdot\Delta\mathbf{U}'^{(N)}dx'-\alpha\int\limits_\sigma(\mathbf{w}_k\cdot\nabla)\mathbf{U}'^{(N)}\cdot\Delta\mathbf{U}'^{(N)}dx'\\
&+\int\limits_\sigma\nabla U_3^{(N-1)}\cdot\mathbf{w}_k(U_3^{(N-1)}-\alpha\Delta U_3^{(N-1)})dx'=f_k'(t),\\
&c_k^{(N)}(0)=u_{0k}',\;\; k=1,\dots,N,\; N\ge2.
\end{aligned}
\right.
\tag{5.52}
$$

As usual,

$$
\begin{aligned}
&f_k'(t)=\int\limits_\sigma\mathbf{f}'\cdot\mathbf{w}_kdx',\quad u_{0k}'=\int\limits_\sigma\mathbf{u}_0'\cdot\mathbf{w}_kdx',\\
&f_{3k}(t)=\int\limits_\sigma f_3w_kdx',\quad v_{0k}=\int\limits_\sigma v_0w_kdx'.
\end{aligned}
$$

The functions $q^{(N)}(t)$ in (5.51) are chosen in order to satisfy the flux condition

$$\int_\sigma U_3^{(N)}(x',t)dx' = \widetilde{F}^{(N)}(t) \quad \forall t \in [0,T], \tag{5.53}$$

where $\widetilde{F}^{(N)} \to F$ in $W_2^1(0,T)$ ($\widetilde{F}^{(N)}$ will be defined later).

First of all we prove an a priori estimates for $\mathbf{U}'^{(N)}$ assuming that $U_3^{(N-1)}$ is known.

Lemma 5.3. *Suppose that* $U_3^{(N-1)} \in \mathring{\mathcal{W}}(\sigma^T)$ *is given. Let* $\mathbf{f}' \in L_2(\sigma^T)$, $\operatorname{curl}\mathbf{f} \in L_2(\sigma^T)$ *and* $\mathbf{u}_0' \in W_2^3(\sigma) \cap \mathring{W}_2^1(\sigma)$. *Then for* $\mathbf{U}'^{(N)} \in \mathring{\mathcal{W}}(\sigma^T)$ *the following estimates*

$$\sup_{t\in[0,T]} \|\mathbf{U}'^{(1)}\|^2_{\mathring{\mathcal{Y}}(\sigma)} + \|\mathbf{U}'^{(1)}\|^2_{\mathring{\mathcal{V}}(\sigma^T)} \le C_1\big(\|\mathbf{f}'\|^2_{L_2(\sigma^T)} + \|\operatorname{curl}\mathbf{f}'\|^2_{L_2(\sigma^T)} + \|\mathbf{u}_0'\|^2_{\mathring{\mathcal{Y}}(\sigma)}\big), \tag{5.54}$$

$$\begin{aligned}&\sup_{t\in[0,T]} \|\mathbf{U}'^{(N)}\|^2_{\mathring{\mathcal{Y}}(\sigma)} + \|\mathbf{U}'^{(N)}\|^2_{\mathring{\mathcal{V}}(\sigma^T)} \le C_2\big(\|\mathbf{f}'\|^2_{L_2(\sigma^T)} + \|\operatorname{curl}\mathbf{f}'\|^2_{L_2(\sigma^T)}\\ &+ \sup_{t\in[0,T]} \|U_3^{(N-1)}\|^4_{\mathring{\mathcal{X}}(\sigma)} + \|U_3^{(N-1)}\|^4_{\mathring{\mathcal{W}}(\sigma^T)} + \|\mathbf{u}_0'\|^2_{\mathring{\mathcal{Y}}(\sigma)}\big), \quad N \ge 2,\end{aligned} \tag{5.55}$$

hold. Here C_2 *is independent of* N.

Proof. Estimate (5.54) follows from the equation (5.50) and the properties of the basis $\{\mathbf{w}_k\}$.

In order to prove (5.55), we use results obtained (see [13], [14]) for the two dimensional equations of the second grate fluid flow

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{U}' - \alpha\Delta\mathbf{U}') - \nu\Delta\mathbf{U}' + \operatorname{curl}(\mathbf{U}' - \alpha\Delta\mathbf{U}') \times \mathbf{U}' + \nabla\widetilde{p} = \mathbf{f}', \\ \operatorname{div}\mathbf{U}' = 0, \\ \mathbf{U}'|_{\partial\sigma} = 0, \quad \mathbf{U}'(x',0) = \mathbf{u}_0'(x'). \end{cases} \tag{5.56}$$

It was proved in [13], [14] that if $\boldsymbol{f}' \in L_2(\sigma^T)$, $\operatorname{curl}\boldsymbol{f}' \in L_2(\sigma^T)$ and $\mathbf{u}_0' \in W_2^3(\sigma)$, then problem (5.56) has the unique weak solution $\mathbf{U}' \in \mathring{\mathcal{V}}(\sigma^T)$ and the following estimate

$$\sup_{t\in[0,T]} \|\mathbf{U}'\|^2_{\mathring{\mathcal{Y}}(\sigma)} + \|\mathbf{U}'\|^2_{\mathring{\mathcal{V}}(\sigma^T)} \le c\big(\|\mathbf{f}'\|^2_{L_2(\sigma^T)} + \|\operatorname{curl}\mathbf{f}'\|^2_{L_2(\sigma^T)} + \|\mathbf{u}_0'\|^2_{\mathring{\mathcal{Y}}(\sigma)}\big)$$

holds. In [13], [14] the solution was found by Galerkin method. Repeating literally the arguments from [13] one gets the following estimate for $\mathbf{U}'^{(N)}$:

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|\mathbf{U}'^{(N)}\|^2_{\mathring{\mathcal{Y}}(\sigma)} + \|\mathbf{U}'^{(N)}\|^2_{\mathring{\mathcal{V}}(\sigma^T)} \le c\big(\|\mathbf{f}'\|^2_{L_2(\sigma^T)} \\
&+\|\mathrm{curl}\,\mathbf{f}'\|^2_{L_2(\sigma^T)} + \|U_3^{(N-1)}\nabla(U_3^{(N-1)} - \alpha\Delta U_3^{(N-1)})\|^2_{L_2(\sigma^T)} \\
&+\|\mathrm{curl}\,\big(U_3^{(N-1)}\nabla(U_3^{(N-1)} - \alpha\Delta U_3^{(N-1)})\big)\|^2_{L_2(\sigma^T)} + \|\mathbf{u}_0'\|^2_{\mathring{\mathcal{Y}}(\sigma)}\big).
\end{aligned}
\tag{5.57}
$$

It is easy to show using Sobolev Embedding Theorem that

$$
\begin{aligned}
&\|U_3^{(N-1)}\nabla(U_3^{(N-1)} - \alpha\Delta U_3^{(N-1)})\|^2_{L_2(\sigma^T)} \\
&\le c\big(\sup_{t\in[0,T]} \|U_3^{(N-1)}\|^4_{\mathring{\mathcal{X}}(\sigma)} + \|U_3^{(N-1)}\|^4_{\mathring{\mathcal{W}}(\sigma^T)}\big).
\end{aligned}
\tag{5.58}
$$

Let us estimate the second term in (5.57) containing $U_3^{(N-1)}$. We have[2]

$$
\begin{aligned}
&\mathrm{curl}(U_3^{(N-1)}\nabla(U_3^{(N-1)} - \alpha\Delta U_3^{(N-1)})) = U_3^{(N-1)}\mathrm{curl}(\nabla(U_3^{(N-1)} - \alpha\Delta U_3^{(N-1)})) \\
&+\nabla U_3^{(N-1)} \times \nabla(U_3^{(N-1)} - \alpha\Delta U_3^{(N-1)}) = \nabla U_3^{(N-1)} \times \nabla(U_3^{(N-1)} - \alpha\Delta U_3^{(N-1)}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_0^t \int_\sigma |\mathrm{curl}\big(U_3^{(N-1)}\nabla(U_3^{(N-1)} - \alpha\Delta U_3^{(N-1)})\big)|^2 dx' d\tau \\
&= \int_0^t \int_\sigma |\nabla U_3^{(N-1)} \times \nabla(U_3^{(N-1)} - \alpha\Delta U_3^{(N-1)})|^2 dx' d\tau \\
&\le c\int_0^t \sup_{x'\in\sigma} |\nabla U_3^{(N-1)}|^2 \|\nabla(U_3^{(N-1)} - \alpha\Delta U_3^{(N-1)})\|^2_{L_2(\sigma^T)} d\tau \\
&\le c\int_0^t \|\nabla U_3^{(N-1)}\|^2_{W_2^2(\sigma)} \|\nabla(U_3^{(N-1)} - \alpha\Delta U_3^{(N-1)})\|^2_{L_2(\sigma^T)} d\tau \\
&\le c \sup_{t\in[0,T]} \|U_3^{(N-1)}\|^2_{\mathring{\mathcal{X}}(\sigma)} \|U_3^{(N-1)}\|^2_{\mathring{\mathcal{W}}(\sigma^T)} \\
&\le c\big(\sup_{t\in[0,T]} \|U_3^{(N-1)}\|^4_{\mathring{\mathcal{X}}(\sigma)} + \|U_3^{(N-1)}\|^4_{\mathring{\mathcal{W}}(\sigma^T)}\big).
\end{aligned}
\tag{5.59}
$$

Estimate (5.55) follows from (5.57), (5.58) and (5.59). □

Remark 5.2. It follows from estimate (5.55) that Cauchy problem for system (5.52) has a unique solution $c_k^{(N)}(t), k = 1, \dots, N$, (see, for example, [83]), and, thus, functions $\mathbf{U}'^{(N)}$ can be uniquely determined from (5.52) if we already know $U_3^{(N-1)}$.

Let us consider now the approximations $(U_3^{(N)}(x', t), q^{(N)}(t))$. Suppose that $\mathbf{U}'^{(N)} \in$

[2] These calculations have sense because the elements w_k of the basis belong to the space $W_2^4(\sigma)$ (see Theorem 2.2).

$\overset{\circ}{\mathcal{V}}(\sigma^T)$ in (5.51) is given and

$$\sup_{t\in[0,T]} \|\mathbf{U}'^{(N)}\|_{\overset{\circ}{\mathcal{Y}}(\sigma)} + \|\mathbf{U}'^{(N)}\|_{\overset{\circ}{\mathcal{V}}(\sigma^T)} < \infty.$$

Because of (5.11) and Sobolev Embedding Theorem it follows that

$$\begin{aligned} &\sup_{t\in[0,T]} \big(\sup_{x'\in\sigma} |\mathbf{U}'^{(N)}| + \sup_{x'\in\sigma} |\nabla\mathbf{U}'^{(N)}|\big) \le c \sup_{t\in[0,T]} \|\mathbf{U}'^{(N)}\|_{W_2^3(\sigma)} \\ &\le c \sup_{t\in[0,T]} \|\mathbf{U}'^{(N)}\|_{\overset{\circ}{\mathcal{Y}}(\sigma)}. \end{aligned} \tag{5.60}$$

Suppose that the vector $\mathbf{U}'^{(N)}$ is "sufficiently small":

$$\sup_{t\in[0,T]} \big(\sup_{x'\in\sigma} |\mathbf{U}'^{(N)}| + \sup_{x'\in\sigma} |\nabla\mathbf{U}'^{(N)}|\big) \le C_0 \sup_{t\in[0,T]} \|\mathbf{U}'^{(N)}\|_{\overset{\circ}{\mathcal{Y}}(\sigma)} \le \delta_0, \tag{5.61}$$

where δ_0 will be determined below.

We look for the approximate solution $(U_3^{(N)}(x',t), q^{(N)}(t))$ in the form

$$(U_3^{(N)}(x',t), q^{(N)}(t)) = (v^{(N)}(x',t), 0) + (V^{(N)}(x',t), q^{(N)}(t)),$$

where $v^{(N)}(x',t) = \sum\limits_{k=1}^{N} z_k^{(N)}(t) w_k(x')$ and $V^{(N)}(x',t) = \sum\limits_{k=1}^{N} y_k^{(N)}(t) w_k(x')$.

Coefficients $z_k^{(N)}(t)$ are found as solutions of the following problem

$$\begin{cases} \dfrac{d}{dt} z_k^{(N)}(t) + \nu \int\limits_\sigma \nabla v^{(N)} \cdot \nabla w_k dx' \\ + \int\limits_\sigma \mathbf{U}'^{(N)} \cdot \nabla w_k (v^{(N)} - \alpha \Delta v^{(N)}) dx' = f_{3k}(t), \\ z_k^{(N)}(0) = v_{0k}, \quad k = 1, \dots, N, \ \ N \ge 1, \end{cases} \tag{5.62}$$

The unique solvability of this Cauchy problem follows from Lemma 5.4 with $\mathbf{W} = \mathbf{U}'^{(N)}$.

Coefficients $y_k^{(N)}(t)$ are found as solutions of the following Cauchy problems

$$\begin{cases} \dfrac{d}{dt} y_k^{(N)}(t) + \nu \int\limits_\sigma \nabla V^{(N)} \cdot \nabla w_k dx' \\ + \int\limits_\sigma \mathbf{U}'^{(N)} \cdot \nabla w_k (V^{(N)} - \alpha \Delta V^{(N)}) dx' = q^{(N)}(t) \int\limits_\sigma w_k dx', \\ y_k^{(N)}(0) = 0, \quad k = 1, \dots, N, \ \ N \ge 1. \end{cases} \tag{5.63}$$

The functions $q^{(N)}(t)$ are chosen so that $V^{(N)}(x',t)$ would satisfy flux condition (5.53), i.e.,

$$\int_\sigma V^{(N)}(x',t)dx' = \widetilde{F}^{(N)}(t) = F(t) - \int_\sigma v^{(N)}(x',t)dx' - \alpha^{(N)}, \tag{5.64}$$

where $\alpha^{(N)} = \int_\sigma (v_0 - v_0^{(N)})dx'$. We notice, that $\alpha^{(N)} \to 0$ as $N \to \infty$, and that there holds the compatibility condition

$$\widetilde{F}^{(N)}(0) = 0 \quad \forall N \geq 1. \tag{5.65}$$

The Cauchy problem (5.63) can be rewritten as

$$\begin{cases} \frac{d}{dt}y_k^{(N)}(t) + \sum_{j=1}^{N}(\frac{\nu}{\alpha} + m_{kj}(t))y_k^{(N)}(t) = q^{(N)}(t)\beta_k, \\ y_k^{(N)}(0) = 0, \quad k = 1, ..., N, \end{cases}$$

where $m_{kj}(t) = -\int_\sigma(\frac{\nu}{\alpha}w_k w_j - (\mathbf{U}'^{(N)} \cdot \nabla)w_j(w_k - \alpha\Delta w_k))dx'$, $\beta_k = \int_\sigma w_k dx'$, or in the vector form

$$\begin{cases} \frac{d}{dt}\mathbf{Y}^{(N)}(t) + (\mathbb{J}^{(N)} + \mathbb{A}^{(N)}(t))\mathbf{Y}^{(N)}(t) = q^{(N)}(t)\boldsymbol{\beta}^{(N)}, \\ \mathbf{Y}^{(N)}(0) = 0, \end{cases} \tag{5.66}$$

where

$$\mathbf{Y}^{(N)}(t) = \begin{pmatrix} y_1^{(N)}(t) \\ ... \\ y_N^{(N)}(t) \end{pmatrix}, \quad \boldsymbol{\beta}^{(N)} = \begin{pmatrix} \beta_1 \\ ... \\ \beta_N \end{pmatrix},$$

$\mathbb{J}^{(N)} = \mathrm{diag}(\frac{\nu}{\alpha}, ..., \frac{\nu}{\alpha})$ - diagonal matrix, $\mathbb{A}^{(N)}$ is $(N \times N)$ matrix with elements $m_{kj}(t)$.

Lemma 5.4. *Suppose that* $q^{(N)} \in L_2(0,T)$ *and* $\mathbf{U}'^{(N)} \in \mathring{\mathcal{V}}(\sigma^T)$ *satisfies* (5.61) *with sufficiently small* δ_0 *and condition* (5.65) *holds. Then there exist a unique solution* $\mathbf{Y}^{(N)} \in W_2^1(0,T)$ *of Cauchy problem* (5.66).

The proof of this lemma literally repeats the proof of Lemma 5.4.

The fundamental matrix $\mathbb{Z}^{(N)}(t)$ of problem (5.66) is the solution of the matrix Cauchy problem

$$\mathbb{Z}^{(N)\prime}(t) + (\mathbb{J}^{(N)} + \mathbb{A}^{(N)}(t))\mathbb{Z}^{(N)}(t) = \mathbb{O}, \quad \mathbb{Z}^{(N)}(0) = \mathbb{E}^{(N)}, \tag{5.67}$$

where $\mathbb{E}^{(N)}$ is the unit matrix and $\mathbb{O}$ is zero matrix. The solution $\mathbf{Y}^{(N)}(t)$ of problem (5.66) can be represented in the form

$$\mathbf{Y}^{(N)}(t) = \int_0^t \mathbb{Z}^{(N)}(t)(\mathbb{Z}^{(N)}(\tau))^{-1}\boldsymbol{\beta}^{(N)}q^{(N)}(\tau)d\tau. \tag{5.68}$$

We find the functions $q^{(N)}(t)$ from flux condition (5.64). Substituting $V^{(N)}(x',t)$ into (5.64) gives

$$\begin{aligned}\widetilde{F}^{(N)}(t) &= \int_\sigma V^{(N)}(x',t)dx' = \sum_{k=1}^N y_k^{(N)}(t)\int_\sigma w_k(x')dx' = \mathbf{Y}^{(N)}(t)\cdot\boldsymbol{\beta}^{(N)} \\ &= \boldsymbol{\beta}^{(N)}\cdot\int_0^t \mathbb{Z}^{(N)}(t)(\mathbb{Z}^{(N)}(\tau))^{-1}\boldsymbol{\beta}^{(N)}q^{(N)}(\tau)d\tau.\end{aligned}$$

Thus, $q^{(N)}(t)$ has to be found as the solution of the Volterra integral equation of the first kind

$$\int_0^t \boldsymbol{\beta}^{(N)}\cdot\mathbb{Z}^{(N)}(t)(\mathbb{Z}^{(N)}(\tau))^{-1}\boldsymbol{\beta}^{(N)}q^{(N)}(\tau)d\tau = \widetilde{F}^{(N)}(t).$$

Differentiating the last equation and using (5.67), we reduce it to the Volterra integral equation of the second kind

$$\begin{aligned}&-\int_0^t \frac{\boldsymbol{\beta}^{(N)}}{|\boldsymbol{\beta}^{(N)}|^2}\cdot(\mathbb{J}^{(N)}+\mathbb{A}^{(N)}(t))\mathbb{Z}^{(N)}(t)(\mathbb{Z}^{(N)}(\tau))^{-1}\boldsymbol{\beta}^{(N)}q^{(N)}(\tau)d\tau \\ &+q^{(N)}(t) = \frac{1}{|\boldsymbol{\beta}^{(N)}|^2}\frac{d}{dt}\widetilde{F}^{(N)}(t)\end{aligned} \tag{5.69}$$

with the kernel

$$\mathcal{K}^{(N)}(t,\tau) = \frac{\boldsymbol{\beta}^{(N)}}{|\boldsymbol{\beta}^{(N)}|^2}\cdot(\mathbb{J}^{(N)}+\mathbb{A}^{(N)}(t))\mathbb{Z}^{(N)}(t)(\mathbb{Z}^{(N)}(\tau))^{-1}\boldsymbol{\beta}^{(N)}.$$

For any fixed N the kernel $\mathcal{K}^{(N)}(t,\tau)$ is bounded for all $0\le\tau\le t$ and, hence, $\mathcal{K}^{(N)}\in L_2(\mathbb{Q}^T)$, $\mathbb{Q}^T=(0,T)\times(0,T)$. Therefore, for any $\frac{d}{dt}\widetilde{F}^{(N)}\in L_2(0,T)$ there exists a unique solution $q^{(N)}\in L_2(0,T)$ of integral equation (5.69) and the following estimate

$$\|q^{(N)}\|_{L_2(0,T)} \le C_N\|\frac{d}{dt}\widetilde{F}^{(N)}\|_{L_2(0,T)} \tag{5.70}$$

holds (see, for example, [84]). The constant C_N in (5.70) depends on the kernel $\mathcal{K}^{(N)}(t,\tau)$, and we cannot say in advance that C_N stay bounded as $N\to\infty$. In the next section we will prove uniform with respect to N estimates for $(V^{(N)}(x',t),q^{(N)}(t))$.

5.4 A priori estimates for inverse problem (5.8)

Lemma 5.5. *Suppose that $\partial\sigma \in C^4$, $\mathbf{U}'^{(N)} \in \overset{\circ}{\mathcal{V}}(\sigma^T)$ is given and satisfies condition* (5.61) *with sufficiently small δ_0, $\widetilde{F}^{(N)} \in W_2^1(0,T)$ and satisfies condition* (5.65). *Then for $(V^{(N)}(x',t), q^{(N)}(t))$ the following estimate*

$$\sup_{t\in[0,T]} \|V^{(N)}\|^2_{\overset{\circ}{\mathcal{X}}(\sigma)} + \|V^{(N)}\|^2_{\overset{\circ}{\mathcal{W}}(\sigma^T)} + \|q^{(N)}\|^2_{L_2(0,T)} \le c\|\widetilde{F}^{(N)}\|^2_{W_2^1(0,T)}. \tag{5.71}$$

holds. Here the constant c is independent of N.

Proof. Multiply equalities (5.63) by $y_k^{(N)}(t)$ and sum them by k from 1 to N:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\sigma (|V^{(N)}|^2 + \alpha|\nabla V^{(N)}|^2)dx' + \nu\int_\sigma |\nabla V^{(N)}|^2 dx' \\
&= q^{(N)}(t)\int_\sigma V^{(N)}dx' + \alpha\int_\sigma (\mathbf{U}'^{(N)}\cdot\nabla)V^{(N)}\Delta V^{(N)}dx' \\
&\le |q^{(N)}(t)\widetilde{F}^{(N)}(t)| + \alpha\sup_{x'\in\sigma}|\mathbf{U}'^{(N)}|\|\nabla V^{(N)}\|_{L_2(\sigma)}\|\Delta V^{(N)}\|_{L_2(\sigma)} \\
&\le \frac{\varepsilon}{2}|q^{(N)}(t)|^2 + \frac{1}{2\varepsilon}|\widetilde{F}^{(N)}(t)|^2 \\
&+ c\delta_0\|\nabla V^{(N)}\|_{L_2(\sigma)}\big(\|\nabla(V^{(N)} - \Delta V^{(N)})\|_{L_2(\sigma)} + \|\nabla V^{(N)}\|_{L_2(\sigma)}\big) \\
&\le \frac{\varepsilon}{2}|q^{(N)}(t)|^2 + \frac{1}{2\varepsilon}|\widetilde{F}^{(N)}(t)|^2 \\
&+ c_1\delta_0\|\nabla V^{(N)}\|^2_{L_2(\sigma)} + c\delta_0\|\nabla(V^{(N)} - \Delta V^{(N)})\|^2_{L_2(\sigma)}.
\end{aligned}$$

Here we have applied Cauchy inequality with $\varepsilon > 0$, inequalities (5.9), (5.61), and the fact that

$$\frac{d}{dt}y_k^{(N)}(t) = \int_\sigma \big(\frac{\partial}{\partial t}U_3^{(N)}w_k + \alpha\nabla\frac{\partial}{\partial t}U_3^{(N)}\cdot\nabla w_k\big)dx'.$$

If $c_1\delta_0 \le \dfrac{\nu}{2}$, then the last inequality yields

$$\begin{aligned}
&\frac{d}{dt}\int_\sigma (|V^{(N)}|^2 + \alpha|\nabla V^{(N)}|^2)dx' + \nu\int_\sigma |\nabla V^{(N)}|^2 dx' \\
&\le c\delta_0\|\nabla(V^{(N)} - \Delta V^{(N)})\|^2_{L_2(\sigma)} + \varepsilon|q^{(N)}(t)|^2 + c|\widetilde{F}^{(N)}(t)|^2.
\end{aligned} \tag{5.72}$$

Denote

$$\Phi^{(N)}(x',t) = (\mathbf{U}'^{(N)}\cdot\nabla)(V^{(N)} - \alpha\Delta V^{(N)}) - \nu\Delta V^{(N)} - q^{(N)}(t). \tag{5.73}$$

Since the eigenfunctions $w_k \in W_2^4(\sigma)$, it follows that $\Phi^{(N)} \in W_2^1(\sigma)$.

Let us rewrite equalities (5.63) in the form

$$\int_\sigma \Big(\frac{\partial}{\partial t}V^{(N)}w_k + \alpha\nabla\frac{\partial}{\partial t}V^{(N)}\cdot\nabla w_k\Big)dx' + \int_\sigma \Phi^{(N)}w_k dx' = 0. \tag{5.74}$$

From this point, following the ideas of [13], [15], we will use the specific form of the basis $\{w_k\}_{k\geq 1}$. Denote by $W^{(N)}(\cdot,t)\in \overset{\circ}{W}{}^1_2(\sigma)\cap W^3_2(\sigma)$ the solution of the following problem

$$\begin{cases} -\alpha\Delta W^{(N)} + W^{(N)} = \Phi^{(N)}, \\ W^{(N)}|_{\partial\sigma} = 0. \end{cases} \tag{5.75}$$

Then

$$\int_\sigma (\alpha\nabla W^{(N)}\cdot\nabla\eta + W^{(N)}\eta)dx' = \int_\sigma \Phi^{(N)}\eta dx' \quad \forall\,\eta\in\overset{\circ}{W}{}^1_2(\sigma). \tag{5.76}$$

Taking in (5.76) $\eta = w_k$, from (5.74) we obtain the relations

$$\int_\sigma \Big(\frac{\partial}{\partial t}V^{(N)}w_k + \alpha\nabla\frac{\partial}{\partial t}V^{(N)}\cdot\nabla w_k\Big)dx' + \int_\sigma (W^{(N)}w_k + \alpha\nabla W^{(N)}\cdot\nabla w_k)dx' = 0.$$

Using the definition of the eigenfunctions w_k (see (5.13)) we can rewrite the last equalities in the form

$$\begin{aligned}\frac{1}{\lambda_k}\int_\sigma \Big(\frac{\partial}{\partial t}V^{(N)}w_k + \alpha\nabla\frac{\partial}{\partial t}V^{(N)}\cdot\nabla w_k + \nabla\big(\frac{\partial}{\partial t}V^{(N)} - \alpha\Delta\frac{\partial}{\partial t}V^{(N)}\big)\cdot\nabla\big(\frac{\partial}{\partial t}w_k - \alpha\Delta\frac{\partial}{\partial t}w_k\big)dx' \\ + \frac{1}{\lambda_k}\int_\sigma \Big(W^{(N)}w_k + \alpha\nabla W^{(N)}\cdot\nabla w_k + \nabla\big(\frac{\partial}{\partial t}W^{(N)} - \alpha\Delta\frac{\partial}{\partial t}W^{(N)}\big)\cdot\nabla\big(\frac{\partial}{\partial t}w_k - \alpha\Delta\frac{\partial}{\partial t}w_k\big)dx' = 0.\end{aligned}$$

Multiplying these relations by $\lambda_k y_k^{(N)}(t)$ and summing from 1 to N yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\sigma \big(|V^{(N)}|^2 + \alpha|\nabla V^{(N)}|^2 + |\nabla(V^{(N)} - \alpha\Delta V^{(N)})|^2\big)dx' \\ &+ \int_\sigma \big(W^{(N)}V^{(N)} + \alpha\nabla W^{(N)}\cdot\nabla V^{(N)}\big)dx' \\ &+ \int_\sigma \nabla(W^{(N)} - \alpha\Delta W^{(N)})\cdot\nabla(V^{(N)} - \alpha\Delta V^{(N)})dx' = 0.\end{aligned}$$

From (5.75), (5.76) it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\sigma \big(|V^{(N)}|^2 + \alpha|\nabla V^{(N)}|^2 + |\nabla(V^{(N)} - \alpha\Delta V^{(N)})|^2\big)dx' \\ &+ \int_\sigma \Phi^{(N)}V^{(N)}dx' + \int_\sigma \nabla(V^{(N)} - \alpha\Delta V^{(N)})\cdot\nabla\Phi^{(N)}dx' = 0.\end{aligned} \tag{5.77}$$

Substituting the expression (5.73) of $\Phi^{(N)}$ into (5.77) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_\sigma \big(|V^{(N)}|^2+\alpha|\nabla V^{(N)}|^2+|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2\big)dx'\\
&+\alpha\int\limits_\sigma \mathbf{U}'^{(N)}\cdot\nabla V^{(N)}\Delta V^{(N)}dx'+\nu\int\limits_\sigma|\nabla V^{(N)}|^2dx'\\
&+\int\limits_\sigma\nabla\big[(\mathbf{U}'^{(N)}\cdot\nabla)(V^{(N)}-\alpha\Delta V^{(N)})\big]\cdot\nabla(V^{(N)}-\alpha\Delta V^{(N)})dx'\\
&+\frac{\nu}{\alpha}\int\limits_\sigma|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2dx'\\
&-\frac{\nu}{\alpha}\int\limits_\sigma\nabla V^{(N)}\cdot\nabla(V^{(N)}-\alpha\Delta V^{(N)})dx'=q^{(N)}(t)\int\limits_\sigma V^{(N)}dx'.
\end{aligned}
\tag{5.78}
$$

The fourth integral in (5.78) contains the term having fourth order derivatives of $V^{(N)}$. However, it can be estimated by the integral containing derivatives of $V^{(N)}$ only up to the third order. The calculations below are correct because elements w_k of the basis belong to the space $W_2^4(\sigma)$. Denote for simplicity $V^{(N)}-\alpha\Delta V^{(N)}=R^{(N)}$ and $\{\nabla\mathbf{U}'^{(N)}\cdot\nabla\}=\nabla U_1^{(N)}\dfrac{\partial}{\partial x_1}+\nabla U_2^{(N)}\dfrac{\partial}{\partial x_2}$. Integrating by parts and applying (5.61) we obtain

$$
\begin{aligned}
&\int\limits_\sigma\nabla\big((\mathbf{U}'^{(N)}\cdot\nabla)(V^{(N)}-\alpha\Delta V^{(N)})\big)\cdot\nabla(V^{(N)}-\alpha\Delta V^{(N)})dx'\\
&=\int\limits_\sigma\nabla\big((\mathbf{U}'^{(N)}\cdot\nabla)R^{(N)}\big)\cdot\nabla R^{(N)}dx'=\int\limits_\sigma\{\nabla\mathbf{U}'^{(N)}\cdot\nabla\}R^{(N)}\cdot\nabla R^{(N)}dx'\\
&+\int\limits_\sigma(\mathbf{U}'^{(N)}\cdot\nabla)\nabla R^{(N)}\cdot\nabla R^{(N)}dx'=\int\limits_\sigma\{\nabla\mathbf{U}'^{(N)}\cdot\nabla\}R^{(N)}\cdot\nabla R^{(N)}dx'\\
&+\frac{1}{2}\int\limits_\sigma(\mathbf{U}'^{(N)}\cdot\nabla)|\nabla R^{(N)}|^2dx'=\int\limits_\sigma\{\nabla\mathbf{U}'^{(N)}\cdot\nabla\}R^{(N)}\cdot\nabla R^{(N)}dx'\\
&-\frac{1}{2}\int\limits_\sigma\nabla\cdot\mathbf{U}'^{(N)}|\nabla R^{(N)}|^2dx'=\int\limits_\sigma\{\nabla\mathbf{U}'^{(N)}\cdot\nabla\}R^{(N)}\cdot\nabla R^{(N)}dx'\\
&\le c\int\limits_\sigma|\nabla\mathbf{U}'^{(N)}||\nabla R^{(N)}|^2dx'\le c\sup_{x'\in\sigma}|\nabla\mathbf{U}'^{(N)}|\int\limits_\sigma|\nabla R^{(N)}|^2dx'\\
&\le c\delta_0\int\limits_\sigma|\nabla R^{(N)}|^2dx'=c\delta_0\int\limits_\sigma|\nabla V^{(N)}-\alpha\nabla\Delta V^{(N)}|^2dx'.
\end{aligned}
\tag{5.79}
$$

Using (5.79) from (5.78) we derive the estimate

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_\sigma\big(|V^{(N)}|^2+\alpha|\nabla V^{(N)}|^2+|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2\big)dx'\\
&+\nu\int\limits_\sigma|\nabla V^{(N)}|^2dx'+\frac{\nu}{\alpha}\int_\sigma|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2dx'\\
&=\alpha\int\limits_\sigma\mathbf{U}'^{(N)}\cdot\nabla\Delta V^{(N)}V^{(N)}dx'+\frac{\nu}{\alpha}\int\limits_\sigma\nabla V^{(N)}\cdot\nabla(V^{(N)}-\alpha\Delta V^{(N)})dx'\\
&+\int\limits_\sigma\nabla\big((\mathbf{U}'^{(N)}\cdot\nabla)(V^{(N)}-\alpha\Delta V^{(N)})\big)\cdot\nabla(V^{(N)}-\alpha\Delta V^{(N)})dx'
\end{aligned}
$$

$$
\begin{aligned}
&+q^{(N)}(t)\int\limits_{\sigma} V^{(N)}dx' \le \alpha \sup_{x'\in\sigma}|\mathbf{U}'^{(N)}|\|\nabla\Delta V^{(N)}\|_{L_2(\sigma)}\|V^{(N)}\|_{L_2(\sigma)}\\
&+\frac{\nu}{\alpha}\|\nabla V^{(N)}\|_{L_2(\sigma)}\|\nabla(V^{(N)}-\alpha\Delta V^{(N)})\|_{L_2(\sigma)} + |q^{(N)}(t)\widetilde{F}^{(N)}(t)|\\
&+c\delta_0\int\limits_{\sigma}|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2dx' \le \big(c_2\delta_0+\frac{\mu}{2}\big)\int\limits_{\sigma}|\nabla(V^{(N)}-\Delta V^{(N)})|^2dx'\\
&+\big(c_3\delta_0+\frac{1}{2\mu}\big)\int\limits_{\sigma}|\nabla V^{(N)}|^2dx' + \frac{\varepsilon}{2}|q^{(N)}(t)|^2+\frac{1}{2\varepsilon}|\widetilde{F}^{(N)}(t)|^2.
\end{aligned}
$$

Taking $\mu=\dfrac{\nu}{4\alpha}$ and assuming that δ_0 is sufficiently small: $c_2\delta_0\le\dfrac{\nu}{4\alpha}$, $c_3\delta_0\le\dfrac{\nu}{2}$, from the latter inequality we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_{\sigma}\big(|V^{(N)}|^2+\alpha|\nabla V^{(N)}|^2+|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2\big)dx'+\nu\int\limits_{\sigma}|\nabla V^{(N)}|^2dx'\\
&+\frac{\nu}{\alpha}\int\limits_{\sigma}|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2dx' \le c\big(|\widetilde{F}^{(N)}(t)|^2+\int\limits_{\sigma}|\nabla V^{(N)}|^2dx'\big)+\varepsilon|q^{(N)}(t)|^2.
\end{aligned}
\tag{5.80}
$$

Integrating inequality (5.72) with respect to t yields

$$
\begin{aligned}
&\int\limits_{\sigma}\big(|V^{(N)}|^2+\alpha|\nabla V^{(N)}|^2\big)dx'+\nu\int\limits_0^t\int\limits_{\sigma}|\nabla V^{(N)}|^2dx'd\tau\\
&\le c\delta_0\int\limits_0^t\int\limits_{\sigma}|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2dx'd\tau+\varepsilon\int\limits_0^t|q^{(N)}(\tau)|^2d\tau+c\int\limits_0^t|\widetilde{F}^{(N)}(\tau)|^2d\tau.
\end{aligned}
\tag{5.81}
$$

On the other hand, integrating inequality (5.80) and applying (5.81) we derive

$$
\begin{aligned}
&\int\limits_{\sigma}\big(|V^{(N)}|^2+\alpha|\nabla V^{(N)}|^2+|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2\big)dx'\\
&+\nu\int\limits_0^t\int\limits_{\sigma}|\nabla V^{(N)}|^2dx'd\tau+\frac{\nu}{\alpha}\int\limits_0^t\int\limits_{\sigma}|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2dx'd\tau\\
&\le c\varepsilon\int\limits_0^t|q^{(N)}(\tau)|^2d\tau+c\int\limits_0^t|\widetilde{F}^{(N)}(\tau)|^2d\tau+c_4\delta_0\int\limits_0^t\int\limits_{\sigma}|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2dx'd\tau.
\end{aligned}
\tag{5.82}
$$

If $c_4\delta_0\le\dfrac{1}{2}$, then from (5.82) follows the estimate

$$
\begin{aligned}
&\int\limits_{\sigma}\big(|V^{(N)}|^2+\alpha|\nabla V^{(N)}|^2+|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2\big)dx'\\
&+\int\limits_0^t\int\limits_{\sigma}\big(\nu|\nabla V^{(N)}|^2+|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2\big)dx'd\tau\\
&\le c\varepsilon\int\limits_0^t|q^{(N)}(\tau)|^2d\tau+c\int\limits_0^t|\widetilde{F}^{(N)}(\tau)|^2d\tau.
\end{aligned}
\tag{5.83}
$$

Multiply equalities (5.63) by $\dfrac{d}{dt}y_k^{(N)}(t)$ and sum them by k from 1 to N:

$$
\begin{aligned}
&\int\limits_\sigma \big(|\frac{\partial}{\partial t}V^{(N)}|^2 + \alpha|\nabla\frac{\partial}{\partial t}V^{(N)}|^2\big)dx' + \frac{\nu}{2}\frac{d}{dt}\int_\sigma|\nabla V^{(N)}|^2dx' \\
&= \int\limits_\sigma (\mathbf{U}'^{(N)}\cdot\nabla)(V^{(N)} - \alpha\Delta V^{(N)})\frac{\partial}{\partial t}V^{(N)}dx' + \int\limits_\sigma q^{(N)}(t)\frac{\partial}{\partial t}V^{(N)}dx' \\
&\leq \frac{1}{2}\sup_{x'\in\sigma}|\mathbf{U}'^{(N)}|^2\int\limits_\sigma|\nabla(V^{(N)} - \alpha\Delta V^{(N)})|^2dx' + \frac{1}{2}\int\limits_\sigma|\frac{\partial}{\partial t}V^{(N)}|^2dx' \\
&+\frac{\varepsilon}{2}|q^{(N)}(t)|^2dx' + \frac{1}{2\varepsilon}|\frac{d}{dt}\widetilde{F}^{(N)}(t)|^2.
\end{aligned}
$$

Integrating this inequality with respect to t and applying (5.61) and (5.83) we get

$$
\begin{aligned}
&\nu\int\limits_\sigma|\nabla V^{(N)}|^2dx' + \int\limits_0^t\int\limits_\sigma\big(|\frac{\partial}{\partial\tau}V^{(N)}|^2 + \alpha|\nabla\frac{\partial}{\partial\tau}V^{(N)}|^2\big)dx'd\tau \\
&\leq c\varepsilon\int\limits_0^t|q^{(N)}(\tau)|^2dx'd\tau + c\int\limits_0^t|\frac{d}{d\tau}\widetilde{F}^{(N)}(\tau)|^2d\tau.
\end{aligned}
\tag{5.84}
$$

Let $\omega\in\mathring{W}_2^1(\sigma)$ be the solution of the Dirichlet problem for the Poisson equation on σ:

$$
\begin{cases}
-\Delta\omega(x') = 1, \\
\omega(x')|_{\partial\sigma} = 0.
\end{cases}
\tag{5.85}
$$

If $\partial\sigma\in C^2$, then $\omega\in W_2^2(\sigma)$, and $\omega(x')$ can be expressed by the Fourier series

$$
\omega(x') = \sum_{k=1}^\infty\gamma_k w_k(x'), \quad \gamma_k = \int\limits_\sigma\omega(x')w_k(x')dx',
$$

which converges in the space $W_2^2(\sigma)$. Multiply (5.63) by γ_k and sum obtained relations from 1 to N:

$$
\begin{aligned}
&\int\limits_\sigma\big(\frac{\partial}{\partial t}V^{(N)}\omega^{(N)}(x') + \alpha\nabla\frac{\partial}{\partial t}V^{(N)}\cdot\nabla\omega^{(N)}\big)dx' + \nu\int\limits_\sigma\nabla V^{(N)}\cdot\nabla\omega^{(N)}dx' \\
&+\int\limits_\sigma(\mathbf{U}'^{(N)}\cdot\nabla)(V^{(N)} - \alpha\Delta V^{(N)})\omega^{(N)}dx' = q^{(N)}(t)\int\limits_\sigma\omega^{(N)}dx',
\end{aligned}
\tag{5.86}
$$

where $\omega^{(N)}(x') = \sum\limits_{k=1}^N\gamma_k w_k(x')$. In virtue of (5.85) and the flux condition (5.53) we have

$$
\int\limits_\sigma\nabla V^{(N)}\cdot\nabla\omega^{(N)}dx' = \int\limits_\sigma\nabla V^{(N)}\cdot\nabla\omega dx' + \int\limits_\sigma\nabla V^{(N)}\cdot\nabla(\omega^{(N)} - \omega)dx'
$$

$$= -\int\limits_{\sigma} V^{(N)}\Delta\omega dx' + \int\limits_{\sigma} \nabla V^{(N)}\cdot\nabla(\omega^{(N)}-\omega)dx'$$
$$= \widetilde{F}^{(N)}(t) + \int\limits_{\sigma} \nabla V^{(N)}\cdot\nabla(\omega^{(N)}-\omega)dx',$$

and, analogously,

$$\int\limits_{\sigma} \nabla\frac{\partial}{\partial t}V^{(N)}\cdot\nabla\omega^{(N)}dx' = \frac{d}{dt}\widetilde{F}^{(N)}(t) + \int\limits_{\sigma} \nabla\frac{\partial}{\partial t}V^{(N)}\cdot\nabla(\omega^{(N)}-\omega)dx'.$$

Therefore, relation (5.86) can be rewritten as

$$\begin{aligned}
&\int\limits_{\sigma} \frac{\partial}{\partial t}V^{(N)}\omega^{(N)}dx' + \alpha\frac{d}{dt}\widetilde{F}^{(N)}(t) + \alpha\int\limits_{\sigma} \nabla\frac{\partial}{\partial t}V^{(N)}\cdot\nabla(\omega^{(N)}-\omega)dx' \\
&+\nu\widetilde{F}^{(N)}(t) + \nu\int\limits_{\sigma} \nabla V^{(N)}\cdot\nabla(\omega^{(N)}-\omega)dx' \\
&+\int\limits_{\sigma}(\mathbf{U}'^{(N)}\cdot\nabla)(V^{(N)}-\alpha\Delta V^{(N)})\omega^{(N)}dx' = q^{(N)}(t)\int\limits_{\sigma}\omega^{(N)}dx'.
\end{aligned}$$

Let us estimate the functions $q^{(N)}(t)$. From the last equality it follows that

$$\begin{aligned}
&\kappa_0 q^{(N)}(t) = \int\limits_{\sigma} \frac{\partial}{\partial t}V^{(N)}\omega^{(N)}dx' + \alpha\frac{d}{dt}\widetilde{F}^{(N)}(t) + \nu\widetilde{F}^{(N)}(t) \\
&+\alpha\int\limits_{\sigma} \nabla\frac{\partial}{\partial t}V^{(N)}\cdot\nabla(\omega^{(N)}-\omega)dx' + \nu\int\limits_{\sigma} \nabla V^{(N)}\cdot\nabla(\omega^{(N)}-\omega)dx' \\
&+\int\limits_{\sigma}(\mathbf{U}'^{(N)}\cdot\nabla)(V^{(N)}-\alpha\Delta V^{(N)})\omega^{(N)}dx' + q^{(N)}(t)\int\limits_{\sigma}(\omega-\omega^{(N)})dx',
\end{aligned}$$

where $\kappa_0 = \int\limits_{\sigma}\omega(x')dx' > 0$. Therefore,

$$\begin{aligned}
&\kappa_0^2\int\limits_0^t |q^{(N)}(\tau)|^2 d\tau \le c\Big(\int\limits_{\sigma}|\omega^{(N)}|^2dx'\int\limits_0^t\int\limits_{\sigma}|\frac{\partial}{\partial\tau}V^{(N)}|^2dx'd\tau + \alpha^2\int\limits_0^t|\frac{d}{d\tau}\widetilde{F}^{(N)}(\tau)|^2d\tau \\
&+\nu^2\int\limits_0^t|\widetilde{F}^{(N)}(\tau)|^2d\tau + \alpha^2\int\limits_{\sigma}|\nabla(\omega^{(N)}-\omega)|^2dx'\int\limits_0^t\int\limits_{\sigma}|\frac{\partial}{\partial\tau}\nabla V^{(N)}|^2dx' \\
&+\nu^2\int\limits_{\sigma}|\nabla(\omega^{(N)}-\omega)|^2dx'\int\limits_0^t\int\limits_{\sigma}|\nabla V^{(N)}|^2dx' \\
&+\big(\sup_{x'\in\bar\sigma}|\omega^{(N)}(x')|^2\big)\sup_{\tau\in[0,t]}\big(\int\limits_{\sigma}|\mathbf{U}'^{(N)}|^2dx'\big)\int\limits_0^t\int\limits_{\sigma}|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2dx'd\tau \\
&+\int\limits_{\sigma}|\nabla(\omega-\omega^{(N)})|^2dx'\int\limits_0^t|q^{(N)}(\tau)|^2d\tau.\Big)
\end{aligned}$$

Since $\|\omega-\omega^{(N)}\|_{W_2^1(\sigma)}\to 0$ as $N\to\infty$, the last inequality and (5.61) yield

$$
\begin{aligned}
&\int_0^t |q^{(N)}(\tau)|^2 d\tau \le c\Big(\int_0^t\int_\sigma |\frac{\partial}{\partial\tau}V^{(N)}|^2 dx'd\tau + \int_0^t |\frac{d}{d\tau}\widetilde{F}^{(N)}(\tau)|^2 d\tau \\
&+\int_0^t \big(|\widetilde{F}^{(N)}(\tau)|^2 d\tau + \int_0^t\int_\sigma |\nabla(V^{(N)}-\alpha\Delta' V^{(N)})|^2 dx'd\tau d\tau\Big).
\end{aligned}
\tag{5.87}
$$

On the other hand, from (5.83), (5.84) we establish

$$
\begin{aligned}
&\int_\sigma \big(|V^{(N)}|^2+(\nu+\alpha)|\nabla V^{(N)}|^2+|\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2\big)dx' \\
&+\nu\int_0^t\int_\sigma |\nabla V^{(N)}|^2 dx'd\tau + \int_0^t\int_\sigma |\nabla(V^{(N)}-\alpha\Delta V^{(N)})|^2 dx'd\tau \\
&+\int_0^t\int_\sigma (|\frac{\partial}{\partial\tau}V|^2+|\nabla\frac{\partial}{\partial\tau}V|^2)dx'd\tau \\
&\le c\int_0^t \big(|\widetilde{F}^{(N)}(\tau)|^2+|\frac{d}{d\tau}\widetilde{F}^{(N)}(\tau)|^2\big)d\tau + c\varepsilon\int_0^t |q^{(N)}(\tau)|^2 d\tau.
\end{aligned}
\tag{5.88}
$$

For sufficiently small ε from (5.87) and (5.88) follows the estimate

$$
\int_0^t |q^{(N)}(\tau)|^2 d\tau \le c\int_0^t \big(|\widetilde{F}^{(N)}(\tau)|^2+|\frac{d}{d\tau}\widetilde{F}^{(N)}(\tau)|^2\big)d\tau. \tag{5.89}
$$

Estimate (5.71) is the consequence of (5.83), (5.84), (5.89) and the definitions of the norms. $\square$

From Lemmas 5.1 and 5.2 follows

Lemma 5.6. *Suppose that $\partial\sigma\in C^4$, $f_3\in W_2^{1,0}(\sigma^T)$, $v_0\in W_2^3(\sigma)\cap\overset{\circ}{W}{}_2^1(\sigma)$, $F\in W_2^1(0,T)$, $\mathbf{U}'^{(N)}\in\overset{\circ}{\mathcal{V}}(\sigma^T)$ are given and $\mathbf{U}'^{(N)}$ satisfy condition (5.61) with sufficiently small δ_0. Assume that the necessary compatibility condition $F(0)=\int_\sigma v_0(x')dx'$ is valid. Then for the approximate solution $(U_3^{(N)}(x',t),q^{(N)}(t))$ the following estimate*

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|U_3^{(N)}\|^2_{\overset{\circ}{\mathcal{X}}(\sigma)}+\|U_3^{(N)}\|^2_{\overset{\circ}{\mathcal{W}}(\sigma^T)}+\|q^{(N)}\|^2_{L_2(0,T)} \\
&\le C_3\big(\|f_3\|^2_{W_2^{1,0}(\sigma^T)}+\|F\|^2_{W_2^1(0,T)}+\|v_0\|^2_{\overset{\circ}{\mathcal{X}}(\sigma)}\big)
\end{aligned}
\tag{5.90}
$$

holds. The constant C_3 in (5.90) is independent of N.

Proof. The approximate solution $(U_3^{(N)},q^{(N)})$ has the form $(U_3^{(N)},q^{(N)})=(v^{(N)},0)+(V^{(N)},q^{(N)})$ (see Section 5.2), where for $(V^{(N)},q^{(N)})$ holds estimate (5.71) and for $v^{(N)}$ -

estimate (5.30). Hence, to prove (5.90) we have only to estimate the norm of $\widetilde{F}^{(N)}(t)$. The functions $\widetilde{F}^{(N)}(t)$ are defined by (5.64). Therefore,

$$
\begin{aligned}
&\|\widetilde{F}^{(N)}\|^2_{W^1_2(0,T)} \le c\big(\|F\|^2_{W^1_2(0,T)} + \int\limits_0^T \big(\|v^{(N)}\|^2_{L_2(\sigma)} + \|\frac{\partial}{\partial t}v^{(N)}\|^2_{L_2(\sigma)}\big)d\tau \\
&+ \int\limits_0^T |\alpha_N|^2 d\tau\big) \le c\big(\|F\|^2_{W^1_2(0,T)} + \|f_3\|^2_{W^1_2(\sigma^T)} + \|v_0\|^2_{\mathring{\mathcal{X}}(\sigma)}\big).
\end{aligned}
\tag{5.91}
$$

Here we have used (5.25) and the obvious estimate

$$
\int\limits_0^T |\alpha_N|^2 d\tau \le T|\alpha_N|^2 = T\big(\int\limits_\sigma \big(v_0 - v_0^{(N)}\big)dx'\big)^2 \le T|\sigma|\|v_0 - v_0^{(N)}\|^2_{L_2(\sigma)} \le c\|v_0\|^2_{L_2(\sigma)}.
$$

Inequality (5.90) follows from (5.25), (5.71) and (5.91). □

Theorem 5.4. *Suppose that* $\partial\sigma \in C^4$, $\mathbf{u}'_0, v_0 \in W^3_2(\sigma) \cap \mathring{W}^1_2(\sigma)$, $F \in W^1_2(0,T)$, $\mathbf{f}' \in L_2(\sigma^T)$, $\operatorname{curl}\mathbf{f}' \in L_2(\sigma^T)$, $f_3 \in W^{1,0}_2(\sigma^T)$ *and that the necessary compatibility condition*

$$
F(0) = \int\limits_\sigma v_0(x')dx'
$$

is valid. Assume that the data satisfy the conditions

$$
\begin{aligned}
&\|\mathbf{f}'\|^2_{L_2(\sigma^T)} + \|\operatorname{curl}\mathbf{f}'\|^2_{L_2(\sigma^T)} + \|\mathbf{u}'_0\|^2_{\mathring{\mathcal{Y}}(\sigma)} \le \mu_0\delta_0^2, \\
&\|F\|^2_{W^1_2(0,T)} + \|f_3\|^2_{W^{1,0}_2(\sigma^T)} + \|v_0\|^2_{\mathring{\mathcal{X}}(\sigma)} \le \mu_0\delta_0^2,
\end{aligned}
\tag{5.92}
$$

where

$$
\mu_0 = \min\big\{\frac{1}{C_1C_0^2}, \frac{1}{2C_2C_0^2}, 1\big\},
\tag{5.93}
$$

C_0 *and* δ_0 *are constants from inequality* (5.61), C_1, C_2 *and* C_3 *are constants from estimates* (5.54), (5.55) *and* (5.90) *respectively,* δ_0 *is "sufficiently small", i.e.,*

$$
C_3\delta_0 \le 1,
\tag{5.94}
$$

and such that Lemma 5.3 *is valid. Then the approximate solution* $\big(\mathbf{U}'^{(N)}, U_3^{(N)}, q^{(N)}\big) \in \mathring{\mathcal{V}}(\sigma^T) \times \mathring{\mathcal{W}}(\sigma^T) \times L_2(0,T)$ *of problem* (5.8) *satisfies the following estimate*

$$
\begin{aligned}
&\sup_{t\in[0,T]}\big(\|\mathbf{U}'^{(N)}\|^2_{\mathring{\mathcal{Y}}(\sigma)}+\|U_3^{(N)}\|^2_{\mathring{\mathcal{X}}(\sigma)}\big)+\|\mathbf{U}'^{(N)}\|^2_{\mathring{\mathcal{V}}(\sigma^T)}+\|U_3^{(N)}\|^2_{\mathring{\mathcal{W}}(\sigma^T)}\\
&+\|q^{(N)}\|^2_{L_2(0,T)}\le c\big(\|\mathbf{f}'\|^2_{L_2(\sigma^T)}+\|\mathrm{curl}\,\mathbf{f}'\|^2_{L_2(\sigma^T)}+\|\mathbf{u}_0'\|^2_{\mathring{\mathcal{Y}}(\sigma)}\\
&+\big(\|f_3\|^2_{W_2^{1,0}(\sigma^T)}+\|v_0\|^2_{\mathring{\mathcal{X}}(\sigma)}+\|F\|^2_{W_2^1(0,T)}\big)\times\\
&\times\big(1+\|f_3\|^2_{W_2^{1,0}(\sigma^T)}+\|v_0\|^2_{\mathring{\mathcal{X}}(\sigma)}+\|F\|^2_{W_2^1(0,T)}\big)\big).
\end{aligned}
\tag{5.95}
$$

Proof. For the approximation $\mathbf{U}'^{(1)}(x',t)$ holds inequality (5.54) and from condition (5.93) we see that $\mathbf{U}'^{(1)}(x',t)$ satisfies inequality (5.61), i.e.,

$$
C_0\sup_{t\in[0,T]}\|\mathbf{U}'^{(1)}\|_{\mathring{\mathcal{Y}}(\sigma)}\le\delta_0.
$$

Therefore, we can find $\big(U_3^{(1)},q^{(1)}\big)$ and by (5.90)

$$
\sup_{t\in[0,T]}\|U_3^{(1)}\|^2_{\mathring{\mathcal{X}}(\sigma)}+\|U_3^{(1)}\|^2_{\mathring{\mathcal{W}}(\sigma^T)}\le C_3\big(\|f_3\|^2_{W_2^{1,0}(\sigma^T)}+\|F\|^2_{W_2^1(0,T)}+\|v_0\|^2_{\mathring{\mathcal{X}}(\sigma)}\big)\le C_3\mu_0\delta_0^2.
$$

Now, applying successively inequalities (5.55) and (5.90), and using (5.92), (5.93), (5.94) we obtain the estimates

$$
\sup_{t\in[0,T]}\|U_3^{(N)}\|^2_{\mathring{\mathcal{X}}(\sigma)}+\|U_3^{(N)}\|^2_{\mathring{\mathcal{W}}(\sigma^T)}\le C_3\mu_0\delta_0^2\quad\forall N\ge1,
$$

$$
C_0^2\sup_{t\in[0,T]}\|\mathbf{U}'^{(N)}\|^2_{\mathring{\mathcal{Y}}(\sigma)}\le\delta_0^2C_0^2C_2\mu_0(1+C_3^2\mu_0\delta_0^2)\le\delta_0^2\qquad\forall N\ge1.
$$

Thus, condition (5.61) is valid for all $N\ge1$. Therefore, Galerkin approximations $\big(\mathbf{U}'^{(N)},U_3^{(N)},\,q^{(N)}\big)$ are defined $\forall\,N\ge1$. Estimate (5.95) follows from (5.55) and (5.90). $\square$

Remark 5.3. In the case when $\mathbf{u}_0'=0$, $\mathbf{f}'=0$ (i.e., the external force and the initial velocity are directed along the axis of the cylinder) from inequalities (5.90) and (5.92) we get

$$
\sup_{t\in[0,T]}\|U_3^{(N)}\|^2_{\mathring{\mathcal{X}}(\sigma)}+\|U_3^{(N)}\|^2_{\mathring{\mathcal{W}}(\sigma^T)}+\|q^{(N)}\|^2_{L_2(0,T)}\le C_3\mu_0\delta_0^2.\tag{5.96}
$$

Substituting (5.96) into inequality (5.55) we obtain

$$
\sup_{t\in[0,T]}\|\mathbf{U}'^{(N)}\|^2_{\mathring{\mathcal{Y}}(\sigma)}+\|\mathbf{U}'^{(N)}\|^2_{\mathring{\mathcal{V}}(\sigma^T)}\le C_2C_3^2\mu_0^2\delta_0^4.\tag{5.97}
$$

Estimates (5.96), (5.97) shows, in particular, that the velocity components $U_1^{(N)}$ and $U_2^{(N)}$ are secondary in comparison with the axial velocity $U_3^{(N)}$.

5.5 Existence and uniqueness of the solution to inverse problem (5.8)

Theorem 5.5. *Suppose that $\partial\sigma \in C^4$, and the data $\mathbf{u}_0(x')$, $\mathbf{f}(x',t)$, $F(t)$ satisfy the condition of Theorem 5.4. Then problem (5.8) admits a unique weak solution $(\mathbf{U}', U_3, q) \in \mathring{\mathcal{V}}(\sigma^T) \times \mathring{\mathcal{W}}(\sigma^T) \times L_2(0,T)$ and the following estimate*

$$
\begin{aligned}
&\sup_{t\in[0,T]} \big(\|\mathbf{U}'\|^2_{\mathring{\mathcal{Y}}(\sigma)} + \|U_3\|^2_{\mathring{\mathcal{X}}(\sigma)}\big) + \|\mathbf{U}'\|^2_{\mathring{\mathcal{V}}(\sigma^T)} + \|U_3\|^2_{\mathring{\mathcal{W}}(\sigma^T)} + \|q\|^2_{L_2(0,T)} \\
&\le c\big(\|\mathbf{f}'\|^2_{L_2(\sigma^T)} + \|\mathrm{curl}\,\mathbf{f}'\|^2_{L_2(\sigma^T)} + \|\mathbf{u}_0'\|^2_{\mathring{\mathcal{Y}}(\sigma)} \\
&+\big(\|f_3\|^2_{W_2^{1,0}(\sigma^T)} + \|v_0\|^2_{\mathring{\mathcal{X}}(\sigma)} + \|F\|^2_{W_2^1(0,T)}\big)\times \\
&\times\big(1 + \|f_3\|^2_{W_2^{1,0}(\sigma^T)} + \|v_0\|^2_{\mathring{\mathcal{X}}(\sigma)} + \|F\|^2_{W_2^1(0,T)}\big)\big).
\end{aligned}
\tag{5.98}
$$

holds.

Proof. Let $\boldsymbol{\eta}^{(M)} = (\boldsymbol{\eta}'^{(M)}, \eta_3^{(M)})$ with

$$
\boldsymbol{\eta}'^{(M)}(x',t) = \sum_{k=1}^{M} d_k(t)\mathbf{w}_k(x'), \qquad \eta_3^{(M)}(x',t) = \sum_{k=1}^{M} g_k(t) w_k(x'),
\tag{5.99}
$$

where $d_k(t)$ and $g_k(t)$ are arbitrary functions from $C^\infty[0,T]$. Multiplying equalities (5.50), (5.52) by $d_k(t)$ and (5.51) by $g_k(t)$, integrating by t, summing these relations by k from 1 to M ($M \le N$), and then summing the obtained integral identities, we obtain

$$
\begin{aligned}
&\int_0^t \int_\sigma \Big(\frac{\partial}{\partial\tau}\mathbf{U}^{(N)} \cdot \boldsymbol{\eta}^{(M)} + \alpha\nabla\frac{\partial}{\partial\tau}\mathbf{U}^{(N)} \cdot \nabla\boldsymbol{\eta}^{(M)}\Big)dx'd\tau \\
&+\nu \int_0^t \int_\sigma \nabla\mathbf{U}^{(N)} \cdot \nabla\boldsymbol{\eta}^{(M)} dx'd\tau + \int_0^t \int_\sigma (\mathbf{U}^{(N)} \cdot \nabla)\mathbf{U}^{(N)} \cdot \boldsymbol{\eta}^{(M)} dx'd\tau \\
&+\alpha \int_0^t \int_\sigma (\mathbf{U}^{(N)} \cdot \nabla)\boldsymbol{\eta}^{(M)} \cdot \Delta\mathbf{U}^{(N)} dx'd\tau - \alpha \int_0^t \int_\sigma (\eta_3^{(M)} \cdot \nabla)\mathbf{U}'^{(N)} \cdot \Delta\mathbf{U}'^{(N)} dx'd\tau \\
&+\alpha \int_0^t \int_\sigma \nabla U_3^{(N-1)} \cdot \boldsymbol{\eta}'^{(M)} (U_3^{(N-1)} - \alpha\Delta U_3^{(N-1)}) dx'd\tau \\
&= \int_0^t \int_\sigma \mathbf{f}^{(N)} \cdot \boldsymbol{\eta}^{(M)} dx'd\tau + \int_0^t \int_\sigma q^{(N)} \eta_3^{(M)} dx'd\tau.
\end{aligned}
\tag{5.100}
$$

From inequality (5.95) it follows that the sequence $\{\mathbf{U}'^{(N)}\}_{N\geq 1}$ is bounded in the space $\overset{\circ}{\mathcal{V}}(\sigma^T)$, $\{U_3^{(N)}\}_{N\geq 1}$ is bounded in $\overset{\circ}{\mathcal{W}}(\sigma^T)$ and $\{q^{(N)}\}_{N\geq 1}$ is bounded in $L_2(0,T)$. Moreover, for almost all $t \in (0,T)$ the sequences $\{\mathbf{U}'^{(N)}(\cdot,t)\}_{N\geq 1}$ and $\{U_3^{(N)}(\cdot,t)\}_{N\geq 1}$ are bounded in $\overset{\circ}{\mathcal{Y}}(\sigma)$ and $\overset{\circ}{\mathcal{X}}(\sigma)$, respectively. Therefore, there exist subsequences $\{\mathbf{U}'^{(N_l)}\}$, $\{U_3^{(N_l)}\}$ and $\{q^{(N_l)}\}$ such that

$$
\begin{aligned}
&\mathbf{U}'^{(N_l)}(\cdot,t) \rightharpoonup \mathbf{U}'(\cdot,t) \quad \text{in } \overset{\circ}{\mathcal{Y}}(\sigma), \ \text{for almost all } t \in (0,T),\\
&U_3^{(N_l)}(\cdot,t) \rightharpoonup U_3(\cdot,t) \quad \text{in } \overset{\circ}{\mathcal{X}}(\sigma) \ \text{for almost all } t \in (0,T),\\
&\mathbf{U}'^{(N_l)} \rightharpoonup \mathbf{U}' \ \text{ in } \overset{\circ}{\mathcal{V}}(\sigma^T), \quad U_3^{(N_l)} \rightharpoonup U_3 \ \text{ in } \overset{\circ}{\mathcal{W}}(\sigma^T),\\
&q^{(N_l)}(t) \rightharpoonup q(t) \quad \text{in } L_2(0,T).
\end{aligned}
$$

Let us fix $\boldsymbol{\eta}^{(M)}$ in (5.100). Using Sobolev Embedding Theorem it is easy to prove that passing in (5.100) to a limit as $N_l \to \infty$ we get the following integral identity

$$
\begin{aligned}
&\int_0^t \int_\sigma \Big(\frac{\partial}{\partial \tau}\mathbf{U}\cdot\boldsymbol{\eta}^{(M)} + \alpha\nabla\frac{\partial}{\partial\tau}\mathbf{U}\cdot\nabla\boldsymbol{\eta}^{(M)}\Big)dx'd\tau\\
&+\nu\int_0^t\int_\sigma \nabla\mathbf{U}\cdot\nabla\boldsymbol{\eta}^{(M)}dx'd\tau + \int_0^t\int_\sigma(\mathbf{U}\cdot\nabla)\mathbf{U}\cdot\boldsymbol{\eta}^{(M)}dx'd\tau\\
&+\alpha\int_0^t\int_\sigma(\mathbf{U}\cdot\nabla)\boldsymbol{\eta}^{(M)}\cdot\Delta\mathbf{U}dx'd\tau - \alpha\int_0^t\int_\sigma(\boldsymbol{\eta}^{(M)}\cdot\nabla)\mathbf{U}\cdot\Delta\mathbf{U}dx'd\tau\\
&=\int_0^t\int_\sigma \mathbf{f}\boldsymbol{\eta}^{(M)}dx'd\tau + \int_0^t\int_\sigma q\eta_3^{(M)}dx'd\tau.
\end{aligned}
\tag{5.101}
$$

The eigenfunctions $\mathbf{w}_k(x')$ constitute basis in $\overset{\circ}{\mathcal{Y}}(\sigma)$ and $w_k(x')$ constitute basis in $\overset{\circ}{\mathcal{X}}(\sigma)$. Therefore, the linear combinations $\boldsymbol{\eta}'^{(M)}(x',t)$ and $\eta_3^{(M)}(x',t)$ (see (5.99)) are dense in $\overset{\circ}{\mathcal{V}}(\sigma^T)$ and $\overset{\circ}{\mathcal{W}}(\sigma^T)$, respectively, and we can conclude that the integral identity (5.101) is valid for arbitrary $\boldsymbol{\eta} = (\boldsymbol{\eta}', \eta_3) \in \overset{\circ}{\mathcal{V}}(\sigma^T) \times \overset{\circ}{\mathcal{W}}(\sigma^T)$. Moreover, U_3 satisfies flux condition (5.46). Hence, by the definition $(\mathbf{U}, q)$ is a weak solution of problem (5.8). Obviously, for the limit functions $\mathbf{U}', U_3, q$ estimate (5.95) remains valid, i.e., we get estimate (5.98).

Let us prove the uniqueness. Assume that $(\mathbf{U}^{[1]}, q^{[1]})$ and $(\mathbf{U}^{[2]}, q^{[2]})$ are two weak solutions of problem (5.8). The differences $\mathbf{V} = \mathbf{U}^{[2]} - \mathbf{U}^{[1]}, S = q^{[2]} - q^{[1]}$ satisfy the integral identity

$$
\begin{aligned}
&\int_0^t \int_\sigma \big(\frac{\partial}{\partial\tau}\mathbf{V}\cdot\boldsymbol{\eta}+\alpha\nabla\frac{\partial}{\partial\tau}\mathbf{V}\cdot\nabla\boldsymbol{\eta}\big)dx'd\tau+\nu\int_0^t\int_\sigma \nabla\mathbf{V}\cdot\nabla\boldsymbol{\eta}dx'd\tau\\
&+\int_0^t\int_\sigma \big((\mathbf{U}^{[1]}\cdot\nabla)\mathbf{V}\cdot\boldsymbol{\eta}+(\mathbf{V}\cdot\nabla)\mathbf{U}^{[2]}\cdot\boldsymbol{\eta}\big)dx'd\tau\\
&+\alpha\int_0^t\int_\sigma \big((\mathbf{V}\cdot\nabla)\boldsymbol{\eta}\cdot\Delta\mathbf{U}^{[1]}+(\mathbf{U}^{[2]}\cdot\nabla)\boldsymbol{\eta}\cdot\Delta\mathbf{V}\big)dx'd\tau\\
&-\alpha\int_0^t\int_\sigma \big((\boldsymbol{\eta}\cdot\nabla)\mathbf{V}\cdot\Delta\mathbf{U}^{[1]}+(\boldsymbol{\eta}\cdot\nabla)\mathbf{U}^{[2]}\cdot\Delta\mathbf{V}\big)dx'd\tau\\
&=\int_0^t S(t)\int_\sigma \eta_3dx'd\tau\quad\forall\boldsymbol{\eta}\in\overset{\circ}{W}{}_2^{1,0}(\sigma^T)\text{ with }\operatorname{div}\boldsymbol{\eta}'=0.
\end{aligned}
\tag{5.102}
$$

Take in (5.102) $\boldsymbol{\eta}=\mathbf{V}$:

$$
\begin{aligned}
&\frac{1}{2}\int_\sigma(|\mathbf{V}|^2+\alpha|\nabla\mathbf{V}|^2)dx'+\nu\int_0^t\int_\sigma|\nabla\mathbf{V}|^2dx'd\tau\\
&+\int_0^t\int_\sigma\big(\mathbf{V}\cdot\nabla)\mathbf{U}^{[2]}\cdot\mathbf{V}+\alpha(\mathbf{U}^{[2]}\cdot\nabla)\mathbf{V}\cdot\Delta\mathbf{V}\\
&-\alpha(\mathbf{V}\cdot\nabla)\mathbf{U}^{[2]}\cdot\Delta\mathbf{V}\big)dx'd\tau=\int_0^t S(\tau)\int_\sigma\big(U_3^{[1]}-U_3^{[2]}\big)dx'd\tau\\
&=\int_0^t S(\tau)\big(F(\tau)-F(\tau)\big)d\tau=0.
\end{aligned}
\tag{5.103}
$$

Let us estimate the third integral (which we denote J) on the left-hand side of the last equality. We have

$$
\big|\int_0^t\int_\sigma\big(\mathbf{V}\cdot\nabla)\mathbf{U}^{[2]}\cdot\mathbf{V}dx'd\tau\big|\le\int_0^t\|\nabla\mathbf{U}^{[2]}\|_{L_2(\sigma)}\|\mathbf{V}\|^2_{L_4(\sigma)}d\tau\le c\int_0^t\|\nabla\mathbf{U}^{[2]}\|_{L_2(\sigma)}\|\nabla\mathbf{V}\|^2_{L_2(\sigma)}d\tau.
$$

Further, integrating by part yields

$$
\begin{aligned}
&\alpha\big|\int_0^t\int_\sigma(\mathbf{U}^{[2]}\cdot\nabla)\mathbf{V}\cdot\Delta\mathbf{V}dx'd\tau\big|\\
&=\alpha\big|\sum_{i=1}^2\big(\int_0^t\int_\sigma\big(\frac{\partial}{\partial x_i}\mathbf{U}^{[2]}\cdot\nabla\big)\mathbf{V}\cdot\frac{\partial}{\partial x_i}\mathbf{V}dx'd\tau+\int_0^t\int_\sigma\big(\mathbf{U}^{[2]}\cdot\nabla\big)\frac{\partial}{\partial x_i}\mathbf{V}\cdot\frac{\partial}{\partial x_i}\mathbf{V}dx'd\tau\big)\big|\\
&=\alpha\big|\sum_{i=1}^2\int_0^t\int_\sigma\big(\frac{\partial}{\partial x_i}\mathbf{U}^{[2]}\cdot\nabla\big)\mathbf{V}\cdot\frac{\partial}{\partial x_i}\mathbf{V}dx'd\tau\big|\le c\int_0^t\|\nabla\mathbf{U}^{[2]}\|_{L^\infty(\sigma)}\|\nabla\mathbf{V}\|^2_{L_2(\sigma)}d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\alpha\Big|\int_0^t\int_\sigma (\mathbf{V}\cdot\nabla)\mathbf{U}^{[2]}\cdot\Delta\mathbf{V}\big)dx'd\tau\Big| = \alpha\Big|\sum_{i=1}^{2}\Big(\int_0^t\int_\sigma\big(\frac{\partial}{\partial x_i}\mathbf{V}\cdot\nabla\big)\mathbf{U}^{[2]}\cdot\frac{\partial}{\partial x_i}\mathbf{V}dx'd\tau\\
&+\int_0^t\int_\sigma\big(\mathbf{V}\cdot\nabla\big)\frac{\partial}{\partial x_i}\mathbf{U}^{[2]}\cdot\frac{\partial}{\partial x_i}\mathbf{V}dx'd\tau\Big)\Big| \le c\int_0^t\|\nabla\mathbf{U}^{[2]}\|_{L_\infty(\sigma)}\|\nabla\mathbf{V}\|^2_{L_2(\sigma)}d\tau\\
&+c\int_0^t\|\partial^2_{x'}\mathbf{U}^{[2]}\|_{L_4(\sigma)}\|\mathbf{V}\|_{L_4(\sigma)}\|\nabla\mathbf{V}\|_{L_2(\sigma)}d\tau\\
&\le c\int_0^t\big(\|\partial^2_{x'}\mathbf{U}^{[2]}\|_{L_\infty(\sigma)}+\|\partial^2_{x'}\mathbf{U}^{[2]}\|_{L_4(\sigma)}\big)\|\nabla\mathbf{V}\|^2_{L_2(\sigma)}d\tau.
\end{aligned}
$$

Therefore,

$$
|J|\le c\sup_{t\in[0,T]}\big(\|\mathbf{U}^{[2]\prime}(\cdot,t)\|_{\mathring{\mathcal{Y}}(\sigma)}+\|U_3^{[2]}(\cdot,t)\|_{\mathring{\mathcal{X}}(\sigma)}\big)\int_0^t\|\nabla\mathbf{V}\|^2_{L_2(\sigma)}d\tau,
$$

and from (5.103) follows the estimate

$$
\begin{aligned}
&\|\nabla\mathbf{V}(\cdot,t)\|^2_{L_2(\sigma)}\\
&\le c\sup_{t\in[0,T]}\big(\|\mathbf{U}^{[2]\prime}(\cdot,t)\|_{\mathring{\mathcal{Y}}(\sigma)}+\|U_3^{[2]}(\cdot,t)\|_{\mathring{\mathcal{X}}(\sigma)}\big)\int_0^t\|\nabla\mathbf{V}(\cdot,\tau)\|^2_{L_2(\sigma)}d\tau.
\end{aligned}
\tag{5.104}
$$

Since (5.98)

$$
\sup_{t\in[0,T]}\big(\|\mathbf{U}^{[2]\prime}(\cdot,t)\|_{\mathring{\mathcal{Y}}(\sigma)}+\|U_3^{[2]}(\cdot,t)\|_{\mathring{\mathcal{X}}(\sigma)}\big)\le const,
$$

in virtue of the Gronwall's inequality, estimate (5.104) implies $\mathbf{V}(x',t)=0\ \forall t\in[0,T]$ and, therefore, $\mathbf{U}^{[1]}=\mathbf{U}^{[2]}$.

Taking now in (5.102) $\boldsymbol{\eta}=(0,\eta_3)$ such that $\int_\sigma\eta_3(x',t)dx'=q^{[1]}(t)-q^{[2]}(t)$, we obtain $0=\int_0^t(q^{[1]}-q^{[2]})^2d\tau$, and $q^{[1]}(t)=q^{[2]}(t)$. $\square$

Remark 5.4. Using the standard arguments (see, for example, [52]) it can be proved that there exists the unique function $\widetilde{p}=\widetilde{p}(x',t)$ such that $\widetilde{p}\in L_2(\sigma^T)$, $\int_\sigma\widetilde{p}(x',t)dx'=0$ and the following integral identity

$$
\begin{aligned}
&\int\limits_0^t \int\limits_\sigma \Big(\frac{\partial}{\partial\tau}\mathbf{U}'\cdot\boldsymbol{\eta}' + \alpha\nabla\frac{\partial}{\partial\tau}\mathbf{U}'\cdot\nabla\boldsymbol{\eta}'\Big)dx'd\tau + \nu\int\limits_0^t\int\limits_\sigma \nabla\mathbf{U}'\cdot\nabla\boldsymbol{\eta}'dx'd\tau\\
&+\int\limits_0^t\int\limits_\sigma(\mathbf{U}'\cdot\nabla)\mathbf{U}'\cdot\boldsymbol{\eta}'dx'd\tau + \alpha\int\limits_0^t\int\limits_\sigma(\mathbf{U}'\cdot\nabla)\boldsymbol{\eta}'\cdot\triangle\mathbf{U}'dx'd\tau\\
&-\alpha\int\limits_0^t\int\limits_\sigma(\boldsymbol{\eta}'\cdot\nabla)\mathbf{U}'\cdot\triangle\mathbf{U}'dx'd\tau + \int\limits_0^t\int\limits_\sigma \nabla U_3\cdot\boldsymbol{\eta}'(U_3-\alpha\triangle U_3)dx'd\tau\\
&=\int\limits_0^t\int\limits_\sigma \widetilde{p}\,\mathrm{div}\,\boldsymbol{\eta}'dx'd\tau + \int\limits_0^t\int\limits_\sigma \mathbf{f}'\cdot\boldsymbol{\eta}'dx'd\tau \quad \forall\,\boldsymbol{\eta}'\in \overset{\circ}{W}{}_2^{1,0}(\sigma^T)
\end{aligned}
$$

is valid. The pressure corresponding to the weak solution of problem (5.8) has the form $p(x,t)=\widetilde{p}(x',t)+q(t)x_3+p_0(t)$, where $p_0(t)$ is an arbitrary function.

Conclusions

The second grade fluid flow problem

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{u}-\alpha\Delta\mathbf{u})-\nu\Delta\mathbf{u}+\mathrm{curl}(\mathbf{u}-\alpha\Delta\mathbf{u})\times\mathbf{u}+\nabla\widetilde{p}=\mathbf{f},\\ \mathrm{div}\mathbf{u}=0,\\ \mathbf{u}|_{S^T}=0, \quad \mathbf{u}(x,0)=\mathbf{u}_0(x),\\ \int\limits_{\sigma} u_n dx'=F(t) \end{cases}$$

was studied in three different unbounded domains:

- the two-dimensional channel,
- the three-dimensional axially symmetric pipe,
- the three-dimensional pipe with an arbitrary cross section.

In the first two cases the existence of a unique unidirectional Poiseuille type solution is proved and the relation between the flux of the velocity field and the pressure drop (the gradient of the pressure) is found.

The analogous results were obtained for the time periodic problem

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{u}-\alpha\Delta\mathbf{u})-\nu\Delta\mathbf{u}+\mathrm{curl}(\mathbf{u}-\alpha\Delta\mathbf{u})\times\mathbf{u}+\nabla\widetilde{p}=\mathbf{f},\\ \mathrm{div}\mathbf{u}=0,\\ \mathbf{u}|_{S^T}=0, \quad \mathbf{u}(x,0)=\mathbf{u}(x,2\pi)\\ \int\limits_{\sigma} u_n dx'=F(t), \quad F(0)=F(2\pi) \end{cases}$$

in the two-dimensional channel.

It is shown that in the three-dimensional pipe with an arbitrary cross section the unidirectional solution does not exists even if data are unidirectional. However, for sufficiently small data in this case exists a unique solution having all three components $\big(u_1(x_1,x_2,t),u_2(x_1,x_2,t),u_3(x_1,x_2,t)\big)$ of the velocity field $\mathbf{u}$ and the velocity components $\big(u_1,u_2\big)$ perpendicular to the x_3-axis of the cylinder are secondary comparing with u_3.

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