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Giedrius Alkauskas

Several problems from number theory

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Scientific supervisor:
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Giedrius Alkauskas

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Prof. habil. dr. Antanas Laurinčikas (Vilniaus universitetas, fizinai mokslai, matematika - 01 P).

Contents

Introduction

0.1 Introduction

This thesis consists of three completely independent chapters. The first chapter deals with the Stieltjes transform of the Minkowski question mark function, the second one investigates functional equations associated to various forms in two or more variables, and finally the third gives an amusing proof of celebrated Fermat's little theorem. Because of independence of chapters, we indulge in being very brief in this introduction, since each chapter has its own self contained introduction.

0.1.1 Actuality

In recent decade, the interest in the Minkowski question mark function $?(x)$ grew significantly. Nevertheless, all previous results concerned $?(x)$ as a function itself. Chapter 1 establishes the result of a completely new kind, which can be thought as a first step in understanding a deep arithmetic and analytic structure of integral transforms of this function.

Further, though results of Chapter 2 are of no exceptional significance, I anticipate that the functional equations associated to certain forms encode a rich arithmetic structure of an underlying variety or field.

Chapter 3 is a mathematical joke, though it contains a rigorous and original proof.

0.1.2 Aims and problems

The aim of Chapter 1 is to find an expression of the dyadic period function in terms of objects which carry a *finite* amount of information, also allowing to use one or several limit processes. By the definition, the dyadic period function is defined via Stieltjes integral, which, in this case, is a rather complicated and ineffective expression. Further, Taylor coefficients of the dyadic period function are real numbers, which, conjecturally, are not arithmetic (by "arithmetic" we mean algebraic numbers, periods, exponential periods, etc). Thus, each of them carries infinite amount of information. Nevertheless, our main result states that there exists a nice expression for this function, which involves only one limit process (infinite sum).

The aim of Chapter 2 to is to show that functional equations related to norm forms are in fact rich mathematical objects (at this stage, only algebra of an underlying field or variety does manifest, but, possibly, arithmetic is of some importance too).

0.1.3 Methods

In Chapter 1 we use methods from complex dynamics, analytic theory of continued fractions, functions in several complex variables, analysis, integral transforms. In Chapter 2 methods from field algebra and arithmetic are being used. In Chapter 3 we use formal power series techniques.

0.1.4 Novelty

All results from Chapters 2 and 3 are new. The proof of the classical theorem in Chapter 3 is also new.

0.1.5 Statements presented for the defence

1. The generating function of moments of the Minkowski question mark function has several analytic expressions, which reveal its nature to a greater extent.
2. Some classes of forms do have non-trivial endomorphisms, and they encode certain algebraic data of the underlying field.
3. Fermat's little theorem is a consequence of one identity used in $p-$adic analysis.

0.1.6 History of the problem and main results

Here we present a short literature overview of the Minkowski question mark function $F(x)$, the main hero of Chapter 1. The history of the problem in Chapter 2 is very short, and it is given in the introductory section of that Chapter. On the other hand, the history of Fermat little theorem is long, and it can be found in any good book on elementary number theory.

The function $F(x)$ (with the awkward name "the question mark function", which is now standard), was introduced by Minkowski in 1904 as an example of continuous function $F:[0,\infty)\rightarrow[0,1)$, which maps rationals to dyadic rationals, and quadratic irrationals to non-dyadic rationals. For non-negative real x it is defined by the expression

$$F([a_0,a_1,a_2,a_3,...])=1-2^{-a_0}+2^{-(a_0+a_1)}-2^{-(a_0+a_1+a_2)}+...,$$

where $x=[a_0,a_1,a_2,a_3,...]$ stands for the representation of x by a (regular) continued fraction. The latter explicit expression for the first time was given by Denjoy in 1938. Our definition slightly differs from the customary - usually one considers a function $?(x)$, defined only for $x\in[0,1]$. Thus, we will make a convention that $?(x):=2F(x)$ for $x\in[0,1]$. For rational x the series terminates at the last nonzero element a_n of the continued fraction. Though being remembered rarely in the first half of the 20th century, this function received a substantial increase in interest in the past two decades; the number of publications is constantly growing. Next section gives a short overview of available literature. Nevertheless, the author of this thesis has a strong conviction that many hidden facts still need to be discovered, and many profound things are encoded in this simple definition. Why this object is so important in number theory, dynamic systems, complex dynamics, ergodic theory and the theory of automorphic forms? To answer, we proceed with concise literature overview.

Denjoy gave an explicit expression of $F(x)$ in terms of continued fraction expansion, given above. He also showed that $?(x)$ is purely singular: the derivative, in terms of the Lebesgue measure, vanishes almost everywhere. Salem [34] proved that $?(x)$ satisfies the Lipschitz condition of order $\frac{\log 2}{2\log\gamma}$, where $\gamma = \frac{1+\sqrt{5}}{2}$, and this is in fact the best possible exponent for the Lipschitz condition. The Fourier-Stieltjes coefficients of $?(x)$, defined as $\int_0^1 e^{2\pi inx}\,\mathrm{d}?(x)$, where also investigated in the same paper (see also [32]). The author, as an application of Wiener's theorem about Fourier series, gives average results on these coefficients without giving an answer to yet unsolved problem whether these coefficients vanish, as $n \to \infty$. Kinney [19] proved that the Hausdorff dimension of growth points of $?(x)$ (denote this set by $\mathcal{A}$) is equal to $\alpha = \frac{1}{2}\big(\int_0^1 \log_2(1+x)\ ?(x)\big)^{-1}$ (see last section of Chapter 1 for a numerical value of this constant). Also, if $x_0 \in \mathcal{A}$, $?(x)$ at a point x_0 satisfies the Lipschitz condition with an exponent α. In [23] Lagarias and Tresser introduced the so called $\mathbb{Q}-$tree: an extension of the Farey tree, which contains all (positive and negative) rationals. Tichy and Uitz [36] extended Kinney's approach (mainly, the calculation of a Hausdorff dimension) to a parametrized class of singular functions related to $?(x)$. Motivated by the investigation of Hermite problem - to represent real cubic irrationals as periodic sequences of integers - Beaver and Garrity [5] introduced a 2-dimensional analogue of $?(x)$. They showed that periodicity of Farey iterations corresponds to a class of cubic irrationals, and that $2-$dimensional analogue of $?(x)$ possesses similar singularity properties. Nevertheless, the Hermite problem remains open. Bower considers the solution of the equation $?(x) = x$, different from $x = 0, \frac{1}{2}$ or 1. There are two them (symmetric with respect to $x = \frac{1}{2}$), the first one is given by $x = 0.42037233_{+}$. Apparently, no closed form formula exists for it. In [11] Dilcher and Stolarsky introduced what they call Stern polynomials. The construction is analogous to the one given in Chapter 1. Nevertheless, in the work cited all polynomials have coefficients 0 and 1, and their structure is compatible with regular continued fraction algorithm, whereas in our case another algorithm is being introduced ($\mathsf{p}-$continued fractions). In [12] Dushistova and Moshchevitin find conditions in order $?'(x) = 0$ and $?'(x) = \infty$ to hold (for certain fixed positive real x) in terms of

$$\limsup_{t\to\infty} \frac{a_0 + a_1 + ... + a_t}{t} \text{ and } \liminf_{t\to\infty} \frac{a_0 + a_1 + ... + a_t}{t}$$

respectively, where $x = [a_0, a_1, a_2, ...]$ is represented by a continued fraction. The nature of singularity of $?(x)$ was clarified by Viader, Paradís and Bibiloni [29]. In particular, the existence of the derivative $?'(x)$ in $\mathbb{R}$ for fixed x forces it to vanish. Some other properties of $?(x)$ are demonstrated in [30]. In [16] Kesseböhmer and Stratmann study various fractal geometric aspects of the Minkowski question mark function $F(x)$. They show that the unit interval can be written as the union of three sets: $\Lambda_0 := \{x : F'(x) = 0\}$, $\Lambda_\infty := \{x : F'(x) = \infty\}$, and $\Lambda_\sim := \{x : F'(x) \text{ does not exist and } F'(x) \neq \infty\}$. Their main result is that the Hausdorff dimensions of these sets are related in the following way:

$$\dim_H(\nu_F) < \dim_H(\Lambda_\sim) = \dim_H(\Lambda_\infty) = \dim_H\big(\mathscr{L}(h_{\mathtt{top}})\big) < \dim_H(\Lambda_0) = 1.$$

Here $\mathscr{L}(h_{\mathtt{top}})$ refers to the level set of the Stern-Brocot multifractal decomposition at the topological entropy $h_{\mathtt{top}} = \log 2$ of the Farey map Q, and $\dim_H(\nu_F)$ denotes the Hausdorff dimension of the measure of maximal entropy of the dynamical system associated with Q. The notions and technique were developed earlier by authors in [17]. The paper [21] deals with the interrelations among the additive continued fraction algorithm, the Farey tree, the Farey shift and the Minkowski question mark function. The motivation for the work [28] is a fact that the function $?(x)$ can be characterized as the unique homeomorphism of the real unit interval that conjugates the Farey map with the tent map. In [28] Panti constructs an n-dimensional analogue of the Minkowski question mark function as the only homeomorphism of an n-simplex that conjugates the piecewise-fractional map associated to the Mönkemeyer continued fraction algorithm with an appropriate tent map. In [7] Bonanno and Isola introduce a class of 1-dimensional maps which can be used to generate the binary trees in different ways, and study their ergodic properties. This leads to studying some random processes (Markov chains and martingales) arising in a natural way in this context. In the course of the paper the authors also introduce a function $\rho(x) = ?\big(\frac{x}{x+1}\big)$, which is, of course, exactly $F(x)$. Okamoto and Wunsch [27] construct yet another generalization of $?(x)$, though their main concern is to introduce a new family of purely singular functions. Meanwhile, the paper by Grabner, Kirschenhofer and Tichy [15], out of all papers in the bibliography list, is the closest in spirit to the current thesis. In order to derive precise error bounds for the so called Garcia entropy of a certain measure, the authors consider the moments of the continuous and singular function

$$F_2([a_1, a_2, ...]) = \sum_{n=1}^{\infty}(-1)^{n-1}3^{-(a_1+...+a_n-1)}(q_n + q_{n-1}),$$

where $q_\star$ stand for a corresponding denominator of the convergent to $[a_1, a_2, ...]$. Lamberger [24] has shown that $F(x)$ and $F_2(x)$ are the first two members of a family (indexed by natural numbers) of mutually singular measures, derived from the subtractive Euclidean algorithm. The latter two papers are very interesting and promising, and the author of this thesis does intend to generalize the results about $F(x)$ to the whole family $F_j(x)$, $j \in \mathbb{N}$.

In Chapter 1 we add a result of completely different flavour. The main result states that in a domain $\{|z| \leq \frac{3}{4}\} \cup \{|z + \frac{9}{7}| \leq \frac{12}{7}\}$ (apparently, $\Re z \leq 1$ is a precise region of convergence) the dyadic period function $G(z)$, defined as

$$G(z) = \int\limits_0^\infty \frac{1}{x+1-z}\,\mathrm{d}F(x) = 2\int\limits_0^1 \frac{x}{1-xz}\,\mathrm{d}F(x),$$

(which is a Stieltjes transform of $F(x)$) can be expressed as a convergent sum of explicit rational function of the form $\mathbf{H}_n(z) = \frac{\mathscr{B}_n(z)}{(z-2)^{n+1}}$, where $\mathscr{B}_n(z)$ are certain polynomials of degree $n-1$ with *rational* coefficients.

0.1.7 Approbation

The results of Chapter 1 were presented at the International Šiauliai conference in honour of prof. A. Laurinčikas' 60th birthday, Šiauliai (August 2008); Max-Planck-Institut für Mathematik, Bonn, Germany (March 2009); Würzburg University number theory seminar, Germany (March 2009); Graz Technical University number theory seminar, Austria (March 2009).

0.1.8 Principal publications

- A generalization of the Rödseth-Gupta theorem on binary partitions, *Lith. Math. J.* **43** (2) (2003), 103-110.
- Dirichlet series associated with strongly $q-$multiplicative functions, *Ramanujan J.* **8** (1) (2004), 13-21.
- Prime and composite numbers as integer parts of powers (with A. Dubickas), *Acta Math. Hungar.* **105** (3) (2004), 249-256.
- Functional equation related to quadratic and norm forms, *Lith. Math. J.* **45** (2) (2005), 123-141.
- An asymptotic formula for the moments of Minkowski question mark function in the interval $[0,1]$, *Lith. Math. J.* **48** (4) (2008), 357-367.
- A curious proof of Fermat's little theorem, *Amer. Math. Monthly* **116** (4) (2009), 362-364.
- Generating and zeta functions, structure, spectral and analytic properties of the moments of the Minkowski question mark function, *Involve* **2** (2) (2009), 121-159.
- The Minkowski question mark function: explicit series for the dyadic period function and moments, *Math. Comp.* (in press).
- The moments of Minkowski question mark function: the dyadic period function, *Glasgow Math. J.* (in press).

Chapter 1

The Minkowski question mark function: explicit series for the dyadic period function and moments

1.1 Introduction and main result

The aim of this chapter to continue investigations on the moments of Minkowski $?(x)$ function, begun in [1], [2] and [3]. The function $?(x)$ ("the question mark function") was introduced by Minkowski as an example of a continuous function $F:[0,\infty)\rightarrow[0,1)$, which maps rationals to dyadic rationals, and quadratic irrationals to non-dyadic rationals. For non-negative real x it is defined by the expression

$$F([a_0,a_1,a_2,a_3,...])=1-2^{-a_0}+2^{-(a_0+a_1)}-2^{-(a_0+a_1+a_2)}+..., \tag{1.1}$$

where $x=[a_0,a_1,a_2,a_3,...]$ stands for the representation of x by a (regular) continued fraction [18]. By tradition, this function is more often investigated in the interval $[0,1]$, and in this case it is normalized in order $F(1)=1$, whereas in our case $F(1)=\frac{1}{2}$. Accordingly, we make a convention that $?(x)=2F(x)$ for $x\in[0,1]$. For rational x, the series terminates at the last nonzero partial quotient a_n of the continued fraction. This function is continuous, monotone and singular [10]. By far not complete overview of the papers written about the Minkowski question mark function or closely related topics (Farey tree, enumeration of rationals, Stern's diatomic sequence, various 1-dimensional generalizations and generalizations to higher dimensions, statistics of denominators and Farey intervals, Hausdorff dimension and analytic properties) can be found in [1]. These works include [6], [7], [9], [10], [12], [14], [15] (this is the only paper where the moments of a certain singular distribution - a close relative of $F(x)$ - were considered), [13], [16], [19], [21], [23], [27], [28] [29], [30], [31], [32], [33], [34], [36]. The internet page [39] contains up-to-date and exhaustive bibliography list of papers related to Minkowski question mark function.

Recently, in Calkin and Wilf [9] defined a binary tree which is generated by the iteration

$$\frac{a}{b} \quad \mapsto \quad \frac{a}{a+b}\ , \quad \frac{a+b}{b},$$

starting from the root $\frac{1}{1}$. The last two authors have greatly publicized this tree, but it was known long ago to physicists and mathematicians (alias, Stern-Brocot or Farey tree). Elementary considerations show that this tree contains every positive rational number once and only once, each being represented in lowest terms. The first four iterations lead to

$$\begin{array}{cccccccc}
\multicolumn{8}{c}{\frac{1}{1}} \\
\multicolumn{4}{c}{\frac{1}{2}} & \multicolumn{4}{c}{\frac{2}{1}} \\
\multicolumn{2}{c}{\frac{1}{3}} & \multicolumn{2}{c}{\frac{3}{2}} & \multicolumn{2}{c}{\frac{2}{3}} & \multicolumn{2}{c}{\frac{3}{1}} \\
\frac{1}{4} & \frac{4}{3} & \frac{3}{5} & \frac{5}{2} & \frac{2}{5} & \frac{5}{3} & \frac{3}{4} & \frac{4}{1}
\end{array} \tag{1.2}$$

It is of utmost importance to note that the nth generation consists of exactly those 2^{n-1} positive rational numbers, whose elements of the continued fraction sum up to n. This fact can be easily inherited directly from the definition. First, if rational number $\frac{a}{b}$ is represented as a continued fraction $[a_0, a_1, ..., a_r]$, then the map $\frac{a}{b} \to \frac{a+b}{b}$ maps $\frac{a}{b}$ to $[a_0+1, a_1..., a_r]$. Second, the map $\frac{a}{b} \to \frac{a}{a+b}$ maps $\frac{a}{b}$ to $[0, a_1+1, ..., a_r]$ in case $\frac{a}{b} < 1$, and to $[0, 1, a_0, a_1, ..., a_r]$ in case $\frac{a}{b} > 1$. This is an important fact which makes the investigations of rational numbers according to their position in the Calkin-Wilf tree highly motivated from the perspective of metric number theory and dynamics of continued fractions.

It is well known that each generation of (1.2) possesses a distribution function $F_n(x)$, and $F_n(x)$ converges uniformly to $F(x)$. The function $F(x)$ as a distribution function (in the sense of probability theory, which imposes the condition of monotonicity) is uniquely determined by the functional equation [1]

$$2F(x) = \begin{cases} F(x-1)+1 & \text{if} \quad x \geq 1, \\ F(\frac{x}{1-x}) & \text{if} \quad 0 \leq x < 1. \end{cases} \tag{1.3}$$

This implies $F(x) + F(1/x) = 1$. The mean value of $F(x)$ has been investigated by several authors, and was proved to be $3/2$.

Lastly, and most importantly, let us point out that, surprisingly, there are striking similarities and parallels between the results proved in [1] and [2] with Lewis'-Zagier's ([25], [26]) results on period functions for Maass wave forms. (see [2] for the explanation of this phenomena).

Just before formulating the main Theorem of this chapter, we provide a short summary of previous results proved by the author about certain natural integral transforms of $F(x)$. Let

$$M_L = \int\limits_0^\infty x^L \, \mathrm{d}F(x), \quad m_L = \int\limits_0^\infty \Big(\frac{x}{x+1}\Big)^L \, \mathrm{d}F(x) = 2\int\limits_0^1 x^L \, \mathrm{d}F(x).$$

Both sequences are of definite number-theoretical significance because

$$M_L=\lim_{n\to\infty}2^{1-n}\sum_{a_0+a_1+\ldots+a_s=n}[a_0,a_1,..,a_s]^L,\quad m_L=\lim_{n\to\infty}2^{2-n}\sum_{a_1+\ldots+a_s=n}[0,a_1,..,a_s]^L,$$

(the summation takes place over rational numbers represented as continued fractions; thus, $a_i\geq 1$ and $a_s\geq 2$). We define the exponential generating functions

$$\begin{aligned}M(t)&=\sum_{L=0}^{\infty}\frac{M_L}{L!}t^L=\int_0^{\infty}e^{xt}\,\mathrm{d}F(x),\\ \mathfrak{m}(t)&=\sum_{L=0}^{\infty}\frac{m_L}{L!}t^L=\int_0^{\infty}\exp\Big(\frac{xt}{x+1}\Big)\,\mathrm{d}F(x)=2\int_0^{1}e^{xt}\,\mathrm{d}F(x).\end{aligned}$$

One directly verifies that $\mathfrak{m}(t)$ is an entire function, and that $M(t)$ is meromorphic function with simple poles at $z=\log 2+2\pi in$, $n\in\mathbb{Z}$. Further, we have

$$M(t)=\frac{\mathfrak{m}(t)}{2-e^t},\quad \mathfrak{m}(t)=e^t\mathfrak{m}(-t).$$

The second identity represents only the symmetry property, given by $F(x)+F(1/x)=1$. The main result about $\mathfrak{m}(t)$ is that it is uniquely determined by the regularity condition $\mathfrak{m}(-t)\ll e^{-\sqrt{\log 2}\sqrt{t}}$, as $t\to\infty$, the boundary condition $\mathfrak{m}(0)=1$, and the integral equation

$$\mathfrak{m}(-s)=(2e^s-1)\int_0^{\infty}\mathfrak{m}'(-t)J_0(2\sqrt{st})\,\mathrm{d}t,\quad s\in\mathbb{R}_+. \tag{1.4}$$

(Here $J_0(\star)$ stands for the Bessel function $J_0(z)=\frac{1}{\pi}\int_0^{\pi}\cos(z\sin x)\,\mathrm{d}x$).

Our primary object of investigations is the generating function of moments. Let $G(z)=\sum_{L=1}^{\infty}m_Lz^{L-1}$. This series converges for $|z|\leq 1$, and the functional equation for $G(z)$ (see below) implies that there exist all derivatives of $G(z)$ at $z=1$, if we approach this point while remaining in the domain $\Re z\leq 1$. Then the integral

$$G(z)=\int_0^{\infty}\frac{1}{x+1-z}\,\mathrm{d}F(x)=2\int_0^{1}\frac{x}{1-xz}\,\mathrm{d}F(x). \tag{1.5}$$

(which is Stieltjes transform of $F(x)$) extends $G(z)$ to the cut plane $\mathbb{C}\setminus(1,\infty)$. The generating function of moments M_L does not exist due to the factorial growth of M_L, but this generating function can still be defined in the cut plane $\mathbb{C}'=\mathbb{C}\setminus(0,\infty)$ by $\int_0^{\infty}\frac{x}{1-xz}\,\mathrm{d}F(x)$. In fact, this integral just equals to $G(z+1)$. Thus, there exist all higher derivatives of $G(z)$ at $z=1$, and $\frac{1}{(L-1)!}\frac{\mathrm{d}^{L-1}}{\mathrm{d}z^{L-1}}G(z)\big|_{z=1}=M_L$, $L\geq 1$. The following result was proved in [1].

Theorem 1.1. *The function $G(z)$, defined initially as a power series, has an analytic continuation to the cut plane $\mathbb{C}\setminus(1,\infty)$ via (1.5). It satisfies the functional equation*

$$\frac{1}{z}+\frac{1}{z^2}G\Big(\frac{1}{z}\Big)+2G(z+1)=G(z), \tag{1.6}$$

and also the symmetry property

$$G(z+1) = -\frac{1}{z^2}G\Big(\frac{1}{z}+1\Big) - \frac{1}{z},$$

(which is a consequence of the main functional equation). Moreover, $G(z)\to 0$, if $z\to\infty$ and the distance from z to a half line $[0,\infty)$ tends to infinity. Conversely, the function having these properties is unique.

Accordingly, this result and the specific appearance of the three term functional equation justifies the name for $G(z)$ as *the dyadic period function*.

We wish to emphasize that the main motivation for previous research was clarification of the nature and structure of the moments m_L. It was greatly desirable to give these constants (emerging as if from geometric chaos) some other expression than the one obtained directly from the Farey (or Calkin-Wilf) tree, which could reveal their structure to greater extent. This is accomplished in the current work. Thus, the main result can be formulated as follows.

Theorem 1.2. *There exist canonical and explicit sequence of rational functions $\mathbf{H}_n(z)$, such that for $\{|z|\leq\frac{3}{4}\}\cup\{|z+\frac{9}{7}|\leq\frac{12}{7}\}$ one has an absolutely convergent series*

$$G(z) = \int\limits_0^\infty \frac{1}{x+1-z}\,\mathrm{d}F(x) = \sum_{n=0}^{\infty}(-1)^n\mathbf{H}_n(z), \quad \mathbf{H}_n(z) = \frac{\mathscr{B}_n(z)}{(z-2)^{n+1}},$$

where $\mathscr{B}_n(z)$ is polynomial with rational coefficients of degree $n-1$. For $n\geq 1$ it has the following reciprocity property:

$$\mathscr{B}_n(z+1) = (-1)^n z^{n-1}\mathscr{B}_n\Big(\frac{1}{z}+1\Big), \quad \mathscr{B}_n(0)=0.$$

The rational function $\mathbf{H}_n(z)$ are defined via implicit and rather complicated recurrence (1.27) (see Section 6). The following table gives initial polynomials $\mathscr{B}_n(z)$.

n	$\mathscr{B}_n(z)$	n	$\mathscr{B}_n(z)$
0	-1	4	$-\frac{2}{27}z^3+\frac{53}{270}z^2-\frac{53}{270}z$
1	0	5	$\frac{4}{81}z^4-\frac{104}{675}z^3+\frac{112}{675}z^2-\frac{224}{2025}z$
2	$-\frac{1}{6}z$	6	$-\frac{8}{243}z^5+\frac{47029}{425250}z^4-\frac{1384}{14175}z^3-\frac{787}{30375}z^2+\frac{787}{60750}z$
3	$\frac{1}{9}z^2-\frac{2}{9}z$	7	$\frac{16}{729}z^6-\frac{1628392}{22325625}z^5+\frac{272869}{22325625}z^4+\frac{5392444}{22325625}z^3-\frac{238901}{637875}z^2+\frac{477802}{3189375}z$

Remark. The constant $\frac{3}{4}$ can be replaced by any constant less than 1.29^{-1} (the latter comes exactly from Lemma 1.27). Unfortunately, our method does not allow to prove an absolute convergence in the disk $|z| \le 1$. In fact, apparently the true region of convergence of the series in question is the half plane $\Re z \le 1$. Take, for example, $z_0 = \frac{2}{3} + 4i$. Then by (2.6) and symmetry property one has

$$G(z_0) = \frac{1}{2}G(z_0 - 1) - \frac{1}{2(z_0-1)^2}G\Big(\frac{1}{z_0-1}\Big) - \frac{1}{2(z_0-1)} =$$
$$-\frac{1}{2(z_0-2)^2}G\Big(\frac{z_0-1}{z_0-2}\Big) - \frac{1}{2(z_0-1)^2}G\Big(\frac{1}{z_0-1}\Big) - \frac{1}{2(z_0-2)} - \frac{1}{2(z_0-1)}.$$

Both arguments under G on the right belong to the unit circle, and thus we can use Taylor series for $G(z)$. Using numerical values of m_L, obtained via the method described in Appendix A.2., we obtain: $G(z_0) = 0.078083_{+} + 0.205424_{+}i$, with all digits exact. On the other hand, the series in Theorem 1.2 for $n = 60$ gives

$$\sum_{n=0}^{60}(-1)^n \mathbf{H}_n(z_0) = 0.078090_{+} + 0.205427_{+}i.$$

Finally, based on the last integral in (1.5), we can calculate $G(z)$ as a Stieltjes integral. If we divide the unit interval into $N = 3560$ equal subintervals, and use Riemann-Stieltjes sum, we get an approximate value $G(z_0) \approx 0.078082_{+} + 0.205424_{+}i$. All evaluations match very well.

Experimental observation 1.3. *We conjecture that the series in Theorem 1.2 converges absolutely for* $\Re z \le 1$.

With a slight abuse of notation, we will henceforth write $f^{(L-1)}(z_0)$ instead of $\frac{\mathrm{d}^{L-1}}{\mathrm{d}z^{L-1}} f(z)\big|_{z=z_0}$.

Corollary 1.4. *The moments m_L can be expressed by the convergent series of rational numbers:*

$$m_L = \lim_{n\to\infty} 2^{2-n} \sum_{a_1+a_2+\ldots+a_s=n} [0, a_1, a_2, ..., a_s]^L = \frac{1}{(L-1)!}\sum_{n=0}^{\infty}(-1)^n \mathbf{H}_n^{(L-1)}(0).$$

The speed of convergence is given by the following estimate: $\Big|\mathbf{H}_n^{(L-1)}(0)\Big| \ll \frac{1}{n^M}$, *for every* $M \in \mathbb{N}$. *The implied constant depends only on L and M.*

Thus, $m_2 = \sum_{n=0}^{\infty}(-1)^n \mathbf{H}'_n(0) = 0.2909264764_{+}$. Regarding the speed, numerical calculations show that in fact the convergence is geometric. Theorem 1.2 in case $z = 1$ gives

$$M_1 = G(1) = 1 + 0 + \sum_{n=0}^{\infty} \frac{1}{6}\Big(\frac{2}{3}\Big)^n = \frac{3}{2},$$

which we already know (see Corollary 1.18; the above is a Taylor series for $M_1(\mathsf{p})$ in powers of $\mathsf{p} - 2$, specialized at $\mathsf{p}_0 = 1$). Geometric convergence would be the consequence of the fact that analytic functions $m_L(\mathsf{p})$ extend beyond $\mathsf{p} = 1$ (see below). This is supported by the phenomena represented as Experimental observation 1.5. Meanwhile, we are able to prove only the given rate. If we were allowed to use the point $z = 1$, Theorem 1.2 would give a convergent series for the moments M_L as well. This is exactly the same as the series in the Corollary 1.4, only one needs to use a point $z = 1$ instead of $z = 0$.

Experimental observation 1.5. *For $L \geq 1$, the series*

$$M_L(\mathsf{p}) = \frac{1}{(L-1)!} \cdot \sum_{n=0}^{\infty} (\mathsf{p}-2)^n \mathbf{H}_n^{(L-1)}(1), \quad M_L(1) = M_L,$$

has exactly $2 - \frac{1}{\sqrt[L]{2}}$ as a radius of convergence.

To this account, Proposition 1.16 endorse this phenomena, which is highly supported by numerical calculations, and which does hold for $L = 1$.

The following two tables give starting values for the sequence $\mathbf{H}'_n(0)$.

n	$\mathbf{H}'_n(0)$	n	$\mathbf{H}'_n(0)$	n	$\mathbf{H}'_n(0)$
0	$\frac{1}{4}$	5	$-\frac{7}{2 \cdot 3^4 \cdot 5^2}$	10	$-\frac{8026531718888633}{2^{12} \cdot 3^9 \cdot 5^7 \cdot 7^4 \cdot 11 \cdot 17^2}$
1	0	6	$-\frac{787}{2^8 \cdot 3^5 \cdot 5^3}$	11	$\frac{797209536976557079423}{2^{11} \cdot 3^{10} \cdot 5^8 \cdot 7^5 \cdot 11^2 \cdot 17^3 \cdot 31}$
2	$\frac{1}{48}$	7	$\frac{238901}{2^7 \cdot 3^6 \cdot 5^4 \cdot 7}$	12	$\frac{4198988799919158293319845971}{2^{14} \cdot 3^{11} \cdot 5^9 \cdot 7^6 \cdot 11^3 \cdot 13 \cdot 17^4 \cdot 31^2}$
3	$-\frac{1}{72}$	8	$-\frac{181993843}{2^{10} \cdot 3^7 \cdot 5^5 \cdot 7^2}$	13	$-\frac{12702956822417247965298252330349561}{2^{10} \cdot 3^{12} \cdot 5^{10} \cdot 7^7 \cdot 11^4 \cdot 13^2 \cdot 17^5 \cdot 31^3}$
4	$\frac{53}{8640}$	9	$\frac{12965510861}{2^6 \cdot 3^8 \cdot 5^6 \cdot 7^3 \cdot 17}$	14	$\frac{72261916360136752928335145486035163954998 99}{2^{16} \cdot 3^{13} \cdot 5^{11} \cdot 7^8 \cdot 11^5 \cdot 13^3 \cdot 17^6 \cdot 31^4}$

n	$\mathbf{H}'_n(0)$
15	$-\frac{129337183009042141853748450730581369733226857443915617}{2^{15} \cdot 3^{14} \cdot 5^{12} \cdot 7^9 \cdot 11^6 \cdot 13^4 \cdot 17^7 \cdot 31^5 \cdot 43 \cdot 127}$
16	$\frac{3125818627577719704107324375271510984275378559830681202898421325 1}{2^{18} \cdot 3^{15} \cdot 5^{13} \cdot 7^{10} \cdot 11^7 \cdot 13^5 \cdot 17^8 \cdot 31^6 \cdot 43^2 \cdot 127^2}$
17	$-\frac{3282520501229639755997762022707321704397776888948469860959830459774414444483}{2^{12} \cdot 3^{16} \cdot 5^{14} \cdot 7^{11} \cdot 11^8 \cdot 13^6 \cdot 17^9 \cdot 31^7 \cdot 43^3 \cdot 127^3 \cdot 257}$

The float values of the last three rational numbers are -0.000025804822076, 0.000018040274062 and -0.000010917558446 respectively. The alternating sum of the elements in the table is $\sum_{n=0}^{N}(-1)^n \mathbf{H}'_n(0) = 0.2909255862_+$ (where $N = 17$), whereas $N = 40$ gives 0.2909264880_+, and $N = 50$ gives 0.2909264784_+. Note that the manifestation of Fermat and Mersenne primes in the denominators at an early stage is not accidental, minding the exact value of the determinant in Lemma 1.22, Chapter 6 (see below). Moreover, the prime powers of every odd prime, which divides the denominator, increase each time by 1 while passing from $\mathbf{H}'_n(0)$ to $\mathbf{H}'_{n+1}(0)$. The pattern for the powers of 2 is more complicated. More thorough research of the linear map in Lemma 1.22 can thus clarify prime decomposition of denominators; numerators remains much more complicated.

As will be apparent later, the result in Theorem 1.2 is derived from the knowledge of $\mathsf{p}-$derivatives of $G(\mathsf{p},z)$ at $\mathsf{p}=2$ (see below). On the other hand, since there are two points ($\mathsf{p}=2$ and $\mathsf{p}=0$) such that all higher $\mathsf{p}-$derivatives of $G(\mathsf{p},z)$ are rational functions in z, it is not completely surprising that the approach through $\mathsf{p}=0$ also gives convergent series for the moments, though in this case we are forced to use Borel summation. At this point, the author does not have a strict mathematical proof of this result (since the function $G(\mathsf{p},z)$ is meanwhile defined only for $\Re\mathsf{p}\geq 1$), though numerical calculations provide overwhelming evidence for its validity.

Experimental observation 1.6. *Define the rational functions (with rational coefficients) $\mathbf{Q}_n(z)$, $n\geq 0$, by*

$$\mathbf{Q}_0(z)=-\frac{1}{2z},\ \textit{and recurrently by}\ \mathbf{Q}_n(z)=\frac{1}{2}\sum_{j=0}^{n-1}\frac{1}{j!}\cdot\frac{\partial^j}{\partial z^j}\mathbf{Q}_{n-j-1}(-1)\cdot\Big(z^j-\frac{1}{z^{j+2}}\Big).$$

Then

$$m_L=\lim_{n\to\infty}2^{2-n}\sum_{a_1+a_2+\ldots+a_s=n}[0,a_1,a_2,...,a_s]^L=$$
$$\frac{1}{(L-1)!}\sum_{r=0}^{\infty}\Big(\sum_{n=0}^{\infty}\frac{\mathbf{Q}_n^{(L-1)}(-1)}{n!}\cdot\int\limits_r^{r+1}t^ne^{-t}\,\mathrm{d}t\Big). \tag{1.7}$$

Moreover,

$$\mathbf{Q}_n(z)=\frac{(z+1)(z-1)\mathscr{D}_n(z)}{z^{n+1}},\quad n\geq 1,$$

where $\mathscr{D}_n(z)$ are polynomials with rational coefficients ($\mathbb{Q}_p$ integers for $p\neq 2$) of degree $2n-2$ with the reciprocity property

$$\mathscr{D}_n(z)=z^{2n-2}\mathscr{D}_n\Big(\frac{1}{z}\Big).$$

Note the order of summation in the series for m_L, since the reason for introducing exponential function is because we use Borel summation. For example,

$$\text{“}1-2+4-8+16-32+\ldots\text{”}\overset{\texttt{Borel}}{=}\sum_{r=0}^{\infty}\Big(\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}\cdot\int\limits_r^{r+1}t^ne^{-t}\,\mathrm{d}t\Big)=\frac{1}{3}.$$

The following table gives initial results.

n	$\mathscr{D}_n(z)$	n	$\mathscr{D}_n(z)$
1	$\frac{1}{4}$	4	$\frac{1}{8}(2z^6 - 3z^5 + 6z^4 - 3z^3 + 6z^2 - 3z + 2)$
2	$\frac{1}{4}(z^2 + 1)$	5	$\frac{1}{4}(z^8 - 2z^7 + 4z^6 - 7z^5 + 4z^4 - 7z^3 + 4z^2 - 2z + 1)$
3	$\frac{1}{4}(z^4 - z^3 + z^2 - z + 1)$	6	$\frac{1}{8}(2z^{10} - 5z^9 + 12z^8 - 20z^7 + 37z^6 -$ $- 20z^5 + 37z^4 - 20z^3 + 12z^2 - 5z + 2)$

The next table gives $\mathbf{Q}'_n(-1) = 2(-1)^n \mathscr{D}_n(-1)$ explicitly: these constants appear in the series defining the first non-trivial moment m_2. Also, since these numbers are $p-$adic integers for $p \neq 2$, there is a hope for the successful implementation of the idea from the last chapter in [2]; that is, possibly one can define moments m_L as $p-$adic rationals as well.

n	$\mathbf{Q}'_n(-1)$	n	$\mathbf{Q}'_n(-1)$	n	$\mathbf{Q}'_n(-1)$	n	$\mathbf{Q}'_n(-1)$
0	$\frac{1}{2}$	8	$\frac{1417}{4}$	16	$\frac{206836175}{64}$	24	$\frac{1685121707817}{32}$
1	$-\frac{1}{2}$	9	$-\frac{8431}{8}$	17	$-\frac{339942899}{32}$	25	$-\frac{92779913448103}{512}$
2	1	10	$\frac{50899}{16}$	18	$\frac{1125752909}{32}$	26	$\frac{80142274019997}{128}$
3	$-\frac{5}{2}$	11	-9751	19	$-\frac{15014220659}{128}$	27	$-\frac{1111839248032133}{512}$
4	$\frac{25}{4}$	12	30365	20	$\frac{25188552721}{64}$	28	$\frac{7740056893342455}{1024}$
5	-16	13	$-\frac{3069719}{32}$	21	$-\frac{170016460947}{128}$	29	$-\frac{13515970598654393}{512}$
6	43	14	$\frac{1227099}{4}$	22	$\frac{1153784184807}{256}$	30	$\frac{47354245650630005}{512}$
7	$-\frac{971}{8}$	15	$-\frac{31719165}{32}$	23	$-\frac{983668214037}{64}$	31	$-\frac{665632101181145115}{2048}$

The final table in this section lists float values of the constants

$$\vartheta_r = \sum_{n=0}^{\infty} \frac{\mathbf{Q}'_n(-1)}{n!} \cdot \int\limits_{r}^{r+1} t^n e^{-t} \,\mathrm{d}t, \quad r \in \mathbb{N}_0, \quad \sum_{r=0}^{\infty} \vartheta_r = m_2,$$

appearing in Borel summation.

r	ϑ_r	r	ϑ_r
0	0.2327797875	6	0.0004701146
1	0.0471561089	7	0.0004980015
2	0.0085133626	8	0.0004005270
3	0.0005892453	9	0.0002722002
4	−0.0001872357	10	0.0001607897
5	0.0002058729	11	0.0000812407

Thus, $\sum_{r=0}^{11} \vartheta_r = 0.2909400155_{+} = m_2 + 0.000013539_{+}$.

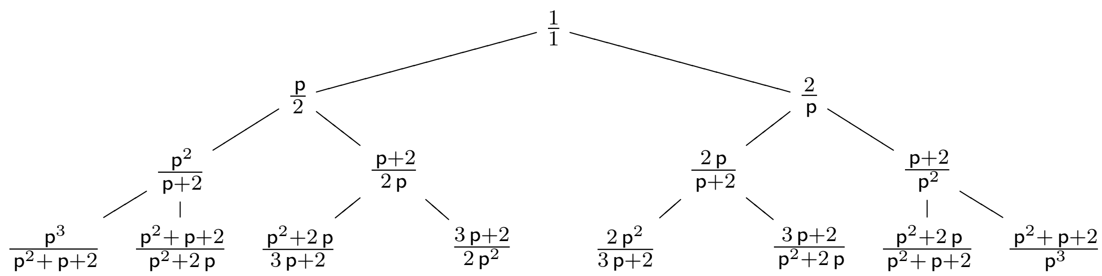
This chapter is organized as follows. In Section 2, for each p, $1 \leq \mathsf{p} < \infty$, we introduce a generalization of the Farey (Calkin-Wilf) tree, denoted by $\mathcal{Q}_{\mathsf{p}}$. This leads to the notion of $\mathsf{p}-$continued fractions and $\mathsf{p}-$Minkowski question mark functions $F_{\mathsf{p}}(x)$. Though p−continued fractions are of independent interest (one could define a transfer operator, to prove an analogue of Gauss-Kuzmin-Lévy theorem, various metric results and introduce structural constants), we confine to the facts which are necessary for our purposes and leave the deeper research for the future. In Section 3 we extend these results to the case of complex p, $|\mathsf{p}-2|\leq 1$. The crucial consequence of these results is the fact that a function $\mathfrak{X}(\mathsf{p},x)$ (which gives a bijection between trees $\mathcal{Q}_1$ and $\mathcal{Q}_{\mathsf{p}}$) is a continuous function in x and an analytic function in p for $|\mathsf{p}-2|\leq 1$. In Section 4 we introduce exactly the same integral transforms of $F_{\mathsf{p}}(x)$ as was done in a special (though most important) case of $F(x)=F_1(x)$. Also, in this section we prove certain relations among the moments. In Section 5 we give the proof of the three term functional equation for $G_{\mathsf{p}}(z)$ and the integral equation for $\mathfrak{m}_{\mathsf{p}}(t)$. Finally, Theorem 1.2 is proved in Section 6. The hierarchy of sections is linear, and all results from previous ones is used in Section 6. Appendix A. contains: derivation for the series (1.7); MAPLE codes to compute rational functions $\mathbf{H}_n(z)$ and $\mathbf{Q}_n(z)$; description of high-precision method to calculate numerical values for the constants m_L; auxiliary lemmas for the Section 3. The chapter also contains graphs of some p−Minkowski question mark functions $F_{\mathsf{p}}(x)$ for real p, and also pictures of locus points of elements of trees $\mathcal{Q}_{\mathsf{p}}$ for certain characteristic values of p.

1.2 p−question mark functions and p−continued fractions

In this section we introduce a family of natural generalizations of the Minkowski question mark function $F(x)$. Let $1\leq\mathsf{p}<2$. Consider the following binary tree, which we denote by $\mathcal{Q}_{\mathsf{p}}$. We start from the root $x=1$. Further, each element ("root") x of this tree generates two "offsprings" by the following rule:

$$x\mapsto\frac{\mathsf{p}x}{x+1},\quad\frac{x+1}{\mathsf{p}}.$$

We will use the notation $\mathcal{T}_{\mathsf{p}}(x)=\frac{x+1}{\mathsf{p}}$, $\mathcal{U}_{\mathsf{p}}(x)=\frac{\mathsf{p}x}{x+1}$. When p is fixed, we will sometimes discard the subscript. Thus, the first four generations lead to



(1.8)

We refer the reader to the paper [13], where authors consider a rather similar construction, though having a different purpose in mind (see also [7]). Denote by $T_n(\mathsf{p})$ the sequence of polynomials, appearing as numerators of fractions of this tree. Thus, $T_1(\mathsf{p})=1$, $T_2(\mathsf{p})=\mathsf{p}$, $T_3(\mathsf{p})=2$. Directly from the definition of this tree we inherit that

$$\begin{aligned}T_{2n}(\mathsf{p}) &= \mathsf{p}T_n(\mathsf{p}) \text{ for } n\geq 1,\\ T_{2n-1}(\mathsf{p}) &= T_{n-1}(\mathsf{p})+\mathsf{p}^{-\epsilon}T_n(\mathsf{p}) \text{ for } n\geq 2,\end{aligned}$$

where $\epsilon = \epsilon(n) = 1$ if n is a power of two, and $\epsilon = 0$ otherwise. Thus, the definition of these polynomials is almost the same as it appeared in [20] (these polynomials were named Stern polynomials by the authors), with the distinction that in [20] everywhere one has $\epsilon = 0$. Naturally, this difference produces different sequence of polynomials.

There are 2^{n-1} positive real numbers in each generation of the tree $\mathcal{Q}_{\mathsf{p}}$, say $a_k^{(n)}$, $1 \leq k \leq 2^{n-1}$. Moreover, they are all contained in the interval $[\mathsf{p}-1, \frac{1}{\mathsf{p}-1}]$. Indeed, this holds for the initial root $x = 1$, and

$$\mathsf{p}-1 \leq x \leq \frac{1}{\mathsf{p}-1} \Leftrightarrow \mathsf{p}-1 \leq \frac{\mathsf{p}x}{x+1} \leq 1,$$
$$\mathsf{p}-1 \leq x \leq \frac{1}{\mathsf{p}-1} \Leftrightarrow 1 \leq \frac{x+1}{\mathsf{p}} \leq \frac{1}{\mathsf{p}-1}.$$

This also shows that the left offspring is contained in the interval $[\mathsf{p}-1, 1]$, while the right one - in the interval $[1, \frac{1}{\mathsf{p}-1}]$. The real numbers appearing in this tree have intrinsic relation with $\mathsf{p}-$*continued fractions algorithm.* The definition of the latter is as follows. Let $x \in (\mathsf{p}-1, \frac{1}{\mathsf{p}-1})$. Consider the following procedure:

$$R_{\mathsf{p}}(x) = \begin{cases} \mathcal{T}^{-1}(x) = \mathsf{p}x - 1, & \text{if} \quad 1 \leq x < \frac{1}{\mathsf{p}-1}, \\ \mathcal{I}(x) = \frac{1}{x}, & \text{if} \quad \mathsf{p}-1 < x < 1, \\ \text{STOP}, & \text{if} \quad x = \mathsf{p}-1. \end{cases}$$

Then each such x can be uniquely represented as $\mathsf{p}-$continued fraction

$$x = [a_0, a_1, a_2, a_3, ...]_{\mathsf{p}},$$

where $a_i \in \mathbb{N}$ for $i \geq 1$, and $a_0 \in \mathbb{N} \cup \{0\}$. This notation means that in the course of iterations $R^{\infty}_{\mathsf{p}}(x)$ we apply $\mathcal{T}^{-1}(x)$ exactly a_0 times, then once $\mathcal{I}$, then we apply $\mathcal{T}^{-1}$ exactly a_1 times, then $\mathcal{I}$, and so on. The procedure terminates exactly for those $x \in (\mathsf{p}-1, \frac{1}{\mathsf{p}-1})$, which are the members of the tree $\mathcal{Q}_{\mathsf{p}}$ ("p-rationals"). Also, direct inspection shows that if procedure does terminate, the last entry $a_s \geq 2$. Thus, we have the same ambiguity for the last entry exactly as is the case with ordinary continued fractions. At this point it is straightforward to show that the nth generation of $\mathcal{Q}_{\mathsf{p}}$ consists of $x = [a_0, a_1, ..., a_s]_{\mathsf{p}}$ such that $\sum_{j=0}^{s} a_j = n$, exactly as in the case $\mathsf{p} = 1$ and tree (1.2).

Now, consider a function $\mathfrak{X}_{\mathsf{p}}(x)$ with the following property: $\mathfrak{X}_{\mathsf{p}}(x) = \overline{x}$, where x is a rational number in the Calkin-Wilf tree (1.2), and $\overline{x}$ is a corresponding number in the tree (1.8). In other words, $\mathfrak{X}_{\mathsf{p}}(x)$ is simply a bijection between these two trees. First, if $x < y$, then $\overline{x} < \overline{y}$. Also, all positive rationals appear in the tree (1.2) and they are everywhere dense in $\mathbb{R}_{+}$. Moreover, $\mathcal{T}$ and $\mathcal{U}$ both preserve order, and $[\mathsf{p}-1, \frac{1}{\mathsf{p}-1})$ is a disjoint union of $\mathcal{T}[\mathsf{p}-1, \frac{1}{\mathsf{p}-1})$ and $\mathcal{U}[\mathsf{p}-1, \frac{1}{\mathsf{p}-1})$. Now it is obvious that the function $\mathfrak{X}_{\mathsf{p}}(x)$ can be extended to a continuous monotone increasing function

$$\mathfrak{X}_{\mathsf{p}}(\star) : \mathbb{R}_{+} \to [\mathsf{p}-1, \frac{1}{\mathsf{p}-1}\Big), \quad \mathfrak{X}_{\mathsf{p}}(\infty) = \frac{1}{\mathsf{p}-1}.$$

Thus,

$$\mathfrak{X}_{\mathsf{p}}\big([a_0, a_1, a_2, a_3...]\big) = [a_0, a_1, a_2, a_3...]_{\mathsf{p}}.$$

As can be seen from the definitions of both trees (1.2) and (1.8), this function satisfies functional equations

$$\begin{aligned}\mathfrak{X}_{\mathsf{p}}(x+1) &= \frac{\mathfrak{X}_{\mathsf{p}}(x)+1}{\mathsf{p}},\\ \mathfrak{X}_{\mathsf{p}}\Big(\frac{x}{x+1}\Big) &= \frac{\mathsf{p}\mathfrak{X}_{\mathsf{p}}(x)}{\mathfrak{X}_{\mathsf{p}}(x)+1},\\ \mathfrak{X}_{\mathsf{p}}\Big(\frac{1}{x}\Big) &= \frac{1}{\mathfrak{X}_{\mathsf{p}}(x)}.\end{aligned} \tag{1.9}$$

The last one (symmetry property) is a consequence of the first two. We are not aware whether this notion of $\mathsf{p}-$continued fractions is new or not. For example,

$$\begin{aligned}\frac{1+\sqrt{1+4\,\mathsf{p}}}{2\,\mathsf{p}} &= [1,1,1,1,1,1,...]_{\mathsf{p}} = \mathfrak{X}_{\mathsf{p}}\Big(\frac{1+\sqrt{5}}{2}\Big),\\ \sqrt{3} &= [4,2,1,10,1,1,2,1,5,1,1,2,1,2,1,1,2,1,3,7,4,...]_{\frac{3}{2}},\\ 2 &= [4,1,1,\overline{2,1,1}]_{\sqrt{2}}.\end{aligned}$$

Now fix p, $1 \leq \mathsf{p} < 2$. The following proposition follows immediately from the properties of $F(x)$.

Proposition 1.7. *There exists a limit distribution of the nth generation of the tree $\mathcal{Q}_{\mathsf{p}}$ as $n \to \infty$, defined as*

$$F_{\mathsf{p}}(x) = \lim_{n\to\infty} 2^{-n+1}\#\{k : a_k^{(n)} < x\}.$$

This function is continuous, $F_{\mathsf{p}}(x) = 0$ for $x \leq \mathsf{p}-1$, $F_{\mathsf{p}}(x) = 1$ for $x \geq \frac{1}{\mathsf{p}-1}$, and it satisfies two functional equations:

$$2\,F_{\mathsf{p}}(x) = \begin{cases} F_{\mathsf{p}}(\,\mathsf{p}x-1)+1, & \text{if } \ 1 \leq x \leq \frac{1}{\mathsf{p}-1},\\ F_{\mathsf{p}}(\frac{x}{\mathsf{p}-x}), & \text{if } \ \mathsf{p}-1 \leq x \leq 1.\end{cases} \tag{1.10}$$

Additionally,

$$F_{\mathsf{p}}(x) + F_{\mathsf{p}}\Big(\frac{1}{x}\Big) = 1 \text{ for } x > 0.$$

The explicit expression for $F_{\mathsf{p}}(x)$ is given by

$$F_{\mathsf{p}}([a_0,a_1,a_2,a_3,...]_{\mathsf{p}}) = 1 - 2^{-a_0} + 2^{-(a_0+a_1)} - 2^{-(a_0+a_1+a_2)} + ...$$

We will refer to the last functional equation as *the symmetry property.* As was said, it is a consequence of the other two, though it is convenient to separate it.

Proof. Indeed, as it is obvious from the observations above, we simply have

$$F_{\mathsf{p}}\big(\mathfrak{X}_{\mathsf{p}}(x)\big) = F(x), \quad x \in [0,\infty).$$

Therefore, two functional equations follow from (1.3) and (1.9). All the other statements are immediate and follow from the properties of $F(x)$. $\square$

Equally important, consider the binary tree (1.8) for $\mathsf{p} > 2$. In this case analogous proposition holds.

Proposition 1.8. *Let $\mathsf{p} > 2$. Then there exists a limit distribution of the nth generation as $n \to \infty$. Denote it by $f_{\mathsf{p}}(x)$ This function is continuous, $f_{\mathsf{p}}(x) = 0$ for $x \leq \frac{1}{\mathsf{p}-1}$, $f_{\mathsf{p}}(x) = 1$ for $x \geq \mathsf{p} - 1$, and it satisfies two functional equations:*

$$2f_{\mathsf{p}}(x) = \begin{cases} f_{\mathsf{p}}(\mathsf{p}x - 1) & \text{if } 1 \leq x \leq \mathsf{p} - 1, \\ f_{\mathsf{p}}(\frac{x}{\mathsf{p}-x}) + 1 & \text{if } \frac{1}{\mathsf{p}-1} \leq x \leq 1, \end{cases}$$

and

$$f_{\mathsf{p}}(x) + f_{\mathsf{p}}\Big(\frac{1}{x}\Big) = 1 \text{ for } x > 0.$$

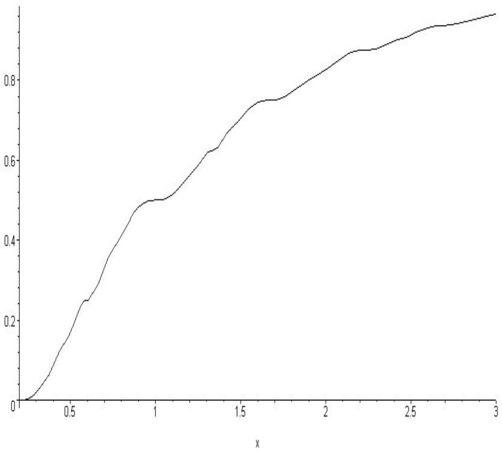
Proof. The proof is analogous to the one of Proposition 1.7, only this time we use equivalences

$$\mathsf{p} - 1 \leq x \leq \frac{1}{\mathsf{p}-1} \Leftrightarrow 1 \leq \frac{\mathsf{p}x}{x+1} \leq \mathsf{p} - 1,$$
$$\mathsf{p} - 1 \leq x \leq \frac{1}{\mathsf{p}-1} \Leftrightarrow \frac{1}{\mathsf{p}-1} \leq \frac{x+1}{\mathsf{p}} \leq \mathsf{p} - 1. \quad \square$$

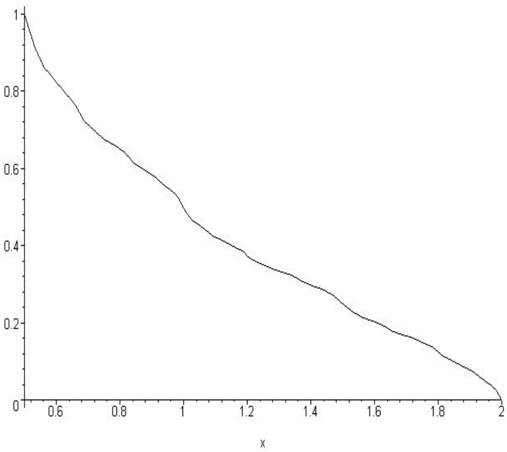
For the sake of uniformity, we introduce $F_{\mathsf{p}}(x) = 1 - f_{\mathsf{p}}(x)$ for $\mathsf{p} > 2$. Then $F_{\mathsf{p}}(x)$ satisfies exactly the same functional equations (1.3), with a slight difference that $F_{\mathsf{p}}(x) = 1$ for $x \leq \frac{1}{\mathsf{p}-1}$ and $F_{\mathsf{p}}(x) = 0$ for $x \geq \mathsf{p} - 1$. Consequently, we will not separate these two cases and all our subsequent results hold uniformly. To this account it should be noted that, for example, in case $\mathsf{p} > 2$ the integral $\int_{\mathsf{p}-1}^{1} \star\, \mathrm{d}\star$ should be understood as $-\int_{1}^{\mathsf{p}-1} \star\, \mathrm{d}\star$. Figure 1 gives graphic images of typical cases for $F_{\mathsf{p}}(x)$.

1.3 Complex case

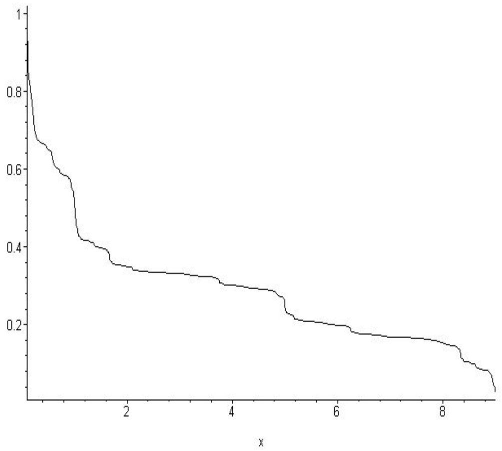
After dealing the case of real p, $1 \leq \mathsf{p} < \infty$, let us consider a tree (1.8) when $\mathsf{p} \in \mathbb{C}$. For our purpose we will concentrate on the case $|\mathsf{p} - 2| \leq 1$. It should be noted that the method which we use allows to extend these result to the case $\Re\mathsf{p} \geq 1$. The question in determining the set in the complex plain where similar results are valid remains open. More importantly, the problem to determine all $\mathsf{p} \in \mathbb{C}$ for which there exists an analytic function $G_{\mathsf{p}}(z)$, which satisfied (1.22), seems to be much harder and interesting. Here and below $[0, \infty]$ stands for a compactification of $[0, \infty)$. In the sequel, the notion of a function $f(z)$ to be analytic in the closed disc $|z - 2| \leq 1$ means that for $z_0 \neq 1$, $|z_0 - 2| \leq 1$, this function is analytic in a certain small neighborhood of z_0. If $z_0 = 1$, this means that there



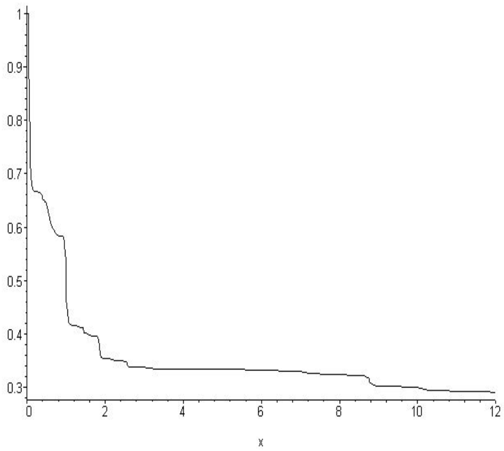
$\mathsf{p} = 1.2,\ x \in [0.2, 3]$



$\mathsf{p} = 3,\ x \in [0.5, 2]$



$\mathsf{p} = 10,\ x \in [0.1, 9]$



$\mathsf{p} = 25,\ x \in [0, 10]$

Figure 1.1: Functions $F_{\mathsf{p}}(x)$

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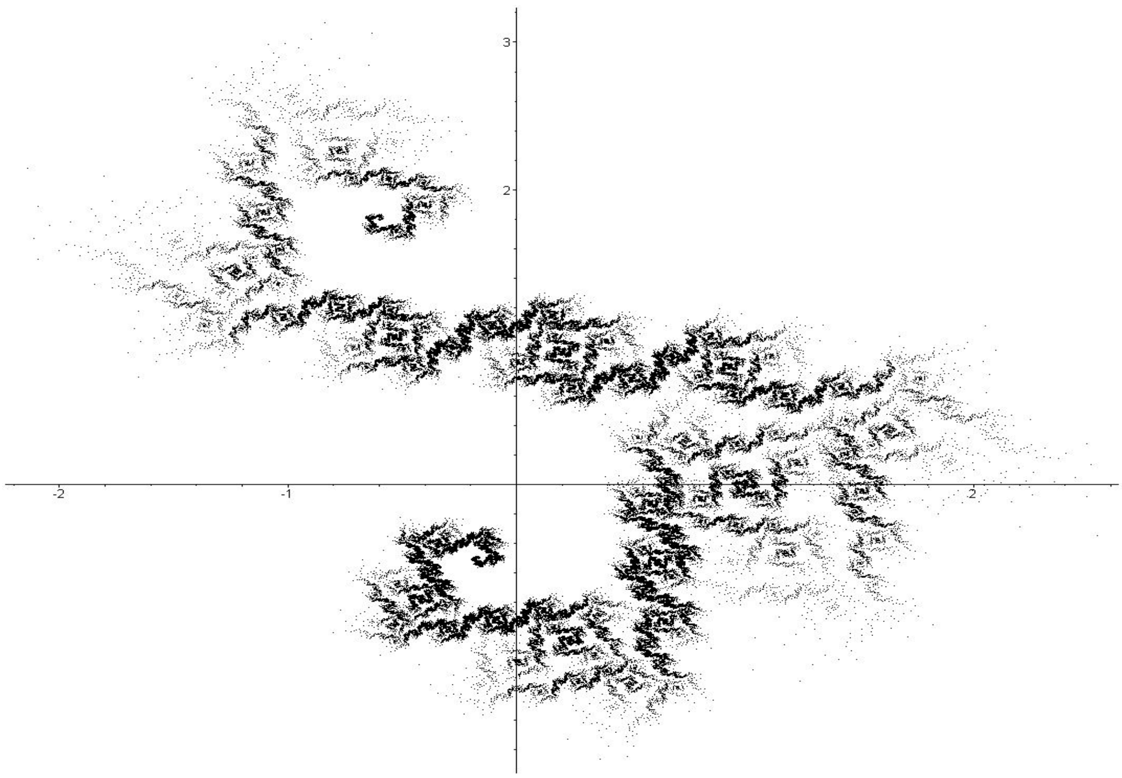

Figure 1.2: $\mathscr{I}_{\mathsf{p}}$, $\mathsf{p} = 0.4 + 1.8i$

exist all higher derivatives, if one approaches the point $z_0 = 1$ while remaining in the disc $|z - 2| \leq 1$.
In this section we prove the following result.

Theorem 1.9. *There exists a unique function* $\mathfrak{X}_{\mathsf{p}}(x) = \mathfrak{X}(\mathsf{p}, x) : \{|\mathsf{p} - 2| \leq 1\} \times \{[0, \infty]\} \to \mathbb{C} \cup \{\infty\}$*, having these properties:*
(i) $\mathfrak{X}(\mathsf{p}, x)$ *satisfies functional equations (1.9);*
(ii) For fixed $\mathsf{p} \neq 1$, $\mathfrak{X}(\mathsf{p}, x) : [0, \infty] \to \mathbb{C}$ *is a continuous function, and the image (denote it by* $\mathscr{I}_{\mathsf{p}}$*) is thus a bounded curve; it is contained in the domain* $\{\mathbb{C} \setminus \{|z + 1| \leq \frac{3}{4}\}$*;*
(iii) For every p, $|\mathsf{p} - 2| \leq 1$, $\mathsf{p} \neq 1$*, in some neighborhood of* p *there exists the derivative* $\frac{\partial}{\partial \mathsf{p}} \mathfrak{X}(\mathsf{p}, x)$*, which is a continuous function for* $x \in [0, \infty]$*;*
(iv) There exist all derivatives $\mathcal{S}_N(x) = \frac{\partial^N}{\partial \mathsf{p}^N} \mathfrak{X}(\mathsf{p}, x)|_{\mathsf{p}=1} : [0, \infty) \to \mathbb{R}$ *(the derivatives are taken inside* $|\mathsf{p} - 2| \leq 1$*). These functions are uniformly continuous for irrational* x *in any finite interval. Moreover,* $\mathcal{S}_N(x) \ll_N x^{N+1}$ *for* $x \geq 1$*, and* $\mathcal{S}_N(x) \ll_N 1$ *for* $x \in (0, 1)$*.*

The curve $\mathscr{I}_{\mathsf{p}}$ has a natural fractal structure: it decomposes into two parts, namely $\frac{\mathscr{I}_{\mathsf{p}}+1}{\mathsf{p}}$ and $\frac{\mathsf{p}\mathscr{I}_{\mathsf{p}}}{\mathscr{I}_{\mathsf{p}}+1}$, with a common point $z = 1$. Additionally, $\mathscr{I}_{\mathsf{p}} = \frac{1}{\mathscr{I}_{\mathsf{p}}}$. As a consequence, $0 \notin \mathscr{I}_{\mathsf{p}}$ for $\mathsf{p} \neq 1$. Figures 2-4 show the images of $\mathscr{I}_{\mathsf{p}}$ for certain characteristic values of p. They are indeed all continuous curves, at least for $\Re \mathsf{p} \geq 1$! Further, Figure 5 shows the image of the curve $\frac{\mathrm{d}}{\mathrm{d}\,\mathsf{p}} \mathfrak{X}(\mathsf{p}, x)|_{\mathsf{p}=1.5+0.5i}$, $x \in [0, \infty]$.

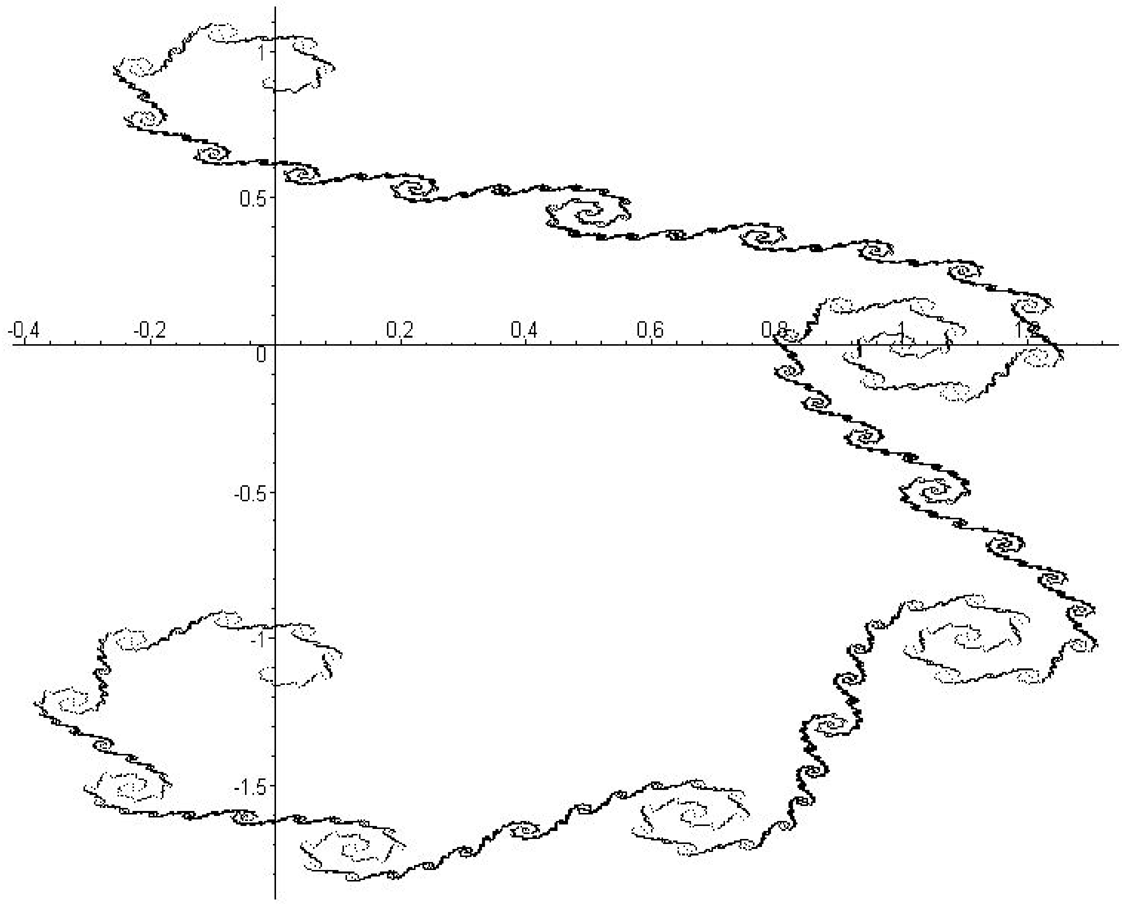

Figure 1.3: $\mathscr{I}_{\mathsf{p}}$, $\mathsf{p} = 1 + 0.9i$. This is a continuous curve!

The investigations of the tree $\mathcal{Q}_{\mathsf{p}}$ deserve a separate paper. I am very grateful to my colleagues Jeffrey Lagarias and Stefano Isola, who sent me various references, also informing about the intrinsic relations of this problem with: Julia sets of rational maps of the Riemann sphere; iterated function systems; forward limit sets of semigroups; various topics from complex dynamics and geometry of discrete groups. Thus, the problem is much more subtle and involved than it appears to be. This poses a difficult question on the limit set of the semigroup generated by transformations $\mathcal{U}_{\mathsf{p}}$ and $\mathcal{T}_{\mathsf{p}}$, or any other two "conjugate" analytic maps of the Riemann sphere (say, two analytic maps $\mathcal{A}$ and $\mathcal{B}$ are "conjugate", if $\mathcal{A}(\alpha) = \alpha$, $\mathcal{B}(\beta) = \beta$, $\mathcal{A}(\beta) = \mathcal{B}(\alpha)$ for some two points α and β on the Riemann sphere). Possibly, certain techniques from complex dynamics do apply here. As pointed out by Curtis McMullen, the property of boundedness of $\mathscr{I}_{\mathsf{p}}$ can be reformulated in a coordinate-free manner. It appears that this curve consists of the closure of the attracting fixed points of the elements of the semigroup $\langle \mathcal{T}_{\mathsf{p}}, \mathcal{U}_{\mathsf{p}} \rangle$. Then the property for the curve being bounded and being bounded away from $z = 0$ means that it does not contain a repelling fixed point of $\mathcal{T}_{\mathsf{p}}$ ($z = \infty$) and a repelling fixed point of $\mathcal{U}_{\mathsf{p}}$ ($z = 0$). It also does not contain neither of the repelling fixed points of the elements of this semigroup. Note that $\mathcal{T}_2(1) = \mathcal{U}_2(1) = 1$, $\mathcal{T}_2'(1) = \mathcal{U}_2'(1) = 1/2$. Thus, there exists a small ball $\mathbf{D}$ around $z = 1$, such that $\mathcal{T}_2(\mathbf{D}) \subset \mathbf{D}$,

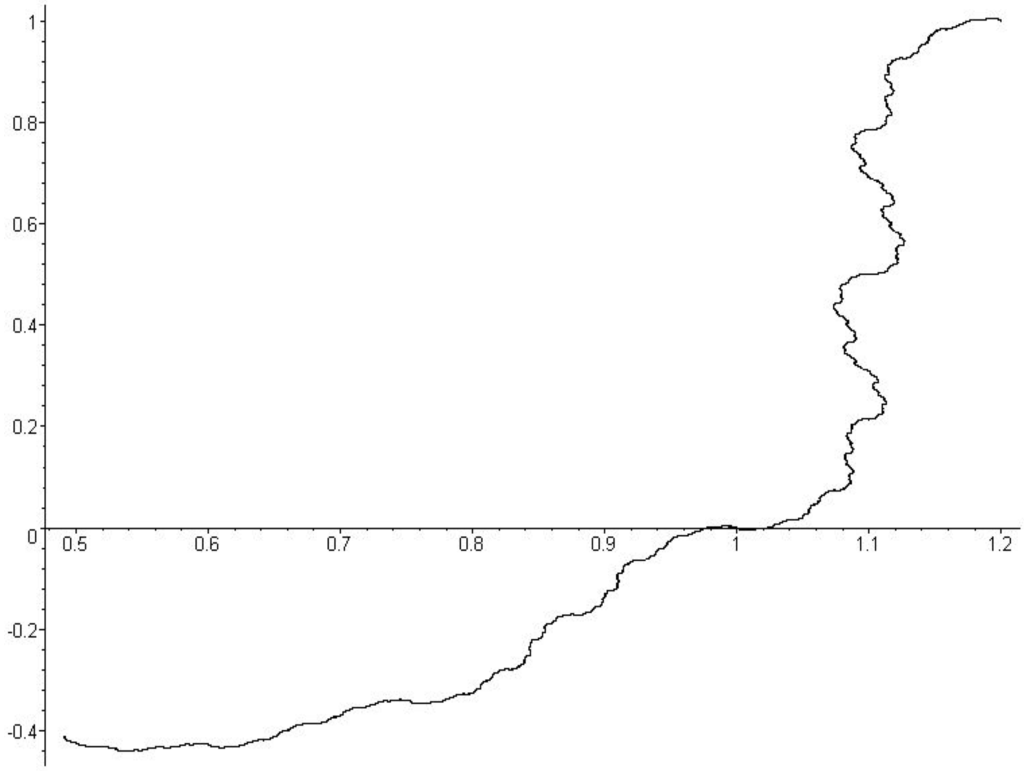

Figure 1.4: $\mathscr{I}_{\mathsf{p}}$, $\mathsf{p} = 1.5 + 0.5i$

$\mathcal{U}_2(\mathbf{D}) \subset \mathbf{D}$, and the last two maps are contractions in $\mathbf{D}$. This strict containment is an open condition on p, and thus there exists a neighborhood of $\mathsf{p} = 2$ such that Theorem 1.9 does hold. I am grateful to Curtis McMullen for this remark: we get the result almost for free. Yet, the full result for $|\,\mathsf{p} - 2| \leq 1$ is needed. This is not a new kind of problem. Some cases of pairs of Möbius transformations were studied. For example, the author in [8] deals with the case of a semigroup generated by two maps $z \mapsto sz \pm 1$, for fixed s, $|s| < 1$, and investigates a closure of a set of all attracting fixed points. For example, for $|s| > 2^{-1/2}$ this set is connected. Further development of this problem can be seen in [35]. On the other hand, the case of one rational map is rather well understood, and it is treated in [4]. Thus, though the machinery of complex dynamics can greatly clarify our understanding of the structure of the curve $\mathscr{I}_{\mathsf{p}}$, we will rather employ the techniques from the analytic theory of continued fractions. The main source is the monograph by H.S. Wall [37]. (Lemmas 1.25, 1.26 and 1.27 can be found in the Appendix A.2.)

Proof of Theorem 1.9. We need the following two results.

Theorem 1.10. *([37] p. 57.) Let v_ν, $\nu \in \mathbb{N}$ be positive numbers such that*

$$v_1 < 1, \quad v_\nu + v_{\nu+1} \leq 1, \ \textit{for}\ \nu \geq 1. \tag{1.11}$$

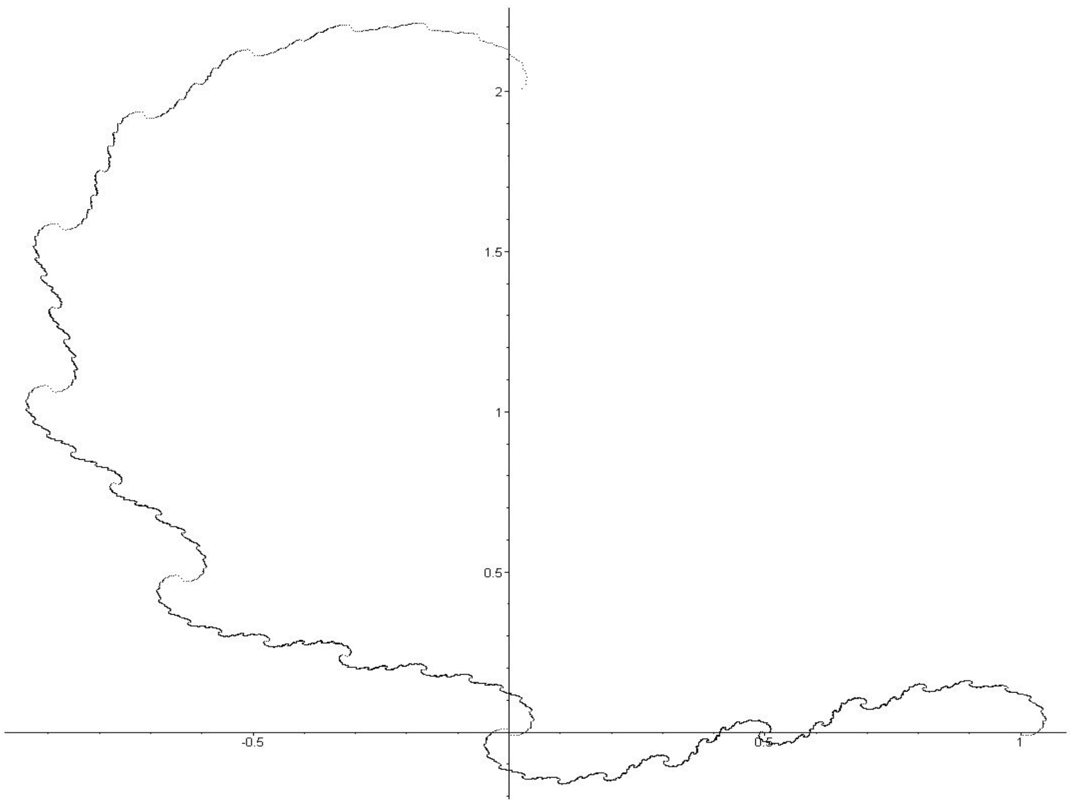

Figure 1.5: $\frac{\mathrm{d}}{\mathrm{d}\,\mathsf{p}}\mathfrak{X}(\,\mathsf{p},[0,\infty])|_{\,\mathsf{p}=\,\mathsf{p}_0},\;\; \mathsf{p}_0 = 1.5+0.5i$

Suppose given complex numbers e_ν, $\nu \in \mathbb{N}$, such that

$$|e_{\nu+1}| - \Re(e_{\nu+1}) \le v_\nu, \quad \nu \ge 1. \tag{1.12}$$

Define the sequence b_ν by the recurrence $b_1 = 1$, $e_{\nu+1} = \frac{1}{b_\nu b_{\nu+1}}$, $\nu \ge 1$. Then the continued fraction

$$\mathcal{F} = \cfrac{1}{1+\cfrac{e_2}{1+\cfrac{e_3}{1+\cfrac{e_4}{\ddots}}}} \tag{1.13}$$

converges if, and only if, (a) some e_ν vanishes, or (b) $e_\nu \ne 0$ for $\nu \ge 2$ and the series $\sum_{\nu=1}^{\infty} |b_\nu|$ diverges. Moreover, if $e_\nu(z) : \mathbf{K}_1 \to \mathbf{K}_2$ are analytic functions of a complex variable, $\mathbf{K}_1$ and $\mathbf{K}_2$ are compact sets, (1.11) and (1.12) are satisfied, and the above series diverges uniformly, then the continued fraction converges uniformly for all $z \in \mathbf{K}_1$.

Theorem 1.11. *([37], p. 60.) If all $v_\nu = \frac{1}{2}$, and the conditions (a) and (b) of Theorem 1.10 hold, then $|\mathcal{F} - 1| \le 1$, $\mathcal{F} \ne 0$.*

For $a, b \in \mathbb{N}$, $\mathsf{p} \in \mathbb{C}$, $|\mathsf{p} - 2| \leq 1$, define rational functions

$$\begin{aligned} W_a(\mathsf{p}) &= \frac{\mathsf{p}^a - 1}{\mathsf{p}^{a+1} - \mathsf{p}^a}, \\ T_{a,b}(\mathsf{p}) &= W_a^{-1}(\mathsf{p}) W_b^{-1}(\mathsf{p})\,\mathsf{p}^{-a} = \frac{(\mathsf{p}-1)^2\,\mathsf{p}^b}{(\mathsf{p}^a - 1)(\mathsf{p}^b - 1)}, \quad T_{a,\infty}(\mathsf{p}) = \frac{(\mathsf{p}-1)^2}{(\mathsf{p}^a - 1)}. \end{aligned}$$

Since, for fixed $\mathsf{p} \neq 1$, $W_a(\mathsf{p}) \to \mathsf{p} - 1$, as $a \to \infty$, then there exist two constants $k_1 = k_1(\mathsf{p})$ and $k_2 = k_2(\mathsf{p})$, such that

$$0 < k_1 \leq |W_a(\mathsf{p})| \leq k_2 < +\infty, \quad a \in \mathbb{N}. \tag{1.14}$$

Let $x \geq 1$, $x = [a_1, a_2, a_3, ...]$, be an irrational number, $a_i \in \mathbb{N}$. Let us consider the continued fraction

$$\mathcal{F}(\mathsf{p}, x) = \mathcal{F}(\mathsf{p}, a_1, a_2, ...) = \cfrac{1}{1 + \cfrac{T_{a_1,a_2}(\mathsf{p})}{1 + \cfrac{T_{a_2,a_3}(\mathsf{p})}{1 + \cfrac{T_{a_3,a_4}(\mathsf{p})}{\ddots}}}}. \tag{1.15}$$

If $x = [a_1, a_2, ..., a_\kappa] \geq 1$ is rational, let us define

$$\mathcal{F}(\mathsf{p}, x) = \mathcal{F}(\mathsf{p}, a_1, a_2, ..., a_\kappa) = \cfrac{1}{1 + \cfrac{T_{a_1,a_2}(\mathsf{p})}{1 + \cfrac{T_{a_2,a_3}(\mathsf{p})}{\ddots \, 1 + \cfrac{}{1 + T_{a_\kappa,\infty}}}}}.$$

From the definition, this continued fraction obeys the following rule

$$\mathcal{F}(\mathsf{p}, a_1, a_2, ...) = \frac{1}{1 + T_{a_1,a_2}(\mathsf{p}) \cdot \mathcal{F}(\mathsf{p}, a_2, a_3 ...)}.$$

We will now apply Theorem 1.10 to $\mathcal{F}(\mathsf{p}, a_1, a_2, a_3, ...)$. Suppose x is irrational. Thus, let $e_\nu = T_{a_{\nu-1},a_\nu}(\mathsf{p})$, $\nu \geq 2$. Let us define constants

$$\mu(a, b) = \sup_{\mathsf{p} \in \mathbb{C}, |\mathsf{p}-2| \leq 1} |T_{a,b}(\mathsf{p})| - \Re(T_{a,b}(\mathsf{p})).$$

By Lemma 1.25, $\mu(a, b) + \mu(b, c) < 0.76$, $a, b, c \in \mathbb{N}$. Further, from the definition in Theorem 1.10 it follows that

$$\begin{aligned} b_{2\nu} &= W_{a_1}(\mathsf{p}) W_{a_{2\nu}}(\mathsf{p})\,\mathsf{p}^{a_{2\nu-1} - ... + a_3 - a_2 + a_1}, \\ b_{2\nu+1} &= W_{a_1}^{-1}(\mathsf{p}) W_{a_{2\nu+1}}(\mathsf{p})\,\mathsf{p}^{a_{2\nu} - ... - a_3 + a_2 - a_1}. \end{aligned} \tag{1.16}$$

It is obvious that the series $\sum_{\nu=1}^{\infty} |b_\nu|$ diverges. Hence, Theorem 1.10 tells that the continued fraction converges, and that for fixed irrational $x = [a_1, a_2, ...] > 1$, $\mathcal{F}(\mathsf{p}_0, a_1, a_2, ...)$ is an analytic function in p_0 in some small neighborhood of p. For rational x this is in fact a rational function.

As it is shown in [37], the νth convergent of the continued fraction (2.12) (denote it by $\frac{A_\nu}{B_\nu}$) is equal to the νth convergent (denote it by $\frac{P_\nu}{Q_\nu}$) of the continued fraction

$$\cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots}}}}.$$

Moreover, since (1.11) and (1.12) are satisfied, we have that, for certain positive constant $k = k(b_1, b_2, b_3)$ ([37], p.55-56),

$$\begin{aligned} |Q_{2\nu}| &\geq k(1 + |b_2| + |b_4| + ..., +|b_{2\nu}|), \\ |Q_{2\nu+1}| &\geq k(1 + |b_3| + |b_5| + ..., +|b_{2\nu+1}|), \\ \Big|\frac{A_{\nu+1}}{B_{\nu+1}} - \frac{A_\nu}{B_\nu}\Big| &= \frac{1}{|Q_\nu Q_{\nu+1}|}. \end{aligned} \tag{1.17}$$

Now we have

Proposition 1.12. *Fix* $\mathsf{p} \in \mathbb{C}$, $|\mathsf{p} - 2| \leq 1$, $\mathsf{p} \neq 1$. *Let* $x = [a_1, a_2, ...] \geq 1$ *be a real number. The function* $\mathcal{F}(\mathsf{p}, x) : [1, \infty] \to \mathbb{C}$ *is continuous.*

Proof. Fix irrational $x > 1$. Let $\delta > 0$, and $y \geq 1$ be such that $|x - y| < \delta$. Then there exists N such that the first N partial quotients of x and y coincide, $N = N(\delta) \to \infty$ as $\delta \to 0$. Consequently, let the corresponding convergents to $\mathcal{F}(\mathsf{p}, x)$ and $\mathcal{F}(\mathsf{p}, y)$ be respectively

$$\begin{aligned} &\frac{A_1}{B_1}, \quad \frac{A_2}{B_2}, \quad ..., \frac{A_N}{B_N}, \quad \frac{A_{N+1}}{B_{N+1}}, \quad \frac{A_{N+2}}{B_{N+2}}, ...; \text{ and} \\ &\frac{A_1}{B_1}, \quad \frac{A_2}{B_2}, \quad ..., \frac{A_N}{B_N}, \quad \frac{A'_{N+1}}{B'_{N+1}}, \quad \frac{A'_{N+2}}{B'_{N+2}} ... \end{aligned}$$

Now, combining (1.14), (1.16) and (1.17) we see that

$$\begin{aligned} |Q_{2\nu}Q_{2\nu+1}| > k^2 k_1^3 k_2^{-1} &\times \Big(|\mathsf{p}|^{a_1} + |\mathsf{p}|^{a_3 - a_2 + a_1} + ... + |\mathsf{p}|^{a_{2\nu-1} - ... + a_3 - a_2 + a_1}\Big) \\ &\times \Big(|\mathsf{p}|^{a_2 - a_1} + |\mathsf{p}|^{a_4 - a_3 + a_2 - a_1} + ... + |\mathsf{p}|^{a_{2\nu} - ... - a_3 + a_2 - a_1}\Big). \end{aligned}$$

Denote $c_1 = k^2 k_1^3 k_2^{-1}$. Let $|\mathsf{p}|^{a_{2\ell-1} - ... + a_3 - a_2 + a_1} = \lambda_\ell$, $1 \leq \ell \leq \nu$. The above inequality and the arithmetic-harmonic mean inequality give

$$\begin{aligned} |Q_{2\nu}Q_{2\nu+1}| &> c_1(\lambda_1 + \lambda_2 + ... + \lambda_\nu) \cdot (|\mathsf{p}|^{a_2}\lambda_1^{-1} + |\mathsf{p}|^{a_4}\lambda_2^{-1} + ... + |\mathsf{p}|^{a_{2\nu}}\lambda_\nu^{-1}) \\ &\geq |\mathsf{p}| c_1(\lambda_1 + \lambda_2 + ... + \lambda_\nu) \cdot (\lambda_1^{-1} + \lambda_2^{-1} + ... + \lambda_\nu^{-1}) \geq |\mathsf{p}| c_1 \nu^2 \quad \nu \geq 1. \end{aligned}$$

Analogously we prove that $|Q_{2\nu-1}Q_{2\nu}| > |\mathsf{p}| c_2 \nu^2$, $\nu \geq 2$. Thus, $|Q_\nu Q_{\nu+1}| > c\nu^2$ for certain real $c > 0$, $\nu \geq 2$. We see that (1.17) yield

$$\Big|\mathcal{F}(\mathsf{p}, x) - \frac{A_N}{B_N}\Big| < \sum_{\nu=N}^{\infty} \frac{1}{|Q_\nu Q_{\nu+1}|} \leq \sum_{\nu=N}^{\infty} \frac{c^{-1}}{\nu^2} < \frac{c^{-1}}{N-1}; \quad \Big|\mathcal{F}(\mathsf{p}, y) - \frac{A_N}{B_N}\Big| < \frac{c^{-1}}{N-1}.$$

This implies $|\mathcal{F}(\mathsf{p},x)-\mathcal{F}(\mathsf{p},y)|<\frac{2c^{-1}}{N-1}$. In case x is rational we argue in a similar way. In this case note that real numbers close to $x=[a_1,a_2,...,a_\kappa]$ are of the form or $[a_1,a_2,...,a_\kappa,T,...]$, either $[a_1,a_2,...,a_\kappa-1,1,T,...]$ for T sufficiently large. The case $x=\infty$ is analogous. This establishes the validity of the Proposition. □

Eventually, for real number $x\geq 0$, $x=[a_0,a_1,a_2,...]$, let us define

$$\mathfrak{X}(\mathsf{p},[a_0,a_1,...])=W_{a_0}(\mathsf{p})+\cfrac{\mathsf{p}^{-a_0}}{W_{a_1}(\mathsf{p})+\cfrac{\mathsf{p}^{-a_1}}{W_{a_2}(\mathsf{p})+\cfrac{\mathsf{p}^{-a_2}}{W_{a_3}(\mathsf{p})+\ddots}}}.$$

After an equivalence transformation ([37], p.19), this can be given an expression

$$\mathfrak{X}(\mathsf{p},[a_0,a_1,...])=W_{a_0}(\mathsf{p})+\mathsf{p}^{-a_0}W_{a_1}^{-1}(\mathsf{p})\cdot\mathcal{F}(\mathsf{p},a_1,a_2,a_3,...). \tag{1.18}$$

From the very construction, this function satisfies the functional equations (1.9), is continuous at $x=1$ and thus is continuous for $x\in[0,\infty]$. Obviously, (1.9) determine the values of $\mathfrak{X}(\mathsf{p},x)$ at rational x uniquely, hence a continuous solution to (1.9) is unique. We are left to show that the image of the curve $\mathscr{I}_\mathsf{p}$ is contained outside the circle $|z+1|\leq\frac{3}{4}$. This is equivalent to the statement that $\frac{\mathsf{p}\mathscr{I}_\mathsf{p}}{\mathscr{I}_\mathsf{p}+1}$ is contained inside the circle $|z-\mathsf{p}|\leq\frac{4\mathsf{p}}{3}$. But the points on $\frac{\mathsf{p}\mathscr{I}_\mathsf{p}}{\mathscr{I}_\mathsf{p}+1}$ are exactly the point on the curve $\mathscr{I}_\mathsf{p}$ with $a_0=0$. Thus, we need to show that

$$|\mathsf{p}^{-1}\mathfrak{X}(\mathsf{p},[0,a_1,a_2,...])-1|=|\mathsf{p}^{-1}W_{a_1}^{-1}\mathcal{F}(\mathsf{p},a_1,a_2,...)-1|\leq\frac{4}{3}. \tag{1.19}$$

Unfortunately, we cannot apply Theorem 1.11 directly to all p, $|\mathsf{p}-2|\leq 1$, since the table above Lemma 1.25 shows that $\mu(1,b)>\frac{1}{2}$ for infinitely many b. The maximum values $\mu(1,b)$ (see the definition of this constant) are produced by points p close to $\chi=2+e^{2\pi i/3}$, or to $\overline{\chi}$. For this reason let us introduce

$$\mu^\star(a,b)=\sup_{\mathsf{p}\in\mathbb{C},|\mathsf{p}-2|\leq 1,|\mathsf{p}-\chi|\geq 0.19,|\mathsf{p}-\overline{\chi}|\geq 0.19}|T_{a,b}(\mathsf{p})|-\Re(T_{a,b}(\mathsf{p})).$$

Then indeed $\mu^\star(a,b)<\frac{1}{2}$ for all $a,b\in\mathbb{N}$. Thus, Theorem 1.11 gives $|\mathcal{F}(\mathsf{p},a_1,a_2,...)-1|\leq 1$, and the statement (1.19) follows from Lemma 1.27. In case $|\mathsf{p}-2|\leq 1$, $|\mathsf{p}-\chi|<0.19$ (or $|\mathsf{p}-\overline{\chi}|<0.19$) we use another theorem by Wall ([37], p. 152), which describes the value region of a continued fraction (2.12), provided elements e_ν belong to the compact domain in the parabolic region $|z|-\Re(ze^{i\phi})\leq 2h\cos^2\frac{\phi}{2}$, for certain fixed $-\pi<\phi<+\pi$, $0<h\leq\frac{1}{4}$. We omit the details. This proves part (ii). In a similar fashion we prove part (iii). Finally, a direct inspection shows that slightly modified proofs remain valid in case $\mathsf{p}=1$, if we define a function to be analytic at $\mathsf{p}=1$, if it possesses all higher $\mathsf{p}-$derivatives, while remaining inside the disc $|\mathsf{p}-2|\leq 1$. □

Definition 1.13. *We define Minkowski* $\mathsf{p}-$*question mark function* $F_{\mathsf{p}}(x) : \mathscr{I}_{\mathsf{p}} \to [0,1]$, *by*

$$F_{\mathsf{p}}(\mathfrak{X}(\mathsf{p}, x)) = F(x), \quad x \in [0, \infty].$$

1.4 Properties of integral transforms of $F_{\mathsf{p}}(x)$

For given p, $|\mathsf{p}-2| \leq 1$, we define

$$\chi_n = \frac{\mathsf{p} + \mathsf{p}^{n-1} - 2}{\mathsf{p}^{n-1}(\mathsf{p}-1)}, \quad \mathscr{I}_n = [\chi_n, \chi_{n+1}] = \mathfrak{X}(\mathsf{p}, [n, n+1]) \text{ for } n \in \mathbb{N}_0.$$

Complex numbers χ_n stand for the analogue of non-negative integers on the curve $\mathscr{I}_{\mathsf{p}}$. In other words, $\chi_n = \mathcal{U}^n(\mathsf{p}-1)$. We consider $\mathscr{I}_n$ as part of the curve $\mathscr{I}_{\mathsf{p}}$ contained between the points χ_n and χ_{n+1}. Thus, $\chi_0 = \mathsf{p}-1$, $\chi_1 = 1$, and the sequence χ_n is "increasing", in the sense that χ_j as a point on a curve $\mathscr{I}_{\mathsf{p}}$ is between χ_i and χ_k if $i < j < k$. Moreover, $\bigcup\limits_{n=0}^{\infty} \mathscr{I}_n \bigcup \{\frac{1}{\mathsf{p}-1}\} = \mathscr{I}_{\mathsf{p}}$.

Proposition 1.14. *Let* $\omega(x) : \mathscr{I}_{\mathsf{p}} \to \mathbb{C}$ *be a continuous function. Then*

$$\int\limits_{\mathscr{I}_{\mathsf{p}}} \omega(x) \,\mathrm{d}\, F_{\mathsf{p}}(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \int\limits_{\mathscr{I}_{\mathsf{p}}} \omega\Big(\frac{x}{\mathsf{p}^{n-1}(x+1)} + \frac{\mathsf{p}^n - 1}{\mathsf{p}^{n+1} - \mathsf{p}^n}\Big) \,\mathrm{d}\, F_{\mathsf{p}}(x).$$

Proof. Indeed, using (1.10) we obtain

$$\int\limits_{\mathscr{I}_{\mathsf{p}}} \omega(x) \,\mathrm{d}\, F_{\mathsf{p}}(x) = \sum_{n=0}^{\infty} \int\limits_{\mathscr{I}_n} \omega(x) \,\mathrm{d}\, F_{\mathsf{p}}(x) = \sum_{n=0}^{\infty} \int\limits_{\mathcal{T}^n(\mathscr{I}_0)} \omega(x) \,\mathrm{d}\, F_{\mathsf{p}}(x) \overset{x \to \mathcal{T}^n x}{=}$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \int\limits_{\mathscr{I}_0} \omega(\mathcal{T}^n x) \,\mathrm{d}\, F_{\mathsf{p}}(x) \overset{x \to \mathcal{U} x}{=} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \int\limits_{\mathscr{I}_{\mathsf{p}}} \omega(\mathcal{T}^n \mathcal{U} x) \,\mathrm{d}\, F_{\mathsf{p}}(x),$$

and this is exactly the statement of the Proposition. □

For $L, T \in \mathbb{N}_0$ let us introduce

$$B_{L,T}(\mathsf{p}) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}\,\mathsf{p}^{Tn}} \Big(\frac{\mathsf{p}^n - 1}{\mathsf{p}^{n+1} - \mathsf{p}^n}\Big)^L.$$

For example,

$$\begin{aligned}
B_{0,T} &= \frac{\mathsf{p}^T}{2\,\mathsf{p}^T - 1}, \quad B_{1,T}(\mathsf{p}) = \frac{\mathsf{p}^T}{(2\,\mathsf{p}^T - 1)(2\,\mathsf{p}^{T+1} - 1)}, \\
B_{2,T}(\mathsf{p}) &= \frac{\mathsf{p}^T(2\,\mathsf{p}^{T+1} + 1)}{(2\,\mathsf{p}^{T+2} - 1)(2\,\mathsf{p}^{T+1} - 1)(2\,\mathsf{p}^T - 1)}, \\
B_{3,T}(\mathsf{p}) &= \frac{\mathsf{p}^T(4\,\mathsf{p}^{2T+3} + 4\,\mathsf{p}^{T+2} + 4\,\mathsf{p}^{T+1} + 1)}{(2\,\mathsf{p}^{T+3} - 1)(2\,\mathsf{p}^{T+2} - 1)(2\,\mathsf{p}^{T+1} - 1)(2\,\mathsf{p}^T - 1)}, \\
B_{4,T}(\mathsf{p}) &= \frac{\mathsf{p}^T(2\,\mathsf{p}^{T+2} + 1)(4\,\mathsf{p}^{2T+4} + 6\,\mathsf{p}^{T+3} + 8\,\mathsf{p}^{T+2} + 6\,\mathsf{p}^{T+1} + 1)}{(2\,\mathsf{p}^{T+4} - 1)(2\,\mathsf{p}^{T+3} - 1)(2\,\mathsf{p}^{T+2} - 1)(2\,\mathsf{p}^{T+1} - 1)(2\,\mathsf{p}^T - 1)}.
\end{aligned}$$

As it is easy to see, $B_{L,T}(\mathsf{p})$ are rational functions in p for $L, T \in \mathbb{N}_0$. Indeed,

$$B_{L,T}(\mathsf{p}) = \frac{1}{(\mathsf{p}-1)^L} \cdot \sum_{n=0}^{\infty} \frac{1}{\mathsf{p}^{Tn} 2^{n+1}} \Big(1 - \frac{1}{\mathsf{p}^n}\Big)^L =$$

$$\frac{1}{2(\mathsf{p}-1)^L} \cdot \sum_{s=0}^{L} (-1)^s \binom{L}{s} \sum_{n=0}^{\infty} \frac{1}{2^n\, \mathsf{p}^{n(s+T)}} = \frac{\mathsf{p}^T}{(\mathsf{p}-1)^L} \cdot \sum_{s=0}^{L} (-1)^s \binom{L}{s} \frac{\mathsf{p}^s}{2\,\mathsf{p}^{s+T} - 1} =$$

$$\frac{\mathsf{p}^T \mathscr{R}_{L,T}(\mathsf{p})}{(2\,\mathsf{p}^{T+L}-1)(2\,\mathsf{p}^{T+L-1}-1) \cdot \ldots \cdot (2\,\mathsf{p}^{T+1}-1)(2\,\mathsf{p}^T - 1)},$$

where $\mathscr{R}_{L,T}(\mathsf{p})$ are polynomials. This follows from the observation that $\mathsf{p} = 1$ is a root of numerator of multiplicity not less than L.

As in case $\mathsf{p} = 1$, our main concern are the moments of distributions $F_{\mathsf{p}}(x)$, which are defined by

$$\begin{aligned} m_L(\mathsf{p}) &= 2 \int_{\mathscr{I}_0} x^L \,\mathrm{d}\, F_{\mathsf{p}}(x) = \int_{\mathscr{I}_{\mathsf{p}}} \Big(\frac{\mathsf{p}x}{x+1}\Big)^L \mathrm{d}\, F_{\mathsf{p}}(x) \\ &= 2 \int_0^1 \mathfrak{X}^L(\mathsf{p}, x)\,\mathrm{d}F(x) = \lim_{n\to\infty} 2^{2-n} \sum_{a_1+a_2+\ldots+a_s=n} [0, a_1, a_2, .., a_s]^L_{\mathsf{p}}., \\ M_L(\mathsf{p}) &= \int_{\mathscr{I}_{\mathsf{p}}} x^L \,\mathrm{d}\, F_{\mathsf{p}}(x). \end{aligned}$$

Thus, if $\sup_{z \in \mathscr{I}_{\mathsf{p}}} |z| = \rho_{\mathsf{p}} > 1$, which is finite for $\Re\,\mathsf{p} \geq 1$, $\mathsf{p} \neq 1$ (see Section 3), then $M_L(\mathsf{p}) \leq \rho^L_{\mathsf{p}}$.

Proposition 1.15. *The function $m_L(\mathsf{p})$ is analytic in the disc $|\,\mathsf{p} - 2| \leq 1$, including its boundary. In particular, if in this disc*

$$\widehat{m}_L(\mathsf{p}) := \frac{m_L(\mathsf{p})}{\mathsf{p}^L} = \sum_{v=0}^{\infty} \eta_{v,L}(\mathsf{p} - 2)^v,$$

then for any $M \in \mathbb{N}$, one has the estimate $\eta_{v,L} \ll v^{-M}$ as $v \to \infty$.

Proof. The function $\mathfrak{X}(\mathsf{p}, x)$ possesses a derivative in p for $\Re\,\mathsf{p} \geq 1$, $|\,\mathsf{p} - 2| \leq 1$, and these are bounded and continuous functions for $x \in \mathbb{R}_+$. Therefore $m_L(\mathsf{p})$ has a derivative. For $\mathsf{p} = 1$, there exists $\frac{\mathrm{d}^M}{\mathrm{d}\,\mathsf{p}^M}\mathfrak{X}(\mathsf{p}, x) \ll x^{M+1}$, and it is a continuous function for irrational x. Additionally, $F'(x) = 0$ for $x \in \mathbb{Q}_+$. This proves the analyticity of $m_L(\mathsf{p})$ in the disc $|\,\mathsf{p} - 2| \leq 1$. Then an estimate for the Taylor coefficients is the standard fact from Fourier analysis. In fact,

$$\eta_{v,L} = \int_0^1 \widehat{m}_L(2 + e^{2\pi i\vartheta}) e^{-2\pi i v\vartheta}\,\mathrm{d}\vartheta.$$

The function $\widehat{m}_L(2 + e^{2\pi i\vartheta}) \in \mathsf{C}^\infty(\mathbb{R})$, hence the iteration of integration by parts implies the needed estimate. □

Proposition 1.16. *Functions $M_L(\mathsf{p})$ and $m_L(\mathsf{p})$ are related via rational functions $B_{L,T}(\mathsf{p})$ in the following way:*

$$M_L(\mathsf{p}) = \sum_{s=0}^{L} m_s(\mathsf{p}) B_{L-s,s}(\mathsf{p}) \binom{L}{s}.$$

Proof. Indeed, this follows from the definitions and Proposition 1.14 in case $\omega(x) = x^L$. □

Let us introduce, following [1] in case $\mathsf{p} = 1$, the following generating functions:

$$\begin{aligned} \mathfrak{m}_{\mathsf{p}}(t) &= \sum_{L=0}^{\infty} m_L(\mathsf{p}) \frac{t^L}{L!} = 2 \int_{\mathscr{I}_0} e^{xt} \, \mathrm{d}\, F_{\mathsf{p}}(x) = \int_{\mathscr{I}_{\mathsf{p}}} \exp\Big(\frac{\mathsf{p}xt}{x+1}\Big) \, \mathrm{d}\, F_{\mathsf{p}}(x); \\ G_{\mathsf{p}}(z) &= \sum_{L=1}^{\infty} \frac{m_L(\mathsf{p})}{\mathsf{p}^L} z^{L-1} = \int_{\mathscr{I}_{\mathsf{p}}} \frac{1}{x+1-z} \, \mathrm{d}F_{\mathsf{p}}(x) = \int_0^{\infty} \frac{1}{\mathfrak{X}(\mathsf{p}, x) + 1 - z} \, \mathrm{d}F(x) \end{aligned} \quad (1.20)$$

The situation $\mathsf{p} = 2$ is particularly important, since all these functions can be explicitly calculated, and it provides the case where all the subsequent results can be checked directly and the starting point in proving Theorem 1.2. Thus,

$$\mathfrak{m}_2(t) = e^t, \quad G_2(z) = \frac{1}{2 - z}.$$

By the definition, expressions $m_L(\mathsf{p})/\mathsf{p}^L$ are Taylor coefficients of $G_{\mathsf{p}}(z)$ at $z = 0$. Differentiation of $L - 1$ times under the integral defining $G_{\mathsf{p}}(z)$, and substitution $z = 1$ gives

$$G_{\mathsf{p}}^{(L-1)}(1) = (L-1)! \int_{\mathscr{I}_{\mathsf{p}}} \frac{1}{x^L} \, \mathrm{d}\, F_{\mathsf{p}}(x) = (L-1)! M_L(\mathsf{p}) \Rightarrow G_{\mathsf{p}}(z+1) = \sum_{L=0}^{\infty} M_L(\mathsf{p}) z^{L-1}, \quad (1.21)$$

with a radius of convergence equal to ρ_{p}^{-1}. As was proved in [1] and mentioned before, in case $\mathsf{p} = 1$ ($\rho_1 = \infty$) this must be interpreted that there exist all derivatives at $z = 1$. The next Proposition shows how symmetry property reflects in $\mathfrak{m}_{\mathsf{p}}(t)$.

Proposition 1.17. *One has*

$$\mathfrak{m}_{\mathsf{p}}(t) = e^{\mathsf{p}t} \mathfrak{m}_{\mathsf{p}}(-t).$$

Proof. Indeed,

$$\mathfrak{m}_{\mathsf{p}}(t) = \int_{\mathscr{I}_{\mathsf{p}}} \exp\Big(\frac{\mathsf{p}xt}{x+1}\Big) \, \mathrm{d}\, F_{\mathsf{p}}(x) = \int_{\mathscr{I}_{\mathsf{p}}} \exp\Big(\mathsf{p}t - \frac{\mathsf{p}t}{x+1}\Big) \, \mathrm{d}\, F_{\mathsf{p}}(x) =$$

$$e^{\mathsf{p}t} \int_{\mathscr{I}_{\mathsf{p}}} \exp\Big(-\frac{\mathsf{p}t}{x+1}\Big) \, \mathrm{d}\, F_{\mathsf{p}}(x) \overset{x \to \frac{1}{x}}{=} e^{\mathsf{p}t} \mathfrak{m}_{\mathsf{p}}(-t). \quad \square$$

This result allows to obtain linear relations among moments $m_L(\mathsf{p})$ and the exact value of the first (trivial) moment $m_1(\mathsf{p})$.

Corollary 1.18. *One has*

$$m_1(\mathsf{p}) = \frac{\mathsf{p}}{2}, \quad M_1(\mathsf{p}) = \frac{\mathsf{p}^2+2}{4\,\mathsf{p}-2}.$$

Proof. Indeed, the last propositions implies

$$m_L(\mathsf{p}) = \sum_{s=0}^{L} \binom{L}{s} (-1)^s m_s(\mathsf{p})\,\mathsf{p}^{L-s}, \quad L \geq 0.$$

For $L=1$ this gives the first statement of the Corollary. Additionally, Proposition 1.16 for $L=1$ reads as

$$M_1(\mathsf{p}) = \frac{\mathsf{p}}{2\,\mathsf{p}-1}\cdot m_1(\mathsf{p}) + \frac{1}{2\,\mathsf{p}-1},$$

and we are done. □

1.5 Three term functional equation

Theorem 1.19. *The function $G_{\mathsf{p}}(z)$ can be extended to analytic function in the domain $\mathbb{C}\setminus(\mathscr{I}_{\mathsf{p}}+1)$. It satisfies the functional equation*

$$\frac{1}{z} + \frac{\mathsf{p}}{z^2} G_{\mathsf{p}}\Big(\frac{\mathsf{p}}{z}\Big) + 2G_{\mathsf{p}}(z+1) = \mathsf{p} G_{\mathsf{p}}(\mathsf{p}z), \ \textit{for } z \notin \frac{\mathscr{I}_{\mathsf{p}}+1}{\mathsf{p}}. \tag{1.22}$$

Its consequence is the symmetry property

$$G_{\mathsf{p}}(z+1) = -\frac{1}{z^2} G_{\mathsf{p}}\Big(\frac{1}{z}+1\Big) - \frac{1}{z}.$$

Moreover, $G_{\mathsf{p}}(z) \to 0$ if dist$(z, \mathscr{I}_{\mathsf{p}}) \to \infty$.
Conversely - the function satisfying this functional equation and regularity property is unique.

Proof. Let $w(x,z) = \frac{1}{x+1-z}$. Then it is straightforward to check that

$$w\Big(\frac{x+1}{\mathsf{p}}, z+1\Big) = \mathsf{p}\cdot w(x, \mathsf{p}z),$$
$$w\Big(\frac{\mathsf{p}}{x+1}, z+1\Big) = -\frac{\mathsf{p}}{z^2} w\Big(x, \frac{\mathsf{p}}{z}\Big) - \frac{1}{z}.$$

Thus, for $|\mathsf{p}-2|\leq 1,\ \mathsf{p}\neq 2$,

$$\begin{aligned}
2G_{\mathsf{p}}(z+1) &= 2\int\limits_{\mathscr{I}_0} w(x,z+1)\,\mathrm{d}\,F_{\mathsf{p}}(x)+2\int\limits_{\mathscr{I}_{\mathsf{p}}\setminus\mathscr{I}_0} w(x,z+1)\,\mathrm{d}\,F_{\mathsf{p}}(x)\\
&= 2\int\limits_{\mathscr{I}_{\mathsf{p}}} w(\frac{\mathsf{p}x}{x+1},z+1)\,\mathrm{d}\,F_{\mathsf{p}}\Big(\frac{\mathsf{p}x}{x+1}\Big)+2\int\limits_{\mathscr{I}_{\mathsf{p}}} w(\frac{x+1}{\mathsf{p}},z+1)\,\mathrm{d}\,F_{\mathsf{p}}\Big(\frac{x+1}{\mathsf{p}}\Big)\\
&= \int\limits_{\mathscr{I}_{\mathsf{p}}} w(\frac{\mathsf{p}}{x+1},z+1)\,\mathrm{d}\,F_{\mathsf{p}}(x)+\int\limits_{\mathscr{I}_{\mathsf{p}}} w(\frac{x+1}{\mathsf{p}},z+1)\,\mathrm{d}\,F_{\mathsf{p}}(x)\\
&= -\frac{1}{z}-\frac{\mathsf{p}}{z^2}G_{\mathsf{p}}\Big(\frac{\mathsf{p}}{z}\Big)+\ \mathsf{p}G_{\mathsf{p}}(\mathsf{p}z).
\end{aligned}$$

(In the first integral we used a substitution $x\to\frac{1}{x}$). The functional equation holds in case $\mathsf{p}=2$ as well, which can be checked directly. The holomorphicity of $G_{\mathsf{p}}(z)$ follows exactly as in case $\mathsf{p}=1$ [1]. All we need is the first integral in (1.20) and the fact that $\mathscr{I}_{\mathsf{p}}$ is a closed set.

As was mentioned, the uniqueness of a function satisfying (1.22) for $\mathsf{p}=1$ was proved in [1]. Thus, the converse implication follows from analytic continuation principle for the function in two complex variables (p,z) (see Lemma 1.23 below, where the proof in case $\mathsf{p}=2$ is presented. Similar argument works for general p). □

Corollary 1.20. *Let $\mathsf{p}\neq 1$, and $\mathscr{C}$ be any closed smooth contour which rounds the curve $\mathscr{I}_{\mathsf{p}}+1$ once in the positive direction. Then*

$$\frac{1}{2\pi i}\oint\limits_{\mathscr{C}} G_{\mathsf{p}}(z)\,\mathrm{d}z=-1.$$

Proof. Indeed, this follows from the functional equation (1.22), as well as from the symmetry property. It is enough to take a sufficiently large circle $\mathscr{C}=\{|z|=R\}$ such that $\mathscr{C}^{-1}+1$ is contained in a small neighborhood of $z=1$, for which $(\mathscr{C}^{-1}+1)\cap(\mathscr{I}_{\mathsf{p}}+1)=\emptyset$. This is possible since $0\notin\mathscr{I}_{\mathsf{p}}$ (see Theorem 1.9). □

We finish with providing an integral equation for $\mathfrak{m}_{\mathsf{p}}(t)$. We indulge in being concise since the argument directly generalizes the one used in [1] to prove the integral functional equation for $\mathfrak{m}(t)$ (in our notation, this is $\mathfrak{m}_1(t)$).

Proposition 1.21. *Let $1\leq\mathsf{p}<\infty$ be real. Then the function $\mathfrak{m}_{\mathsf{p}}(t)$ satisfies the boundary condition $\mathfrak{m}_{\mathsf{p}}(0)=1$, regularity property $\mathfrak{m}_{\mathsf{p}}(-t)\ll e^{-\sqrt{t\log 2}}$, and the integral equation*

$$\mathfrak{m}_{\mathsf{p}}(-s)=\int\limits_0^{\infty}\mathfrak{m}'_{\mathsf{p}}(-t)\Big(2e^sJ_0(2\sqrt{\mathsf{p}st})-J_0(2\sqrt{st})\Big)\,\mathrm{d}t,\quad s\in\mathbb{R}_+.$$

For instance, in the case $\mathsf{p}=1$ this reduces to (1.4), and in the case $\mathsf{p}=2$ this reads as

$$2e^{s}\int\limits_{0}^{\infty} e^{-t}J_0(2\sqrt{2st})\,\mathrm{d}t = 2e^{s}e^{-2s} = e^{-s}+e^{-s} = e^{-s}+\int\limits_{0}^{\infty} e^{-t}J_0(2\sqrt{st})\,\mathrm{d}t,$$

which is an identity [38].

Proof. Indeed, the functional equation for $G_{\mathsf{p}}(z)$ in the region $\Re z<-1$ in terms of $\mathfrak{m}'_{\mathsf{p}}(t)$ reads as

$$\frac{1}{z}=\int\limits_{0}^{\infty}\mathfrak{m}'_{\mathsf{p}}(-t)\Big(\frac{2}{z+1}e^{\frac{\mathsf{p}t}{z+1}}+\frac{1}{z}e^{tz}-\frac{1}{z}e^{\frac{t}{z}}\Big)\,\mathrm{d}t.$$

Now, multiply this by e^{-sz} and integrate over $\Re z=-\sigma<-1$, where $s>0$ is real. All the remaining steps are exactly the same as in [1]. $\square$

Remark. If $\mathsf{p}\neq 1$, the regularity bound is easier than in case $\mathsf{p}=1$. Take, for example, $1<\mathsf{p}<2$. Then

$$|\mathfrak{m}_{\mathsf{p}}(t)|\leq\int\limits_{\mathsf{p}-1}^{\frac{1}{\mathsf{p}-1}}\Big|\exp\Big(\frac{\mathsf{p}xt}{x+1}\Big)\Big|\,\mathrm{d}\,F_{\mathsf{p}}(x)<\int\limits_{\mathsf{p}-1}^{\frac{1}{\mathsf{p}-1}}e^{t}\,\mathrm{d}\,F_{\mathsf{p}}(x)=e^{t}.$$

Thus, Proposition 1.17 gives $|\mathfrak{m}_{\mathsf{p}}(-t)|<e^{(1-\mathsf{p})t}$. The same argument shows that for $\mathsf{p}>2$ we have $|\mathfrak{m}_{\mathsf{p}}(-t)|<e^{-t}$.

1.6 The proof: approach through $\mathsf{p}=2$

Let us rewrite the functional equation for $G_{\mathsf{p}}(z)=G(\mathsf{p},z)$ as

$$\frac{1}{z}+\frac{\mathsf{p}}{z^2}G\Big(\mathsf{p},\frac{\mathsf{p}}{z}\Big)+2G(\mathsf{p},z+1)=\mathsf{p}G(\mathsf{p},\mathsf{p}z). \tag{1.23}$$

Direct induction shows that the following "chain-rule" holds:

$$\frac{\partial^n}{\partial\,\mathsf{p}^n}\Big(\mathsf{p}G(\mathsf{p},\mathsf{p}z)\Big)=\sum_{i+j=n}\binom{n}{j}\mathsf{p}\frac{\partial^i\,\partial^j}{\partial\,\mathsf{p}^i\,\partial z^j}G(\mathsf{p},\mathsf{p}z)z^j+\sum_{i+j=n-1}n\binom{n-1}{j}\frac{\partial^i\,\partial^j}{\partial\,\mathsf{p}^i\,\partial z^j}G(\mathsf{p},\mathsf{p}z)z^j, \tag{1.24}$$

where in the summation it is assumed that $i,j\geq 0$.

Now we will provide rigorous calculations which yield explicit series for $G(\mathsf{p},z)$ in terms of powers of $(\mathsf{p}-2)$ and certain rational functions. The function $G(\mathsf{p},z)$ is analytic in $\{|\mathsf{p}-2|\leq 1\}\times\{|z|\leq\frac{3}{4}\}$. This follows from Theorem 1.9 and integral representation (1.20). Thus, for $\{|\mathsf{p}-2|<1\}\times\{|z|\leq\frac{3}{4}\}$ it has a Taylor expansion

$$G(\mathsf{p},z)=\sum_{L=1}^{\infty}\sum_{v=0}^{\infty}\eta_{v,L}\cdot z^{L-1}(\mathsf{p}-2)^{v}. \tag{1.25}$$

Moreover, the function $G(2+e^{2\pi i\vartheta}, \frac{3}{4}e^{2\pi i\varphi}) \in \mathsf{C}^{\infty}(\mathbb{R}\times\mathbb{R})$, and it is double-periodic. Thus,

$$\eta_{v,L} = \Big(\frac{4}{3}\Big)^{L-1}\int\limits_0^1\int\limits_0^1 G(2+e^{2\pi i\vartheta}, \frac{3}{4}e^{2\pi i\varphi})e^{-2\pi i v\vartheta-2\pi i(L-1)\varphi}\,\mathrm{d}\vartheta\,\mathrm{d}\varphi, \quad v\geq 0, \quad L\geq 1.$$

A standard trick from Fourier analysis (using iteration of integration by parts) shows that $\eta_{v,L} \ll_M (4/3)^L\cdot(Lv)^{-M}$ for any $M\in\mathbb{N}$. Thus, (1.25) holds for $(\mathsf{p},z)\in\{|\mathsf{p}-2|\leq 1\}\times\{|z|\leq 3/4\}$.

Our idea is a simple one. Indeed, let us look at (1.20). This implies the Taylor series for $m_L(\mathsf{p})/\mathsf{p}^L = \sum_{v=0}^{\infty}\eta_{v,L}(\mathsf{p}-2)^v$, convergent in the disc $|\mathsf{p}-2|\leq 1$. Due to the absolute convergence, the order of summation in (1.25) is not essential. This yields

$$G(\mathsf{p},z) = \sum_{v=0}^{\infty}(\mathsf{p}-2)^v\Big(\sum_{L=1}^{\infty}\eta_{v,L}\cdot z^{L-1}\Big).$$

Therefore, let

$$\frac{1}{n!}\frac{\partial^n}{\partial\,\mathsf{p}^n}G(\mathsf{p},z)\Big|_{\mathsf{p}=2} = \mathbf{H}_n(z) = \sum_{L=1}^{\infty}\eta_{n,L}\cdot z^{L-1}.$$

We already know that $\mathbf{H}_0(z)=\frac{1}{2-z}$. Though $m_L(\mathsf{p})$ are obviously highly transcendental (and mysterious) functions, the series for $\mathbf{H}_n(z)$ is in fact a rational function in z, and this is the main point of our approach. Moreover, we will show that

$$\mathbf{H}_n(z) = \frac{\mathscr{B}_n(z)}{(z-2)^{n+1}},$$

where $\mathscr{B}_n(z)$ is a polynomial with rational coefficients of degree $n-1$ with the reciprocity property $\mathscr{B}_n(z+1) = (-1)^n z^{n-1}\mathscr{B}_n(\frac{1}{z}+1)$, $\mathscr{B}_n(0)=0$. We argue by induction on n. First we need an auxiliary lemma.

Let $\mathbb{Q}[z]_{n-1}$ denote the linear space of dimension n of polynomials of degree $\leq n-1$ with rational coefficients. Consider a following linear map $\mathcal{L}_{n-1}:\mathbb{Q}[z]_{n-1}\to\mathbb{Q}[z]_{n-1}$, defined by

$$\mathcal{L}_{n-1}(P)(z) = P(z+1) - \frac{1}{2^{n+1}}P(2z) + \frac{(-1)^{n+1}}{2^{n+1}}P\Big(\frac{2}{z}\Big)z^{n-1}.$$

Lemma 1.22. *$det(\mathcal{L}_{n-1})\neq 0$. Accordingly, $\mathcal{L}_{n-1}$ is the isomorphism.*

Remark. Let $m=\big[\frac{n}{2}\big]$. Then it can be proved that indeed $\det(\mathcal{L}_{n-1}) = \frac{\prod_{i=1}^{m}(4^i-1)}{2^{m^2+m}}$.

Proof. Suppose $P\in\ker(\mathcal{L}_{n-1})$. Then a rational function $\mathbf{H}(z)=\frac{P(z)}{(z-2)^{n+1}}$ satisfies the three term functional equation

$$\mathbf{H}(z+1) - \mathbf{H}(2z) + \mathbf{H}\Big(\frac{2}{z}\Big)\frac{1}{z^2} = 0, \quad z\neq 1. \tag{1.26}$$

Also, $\mathbf{H}(z)=o(1)$, as $z\to\infty$. Now the result follows from the next

Lemma 1.23. *Let $\Upsilon(z)$ be any analytic function in the domain $\mathbb{C}\setminus\{1\}$. Then if $\mathbf{H}(z)$ is a solution of the equation*

$$\mathbf{H}(z+1)-\mathbf{H}(2z)+\mathbf{H}\Big(\frac{2}{z}\Big)\frac{1}{z^2}=\Upsilon(z),$$

such that $\mathbf{H}(z)\to 0$ as $z\to\infty$, $\mathbf{H}(z)$ is analytic in $\mathbb{C}\setminus\{2\}$, then such $\mathbf{H}(z)$ is unique.

Proof. All we need is to show that with the imposed diminishing condition, homogeneous equation (1.26) admits only the solution $\mathbf{H}(z)\equiv 0$. Indeed, let $\mathbf{H}(z)$ be such a solution. Put $z\to 2^n z+1$. Thus,

$$\mathbf{H}(2^n z+2)-\mathbf{H}(2^{n+1}z+2)+\frac{1}{(2^n z+1)^2}\mathbf{H}\Big(\frac{2}{2^n z+1}\Big)=0.$$

This is valid for $z\neq 0$ (since $\mathbf{H}(z)$ is allowed to have a singularity at $z=2$). Now sum this over $n\geq 0$. Due to the diminishing assumption, one gets (after additional substitution $z\to z-2$)

$$\mathbf{H}(z)=-\sum_{n=0}^{\infty}\frac{1}{(2^n z-2^{n+1}+1)^2}\mathbf{H}\Big(\frac{2}{2^n z-2^{n+1}+1}\Big).$$

For clarity, put $z\to -z$ and consider a function $\widehat{\mathbf{H}}(z)=\mathbf{H}(-z)$. Thus,

$$\widehat{\mathbf{H}}(z)=-\sum_{n=0}^{\infty}\frac{1}{(2^n z+2^{n+1}-1)^2}\widehat{\mathbf{H}}\Big(\frac{2}{2^n z+2^{n+1}-1}\Big).$$

Consider this for $z\in[0,2]$. As can be easily seen, then all arguments on the right also belong to this interval. We want to prove the needed result simply by applying the maximum argument. The last identity is still insufficient. For this reason consider its second iteration. This produces a series

$$\widehat{\mathbf{H}}(z)=\sum_{n,m=0}^{\infty}\frac{1}{(2^{n+m+1}z+2^{n+m+2}-2^n z-2^{n+1}+1)^2}\widehat{\mathbf{H}}\Big(\omega_m\circ\omega_n(z)\Big),$$

where $\omega_n(z)=\frac{2}{2^n z+2^{n+1}-1}$. As said, $\omega_m\circ\omega_n(z)\in[0,2]$ for $z\in[0,2]$. Since a function $\widehat{\mathbf{H}}(z)$ is continuous in the interval $[0,2]$, let $z_0\in[0,2]$ be such that $M=|\widehat{\mathbf{H}}(z_0)|=\sup_{z\in[0,2]}|\widehat{\mathbf{H}}(z)|$. Consider the above expression for $z=z_0$. Thus,

$$M=|\widehat{\mathbf{H}}(z_0)|\leq\sum_{n,m=0}^{\infty}\Big|\frac{1}{(2^{n+m+1}z_0+2^{n+m+2}-2^n z_0-2^{n+1}+1)^2}\widehat{\mathbf{H}}\Big(\omega_m\circ\omega_n(z_0)\Big)\Big|\leq$$

$$M\sum_{n,m=0}^{\infty}\frac{1}{(2^{n+m+2}-2^{n+1}+1)^2}=0.20453_{+}M.$$

This is contradictory unless $M=0$. By the principle of analytic continuation, $\mathbf{H}(z)\equiv 0$, and this proves the Lemma. □

Remark. Direct inspection of the proof reveals that the statement of Lemma still holds with a weaker assumption that $\mathbf{H}(z)$ is real-analytic function on $(-\infty, 0]$.

Now, let us differentiate (1.23) n times with respect to p, use (1.24) and afterwards substitute $\mathsf{p} = 2$. This gives

$$\sum_{j=1}^{n} \frac{2}{j!} \frac{\partial^j}{\partial z^j} \mathbf{H}_{n-j}(2z) z^j + \sum_{j=0}^{n-1} \frac{1}{j!} \frac{\partial^j}{\partial z^j} \mathbf{H}_{n-j-1}(2z) z^j -$$

$$\sum_{j=1}^{n} \frac{2}{j!} \frac{\partial^j}{\partial z^j} \mathbf{H}_{n-j}\Big(\frac{2}{z}\Big) \frac{1}{z^{j+2}} - \sum_{j=0}^{n-1} \frac{1}{j!} \frac{\partial^j}{\partial z^j} \mathbf{H}_{n-j-1}\Big(\frac{2}{z}\Big) \frac{1}{z^{j+2}} =$$

$$2\mathbf{H}_n(z+1) - 2\mathbf{H}_n(2z) + 2\mathbf{H}_n\Big(\frac{2}{z}\Big)\frac{1}{z^2}. \tag{1.27}$$

We note that this implies the reciprocity property

$$\mathbf{H}_n(z+1) = -\frac{1}{z^2}\mathbf{H}_n\Big(\frac{1}{z}+1\Big), \quad n \geq 1.$$

A posteriori, this clarifies how the identity $F(x) + F(1/x) = 1$ reflects in the series for $G(z)$, as stated in Theorem 1.2: reciprocity property (non-homogeneous for $n = 0$ and homogeneous for $n \geq 1$) is reflected in each of the summands separately, whereas the three term functional equation heavily depends on inter-relations among $\mathbf{H}_n(z)$.

Now, suppose we know all $\mathbf{H}_j(z)$ for $j < n$.

Lemma 1.24. *The left hand side of the equation (1.27) is of the form*

$$l.h.s. = \frac{\mathscr{J}_n(z)}{(z-1)^{n+1}},$$

where $\mathscr{J}_n(z) \in \mathbb{Q}[z]_{n-1}$.

Proof. First, as it is clear from the appearance of l.h.s., we need to verify that z does not divide a denominator, if l.h.s. is represented as a quotient of two co-prime polynomials. Indeed, using the symmetry property in (1.23) for the term $G(\mathsf{p}, \frac{\mathsf{p}}{z})$, we obtain the three term functional equation of the form

$$-\frac{1}{\mathsf{p}-z} - \frac{\mathsf{p}}{(\mathsf{p}-z)^2} G\Big(\mathsf{p}, \frac{\mathsf{p}}{\mathsf{p}-z}\Big) + 2G(\mathsf{p}, z+1) = \mathsf{p}G(\mathsf{p}, \mathsf{p}z).$$

Let us perform the same procedure which we followed to arrive at the equation (1.27). Thus, differentiation n times with respect to p and substitution $\mathsf{p} = 2$ gives the expression of the form

$$\text{l.h.s.}_2 = 2\mathbf{H}_n(z+1) - 2\mathbf{H}_n(2z) - 2\mathbf{H}_n\Big(\frac{2}{2-z}\Big)\frac{1}{(2-z)^2},$$

where lh.s.$_2$ is expressed in terms of $\mathbf{H}_j(z)$ for $j < n$. Nevertheless, this time the common denominator of l.h.s.$_2$ is of the form $(z-1)^{n+1}(z-2)^{n+2}$. As a corollary, z does not divide it. Finally, due to the reciprocity property, for $n \geq 1$ one has

$$\mathbf{H}_n\Big(\frac{2}{2-z}\Big)\frac{1}{(2-z)^2} = -\mathbf{H}_n\Big(\frac{2}{z}\Big)\frac{1}{z^2}.$$

This shows that actually l.h.s. = l.h.s.$_2$, and therefore if this is expressed as a quotient of two polynomials in lowest terms, the denominator is a power of $(z-1)$. Finally, it is obvious that this exponent is exactly $n+1$, and one easily verifies that deg $\mathscr{J}_n(z) \leq n-1$. (Possibly, $\mathscr{J}_n(z)$ can be divisible by $(z-1)$, but this does not have an impact on the result). $\square$

Proof of Theorem 1.2. Now, using Lemma 1.22, we inherit that there exists a unique polynomial $\mathscr{B}_n(z)$ of degree $\leq n-1$ such that $\mathscr{B}_n(z) = \frac{1}{2}\mathcal{L}_{n-1}^{-1}(\mathscr{J}_n)(z)$. Summarizing, $\mathbf{H}_n(z) = \frac{\mathscr{B}_n(z)}{(z-2)^{n+1}}$ solves the equation (1.27). On the other hand, Lemma 1.23 implies that the solution of (1.27) we obtained is indeed the unique one. This reasoning proves that for $|\,\mathsf{p}-2| \leq 1$, $|z| \leq 3/4$ we have the series

$$G(\,\mathsf{p}, z) = \sum_{n=0}^{\infty}(\,\mathsf{p}-2)^n \cdot \mathbf{H}_n(z).$$

This finally establishes the validity of Theorem 1.2. Note also that each summand satisfies the symmetry property. The series converges absolutely for any z, $|z| \leq 3/4$, and if this holds for z, the same does hold for $\frac{z}{z-1}$, which gives the circle $|z+9/7| \leq 12/7$. $\square$

Curiously, one could formally verify that the function defined by this series does indeed satisfy (1.22). Indeed, using (1.27), we get:

$$2G(\,\mathsf{p}, z+1) = 2\mathbf{H}_0(z+1) + 2\sum_{n=1}^{\infty}(\,\mathsf{p}-2)^n\mathbf{H}_n(z+1) =$$

$$2\mathbf{H}_0(z+1) + \sum_{n=1}^{\infty}(\,\mathsf{p}-2)^n\Bigg(\sum_{j=0}^{n}\frac{2}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_{n-j}(2z)z^j + \sum_{j=0}^{n-1}\frac{1}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_{n-j-1}(2z)z^j -$$
$$\sum_{j=0}^{n}\frac{2}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_{n-j}\Big(\frac{2}{z}\Big)\frac{1}{z^{j+2}} - \sum_{j=0}^{n-1}\frac{1}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_{n-j-1}\Big(\frac{2}{z}\Big)\frac{1}{z^{j+2}}\Bigg)$$

Denote $n-j=s$. Then interchanging the order of summation for the first term of the sum in the brackets, we obtain:

$$2\sum_{n=1}^{\infty}(\,\mathsf{p}-2)^n\sum_{j=0}^{n}\frac{1}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_{n-j}(2z)z^j = 2\sum_{s=0}^{\infty}\sum_{j=0}^{\infty}(\,\mathsf{p}-2)^{j+s}\frac{1}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_s(2z)z^j - 2\mathbf{H}_0(2z) =$$
$$2\sum_{s=0}^{\infty}(\,\mathsf{p}-2)^s\mathbf{H}_s(2z+(\,\mathsf{p}-2)z) - 2\mathbf{H}_0(2z) = 2G(\,\mathsf{p},\,\mathsf{p}z) - 2\mathbf{H}_0(2z).$$

The same works for the second sum:

$$\sum_{n=1}^{\infty}(\,\mathsf{p}-2)^n\sum_{j=0}^{n-1}\frac{1}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_{n-j-1}(2z)z^j = (\,\mathsf{p}-2)G(\,\mathsf{p},\,\mathsf{p}z).$$

We perform the same interchange of summation for the second and the third summand respectively. Thus, this yields

$$2G(\mathsf{p}, z+1) = \mathsf{p}G(\mathsf{p}, \mathsf{p}z) - \frac{\mathsf{p}}{z^2}G\Big(\mathsf{p}, \frac{\mathsf{p}}{z}\Big) + 2\mathbf{H}_0(z+1) - 2\mathbf{H}_0(2z) + \frac{2}{z^2}\mathbf{H}_0\Big(\frac{2}{z}\Big) = \mathsf{p}G(\mathsf{p}, \mathsf{p}z) - \frac{\mathsf{p}}{z^2}G\Big(\mathsf{p}, \frac{\mathsf{p}}{z}\Big) - \frac{1}{z}.$$

On the other hand, it is unclear how one can make this argument to work. This would require rather detailed investigation of the linear map $\mathcal{L}_{n-1}$ and recurrence (1.27), and this seems to be very technical.

1.7 Appendix

1.7.1 Approach through $\mathsf{p} = 0$

With a slight abuse of notation, we will use the expression $\frac{\partial^s}{\partial \mathsf{p}^s}G(0, z)$ to denote $\frac{\partial^s}{\partial \mathsf{p}^s}G(\mathsf{p}, z)\big|_{\mathsf{p}=0}$ for $s \in \mathbb{N}_0$. Though the function $G(\mathsf{p}, z)$ is defined only for $\Re\,\mathsf{p} \geq 1$, $z \notin (\mathscr{I}_{\mathsf{p}} + 1)$, assume that we are able to prove that it is analytic in p in a certain wider domain containing a disc $|\mathsf{p}| < \varpi$, $\varpi > 0$. These are only formal calculations, but they unexpectedly yield series (1.7) (see Section 1), and numerical calculations do strongly confirm the validity of it.

Thus, substitution $\mathsf{p} = 0$ into (1.23) gives $G(0, z) = \frac{1}{2(1-z)}$. Partial differentiation of (1.23) with respect to p, and consequent substitution $\mathsf{p} = 0$ gives

$$\frac{1}{z^2}G(0,0) + 2\frac{\partial}{\partial \mathsf{p}}G(0, z+1) = G(0,0) \Rightarrow \frac{\partial}{\partial \mathsf{p}}G(0, z) = \frac{(z-1)^2 - 1}{4(z-1)^2}.$$

In the same fashion, differentiating the second time, we obtain $\frac{\partial^2}{\partial \mathsf{p}^2}G(0, z) = \frac{(z-1)^4-1}{2(z-1)^3}$. In general, differentiating (1.23) $n \geq 1$ times with respect to p, using (1.24), and substituting $\mathsf{p} = 0$, we obtain:

$$2\frac{\partial^n}{\partial \mathsf{p}^n}G(0, z+1) = \sum_{i+j=n-1} n\binom{n-1}{j}\frac{\partial^i\,\partial^j}{\partial \mathsf{p}^i\,\partial z^j}G(0,0)\Big(z^j - \frac{1}{z^{j+2}}\Big).$$

Let

$$\frac{1}{n!} \cdot \frac{\partial^n}{\partial \mathsf{p}^n}G(0, z) = \overline{\mathbf{Q}}_n(z).$$

Then

$$2\overline{\mathbf{Q}}_n(z+1) = \sum_{j=0}^{n-1}\frac{1}{j!}\frac{\partial^j}{\partial z^j}\overline{\mathbf{Q}}_{n-j-1}(0)\Big(z^j - \frac{1}{z^{j+2}}\Big).$$

Consequently, we have a recurrent formula to compute rational functions $\overline{\mathbf{Q}}(z)$. Let $\mathbf{Q}_n(z) = \overline{\mathbf{Q}}_n(z+1)$. Thus,

$$\mathbf{Q}_n(z) = \frac{(z+1)(z-1)\mathscr{D}_n(z)}{z^{n+1}}, \quad n \geq 1,$$

where $\mathscr{D}_n$ are polynomials of degree $2n-2$ with the reciprocity property $\mathscr{D}_n(z) = z^{2n-2}\mathscr{D}_n\left(\frac{1}{z}\right)$ (this is obvious from the recurrence relation which defines $\mathbf{Q}_n(z)$). Moreover, the coefficients of $\mathscr{D}_n$ are $\mathbb{Q}_p$ integers for any prime $p \neq 2$. These calculations yield a following formal result:

$$G(\mathsf{p}, z) \text{``} = \text{''} \sum_{n=0}^{\infty} \mathsf{p}^n \cdot \mathbf{Q}_n(z-1) = \sum_{n=0}^{\infty} \mathsf{p}^n \frac{z(z-2)\mathscr{D}_n(z-1)}{(z-1)^{n+1}}.$$

This produces the "series" for the second and higher moments of the form

$$m_2(\mathsf{p}) = \mathsf{p}^2 \cdot \sum_{n=0}^{\infty} \mathsf{p}^n \mathbf{Q}'_n(-1).$$

In particular, inspection of the table in Section 1 (where the initial values for $\mathbf{Q}'_n(-1)$ are listed) shows that this series for $\mathsf{p} = 1$ does not converge. However, the Borel sum is properly defined and it converges exactly to the value m_2. This gives empirical evidence for the validity of (1.7). The principles of Borel summation also suggest the mysterious fact that indeed $G(\mathsf{p}, z)$ analytically extends to the interval $\mathsf{p} \in [0, 1]$.

Additionally, numerical calculations reveal the following fact: the sequence $\sqrt[n]{|\mathbf{Q}'_n(-1)|}$ is monotonically increasing (apparently, tends to ∞), while $\frac{1}{n}\log|\mathbf{Q}'_n(-1)| - \log n$ monotonically decreases (apparently, tends to $-\infty$). Thus,

$$A^n < |\mathbf{Q}'_n(-1)| < (cn)^n,$$

for $c = 0.02372$ and $A = 3.527$, $n \geq 150$. We do not have enough evidence to conjecture the real growth of this sequence. If $c = c(n) \to 0$, as $n \to \infty$, then the function

$$\Lambda(t) = \sum_{n=0}^{\infty} \frac{\mathbf{Q}'_n(-1)}{n!} t^n$$

is entire, and in case $L = 2$, result (1.7) is equivalent to the fact that

$$\int_0^{\infty} \Lambda(t) e^{-t}\,\mathrm{d}t = m_2.$$

1.7.2 Auxiliary lemmas

These lemmas are needed in Section 3. For $a, b \in \mathbb{N}$, $\mathsf{p} \in \mathbb{C}$, $|\mathsf{p} - 2| \leq 1$, define rational functions

$$W_a(\mathsf{p}) = \frac{\mathsf{p}^a - 1}{\mathsf{p}^{a+1} - \mathsf{p}^a}, \quad T_{a,b}(\mathsf{p}) = W_a^{-1}(\mathsf{p}) W_b^{-1}(\mathsf{p})\,\mathsf{p}^{-a} = \frac{(\mathsf{p}-1)^2\,\mathsf{p}^b}{(\mathsf{p}^a - 1)(\mathsf{p}^b - 1)}.$$

Let us define constants

$$\mu(a, b) = \sup_{\mathsf{p} \in \mathbb{C}, |\mathsf{p}-2| \leq 1} |T_{a,b}(\mathsf{p})| - \Re(T_{a,b}(\mathsf{p})).$$

The following table provides some initial values for constants $\mu(a, b)$, computed numerically.

$b \setminus a$	1	2	3	4	5	6
1	0.25000000	0.01250000	0.00780868	0.03343231	0.05778002	0.07712952
2	0.29846114	0.03125000	0.00159908	0.01212467	0.02539758	0.03645721
3	0.35999295	0.05097235	0.00647895	0.00676996	0.01624300	0.02437494
4	0.41433340	0.07007201	0.01316542	0.00500146	0.01287728	0.01963810
5	0.45590757	0.08747624	0.02069451	0.00437252	0.01163446	0.01781467
6	0.48390408	0.10255189	0.02845424	0.00812804	0.01125132	0.01728395
7	0.49985799	0.11503743	0.03601828	0.01200557	0.01120308	0.01729854
8	0.50642035	0.12494927	0.04309384	0.01611126	0.01125789	0.01748823
9	0.50681483	0.13248892	0.04949922	0.02025219	0.01132055	0.01767914
10	0.50452450	0.13796512	0.05514483	0.02427779	0.01136245	0.01780892
11	0.50218322	0.14173414	0.06001269	0.02807992	0.01138335	0.01787452
12	0.50070286	0.14415527	0.06413550	0.03158969	0.01139099	0.01789618
13	0.49999979	0.14555794	0.06757752	0.03477145	0.01139235	0.01789583
14	0.49977304	0.14622041	0.07041891	0.03761547	0.01139159	0.01788837
15	0.49977361	0.14636154	0.07274403	0.04013040	0.01139057	0.01788111
...	...	...	...	...	...	...
∞	0.50000000	0.12500000	0.05479177	0.03097495	0.01138938	0.01787406

Note that there exists $\lim_{b\to\infty} \mu(a,b)$, and $\mu(a,b) \to 0$ uniformly in b, as $a \to \infty$. Thus, the table above and some standard evaluations give the following

Lemma 1.25. *Let $a, b, c \in \mathbb{N}$. Then*

$$\mu(a,b) + \mu(b,c) \le \mu(1,1) + \mu(1,9) < 0.76. \quad \square$$

Lemma 1.26. *There exists an absolute constant $c > 0$ such that for all $\mathsf{p} \in \mathbb{C}$, $\Re \mathsf{p} \ge 1$, and all $a \in \mathbb{N}$, on has $\left|\frac{\mathsf{p}^a - 1}{\mathsf{p}-1}\right| > c$.*

Proof. Consider a contour, consisting of the segment $[1 - iT, 1 + iT]$, and a semicircle $1 + Te^{i\phi}$, $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$. For sufficiently large T, $\frac{\mathsf{p}^a-1}{\mathsf{p}-1}$ will be large on the semicircle. Moreover, this function never vanishes inside or on the contour. Thus, from the maximum-minimum principle, its minimal absolute value is obtained on the segment $[1 - iT, 1 + iT]$. Thus, let $\mathsf{p} = \frac{1}{\cos\psi} e^{i\psi}$, $-\frac{\pi}{2} < \psi < \frac{\pi}{2}$. Without loss of generality, let $\psi \ge 0$. Consider the case $\frac{\pi}{2a} \le \psi < \frac{\pi}{2}$. Then

$$\left|\frac{\mathsf{p}^a - 1}{\mathsf{p} - 1}\right|^2 = \frac{\frac{1}{\cos^{2a}\psi} - \frac{2\cos a\psi}{\cos^a\psi} + 1}{\frac{1}{\cos^2\psi} - 1} \ge \frac{\frac{1}{\cos^{2a}\psi} - \frac{2}{\cos^a\psi} + 1}{\frac{1}{\cos^2\psi} - 1} = \frac{(\rho^a - 1)^2}{\rho^2 - 1} := Y(\rho), \quad \rho = \frac{1}{\cos\psi}.$$

The function $Y(\rho)$ is an increasing function in ρ for $\rho \ge 1$. It is obvious that we may consider a case of a sufficiently large. Thus,

$$\begin{aligned}\Big|\frac{\mathsf{p}^a-1}{\mathsf{p}-1}\Big|^2 &\geq Y\Big(\frac{1}{\cos\frac{\pi}{2a}}\Big)=\frac{\big(\frac{1}{\cos^a\frac{\pi}{2a}}-1\big)^2}{\tan^2\frac{\pi}{2a}}\\ &= \frac{\Big((1+\frac{\pi^2}{8a^2}+\frac{\mathcal{O}(1)}{a^3})^a-1\Big)^2}{\frac{\pi^2}{4a^2}+\frac{\mathcal{O}(1)}{a^3}}=\frac{\frac{\pi^4}{64a^2}+\frac{\mathcal{O}(1)}{a^3}}{\frac{\pi^2}{4a^2}+\frac{\mathcal{O}(1)}{a^3}}=\frac{\pi^2}{16}+\frac{\mathcal{O}(1)}{a}.\end{aligned}$$

Let now $0\leq\psi<\frac{\pi}{2a}$. First, consider a function $\frac{1}{y}\log\cos(y\psi):=V(y)$. It is a decreasing function for $0<y<\frac{\pi}{2\psi}$. Indeed, this is equivalent to the inequality

$$-\tan x\cdot x-\log\cos x<0,\text{ for }0<x<\frac{\pi}{2}.$$

The function on the left is itself a decreasing function, with maximum value attained at $x=0$. Thus, $V(1)\geq V(a)$, which means $\cos a\psi\leq\cos^a\psi$, and this gives

$$\Big|\frac{\mathsf{p}^a-1}{\mathsf{p}-1}\Big|^2\geq\frac{\frac{1}{\cos^{2a}\psi}-1}{\frac{1}{\cos^2\psi}-1}\geq 1.\quad\square$$

Therefore, Lemma 1.26 implies that the function $\mathsf{p}^{-1}W_a^{-1}(\mathsf{p})$ is uniformly bounded:

$$\sup_{a\in\mathbb{N},|\mathsf{p}-2|\leq 1}|\mathsf{p}^{-1}W_a^{-1}(\mathsf{p})|=c_0<+\infty.$$

This shows the validity of the following Lemma (apart from a numerical bound, which is the outcome of computer calculations).

Lemma 1.27. *One has*

$$\sup_{|\mathsf{p}-2|\leq 1,a\in\mathbb{N},|z-1|\leq 1}|\mathsf{p}^{-1}W_a^{-1}(\mathsf{p})z-1|<1.29.\quad\square$$

1.7.3 Numerical values for the moments

Unfortunately, Corollary 1.4 is not very useful in finding exact decimal digits of m_2. In fact, the vector $(m_1,m_2,m_3...)$ is the solution of an (infinite) system of linear equations, which encodes the functional equation (2.6) (see [1], Proposition 6). Namely, if we denote $c_L=\sum_{n=1}^{\infty}\frac{1}{2^n n^L}=\mathrm{Li}_L(\frac{1}{2})$, we have a linear system for m_s which describes the coefficients m_s uniquely:

$$m_s=\sum_{L=0}^{\infty}(-1)^L c_{L+s}\binom{L+s-1}{s-1}m_L,\quad s\geq 1.$$

Note that this system is not homogeneous ($m_0=1$). We truncate this matrix at sufficiently high order to obtain float values. The accuracy of this calculation can be checked on the test value $m_1=0.5$. This approach yields (for the matrix of order 325):

$$\begin{aligned}m_2 &= 0.290926476429308736380697762739120290080437102195594366549 2_{+},\\ m_3 &= 0.186389714643963104571046644108680435120655653293391549823 8_{+}\\ m_4 &= 0.126992258407443135202892227880211638841185145761725718101 6_{+}.\end{aligned}$$

with all 58 digits exact (note that $3m_2 - 2m_3 = 0.5$). In fact, the truncation of this matrix at an order 325 gives rather accurate values for m_L for $1 \leq L \leq 125$, well in correspondence with an asymptotic formula [3]

$$m_L = \sqrt[4]{4\pi^2 \log 2} \cdot c_0 \cdot L^{1/4} \mathsf{C}^{\sqrt{L}} + O(\mathsf{C}^{\sqrt{L}} L^{-1/4}), \tag{1.28}$$

where $c_0 = \int_0^1 2^x(1 - F(x))\,\mathrm{d}x = 1.030199563382_+$, and $\mathsf{C} = e^{-2\sqrt{\log 2}}$. So obtained numerical values for higher moments tend to deviate from this expression rather quickly.

Kinney [19] proved that the Hausdorff dimension of growth points of $?(x)$ is equal to

$$\alpha = \frac{1}{2}\Big(\int\limits_0^1 \log_2(1+x)\,\mathrm{d}?(x)\Big)^{-1}.$$

Lagarias [22] gives the following estimates: $0.8746 < \alpha < 0.8749$. Tichy and Uitz [36] calculated $\alpha \approx 0.875$. Paradis et al. [29] give the value $\alpha \approx 0.874832$. We have (note that $?(1-x)+?(x) = 1$):

$$A := \int\limits_0^1 \log(1+x)\,\mathrm{d}?(x) = \int\limits_0^1 \log\Big(1 - \frac{1-x}{2}\Big)\,\mathrm{d}?(x) + \int\limits_0^1 \log 2\,\mathrm{d}?(x) =$$

$$-\sum_{L=1}^{\infty} \frac{1}{L \cdot 2^L} \int\limits_0^1 (1-x)^L\,\mathrm{d}?(x) + \log 2 = -\sum_{L=1}^{\infty} \frac{m_L}{L \cdot 2^L} + \log 2.$$

Thus, we are able to present much more precise result:

$$\alpha = \frac{\log 2}{2A} = 0.874716305108211142215152904219159757...$$

with all 36 digits exact. The author of this thesis have contacted the authors of [29] inquiring about the error bound for the numerical value of α they obtained. It appears that for this purpose 10 generations of (1.2) were used. The authors of [29] were very kind in agreeing to perform the same calculations with more generations. Thus, if one uses 18 generations, this gives $0.874716 < \alpha < 0.874719$.

Additionally, the constant c_0 in (1.28) is given by

$$c_0 = \int\limits_0^1 2^x(1 - F(x))\,\mathrm{d}x = \frac{\mathfrak{m}(\log 2)}{2\log 2} = \frac{1}{2}\sum_{L=0}^{\infty} \frac{m_L}{L!}(\log 2)^{L-1}.$$

This series is fast convergent, and we obtain

$$c_0 = 1.030199563382694623156004112564478676694158859182 40...$$

1.7.4 Rational functions $\mathbf{H}_n(z)$

The following is MAPLE code to compute rational functions $\mathbf{H}_n(z)$=`h[n]` and coefficients $\mathbf{H}'_n(0)$=`alpha[n]` for $0 \leq n \leq 50$.

```
> restart;
> with(LinearAlgebra):
> U:=50:
> h[0]:=1/(2-z):
> for n from 1 to U do
>     j[n]:=1/2*simplify(
>         add( unapply(diff(h[n-j],z$j),z)(2*z)*2/j!*(z^(j)),j=1..n)+
>         add( unapply(diff(h[n-j-1],z$j),z)(2*z)*1/j!*(z^(j)),j=1..n-1)+
>              unapply(h[n-1],z)(2*z)    ):
>     k[n]:=simplify((z-1)^(n+1)*(unapply(j[n],z)(z)-
>                                 unapply(j[n],z)(1/z)/z^2)):
>     M[n,1]:=Matrix(n,n):M[n,2]:=Matrix(n,n): M[n,3]:=Matrix(n,n):
>           for tx from 1 to n do for ty from tx to n do
>                     M[n,1][ty,tx]:=binomial(n-tx,n-ty)
>                          end do: end do:
>           for tx from 1 to n do M[n,2][tx,tx]:=2^(n-tx) end do:
>           for tx from 1 to n do M[n,3][tx,n+1-tx]:=2^(tx-1) end do:
>     Y[n]:=M[n,1]-1/2^(n+1)*M[n,2]+(-1)^(n+1)/2^(n+1)*M[n,3]:
>     A[n]:=Matrix(n,1):
>           for tt from 1 to n do A[n][tt,1]:=coeff(k[n],z,n-tt) end do:
>     B[n]:=MatrixMatrixMultiply(MatrixInverse(Y[n]),A[n]):
>     h[n]:=add(z^(n-s)*B[n][s,1](s,1),s=1..n)/(z-2)^(n+1):
>            end do:
>
> for n from 0 to U do alpha[n]:=unapply(diff(h[n],z$1),z)(0) end do;
```

It causes no complications to compute `h[n]` on a standard home computer for $0 \leq n \leq 60$, though the computations heavily increase in difficulty for $n > 60$.

1.7.5 Rational functions $\mathbf{Q}_n(z)$

This program computes $\mathbf{Q}_n(z) =$`q[n]` and the values
$\mathbf{Q}'_n(-1) =$`beta[n]` for $0 \leq n \leq 50$.

```
> restart;
>q[0]:=-1/(2*z);
>N:=50:
>q[1]:=simplify(1/2*unapply(q[0],z)(-1)*(1-1/z^2)):
> for n from 1 to N do
>   q[n]:=1/2*simplify(
>       add(unapply(diff(q[n-j-1],z$j),z)(-1)/j!*(z^(j)-1/z^(j+2)),j=1..n-1)+
```

```
>       unapply(q[n-1],z)(-1)*(1-1/z^2)
>                      ):
                end do:
> for w from 0 to N do beta[w]:=unapply(diff(q[w],z$1),z)(-1) end do;
```

Chapter 2

Functional equation related to forms

2.1 The formulation of the problem

The questions considered in this article find there origin in the following problem, which appeared in "American Mathematical Monthly" (1991) **8**, Problem E 3458:

Find all functions $f : \mathbb{N} \to \mathbb{C}$, satisfying the following integer functional equation:

$$f(n^2 + m^2) = f^2(n) + f^2(m) \text{ for all } n, m \in \mathbb{N}, \quad n, m \geq n_0. \tag{2.1}$$

If $n_0 = 0$, the problem is simply an exercise in elementary number theory (for the discussion of this see the popular article [44]). The case $n_0 = 1$ can be dealt in the similar manner, though the proof is longer and more tedious. For arbitrary n_0 the problem can be solved using another method: first we need to derive from (2.1) (or related equation) a linear recurrence relation satisfied by $f^2(n)$, and then to employ the formal generating power series. This related functional equation appears naturally: if the function f satisfies (2.1), it certainly implies at least that $f^2(n) + f^2(m)$ depends only on the value of $n^2 + m^2$. Hence, first task is to solve related (and, as it appears, much more interesting) problem:

Find all functions $f : \mathbb{N} \to \mathbb{C}$, satisfying the following integer functional equation:

$$a^2 + b^2 = c^2 + d^2 \Rightarrow f^2(a) + f^2(b) = f^2(c) + f^2(d) \text{ for integers } a, b, c, d \geq n_0. \tag{2.2}$$

These sort of questions are discussed in paper [42]. The author introduces the notion of $(a, c)-$ square additive function: a function $G : \mathbb{N} \to \mathbb{R}$ is called $(a, c)-$square additive, if the following is satisfied:

$$ax^2 + cy^2 = au^2 + cv^2 \Rightarrow aG(x) + cG(y) = aG(u) + cG(v) \text{ for all } x, y, u, v \in \mathbb{N}. \tag{2.3}$$

Here we may also demand that the relation is satisfied only for $x, y, u, v \geq n_0$ - this does not give any new essential difficulties in the solution. The $(1, 1)$- square additive functions

are called simply square additive. Then this condition is, of course, nothing else than (2.2) for $G(n) = f^2(n)$ (without loss of generality, complex numbers can certainly be replaced by real, due to linearity). The author proves, for example, among other things, that:
(i) There are six linearly independent square additive functions;
(ii) Every square additive function satisfies the linear recurrence $G(x+12) = G(x+9) + G(x+8) + G(x+7) - G(x+5) - G(x+4) - G(x+3) + G(x)$ and no linear recurrence of degree less than 12 is satisfied by all square-additive functions.
(iii) If $a > c$ are coprime positive integers, then the set of $(a,c)-$ square additive functions forms a finite dimensional vector space over $\mathbb{R}$;
The first two propositions are already contained in the solution of the problem in discussion in "American Mathematical Monthly". With the kind permission of the author U. Zannier, we reproduce it here, since this unpublished manuscript was written 6 years earlier. The author of [42] also introduces the notion of Pythagorean or $P-$additive functions. A function $G : \mathbb{N} \to \mathbb{R}$ is called a $P-$additive, if $z^2 = x^2 + y^2 \Rightarrow G(z) = G(x) + G(y)$ for integers x, y, z. Then there are at least 17 linearly independent $P-$additive functions. To the addition of the results in [42], in the paper [41] it is proved that if G is a $P-$additive and periodic, then only the primes $2, 3, 5$ and 13 can be periods. This question also plays a certain role in our generalisation for the above task to other quadratic forms (mainly deciding what are the possible periods of the function $g(n)$ in Theorem 3.3 - see below).

These questions give therefore impetus for posing more general problems. Here incidentally only quadratic forms are involved. Naturally, we can ask the same question for integer forms in several variables. In some cases we confine ourselves to norm forms in number fields. There are some reasons for that, and these are explained in the end. Hence we can formulate two main problems:

Problem 1. *Let $T(a_1, a_2, ..., a_n)$ be a norm form in some integral basis of some proper field extension of $\mathbb{Q}$ of degree n. Find all functions $f : \mathbb{Z} \to \mathbb{C}$, such that*

$$f(T(a_1, a_2, ..., a_n)) = T(f(a_1), f(a_2), ..., f(a_n)). \tag{2.4}$$

Problem 2. *Let $T(a_1, a_2, ..., a_n)$ be a norm form in some integral basis of some proper field extension of $\mathbb{Q}$ of degree n. Find all functions $f : \mathbb{Z} \to \mathbb{C}$, such that*

$$T(f(a_1), f(a_2), ..., f(a_n)) \text{ depends only on the value of } T(a_1, a_2, ..., a_n). \tag{2.5}$$

Here we choose norm forms as slightly more convenient. In fact, the second question can be formulated for any irreducible quadratic form with all three coefficients coprime.
This chapter is organised as follows. In Section 2 we give the full solution for both questions for the form $X^2 + Y^2$. In Section 3 we introduce one special type of quadratic forms $q^2X^2 + (q^2 - 2p^2)XY + 2p^2Y^2$, prove one result on solutions of functional equation

$G(aX+bY)+G(bX-aY)=G(aX-bY)+G(bX+aY)$, where $G:\mathbb{Z}\to\mathbb{C}$. This result allows us to deduce some important information for Problem 2 for these forms, though it is interesting in itself. In the second paper on this subject we will apply these results about Problem 2 to some concrete examples, such as, quadratic form $X^2+XY+2Y^2$. In Section 4 the outline for possible solution for general quadratic norm forms is presented, though its implementation would lead to some tiresome calculations. Section 5 is devoted for the complete solution of the first problem for one cubic field, generated by the root α of the polynomial X^3-3X+1, and with integral basis $\{1,\alpha,\alpha'\}$, where α' is another root of this polynomial. Finally, in Section 6 we give corresponding conjecture about the first problem, having enough evidence for this. Further, several other questions concerning the second problem are given, and moreover several remarks concerning our choice of norm forms are presented. For reference we formulate the second question for the quadratic forms separately.

Problem 2'. *Find all functions $f:\mathbb{Z}\to\mathbb{C}$, such that for all integers X,Y,W,Z*

$$\begin{aligned} uX^2+vXY+wY^2 &= uZ^2+vZW+wW^2 \Rightarrow \\ uf^2(X)+vf(X)f(Y)+wf^2(Y) &= uf^2(Z)+vf(Z)f(W)+wf^2(W), \end{aligned} \tag{2.6}$$

where $uX^2+vXY+wY^2$ is an irreducible quadratic form.

2.2 Solution for one Gaussian quadratic form

As mentioned in the Section 1, here we reproduce without changes the solution for both questions for a special Gaussian form X^2+Y^2. We use the unpublished manuscript of U. Zannier (1992), with his kind permission. The result of the Proposition 3.1 is contained in [42], though the latter paper was published 6 years later. This result is also in fact a partial case of Theorem 3.3.

Proposition 2.1. *Let a function $G:\mathbb{N}\to\mathbb{C}$ satisfies the relation: whenever $a^2+b^2=c^2+d^2$ for integers $a,b,c,d\geq n_0$, one has $G(a)+G(b)=G(c)+G(d)$. Then*

$$G(n)=An^2+B+C(-1)^n+D(i^n+i^{-n})+E\Big(\frac{n}{3}\Big)^2+F\Big(\frac{n}{5}\Big),$$

where A,B,C,D,E,F are some complex constants.

Here $\left(\frac{n}{p}\right)$ stands for a usual Legendre symbol, and we use a square of it for brevity simply to express that it is 1 unless $p|n$, when it is 0.

Proof. We have the following identity: $(2n+r)^2+(n-2r)^2=(2n-r)^2+(n+2r)^2$. This identity implies, for large n and fixed r:

$$G(2n+r)+G(n-2r)=G(2n-r)+G(n+2r). \tag{2.7}$$

Set first $r = 1$, then $n-1$, n, $n+1$ in place of n and sum the resulting three equations, obtaining $G(2n{+}3){+}G(n{-}1){+}G(n{-}2){+}G(n{-}3){=}G(2n{-}3){+}G(n{+}1){+}G(n{+}2){+}G(n{+}3)$. set $r = 3$ in (2.7) and subtract from the last equality. We get

$$\begin{aligned} G(n+6) &+ G(n-1) + G(n-2) + G(n-3) = \\ G(n-6) &+ G(n+1) + G(n+2) + G(n+3). \end{aligned}$$

Let $H(x) = \sum_{n=0}^{\infty} G(n)x^n$. Thus, we are working in the field $\mathbb{C}((x))$. The above identity means that $P(x)H(x)$ is a polynomial, where

$$P(x) = x^{12} - x^9 - x^8 - x^7 + x^5 + x^4 + x^3 - 1 = (x^3 - 1)(x^4 - 1)(x^5 - 1).$$

Hence, $H(x)$ is a rational function with denominator $P(x)$. Expanding it into partial fractions we readily obtain, in view of the factorisation of $P(x)$, that for large n the following holds:

$$G(n) = An^2 + Ln + B + h_3(n) + h_4(n) + h_5(n), \tag{2.8}$$

where $h_j(n)$ is a periodic function of period j. Plugging this into (2.7) we readily find $L = 0$ and, due to the fact that $3, 4, 5$ are coprime in pairs, we deduce that h_j satisfies (2.7) in place of G, for $j = 3, 4, 5$. A brief inspection shows the exact appearance of each of these periodic functions, which completes the proof of Proposition 3.1, and therefore the second question for a form $X^2 + Y^2$. Using this result, we can solve the first question.

Proposition 2.2. *Let $f : \mathbb{N} \to \mathbb{C}$ satisfies $f(a^2 + b^2) = f^2(a) + f^2(b)$ for all integers $a, b \geq n_0$. Then, for large n, either $f^2(n) = n^2$, either $f(n) \equiv 0$ or $f^2(n) \equiv 1/4$.*

Proof. Formula (2.8) together with $L = 0$ are anyway sufficient for the proof of this statement, the precise form of h_j being immaterial for our purpose. In fact, function $G(n) = f^2(n)$ satisfies the above hypotheses and whence, for large n

$$f^2(n) = An^2 + h(n), \tag{2.9}$$

$h(n)$ being a periodic function with period 60 (we can of course incorporate the constant B into this periodic function). Now, fix $b, c \geq n_0$. Set then $a = 60n + c$. Due to the condition on f and (2.9), the function in n

$$P(n) = f^2(a^2 + b^2) = A((60n + c)^2 + b^2)^2 + h(a^2 + b^2)$$

is the square of the polynomial in n

$$Q(n) = f^2(a) + f^2(b) = A(60n + c)^2 + h(c) + f^2(b).$$

Suppose now, that f is unbounded, that is, $A \neq 0$. Denoting the first square in the expression of P by R^2, we derive $(\sqrt{A}R)^2 + h(a^2 + b^2) = Q^2$. Since both R and Q are quadratic polynomials, we verily derive $h(a^2 + b^2) = 0$, whence

$$A(x^2 + b^2)^2 = (Ax^2 + h(c) + f^2(b))^2.$$

So $A = 1$ and, since the left hand side is independent of c, h must be constant, necessarily equal to 0 (since already $h(a^2 + b^2) = 0$). This completes the proof for the case $A \neq 0$. Formula (2.8) enables one to deal also with the case $A = 0$: straightforward inspection shows that then $f^2(n)$ is necessarily a constant (0 or 1/4) for large n. This completes the proof of Proposition 3.2, and hence the first problem for the form $X^2 + Y^2$.

2.3 One special type of quadratic forms

As mentioned, in [42] it is proved that all functions $G : \mathbb{N} \to \mathbb{C}$, satisfying (2.3) with a and c coprime, form a finite dimensional vector space over $\mathbb{C}$. Here we prove analogous result for one special type of quadratic forms. We will begin with introductory notes explaining why we deal here with this special type.

Let quadratic form $uX^2 + vXY + wY^2$ with $v \neq 0$ be given, satisfying the identity

$$u(aX + bY)^2 - v(aX + bY)(aX - bY) + w(aX - bY)^2 = uX^2 + vXY + wY^2 \quad (2.10)$$

for certain rational a and b. If function f satisfies (2.6), then it implies $f(X)f(-Y) = f(X)f(-Y)$ (since $u+v+w \neq 0$, otherwise $v^2 - 4uv$ is a perfect square). Then, first taking the equation

$$f^2(aX+bY)+vf(-aX-bY)f(aX-bY)+wf^2(aX+aY) = uf^2(X)+vf(X)f(Y)+wf^2(Y),$$

second, the same equation with $-Y$ instead of Y, and third, two more equations obtained from these exchanging the roles of X and Y, and adding all four with suitable sign, we thus obtain (minding identity $f^2(X) = f^2(-X)$):

$$(u - w)\Big(f^2(aX + bY) - f^2(aX - bY) - f^2(bX + aY) + f^2(bX - aY)\Big) = 0. \quad (2.11)$$

Thus, unless $u - w = 0$, we will be able to treat this identity in the similar manner as in Section 2, first multiplying of course (2.10) by a square of the common denominator of a and b.

In order to exist such rational a and b, satisfying the identity (2.10), the following system of linear equations should have non-zero solution:

$$\begin{cases} (a^2 - 1)u - a^2v + a^2w &= 0 \\ b^2u + b^2v + (b^2 - 1)w &= 0 \\ 2abu - v - 2abw &= 0. \end{cases}$$

The determinant is $-(2ab-1)(2ab+a+b+1)(2ab-a-b+1)$. If second or third multiplier is 0, then corresponding solutions u, v and w give quadratic form, which is degenerate. The remaining case $a = 1/2b$ gives $u = r$, $v = (1 - 2b^2)r$ and $w = 2b^2r$. Since here $u \neq w$, thus we obtain a special type of quadratic forms

$$q^2X^2 + (q^2 - 2p^2)XY + 2p^2Y^2,$$

(p and q are coprime integers, q odd) for which this trick works with $a = q^2$ and $b = 2p^2$: that is, starting from the identity

$$\begin{aligned} q^2(q^2X+2p^2Y)^2 \quad &- \quad (q^2-2p^2)(q^2X+2p^2Y)(q^2X-2p^2Y)+2p^2(q^2X-2p^2Y)^2 = \\ q^2(2pqX)^2 \quad &+ \quad (q^2-2p^2)(2pqX)(2pqY)+2p^2(2pqY)^2, \end{aligned}$$

we thus obtain in the above fashion the identity (2.11). This type includes, for example, quadratic forms, $X^2 - XY + 2Y^2$ with discriminant $D = -7$; $9X^2 + XY + 8Y^2$ with $D = -71$; $X^2 - 7XY + 8Y^2$ with $D = 17$. These forms are not generally norm forms for negative discriminant, excluding the first case, though, as mentioned, Problem 2' can be formulated for them as well.

Now let $f^2(x) = G(x)$. Thus, we have the functional equation:

$$G(aX+bY)+G(bX-aY) = G(aX-bY)+G(bX+aY). \tag{2.12}$$

Theorem 2.3. *Suppose a function $G : \mathbb{N} \to \mathbb{C}$ satisfies (2.12) for certain coprime positive integers a and b, one of them being even. Then $G(n) = g(n) + An^2$, where A is a complex constant and $g(n)$ is a periodic function with a period depending only on a and b.*

The proof of this theorem is contained in the three next Lemmas. This result is interesting in itself, nevertheless, we will apply it to our needs, which is the following corollary.

Corollary 2.4. *If the quadratic form in Problem 2' has a form*

$$q^2X^2 + (q^2-2p^2)XY + 2p^2Y^2,$$

where p and q are coprime integers, q is odd, then the function $f : \mathbb{Z} \to \mathbb{C}$, satisfying (2.6), has a form $f^2(n) = An^2 + g(n)$, where A a complex constant, and g is a periodic function with period depending only on p and q. Therefore $f^2(n)$ belongs to finite dimensional vector space over $\mathbb{C}$.

Naturally, since the condition (2.6) for $v \neq 0$ is not linear, we cannot claim that solutions form a vector space.

Proof. The proof of the Corollary is nothing else but considerations in the beginning of this section.

Lemma 2.5. *Suppose the complex function $G : \mathbb{N} \to \mathbb{C}$ satisfies (2.12) with certain coprime positive integers a and b, not both equal to 1. Then the generating power series $H(x) := \sum_{n=0}^{\infty} G(n)x^n$ is a rational function $\frac{P(x)}{Q(x)}$, where $P(x) \in \mathbb{C}[x]$ and $Q(x)$ is an integer monic polynomial with free coefficient ± 1.*

Proof. Take in this identity $Y = sa$, for some fixed s, $X = n + sbk$, and sum both expressions for $k = 1$ and $k = -1$. Thus, we obtain:

$$\begin{aligned} &G(an+2sab)+G(bn-sa^2+sb^2)+G(bn-sa^2-sb^2) = \\ &\quad G(an-2sab)+G(bn+sa^2+sb^2)+G(bn+sa^2-sb^2). \end{aligned}$$

Now subtracting from this expression (2.12) with $X = n$ and $Y = 2sa$, we obtain:

$$G(bn + 2sa^2) + G(bn - sa^2 + sb^2) + G(bn - sa^2 - sb^2) = \\ G(bn - 2sa^2) + G(bn + sa^2 + sb^2) + G(bn + sa^2 - sb^2). \qquad (2.13)$$

And, naturally, we can get the similar identity, with a and b exchanged; hence we may assume in the above $b > a$. Then $a^2 + b^2 > 2a^2$ and it is coprime with b, hence the product $s(a^2 + b^2)$ can have any residue modulo b. And therefore we have proved that (2.12) implies the recurrence relation which holds for any natural n and w:

$$G(bn + w) = \sum_{i=1}^{T} A_{i,w} G(bn + w - i); \qquad (2.14)$$

here T is some fixed integer (depending only on a and b), and $A_{i,w}$ are some integers from the set $\{-1, 0, 1\}$, depending actually on i and only $w \pmod b$. We see also from (2.13) that the last nonzero term in (2.14) is $G(bn' - w)$ (this fact later will imply that $Q(x)$ is monic).
Finally denote the formal power series $\sum_{n=0}^{\infty} G(bn + w)x^{bn+w} := H_w(x)$. Hence, we are working in the field $\mathbb{C}((x))$. Take now in (2.14) w any in the range $0, 1, ..., b - 1$, multiply this equality by x^{bn+w} and sum over all nonnegative n such that $bn + w - T \geq 0$. Thus, we obtain the following system of linear equations:

$$H_w(x) + p_w(x) = \sum_{w'=0}^{b-1} P_{w,w'}(x) H_{w'}(x), \quad w = 0, 1, ..., b - 1. \qquad (2.15)$$

where $p_w \in \mathbb{C}[x]$ and $P_{w,w'} \in \mathbb{Z}[x]$. Further, for each w one and only one of $P_{w,w'}$ has a maximal degree with leading coefficient $+1$; namely, $P_{w,b-w}$. Moreover, in the matrix of this linnear system only diagonal terms have free coefficients, equal to -1.
Hence, if such a function $G(n)$ exists, the corresponding power series necessarily satisfy the system of linear equations, hence all H_w are in fact rational functions, and whence

$$H(x) = \sum_{n=0}^{\infty} G(n)x^n = \sum_{w=0}^{b-1} H_w(x) = \frac{P(x)}{Q(x)}.$$

$Q(x)$ is a $\mathbb{Z}[x]$-factor of the determinant of the above system, and since in each row only $P_{w,b-w}$ has a maximal degree and is monic, the determinant and hence $Q(x)$ itself is monic. More importantly, by the remarks above, the free coefficient of $Q(x)$ is ± 1, which finishes the proof of Lemma 3.5.

It can be traced by a more thorough inspection of the proof of the Lemma 3.5 that the main part of this rational function is a polynomial with degree d less that $2b(a^2 + b^2)$. Now separating this main part and expanding the proper rational function into simple fractions, we therefore obtain a finite sum:

$$G(n) = \sum_{i,k} C_{i,k} n^i \xi_k^n, \text{ for } n \geq d, \qquad (2.16)$$

where ξ_k are reciprocals of the roots of $Q(x)$; hence, algebraic integers (moreover, units), and $C_{i,k} \neq 0$ are some complex constants.

Lemma 2.6. *Let function $G : \mathbb{N} \to \mathbb{C}$ defined by* (2.16)*, satisfies* (2.12)*. Then all ξ_k are the roots of unity.*

Proof. Suppose some of ξ_k have absolute value greater than 1. Choose all with the maximal absolute value $r > 1$, and let ξ_k, $1 \leq k \leq T$, be all of them with maximal $i = I$. Choose in (2.12) $X = k$ and $Y = l$ in such manner to that all $d_1 = ak + bl$, $d_2 = bk - al$, $d_3 = ak - bl$ and $d_4 = bk + al$ are distinct and positive. For this it is sufficient that $(k \pm l)a \neq (k \mp l)b$ and $\frac{k}{l} > \max\{\frac{b}{a}, \frac{a}{b}\}$. Since a and b are coprime, the greatest of these d_i (say, $d_1 = L$) can attain any sufficiently large integral value (say, all values $L \geq S$). Substitute (2.16) into (2.12) with $X = kn$ and $Y = ln$. Consider part of this sum

$$\sum_{k=1}^{T} (nL)^I C_{I,k} \cdot \xi_k^{Ln} := E_n.$$

Let $\arg(\xi_k) = \phi_k$. If $E_{n_0} \neq 0$ for some n_0 exceeding our bound, then choosing arbitrarily big n such that $nn_0\phi_k = n_0\phi_k + \varepsilon_{n,k} \pmod{2\pi}$, $\varepsilon_{n,k} \to 0$, we thus obtain $|E_{nn_0}| > \delta(n^I r^{nn_0})$, and since then E_{nn_0} is a dominant term, the identity (2.12) cannot be satisfied (such a choice of n is always possible - see the proof is below). Thus, $E_n = 0$ for all n exceeding our bound:

$$\sum_{k=1}^{T} C_{I,k} \cdot \xi_k^{Ln} = 0.$$

It is easy to see that there exists such $L \geq S$ such that all $L\phi_k$ as an angles are arbitrarily close to 0. In fact, consider all $T-$tuples $\mathbf{a}_L = (L\phi_1/2\pi, L\phi_2/2\pi, ..., L\phi_T/2\pi) \pmod 1$, $L \in \mathbb{N}$, $L \geq S$, as a points in $T-$dimensional unit cube. Let $\mathcal{C}$ be a closure of this set. Hence, for every ε there exists finite integer N such that each $\mathbf{b} \in \mathcal{C}$ is at a distance at most ε from at least one $\mathbf{a}_{L'}$, $L' = S, S+1, S+N$. This is valid hence also for $\mathbf{b} = \mathbf{a}_L$. Taking $L \geq 2S + N$, and finding such L' we get that $L - L' \geq S$ and $\mathbf{a}_{L-L'}$ is close to some vertex of the unit cube. And therefore $\mathbf{a}_{L-L'+1}$ is arbitrarily close to $\mathbf{a}_1$. Since in our case all ϕ_i are different, we can choose $L \geq S$ such that $\mathbf{a}_L$ will have all coordinates also different.
Now take in the above equality such L that all ξ_k^L are different, and let n attain T consecutive sufficiently large values, and consider this as a system of linear equations for $C_{I,k}$, $1 \leq k \leq T$. The corresponding determinant will be the nonzero multiple of Vandermonde determinant $\det(\xi_k^{Ln})_{k,n=1}^T$, hence it is non-zero, and therefore all $C_{I,k}$ are zeros - a contradiction. Hence, all algebraic integers in (2.16) satisfy $|\xi_k| \leq 1$.
To finish, suppose some ξ in (2.16) has a conjugate ξ', for which $|\xi'| > 1$. Let $\mathbf{L}$ be a normal closure of $\mathbb{Q}(\xi)$. Consider the automorphism of $\mathbf{L}$, which maps ξ to ξ'. Extend this automorphism to $\mathbb{C}$ and denote it by σ (such an extension is always possible, see [43], chapter VIII). Applying σ to the equality (2.12), we see that $G'(n) := \sigma G(n)$ satisfies the same relation, and applying σ for (2.16) we therefore obtain similar expression for $G'(n)$, only each $C_{i,k}$ is replaced by $\sigma C_{i,k}$, and ξ_k by $\sigma\xi_k$. Here $|\sigma\xi| > 1$, which, as we have seen,

cannot occur (here we use a trivial fact that σ maps non-zeros to non-zeros). Therefore, all algebraic integers in (2.16) have conjugates only on or inside the unit circle, and therefore Kronecker's theorem (see [40]) implies that they are roots of unity. Lemma 3.6 is proved.
Therefore, the expression (2.16) can be simplified to

$$G(n) = \sum_{i=0}^{I} n^i g_i(n), \tag{2.17}$$

where g_i are periodic functions with a finite period.

Lemma 2.7. *If the function $G : \mathbb{N} \to \mathbb{C}$ of the form (2.17) satisfies (2.12) with a and b coprime positive integers, one of these being even, then $G(n) = g(n) + An^2$, where $g(n)$ is a periodic function with a finite period, and A is a complex constant.*

Proof. Suppose the last nonzero periodic function in (2.17) is g_I, $I \geq 3$. Let the period of g_I be M. Plugging (2.17) into (2.12), consider one part

$$\begin{aligned} W(X,Y) := (aX+bY)^I g_I(aX+bY) \quad &+ \quad (bX-aY)^I g_I(bX-aY) - \\ (aX-bY)^I g_I(aX-bY) \quad &- \quad (bX+aY)^I g_I(bX+aY). \end{aligned}$$

When X and Y run through $X \equiv X_0 \pmod M$ and $Y \equiv Y_0 \pmod M$, the second multipliers in the expression of $W(X,Y)$ are then constant, say h_1, h_2, h_3 and h_4. Then $W(X,Y)$ is a homogeneous polynomial of degree I, and unless it is zero, it is a dominant term in the obtained expression, and we get a contradiction. Hence

$$(aX+bY)^I h_1 + (bX-aY)^I h_2 - (aX-bY)^l h_3 - (bX+aY)h_4 \equiv 0.$$

This is valid for $X \equiv X_0 \pmod M$ and $Y \equiv Y_0 \pmod M$, but since it is a polynomial, all its coefficients should be zero.
Therefore, we have $I+1$ linear conditions for four unknowns h_i, $1 \leq i \leq 4$. Since $I \geq 3$, choose first four of them. Whence $\mathcal{A}z = \mathbf{o}$, where z is a column $(h_1, h_2, h_3, h_4)^T$, $\mathbf{o}$ is a column $(0,0,0,0)^T$ and $\mathcal{A}$ is a 4×4 matrix:

$$\mathcal{A} = \begin{pmatrix} a^I & b^I & -a^I & -b^I \\ a^{I-1}b & -b^{I-1}a & a^{I-1}b & -b^{I-1}a \\ a^{I-2}b^2 & b^{I-2}a^2 & -a^{I-2}b^2 & -b^{I-2}a^2 \\ a^{I-3}b^3 & -b^{I-3}a^3 & a^{I-3}b^3 & -b^{I-3}a^3 \end{pmatrix}.$$

The determinant is $4a^{2I-4}b^{2I-4}(a-b)^2(a+b)^2(a^2+b^2)^2 \neq 0$, hence all $h_i = 0$.
In particular, $h_1 = g_I(aX+bY) = 0$, and since this argument can attain any residue modulo M, this implies $g_I(X) \equiv 0$ - a contradiction, whence $I \leq 2$.
If $I = 2$, we obtain the similar system of three linear equations for h_i, with the matrix, consisting of first three rows of $\mathcal{A}$ with $I = 2$. Since this matrix has rank 3, the space of solutions has rank 1 and solving we obtain:

$$g_2(aX+bY) = g_2(bX-aY) = g_2(aX-bY) = g_2(bX+aY) \text{ for all } X, Y.$$

Let g_2 has a period M. Choose any residue w_0 modulo M, sufficiently large $w \equiv w_0 \pmod{M}$ and a', b' such that $ab' + ba' = w$. Let $X = b' + as$ and $Y = a' + bs$. From the first equality above we obtain:

$$g_2(w + s(a^2 + b^2)) = g_2(bb' - aa'), \quad s \in \mathbb{N},$$

hence $a^2 + b^2$ is a period. From the second equality in the same manner we get that $b^2 - a^2$ is also a period. Since in our case a and b are coprime and one is even, $a^2 + b^2$ and $b^2 - a^2$ are also coprime, whence we get that 1 is also a period, and therefore $g_2(n) \equiv A$.
And so, $G(n) = g(n) + h(n)n + An^2$, and suppose M is the smallest period of $h(n)$. Plugging this again in (2.12), we see that squares vanish, and similar considerations show that

$$\begin{cases} ah_1 + bh_2 - ah_3 - bh_4 = 0 \\ bh_1 - ah_2 + bh_3 - ah_4 = 0, \end{cases}$$

for $h_1 = h(aX + bY)$, $h_2 = h(bX - aY)$, $h_3 = h(aX - bY)$ and $h_4 = h(bX + aY)$. Suppose from symmetry that $a > b$. Now, denote $M(M, b)^{-1}$ by M', where (M, b) stands as usually for the greatest common divisor. Let in the above equalities $Y = M's$. Since $M'b$ is divisible by M, and M is a period, the first equality implies $h(bX - aM's) = h(bX + aM's)$. The second equality gives $2bh(aX) = ah(bX - aM's) + ah(bX + aM's)$, and therefore

$$bh(aX) = ah(bX + aM's).$$

Further, since b and aM' are coprime, the argument $bX + aM's$ can attain any residue modulo M for X, s varying. Choose such w that $T = |h(w)|$ is maximal, and let X and s satisfy $bX + aM's \equiv w \pmod{M}$. Then we get $bT \geq aT$, and since $a > b$, verily $T = 0$ and $h(n) \equiv 0$.

To finish the proof of Theorem 3.3, we need to prove the last statement, namely, about the period of this periodic function $g(n)$. But all roots of unity appearing in its Fourier expansion are the roots of the determinant of the linear system (2.15), and the latter depends only on a and b.

It seems plausible that in fact $g(n)$ can be decomposed into three periodic functions $g(n) = g_{2ab}(n) + g_{a^2+b^2}(n) + g_{|a^2-b^2|}(n)$, where g_i is periodic with period i (this is of course somehow stronger statement than to say that $g(n)$ has a period $2ab(a^2 + b^2)|a^2 - b^2|$). We do not give a proof of this here, since the concrete examples will be studied in the continuation of this chapter. Also in the formulation of Theorem 3.3 the additional conditions $X, Y \geq n_0$, and the same condition for all arguments appearing there, e.g. $bX - aY \geq n_0$, will not make the proof more difficult. It is obvious from the more detailed inspection of the proof of all three Lemmas. When, for example, we are dealing with periodic functions, small arguments can be replaced by arbitrarily large.

2.4 Outline for a general quadratic form

In this section we are dealing with quadratic forms and Problem 2'. Our aim is to derive for a general quadratic form and a function f, satisfying (2.6), a similar statement, analogous to the Corollary of the Theorem 3.3. Here we give a short outline for a possible solution, and confine ourselves to norm forms in quadratic extensions.

Hence, let for simplicity $P \equiv 2 \pmod 4$ or $P \equiv 3 \pmod 4$, and $\alpha = a + c\sqrt{P}$ and $\beta = b + d\sqrt{P}$ be given integral basis of the field $\mathbf{K} = \mathbb{Q}(\sqrt{P})$, $ad - bc = 1$. Let $(\alpha', \beta') = (\alpha, \beta)A$; here and in the sequel α' means the conjugate of α under the non-trivial automorphism of $\mathbf{K}$. Then

$$A = \begin{pmatrix} i & k \\ j & -i \end{pmatrix},$$

where $i = ad + bc$, $j = -2ac$, $k = 2bd$. Further, let $\mathrm{Tr}\alpha^2 = e = 2a^2 + 2c^2P$, $\mathrm{Tr}(\alpha\beta) = f = 2ab + 2cdP$ and $\mathrm{Tr}\beta^2 = g = 2b^2 + 2d^2P$. Then $eg - f^2 = \mathrm{disc}(\mathbf{K})$, also $\mathrm{Tr}\alpha = 2a$, $\mathrm{Tr}\beta = 2b$. Let $\mathcal{N}(X\alpha + Y\beta) = uX^2 + vXY + wY^2$. We will follow the pattern of the proof in the Section 3. First task is by applying linear combinations of norm form for certain values of X and Y to "eliminate" middle terms. Hence, we need some parametrisation of solutions of the following system of equations in $A, B, ..., H$:

$$\begin{cases} uA^2 + vAB + wB^2 = uC^2 + vCD + wD^2 \\ uD^2 + vCD + wC^2 = uE^2 + vEF + wF^2 \\ uF^2 + vEF + wE^2 = uG^2 + vGH + wH^2 \\ uH^2 + vGH + wG^2 = uB^2 + vAB + wA^2 \end{cases} \tag{2.18}$$

For $A\alpha + B\beta$ and $C\alpha + D\beta$ to have equal norms, it is sufficient that $A\alpha + B\beta = \pi_1\sigma_1$ and $C\alpha + D\beta = \pi_1\sigma_1'$ for some algebraic integers $\pi_1, \sigma_1 \in \mathbf{K}$. (In view of Hilbert's theorem 90 (see [43], p. 288), such decomposition is also necessary, at least with fractional π_1, σ_1). Therefore, it is sufficient the existence of algebraic integers π_i, σ_i, $1 \le i \le 4$, such that:

$$\begin{cases} A\alpha + B\beta = \pi_1\sigma_1, & C\alpha + D\beta = \pi_1\sigma_1' \\ D\alpha + C\beta = \pi_2\sigma_2, & E\alpha + F\beta = \pi_2\sigma_2' \\ F\alpha + E\beta = \pi_3\sigma_3, & G\alpha + H\beta = \pi_3\sigma_3' \\ H\alpha + G\beta = \pi_4\sigma_4, & B\alpha + A\beta = \pi_4\sigma_4'. \end{cases} \tag{2.19}$$

Let consider all $\pi_i = p_i\alpha + q_i\beta$ fixed, and $\sigma_i = x_i\alpha + y_i\beta$ to be unknown. Each equality above gives two linear conditions for $A, B, ..., H$ and x_i, y_i; therefore, totally 16 unknowns. To simplify, add, for example, first two, further first and conjugate of the second, and finally take the trace of the first. Therefore we have three equations:

$$\begin{cases} (A + C)\alpha + (B + D)\beta = \pi_1\mathrm{Tr}\sigma_1 \\ (A + iC + kD)\alpha + (B + jC - iD)\beta = \sigma_1\mathrm{Tr}\pi_1 \\ 2Aa + 2Bb = \mathrm{Tr}(\pi_1\sigma_1). \end{cases}$$

This gives five linear conditions instead of four, but they are dependant. We will choose four of them, equivalent to the initial. In fact, since both a and b cannot be 0, without loss of generality we can consider $\mathrm{Tr}(\beta) = 2b \neq 0$. Hence, if T is a matrix

$$T = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & i & k \\ 2a & 2b & 0 & 0 \end{pmatrix},$$

then $\det(T) = 2ak - 2bi + b = \mathrm{Tr}(\beta') + 2b = 4b \neq 0$. Therefore, the first two equalities of (2.19) are equivalent to the following system of linear equations:

$$\begin{aligned} A + C - 2p_1 a x_1 - 2p_1 b y_1 &= 0 \\ B + D - 2q_1 a x_1 - 2q_1 b y_1 &= 0 \\ A + iC + kD - (2p_1 a + 2q_1 b) x_1 &= 0 \\ 2Aa + 2Bb - (ep_1 + fq_1) x_1 - (fp_1 + gq_1) y_1 &= 0. \end{aligned}$$

In the similar fashion, each pair of equations in (2.19) gives four linear equations. Therefore, finally we get

$$(A, B, ..., H, x_1, y_1, ..., x_4, y_4)\mathcal{R} = (0, 0, ..., 0),$$

where $\mathcal{R}$ is 16×16 square matrix. If we consider p's, q's, x's and y's to be rational, then, except possible cases when some π_i is a rational multiple of α, we can always achieve all $q_i = 1$ (by replacing a pair π_i, σ_i in (2.19) by a pair $\sigma_i q_i$ and π_i / q_i). Hence the determinant of $\mathcal{R}$ is a cyclic polynomial in p_1, p_2, p_3 and p_4 of degree at most 8, and degree at most 2 in each p_i. This polynomial depends only on a, b, c, d and P. Suppose now we are able to choose such rational p's that the matrix $\mathcal{R}$ has a rank at most 14. Then, if a function $\mathbb{Z} \to \mathbb{C}$ satisfies (2.6), adding all equations corresponding to equalities in (2.18), we would obtain:

$$(u - w)\Big(f^2(A) + f^2(D) + f^2(F) + f^2(H) - f^2(B) - f^2(C) - f^2(E) - f^2(G)\Big) = 0,$$

where $A, B, ..., H$ are linear forms in two variables, and the analogue of Theorem 3.3 would imply that necessarily $f^2(n) = An^2 + g(n)$, where A is a complex constant and $g(n)$ is a periodic function. Hence, to implement this, first we need to choose suitable p's. In several special choices of a, b, c and d this can be done. Unfortunately, we are unable to give a more exhaustive treatment of this here, and the corresponding investigations will be presented in the continuation of this chapter.

2.5 One cubic field

As mentioned in the introduction, here we will deal with Problem 1 for one normal cubic field. This method allows to solve this problem for quadratic norm forms, such as $X^2 + Y^2$, $X^2 + 5Y^2$, $X^2 - 6Y^2$, $X^2 + XY + 2Y^2$. We skip this, since in first three cases or even in the case $X^2 + DY^2$ it can be solved using the result in [42], and the method is similar as the proof of Proposition 3.2. We also skip the last case, since proof uses the same induction as in the cubic field case, which we will present now.

Consider the polynomial $h(X) = X^3 - 3X + 1$. It has the discriminant 81, and since it is the perfect square, the splitting field of $h(X)$ is cubic. Since $\mathrm{disc}(h) > 0$, all roots are real. Let α be one of them. Then the Galois group $\mathrm{Gal}(\mathbf{K}/\mathbb{Q})$ is cyclic of order 3, where $\mathbf{K} = \mathbb{Q}(\alpha)$. We will show that $\{1, \alpha, \alpha^2\}$ is the integral basis of the ring of integers $\mathcal{O}_\mathbf{K}$ in $\mathbf{K}$. Since $81 = \mathrm{disc}(h(X)) = D(1, \alpha, \alpha^2) = \mathrm{disc}(\mathcal{O}_\mathbf{K}/\mathbb{Z}) \cdot (\mathcal{O}_\mathbf{K} : \mathbb{Z}[\alpha])^2$ and $\mathrm{disc}(\mathcal{O}_\mathbf{K}/\mathbb{Z}) > 1$, we should only verify that $(\mathcal{O}_\mathbf{K} : \mathbb{Z}[\alpha])$ is not equal to 3. Suppose it is. Let ω_1, ω_2 and ω_3 be an integral basis of $\mathcal{O}_\mathbf{K}$. Then there exists integer square matrix A of order 3 and with determinant 3 such that $(\omega_1, \omega_2, \omega_3) \cdot A = (1, \alpha, \alpha^2)$. Changing integral basis ω and matrix A, we can achieve it to be of the Hermite normal form (see [43], p. 35 for details). Hence, it has one of the three following forms:

$$\begin{pmatrix} 1 & 0 & \kappa \\ 0 & 1 & \eta \\ 0 & 0 & 3 \end{pmatrix}; \quad \begin{pmatrix} 1 & \delta & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

where η, κ and δ are integers from the set $\{-2, -1, 0\}$. Rename the new integral basis again $\{\omega_1, \omega_2, \omega_3\}$. Then in the third case we get that $3\omega_1 = 1$, hence $\frac{1}{3}$ is an integer, which is not. In the first case we get $\omega_1 = 1$, $\omega_2 = \alpha$ and $\alpha^2 = \kappa\omega_1 + \eta\omega_2 + 3\omega_3 = \kappa + \eta\alpha + 3\omega_3$. Therefore $\frac{1}{3}(\alpha^2 + \eta\alpha + \kappa)$ is an algebraic integer for certain κ and η, $0 \le \kappa, \eta \le 2$, and we should only verify that it is not. Suppose it is. Then

$$\mathcal{N}_{\mathbf{K}/\mathbb{Q}}(\alpha^2 + \eta\alpha + \kappa) = 1 + 3\eta + 9\kappa + 6\kappa^2 + 3\eta\kappa - 3\eta^2\kappa - \eta^3 + \kappa^3 \equiv 0 (\bmod 27).$$

Now, simple check shows that no pair (κ, η), $0 \le \kappa, \eta \le 2$ satisfies this congruence.
In the second case $\omega_1 = 1$ and $\alpha = \delta\omega_1 + 3\omega_2$, hence $\frac{1}{3}(\alpha + \delta)$ is an algebraic integer for certain δ, $0 \le \delta \le 2$. Therefore $\mathcal{N}_{\mathbf{K}/\mathbb{Q}}(\alpha + \delta) = \delta^3 - 3\delta + 1 \equiv 0 (\bmod 27)$ for some δ, $0 \le \delta \le 2$, which is not satisfied. Hence, $\{1, \alpha, \alpha^2\}$ is an integral basis for $\mathcal{O}_\mathbf{K}$ and $\mathrm{disc}(\mathcal{O}_\mathbf{K}/\mathbb{Z}) = 81$. The polynomial $h(X)$ factors over $\mathbb{Q}(\alpha)$ as $(X - \alpha)(X - (\alpha^2 - 2))(X - (-\alpha^2 - \alpha + 2))$. Let $\alpha' = \alpha^2 - 2$, $\alpha'' = -\alpha^2 - \alpha + 2$. Therefore $\{1, \alpha, \alpha'\}$ is also the integral basis, and henceforth we fix this one. The norm form in this basis is

$$T(a, b, c) = \mathcal{N}(a + b\alpha + c\alpha') = a^3 - b^3 - c^3 - 3ab^2 - 3ac^2 + 3abc + 6b^2c - 3bc^2. \quad (2.20)$$

Thus, we have the functional equation for the function $f : \mathbb{Z} \to \mathbb{C}$:

$$T(f(a), f(b), f(c)) = f(T(a, b, c)). \quad (2.21)$$

And so, here we will proof the following statement, solves Problem 1 for this cubic field.

Proposition 2.8. *Let a form T be defined by* (2.20)*. Then all functions $f : \mathbb{Z} \to \mathbb{C}$, satisfying* (2.21)*, are these: $f(n) = n$, $f(n) = -n$, $f(n) \equiv 0$, $f(n) \equiv i$ or $f(n) \equiv -i$.*

Proof. We have: $\mathcal{N}(1+\alpha) = -3$ and $\mathcal{N}(n+m\alpha) = n^3 - 3nm^2 - m^3$, and also

$$\mathcal{N}((1+\alpha)(n+m\alpha)) = \mathcal{N}((1+\alpha')(n+m\alpha)) = \mathcal{N}((1+\alpha'')(n+m\alpha)),$$

and this in terms of T gives following identities:

$$T(n+2m, n+m, m) = T(n-m, n+m, m) = T(n-m, n-2m, -2m);$$

and a special case ($m = 1$):

$$T(n+2, n+1, 1) = T(n-1, n+1, 1) = T(n-1, n-2, -2). \tag{2.22}$$

Now, the equation (2.21) with $a = b = c = 0$ gives $f(0) = -f^3(0)$. That is, $f(0) = 0$, $f(0) = i$ or $f(0) = -i$. The last two cases lead to the solutions $f(n) \equiv i$ and $f(n) \equiv -i$ respectively. We will skip the proofs, since the method is similar to the proof what follows. Suppose $f(0) = 0$. Then $b = c = 0$ gives $f(a^3) = f^3(a)$, and for $a = 1$ we derive $f(1) = 0$ (which leads to the solution $f(n) \equiv 0$), $f(1) = 1$ or $f(1) = -1$. Now, if f satisfies (2.21), then also $-f$ does, since T is the form of odd degree. And therefore without loss of generality we can assume $f(1) = 1$. Substitution $a = a$, $b = -a$, $c = 0$ gives $f(-a^3) = f^3(a) - f^3(-a) - 3f(a)f^2(-a)$; and since $f(-a^3) = f^3(-a)$, we obtain:

$$f^3(a) - 3f(a)f^2(-a) - 2f^3(-a) = 0.$$

This implies that, in case $f(-a) = 0$, then also we have $f(a) = 0$; and in case $f(-a) \neq 0$ the ratio $\frac{f(a)}{f(-a)} = Y$ satisfies the equation $Y^3 - 3Y - 2 = 0$. Hence, it is equal to -1 or 2. The last is impossible, since $\frac{f(-a)}{f(a)} = \frac{1}{2}$ by the same reason cannot occur. And therefore $f(-a) = -f(a)$ in all cases.

Further, $a = b = 1$, $c = 0$ gives $f(-3) = -3$, and then also $f(3) = 3$. The first and third terms of the identity (2.22) for $n = -1$ give: $-3 = T(1, 0, 1) = T(-2, -3, -2) = -T(2, 3, 2)$. Therefore $T(2, 3, 2) = 3$ and $f(T(2, 3, 2)) = T(f(2), 3, f(2)) = f(3) = 3$. Let $f(2) = w$. Then the last equation yields: $w^3 - 9w + 10 = 0$. On the other hand, $T(2, 1, 0) = 1 \Rightarrow f(T(2, 1, 0)) = T(f(2), 1, 0) = f(1) = 1$. This gives another cubic equation for w: $w^3 - 3w - 2 = 0$. Since w must satisfy both equations, the only possibility is $w = 2$. Additionally $f(-2) = -2$.

Now we need only $f(4)$. Note that $T(4, 2, 0) = 2^3 T(2, 1, 0) = 8$, and hence $8 = f^3(2) = f(8) = f(T(4, 2, 0)) = T(f(4), 2, 0)$, and so $\kappa = f(4)$ satisfies $\kappa^3 - 12\kappa - 16 = 0$. Hence, $f(4) = 4$ or $f(4) = -2$. That suffices, for later we will show that in fact the last does not occur.

Thus, we will finish the proof using induction. Suppose, we have proved that $f(n) = n$ for all $|n| \leq M$. The statement is true for $M = 3$. Then the first equality of (2.22) with $n = M - 1$ gives $T(f(M+1), M, 1) = T(f(M+1), f(M), f(1)) = f(T(M+1, M, 1)) = f(T(M-2, M, 1)) = T(f(M-2), f(M), f(1)) = T(M-2, M, 1) = -3M^3 + 9M^2 - 3$. Let $f(M+1) = \Delta$. Then this gives the cubic equation for Δ:

$$\Delta^3 + \Delta(-3M^2 + 3M - 3) + (2M^3 - 3M^2 - 3M + 2) = 0.$$

This factors as

$$(\Delta - (M+1))(\Delta - (M-2))(\Delta + (2M-1)) = 0.$$

In particular, $f(4) = 4$, 1 or -5, but we have already obtained that $f(4) = 4$ or -2, hence $f(4) = 4$. We have proved the inductive step for $M = 3$, and let $M \geq 4$.
Now, in the same manner we will obtain the cubic equation for Δ from the second equality of (2.22) with $n = -M+1$. Then $T(M, M-2, -1) = -T(-M, -M+2, 1) = -T(-M, -M-1, -2) = T(M, M+1, 2)$. Similarly, this gives $T(M, \Delta, 2) = T(f(M), f(M+1), f(2)) = f(T(M, M+1, 2)) = f(T(M, M-2, -1)) = T(M, M-2, -1) = -3M^3 + 9M^2 - 9$, and hence this implies

$$\Delta^3 + \Delta^2(3M-12) + \Delta(-6M+12) + (-4M^3 + 9M^2 + 12M - 1) = 0.$$

This expression factors as

$$(\Delta - (M+1))(\Delta^2 + \Delta(4M-11) + (4M^2 - 13M + 1)) = 0.$$

The discriminant of the second factor is equal to $-36M + 117 < 0$ (for $M \geq 4$), and so it is irreducible.
Finally, Δ must satisfy both equation we have obtained, and therefore $\Delta = M+1$, i.e. $f(M+1) = M+1$, $f(-M-1) = -M-1$. The inductive step is proved.
Summarising, for the field $\mathbb{Q}(\alpha)$ and a fixed integral basis $\{1, \alpha, \alpha'\}$, the only functions $f : \mathbb{Z} \to \mathbb{C}$, satisfying (2.21), are these: $f(n) \equiv 0$, $f(n) \equiv i$, $f(n) \equiv -i$, $f(n) = n$ and $f(n) = -n$, which finishes the proof of Proposition 3.8.

2.6 Conclusion

Before formulating the conjecture, we need one auxiliary Lemma. It is not crucial for us, but it allows to formulate the conjecture more clearly. Such Lemma might have appeared in the literature, but we could not find a relevant reference.

Lemma 2.9. *Let* $\mathbf{K}$ *be a finite extension of* $\mathbb{Q}$ *of degree* $n \geq 2$. *Let* $\omega_1, \omega_2, ..., \omega_n$ *be a basis of* $\mathbf{K}$ *as a vector space over* $\mathbb{Q}$. *Let a norm form in this basis be* $\mathcal{N}(a_1\omega_1 + a_2\omega_2 + ... + a_n\omega_n) = T(a_1, a_2, ..., a_n)$, *where* $\mathcal{N} = \mathcal{N}_{\mathbf{K}/\mathbb{Q}}$. *If* $T(a_1, a_2, ..., a_n) = T'(a_1^s, a_2^s, ..., a_n^s)$ *for some rational form* T' *and natural* s, *then* $s = 1$ *or* $s = 2$.

Proof. We have that $\{1, \frac{\omega_2}{\omega_1}, \frac{\omega_3}{\omega_1}, ..., \frac{\omega_n}{\omega_1}\}$ is also a vector space basis. In fact, for any $\gamma \in \mathbf{K}$, the product $\gamma\omega_1$ can be uniquely expressed as $\gamma\omega_1 = \sum_{i=1}^n r_i\omega_i$, $r_i \in \mathbb{Q}$. Therefore, there is a unique expression $\gamma = \sum_{i=1}^n r_i \frac{\omega_i}{\omega_1}$. Let $\gamma_i = \frac{\omega_i}{\omega_1}$. Thus, $\gamma_1 = 1$. Suppose, the assumption of Lemma is satisfied with some $s \geq 3$. Then also $n \geq 3$. In this case $\mathcal{N}(a_1\omega_1 + a_2\omega_2 + a_3\omega_3) = T''(a_1^s, a_2^s, a_3^s)$ for all rational a_1, a_2, a_3. Let $a_1 = r$, and we fix $a_2 = a$ and $a_3 = b$, not both equal to 0. Let $\lambda = a\gamma_2 + b\gamma_3$. Then

$$\mathcal{N}(r + \lambda) = \mathcal{N}(r + a\gamma_2 + b\gamma_3) = \mathcal{N}(r\omega_1 + a\omega_2 + b\omega_3)\mathcal{N}(\omega_1)^{-1} = CT''(r^s, a^s, b^s) = H_\lambda(r^s),$$

where $H_\lambda = H$ is a rational polynomial in one variable. Now let $T_\lambda(X) = T(X) \in \mathbb{Q}[X]$ be the minimal monic polynomial of λ, $\big[\mathbb{Q}(\lambda) : \mathbb{Q}\big] = d_\lambda = d$, $\big[\mathbf{K} : \mathbb{Q}(\lambda)\big] = c_\lambda = c$. Then $dc = n$. Thus, in these notations, $\mathcal{N}(\lambda) = \big((-1)^d T(0)\big)^c = (-1)^n T^c(0)$. The number $r + \lambda$ is a root of irreducible monic polynomial $T(X - r)$; therefore, $\mathcal{N}(r + \lambda) = (-1)^n T^c(-r)$. But we know that the last is equal to $H(r^s)$. Since both are polynomials, and r is arbitrary rational number, they are equal:

$$(-1)^n T^c(-X) = H(X^s).$$

Clearly, $T(X)$ has a nonzero constant term. Then it is easy to see that $T(X)$ is also of the form $G(X^s)$. In fact, if it is not, let $q \in \mathbb{Q}$ be the constant term of $T(-X)$, and pX^t, $p \in \mathbb{Q}$, $p \neq 0$, be the term of the smallest degree, for which s does not divide t. Then $T^c(-X)$ contains a term $cq^{c-1}pX^t$. A contradiction. Therefore, provided that $s \geq 3$, we have proved the following:

For all $a, b \in \mathbb{Q}$, not both 0, a number $\lambda = a\gamma_2 + b\gamma_3$ is a root of irreducible monic polynomial of the form $G_\lambda(X^s)$, where $G_\lambda(X) \in \mathbb{Q}[X]$.

Let $\sigma_1, \sigma_2, ..., \sigma_n$ be different embeddings of $\mathbf{K}$ into $\overline{\mathbb{Q}}$ - some algebraic closure of $\mathbb{Q}$. We will use exponential notation: $\sigma : \alpha \to \alpha^\sigma$. Then the polynomial

$$\prod_{l=1}^{n}(X - (a\gamma_2^{\sigma_l} + b\gamma_3^{\sigma_l}))$$

is a power of $G(X^s)$, therefore, it is also of the form $G'(X^s)$. In particular, since $s \geq 3$, the coefficients at X^{n-1} and X^{n-2} are 0, and we have :

$$\sum_{l=1}^{n}(a\gamma_2^{\sigma_l} + b\gamma_3^{\sigma_l}) = a\mathrm{Tr}(\gamma_2) + b\mathrm{Tr}(\gamma_3) = 0,$$

where $\mathrm{Tr} = \mathrm{Tr}_{\mathbf{K}/\mathbb{Q}}$; and also

$$\sum_{l\neq k}^{n}(a\gamma_2^{\sigma_l} + b\gamma_3^{\sigma_l}) \cdot (a\gamma_2^{\sigma_k} + b\gamma_3^{\sigma_k}) = a^2\sum_{l\neq k}\gamma_2^{\sigma_l}\gamma_2^{\sigma_k} + 2ab\sum_{l\neq k}\gamma_2^{\sigma_l}\gamma_3^{\sigma_k} + b^2\sum_{l\neq k}\gamma_3^{\sigma_l}\gamma_3^{\sigma_k} = 0.$$

Since a and b are arbitrary rational numbers (not both equal to 0), then the two summands in the first and the three summands in the second equality are all equal to 0. Hence,

$$\mathrm{Tr}(\gamma_2) = 0, \quad \mathrm{Tr}(\gamma_3) = 0.$$

Additionally,

$$\mathrm{Tr}(\gamma_2^2) = (\mathrm{Tr}(\gamma_2))^2 - \sum_{l\neq k}\gamma_2^{\sigma_l}\gamma_2^{\sigma_k} = 0, \quad \mathrm{Tr}(\gamma_3^2) = 0;$$

and also

$$\mathrm{Tr}(\gamma_2\gamma_3) = \mathrm{Tr}(\gamma_2)\mathrm{Tr}(\gamma_3) - \sum_{l\neq k}\gamma_2^{\sigma_l}\gamma_3^{\sigma_k} = 0.$$

Obviously, indexes 2 and 3 can be replaced by any pair $\{i,j\}$, $2 \le i,j \le n$, $i \ne j$. The last equality is also true for indexes 1 and $i \ge 2$, since $\gamma_1 = 1$.
Eventually, taking this into account, we obtain that the matrix $\left(\mathrm{Tr}(\gamma_i\gamma_j)\right)_{i,j=1}^n$ has only one nonzero entry, that is, $\mathrm{Tr}(\gamma_1^2) = \mathrm{Tr}(1) = n$. And therefore it has determinant 0, and since $\gamma_1, \gamma_2, ..., \gamma_n$ is basis of $\mathbf{K}$ as a vector space over $\mathbb{Q}$, the last contradicts to the fact that $\langle \delta_1, \delta_2 \rangle := \mathrm{Tr}(\delta_1\delta_2)$ is a nondegenerate bilinear form in $\mathbf{K}$ (see [43], p. 286). For the completeness we give a short proof. Any $\delta \in \mathbf{K}$ is a $\mathbb{Q}$-linear combination of γ_i. Thus, $\mathrm{Tr}(\gamma_2\delta) = 0 \Rightarrow \mathrm{Tr}(\gamma_2\mathbf{K}) = 0 \Rightarrow \mathrm{Tr}(\mathbf{K}) = 0$, which contradicts to $\mathrm{Tr}(1) = n$. Thus, $s \le 2$ and Lemma is proved.

And so, now we are ready to proceed with the following conjecture. As mentioned, this statement is true for quadratic norm forms $X^2 + DY^2$. The statement is also correct for the cubic form in the Section 5. Also the Proposition 2 of Section 2 corresponds to the second half of this conjecture in case of the form X^2+Y^2, and this can be extended without difficulty to the forms $X^2 + DY^2$, replacing the identity $(2n+r)^2 + (n-2r)^2 = (2n-r)^2 + (n+2r)^2$ by $(n+D)^2 + D(n-1)^2 = (n-D)^2 + D(n+1)^2$, which at once gives the desired linear recurrence relation. And, naturally, the "if" part of the conjecture is trivial.

Conjecture 2.10. *Let $f : \mathbb{Z} \to \mathbb{C}$ be any function. Let $\mathbf{K}$ be any proper finite extension of $\mathbb{Q}$ of degree n. Fix any integral basis of the ring of integers $\mathcal{O}_{\mathbf{K}} : \omega_1, \omega_2, ..., \omega_n$, and denote a norm form $\mathcal{N}(a_1\omega_1 + ... + a_n\omega_n) = T(a_1, ..., a_n)$. define $\Delta = T(1, 1, ..., 1)$ (which is therefore non-zero). Then the relation (2.4) is satisfied if and only if $f(m) \equiv 0$, $f(m) \equiv \Delta^{-\frac{1}{n-1}}$ (any, but fixed value of this radical, therefore, totally $n-1$ values) or $f(m) = \zeta m$ for some fixed ζ, $\zeta^{n-1} = 1$, $m \in \mathbb{Z}$.*
Moreover, the statement remains true if (2.4) is satisfied only for all $a_i \in \mathbb{Z}$, $|a_i| \ge N$ for all i, $1 \le i \le n$, and some fixed positive integer N.

Remark. In case $T(a_1, a_2, ..., a_n) = T'(a_1^2, a_2^2, ..., a_n^2)$ for some form T', we consider only "essentially different" solutions, which is defined in such manner. Two functions f are said to be "essentially equal", if they differ (probably) by the sign on the terms, which are not expressible as values of T with integer a_i. We needed the Lemma 3.9 for such purpose. In case there existed a norm form $T'(a_1^s, a_2^s, ..., a_n^s)$ for some $s \ge 3$, we would need to modify the notion of "essential equality" for every s. Fortunately, it can not happen.

It is easy to explain why empirically it should be true. The examples with the quadratic and cubic fields show that we can always expect to calculate some first values of $f(n)$ by *ad hoc* method. Moreover, for the extension of degree n (at least in Galois case) we could simply write the expression of type (2.22) with $n-1$ equalities. Generally, in the inductive step we have $(n-1)$ equations, which are satisfied by the same complex number, and these are polynomials of degree n. And so it is hardly expectable that these weakly related polynomials have two common roots.

What concerns Question 1, we conclude with few remarks concerning our choice of norm forms. First, norm form $T(a_1, a_2, ..., a_s)$ is irreducible as polynomial in $\mathbb{Z}[a_1, a_2, ..., a_s]$, and for reducible forms the equivalent statement in general is false.
Example 1. Consider a reducible form $W(X,Y) = X^2 - Y^2$. Then the equation

$$f(X^2 - Y^2) = f^2(X) - f^2(Y)$$

is also satisfied by the primitive character modulo 4, that is:

$$f(X) = \begin{cases} 0 & \text{if } X \equiv 0 \pmod 2, \\ 1 & \text{if } X \equiv 1 \pmod 4, \\ -1 & \text{if } X \equiv 3 \pmod 4. \end{cases}$$

Second, if we consider irreducible polynomials, which are not forms, this statement in general is also wrong.
Example 2. Let $W(a,b) = ab + a + b = (1+a)(1+b) - 1$ (the simplest case of the formal group law). Then for any nonnegative integer q and a function $f : \mathbb{Z} \to \mathbb{Z}$, $f(X) = (1+X)^q - 1$ we have:

$$f(W(X,Y)) = f((1+X)(1+Y) - 1) = (1+X)^q(1+X)^q - 1 = W(f(X), f(Y)).$$

More generally, let $\Gamma : \mathbb{Z} \to \mathbb{C}$ be any strongly multiplicative function. That is, given any complex number $\Gamma(p)$ for each prime $p \in \mathbb{N}$, $\Gamma(1) = 1$, $\Gamma(-1) = \pm 1$, $\Gamma(0) = 0$, we define $\Gamma(X) = \Gamma(\text{sgn}X) \prod_{i=1}^{r} \Gamma^{s_i}(p_i)$, if $X = \pm \prod_{i=1}^{r} p^{s_i}$ is a canonical expression of X. Then all complex valued functions $f(X)$, $X \in \mathbb{Z}$, satisfying

$$f(W(X,Y)) = W(f(X), f(Y)), \text{ for all } X, Y \in \mathbb{Z},$$

are given by $f(X) = \Gamma(X+1) - 1$.

Third, the key point in proving Conjecture in special cases is the presence of the relation of the type (2.22) - that is, integers in number field form a linear algebraic group. This fails for certain forms, which are not norm forms.
Example 3. Consider $W(X,Y) = X^3 + 2Y^3$. If there existed linear polynomials $g_1(X,Y)$, $g_2(X,Y)$, $g_3(X,Y)$ and $g_4(X,Y) \in \mathbb{Z}[X,Y]$ such that $g_1^3 + 2g_2^3 \equiv g_3^3 + 2g_4^3$, $g_1 \neq g_3$, $g_2 \neq g_4$, then the same would hold for some one variable linear functions, which can be verified to be wrong.
Fourth, the degree of the norm form is equal to the number of variables. Is it the "breaking point"? In other words, it is very likely that for irreducible form $L(a_1, a_2, ..., a_s)$, which has the degree n, $n \geq 2$, less than the number of variables ($n < s$), the equivalent statement remains correct. Then, on the other hand, it is natural to ask the following:

Question 1. *Does there exist an irreducible integer form $M(a_1, a_2, ..., a_s)$ of degree n greater than the number of variables ($n > s$), such that*

$$f(M(a_1, a_2, ..., a_s)) = M(f(a_1), f(a_2), ..., f(a_s))$$

for some function $f : \mathbb{Z} \to \mathbb{C}$, which is not a constant and not of the form $f(m) = \zeta m$?

The Problem 2 seems to be more interesting. Here it is reasonable to ask the following:

Question 2. *Is it true that the relation* (2.6) *necessarily implies $f^2(n) = An^2 + g(n)$, with g being a periodic function?*

In general case of Problem 2 and relation (2.5), still we do not have any evidence that this necessarily yields $f^n(a) = Aa^n + g(a)$ with g being a periodic function.

Chapter 3

A curious proof of Fermat's little theorem

Fermat's little theorem states that for p prime and $a \in \mathbb{Z}$, p divides $a^p - a$. This result is of huge importance in elementary and algebraic number theory. For instance, with its help we obtain the so-called Frobenius automorphism of a finite field $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$.

This theorem has many interesting and sometimes unexpected proofs. One classical proof is based upon properties of binomial coefficients. In fact, $(d+1)^p - d^p - 1 = \sum_{i=1}^{p-1} \binom{p}{i} d^i$. Since $\binom{p}{i} = \frac{p!}{i!(p-i)!}$ is divisible by p for $1 \leq i \leq p-1$, then $(d+1)^p - d^p - 1$ is divisible by p. Summing this over $d = 1, 2, ..., a-1$, we obtain the desired result. Another classical proof is based upon Lagrange's theorem, which states that the order of an element of a finite group divides the group order. Applying this theorem to the multiplicative group of a finite field $\mathbb{F}_p$ we obtain the result immediately. Several other proofs can be found at [47]. Nevertheless, in all of these proofs one or another analogue of the Euclidean algorithm (hence arithmetic) is being used.

In this short note we present a curious proof which was found as a side result of another, unrelated problem (which is the case, maybe, with many such "curious" proofs). Surprisingly, arithmetic, group theory, and the properties of binomial coefficients do not manifest at all.

Let $f(x) = 1 - x - dx^2 + \sum_{k \geq 3} a_k x^k$ be any formal power series in $\mathbb{Q}$, with coefficients in $\mathbb{Z}$. It is well known that this series can be represented in a unique way as a formal product of the following form:

$$f(x) = \prod_{k \geq 1} (1 - m_k x^k),$$

where the coefficients m_k are integers. This result can be found in [46], but the proof is simple and straightforward. In fact, for $k = 1$ and $k = 2$ we have a unique choice $m_1 = 1$ and $m_2 = d$. Suppose $N \geq 3$ and we have already chosen m_k for $k \leq N-1$. Then $\prod_{k=1}^{N-1} (1 - m_k x^k) = 1 - x - dx^2 + \sum_{k=3}^{N-1} a_k x^k + Cx^N +$ "higher terms", where C is a certain integer which depends only on m_k for $1 \leq k \leq N-1$. Therefore, the unique choice for m_N

is $m_N = C - a_N$. In a similar fashion, since $\frac{1}{f(x)} = 1 + x + (d+1)x^2 + \sum_{k\geq 3} b_k x^k$ is also a formal integer power series, it can be represented in a unique way as a product

$$\frac{1}{f(x)} = (1+x)(1+(d+1)x^2)\prod_{k\geq 3}(1 - n_k x^k),$$

where n_k are integers as well, $n_1 = -1$, and $n_2 = -(d+1)$.

Recall that the logarithmic derivative of a power series $g(x)$, denoted by $(\ln g(x))'$, is defined to be the power series $g'(x)/g(x)$. It is not hard to prove that for any two formal power series $g(x)$ and $h(x)$, $(\ln g(x) \cdot h(x))' = (\ln g(x))' + (\ln h(x))'$. Indeed, this property reduces to the Leibniz rule

$$(g(x) \cdot h(x))' = g'(x)h(x) + g(x)h'(x).$$

This is verified simply by comparing the corresponding coefficients. Note also that the binomial theorem is not used in the proof.

Now take the formal logarithmic derivative of $f(x)$. We obtain:

$$-x\Big(\ln f(x)\Big)' = \sum_{k\geq 1} \frac{k m_k x^k}{1 - m_k x^k} = \sum_{N\geq 1} x^N \sum_{s|N} m_{N/s}^s \frac{N}{s}.$$

In a similar fashion,

$$-x\Big(\ln \frac{1}{f(x)}\Big)' = x(\ln f(x))' = \sum_{N\geq 1} x^N \sum_{s|N} n_{N/s}^s \frac{N}{s}.$$

Therefore, we have interesting identities among the terms of two infinite sequences:

$$\sum_{s|N} m_{N/s}^s \frac{N}{s} = -\sum_{s|N} n_{N/s}^s \frac{N}{s}, \quad N \in \mathbb{N}. \tag{3.1}$$

We can easily prove by induction that this implies $m_k = -n_k$ for odd k, but not for the terms with even indices! Thus, a consequence of this reasoning is the fact that any infinite sequence of integers $\{m_k, k \in \mathbb{N}\}$ with $m_1 = \pm 1$ has an "inverse" sequence of integers $\{n_k, k \in \mathbb{N}\}$ with $n_1 = \mp 1$. Consequently, all such sequences split into mutually inverse pairs. It is rather tempting to try to express an inverse of a certain sequence for which the infinite product has a rich mathematical content. For example, let us take $m_k = 1$ for $k \in \mathbb{N}$. Hence, we have a product

$$(x, x)_\infty = \prod_{k=1}^{\infty}(1 - x^k).$$

It is well known that $(x, x)_\infty^{-1} = \sum\limits_{n=0}^{\infty} p(n)x^n$, where $p(n)$ is Ramanujan's partition function. Using the recurrence (3.1) we can compute the sequence $\widetilde{n}_k = -n_k$. As mentioned, $\widetilde{n}_k = 1$ for k odd, and terms of this sequence with even indices begin with

$$2, 4, 0, 14, -4, -8, -16, 196, -54, -92, -184, 144, -628, -1040, -2160, 41102...$$

Therefore,

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty}(1 + \widetilde{n}_k x^k).$$

Let us return to our case. Recall that $m_2 = d$ and $n_2 = -(d+1)$. Hence, when $N = 2p$, where $p > 2$ is a prime, (3.1) reads as:

$$2p \cdot m_{2p} + p \cdot m_p^2 + 2d^p + 1 = -2p \cdot n_{2p} - p \cdot n_p^2 + 2(d+1)^p - 1.$$

Thus, p divides $(d+1)^p - d^p - 1$. Summing this over $d = 1, 2, ..., a-1$, we finally obtain $p | a^p - a$. Quite unexpected!

Likewise, expand the following function into a formal infinite product:

$$f(x) = 1 - x - \sum_{n=1}^{\infty} d^n x^{n+1} = \prod_{n=1}^{\infty}(1 - a_n x^n).$$

Since $f(x) = \frac{1-(d+1)x}{1-dx}$, after taking the logarithmic derivative, we obtain:

$$-x\Big(\ln f(x)\Big)' = \sum_{N=1}^{\infty}\Big((d+1)^N - d^N\Big)x^N = \sum_{N \geq 1} x^N \sum_{s|N} a_{N/s}^s \frac{N}{s}.$$

As a direct consequence, $a_p = \frac{(d+1)^p - d^p - 1}{p}$, which implies that $\frac{(d+1)^p - d^p - 1}{p}$ is an integer. Possible variations on this theme unexpectedly produce other congruences and identities. Recall that a prime number p is said to be a Wieferich prime if and only if $2^{p-1} \equiv 1 \pmod{p^2}$. Examples are $p = 1093$ and $p = 3511$, with no others in the range $p < 4 \cdot 10^{12}$. In the last example with $d = 1$, *all* the numbers $a_p = \frac{2^p - 2}{p}$ appear simultaneously in the infinite product defining $\frac{1-2x}{1-x}$, and as the proof of the algorithm used to expand a formal power series into an infinite product suggests, strangely enough, the coefficients a_N are defined inductively on N without a distinction between prime and composite values of N. Possibly, more profound research of this product could clarify our understanding of these exceptional Wieferich primes.

Conclusions

1. Let $F(x)$ be the Minkowski question mark function. Then its Stieljes transform (called the *the dyadic period function*) is a rich mathematical object, its analytic properties are deep, and this function has a representation in "almost finite" form. On the other hand, this result is only the first step in understanding the analytic structure of the dyadic period function, and it poses many questions.
2. Some classes of forms posses what can be called *pseudo-endomorphisms*. These are algebraic objects, though possibly arithmetic is hidden as well.
3. Fermat's little theorem has many interesting proofs: arithmetic, those arising from dynamical systems, combinatorics. We present one as well, arising from the theory of formal power series.

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