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REGRESSION AND DEGRADATION MODELS IN RELIABILITY THEORY AND SURVIVAL ANALYSIS

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Notation

$F(t)$	Cumulative distribution function;
$f(t)$	Probability density function;
$S(t)$	Survival function;
n	Sample size;
T_i	Failure time
x	Explanatory variable;
$\lambda(t)$	Hazard rate function;
$\Lambda(t)$	Cumulative hazard rate function
$T_{11}, ..., T_{1n_1}$	Failure times of n_1 units tested in "hot" conditions;
$T_{21}, ..., T_{2n_2}$	Failure times of n_2 units tested in "warm" conditions;
μ	Mean failure time;
σ^2	Variance;
r	Scale parameter;
H_0	Hypothesis;
α, β, ν, μ	Parameters;
$K_j(t)$	Cumulative distribution function of redundant system;
$k_j(t)$	Probability density function of redundant system;
$(\underline{K}_j(t), \overline{K}_j(t))$	Confidence interval;
I	Fisher information matrix;
I^{-1}	Inverse of the Fisher information matrix;
$L(r, \theta)$	Likelihood function;
$l(r, \theta)$	Loglikelihood function;
$Z(t)$	Degradation process;
z_0	Critical level;
$T^{(0)}$	Moment of the non-traumatic failure;
$T^{(k)}$	Moment of the traumatic failure of the kth mode;
$S^{(0)}$	Survival function of the non-traumatic failure;

$S^{(k)}$	Survival function of the traumatic failure of the kth mode;
$\tilde{\lambda}^{(k)}(t\|Z)$	Conditional failure rate;
P	Probability;
C_i	Censoring time of the ith unit .

Contents

Introduction

To warrant high reliability of key components of reliability systems, stand-by units are used. If any component fails then a stand-by unit operates instead of the failed component.

If the stand-by units are functioning in the same "hot" conditions as the main unit then usually after switching the reliability of the stand-by units does not change. But "hot" redundancy has disadvantages because any of stand-by units fails earlier than the main one with the probability 0.5.

If the stand-by units are not operating until the failure of the main unit ("cold" reserving), it is possible that during and after commuting the failure rate increases because the stand-by unit is not "warmed" enough. So "warm" reserving is sometimes used: stand-by units function under lower stress than the main one. In such a case the probability of the failure of the stand-by unit is smaller than that of the main unit and it is also possible that switching is fluent, i.e. switching from "warm" to "hot" conditions does not do any damage to units.

The definition of "fluent switching" as statistical hypothesis on the conditional distribution of the failure time of the system after the switch is given. Well known survival regression models such as Sedyakin's and accelerated failure time (AFT) model are used.

Goodness-of-fit tests for obtained redundant systems reliability models are proposed. Asymptotic properties of proposed test statistics are investigated.

Parametric and non-parametric estimation procedures for the reliability of such systems are given. Properties of the proposed parameter estimators are obtained.

Failures of highly reliable units are rare. One way of obtaining a complementary reliability information is to do accelerated life testing (ALT), i.e. to use higher level of experimental factors, hence to obtain failures quickly. Another way of obtaining complementary reliability information is to measure some parameters which characterize the aging or degradation of the product in time.

Statistical inference from ALT is possible if failure time regression models relating failure time distribution with external explanatory variables (covariates, stresses) influencing the reliability are well chosen. Statistical inference from failure time-degradation data with covariates needs even more complicated models relating failure time distribution not only with external but also with internal explanatory variables (degradation, wear) which explain the state of units before the failures. In the last case models for degradation process distribution are needed, too.

Hence, the second direction of the work is modelling and statistical estimation of the reliability of systems or units in the case when joint failure time and degradation regression data are available.

The modified maximum likelihood method for estimation of failure process and degradation process parameters using simultaneous degradation and multi-mode failure time regression data is introduced.

Estimators of various reliability characteristics of the units related to traumatic and non-traumatic failures are given.

Examples when the degradation process is modelled by time scaled gamma process, path processes, shock processes with the number of shocks modelled by non-homogenous Poisson process are considered.

Actuality

There are many publications on probabilistic modelling of redundant systems reliability given the reliability of the system components. Applying of these results in real analysis of system reliability is possible if the probability distribution of the components is known. So a very actual problem is the estimation of the redundant system reliability and the properties of the estimators using estimators the reliability of the components.

Methods of accelerated life testing and degradation process analysis separately are well developed but joint modelling and statistical analysis of simultaneous failure time-degradation data with covariates is very recent research direction. The last international conferences "Mathematical methods in reliability " (2005, 2007, 2009) show increasing interest in this direction.

Aims and problems

The main problems are the following:
1. to formulate mathematical definition of stand-by unit fluent switching from "warm" to "hot" conditions;
2. to construct tests for general "fluent switching hypothesis" formulated using Sedyakin's "reliability principle" and for particular fluent switching hypothesis formulated using accelerated failure time model;
3. to investigate asymptotic properties of the test statistics;
4. to construct parametric and nonparametric estimators of the cumulative distribution function of redundant system using reliability data of components tested under different stresses;
5. to investigate asymptotic properties of the parametric and nonparametric estimators;
6. to construct asymptotic confidential intervals for cumulative distribution function of redundant system;
7. to investigate finite sample properties of the parametric and nonparametric estimators by simulation;

8. to formulate general simultaneous failure time and degradation regression data models;
9. to modify maximum likelihood method for estimation of failure process and degradation process parameters using simultaneous degradation and multi-mode failure time regression data using predictors of degradation processes;
10. to investigate the structure of modified likelihood function when the degradation process is modelled by time scaled gamma process, path processes, shock processes with the number of shocks modelled by non-homogenous Poisson process.

Methods

Counting process techniques, delta method, parametric and non-parametric estimation methods, limit theorems for the sequences of random variables and stochastic processes, numeric and simulation methods were used.

Novelty

All results of the thesis are new.

Statements presented for the defence

1. Mathematical definition of stand-by unit fluent switching from "warm" to "hot" conditions is formulated.
2. Goodness-of-fit test for a general "fluent switching hypothesis" based on Sedyakin's principle is constructed.
3. Goodness-of-fit test for a "fluent switching hypothesis" based on accelerated failure time model is constructed.
4. Asymptotic properties of the two test statistics are investigated ;
5. Parametric and nonparametric estimators of the cumulative distribution function of redundant system using reliability data of components tested in "hot" and "warm" conditions are constructed;
6. Asymptotic properties of the parametric and nonparametric estimators are investigated;
7. Asymptotic confidence intervals for cumulative distribution function of redundant system are constructed.
8. Finite sample properties of the parametric and nonparametric estimators are investigated by simulation.
9. General simultaneous failure time and degradation regression data models are formulated.
10. Maximum likelihood method for estimation of failure process and degradation process parameters using simultaneous degradation and multi-mode failure time regression data is modified using predictors of degradation processes.
11. The structure of modified likelihood function when the degradation process is

modelled by time scaled gamma process, path processes and shock processes is investigated.

History of the problem

Traditional life data analysis involves analyzing times-to-failure data (of a product, of a system or component) obtained under normal operating conditions in order to quantify the life characteristics of the product, system or component. Failures of highly reliable units are rare, for example, the lifetime of semiconductors is very long, and to test devices under usual conditions would require far too much test time and excessively large sample size. So other information should be used in addition to failure-time data, which could be censored.

One way of obtaining a complementary reliability information is to use higher level of experimental factors, stresses or covariates (such as temperature, voltage or pressure) to increase the number of failures and, hence to obtain reliability information quickly. This procedure provides the methods known today as the accelerated life testing (ALT). These methods were developed by many researchers, see wide surveys of the models and methods in Singpurwalla (also can be found in [6], [26], [28], [32], [37], [40]).

The first part of this work is the first attempt in the scientific literature to apply known models of ALT to the statistical analysis of redundant systems with stand-by units in "warm" operating conditions. Stand-by units operate in "warm" conditions when the main unit functions and in "hot" conditions after their switch on after the failure of the main unit, so the models of ALT can be applied. The problem is that in ALT the moments of stress level change is usually planned and fixed before the experiment. In the analysis of redundant systems the moments of stress change are random and are related with the failure of the main unit. So special goodness-of-fit tests for the redundant system models are needed. Another particularity is the following. In ALT identic units are tested in various stress conditions and inference about the reliability of these units in usual stress conditions is done. In the case of redundant systems components of the system function in different stress conditions and the inference about the reliability of the system, not of the components must be done.

Differently from ALT another way of obtaining complementary reliability information is to measure some parameters which characterize the aging or wear of the product in time. In analysis of longevity of highly reliable complex industrial or biological systems, the degradation processes provide an important additional information about the aging, degradation and deterioration of systems, and from this point of view these degradation data are really a very rich source of reliability information and often offer many advantages over failure time data. Degradation is the natural response for some tests, and it is natural also that with degradation data it is possible to make useful reliability and statistical inference , even with no observed failure.

Statistical inference from ALT is possible if failure time regression models relating failure time distribution with external explanatory variables (covariates, stresses) influencing the reliability are well chosen. Statistical inference from failure time-degradation data with covariates needs even more complicated models relating failure time distribution not only with external but also with internal explanatory variables (degradation, wear) which explain the state of units before the failures. In the last case models for degradation process distribution are needed, too.

The most applied stochastic processes used as degradation models are general path models ([7], [8], [27], [29], [30]) and time scaled stochastic processes with stationary and independent increments such as the gamma process ([5], [21], [34]), compound Poisson process ([14], [18], [19], [20], [42], [43]), and Wiener process with drift ([11], [12], [13], [44], [45], [46], [22], [23], [33]). Harlamov [17] discusses inverse gamma-process as a wear model. Zacks [49] discusses general compound renewal damage processes.

If joint failure time and degradation data are available models and estimation methods for analysis of such data are needed. Excellent introductions to failure time-degradation models is given by Singpurwalla [38], Yashin and Manton [48]. More recent developments can be found in Bagdonavičius and Nikulin [3], Finkelstein [15], Lehmann [24], Yashin [47]. Methods of estimation from failure time-degradation data may be found in Bagdonavičius and Nikulin [5], [6], Lehmann [22], [24], [25], Bagdonavičius *et al* [7], [8], Lawless and Crowder [21], Couallier [9].

The second direction of this work is to formulate general simultaneous failure time and degradation regression data models and give methods of estimation.

Approbation

The results of the thesis were presented at the 6^{th} St. Petersburg Workshop on Simulation, St. Petersburg, Russia, 28 June - 4 July, 2009; at the Second International Conference, ALT 2008, Bordeaux, France, 9–11 June 2008; at the Mathematical Methods in Reliability Conference, Glasgow, Scotland, 1–4 July 2007; as well as at the conferences of Lithuanian Mathematical Society (2008, 2009).

Principal publications

The main results of the thesis are published in the following papers:

1. V. Bagdonavičius, I. Masiulaitytė, M. Nikulin, Statistical Analysis of Redundant System with one Stand-by Unit. In Mathematical Methods in Survival Analysis, Reliability and Quality of Life, (eds. C.Huber, N.Limnios, M.Mesbah, M.Nikulin) , ISTE&Wiley : London 2008, p. 179-189.

2. V. Bagdonavičius, I. Masiulaitytė, M. Nikulin, Statistical analysis of redundant systems with 'warm' stand-by units. Stochastics An International Journal of Probability and Stochastic Processes, Volume 80, Issue 2 and 3, 2008, p. 115 - 128.

3. V. Bagdonavičius, I. Masiulaitytė, M. Nikulin, Statistical analysis of reliability of a redundant system with one operating unit and one stand-by unit in warm operating state. Proceeding of ALT, 2008.

4. V. Bagdonavičius, I. Masiulaitytė, M. Nikulin, Asymptotic properties of redundant systems reliability estimators. In: Advances in Degradation Modelling Applications to Reliability, Survival Analysis, and Finance. M.S. Limnios, N. Balakrishnan, N.; Kahle, W.; Huber-Carol, C.(Editors). Birkhauser, 2010, p. 293-310 ISBN: 978-0-8176-4923-4.

5. V. Bagdonavičius, I. Masiulaitytė, M. Nikulin, Reliability estimation from failure-degradation data with covariates. In: Advances in Degradation Modelling Applications to Reliability, Survival Analysis, and Finance. M.S. Limnios, N. Balakrishnan, N.; Kahle, W.; Huber-Carol, C.(Editors). Birkhauser, 2010, p.275-292 ISBN: 978-0-8176-4923-4.

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I owe my loving thanks to my parents and friends. Without their encouragement and understanding it would have been impossible for me to finish this work.

Structure of the Thesis

The thesis consists from introduction, three chapters and conclusions. In addition, the notation of the function is presented in the thesis. Volume of work is 97 pages.

In the Chapter 1 the main issues and results which other authors analyzed that theme are presented.

In the Chapter 2 redundant system with one main unit and $m-1$ stand-by units operating in "warm" conditions are analysed. Goodness-of-fit tests for a general stand-by unit "fluent switching hypothesis" based on Sedyakin's principle and for "fluent switching hypothesis" based on accelerated failure time model are constructed. Parametric and nonparametric estimators and properties of the estimators of the cumulative distribution function of redundant system using reliability data of components tested in "hot" and "warm" conditions are presented. Asymptotic confidential intervals for cumulative distribution function of redundant system are constructed and investigated by simulation.

In the Chapter 3 general simultaneous failure time and degradation regression data models are presented. Maximum likelihood method for estimation of failure process and degradation process parameters is given.

Chapter 1

Accelerated life models

1.1 Introduction

Accelerated life models relate the lifetime distribution to the explanatory variable (stress, covariate, regressor). This distribution can be defined by the survival, cumulative distribution or probability density functions. Nevertheless, the sense of accelerated life models is best seen if they are formulated in terms of hazard rate function.

Suppose at first that the explanatory variable is a deterministic time function:

$$x(\cdot) = (x_1(\cdot), ..., x_m(\cdot))^T \ : [0, \infty) \to B \in \mathbf{R}^m.$$

If $x(\cdot)$ is constant in time, we shall write x instead of $x(\cdot)$ in all formulas.

Denote informally by $T_{x(\cdot)}$ the failure-time under $x(\cdot)$ and by

$$S_{x(\cdot)}(t) = \mathbf{P}\{T_{x(\cdot)} > t\}, \quad F_{x(\cdot)}(t) = \mathbf{P}\{T_{x(\cdot)} \le t\}, \quad f_{x(\cdot)}(t) = -S'_{x(\cdot)}(t),$$

the survival, cumulative distribution and probability density function, respectively.

The hazard (rate) function under $x(\cdot)$ is

$$\lambda_{x(\cdot)}(t) = \lim_{h \downarrow 0} \frac{1}{h} \mathbf{P}\{T_{x(\cdot)} \in [t, t+h) \mid T_{x(\cdot)} \ge t\} = -\frac{S'_{x(\cdot)}(t)}{S_{x(\cdot)}(t)}.$$

Denote by

$$\Lambda_{x(\cdot)}(t) = \int_0^t \lambda_{x(\cdot)}(u) du = -ln\{S_{x(\cdot)}(t)\}$$

the cumulative hazard (function) under $x(\cdot)$.

Each specified accelerated life model relates the hazard rate (or other function) to the explanatory variable in some particular way.

If the explanatory variable is a stochastic process $X(t)$, $t \ge 0$, and $T_{X(\cdot)}$ is the failure-time under $X(\cdot)$, then denote by

$$S_{x(\cdot)}(t) = \mathbf{P}\{T_{X(\cdot)} > t | X(s) = x(s), \, 0 \le s \le t\},$$

$$\lambda_{x(\cdot)}(t) = -S'_{x(\cdot)}(t)/S_{x(\cdot)}(t), \quad \Lambda_{x(\cdot)}(t) = -\ln\{S_{x(\cdot)}(t)\}$$

the conditional survival, hazard rate and cumulative hazard rate functions. In this case the definitions of models should be understood in terms of these conditional functions.

To be short we shall use the word the stress for the explanatory variable in this chapter.

Suppose that $F_1(t)$ and $F_2(t)$ are the c.d.f. of units functionning under accelerated and usual stresses, respectively.

We suppose that for all positive t the following inequality is valid:

$$F_2(t) < F_1(t), \tag{1.1}$$

and we can find some function $g(t)$ which satisfies the equation:

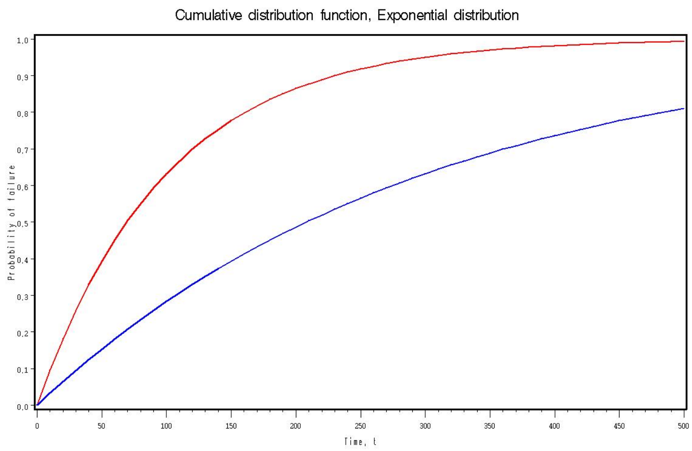
$$F_2(t) = F_1(g(t)). \tag{1.2}$$

Below the different expression of function $g(t)$ is given. The c.d.f. F_1 and F_2 are from the same distribution family.

Example 1.1.1. Exponential distribution

Suppose two cumulative distribution function of exponential distribution have following form:

$$F_1(t) = 1 - e^{-\lambda_1 t}, \quad F_2(t) = 1 - e^{-\lambda_2 t}.$$



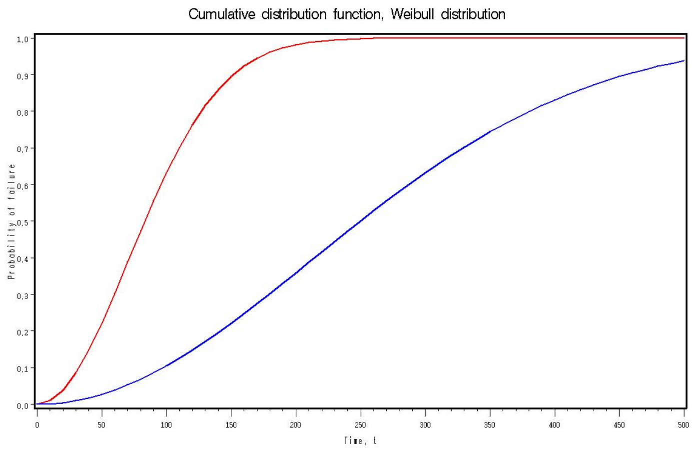
Graph 1.1. Cumulative distribution function

The function $g(t)$ which satisfied (1.2) has the following form:

$$g(t) = \frac{\lambda_2}{\lambda_1} t. \tag{1.3}$$

Example 1.1.2. Weibull distribution Suppose two cumulative distribution function of weibull distribution have following form:

$$F_1(t) = 1 - \exp\left(-\left(\frac{t}{\beta_1}\right)^{\alpha_1}\right), \quad F_1(t) = 1 - \exp\left(-\left(\frac{t}{\beta_2}\right)^{\alpha_2}\right).$$



Graph 1.2. Cumulative distribution function

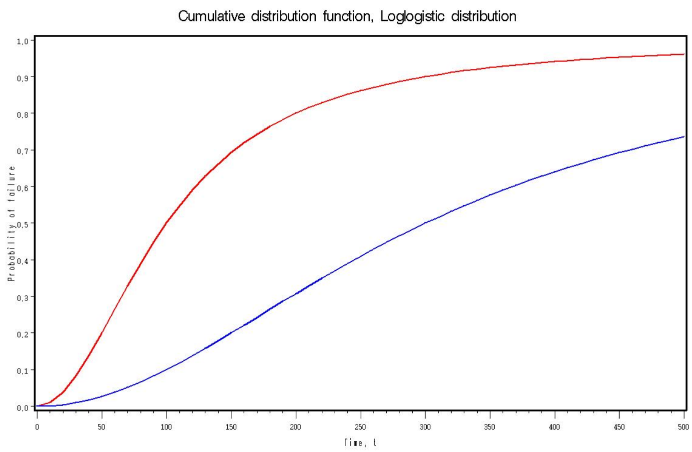
The function $g(t)$ which satisfied (1.2) has the following form:

$$g(t) = \left(\frac{\beta_1^{\alpha_1}}{\beta_2^{\alpha_2}}\right)^{1/\alpha_1} t^{\alpha_2/\alpha_1}. \tag{1.4}$$

If $\alpha_1 = \alpha_2 = 1$ then (1.4) has the following form: $g(t) = \left(\frac{\beta_1}{\beta_2}\right) t$. If $\alpha_1 = 1$ and $\alpha_2 = 2$ then (1.4) has the following form: $g(t) = \left(\frac{\beta_1}{\beta_2^2}\right) t^2$. If $\alpha_1 = \alpha_2 = 2$ then (1.4) has the following form: $g(t) = \left(\frac{\beta_1}{\beta_2}\right) t$.

Example 1.1.3. Loglogistic distribution Suppose two cumulative distribution function of loglogistic distribution have following form:

$$F_1(t) = \frac{1}{1 + \left(\frac{t}{\mu_1}\right)^{\nu_1}}, \quad F_2(t) = \frac{1}{1 + \left(\frac{t}{\mu_2}\right)^{\nu_2}}.$$



Graph 1.3. Cumulative distribution function

The function $g(t)$ which satisfied (1.2) has the following form:

$$g(t) = \mu_1 \left(\frac{t}{\mu_2} \right)^{\nu_2/\nu_1}. \tag{1.5}$$

If $\nu_1 = \nu_2 = 1$ then (1.5) equation is

$$g(t) = \frac{\mu_1}{\mu_2} t.$$

In the examples the function $g(t)$ is linear function, which in general can be defined:

$$g(t) = r \cdot t. \tag{1.6}$$

If the function $g(t)$ has the form (1.5) then obtain $F_2(t) = F_1(r \cdot t)$.

The kth moments are:

$$\alpha_1^{(k)} = \int_0^\infty t^k dF_1(t), \tag{1.7}$$

$$\alpha_2^{(k)} = \int_0^\infty t^k dF_2(t) = \int_0^\infty t^k dF_1(g(t)) = \int_0^\infty t^k dF_1(r \cdot t), \tag{1.8}$$

and the central moments are:

$$\mu_1^{(k)} = \int_0^\infty (t - \alpha_1^{(1)})^k dF_1(t), \tag{1.9}$$

$$\mu_2^{(k)} = \int_0^\infty (t - \alpha_2^{(1)})^k dF_2(t) = \int_0^\infty (t - \alpha_2^{(1)})^k dF_1(g(t)) = \int_0^\infty (t - \alpha_2^{(1)})^k dF_1(r \cdot t). \tag{1.10}$$

Using property of the moment $\mu^{(k)}(r \cdot t) = r^k \mu^{(k)}(t)$ we obtain:

$$\frac{\alpha_1^{(k)}}{\alpha_2^{(k)}} = \frac{\mu_1^{(k)}}{\mu_2^{(k)}} = r^k = \left(\frac{g(t)}{t} \right)^k. \tag{1.11}$$

In this chapter we describe simplest accelerated life models given the literature.

1.2 Generalized Sedyakin's model

1.2.1 Definition of the model

Accelerated life models could be at first formulated for constant explanatory variables. Nevertheless, before formulating them, let us consider a general method for generalizing such models to the case of time-varying stresses.

In 1966 N. M. Sedyakin [36] formulated the *physical principle in reliability.* The idea is the following. For two identical populations of units functioning under different stresses x_1 and x_2, two moments t_1 and t_2 are equivalent if the probabilities of survival until these moments are equal:

$$\mathbf{P}\{T_{x_1} > t_1\} = S_{x_1}(t_1) = S_{x_2}(t_2) = \mathbf{P}\{T_{x_2} > t_2\}.$$

If after these equivalent moments the units of both groups are observed under the same the stress x_2 , i.e. the first population is observed under the step-the stress

$$x(\tau) = \begin{cases} x_1, & 0 \leq \tau < t_1, \\ x_2, & \tau \geq t_1, \end{cases}$$

and the second all time under the constant stress x_2, then for all $s > 0$

$$\lambda_{x(\cdot)}(t_1 + s) = \lambda_{x_2}(t_2 + s).$$

Using the idea of Sedyakin, Bagdonavičius [1] generalized the model to the case of any time-varying stresses by supposing that the hazard rate $\lambda_{x(\cdot)}(t)$ at any moment t is a function of the value of the stress at this moment and of the probability of survival until this moment. It is formalized by the following definition.

Definition 1.2.1 *The generalized Sedyakin's (GS) model holds on a set of stresses E if there exists a positive on $E \times \mathbf{R}^+$ function g such that for all $x(\cdot) \in E$*

$$\lambda_{x(\cdot)}(t) = g\left(x(t), S_{x(\cdot)}(t)\right). \tag{1.12}$$

Equivalently, the model can be written in the form

$$\lambda_{x(\cdot)}(t) = g_1\left(x(t), \Lambda_{x(\cdot)}(t)\right). \tag{1.13}$$

with $g_1(x, s) = g(x, exp\{-s\})$.

1.2.2 GS model for step-stresses

The mostly used time-varying stresses in accelerated life testing (ALT) are step-stresses: units are placed on test at an initial low stress and if they do not fail in a predetermined time t_1, the stress is increased. If they do not fail in a predetermined time $t_2 > t_1$, the stress is increased once more, and so on. Thus, step-stresses have the form

$$x(u) = \begin{cases} x_1, & 0 \leq u < t_1, \\ x_2, & t_1 \leq u < t_2, \\ \cdots & \cdots \\ x_m, & t_{m-1} \leq u < t_m, \end{cases} \tag{1.14}$$

where $x_1, \cdots, x_m$ are constant stresses. If $m = 1$, the step-stress is called *simple*.

Sets of step-stresses of the form (1.14) will be usually denoted by E_m.

Let us consider the meaning of the rule (1.13) for the step-stresses.

Let E_1 be a set of constant stresses and E_2 be a set of simple step-stresses of the form

$$x(u) = \begin{cases} x_1, & 0 \le u < t_1, \\ x_2, & u \ge t_1, \end{cases} \tag{1.15}$$

where $x_1,\ x_2 \in E_1$.

Proposition 1.2.1 (Bagdonavičius [1]). *If the GS model holds on E_2 then the survival function and the hazard rate under the stress $x(\cdot) \in E_2$ verify the equalities*

$$S_{x(\cdot)}(t) = \begin{cases} S_{x_1}(t), & 0 \le t < t_1, \\ S_{x_2}(t - t_1 + t_1^*), & t \ge t_1, \end{cases} \tag{1.16}$$

and

$$\lambda_{x(\cdot)}(t) = \begin{cases} \alpha_{x_1}(t), & 0 \le t < t_1, \\ \alpha_{x_2}(t - t_1 + t_1^*), & t \ge t_1, \end{cases} \tag{1.17}$$

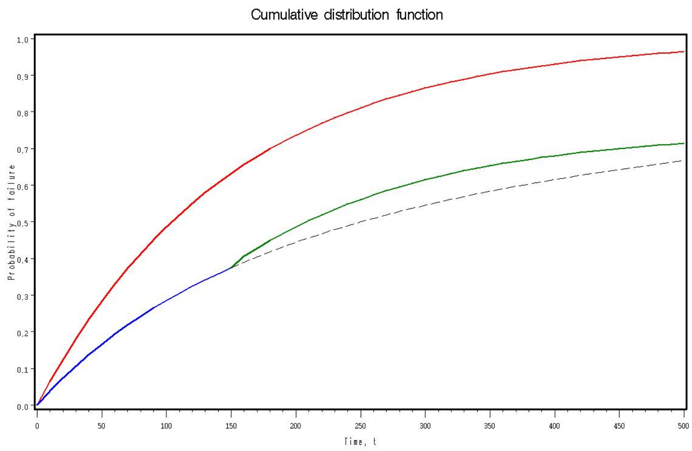
respectively; the moment t_1^* is determined by the equality $S_{x_1}(t_1) = S_{x_2}(t_1^*)$.

Corollary 1.2.1 *Under conditions of the Proposition 1.2.1 for all $s \ge 0$*

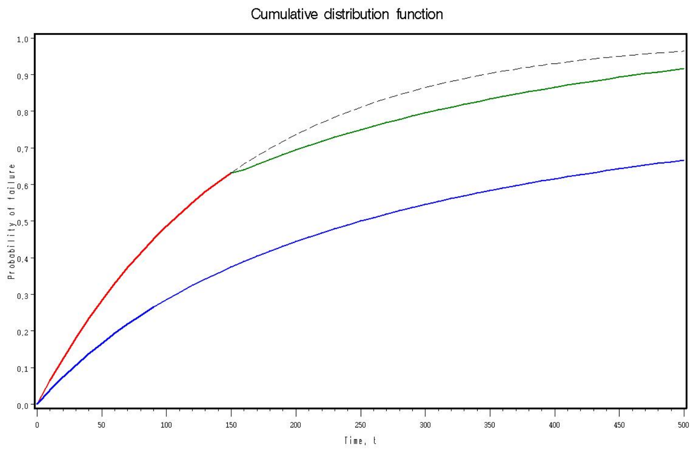
$$\lambda_{x(\cdot)}(t_1 + s) = \lambda_{x_2}(t_1^* + s).$$

It is the model of Sedyakin [36].

In terms of cumulative distribution functions the model graphically can be presented by following in the following figures :



Graph 1.4. Cumulative distribution function: $F_{x_2}(t)$ – red, $F_{x_1}(t)$ – blue, $F_{x(\cdot)}(t)$ – green



Graph 1.5. Cumulative distribution function: $F_{x_1}(t)$ – red, $F_{x_2}(t)$ – blue, $F_{x(\cdot)}(t)$ – green

Let us consider a set E_m of more general stepwise stresses of the form (1.13). Set $t_0 = 0$. We shall show that the rule of time-shift holds and for the general step-stress.

Proposition 1.2.2 (Bagdonavičius [4]). *If the GS model holds on E_m then the survival function $S_{x(\cdot)}(t)$ verifies the equalities:*

$$S_{x(\cdot)}(t) = S_{x_i}(t - t_{i-1} + t^*_{i-1}), \quad \textit{if} \quad t \in [t_{i-1}, t_i),\ (i = 1, 2, \dots, m), \tag{1.18}$$

where where t^*_i verify the equations

$$S_{x_1}(t_1) = S_{x_2}(t^*_1), \dots, S_{x_i}(t_i - t_{i-1} + t^*_{i-1}) = S_{x_{i+1}}(t^*_i),\ (i = 1, \dots, m-1). \tag{1.19}$$

N.M. Sedyakin called his model the physical principle in reliability meaning that this model is very wide. Nevertheless, this model and it's generalization can be not appropriate in situations of periodic and quick change of the stress level or when switch-up's of the stress from one level to the another can imply failures or shorten the life.

1.3 Accelerated failure time model

1.3.1 Definition of the model for constant stresses

Suppose that under different constant stresses the survival functions differ only in scale: for any $x \in E_1$

$$S_x(t) = G\{r(x)\, t\}, \tag{1.20}$$

where the survival function G does not depend on x.

Applicability of this model in accelerated life testing was first noted by Pieruschka [35]. It is the most simple and the mostly used model in FTR data analysis and ALT.

Under the AFT model the distribution of the random variable

$$R = r(x)T_x$$

does not depend on $x \in E_1$ and it's survival function is G. Denote by m, σ^2 and t_p the mean, the variance and the p-quantile of R, respectively.

The AFT model implies that

$$\mathbf{E}(T_x) = m/r(x), \quad \mathbf{Var}(T_x) = \sigma^2/r^2(x), \quad t_x(p) = t_p/r(x),$$

where $t_x(p)$ is the p-quantile of T_x.

The coefficient of variation

$$\frac{\mathbf{E}(T_x)}{\sqrt{\mathbf{Var}(T_x)}} = \frac{m}{\sigma}$$

does not depend on x.

The survival functions under any $x_1, x_2 \in E_1$ are related in the following way:

$$S_{x_2}(t) = S_{x_1}\{\rho(x_1, x_2)\, t\},$$

where $\rho(x_1, x_2) = r(x_2)/r(x_1)$.

Set $\varepsilon = \ln\{r(x)\} + \ln\{T_x\}$, $a(x) = -\ln\{r(x)\}$. Then

$$\ln\{T_x\} = a(x) + \varepsilon.$$

The distribution of the random variable ε does not depend on x. The last equality implies that

$$\mathbf{Var}(\ln T_x) = \mathbf{Var}(\varepsilon).$$

The variance of the random variable $\ln\{T_x\}$ does not depend on x.

1.3.2 Definition of the model for time-varying stresses

The model (1.20) is generalized to the case of time-varying stresses by supposing that the GS model also holds, i.e. the hazard rates under time-varying stresses are obtained from the hazard rates under constant stresses by the rule (1.13).

Proposition 1.3.1 (Bagdonavičius [4]). *The GS model with the survival functions (1.20) on E_1 holds on $E \supset E_1$ if there exist a positive on E function r and a positive on $[0, \infty)$ function q such that for all $x(\cdot) \in E$*

$$\lambda_{x(\cdot)}(t) = r\{x(t)\}\, q\{S_{x(\cdot)}(t))\}. \tag{1.21}$$

Proposition 1.3.1 suggests the following model.

Definition 1.3.1 *The accelerated failure time (AFT) model holds on E if there exists a positive on E function r and a positive on $[0, \infty)$ function q such that for all $x(\cdot) \in E$* the formula (1.21) holds.

Under the AFT model the hazard rate $\lambda_{x(\cdot)}(t)$ at any moment t is proportional to a function of the stress applied at this moment and to a function of the probability of survival until t under $x(\cdot)$.

Let us find the expression of the survival function under time-varying stresses.

Proposition 2.6. (Bagdonavičius [4]). *Suppose that the integral*

$$\int_0^x \frac{dv}{q(v)} \tag{1.22}$$

converges for all $x \geq 0$.

The AFT model holds on a set of stresses E if there exists a survival function G such that for all $x(\cdot) \in E$

$$S_{x(\cdot)}(t) = G\left(\int_0^t r\{x(u)\}\, du\right). \tag{1.23}$$

If the AFT model holds on E_2 then the survival function under any stress $x(\cdot) \in E_2$ of the form (1.15) verifies the equality

$$S_{x(\cdot)}(t) = \begin{cases} S_{x_1}(t), & 0 \leq \tau < t_1, \\ S_{x_2}(t - t_1 + t_1^*), & \tau \geq t_1, \end{cases} \tag{1.24}$$

where

$$t_1^* = \frac{r(x_1)}{r(x_2)} t_1. \tag{1.25}$$

1.3.3 Relations between the means and the quantiles

Suppose that $x(\cdot)$ is a time-varying stress. Denote by $t_{x(\cdot)}(p)$ the p-quantile of the random variable $T_{x(\cdot)}$, and by $x_\tau = x(\tau)\mathbf{1}_{\{t\geq 0\}}$ a constant stress equal to the value of time-varying stress $x(\cdot)$ at the moment τ.

Proposition 1.3.3 (Bagdonavičius [4]). *Suppose that the AFT model holds on E and $x(\cdot)$, $x_t \in E$ for all $t \geq 0$. Then*

$$\int_0^{t_{x(\cdot)}(p)} \frac{d\tau}{t_{x_\tau}(p)} = 1. \tag{1.26}$$

If the means $\mathbf{E}(T_{x(\cdot)})$, $\mathbf{E}(T_{x_\tau})$ exist then

$$\mathbf{E}\left(\int_0^{T_{x(\cdot)}} \frac{d\tau}{\mathbf{E}(T_{x_\tau})}\right) = 1. \tag{1.27}$$

The model (1.27) is the *model of Miner* .

Corollary 1.3.1 *For the stress of the form (1.14) the formula (1.27) implies the equality*

$$\sum_{k=1}^m \frac{\mathbf{E}(T_k)}{\mathbf{E}(T_{x_k})} = 1, \tag{1.28}$$

where
$$T_k = \begin{cases} 0, & T_{x(\cdot)} < t_{k-1}, \\ T_{x(\cdot)} - t_{k-1}, & t_{k-1} \le T_{x(\cdot)} < t_k, \\ t_k - t_{k-1}, & T_{x(\cdot)} \ge t_k, \end{cases}$$
is the life in the interval $[t_{k-1}, t_k)$ *for the unit tested under the stress* $x(\cdot)$.

The formula (1.26) implies that for $t_{x(\cdot)}(p) \in [t_{k-1}, t_k)$ *the following equality holds:*
$$\sum_{i=1}^{k-1} \frac{t_i - t_{i-1}}{t_{x_i}(p)} + \frac{t_{x(\cdot)}(p) - t_{k-1}}{t_{x_k}(p)} = 1. \tag{1.29}$$
In the case $m = 2$, the formula (1.28) can be written in the form
$$\frac{\mathbf{E}(T_1)}{\mathbf{E}(T_{x_1})} + \frac{\mathbf{E}(T_2)}{\mathbf{E}(T_{x_2})} = 1, \tag{1.30}$$
and the formula (1.29) can be written in the form
$$\frac{t_1}{t_{x_1}(p)} + \frac{t_{x(\cdot)}(p) - t_1}{t_{x_2}(p)} = 1. \tag{1.31}$$
So
$$\mathbf{E}(T_{x_1}) = \frac{\mathbf{E}(T_1)}{1 - \frac{\mathbf{E}(T_2)}{\mathbf{E}(T_{x_2})}}, \tag{1.32}$$
and
$$t_{x_1}(p) = \frac{t_1}{1 - \frac{t_{x(\cdot)}(p) - t_1}{t_{x_2}(p)}}, \text{ if } t_{x(\cdot)}(p) \ge t_1. \tag{1.33}$$
Thus, if the AFT model holds on E_2 then $\mathbf{E}(T_1)$, $\mathbf{E}(T_2)$ and $\mathbf{E}(T_{x_2})$ determine $\mathbf{E}(T_{x_1})$, and $t_{x(\cdot)}(p)$ and $t_{x_2}(p)$ determine $t_{x_1}(p)$.

1.4 Proportional hazards model

1.4.1 Definition of the model for constant stresses

In survival analysis the mostly used model describing the influence of covariates on the lifetime distribution is the *proportional hazards* (PH) or Cox model, introduced by D. Cox [10].

Suppose that under different constant stresses $x \in E_1$ the hazard rates are proportional to a baseline hazard rate:
$$\lambda_x(t) = r(x)\, \lambda(t). \tag{1.34}$$
For $x \in E_1$ the survival functions have the form
$$S_x(t) = S^{r(x)}(t) = \exp\{-r(x)\Lambda(t)\}, \tag{1.35}$$
where
$$S(t) = \exp\left\{-\int_0^t \lambda(u)du\right\}, \quad \Lambda(t) = \int_0^t \lambda(u)du = -\ln S(t).$$

1.4.2 Definition of the model for time-varying stresses

In the statistical literature the following formal generalization of the PH model to the case of time-varying stresses is used.

Definition 1.4.1 *The proportional hazards (PH) model holds on a set of stresses E if for all $x(\cdot) \in E$*

$$\lambda_{x(\cdot)}(t) = r\{x(t)\}\ \lambda(t), \tag{1.36}$$

This definition implies that

$$\Lambda_{x(\cdot)}(t) = \int_0^t r\{x(u)\} d\Lambda(u). \tag{1.37}$$

In terms of survival functions the PH model is written :

$$S_{x(\cdot)}(t) = \exp\left\{-\int_0^t r\{x(u)\} d\Lambda(u)\right\}. \tag{1.38}$$

The AFT, PH a are rather restrictive. More given in Bagdonavičius and Nikulin (2002).

1.5 Wiener process

A Wiener process (or Brownian motion; notation W_t or W) is a time-continuous process with the properties

1. $W_0 = 0$.
2. $W_t \sim N(0, t)$ for all $t \geq 0$. That is, for each t the random variable W_t is distributed normally with mean $\mathbf{E}(W_t) = 0$and variance $\mathbf{Var}(W_t) = \mathbf{E}(W_t^2) = t$.
3. All increments $\Delta W_t := W_{t+\Delta t} - W_t$ on the nonoverlapping time intervals are independent. That is, the displacements $W_{t_2} - W_{t_1}$ and $W_{t_4} - W_{t_3}$ are independent for all $0 \leq t_1 < t_2 \leq t_3 < t_4$.
4. W_t depends continuously on t.

Generally for $0 \leq s < t$ the property $W_t - W_s \sim N(0, t - s)$holds, in particular

$$\mathbf{E}(W_t - W_s) = 0, \quad \mathbf{Var}(W_t - W_s) = \mathbf{E}((W_t - W_s)^2) = t - s. \tag{1.39}$$

1.6 Wiener process with drift

A stochastic process $\{W(t), t \geq 0\}$ is called a Wiener-process with drift if it has the following properties:

1. $W(0) = 0$;

2. $\{W(t), t \geq 0\}$ has stationary, independent increments,

3. Every increment $W(t) - W(s)$ has a normal distribution with expected value $\mu(t-s)$ and variance $\sigma^2|t-s|$.

Equivalently, $\{W(t), t \geq 0\}$ is a Winer process with drift if

$$W(t) = \mu t + X(t),$$

where $\{X(t), t \geq 0\}$ is a Wiener process with $\sigma^2 = Var(X(1))$. The constant μ is called drift parameter.

1.7 Gamma process

Stochastic process is a gamma process with the shape parameter ν and the scale parameter σ, denoted by $Z(t)\ inG(\nu(t), 1/\sigma)$, if

1. $Z(0) = 0$;

2. $Z(t)$ has independent increments, i. e. for any $0 < t_1 < ... < t_m$ the random variables $Z(t_1), Z(t_2) - Z(t_1), ..., Z(t_m) - Z(t_{t-1})$ are independent;

3. the distribution of $Z(t) - Z(s)$ is gamma with density

$$p_{Z(t)-Z(s)}(x) = \frac{1}{\Gamma(\nu(t) - \nu(s)} x^{\nu(t)-\nu(s)-1} \sigma^{-(\nu(t)-\nu(s))} e^{-x/\sigma}, x \geq 0.$$

The gamma process is non-decreasing and its increments $\Delta Z(t) = Z(t + \Delta t) - Z(t)$ are from the same family of gamma distributions.

Chapter 2

Statistical analysis of redundant systems

2.1 Redundant system with one main and one stand-by unit

Consider a redundant system with one operating and one stand-by unit. If the main unit fails then the stand-by unit (if it is not failed yet) is commuted and operates instead of the main one. We suppose that commuting is momentary and there are no repairs.

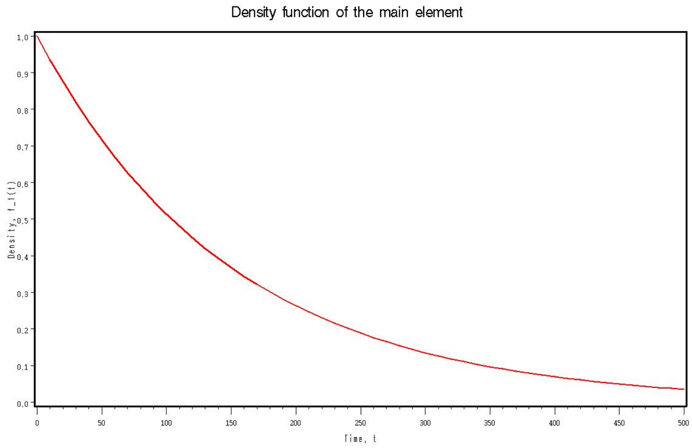
If the stand-by unit is not functioning until the failure of the main unit ("cold" reserving), it is possible that during and after commuting the failure rate increases because the stand-by unit is not "warmed" enough [39]. The probability density function of the main element presented in the graph.

If the stand-by unit is functioning in the same "hot" conditions as the main unit then usually after commuting the reliability of the stand-by unit does not change. But "hot" redundancy has disadvantages because the stand-by unit fails earlier than the main one with the probability 0.5. So "warm" reserving is sometimes used [41]: the stand by unit functions under lower stress than the main one. In such a case the probability of the failure of the stand-by unit is smaller than that of the main unit and it is also possible that commuting is fluent. So the main problem is to verify the hypothesis that the switch on from "warm" to "hot" conditions does not do some damage to units.

Let us formulate the hypothesis strictly.

2.1.1 The models

Suppose that in "hot" conditions the failure times of the main and the stand-by units are absolutely continuous and have the same c.d.f. F_1 and the probability density function f_1 (see Graph 2.1.), the failure time T_2 of the stand-by element has the c.d.f. F_2 and the probability density f_2. Similarly, in "warm" conditions the c.d.f. is F_2 and the p.d.f is f_2.



Graph 2.1. Density function of the main element

The failure time of the system is $T = max(T_1, T_2)$.

Denote by

$$f_2^{(y)}(x) = f_{T_2|T_1=y}(x)$$

the conditional p.d.f. of T_2 given that the main unit fails at the moment y. If $x \le y$ then $f_2^{(y)}(x) = f_2(x)$.

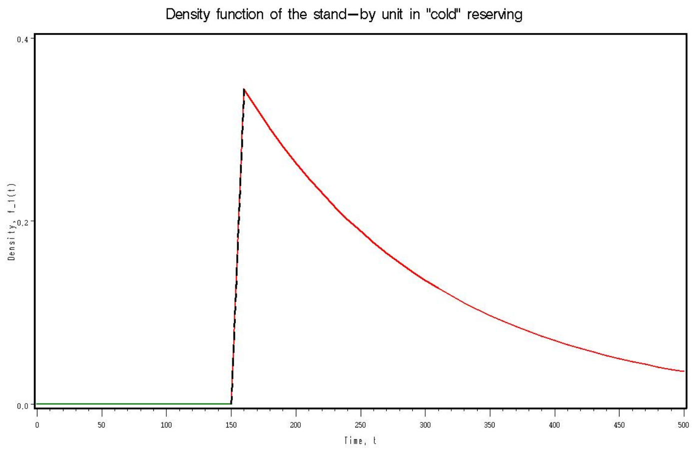
The c.d.f. of the system failure time T is

$$F(t) = P(T_1 \le t, T_2 \le t) = \int_0^t P(T_2 \le t | T_1 = y) f_1(y) dy =$$

$$= \int_0^t \left\{ \int_0^y f_2(x)dx + \int_y^t f_2^{(y)}(x)dx \right\} f_1(y)dy. \qquad (2.1)$$

When stand-by is "cold" then $f_2(x) = 0$ for $x \le y$ and $f_2^{(y)}(x) = f_1(x - y)$ for $x > y$, so

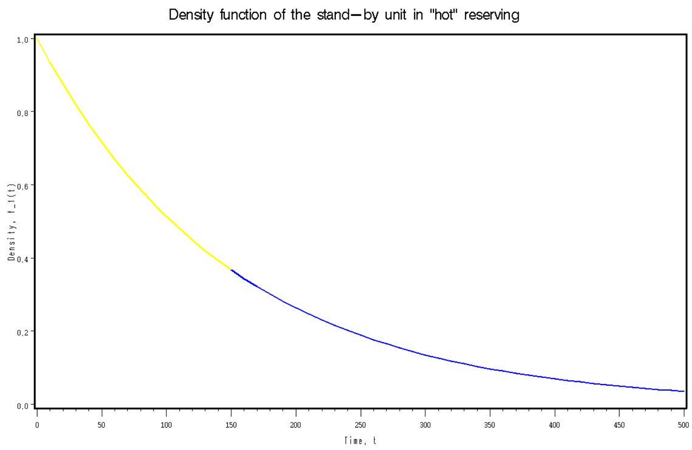
$$F(t) = \int_0^t \left\{ \int_y^t f_1(x - y)dx \right\} f_1(y)dy = \int_0^t F_1(t - y)dF_1(y).$$



Graph 2.2. Density function of the stand-by unit in "cold" reserving

When stand-by is "hot" then $f_2^{(y)}(x) = f_2(x) = f_1(x)$,
so

$$F(t) = \int\limits_0^t \left(\int_0^y f_2(x)dx + \int\limits_y^t f_2(x)dx \right) f_1(y)dy = [F_1(t)]^2.$$



Graph 2.3. Density function of the stand-by unit in "hot" reserving

In the case of "warm" reserving the following hypothesis is assumed:

$$H_0 : f_2^{(y)}(x) = f_1(x + g(y) - y), \quad \text{for all} \quad x \geq y \geq 0, \tag{2.2}$$

where $g(y)$ is the moment which in "hot" conditions corresponds to the moment y in "warm" conditions in the sense that

$$F_1(g(y)) = P(T_1 \leq g(y)) = P(T_2 \leq y) = F_2(y).$$

We suppose that the c.d.f. F_i are continuous and increasing on $(0, \infty)$. In such a case

$$g(y) = F_1^{-1}(F_2(y)).$$

Conditionally (given $T_1 = y$) the hypothesis corresponds to the Sediakin's model [36]. In [2] a goodness-of-fit test of logrank-type for Sediakin's model using experiments with fixed switch off moments is proposed (see also [6]). In the situation considered here the switch off moments are random. So we need a goodness-of-fit test for the model (2.2).

The formula (2.1) implies that under the hypothesis H_0

$$F(t) = \int_0^t \left\{ F_2(y) + \int_y^t f_1(x + g(y) - y)dx \right\} f_1(y)dy =$$

$$= \int_0^t \{F_2(y) + F_1(t + g(y) - y) - F_1(g(y))\} f_1(y)dy = \int_0^t F_1(t + g(y) - y)dF_1(y),$$

so

$$F(t) = \int_0^t F_1(t + g(y) - y)dF_1(y). \tag{2.3}$$

In particular, if we suppose that the distribution of the units functioning in "warm" and "hot" conditions differ only in scale, i.e.

$$F_2(t) = F_1(rt), \tag{2.4}$$

for some $r > 0$, then $g(y) = ry$.

In such a case instead of the hypothesis H_0 narrower hypothesis

$$H_0^* : \exists\, r > 0 : \quad f_2^{(y)}(x) = f_1(x + ry - y), \quad \text{for all} \quad x \geq y \geq 0, \tag{2.5}$$

is to be verified. Conditionally (given $T_1 = y$) the hypothesis corresponds to the accelerated failure time (AFT) model [1], [31]. In [4] a goodness-of-fit test for AFT model using experiments with fixed switch moments is proposed (see also [6]). So we need a test for the hypothesis H_0^*.

2.1.2 Goodness-of-fit test for the hypothesis H_0^*

Suppose that the following data are available :

a) the failure times $T_{11}, \ldots, T_{1n_1}$ of n_1 units tested in "hot" conditions;

b) the failure times $T_{21}, \ldots, T_{2n_2}$ of n_2 units tested in "warm" conditions;

c) the failure times $T_1, \ldots, T_n$ of n redundant systems (with "warm" stand-by units).

The tests are based on the difference of two estimators of the c.d.f. $F(t)$ of the system. The first is the empirical distribution function obtained from the data $T_1, \ldots, T_n$:

$$\hat{F}^{(1)}(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{T_i \leq t\}}. \tag{2.6}$$

The second uses the data $T_{11}, \ldots, T_{1n_1}$ and $T_{21}, \ldots, T_{2n_2}$ and is based on the formula (2.3):

$$\hat{F}^{(2)}(t) = \int_0^t \hat{F}_1(t + \hat{g}(y) - y)d\hat{F}_1(y),$$

where (if we test the hypothesis H_0)

$$\hat{g}(y) = \hat{F}_1^{-1}(\hat{F}_2(y)), \quad \hat{F}_j(t) = \frac{1}{n_j}\sum_{i=1}^{n_j} \mathbf{1}_{\{T_{ji}\le t\}}, \quad \hat{F}_1^{-1}(y) = \inf\{s : \hat{F}_1(s) \ge t\}, \tag{2.7}$$

or (if we test the hypothesis H_0^*)

$$\hat{g}(y) = \hat{r}y, \quad \hat{r} = \frac{\hat{\mu}_1}{\hat{\mu}_2}, \quad \hat{\mu}_j = \frac{1}{n_j}\sum_{i=1}^{n_j} T_{ji}. \tag{2.8}$$

The tests are based on the statistic

$$X = \sqrt{n}\int_0^\infty (\hat{F}^{(1)}(t) - \hat{F}^{(2)}(t))dt. \tag{2.9}$$

In the following sense it is analogous to the Student's t-test for comparing the means of two populations. Indeed, the mean failure time of the system with c.d.f. F is

$$\mu = \int_0^\infty [1 - F(s)]ds,$$

so the statistic (2.9) is the normed difference of two estimators (the second being not the empirical mean) of the mean μ. Student's t-test is based on the difference of empirical means of two populations.

It will be shown that in the case of both hypothesis H_0 and H_0^* the limit distribution (as $n_i/n \to l_i \in (0,\infty)$, $n \to \infty$) of the statistic X is normal with zero mean and finite variance σ^2.

Let us find the asymptotic distribution of the statistic (2.9).

Theorem 2.1.1 *Suppose that $n_i/n \to l_i \in (0,\infty)$, $n \to \infty$ and the densities $f_i(x)$, $i = 1,2$ are continuous and positive on $(0,\infty)$. Then under H_0^* the statistic (2.9) converges in distribution to the normal law $N(0,\sigma^2)$, where*

$$\sigma^2 = \mathbf{Var}(T_i) + \frac{1}{l_1}\mathbf{Var}(H(T_{1i})) + \frac{c^2r^2}{l_2}\mathbf{Var}(T_{2i}), \tag{2.10}$$

$$H(x) = x[c + r - 1 - F_1(x/r) - rF_2(x)] + r\mathbf{E}(\mathbf{1}_{\{T_{1i}\le x/r\}}T_{1i}) + r\mathbf{E}(\mathbf{1}_{\{T_{2i}\le x\}}T_{2i}),$$
$$c = \frac{1}{\mu_2}\int_0^\infty y[1 - F_2(y)]dF_1(y).$$

Proof. The limit distribution of the empirical distribution functions is well known:

$$\sqrt{n}(\hat{F}_i - F_i) \xrightarrow{\mathcal{D}} U_i, \quad \sqrt{n}(\hat{F}^{(1)} - F) \xrightarrow{\mathcal{D}} U \tag{2.11}$$

on $D[0,\infty)$, where $\xrightarrow{\mathcal{D}}$ means weak convergence, U_1, U_2 and U are independent Gaussian martingales with $U_i(0) = U(0) = 0$ and the covariances

$$\mathbf{cov}(U_i(s_1), U_i(s_2)) = \frac{1}{l_i}\, F_i(s_1 \wedge s_2)(1 - F_i(s_1 \vee s_2)),$$

$$\mathbf{cov}(U(s_1), U(s_2)) = F(s_1 \wedge s_2)(1 - F(s_1 \vee s_2)).$$

The Glivenko-Cantelli theorem states that the empirical distribution functions converge in probability two the c.d.f. uniformly on $\mathbf{R}$:

$$\underset{x\in\mathbf{R}}{sup}\,|(\hat{F}_i(x) - F_i(x)| \xrightarrow{P} 0, \quad \underset{x\in\mathbf{R}}{sup}\,|(\hat{F}(x) - F(x)| \xrightarrow{P} 0.$$

Under the hypothesis H_0^* the difference of the two estimators of the distribution function F can be written as follows:

$$\hat{F}^{(1)}(t) - \hat{F}^{(2)}(t) = \hat{F}^{(1)}(t) - F(t) - \int_0^t \hat{F}_1(t + \hat{g}(y) - y)d\hat{F}_1(y) + \int_0^t F_1(t + g(y) - y)\times$$

$$\times dF_1(y) = \hat{F}^{(1)}(t) - F(t) - \int_0^t [F_1(t + \hat{g}(y) - y) - F_1(t + g(y) - y)]dF_1(y) -$$

$$-\int_0^t [(\hat{F}_1(t+\hat{g}(y)-y) - \hat{F}_1(t+g(y)-y)) - (F_1(t+\hat{g}(y)-y) - F_1(t+g(y)-y))]dF_1(y) -$$

$$-\int_0^t [\hat{F}_1(t + \hat{g}(y) - y) - \hat{F}_1(t + g(y) - y)](d\hat{F}_1(y) - dF_1(y)) -$$

$$-\int_0^t [\hat{F}_1(t + g(y) - y) - F_1(t + g(y) - y)]dF_1(y) -$$

$$-\int_0^t [\hat{F}_1(t + g(y) - y) - F_1(t + g(y) - y)](d\hat{F}_1(y) - dF_1(y)) -$$

$$-\int_0^t F_1(t + g(y) - y)(d\hat{F}_1(y) - dF_1(y)).$$

The statistic (2.9) can be written

$$X = \int_0^\infty \sqrt{n}[\hat{F}^{(1)}(t) - F(t)]dt -$$

$$-\int_0^\infty dt \int_0^t \sqrt{n}[F_1(t + \hat{g}(y) - y) - F_1(t + g(y) - y)]\, dF_1(y) -$$

$$-\int_0^\infty dt \int_0^t \sqrt{n}[\hat{F}_1(t + g(y) - y) - F_1(t + g(y) - y)]\, dF_1(y) -$$

$$-\int_0^\infty dt \int_0^t F_1(t + g(y) - y)d\{\sqrt{n}[\hat{F}_1(y) - F_1(y)]\} + o_P(1); \qquad (2.12)$$

here $o_P(1)$ denote a random variable which converges in probability to zero: $o_P(1) \xrightarrow{P} 0$.

Set $\sigma_j^2 = \mathbf{Var}(T_{ji})$, $j = 1, 2$. The convergence

$$\sqrt{n}(\hat{\mu}_j - \mu_j) = -\sqrt{n}\int_0^\infty [\hat{F}_j(y) - F_j(y)]dy \xrightarrow{\mathcal{D}} Y_j = -\int_0^\infty U_j(y)dy \sim N(0, \sigma_j^2/l_i)$$

implies the convergence

$$\sqrt{n}(\hat{r}-r) \xrightarrow{\mathcal{D}} Y = \frac{1}{\mu_2}(Y_1 - rY_2) \sim N(0, \frac{\sigma_1^2(l_1+l_2)}{l_1 l_2 \mu_2^2}). \tag{2.13}$$

The formulas (2.11)-(2.13) imply

$$\int_0^\infty \sqrt{n}[\hat{F}^{(1)}(t) - F(t)]dt \xrightarrow{\mathcal{D}} \int_0^\infty U(t)dt,$$

$$\int_0^\infty dt \int_0^t \sqrt{n}[F_1(t+\hat{r}y-y) - F_1(t+ry-y)]\,dF_1(y) \xrightarrow{\mathcal{D}}$$

$$\xrightarrow{\mathcal{D}} Y\int_0^\infty dt \int_0^t y f_1(t+ry-y)\,dF_1(y) = Y\int_0^\infty y\,dF_1(y)\int_y^\infty f_1(t+ry-y)\,dt =$$

$$= Y\int_0^\infty y[1-F_1(ry)]\,dF_1(y) = cY_1 - rcY_2 = -c\int_0^\infty U_1(y)dy + rc\int_0^\infty U_2(y)dy,$$

$$\int_0^\infty dt \int_0^t \sqrt{n}[\hat{F}_1(t+ry-y) - F_1(t+ry-y)]\,dF_1(y) \xrightarrow{\mathcal{D}}$$

$$\xrightarrow{\mathcal{D}} \int_0^\infty dt\int_0^t U_1(t+ry-y)dF_1(y) = \int_0^\infty dF_1(y)\int_{ry}^\infty U_1(u)du =$$

$$= \int_0^\infty U_1(u)F_1(u/r)du,$$

$$\int_0^\infty dt\int_0^t F_1(t+ry-y)d\{\sqrt{n}[\hat{F}_1(y) - F_1(y)]\} \xrightarrow{\mathcal{D}}$$

$$\xrightarrow{\mathcal{D}} \int_0^\infty dt\int_0^t F_1(t+ry-y)dU_1(y) =$$

$$= \int_0^\infty [F_1(rt)U_1(t) - \int_0^t U_1(y)\,dF_1(t+ry-y)]dt = \int_0^\infty F_2(t)U_1(t)\,dt -$$

$$-\int_0^\infty dt\int_t^{rt} U_1((v-t)/(r-1))dF_1(v) = \int_0^\infty F_2(t)U_1(t)\,dt +$$

$$+\int_0^\infty dF_1(v)\int_t^{v/r} U_1((v-t)/(r-1))dt = \int_0^\infty F_2(t)U_1(t)\,dt +$$

$$+(1-r)\int_0^\infty dF_1(v)\int_0^{v/r} U_1(u))du = \int_0^\infty F_2(t)U_1(t)\,dt +$$

$$+(r-1)\int_0^\infty [1-F_1(ru)]U_1(u))du = \int_0^\infty U_1(y)[rF_2(y) - r + 1]dy.$$

We obtained

$$X \xrightarrow{\mathcal{D}} V_1 + V_2 + V_3,$$

where

$$V_1 = \int_0^\infty U(y)dy, \quad V_2 = \int_0^\infty h(y)U_1(y)dy,$$

$$h(y) = c + r - 1 - F_1(y/r) - rF_2(y), \quad V_3 = -rc\int_0^\infty U_2(y)dy.$$

We shall show that if $G(x) = \int_0^t g(u)du$ then

$$\mathbf{Var}(G(T_i)) = \mathbf{Var}(\int_0^\infty U(y)dG(y)). \tag{2.14}$$

Indeed, set $S(x) = 1 - F(x)$.

$$\mathbf{Var}(G(T_i)) = -\int_0^\infty G^2(x)dS(x) - (\int_0^\infty G(x)dS(x))^2 = 2\int_0^\infty G(x)S(x)dG(x)-$$

$$-(\int_0^\infty S(x)dG(x))^2 = 2\int_0^\infty S(x)dG(x)\int_0^x dG(y)-$$

$$-2\int_0^\infty S(x)dG(x)\int_0^x S(y)dG(y) = 2\int_0^\infty S(x)dG(x)\int_0^x F(y)dG(y) =$$

$$= 2\int_0^\infty\int_0^x EU(x)U(y)dG(x)dG(y) = \mathbf{Var}(\int_0^\infty U(y)dG(y)).$$

Analogous to (2.14) equalities are true replacing the r.v. T_i by T_{ji} and the random processes U by U_j, $j = 1, 2$. These equalities imply that the variances of the random variables V_i are:

$$\mathbf{Var}(V_1) = \mathbf{Var}(T_i), \quad \mathbf{Var}(V_3) = \frac{c^2r^2}{l_2}\mathbf{Var}(T_{2i})$$

$$\mathbf{Var}(V_2) = \frac{2}{l_1}\int_0^\infty [1 - F_1(y)]h(y)dy\int_0^y F_1(z)h(z)dz = \frac{1}{l_1}\mathbf{Var}(H(T_{1i})),$$

where

$$H(x) = \int_0^x h(y)dy = x[c + r - 1 - F_1(x/r) - rF_2(x)] + \int_0^x ydF_1(y/r)+$$

$$+r\int_0^x ydF_2(y) = x[c + r - 1 - F_1(x/r) - rF_2(x)]+$$

$$+r\mathbf{E}(\mathbf{1}_{\{T_{1i}\le x/r\}}T_{1i}) + r\mathbf{E}(\mathbf{1}_{\{T_{2i}\le x\}}T_{2i}).$$

The proof is complete.

A consistent estimator of the variance σ^2 is

$$\hat\sigma^2 = \frac{1}{n}\sum_{i=1}^n (T_i - \hat\mu)^2 + \frac{n}{n_1^2}\sum_{i=1}^{n_1}[\hat H(T_{1i}) - \hat{\bar{H}}]^2 + \frac{\hat c^2\hat r^2 n}{n_2^2}\sum_{i=1}^{n_2}(T_{2i} - \hat\mu_2)^2,$$

where

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} T_i, \quad \hat{c} = \frac{1}{\hat{\mu}_2}\int_0^\infty y[1-\hat{F}_2(y)]d\hat{F}_1(y) = \frac{1}{\hat{\mu}_2 n_1}\sum_{i=1}^{n_1} T_{1i}[1-\hat{F}_2(T_{1i})],$$

$$\hat{H}(x) = x[\hat{c}+\hat{r}-1-\hat{F}_1(x/\hat{r})-\hat{r}\hat{F}_2(x)] + \frac{\hat{r}}{n_1}\sum_{i=1}^{n_1}\mathbf{1}_{\{T_{1i}\le x/\hat{r}\}}T_{1i} + \frac{\hat{r}}{n_2}\sum_{i=1}^{n_2}\mathbf{1}_{\{T_{2i}\le x\}}T_{2i},$$

$$\hat{\hat{H}} = \frac{1}{n_1}\sum_{i=1}^{n_1}\hat{H}(T_{1i}).$$

The statistic

$$X = \sqrt{n}\int_0^\infty (\hat{F}^{(1)}(t) - \hat{F}^{(2)}(t))dt$$

can be written

$$X = -\sqrt{n}\int_0^\infty t\, d(\hat{F}^{(1)}(t) - \hat{F}^{(2)}(t)).$$

Note that

$$\hat{F}^{(2)}(t) = \int_0^t \hat{F}_1(t+\hat{r}y-y)d\hat{F}_1(y) = \frac{1}{n_1}\sum_{i=1}^{n_1}\hat{F}_1(t+\hat{r}T_{1i}-T_{1i})\mathbf{1}_{\{T_{1i}\le t\}} =$$

$$= \frac{1}{n_1^2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_1}\mathbf{1}_{\{T_{1j}\le t+\hat{r}T_{1i}-T_{1i}\}}\mathbf{1}_{\{T_{1i}\le t\}} = \frac{1}{n_1^2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_1}\mathbf{1}_{\{\max(T_{1i},T_{1j}-\hat{r}T_{1i}+T_{1i})\le t\}}.$$

So

$$X = -\sqrt{n}\hat{\mu} + \frac{\sqrt{n}}{n_1^2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_1}\max(T_{1i}, T_{1j}-\hat{r}T_{1i}+T_{1i}).$$

The test statistic has the form

$$T = \frac{X}{\hat{\sigma}},$$

where $\hat{\sigma}$. The distribution of the statistic T is approximated by the standard normal distribution.

The test. The hypothesis H_0^* is rejected with the asymptotic significance value α if $\mid T \mid > z_{\alpha/2}$, where $z_{\alpha/2}$ is the $\alpha/2$-critical value of the standard normal distribution.

2.1.3 Goodness-of-fit test for the hypothesis H_0

As in the case of the hypothesis H_0^* the test for the hypothesis H_0 is based on the statistic (2.9). Let us find the asymptotic distribution of this statistic under the hypothesis H_0.

Theorem 2.1.2 *Suppose that $n_i/n \to l_i \in (0,\infty)$, $n \to \infty$ and the densities $f_i(x)$, $i = 1,2$ are continuous and positive on $(0,\infty)$. Then under H_0 the statistic (2.9) converges in distribution to the normal law $N(0,\sigma^2)$, where*

$$\sigma^2 = \mathbf{Var}(T_i) + \frac{1}{l_1}\mathbf{Var}(H(T_{1i})) + \frac{1}{l_2}\mathbf{Var}(Q(T_{2i}))$$

where

$$H(x) = Q(x) - xF_1(g^{-1}(x)) + g(x)[1 - F_2(x)] + \mathbf{E}(\mathbf{1}_{\{g(T_{1i})\le x\}}g(T_{1i})) +$$

$$+\mathbf{E}(\mathbf{1}_{\{T_{2i}\le x\}}g(T_{2i})) - x, \quad Q(x) = \mathbf{E}\{\mathbf{1}_{\{T_{1i}\le x\}}[1 - F_2(T_{1i})]/f_1(g(T_{1i}))\}.$$

Proof. Similarly as in Theorem 2.1.1 we obtain

$$\int_0^\infty \sqrt{n}[\hat{F}^{(1)}(t) - F(t)]dt \xrightarrow{\mathcal{D}} \int_0^\infty U(t)dt,$$

$$\int_0^\infty dt \int_0^t \sqrt{n}[F_1(t + \hat{g}(y) - y) - F_1(t + g(y) - y)]\, dF_1(y) \xrightarrow{\mathcal{D}}$$

$$\xrightarrow{\mathcal{D}} -\int_0^\infty \frac{U_1(g(y)) - U_2(y)}{f_1(g(y))} f_1(y)[1 - F_2(y)]dy,$$

$$\int_0^\infty dt \int_0^t \sqrt{n}[\hat{F}_1(t + g(y) - y) - F_1(t + g(y) - y)]\, dF_1(y) \xrightarrow{\mathcal{D}}$$

$$\xrightarrow{\mathcal{D}} \int_0^\infty U_1(u)F_1(g^{-1}(u))du,$$

$$\int_0^\infty dt \int_0^t F_1(t + g(y) - y)d\{\sqrt{n}[\hat{F}_1(y) - F_1(y)]\} \xrightarrow{\mathcal{D}}$$

$$\xrightarrow{\mathcal{D}} \int_0^\infty dt \int_0^t F_1(t + g(y) - y)dU_1(y) = \int_0^\infty F_2(t)U_1(t)dt -$$

$$-\int_0^\infty U_1(y)[1 - F_2(y)]d(g(y) - y) = \int_0^\infty U_1(y)dy - \int_0^\infty U_1(y)[1 - F_2(y)]dg(y).$$

We obtained

$$X \xrightarrow{\mathcal{D}} V_1 + V_2 + V_3,$$

where

$$V_1 = \int_0^\infty U(y)dy, \quad V_2 = \int_0^\infty h(y)U_1(y)dy,$$

$$h(y) = \frac{f_1(y)}{f_1(g(y))}[1 - F_2(y)] - F_1(g^{-1}(y)) - 1 + g'(y)[1 - F_2(y)].$$

$$V_3 = -\int_0^\infty \frac{U_2(y)}{f_1(g(y))}[1 - F_2(y)]dF_1(y).$$

The variances of the random variables V_1 and V_3 are

$$\mathbf{Var}(V_1) = \mathbf{Var}(T_i), \quad \mathbf{Var}(V_3) = \frac{1}{l_2}\mathbf{Var}(Q(T_{2i}));$$

here

$$Q(x) = \int_0^\infty \frac{1 - F_2(y)}{f_1(g(y))} dF_1(y) = \mathbf{E}\{\mathbf{1}_{\{T_{1i} \le x\}}[1 - F_2(T_{1i})]/f_1(g(T_{1i}))\}.$$

The variance of the random variable V_2 is

$$\mathbf{Var}(V_2) = \frac{1}{l_1}\mathbf{Var}(H(T_{1i})),$$

where

$$H(x) = \int_0^x \frac{[1 - F_2(y)]}{f_1(g(y))} dF_1(y) - \int_0^x F_1(g^{-1}(y))dy - x + \int_0^x [1 - F_2(y)]dg(y) =$$

$$= Q(x) - xF_1(g^{-1}(x)) + \int_0^x y\, dF_1(g^{-1}(y)) - x + [1 - F_2(x)]g(x) + \int_0^x g(y)dF_2(y) =$$

$$= Q(x) - xF_1(g^{-1}(x)) + g(x)[1 - F_2(x)] - x + \mathbf{E}(\mathbf{1}_{\{g(T_{1i}) \le x\}} g(T_{1i})) + \mathbf{E}(\mathbf{1}_{\{T_{2i} \le x\}} g(T_{2i})).$$

The proof is complete.

A consistent estimator of the variance σ^2 is

$$\hat\sigma^2 = \frac{1}{n}\sum_{i=1}^n (T_i - \hat\mu)^2 + \frac{n}{n_1^2}\sum_{i=1}^{n_1}[\hat H(T_{1i}) - \hat{\hat H}]^2 + \frac{n}{n_2^2}\sum_{i=1}^{n_2}[\hat Q(T_{2i}) - \hat{\hat Q}]^2,$$

where

$$\hat H(x) = \hat Q(x) - x\hat F_1(\hat g^{-1}(x)) + \hat g(x)[1 - \hat F_2(x)] - x + \frac{1}{n_1}\sum_{i=1}^{n_1} \mathbf{1}_{\{\hat g(T_{1i}) \le x\}} \hat g(T_{1i}) +$$

$$+ \frac{1}{n_2}\sum_{i=1}^{n_2} \mathbf{1}_{\{T_{2i} \le x\}} \hat g(T_{2i}), \quad \hat Q(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} \mathbf{1}_{\{T_{1i} \le x\}}[1 - \hat F_2(T_{1i})]/\hat f_1(\hat g(T_{1i})),$$

$$\hat g^{-1}(x) = \hat F_2^{-1}(\hat F_1(x)), \quad \hat{\hat H} = \frac{1}{n_1}\sum_{i=1}^{n_1} \hat H(T_{1i}), \quad \hat{\hat Q} = \frac{1}{n_2}\sum_{i=1}^{n_2} \hat Q(T_{2i}),$$

the density f_1 is estimated by the kernel estimator

$$\hat f_1(x) = \frac{1}{n}\sum_{i=1}^n \frac{1}{h} K\left(\frac{x - X_{1i}}{h}\right),$$

$$X = -\sqrt{n}\hat\mu + \frac{\sqrt{n}}{n_1^2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_1} \max(T_{1i}, T_{1j} - \hat g(T_{1i}) + T_{1i}).$$

The test statistic has the form

$$T = \frac{X}{\hat{\sigma}},$$

where $\hat{\sigma}$. The distribution of the statistic T is approximated by the standard normal distribution.

The test. The hypothesis H_0 is rejected with the asymptotic significance value α if $\mid T \mid > z_{\alpha/2}$, where $z_{\alpha/2}$ is the $\alpha/2$-critical value of the standard normal distribution.

2.1.4 Simulations: power of the tests

We investigated the power of the proposed goodness-of-fit tests when the distribution of the units in "warm" and "hot" conditions is exponential, Weibull and loglogistic.

Let us consider the following alternative hypothesis $\tilde{H}_0^*$:

$$f_2^{(y)}(x) = f_1[x + F_1^{-1}(F_2(y) + p(1 - F_2(y)) - y], \quad 0 < p < 1.$$

It means that at the switching time y the c.d.f. of the stand-by unit has a jump of size $p(1 - F_2(y))$.

Set

$$g_p(y) = F_1^{-1}(F_2(y) + p(1 - F_2(y)).$$

Under the alternative the c.d.f. of the stand-by system is

$$F(t) = \int_0^t F_1(t + g_p(y) - y) dF_1(y) = F_1(t) - \int_0^t S_1(t + g_p(y) - y) dF_1(y). \quad (2.15)$$

Example 2.1.1. Simulated exponential distribution:

$$T_{1j} \sim \mathcal{E}(\lambda_1), \quad T_{2j} \sim \mathcal{E}(\lambda_2), \quad \lambda_2 = r\lambda_1.$$

The c.d.f. of T_{ij} for all $t \geq 0$ is $F_i(t) = 1 - e^{-\lambda_i t}$.

In this case the hypotheses H_0 and H_0^* coincide and under these hypotheses the c.d.f. of the redundant system is

$$F(t) = 1 - \frac{\lambda_2 + \lambda_1}{\lambda_2} e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2} e^{-(\lambda_1 + \lambda_2)t}.$$

Under the exponential distribution and under the alternative hypothesis $\tilde{H}_0^*$ we have

$$g_p(y) = \frac{\lambda_2}{\lambda_1} y - \frac{1}{\lambda_1} \ln(1 - p),$$

and the c.d.f. of the redundant system is

$$F(t) = 1 - e^{-\lambda_1 t} - \lambda_1 (1 - p) e^{-\lambda_1 t} \int_0^t e^{-\lambda_2 y} dy =$$

$$= 1 - e^{-\lambda_1 t} - (1-p)\frac{\lambda_1}{\lambda_2} e^{-\lambda_1 t}(1 - e^{-\lambda_2 t}).$$

In this example the test for the hypothesis H_0^* is considered, so the following values of the parameters were used:

$$\lambda_1 = 1/100, \quad \lambda_2 = 1/300, \quad r = 1/3.$$

The distribution function of a redundant system in this case is

$$F(t) = 1 - 4e^{-t/100} + 3e^{-(4/300)t}.$$

The hypothesis H_0^* was tested using $\alpha = 0.05$ asymptotic significance level and several values of the sample size n, $\quad n = n_1 = n_2$. The number of replications was 3000.

Table 2.1. Significance level of the test

Sample size	Significance level (%)
50	7.5
100	5.3
200	5.1

Under the Exponential distribution and under the alternative hypothesis $\tilde{H}_0^*$ we have

$$F(t) = 1 - e^{-t/100} - (1-p)3e^{-t/100}(1 - e^{-t/300}).$$

Table 2.2. Power of the test

Sample size \ Constant	0.1	0.25	0.5	0.75	0.9
50	5	8	24	45	79
100	6	12	32	85	100
200	15	39	88	100	100

Example 2.1.2. Simulated distribution: Weibull:

$$T_{1j} \sim W(\alpha_1, \beta_1), \quad T_{2j} \sim W(\alpha_2, \beta_2),$$

The c.d.f. of T_{ij} for all $t \geq 0$ is $F_i(t) = 1 - e^{-(t/\beta_i)^{\alpha_i}}$.

Under the hypothesis H_0 the function g is $g(t) = \beta_1(t/\beta_2)^{\alpha_2/\alpha_1}$ and the c.d.f. of the redundant system is

$$F(t) = F_1(t) - \frac{\alpha_1}{\beta_1^{\alpha_1}} \int_0^t y^{\alpha_1 - 1} e^{-(y/\beta_1)^{\alpha_1} - [(t-y)/\beta_1 + (y/\beta_2)^{\alpha_2/\alpha_1}]^{\alpha_1}} dy.$$

The hypothesis H_0 coincides with the hypothesis H_0^* if $\alpha_1 = \alpha_2$. In such a case $g(t) = rt$ and $r = \beta_1/\beta_2$.

Under the Weibull distribution and under the alternative hypothesis $\tilde{H}_0^*$ we have

$$g_p(y) = \beta_1[-\ln(1-p) + (y/\beta_2)^{\alpha_2}]^{1/\alpha_1},$$

and the c.d.f. of the redundant system is

$$F(t) = F_1(t) -$$

$$-\frac{\alpha_1}{\beta_1^{\alpha_1}} \int_0^t y^{\alpha_1 - 1} e^{-(y/\beta_1)^{\alpha_1}} e^{-[(t-y)/\beta_1 + (-\ln(1-p) + (y/\beta_2)^{\alpha_2})^{1/\alpha_1}]^{\alpha_1}} dy.$$

In this example the test for the hypothesis H_0^* is considered, so the following values of the parameters were used:

$$\alpha_1 = \alpha_2 = 2, \quad \beta_1 = 100, \quad \beta_2 = 300.$$

The distribution function of a redundant system in this case is

$$F(t) = 1 - e^{-t^2/\beta_1^2} - \frac{2}{\beta_1^2} \int_0^t y e^{-1/\beta_1^2 \left(t^2 - \frac{4}{3} ty + \frac{13}{9} y^2\right)} dy.$$

The hypothesis H_0^* was tested using $\alpha = 0.05$ asymptotic significance level and several values of the sample size n, $\quad n = n_1 = n_2$. The number of replications was 3000.

Table 2.3. Significance level of the test

Sample size	Significance level (%)
50	4.7
100	5.3
200	5.0

Under the alternative hypothesis $\tilde{H}_0^*$ the c.d.f. of the redundant system is

$$F(t) = 1 - e^{-t^2/\beta_1^2} - \frac{2}{\beta_1^2} \int_0^t y e^{-y^2/\beta_1^2} e^{-\left((t-y)/\beta_1 + \sqrt{-\log(1-p) + \frac{y^2}{9\beta_1^2}}\right)^2} dy.$$

Table 2.4. Power of the test

Sample size \ Constant	0.1	0.25	0.5	0.75	0.9
50	9	21	30	71	88
100	14	35	45	76	95
200	25	58	74	100	100

Example 2.1.3. Continuing Example 2.1.2, instead of the hypothesis H_0^* we considered the hypothesis H_0, taking different values of α_i :

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \beta_1 = 100, \quad \beta_2 = 300.$$

Since the Gauss error function is

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

and relation between the Gauss error function erf and the c.d.f. of the standard normal distribution Φ is $erf(x) = 2\Phi(x\sqrt{2}) - 1$, the distribution function of a redundant system is

$$F(t) = 1 - e^{-\frac{t}{\beta_1}} - \frac{1}{\beta_1} e^{-\frac{t}{\beta_1}} \int_0^t e^{-y^2/(9\beta_1^2)} dy =$$
$$= 1 - e^{-t/\beta_1}[1 + 3\sqrt{\pi}(\Phi(t\sqrt{2}/(3\beta_1)) - 0.5)].$$

The hypothesis H_0 was tested using 5 per cent significance level and several values of the sample size n, $\quad n = n_1 = n_2$. Number of replications was 3000.

Table 2.5. Significance level of the test

Sample size	Significance level (%)
50	5.6
100	5.2
200	5.1

Taking into account that

$$F_1^{-1}(u) = -\beta_1 \ln(1-u), \quad g_p(y) = -\beta_1 \ln(1-p) + \frac{y^2}{9\beta_1},$$

we obtain that the c.d.f. of the redundant system under the alternative hypothesis H_0 is

$$F(t) = 1 - e^{-t/\beta_1} - (1-p)e^{-t/\beta_1} \frac{1}{\beta_1} \int_0^t e^{-\frac{y^2}{9\beta_1^2}} dy =$$
$$= 1 - e^{-t/\beta_1}[1 + 3(1-p)\sqrt{\pi}(\Phi(t\sqrt{2}/(3\beta_1)) - 0.5)].$$

Table 2.6. Power of the test

Sample size \ Constant	0.1	0.25	0.5	0.75	0.9
50	7	11	45	79	93
100	15	33	56	92	100
200	24	45	77	100	100

Example 2.1.4. Simulated distribution: loglogistic:

$$T_{1j} \sim L(\alpha_1, \beta_1), \quad T_{2j} \sim L(\alpha_2, \beta_2),$$

The c.d.f. of T_{ij} for all $t \geq 0$ is $F_i(t) = 1 - \frac{1}{(1+(t/\beta_i)^{\alpha_i})}$.

Under the hypothesis H_0 the function g is $g(t) = \beta_1(t/\beta_2)^{\alpha_2/\alpha_1}$ and the c.d.f. of the redundant system is

$$F(t) = F_1(t) - -\frac{\alpha_1}{\beta_1^{\alpha_1}} \int_0^t \frac{y^{\alpha_1 - 1}}{(1 + (y/\beta_1)^{\alpha_1})^2} * \frac{dy}{1 + [(t-y)/\beta_1 + (y/\beta_2)^{\alpha_2/\alpha_1}]^{\alpha_1}}.$$

Under the loglogistic distribution and under the alternative hypothesis $\tilde{H}_0^*$ we have

$$g_p(y) = \beta_1 \left(\frac{p + (y/\beta_2)^{\alpha_2}}{1-p} \right)^{1/\alpha_1},$$

and the c.d.f. of the redundant system is

$$F(t) = F_1(t) -$$

$$\frac{\alpha_1}{\beta_1^{\alpha_1}} \int_0^t \frac{y^{\alpha_1 - 1}}{(1 + (y/\beta_1)^{\alpha_1})^2} \frac{1}{1 + \left(\frac{t-y}{\beta_1} + \left(\frac{p + (y/\beta_2)^{\alpha_2}}{1-p} \right)^{1/\alpha_1} \right)^{\alpha_1}} dy.$$

In this example the test for the hypothesis H_0^* is considered, so the following values of the parameters were used:

$$\alpha_1 = \alpha_2 = 2, \quad \beta_1 = 100, \quad \beta_2 = 300.$$

The distribution function of a redundant system in this case is

$$F(t) = 1 - \frac{1}{1 + \frac{t^2}{\beta_1^2}} - \frac{2}{\beta_1^2} \int_0^t \frac{1}{1 + \left(\frac{t}{\beta_1} - \frac{2y}{3\beta_1} \right)^2} * \frac{y}{\left(1 + \frac{y^2}{\beta_1^2} \right)^2} dy.$$

The hypothesis H_0^* is tested using 5 per cent significance level and several values of the sample size n, $\quad n = n_1 = n_2$. Number of replication was 3000.

Table 2.7. Significance level of the test

Sample size	Significance level (%)
50	8.1
100	6.3
200	5.2

Under the alternative hypothesis H_0^* the c.d.f. of the redundant system is

$$F(t) = 1 - \frac{1}{1 + \left(\frac{y}{\beta_1} \right)^2} -$$

$$-\frac{2}{\beta_1^2} \int_0^t \frac{y}{\left(1 + \left(\frac{y}{\beta_1} \right)^2 \right)^2} \frac{1}{1 + \left(\frac{t-y}{\beta_1} + \left[\frac{p + (y/\beta_2)^2}{1-p} \right]^{1/2} \right)^2} dy.$$

Table 2.8. Power of the test

Sample size \ Constant	0.1	0.25	0.5	0.75	0.9
50	14	42	60	100	100
100	21	57	73	100	100
200	29	62	100	100	100

Example 2.1.5. Continuing Example 2.1.4., instead of the hypothesis H_0^* we considered the hypothesis H_0, taking different values of α_i :

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \beta_1 = 100, \quad \beta_2 = 300.$$

The distribution function of a redundant system in this case is

$$F(t) = 1 - \frac{1}{1+\frac{t}{\beta_1}} - \frac{1}{\beta_1}\int_0^t \frac{1}{1+\left(\frac{t-y}{\beta_1}+\left(\frac{y}{3\beta_1}\right)^2\right)} * \frac{1}{\left(1+\frac{y}{\beta_1}\right)^2} dy.$$

The hypothesis H_0 is tested using 5 per cent significance level and several values of the sample size n, $\quad n = n_1 = n_2$. Number of replication was 3000.

Table 2.9. Significance level of the test

Sample size	Significance level (%)
50	7.7
100	6.5
200	5.4

Under the alternative hypothesis H_0 the c.d.f. of the redundant system is

$$F(t) = 1 - \frac{1}{1+\frac{y}{\beta_1}} - \frac{1}{\beta_1}\int_0^t \frac{y}{\left(1+\frac{y}{\beta_1}\right)^2} \frac{1}{1+\left(\frac{t-y}{\beta_1}+\left[\frac{p+(y/\beta_2)^2}{1-p}\right]\right)} dy.$$

Table 2.10. Power of the test

Sample size \ Constant	0.1	0.25	0.5	0.75	0.9
50	9	27	39	64	75
100	15	41	52	71	85
200	23	49	79	97	100

2.2 Redundant system with one main and $(m-1)$ stand-by units

Let us consider a system of m units: one main unit and $m-1$ stand-by units. We shall use notation $S(1, m-1)$ for such systems.

Denote by T_1, F_1 and f_1 the failure time, the c.d.f. and the probability density function of the main unit. The failure times of the stand-by units are denoted by $T_2, \ldots, T_m$. In "hot" conditions their distribution functions are also F_1. In "warm" conditions the c.d.f. of T_i is F_2 and the p.d.f is f_2, $i = 2, \ldots, m$. If a stand-by unit is switched from "warm" to "hot" conditions, its c.d.f. is different from F_1 and F_2.

The failure time of the system $S(1, m-1)$ is $T^{(m)} = T_1 \vee T_2 \vee \cdots \vee T_m$. As $T^{(m)} = (T_1 \vee T_2 \vee \cdots \vee T_{m-1}) \vee T_m$, we can consider this system as a system $S(1,1)$ with one main element (which itself is a system $S(1, m-2)$) and one stand-by element.

Denote by K_j and k_j the c.d.f. and the p.d.f. of $T^{(j)}$, respectively, $(j = 2, \ldots, m)$, $K_1 = F_1$, $k_1 = f_1$. The c.d.f K_j can be written in terms of the c.d.f K_{j-1} and F_1:

$$K_j(t) = \mathbf{P}(T^{(j)} \leq t) = \mathbf{P}(T^{(j-1)} \leq t, T_j \leq t) = \int_0^t \mathbf{P}(T_j \leq t | T^{(j-1)} = y) dK_{j-1}(y). \tag{2.16}$$

We generalize (2.2) modelling the conditional distribution $\mathbf{P}(T_j \leq t | T^{(j-1)} = y)$ and define the following hypothesis.

Hypothesis H_0:

$$f_{T_j | T^{(j-1)} = y}(t) = \begin{cases} f_2(t) & \text{if} \quad t \leq y, \\ f_1(t + g(y) - y) & \text{if} \quad t > y; \end{cases} \tag{2.17}$$

here (as in the case of the hypothesis (2.2))

$$g(y) = F_1^{-1}(F_2(y)).$$

The formulas (2.16) and (2.17) imply the equality

$$K_j(t) = \int_0^t F_1(t + g(y) - y) dK_{j-1}(y). \tag{2.18}$$

So the cumulative distribution function of the system with $m-1$ stand-by units is defined recurrently using formula (2.18) $(j = 2, \ldots, m)$.

In particular, if we suppose that the hypothesis H_0 is true and the distribution of units functioning in "warm" and "hot" conditions differ only in scale, i.e.

$$F_2(t) = F_1(rt), \tag{2.19}$$

for all $t \geq 0$ and some $r > 0$, then $g(y) = ry$. So we define the following hypothesis.

Hypothesis H_0^*:

$$f_{T_j | T^{(j-1)} = y}(t) = \begin{cases} f_2(t) & \text{if} \quad t \leq y, \\ f_1(t + ry - y) & \text{if} \quad t > y. \end{cases} \tag{2.20}$$

Under the model (2.20) the cumulative distribution function of the system is obtained using recurrent formulas

$$K_j(t) = \int_0^t F_1(t + ry - y) dK_{j-1}(y). \tag{2.21}$$

If switching from "warm" to "hot" conditions does not damage units in the system $S(1,1)$ then it is natural that this is true for the system $S(1, m-1)$, $m > 2$. So it is sufficient to use goodness-of-fit tests for the hypotheses H_0 and H_0^* when only one stand-by unit is used. Such tests were given in Chapter 2.1.

In what follows we suppose that one of the hypothesis H_0 or H_0^* is verified and we shall consider nonparametric and parametric estimation methods for redundant systems reliability estimation using data from units reliability trials.

2.2.1 Nonparametric estimation

Suppose that the hypothesis H_0^* is true and the following data are available:

a) complete ordered sample $T_{11}, \ldots, T_{1n_1}$ of the failure times of n_1 units tested in "hot" conditions;

b) the ordered first m_2 failure times $T_{21}, \ldots, T_{2m_2}$ obtained by testing of n_2 units up to the time t_1 in "warm" conditions.

The second sample is censored because the time to obtain complete data in "warm" conditions may be long.

Set

$$N_1(t) = \sum_{i=1}^{n_1} \mathbf{1}_{\{T_{1i} \le t\}}, \quad N_2(t) = \sum_{i=1}^{n_2} \mathbf{1}_{\{T_{2i} \le t, t \le t_1\}},$$

$$Y_1(t) = \sum_{i=1}^{n_1} \mathbf{1}_{\{T_{1i} \ge t\}}, \quad Y_2(t) = \sum_{i=1}^{n_2} \mathbf{1}_{\{T_{2i} \ge t, t \le t_1\}}.$$

Note that the random variables T_{1i}/r and T_{2i} can be interpreted as order statistics from samples of size n_1 and n_2, respectively, from the population having the c.d.f F_2. So if we denote

$$\tilde{N}_1(t) = \sum_{i=1}^{n_1} \mathbf{1}_{\{T_{1i}/r \le t\}} = N_1(rt), \tilde{N}_2(t) = N_2(t),$$

$$\tilde{Y}_1(t) = \sum_{i=1}^{n_1} \mathbf{1}_{\{T_{1i}/r \ge t\}} = Y_1(rt), \quad \tilde{Y}_2(t) = Y_2(t),$$

then the following Nelson-Aalen type estimator (still depending on r) of the cumulative hazard function $\Lambda_2 = -\ln S_2$ can be considered:

$$\tilde{\Lambda}_2(t, r) = \int_0^t \frac{d\tilde{N}_1(u) + d\tilde{N}_2(u)}{\tilde{Y}_1(u) + \tilde{Y}_2(u)} = \int_0^t \frac{dN_1(ru) + dN_2(u)}{Y_1(ru) + Y_2(u)}. \tag{2.22}$$

Taking into consideration the fact that the difference

$$M_2(t) = N_2(t) - \int_0^t Y_2(u) d\Lambda_2(u)$$

is a martingale on $[0, t_1]$ with respect to the filtration generated by the data, and $\mathbf{E}M_2(t_1) = 0$, the parameter r can be estimated using the estimating function

$$\tilde{U}(r) = N_2(t_1) - \int_0^{t_1} Y_2(u) d\tilde{\Lambda}_2(u, r) = \tag{2.23}$$

$$= N_2(t_1) - \int_0^{rt_1} \frac{Y_2(v/r) dN_1(v)}{Y_1(v) + Y_2(v/r)} - \int_0^{t_1} \frac{Y_2(u) dN_2(u)}{Y_1(ru) + Y_2(u)}.$$

$\tilde{U}(r)$ is a non-increasing step function,

$$\tilde{U}(0+) = N_2(t_1) - \int_0^{t_1} \frac{Y_2(u) dN_2(u)}{n_1 + Y_2(u)} > 0, \quad \tilde{U}(+\infty) = -\int_0^{\infty} \frac{n_2 dN_1(v)}{Y_1(v) + n_2} < 0,$$

so the parameter r is estimated by the statistic

$$\hat{r} = \tilde{U}^{-1}(0) = sup\{r : \tilde{U}(r) > 0\}. \tag{2.24}$$

The c.d.f. K_m of the redundant system is estimated using the following recurrent equations ($j = 2, \ldots, m$):

$$\hat{K}_1(t) = \hat{F}_1(t),$$

$$\hat{K}_j(t) = \hat{K}_{j-1}(t) - \int_0^t \hat{S}_1(t + \hat{r}y - y) d\hat{K}_{j-1}(y) =$$

$$= \hat{K}_{j-1}(t) - \sum_{i:T_{1i} \le t} \hat{S}_1(t + \hat{r}T_{1i} - T_{1i})(\hat{K}_{j-1}(T_{1i}) - \hat{K}_{j-1}(T_{1,i-1}));$$

here $\hat{S}_1 = 1 - \hat{F}_1$. The estimator of the mean failure time μ of the system is

$$\hat{\mu} = \int_0^{\infty} t d\hat{K}_m(t) = \sum_{i=1}^{n_1} T_{1i} \, [\hat{K}_m(T_{1i}) - \hat{K}_m(T_{1,i-1})].$$

The following alternative estimators of the c.d.f. F_i may be considered.

The estimators of the cumulative hazards Λ_1 and Λ_2 are

$$\Lambda_1^*(t) = \Lambda_2^*(t/\hat{r}, \hat{r}) = \int_0^t \frac{dN_1(u)}{Y_1(u) + Y_2(u/\hat{r})} + \int_0^{t/\hat{r}} \frac{dN_2(u)}{Y_1(\hat{r}u) + Y_2(u)} =$$

$$= \sum_{T_{1i} \le t} \frac{1}{Y_1(T_{1i}) + Y_2(T_{1i}/\hat{r})} + \sum_{T_{2i} \le t/\hat{r}} \frac{1}{Y_1(\hat{r}T_{2i}) + Y_2(T_{2i})},$$

$$\Lambda_2^*(t) = \Lambda_1^*(\hat{r}t).$$

The estimators of the c.d.f. $S_i = 1 - F_i$ are the product integrals of the estimators $\Lambda_1^*(t)$ and $\Lambda_2^*(t)$, so

$$F_1^*(t) = 1 - \mathcal{\pi}_{0 \le s \le t}(1 - d\hat{\Lambda}_1(s)) =$$

$$= 1 - \prod_{T_{1i} \le t} \left(1 - \frac{1}{Y_1(T_{1i}) + Y_2(T_{1i}/\hat{r})}\right) \prod_{T_{2i} \le t/\hat{r}} \left(1 - \frac{1}{Y_1(\hat{r}T_{2i}) + Y_2(T_{2i})}\right),$$

$$F_2^*(t) = F_1^*(\hat{r}t).$$

Mixing all moments T_{1i} and $\hat{r}T_{2j}$ and ordering them, we obtain the sequence of random variables $T_1 \le \dots \le T_{n_1+m_2}$. The estimators $F_1^*(t)$ and $F_2^*(t)$ can be written:

$$F_1^*(t) = 1 - \prod_{T_i \le t} \left(1 - \frac{1}{Y_1(T_i) + Y_2(T_i/\hat{r})}\right), \quad F_2^*(t) = F_1^*(\hat{r}t).$$

The c.d.f. K_m of the redundant system is estimated using the following recurrent equations ($j = 2, \dots, m$):

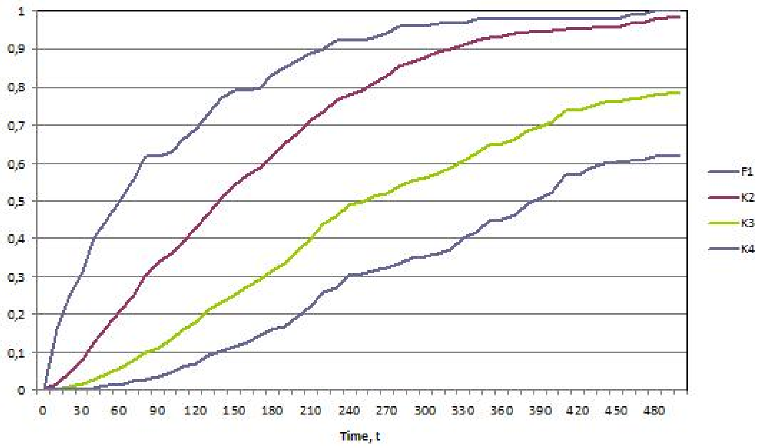
$$K_j^*(t) = K_{j-1}^*(t) - \int_0^t \hat{S}_1(t + \hat{r}y - y) dK_{j-1}^*(y) =$$

$$= K_{j-1}^*(t) - \sum_{i: T_i \le t} S_1^*(t + \hat{r}T_i - T_i)(K_{j-1}^*(T_i) - K_{j-1}^*(T_{i-1}));$$

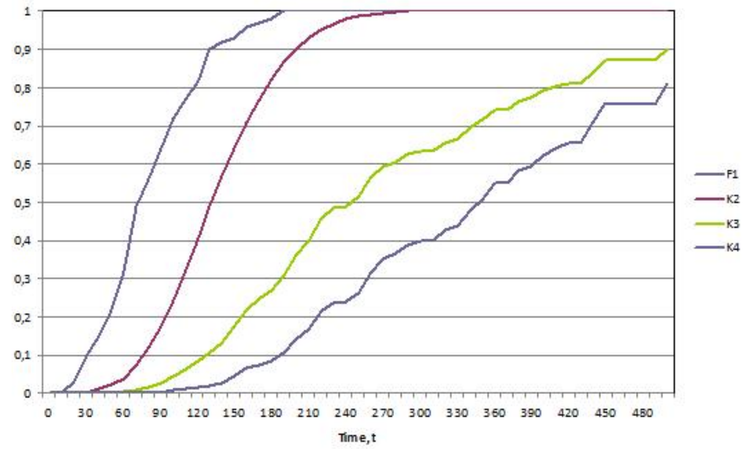
here $S_1^* = 1 - F_1^*$. The estimator of the mean failure time μ of the system is

$$\mu^* = \int_0^\infty t dK_m^*(t) = \sum_{i=1}^{n_1+m_2} T_i \left[K_m^*(T_i) - K_m^*(T_{i-1})\right].$$

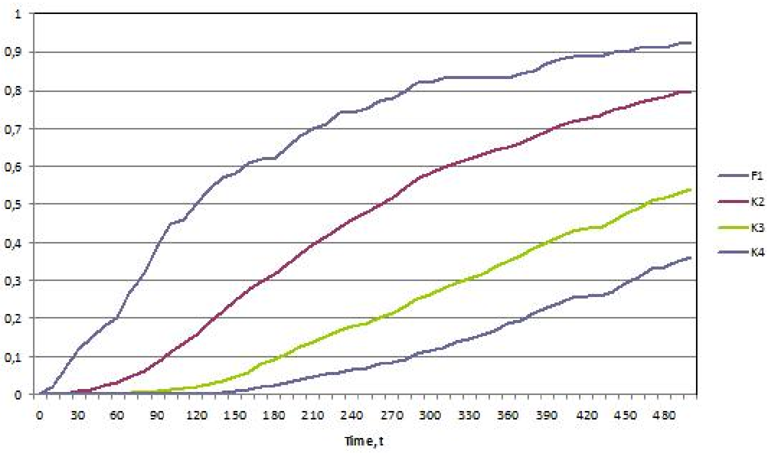
The graphs of the trajectories of the estimators of the c.d.f. F_1 and K_m ($m = 2, 3, 4$), in the case of complete samples and different distributions are presented in Graph 2.4, 2.5 and 2.6. Increasing the number of stand-by units increases the reliability of the redundant system.



Graph 2.4. Graphs of the trajectories of the nonparametric estimators $\hat{F}_1, \hat{K}_i$ (Exponential distribution)



Graph 2.5. Graphs of the trajectories of the nonparametric estimators $\hat{F}_1, \hat{K}_i$ (Weibull distribution)



Graph 2.6. Graphs of the trajectories of the nonparametric estimators $\hat{F}_1, \hat{K}_i$ (Loglogistic distribution)

2.2.2 Parametric estimation

Suppose that in hot conditions the c.d.f. $F_1(t;\theta)$ is absolutely continuous and depends on finite dimensional parameter $\theta \in \Theta \subset \mathbf{R}^k$. Set $\gamma = (r, \theta^T)^T$.

The maximum likelihood estimator $\gamma^* = (r^*, (\theta^*)^T)^T$ of the parameter γ maximizes the loglikelihood function

$$\ell(\gamma) = \sum_{i=1}^{n_1} \ln f_1(T_{1i};\theta) + m_2 \ln r + \sum_{i=1}^{m_2} \ln f_1(rT_{2i};\theta) + (n_2 - m_2) \ln S_1(rt_1;\theta). \quad (2.25)$$

Under H_0^* for any $t \geq 0$ and $j \geq 2$ the c.d.f. $K_j(t)$ is estimated recurrently:

$$\hat{K}_j(t) = \int_0^t F_1(t + r^* y - y; \theta^*) d\hat{K}_{j-1}(y), \quad \hat{K}_1(t) = F_1(t;\theta^*). \quad (2.26)$$

2.3 Asymptotic distribution of $\hat{K}_j$ and confidence intervals for $K_j(t)$

We need for an asymptotic distribution of the estimator of the c.d.f $K_m(t)$ of the redundant system to construct confidential intervals for $K_m(t)$. Suppose that

$$\frac{n_i}{n} = l_i + O(\frac{1}{n}), \quad l_i \in (0,1), \quad \text{as} \quad n = n_1 + n_2 \to \infty.$$

2.3.1 Nonparametric case

The limit distribution of the empirical distribution functions is well known:

$$\sqrt{n}(\hat{F}_i - F_i) \xrightarrow{\mathcal{D}} U_i \quad (2.27)$$

on $D(A_i)$, where $D(A_i)$ is the space of cadlag functions with supremum norm metric, $\xrightarrow{\mathcal{D}}$ means weak convergence, $A_1 = [0, \infty)$, $A_2 = [0, t_1]$, U_1, U_2 are independent Gaussian martingales with $U_i(0) = 0$ and the covariances

$$\mathbf{cov}(U_i(u), U_i(v)) = \frac{1}{l_i} F_i(u \wedge v) S_i(u \vee v). \quad (2.28)$$

Using (2.27) we get

$$\sqrt{n}(\hat{S}_i - S_i) \xrightarrow{\mathcal{D}} -U_i \quad (2.29)$$

Let us find the asymptotic distribution of the estimator $\hat{r}$ defined by (2.24). Denote by $r_0 \in (0,1)$ the true value of r. Under the hypothesis H_0^* it is the ratio of the mean failure times μ_1 and μ_2 of units functioning in "hot" and "warm" conditions, respectively.

Lemma 2.3.1 *Suppose that the c.d.f. F_1 is absolutely continuous with positive p.d.f. f_1 on $(0,\infty)$ and the equality $F_2(t) = F_1(r_0 t)$ is true for all $t \geq 0$. If*

$$A = -\frac{1}{r_0}\int_0^{r_0 t_1} u f_1(u) d\Lambda_1(u) - t_1 f_1(r_0 t_1) \neq 0, \tag{2.30}$$

then

$$\sqrt{n}(\hat{r} - r_0) \xrightarrow{d} Y = -\frac{W(r_0)}{A}, \tag{2.31}$$

where

$$W(r_0) = -\int_0^{t_1} [U_1(r_0 u) - U_2(u)] d\Lambda_2(u) - U_1(r_0 t_1) + U_2(t_1), \tag{2.32}$$

Proof. Set $\hat{U}(r) = \frac{n}{n_1 n_2}\tilde{U}(r)$, $\hat{S}_i = 1 - \hat{F}_i$, where $\tilde{U}(r)$ is defined by (2.23). For any $r > 0$

$$\hat{U}(r) = \frac{n}{n_1 n_2}\left[-\int_0^{t_1} \frac{Y_2(u) dN_1(ru)}{Y_1(ru) + Y_2(u)} + \int_0^{t_1}\left(1 - \frac{Y_2(u)}{Y_1(ru) + Y_2(u)}\right) dN_2(u)\right] =$$

$$= \int_0^{t_1} \frac{\hat{S}_2(u-) d\hat{S}_1(ru)}{\frac{n_1}{n}\hat{S}_1(ru-) + \frac{n_2}{n}\hat{S}_2(u-)} - \int_0^{t_1} \frac{\hat{S}_1(ru-) d\hat{S}_2(u)}{\frac{n_1}{n}\hat{S}_1(ru-) + \frac{n_2}{n}\hat{S}_2(u-)} =:$$

$$= \int_0^{t_1} \hat{Z}_2(u,r) d\hat{S}_1(ru) - \int_0^{t_1} \hat{Z}_1(u,r) d\hat{S}_2(u),$$

$$\hat{Z}_2(u,r) = \frac{n}{n_2}(1 - \frac{n_1}{n}\hat{Z}_1(u,r)).$$

The convergence $\sup\limits_{u \in A_i} |\hat{S}_i(u) - S_i(u)| \xrightarrow{P} 0$, $\sup\limits_{u \in A_i} |\hat{S}_i(u-) - S_i(u)| \xrightarrow{P} 0$ implies that $\hat{U}(r) \xrightarrow{P} U(r)$, where

$$U(r) = \int_0^{t_1} \frac{S_2(u) dS_1(ru)}{l_1 S_1(ru) + l_2 S_2(u)} - \int_0^{t_1} \frac{S_1(ru) dS_2(u)}{l_1 S_1(ru) + l_2 S_2(u)} =:$$

$$=: \int_0^{t_1} Z_2(u,r) dS_1(ru) - \int_0^{t_1} Z_1(u,r) dS_2(u), \tag{2.33}$$

$$Z_2(u,r) = \frac{1}{l_2}(1 - l_1 Z_1(u,r)).$$

Using the equality $S_1(r_0 u) = S_2(u)$, we obtain $U(r_0) = 0$.

Using the convergence $\sqrt{n}(\hat{S}_i - S_i) \xrightarrow{D} -U_i$ and the functional delta method we obtain

$$\sqrt{n}(\hat{Z}_1 - Z_1)(u,r) = \sqrt{n}\left(\frac{\hat{S}_1(ru)}{l_1\hat{S}_1(ru) + l_2\hat{S}_2(u)} - \frac{S_1(ru)}{l_1 S_1(ru) + l_2 S_2(u)}\right) \xrightarrow{d}$$

$$\xrightarrow{d} l_2 \left(\frac{-U_1(ru)S_2(u) + U_2(u)S_1(ru)}{(l_1 S_1(ru) + l_2 S_2(u))^2} \right) =: U_1^*(u, r) \tag{2.34}$$

and

$$\sqrt{n}(\hat{Z}_2 - Z_2)(u, r) = \sqrt{n} \left(\frac{\hat{S}_2(u)}{l_1 \hat{S}_1(ru) + l_2 \hat{S}_2(u)} - \frac{S_2(u)}{l_1 S_1(ru) + l_2 S_2(u)} \right) \xrightarrow{d}$$

$$\xrightarrow{d} l_1 \left(\frac{U_1(ru)S_2(u) - U_2(u)S_1(ru)}{(l_1 S_1(ru) + l_2 S_2(u))^2} \right) = -\frac{l_1}{l_2} U_1^*(u, r) =: U_2^*(v, r) \tag{2.35}$$

on $D([0, t_1] \times [0, 1])$. Note that

$$U_1^*(u, r_0) = l_2 \left(\frac{-U_1(r_0 u)S_2(u) + U_2(u)S_1(r_0 u)}{(l_1 S_1(r_0 u) + l_2 S_2(u))^2} \right) =$$

$$= l_2 \left(\frac{U_2(u)S_2(u) - U_1(r_0 u)S_2(u)}{(l_1 S_2(u) + l_2 S_2(u))^2} \right) = l_2 \frac{U_2(u) - U_1(r_0 u)}{S_2(u)},$$

$$Z_1(u, r_0) = \frac{S_1(r_0 u)}{l_1 S_1(r_0 u) + l_2 S_2(u)} = \frac{S_2(u)}{l_1 S_2(u) + l_2 S_2(u)} \equiv 1,$$

$$Z_2(u, r_0) = \frac{S_2(r_u)}{l_1 S_1(r_0 u) + l_2 S_2(u)} = \frac{S_2(u)}{l_1 S_2(u) + l_2 S_2(u)} \equiv 1.$$

By the functional delta method for stochastic integrals (see Theorem A.0.2) and using (2.34), (2.35), (2.29) we have

$$\sqrt{n}(\hat{U}(r) - U(r)) =$$

$$\sqrt{n} \left(\int_0^{t_1} \hat{Z}_2 d\hat{S}_1(ru) - \int_0^{t_1} \hat{Z}_1 d\hat{S}_2(u) - \int_0^{t_1} Z_2 dS_1(ru) + \int_0^{t_1} Z_1 dS_2(u) \right) \xrightarrow{d}$$

$$\xrightarrow{d} W(r) := \int_0^{t_1} (U_2^*(u, r) dS_1(ru) - Z_2(u, r) dU_1(ru)) -$$

$$- \int_0^{t_1} (U_1^*(u, r) dS_2(u) - Z_1(u, r) dU_2(u)) \tag{2.36}$$

on $[0, 1]$.

By the functional delta method (see Theorem A.0.3) and using (2.24), (2.36)we get

$$\sqrt{n}(\hat{r} - r_0) = \sqrt{n}(\tilde{U}^{-1}(r) - \tilde{U}^{-1}(r_0)) =$$

$$= \sqrt{n} \left(\frac{n_1 n_2}{n} \hat{U}^{-1}(r) - \frac{n_1 n_2}{n} \hat{U}^{-1}(r_0) \right) \xrightarrow{d}$$

$$\xrightarrow{d} Y = -\frac{W(r_0)}{U'(r_0)} = -\frac{W(r_0)}{A};$$

here

$$W(r_0) = -\int\limits_0^{t_1} [U_1(r_0u) - U_2(u)]d\Lambda_2(u) - U_1(r_0t_1) + U_2(t_1).$$

Using the equality

$$U(r) = \int\limits_0^{rt_1} \frac{S_2(v/r)dS_1(v)}{l_1S_1(v) + l_2S_2(v/r)} - \int\limits_0^{t_1} \frac{S_1(ru)dS_2(u)}{l_1S_1(ru) + l_2S_2(u)},$$

we obtain the derivative

$$U'(r) = -t_1\frac{S_2(t_1)f_1(rt_1)}{l_1S_1(rt_1) + l_2S_2(t_1)} + \int\limits_0^{rt_1} \frac{l_1vf_2(v/r)S_1(v)dS_1(v)}{r^2(l_1S_1(v) + l_2S_2(v/r))^2} +$$

$$+\int\limits_0^{t_1} \frac{l_2uf_1(ru)S_2(u)dS_2(u)}{(l_1S_1(v) + l_2S_2(v/r))^2}, \quad U'(r_0) = -t_1f_1(t_1) - \frac{1}{r_0}\int\limits_0^{r_0t_1} vf_1(v)d\Lambda_1(v).$$

The proof is complete.

Remark 2.3.1. *If samples are complete and* $tf_1(t) \to 0$ *as* $t \to \infty$ *then*

$$W(r_0) = -\int\limits_0^{\infty} [U_1(r_0u) - U_2(u)]d\Lambda_2(u), \quad A = -\frac{1}{r_0}\int_0^\infty uf_1(u)d\Lambda_1(u), \tag{2.37}$$

and

$$\sqrt{n}(\hat{r} - r_0) \xrightarrow{\mathcal{D}} Y = -\frac{W}{A} \sim N\left(0, \frac{1}{l_1l_2A^2}\right). \tag{2.38}$$

Proof. By (2.27) $U_i(v) \sim N(0, \frac{1}{l_1}F_1(v)S_1(v))$ and $S_1(v) \to 0$ as $v \to \infty$. So $U_i(v) \xrightarrow{P} 0$ as $v \to \infty$ and the formula (2.31) implies the first formula in (2.36). The condition $tf_1(t) \to 0$ and the formula (2.29) imply the second formula in (2.36).

Using the equality $cov(U_i(s_1), U_i(s_2)) = \frac{1}{\ell_i}F_i(s_1 \wedge s_2)(1 - F_i(s_1 \vee s_2)), \quad i = 1, 2,$ we obtain the variance

$$V(W(r_0)) = \mathbf{E}\left(\int_0^\infty (U_1(r_0u) - U_2(u))d\Lambda_2(u)\right)^2 =$$

$$= \int_0^\infty d\Lambda_2(u)\int_0^\infty \mathbf{E}\left((U_1(r_0u) - U_2(u)(U_1(r_0v) - U_2(v))\right)d\Lambda_2(v) =$$

$$= \int_0^\infty d\Lambda_2(u)\int_0^\infty \left(\mathbf{E}U_1(r_0u)U_1(r_0v) + \mathbf{E}U_2(u)U_2(v)\right)d\Lambda_2(v) =$$

$$= 2\int_0^\infty d\Lambda_2(u)\int_0^u \left(\frac{S_1(r_0u)F_1(r_0v)}{\ell_1} + \frac{S_2(u)F_2(v)}{\ell_2}\right)d\Lambda_2(v) =$$

$$= 2\int_0^\infty d\Lambda_2(u)\int_0^u \left(\frac{S_2(u)F_2(v)}{\ell_1}+\frac{S_2(u)F_2(v)}{\ell_2}\right)d\Lambda_2(v) =$$

$$= \frac{2}{\ell_1\ell_2}\int_0^\infty S_2(u)d\Lambda_2(u)\int_0^u F_2(v)d\Lambda_2(v) =$$

$$= \frac{2}{\ell_1\ell_2}\int_0^\infty \frac{S_2(u)dS_2(u)}{S_2(u)}\int_0^u \frac{F_2(v)dS_2(v)}{S_2(v)} =$$

$$= \frac{2}{\ell_1\ell_2}\int_0^\infty dS_2(u)\int_0^u \frac{[1-S_2(v)]dS_2(v)}{S_2(v)} =$$

$$= -\frac{2}{\ell_1\ell_2}\int_0^\infty (lnS_2(u)-S_2(u)+1)dS_2(u) = \left|S_2(u)=x\right| = \frac{1}{\ell_1\ell_2}.$$

The proof is complete.

Theorem 2.3.2 *If F_1 is continuously differentiable on $[0,\infty)$ then under H_0^* for any $t>0$ and any natural $j\geq 2$*

$$\sqrt{n}(\hat{K}_j(s)-K_j(s)) \xrightarrow{\mathcal{D}}$$

$$W_j(s) = \int_0^s U_1(s+r_0y-y)dK_{j-1}(y)+\mu^{(j-1)}(s)Y+\int_0^s F_1(s+r_0y-y)dW_{j-1}(y) \quad (2.39)$$

on $D[0,t]$, where $W_1(s)=U_1(s)$, $\mu^{(j-1)}(s)=\int_0^s yf_1(s+r_0y-y)dK_{j-1}(y)$.

Proof. Let us prove that under conditions of Lemma 2.3.1 for any $t\geq 0$

$$\sqrt{n}(\hat{F}_1(t+\hat{r}y-y)-F_1(t+r_0y-y)) \xrightarrow{d} U_1(t+r_0y-y)+yf_1(t+r_0y-y)Y \quad (2.40)$$

on $D[0,t]$. It is sufficient to verify the conditions of the Theorem A.0.4.

Fix $\varepsilon:\ 0<\varepsilon<\min(r_0,r_0t)$ and $\tau: t<\tau<t+\varepsilon$. Then

1) $x=F_1$ is continuously differentiable on $[0,\tau]$;

2) $\varphi(y,r)=t+ry-y$ is continuous on $[0,\tau]\times U_\varepsilon(r_0)$, non-increasing in y and $\varphi(0,r_0)=t<\tau$, $\varphi(\tau,r_0)=t+r_0\tau-\tau>-\varepsilon+r_0\tau>0$;

3) $X^n=\hat{F}_1\in D[0,\tau]$ is a sequence of stochastic processes such that

$$\sqrt{n}(X^n-x)\xrightarrow{D} Z=U_1$$

on $D[0,\tau]$, where $Z=U_1$ is a continuous on $[0,\tau]$ stochastic process;

4) $\hat{r}$ is a sequence of random variables such that

$$\sqrt{n}(\hat{r}-r_0)\xrightarrow{D} Y.$$

So all conditions of the Theorem A.0.4 are verified. This theorem implies the convergence (2.39) on $D[0,\tau]$ and consequently on $D[0,t]$ because $x'(\varphi(y,r))=f_1(t+r_0y-y)$, $\varphi'_r(y,r_0)=y$.

We prove the theorem by induction. If $j = 2$ then using (2.26), (2.40), the functional delta method for integrals and the estimator of (2.18)

$$\hat{K}_2(s) = \int_0^s \hat{F}_1(s + \hat{r}y - y)d\hat{K}_1(y), \quad \hat{K}_1(y) = \hat{F}_1(y)$$

we obtain

$$\sqrt{n}(\hat{K}_2(s) - K_2(s)) =$$

$$\sqrt{n}\left(\int_0^s \hat{F}_1(s + \hat{r}y - y)d\hat{F}_1(y) - \int_0^s F_1(s + r_0 y - y)dF_1(y)\right) \xrightarrow{\mathcal{D}}$$

$$\xrightarrow{\mathcal{D}} \int_0^s U_1(s+r_0y-y)dF_1(y) + \int_0^s yf_1(s+r_0y-y)Y dF_1(y) + \int_0^s F_1(s+r_0y-y)dU_1(y) =$$

$$= \int_0^s U_1(s + r_0y - y)dF_1(y) + \mu^{(1)}(s)Y + \int_0^s F_1(s + r_0y - y)dU_1(y) = W_2(s) \quad (2.41)$$

on $D[0, t]$. So (2.39) is true for $j = 2$.

If $j = 3$ then by the functional delta method for integrals and using the estimator

$$\hat{K}_3(s) = \int_0^s \hat{F}_1(s + \hat{r}y - y)d\hat{K}_2(y),$$

we obtain

$$\sqrt{n}(\hat{K}_3(s) - K_3(s)) =$$

$$= \sqrt{n}\left(\int_0^s \hat{F}_1(s + \hat{r}y - y)d\hat{K}_2(y) - \int_0^s F_1(s + r_0y - y)dK_2(y)\right) \xrightarrow{\mathcal{D}}$$

$$\xrightarrow{\mathcal{D}} \int_0^s U_1(s+r_0y-y)dK_2(y) + \int_0^s yf_1(s+r_0y-y)Y dK_2(y) + \int_0^s F_1(s+r_0y-y)dW_2(y) =$$

$$= \int_0^s U_1(s + r_0y - y)dK_2(y) + \mu^{(2)}(s)Y + \int_0^s F_1(s + r_0y - y)dW_2(y) = W_3(s) \quad (2.42)$$

on $D[0, t]$. So (2.39) is true for $j = 3$.

Supposing that (2.38) is true for $j = l$ and using the functional delta method for integrals we obtain the result for $j = l + 1$:

$$\sqrt{n}(\hat{K}_{l+1}(s) - K_{l+1}(s)) =$$

$$\sqrt{n}\left(\int_0^s \hat{F}_1(s + \hat{r}y - y)d\hat{K}_l(y) - \int_0^s F_1(s + r_0y - y)dK_l(y)\right) \xrightarrow{\mathcal{D}}$$

$$\xrightarrow{\mathcal{D}} \int_0^s U_1(s+r_0y-y)dK_l(y) + \int_0^s yf_1(s+r_0y-y)Y dK_l(y) + \int_0^s F_1(s+r_0y-y)dW_l(y) =$$

$$= \int_0^s U_1(s + r_0y - y)dK_l(y) + \mu^{(l)}(s)Y + \int_0^s F_1(s + r_0y - y)dW_l(y) \quad (2.43)$$

on $D[0, t]$.

The proof is complete.

The asymptotic variance of $\sqrt{n}(\hat{K}_j(t) - K_j(t))$, $j \geq 2$ might be estimated recurrently, using the equation (2.39): the covariances

$$\mathbf{Cov}(W_j(s), W_j(t)) = \mathbf{E}(W_j(s), W_j(t))$$

can be written in terms of the covariances

$$\mathbf{E}(W_{j-1}(u)W_{j-1}(v)), \quad \mathbf{E}(W_{j-1}(u)U_1(v)), \quad \mathbf{E}(W_{j-1}(u)U_2(v)),$$

$$\mathbf{E}(U_1(u)U_1(v)), \quad \mathbf{E}(U_2(u)U_2(v)).$$

Note that for $j = 2$ these covariances are

$$\mathbf{E}(W_1(u)W_1(v)) = \mathbf{E}(W_1(u)U_1(v)) = \mathbf{E}(U_1(u)U_1(v)) = \frac{1}{l_1} F_1(u \wedge v)S_1(u \vee v),$$

$$\mathbf{E}(W_1(u)V_2(v)) = \mathbf{E}(U_1(u)U_2(v)) = 0, \quad \mathbf{E}(U_2(u)U_2(v)) = \frac{1}{l_2} F_2(u \wedge v)S_2(u \vee v).$$

Let us find the asymptotic variance of $\sqrt{n}(\hat{K}_2(t) - K_2(t))$ which coincides with the variance of $W_2(t)$.

Suppose first that samples are complete. In the following we skip the index in r_0 .

Using (2.37) we get

$$W_2(t) = \int_0^t U_1(t + ry - y)dF_1(y) + \mu^{(1)}(t)Y + \int_0^t F_1(t + ry - y)dU_1(y) =$$

$$= F_2(t)U_1(t) + \int_0^t U_1(t + ry - y)dF_1(y) - \int_0^t U_1(y)dF_1(t + ry - y) +$$

$$+ \frac{\mu(t)}{A} \left(\int_0^\infty U_1(ry)d\Lambda_2(y) - \int_0^\infty U_2(y)d\Lambda_2(y) \right) = (V_1 + V_2 + V_3 + V_4)(t). \quad (2.44)$$

$$\mu(t) = \mu^{(1)}(t) = \int_0^t y f_1(t + ry - y)dF_1(y), \quad A = -\frac{1}{r} \int_0^\infty u f_1(u)d\Lambda_1(u). \quad (2.45)$$

The random variable $W_2(t)$ has zero mean. Set

$$\nu(t) = \int_0^t F_1(t + ry - y)\, dF_1(y).$$

Taking into account that $0 < r < 1$ and using the equality (2.27) for any $t \geq 0$ we obtain the variances and the covariances of the random variables $V_i(t)$ multiplied by l_1:

$$l_1 \mathbf{E} V_1^2(t) = F_2^2(t)F_1(t)S_1(t),$$

$$l_1 \mathbf{E} V_2^2(t) = l_1 \int_0^t \int_0^t \mathbf{E} U_1(t + ry - y)U_1(t + rz - z)dF_1(y)dF_1(z) =$$

$$= 2 \int_0^t \left(\int_0^y S_1(t + rz - z)dF_1(z) \right) F_1(t + ry - y)dF_1(y) =$$

$$= 2\int_0^t F_1(z)F_1(t+rz-z)dF_1(z) - \left(\int_0^t F_1(t+rz-z)dF_1(z)\right)^2 =$$

$$= 2\int_0^t F_1(z)F_1(t+rz-z)dF_1(z) - \nu^2(t),$$

$$l_1\mathbf{E}V_3^2(t) = l_1\int_0^t\int_0^t \mathbf{E}U_1(y)U_1(z)dF_1(t+ry-y)dF_1(t+rz-z) =$$

$$= 2\int_0^t \left(\int_0^y F_1(z)dF_1(t+rz-z)\right) S_1(y)dF_1(t+ry-y) =$$

$$= 2F_2(t)\int_0^t F_1(z)dF_1(t+rz-z) - 2\int_0^t F_1(z)F_1(t+rz-z)dF_1(t+rz-z) -$$

$$- \left(\int_0^t F_1(z)dF_1(t+rz-z)\right)^2 = 2F_2(t)(F_1(t)F_2(t) -$$

$$-\nu(t)) - 2\int_0^t F_1(z)F_1(t+rz-z)dF_1(t+rz-z) - (F_1(t)F_2(t)-\nu(t))^2,$$

$$l_1\mathbf{E}V_4^2(t) = l_1\mu^2(t)\mathbf{E}Y^2 = \frac{\mu^2(t)}{l_2A^2},$$

$$l_1\mathbf{E}V_1(t)V_2(t) = l_1F_2(t)\int_0^t \mathbf{E}U_1(t+ry-y)U_1(t)dF_1(y) =$$

$$= S_1(t)F_2(t)\int_0^t F_1(t+ry-y)dF_1(y) = S_1(t)F_2(t)\nu(t),$$

$$l_1\mathbf{E}V_1(t)V_3(t) = -l_1F_2(t)\int_0^t \mathbf{E}U_1(y)U_1(t)dF_1(t+ry-y) =$$

$$= -S_1(t)F_2(t)\int_0^t F_1(y)dF_1(t+ry-y) = -S_1(t)F_2(t)(F_1(t)F_2(t)-\nu(t)),$$

$$l_1\mathbf{E}V_1(t)V_4(t) = l_1\frac{F_2(t)\mu(t)}{A}[\int\limits_0^{t/r} \mathbf{E}U_1(ry)U_1(t)d\Lambda_2(y) + \int\limits_{t/r}^{\infty} \mathbf{E}U_1(ry)U_1(t)d\Lambda_2(y)] =$$

$$= \frac{F_2(t)\mu(t)}{A}[S_1(t)\int\limits_0^{t/r} F_1(ry)d\Lambda_2(y) + F_1(t)\int\limits_{t/r}^{\infty} S_1(ry)d\Lambda_2(y)] =$$

$$= -\frac{F_2(t)\mu(t)}{A}[S_1(t)\int\limits_0^{t/r}[\frac{1}{S_2(y)} - 1]dS_2(y) + F_1(t)\int\limits_{t/r}^{\infty} dS_2(y)] =$$

$$= -F_2(t)\frac{\mu(t)}{A}S_1(t)\ln S_1(t),$$

$$-l_1\mathbf{E}V_2(t)V_3(t) = l_1\int_0^t\int_0^t \mathbf{E}U_1(t+ry-y)U_1(z)dF_1(y)dF_1(t+rz-z) =$$

$$= \int_0^t \left(S_1(t+ry-y)\int_0^{t+ry-y} F_1(z)dF_1(t+rz-z) + \right.$$

$$\left. + F_1(t+ry-y)\int_{t+ry-y}^t S_1(z)dF_1(t+rz-z)\right) dF_1(y) =$$

$$= \int_0^t \left(\int_0^{t+ry-y} F_1(z)dF_1(t+rz-z)\right) dF_1(y) -$$

$$-\int_0^t \left(\int_0^{t+ry-y} F_1(z)dF_1(t+rz-z)\right) F_1(t+ry-y)dF_1(y) +$$

$$+\int_0^t \left(\int_{t+ry-y}^t dF_1(t+rz-z)\right) F_1(t+ry-y)dF_1(y) -$$

$$-\int_0^t \left(\int_{t+ry-y}^t F_1(z)dF_1(t+rz-z)\right) F_1(t+ry-y)dF_1(y) =$$

$$= -\int_{rt}^t \left(\int_0^u F_1(z)dF_1(t+rz-z)\right) dF_1((t-u)/(1-r)) +$$

$$+\int_0^t \left(F_1(rt) - F_1(t+(r-1)(t+ry-y))\right) F_1(t+ry-y)dF_1(y) -$$

$$-\int_0^t \left(\int_0^t F_1(z)dF_1(t+rz-z)\right) F_1(t+ry-y)dF_1(y) =$$

$$= -\int_0^{rt} \left(\int_{rt}^t dF_1((t-u)/(1-r))\right) F_1(z)dF_1(t+rz-z) -$$

$$-\int_{rt}^t \left(\int_z^t dF_1((t-u)/(1-r))\right) F_1(z)dF_1(t+rz-z) +$$

$$+\int_0^t \left(F_1(rt) - F_1(t+(r-1)(t+ry-y))\right) F_1(t+ry-y)dF_1(y) -$$

$$-\int_0^t F_1(t+ry-y)dF_1(y)\int_0^t F_1(z)dF_1(t+rz-z) =$$

$$= F_1(t)\int_0^{rt} F_1(z)dF_1(t+rz-z) +$$

$$+\int_{rt}^t F_1((t-z)/(1-r))F_1(z)dF_1(t+rz-z) + F_2(t)\int_0^t F_1(t+ry-y)dF_1(y) -$$

$$-\int_0^t F_1(t+(r-1)(t+ry-y))F_1(t+ry-y)dF_1(y) -$$

$$-\int_0^t F_1(t+ry-y)dF_1(y)\int_0^t F_1(z)dF_1(t+rz-z) =$$

$$= F_1(t)\int_0^{rt} F_1(z)dF_1(t+rz-z)+$$

$$+\int_{rt}^{t} F_1((t-y)/(1-r))F_1(y)dF_1(t+ry-y)+F_2(t)\nu(t)+$$

$$+\int_{rt}^{t} F_1(t+rz-z))F_1(z)dF_1((t-z)/(1-r))-\nu(t)(F_1(t)F_2(t)-\nu(t),$$

$$l_1\mathbf{E}V_2(t)V_4(t) = l_1\frac{\mu(t)}{A}\int\limits_0^t\int\limits_0^\infty \mathbf{E}U_1(t+ry-y)U_1(rz)dF_1(y)d\Lambda_2(z) =$$

$$= -\frac{\mu(t)}{A}\left[\int\limits_0^t S_1(t+ry-y)dF_1(y)\int\limits_0^{(t+ry-y)/r}\left[\frac{1}{S_2(y)}-1\right]dS_2(y)+\right.$$

$$\left.+\int\limits_0^t F_1(t+ry-y)dF_1(y)\int\limits_{(t+ry-y)/r}^{\infty} dS_2(y)\right] =$$

$$= -\frac{\mu(t)}{A}\left[\int\limits_0^t S_1(t+ry-y)\left[\ln S_1(t+ry-y)-S_1(t+ry-y)+1\right]dF_1(y)-\right.$$

$$\left.-\int\limits_0^t F_1(t+ry-y)S_1(t+ry-y)dF_1(y)\right] =$$

$$= -\frac{\mu(t)}{A}\int_0^t S_1(t+ry-y)\ln S_1(t+ry-y)dF_1(y),$$

$$l_1\mathbf{E}V_3(t)V_4(t) = -l_1\frac{\mu(t)}{A}\mathbf{E}\int\limits_0^t U_1(y)dF_1(t+ry-y)\left[\int\limits_0^{y/r}+\int\limits_{y/r}^{\infty}\right]U_1(rz)d\Lambda_2(z) =$$

$$= \frac{\mu(t)}{A}\left[\int\limits_0^t S_1(y)dF_1(t+ry-y)\int\limits_0^{y/r}\left[\frac{1}{S_2(z)}-1\right]dS_2(z)+\right.$$

$$\left.+\int\limits_0^t F_1(y)dF_1(t+ry-y)\int\limits_{y/r}^{\infty} dS_2(z)\right] = \frac{\mu(t)}{A}\int_0^t S_1(y)\ln S_1(y)dF_1(t+ry-y).$$

So the variance $\mathbf{Var}(W_2(t))$ is defined by the following formula:

$$l_1\mathbf{Var}(W_2(t)) = -F_1(t)F_2^2(t)-4\nu^2(t)+\int_0^t F_1(t+ry-y)[F_1(t+ry-y)+2F_1(y)]\,dF_1(y)+$$

$$+2F_1(t)\nu(rt)+2\int_{rt}^{t}F_1(t+ry-y)F_1((t-y)/(1-r))\,dF_1(y)+\frac{\mu^2(t)}{l_2A^2}+$$
$$+\frac{2\mu(t)}{A}\left[\nu(t)+\int_0^t[F_1(t+ry-y)\ln S_1(y)-\right.$$
$$\left.-\,S_1(t+ry-y)\ln S_1(t+ry-y)]dF_1(y))\right].$$

Set

$$Z_{1i}=\hat F_1(t+(\hat r-1)T_{1i}-),\quad \hat F_1(t-)=\frac{1}{n_1}\sum_{i=1}^{n_1}\mathbf{1}_{\{T_{1i}<t\}},\quad Z_{2i}=\hat F_1(\frac{t-T_{1i}}{1-\hat r}-),$$

$$Z_{3i}=\hat F_1(T_{1i}-),\quad Z_{4i}=\hat f_1(t+(\hat r-1)T_{1i}-),\quad \hat\mu(t)=\frac{1}{n_1}\sum_{T_{1i}\le t}T_{1i}Z_{4i},\quad Z_{5i}=\hat f_1(T_{1i}-).$$

The variance $\mathbf{Var}(W_2(t))$ is estimated using the statistic

$$\frac{n_1}{n}\hat{\mathbf{Var}}(W_2(t))=-\hat F_1(t)\hat F_2^2(t)-4\hat\phi_1^2(t)+\hat\phi_2(t)+\frac{n\hat\mu^2(t)}{n_2\hat A^2}+\frac{2\hat\mu(t)}{\hat A}\hat\phi_3(t);$$

here

$$\hat\phi_1(t)=\frac{1}{n_1}\sum_{T_{1i}\le t}Z_{1i},\quad \hat A=-\frac{1}{\hat r n_1}\sum_{i=1}^{n_1}\frac{T_{1i}Z_{5i}}{1-Z_{3i}},$$
$$\hat\phi_2(t)=\frac{1}{n_1}\sum_{T_{1i}\le t}Z_{1i}[Z_{1i}+2Z_{3i}+2\hat F_1(t)\mathbf{1}_{\{T_{1i}\le\hat r t\}}+2Z_{2i}\mathbf{1}_{\{T_{1i}>\hat r t\}}],$$
$$\hat\phi_3(t)=\frac{1}{n_1}\sum_{T_{1i}\le t}[Z_{1i}(1+\ln(1-Z_{3i}))-(1-Z_{1i})\ln(1-Z_{1i})].$$

So the variance $\sigma^2_{\hat K_2}$ of the estimator $\hat K_2(t)$ is estimated by

$$\hat\sigma^2_{\hat K_2}=\frac{1}{n}\hat{\mathbf{Var}}(W_2(t))=\frac{1}{n_1}\left(\frac{n_1}{n}\hat{\mathbf{Var}}(W_2(t))\right).$$

The asymptotic $1-\alpha$ confidence interval for $K_2(t)$ is

$$\hat K_2(t)\pm\hat\sigma_{\hat K_2}z_{1-\alpha/2}. \tag{2.46}$$

Alternative asymptotic confidence interval of the form $(\underline{K}_2(t),\overline{K}_2(t))$, where

$$\underline{K}_2(t)=\left(1+\frac{1-\hat K_2(t)}{\hat K_2(t)}\exp\left\{\frac{\hat\sigma_{\hat K_2}z_{1-\alpha/2}}{\sqrt{\hat K_2(t)(1-\hat K_2(t))}}\right\}\right)^{-1},$$

$$\overline{K}_2(t)=\left(1+\frac{1-\hat K_2(t)}{\hat K_2(t)}\exp\left\{-\frac{\hat\sigma_{\hat K_2}z_{1-\alpha/2}}{\sqrt{\hat K_2(t)(1-\hat K_2(t))}}\right\}\right)^{-1}, \tag{2.47}$$

can be considered.

Remark 2.3.2. *In the case of censoring the expression in parenthesis of the term* V_4 *in (2.44) is replaced by* $\int_0^{t_1}[U_1(ru) - U_2(u)d\Lambda_2(u) + U_1(rt_1) + U_2(t_1)$*, so only minor modifications are needed.*

Example 2.3.1. ***Exponential distribution***: $S_1(t) = e^{-\lambda t}$**.** We investigated finite sample confidence level of the proposed asymptotic confidence intervals. The failure times T_{1j} and T_{2j} were simulated from exponential distribution:

$$T_{1j} \sim E(\lambda_1), \quad T_{2j} \sim E(\lambda_2), \quad \lambda_1 = \frac{1}{100}, \quad \lambda_2 = \frac{1}{300}, \quad t_1 = 500.$$

The number of replications was 2000. For various values of t the proportions of confidence interval realizations covering the true value of the distributional function $K_2(t)$ are given below:

Table 2.11. Confidence level for finite samples ($n1 = n2 = 100$)

Time, t	50	100	200	300	400	500
$K_2(t)$	0.114	0.319	0.667	0.856	0.941	0.977
Confidence level (%)	94.5	91.9	91.8	90.4	90.7	89.2

Example 2.3.2. ***Weibull distribution***: $S_1(t) = e^{-(t/\beta)^\alpha}$**.** The failure times T_{1j} and T_{2j} were simulated from exponential distribution:

$$T_{1j} \sim W(\alpha_1, \beta_1), \quad T_{2j} \sim \ W(\alpha_1, \beta_1),$$

$$\alpha_1 = \alpha_2 = 2, \quad \beta_1 = 100, \quad \beta_2 = 300, \quad t_1 = 500.$$

The number of replications was 2000. For various values of t the proportions of confidence interval realizations covering the true value of the distributional function $K_2(t)$ are given below:

Table 2.12. Confidence level for finite samples ($n1 = n2 = 100$)

Time, t	50	100	200	300	400	500
$K_2(t)$	0.018	0.194	0.822	0.992	0.999	1.000
Confidence level (%)	88.9	91.7	90.7	90.0	89.4	89.9

Example 2.3.3. ***Loglogistic distribution***: $S_1(t) = \frac{1}{1+(t/\beta)^\alpha}$**.** The failure times T_{1j} and T_{2j} were simulated from exponential distribution:

$$T_{1j} \sim L(\alpha_1, \beta_1), \quad T_{2j} \sim \ L(\alpha_1, \beta_1),$$

$$\alpha_1 = \alpha_2 = 2, \quad \beta_1 = 100, \quad \beta_2 = 300, \quad t_1 = 500.$$

The number of replications was 2000. For various values of t the proportions of confidence interval realizations covering the true value of the distributional function $K_2(t)$ are given below:

Table 2.13. Confidence level for finite samples ($n1 = n2 = 100$)

Time, t	50	100	200	300	400	500
$K_2(t)$	0.016	0.138	0.517	0.743	0.851	0.905
Confidence level (%)	92.1	91.7	91.3	89.2	90.5	91.5

2.3.2 Parametric case

Let us consider the parametric estimator (2.25). Denote by $I_n(\gamma) = -\mathbf{E}\ddot{\ell}(\gamma)$ the Fisher information matrix and suppose that $\frac{1}{n}I_n(\gamma) \to i(\gamma)$. Under classical assumptions on the family of distributions $f_1(t,\theta)$ the maximum likelihood estimator γ^* is asymptotically normal:

$$\sqrt{n}(\gamma^* - \gamma) \xrightarrow{d} Y = (Y_1, Y_2^T)^T \sim N_{k+1}(0, i^{-1}(\gamma)).$$

Y_1 is one-dimensional, Y_2 – k-dimensional.

Using delta method we obtain:

$$\sqrt{n}(\hat{K}_2(t) - K_2(t)) \xrightarrow{\mathcal{D}} W_2(t) = Y^T C_2(t;\gamma),$$

where

$$C_2(t;\gamma) = (C_{21}(t;\gamma), C_{22}^T(t;\gamma))^T, \quad C_{21}(t;\gamma) = \int_0^t \frac{\partial}{\partial r} F_1(t + ry - y;\theta) dF_1(y;\theta),$$

$$C_{22}(t;\gamma) = \int_0^t \frac{\partial}{\partial \theta} F_1(t + ry - y;\theta) dF_1(y;\theta) + F_1(t + ry - y;\theta) d(\frac{\partial}{\partial \theta} F_1(y;\theta)).$$

The random variable $W_2(t)$ is linear function of Y.

If $j \geq 2$ then

$$\sqrt{n}(\hat{K}_j(t) - K_j(t)) \xrightarrow{\mathcal{D}} W_j(t).$$

Let us prove by recurrence that the random variable $W_j(t)$, $j \geq 2$, is also linear function of Y:

$$W_j(t) = Y^T C_j(t;\gamma), \quad C_j(t;\gamma) \in (C[0,t])^{k+1}.$$

We showed that it is true for $k = 2$. By functional delta method for integrals (Theorem 1.3.2) and using the assumption that the statement is true for W_{j-1} we obtain

$$W_j(t) = Y^T (\int_0^t \frac{\partial}{\partial \gamma} F_1(t + ry - y;\theta) dK_{j-1}(y;\gamma) + F_1(t + ry - y;\theta) dC_{j-1}(t;\gamma)).$$

So the variance

$$\mathbf{Var}(W_j(t)) = \mathbf{Var}(C_j(t;\gamma)^T Y) = C_j^T(t;\gamma) i^{-1}(\gamma) C_j(t;\gamma)$$

is estimated by $nC_2^T(t;\hat{\gamma})I^{-1}(\hat{\gamma})C_j(t;\hat{\gamma})$, and the variance $\sigma^2_{\hat{K}_j(t)}$ of the estimator $\hat{K}_j(t)$ is estimated by

$$\hat{\sigma}^2_{\hat{K}_j(t)} = C_j^T(t;\hat{\gamma})I^{-1}(\hat{\gamma})C_j(t;\hat{\gamma}).$$

The matrix $I(\hat{\gamma})$ may be replaced by $-\ddot{\ell}(\hat{\gamma})$.

The asymptotic $1-\alpha$ confidence interval for $K_j(t)$ is

$$\hat{K}_j(t) \pm \hat{\sigma}_{\hat{K}_j(t)} z_{1-\alpha/2}, \tag{2.48}$$

or, alternatively, $(\underline{K}_j(t), \overline{K}_j(t))$, where

$$\underline{K}_j(t) = \left(1 + \frac{1-\hat{K}_j(t)}{\hat{K}_j(t)} \exp\left\{\frac{\hat{\sigma}_{\hat{K}_j} z_{1-\alpha/2}}{\sqrt{\hat{K}_j(t)(1-\hat{K}_j(t))}}\right\}\right)^{-1},$$

$$\overline{K}_j(t) = \left(1 + \frac{1-\hat{K}_j(t)}{\hat{K}_j(t)} \exp\left\{-\frac{\hat{\sigma}_{\hat{K}_j} z_{1-\alpha/2}}{\sqrt{\hat{K}_j(t)(1-\hat{K}_j(t))}}\right\}\right)^{-1}. \tag{2.49}$$

Example 2.3.4. *Exponential distribution*: $S_1(t) = e^{-\lambda t}$.

Let us consider the case of complete samples. By (2.25) the loglikelihood function has the form

$$l(r;\theta) = \sum_{i=1}^{n_1} \ln f_1(T_{1i};\theta) + n_2 \ln r + \sum_{j=1}^{n_2} \ln f_1(rT_{2j};\theta).$$

In the case of exponential distribution

$$f_1(t;\lambda) = \lambda e^{-\lambda t}; \quad \ln f_1(t;\lambda) = \ln\lambda - \lambda t,$$

so

$$l(r;\lambda) = n_1 \ln\lambda - \lambda\sum_{i=1}^{n_1} T_{1i} + n_2 \ln r + n_2 \ln\lambda - \lambda r \sum_{j=1}^{n_2} T_{2j} =$$

$$= n\ln\lambda + n_2 \ln r - \lambda(\sum_{i=1}^{n_1} T_{1i} + r\sum_{j=1}^{n_2} T_{2j}).$$

Equating the score function to zero we obtain the system of equations

$$\dot{\ell}_r = \frac{\partial l}{\partial r} = \frac{n_2}{r} - \lambda \sum_{j=1}^{n_2} T_{2j} = 0, \quad \dot{\ell}_\lambda = \frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n_1} T_{1i} - r\sum_{j=1}^{n_2} T_{2j} = 0.$$

So the estimators of the parameters r and λ are:

$$\hat{r} = \frac{\overline{T_1}}{\overline{T_2}}; \quad \hat{\lambda} = \frac{1}{\hat{r}\overline{T_2}} = \frac{1}{\overline{T_1}}.$$

Second partial derivatives are

$$\frac{\partial^2 l}{\partial r^2} = -\frac{n_2}{r^2}; \quad \frac{\partial^2 l}{\partial \lambda^2} = -\frac{n}{\lambda^2}; \quad \frac{\partial^2 l}{\partial \lambda \partial r} = -\sum_{j=1}^{n_2} T_{2j},$$

so the Fisher information matrix and the inverse of the Fisher matrix are

$$I(r;\lambda) = \begin{pmatrix} \frac{n_2}{r^2} & \frac{n_2}{\lambda r} \\ \frac{n_2}{\lambda r} & \frac{n}{\lambda^2} \end{pmatrix}; \quad I^{-1}(r;\lambda) = \begin{pmatrix} \frac{n\hat{r}^2}{n_1 n_2} & -\frac{\hat{\lambda}\hat{r}}{n_1} \\ -\frac{\hat{\lambda}\hat{r}}{n_1} & \frac{\hat{\lambda}^2}{n_1} \end{pmatrix}.$$

If n_1 and n_2 are large then the distribution of the estimator $(\hat{r}, \hat{\lambda})$ is approximated by the normal $N((r,\lambda), I^{-1}(r,\lambda))$.

Taking into consideration the equality

$$K_j(t) = \prod_{i=1}^{j-1}\left(1 + \frac{1}{ir}\right)\sum_{i=0}^{j-1}(-1)^i C_{j-1}^i \frac{1 - e^{-\lambda(1+ir)t}}{1+ir}.$$

the weights $C_j = (C_{j1}, C_{j2})^T$ can be computed:

$$C_{j1}(t;r,\lambda) = \frac{\partial K_j(t)}{\partial r} = \prod_{i=1}^{j-1}\left(1 + \frac{1}{ir}\right)\left[-\frac{1}{r}\sum_{i=1}^{j-1}\frac{1}{1+ir}\sum_{i=0}^{j-1}(-1)^i C_{j-1}^i \frac{1 - e^{-\lambda(1+ir)t}}{1+ir} + \right.$$

$$\left. + \sum_{i=0}^{j-1}(-1)^i C_{j-1}^i \frac{e^{-\lambda(1+ir)t}[i\lambda t(1+ir) + i] - i}{(1+ir)^2}\right],$$

$$C_{j2}(t;r,\lambda) = \frac{\partial K_j(t)}{\partial \lambda} = t\prod_{i=1}^{j-1}\left(1 + \frac{1}{ir}\right)\sum_{i=0}^{j-1}(-1)^i C_{j-1}^i e^{-\lambda(1+ir)t}.$$

The estimator of variance of the estimator $\hat{K}_j(t)$ is

$$\hat{\sigma}^2_{\hat{K}_j(t)} = C_j^T(t;\hat{r},\hat{\lambda}) I^{-1}(\hat{r},\hat{\lambda}) C_j(t;\hat{r},\hat{\lambda}).$$

and the asymptotic $1-\alpha$ confidence interval for $K_j(t)$ has the form (2.48) or, alternatively, (2.49).

In the case $j = 2$ the estimator of the function

$$K_2(t) = 1 - e^{-\lambda t} + \frac{1}{r}(e^{-\lambda(1+r)t} - e^{-\lambda t}) = F_1(t) - \frac{S_1(t)F_2(t)}{r};$$

is

$$\hat{K}_2(t) = \hat{F}_1(t) - \frac{\hat{S}_1(t)\hat{F}_2(t)}{\hat{r}} = 1 - e^{-\hat{\lambda}t} + \frac{1}{\hat{r}}(e^{-\hat{\lambda}(1+\hat{r})t} - e^{-\hat{\lambda}t}).$$

So

$$C_{21}(t;r,\lambda) = \frac{S_1(t)}{r^2}(F_2(t) - r\lambda t S_2(t)), \quad C_{22}(t;r,\lambda) = \frac{(1+r)t}{r}S_1(t)F_2(t).$$

So we obtained

$$C_2(t;r,\lambda) = (C_{21}(t;r,\lambda), C_{22}(t;r,\lambda))^T =$$

$$= \left(\frac{S_1(t)}{r^2}(F_2(t) - r\lambda t S_2(t)), \quad \frac{(1+r)t}{r} S_1(t) F_2(t) \right).$$

$$V\hat{K}_2(t) \approx \left(\frac{\partial K_2}{\partial r}, \frac{\partial K_2}{\partial \lambda} \right) I^{-1}(\lambda, r) \begin{pmatrix} \frac{\partial K_2}{r} \\ \frac{\partial K_2}{\lambda} \end{pmatrix}$$

The estimator of the variance of the estimator $\hat{K}_2(t)$ is

$$\hat{\sigma}^2_{\hat{K}_2(t)} = C_2^T(t;\hat{r},\hat{\lambda}) I^{-1}(\hat{r},\hat{\lambda}) C_2(t;\hat{r},\hat{\lambda}) =$$

$$= \frac{\hat{S}_1^2(t)}{n\hat{l}_1\hat{l}_2 r^2} \left(\hat{l}_1 \left[\hat{F}_2(t) - \hat{\lambda}\hat{r}t\hat{S}_2(t) \right]^2 + \hat{l}_2 \left[(1 - \hat{\lambda}t)\hat{F}_2(t) - \hat{\lambda}\hat{r}t \right]^2 \right);$$

here $\hat{l}_i = n_i/n$.

We found by simulation finite sample confidence levels of the intervals obtained using asymptotic formulas with $1 - \alpha = 0.9$. The failure times T_{1j} and T_{2j} were simulated from exponential distribution with following parameters:

$$T_{1j} \sim E(\lambda_1), \quad T_{2j} \sim E(\lambda_2),$$

$$\lambda_1 = \frac{1}{100}, \quad \lambda_2 = \frac{1}{300}.$$

The number of replications was 2000. For various values of t the proportions of confidence interval realizations covering the true value of the distributional function $K_2(t)$ are given below:

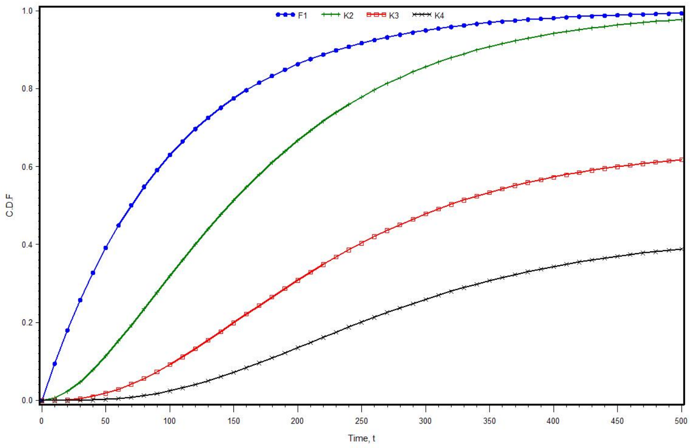
Table 2.14. Confidence level for finite samples ($n1 = n2 = 100$)

Time, t	50	100	200	300	400	500
$K_2(t)$	0.114	0.319	0.667	0.856	0.941	0.977
Confidence level (%)	89.9	89.4	88.9	90.2	90.0	91.5

For various values of t the proportions of confidence interval realizations covering the true value of the distributional function $K_3(t)$ are given in Table 2.15.

Table 2.15. Confidence level for finite samples ($n_1 = n_2 = 100$)

Time, t	50	100	200	300	400	500
$K_3(t)$	0.018	0.092	0.309	0.479	0.573	0.617
Confidence level (%)	88.6	91.8	92.8	90.8	89.9	90.3



Graph 2.7. Graphs of the trajectories of the parametric estimators $\hat{F}_1, \hat{K}_2, \hat{K}_3, \hat{K}_4$, (Exponential distribution)

Example 2.3.5. ***Weibull distribution*:** $S_1(t) = e^{-(t/\mu)^\nu}$**.** Let us consider the case of complete samples. By (2.25) the loglikelihood function has the form

$$l(r;\theta) = \ln L(r,\theta) = \sum_{i=1}^{n_1} \ln f_1(T_{1i};\theta) + n_2 \ln r + \sum_{j=1}^{n_2} \ln f_1(rT_{2j};\theta);$$

here

$$f_1(t;\nu,\mu) = \frac{\nu}{\mu}\left(\frac{t}{\mu}\right)^{\nu-1} e^{-\left(\frac{t}{\mu}\right)^\nu};$$

$$\ln f_1(t;\nu,\mu) = \ln\nu - \ln\mu + (\nu-1)(\ln t - \ln\mu) - \left(\frac{t}{\mu}\right)^\nu =$$

$$= \ln\nu - \nu\ln\mu + (\nu-1)\ln t - \left(\frac{t}{\mu}\right)^\nu;$$

$$\sum_{i=1}^{n_1} \ln f_1(T_{1i};\nu,\mu) = \sum_{i=1}^{n_1}\left(\ln\nu - \nu\ln\mu + (\nu-1)\ln T_{1i} - \left(\frac{T_{1i}}{\mu}\right)^\nu\right) =$$

$$= n_1\ln\nu - n_1\nu\ln\mu + (\nu-1)\sum_{i=1}^{n_1}\ln T_{1i} - \sum_{i=1}^{n_1}\left(\frac{T_{1i}}{\mu}\right)^\nu;$$

So in the case of Weibull distribution the loglikelihood function is

$$l(r;\theta) = n(\ln\nu - \nu\ln\mu) + \nu n_2\ln r + (\nu-1)\left(\sum_{i=1}^{n_1}\ln T_{1i} + \sum_{j=1}^{n_2}\ln T_{2j}\right) -$$

$$-\frac{1}{\mu^\nu}\left(\sum_{i=1}^{n_1} T_{1i}^\nu + r^\nu\sum_{j=1}^{n_2} T_{2j}^\nu\right).$$

Equating the score function to zero the following system of equations is obtained:

$$\dot{\ell}_r = \frac{\partial l}{\partial r} = \frac{n_2 \nu}{r} - \frac{\nu r^{\nu-1}}{\mu^\nu} \sum_{j=1}^{n_2} T_{2j}^\nu = 0;$$

$$\dot{\ell}_\nu = \frac{\partial l}{\partial \nu} = \frac{n}{\nu} - n \ln \mu + n_2 \ln r + \left(\sum_{i=1}^{n_1} \ln T_{1i} + \sum_{j=1}^{n_2} \ln T_{2j} \right) +$$

$$+ \frac{\ln \mu}{\mu^\nu} \left(\sum_{i=1}^{n_1} T_{1i}^\nu + r^\nu \sum_{j=1}^{n_2} T_{2j}^\nu \right) - \frac{1}{\mu^\nu} \left(\sum_{i=1}^{n_1} T_{1i}^\nu \ln T_{1i} + r^\nu \ln r \sum_{j=1}^{n_2} T_{2j}^\nu + \right.$$

$$\left. + r^\nu \sum_{j=1}^{n_2} T_{2j}^\nu \ln T_{2j} \right) = \frac{n}{\nu} - n \ln \mu + n_2 \ln r + \sum_{i=1}^{n_1} \ln T_{1i} + \sum_{j=1}^{n_2} \ln T_{2j} +$$

$$+ \frac{1}{\mu^\nu} \left(\ln \mu \left(\sum_{i=1}^{n_1} T_{1i}^\nu + r^\nu \sum_{j=1}^{n_2} T_{2j}^\nu \right) - \sum_{i=1}^{n_1} T_{1i}^\nu \ln T_{1i} - r^\nu \ln r \sum_{j=1}^{n_2} T_{2j}^\nu - \right.$$

$$\left. - r^\nu \sum_{j=1}^{n_2} T_{2j}^\nu \ln T_{2j} \right) = 0;$$

$$\dot{\ell}_\mu = \frac{\partial l}{\partial \mu} = - \frac{n \nu}{\mu} + \frac{\nu}{\mu^{\nu+1}} \left(\sum_{i=1}^{n_1} T_{1i}^\nu + r^\nu \sum_{j=1}^{n_2} T_{2j}^\nu \right) = 0.$$

Resolving this system of equations we obtain that the estimators $\hat{\mu}$ and $\hat{r}$ are explicit functions of the estimator $\hat{\nu}$:

$$\hat{\mu} = \left(\frac{\sum\limits_{i=1}^{n_1} T_{1i}^{\hat{\nu}}}{n_1} \right)^{1/\hat{\nu}}, \quad r = \left(\frac{n_2}{n_1} \frac{\sum\limits_{i=1}^{n_1} T_{1i}^{\hat{\nu}}}{\sum\limits_{j=1}^{n_2} T_{2j}^{\hat{\nu}}} \right)^{1/\hat{\nu}}.$$

The estimator ν satisfied the equation:

$$\frac{n}{\hat{\nu}} + \sum_{i=1}^{n_1} \ln T_{1i} + \sum_{j=1}^{n_2} \ln T_{2j} - n_1 \frac{\sum_{i=1}^{n_1} T_{1i}^\nu \ln T_{1i}}{\sum_{i=1}^{n_1} T_{1i}} - n_2 \frac{\sum_{j=1}^{n_2} T_{2j}^\nu \ln T_{2j}}{\sum_{j=1}^{n_2} T_{2j}}.$$

Second partial derivatives of the loglikelihood function are

$$\ddot{\ell}_{r^2} = \frac{\partial^2 l}{\partial r^2} = - \frac{n_2 \nu}{r^2} - \frac{\nu(\nu-1) r^{\nu-2}}{\mu^\nu} \sum_{j=1}^{n_2} T_{2j}^\nu;$$

$$\ddot{\ell}_{r\nu} = \frac{\partial^2 l}{\partial r \partial \nu} = \frac{n_2}{r} - \frac{r^{\nu-1} + \nu r^{\nu-1} \ln r - \nu r^{\nu-1} \ln \mu}{\mu^\nu} \sum_{j=1}^{n_2} T_{2j}^\nu - \frac{\nu r^{\nu-1}}{\mu^\nu} \sum_{j=1}^{n_2} T_{2j}^\nu \ln T_{2j};$$

$$\ddot{\ell}_{r\mu} = \frac{\partial^2 l}{\partial r \partial \mu} = \frac{\nu^2 r^{\nu-1}}{\mu^{\nu+1}} \sum_{j=1}^{n_2} T_{2j}^{\nu};$$

$$\ddot{\ell}_{\nu^2} = \frac{\partial^2 l}{\partial \nu^2} = -\frac{n}{\nu^2} - \frac{1}{\mu^\nu} \ln\mu \left(\ln\mu \left(\sum_{i=1}^{n_1} T_{1i}^{\nu} + r^\nu \sum_{j=1}^{n_2} T_{2j}^{\nu} \right) - \right.$$

$$\left. - \sum_{i=1}^{n_1} T_{1i}^{\nu} \ln T_{1i} - r^\nu \ln r \sum_{j=1}^{n_2} T_{2j}^{\nu} - r^\nu \sum_{j=1}^{n_2} T_{2j}^{\nu} \ln T_{2j} \right) +$$

$$+ \frac{1}{\mu^\nu} \left(\ln\mu \left(\sum_{i=1}^{n_1} T_{1i}^{\nu} \ln T_{1i} + r^\nu \ln r \sum_{j=1}^{n_2} T_{2j}^{\nu} + r^\nu \sum_{j=1}^{n_2} T_{2j}^{\nu} \ln T_{2j} \right) - \right.$$

$$- \sum_{i=1}^{n_1} T_{1i}^{\nu} \ln^2 T_{1i} - r^\nu \ln^2 r \sum_{j=1}^{n_2} T_{2j}^{\nu} - r^\nu \ln r \sum_{j=1}^{n_2} T_{2j}^{\nu} \ln T_{2j} -$$

$$\left. - r^\nu \ln r \sum_{j=1}^{n_2} T_{2j}^{\nu} \ln T_{2j} - r^\nu \sum_{j=1}^{n_2} T_{2j}^{\nu} \ln^2 T_{2j} \right);$$

$$\ddot{\ell}_{\mu\nu} = \frac{\partial^2 l}{\partial \mu \partial \nu} = -\frac{n}{\mu} + \frac{\mu^{\nu+1} - \nu\mu^{\nu+1}\ln\mu}{\mu^{2(\nu+1)}} \left(\sum_{i=1}^{n_1} T_{1i}^{\nu} + r^\nu \sum_{j=1}^{n_2} T_{2j}^{\nu} \right) +$$

$$+ \frac{\nu}{\mu^{\nu+1}} \left(\sum_{i=1}^{n_1} T_{1i}^{\nu} \ln T_{1i} + r^\nu \ln r \sum_{j=1}^{n_2} T_{2j}^{\nu} + r^\nu \sum_{j=1}^{n_2} T_{2j}^{\nu} \ln T_{2j} \right);$$

$$\ddot{\ell}_{\mu^2} = \frac{n\nu}{\mu^2} - \frac{\nu(\nu+1)}{\mu^{\nu+2}} \left(\sum_{i=1}^{n_1} T_{1i}^{\nu} + r^\nu \sum_{j=1}^{n_2} T_{2j}^{\nu} \right).$$

The random variables $X_{1i} = \left(\frac{T_{1i}}{\mu}\right)^\nu$ have the standard exponential distribution, i.e. $X_{1i} \sim \mathcal{E}(1)$. We obtain

$$T_{1i} = \mu X_{1i}^{1/\nu}, \quad T_{1i}^{\nu} = \mu^\nu X_{1i},$$

$$T_{1i}^{\nu} \ln T_{1i} = \mu^\nu X_{1i}(\ln\mu + \frac{1}{\nu}\ln X_{1i}) = \mu^\nu \ln\mu\, X_{1i} + \frac{\mu^\nu}{\nu} X_{1i} \ln X_{1i};$$

$$T_{1i}^{\nu} \ln^2 T_{1i} = \mu^\nu \ln^2\mu\, X_{1i} + \frac{2\mu^\nu}{\nu} \ln\mu\, X_{1i} \ln X_{1i} + \frac{\mu^\nu}{\nu^2} X_{1i} \ln^2 X_{1i};$$

Taking into account that $X_{1i} \sim \varepsilon(1)$ we have

$$\mathbf{E}X_{1i} = 1, \quad \mathbf{E}X_{1i}\ln X_{1i} = \int_0^\infty x \ln x e^{-x} dx = \Gamma'(2),$$

$$\mathbf{E}X_{1i}^{\nu} \ln^2 X_{1i} = \int_0^\infty x e^{-x} \ln^2 x dx = \Gamma''(2);$$

because

$$\Gamma(a)=\int_0^\infty x^{a-1}e^{-x}dx;\quad \Gamma^{(k)}(a)=\int_0^\infty x^{a-1}e^{-x}\ln^k x dx.$$

So

$$\mathbf{E}T_{1i}^\nu=\mu^\nu,$$

$$\mathbf{E}T_{1i}^\nu\ln T_{1i}=\mathbf{E}\left(\frac{\mu^\nu}{\nu}X_{1i}\ln X_{1i}+\mu^\nu\ln\mu X_{1i}\right)=$$

$$=\frac{\mu^\nu}{\nu}\left[\Gamma'(2)+\nu\ln\mu\right];$$

$$\mathbf{E}T_{1i}^\nu\ln^2 T_{1i}=\frac{\mu^\nu}{\nu^2}\left[\Gamma''(2)+2\nu\Gamma'(2)\ln\mu+\nu^2\ln^2\mu\right].$$

The random variables rT_{2j} and T_{1i} have the same distribution, so

$$\mathbf{E}T_{2i}^\nu=\mathbf{E}\frac{1}{r^\nu}\left(rT_{2i}\right)^\nu=\left(\frac{\mu}{r}\right)^\nu,$$

$$T_{2j}^\nu\ln T_{2j}=\frac{1}{r^\nu}(rT_{2j})^\nu[\ln(rT_{2j})-\ln r]=\frac{1}{r^\nu}\left((rT_{2j})^\nu\ln(rT_{2j})-\ln r(rT_{2j})^\nu\right);$$

$$\mathbf{E}T_{2j}^\nu\ln T_{2j}=\frac{1}{r^\nu}\left(\frac{\mu^\nu}{\nu}(\Gamma'(2)+\nu\ln\mu)-\mu^\nu\ln r\right)=\frac{\mu^\nu}{r^\nu\nu}\left(\Gamma'(2)+\nu\ln\frac{\mu}{r}\right).$$

$$\mathbf{E}T_{2j}^\nu\ln^2 T_{2j}=\frac{1}{\nu^2}\left(\frac{\mu}{r}\right)^\nu\left[\Gamma''(2)+2\nu\Gamma'(2)\ln\frac{\mu}{r}+\nu^2\ln^2\frac{\mu}{r}\right].$$

Using the obtained means we compute the second partial derivatives of the loglikelihood function:

$$\mathbf{E}\ddot{\ell}_{r^2}=-\mathbf{E}\left[\frac{n_2\nu}{r^2}+\frac{\nu(\nu-1)r^{\nu-2}}{\mu^\nu}\sum_{j=1}^{n_2}T_{2j}^\nu\right]=$$

$$=-\left[\frac{n_2\nu}{r^2}+\frac{\nu(\nu-1)r^{\nu-2}}{\mu^\nu}n_2\left(\frac{\mu}{r}\right)^\nu\right]=-\frac{\nu^2n_2}{r^2};$$

$$\mathbf{E}\ddot{\ell}_{r\mu}=\mathbf{E}\left[\frac{\nu^2r^{\nu-1}}{\mu^{\nu+1}}\sum_{j=1}^{n_2}T_{2j}^\nu\right]=\frac{\nu^2r^{\nu-1}n_2}{\mu^{\nu+1}}\left(\frac{\mu}{r}\right)^\nu=\frac{n_2\nu^2}{r\mu};$$

$$\mathbf{E}\ddot{\ell}_{r\nu}=\mathbf{E}\left[\frac{n_2}{r}-\left(\frac{r^{\nu-1}+\nu r^{\nu-1}\ln r-\nu r^{\nu-1}\ln\mu}{\mu^\nu}\right)\sum_{j=1}^{n_2}T_{2j}^\nu-\frac{\nu r^{\nu-1}}{\mu^\nu}\sum_{j=1}^{n_2}T_{2j}^\nu\ln T_{2j}\right]=$$

$$=\frac{n_2}{r}-\frac{r^{\nu-1}+\nu r^{\nu-1}\ln r-\nu r^{\nu-1}\ln\mu}{\mu^\nu}n_2\left(\frac{\mu}{r}\right)^\nu-\frac{\nu r^{\nu-1}}{\mu^\nu}\frac{n_2}{\nu}\left(\frac{\mu}{r}\right)^\nu[\Gamma'(2)+\nu\ln\frac{\mu}{r}]=$$

$$=\frac{n_2}{r}-\frac{(1+\nu\ln r-\nu\ln\mu)n_2}{r}-\frac{(\Gamma'(2)+\nu\ln\frac{\mu}{r})n_2}{r}=-\frac{n_2}{r}\Gamma'(2);$$

$$\mathbf{E}\ddot{\ell}_{\mu^2}=\mathbf{E}\left[\frac{n\nu}{\mu^2}-\frac{\nu(\nu+1)}{\mu^{\nu+2}}\left(\sum_{i=1}^{n_1}T_{1i}^\nu+r^\nu\sum_{j=1}^{n_2}T_{2j}^\nu\right)\right]=$$

$$= \frac{n\nu}{\mu^2} - \frac{\nu(\nu+1)}{\mu^{\nu+2}}\left(n_1\mu^\nu + r^\nu n_2\left(\frac{\mu}{r}\right)^\nu\right) = \frac{n\nu}{\mu^2} - \frac{n\nu(\nu+1)}{\mu^2} = -\frac{n\nu^2}{\mu^2};$$

$$\mathbf{E}\ddot{\ell}_{\mu\nu} = \mathbf{E}\left[-\frac{n}{\mu} + \frac{1-\nu\ln\mu}{\mu^{\nu+1}}\left(\sum_{i=1}^{n_1} T_{1i}^\nu + r^\nu\sum_{j=1}^{n_2} T_{2j}^\nu\right) + \right.$$

$$\left. + \frac{\nu}{\mu^{\nu+1}}\left(\sum_{i=1}^{n_1} T_{1i}^\nu \ln T_{1i} + r^\nu \ln r\sum_{j=1}^{n_2} T_{2j}^\nu + r^\nu\sum_{j=1}^{n_2} T_{2j}^\nu \ln T_{2j}\right)\right] =$$

$$= -\frac{n}{\mu} + \frac{1-\nu\ln\mu}{\mu^{\nu+1}}(n\mu^\nu) +$$

$$+ \frac{\nu}{\mu^{\nu+1}}\left(\frac{n_1\mu^\nu}{\nu}[\Gamma'(2) + \nu\ln\mu] + r^\nu \ln r n_2\left(\frac{\mu}{r}\right)^\nu + r^\nu\frac{n_2}{\nu}\left(\frac{\mu}{r}\right)^\nu[\Gamma'(2) + \nu\ln\frac{\mu}{r}]\right) =$$

$$= -\frac{n}{\mu} + \frac{n(1-\nu\ln\mu)}{\mu} + \frac{1}{\mu}\left(n_1[\Gamma'(2)+\nu\ln\mu] + \nu n_2\ln r + n_2[\Gamma'(2)+\nu\ln\frac{\mu}{r}]\right) =$$

$$= \frac{n\Gamma'(2)}{\mu};$$

$$\mathbf{E}\ddot{\ell}_{\nu^2} = \mathbf{E}\left[-\frac{n}{\nu^2} - \frac{1}{\mu^\nu}\ln\mu\left(\ln\mu\,(\sum_{i=1}^{n_1} T_{1i}^\nu + r^\nu\sum_{j=1}^{n_2} T_{2j}^\nu\,) - \sum_{i=1}^{n_1} T_{1i}^\nu \ln T_{1i} - \right.\right.$$

$$\left. r^\nu \ln r\sum_{j=1}^{n_2} T_{2j}^\nu - r^\nu\sum_{j=1}^{n_2} T_{2j}^\nu \ln T_{2j}\right) +$$

$$+ \frac{1}{\mu^\nu}\left(\ln\mu\,(\sum_{i=1}^{n_1} T_{1i}^\nu\ln T_{1i} + r^\nu\ln r\sum_{j=1}^{n_2} T_{2j}^\nu + r^\nu\sum_{j=1}^{n_2} T_{2j}^\nu\ln T_{2j}\,) - \right.$$

$$-\sum_{i=1}^{n_1} T_{1i}^\nu \ln^2 T_{1i} - r^\nu\ln^2 r\sum_{j=1}^{n_2} T_{2j}^\nu - r^\nu\ln r\sum_{j=1}^{n_2} T_{2j}^\nu\ln T_{2j} -$$

$$\left.\left. -r^\nu\ln r\sum_{j=1}^{n_2} T_{2j}^\nu\ln T_{2j} - r^\nu\sum_{j=1}^{n_2} T_{2j}^\nu\ln^2 T_{2j}\right)\right] = -\frac{n}{\nu^2}(1+\Gamma''(2)).$$

Then the Fisher information matrix is

$$I(r;\mu,\nu) = \begin{pmatrix} \frac{n_2\nu^2}{r^2} & -\frac{n_2\nu^2}{r\mu} & \frac{n_2}{r}\Gamma'(2) \\ -\frac{n_2\nu^2}{r\mu} & \frac{n\nu^2}{\mu^2} & -\frac{n}{\mu}\Gamma'(2) \\ \frac{n_2}{r}\Gamma'(2) & -\frac{n}{\mu}\Gamma'(2) & \frac{n}{\nu^2}(1+\Gamma''(2)) \end{pmatrix}.$$

$$|I| = \frac{n_2\nu^2}{r^2}\frac{n\nu^2}{\mu^2}\frac{n}{\nu^2}(1+\Gamma''(2)) + \frac{n_2\nu^2}{r\mu}\frac{n}{\mu}\Gamma'(2)\frac{n_2}{r}\Gamma'(2) + \frac{n_2\nu^2}{r\mu}\frac{n}{\mu}\Gamma'(2)\frac{n_2}{r}\Gamma'(2) -$$

$$-\frac{n_2}{r}\Gamma'(2)\frac{n\nu^2}{\mu^2}\frac{n_2}{r}\Gamma'(2) - \frac{n_2\nu^2}{r\mu}\frac{n_2\nu^2}{r\mu}\frac{n}{\nu^2}(1+\Gamma''(2)) - \frac{n_2\nu^2}{r^2}\frac{n}{\mu}\Gamma'(2)\frac{n}{\mu}\Gamma'(2) =$$

$$= \frac{n_2\nu^2}{r^2}\frac{n^2}{\mu^2}(1+\Gamma''(2)) + \frac{n_2^2 n\nu^2}{\mu^2 r^2}(\Gamma'(2))^2 + \frac{n_2^2\nu^2 n}{r^2\mu^2}(\Gamma'(2))^2 -$$

$$-\frac{n_2^2 n\nu^2}{r^2\mu^2}(\Gamma'(2))^2 - \frac{n_2^2\nu^2 n}{r^2\mu^2}(1+\Gamma''(2)) - \frac{n_2 n^2\nu^2}{r^2\mu^2}(\Gamma'(2))^2 =$$

$$= \frac{n_1 n_2 n\nu^2}{r^2\mu^2}\left(1+\Gamma''(2)-[\Gamma'(2)]^2\right),$$

The inverse of the Fisher information matrix is

$$I^{-1} = \begin{pmatrix} \frac{nr^2}{n_1 n_2\nu^2} & \frac{r\mu}{n_1\nu^2} & 0 \\ \frac{r\mu}{n_1\nu^2} & \frac{\mu^2 n[1+\Gamma''(2)]-\mu^2 n_2[\Gamma'(2)]^2}{n_1 n\nu^2\left(1+\Gamma''(2)-[\Gamma'(2)]^2\right)} & \frac{\mu\Gamma'(2)}{n\left(1+\Gamma''(2)-[\Gamma'(2)]^2\right)} \\ 0 & \frac{\mu\Gamma'(2)}{n\left(1+\Gamma''(2)-[\Gamma'(2)]^2\right)} & \frac{\nu^2}{n\left(1+\Gamma''(2)-[\Gamma'(2)]^2\right)} \end{pmatrix}.$$

The c.d.f. K_2 has the form

$$K_2(t) = F_1(t) - \int_0^t S_1(t+ry-y)dF_1(y) =$$

$$= 1 - e^{-\left(\frac{t}{\mu}\right)^\nu} - \frac{\nu}{\mu}\int_0^t \left(\frac{y}{\mu}\right)^{\nu-1} e^{-\left(\frac{t+ry-y}{\mu}\right)^\nu - \left(\frac{y}{\mu}\right)^\nu} dy,$$

and the functions C_{2i} are

$$C_{21}(t) = \frac{\partial K_2(t)}{\partial r} = \frac{\nu^2}{\mu}\int_0^t \left(\frac{y}{\mu}\right)^\nu \left(\frac{t+ry-y}{\mu}\right)^{\nu-1} e^{-\left(\frac{t+ry-y}{\mu}\right)^\nu - \left(\frac{y}{\mu}\right)^\nu} dy,$$

$$C_{22}(t) = \frac{\partial K_2(t)}{\partial \mu} = -\frac{\nu}{\mu}\left(\frac{t}{\mu}\right)^\nu e^{-\left(\frac{t}{\mu}\right)^\nu} +$$

$$+\frac{\nu^2}{\mu^2}\int_0^t \left(\frac{y}{\mu}\right)^{\nu-1}\left[1 - \left(\frac{y}{\mu}\right)^\nu - \left(\frac{t+ry-y}{\mu}\right)^\nu\right] e^{-\left(\frac{t+ry-y}{\mu}\right)^\nu - \left(\frac{y}{\mu}\right)^\nu} dy,$$

$$C_{23}(t) = \frac{\partial K_2(t)}{\partial \nu} = \left(\frac{t}{\mu}\right)^\nu e^{-\left(\frac{t}{\mu}\right)^\nu} \ln\left(\frac{t}{\mu}\right) - \frac{1}{\mu}\int_0^t \left(\frac{y}{\mu}\right)^{\nu-1} e^{-\left(\frac{t+ry-y}{\mu}\right)^\nu - \left(\frac{y}{\mu}\right)^\nu} dy +$$

$$+\frac{\nu}{\mu}\int_0^t \left(\frac{y}{\mu}\right)^{\nu-1}\left[\left(\frac{y}{\mu}\right)^\nu \ln\left(\frac{y}{\mu}\right) + \left(\frac{t+ry-y}{\mu}\right)^\nu \ln\left(\frac{t+ry-y}{\mu}\right)\right] \times$$

$$\times e^{-\left(\frac{t+ry-y}{\mu}\right)^\nu - \left(\frac{y}{\mu}\right)^\nu} dy - \frac{\nu}{\mu}\int_0^t \left(\frac{y}{\mu}\right)^{\nu-1} \ln\left(\frac{y}{\mu}\right) e^{-\left(\frac{t+ry-y}{\mu}\right)^\nu - \left(\frac{y}{\mu}\right)^\nu} dy,$$

We have

$$\hat{\sigma}^2_{\hat{K}_2(t)} = C_2^T(t;\hat{r},\hat{\mu},\hat{\nu}) I^{-1}(\hat{r},\hat{\mu},\hat{\nu}) C_2(t;\hat{r},\hat{\mu},\hat{\nu}),$$

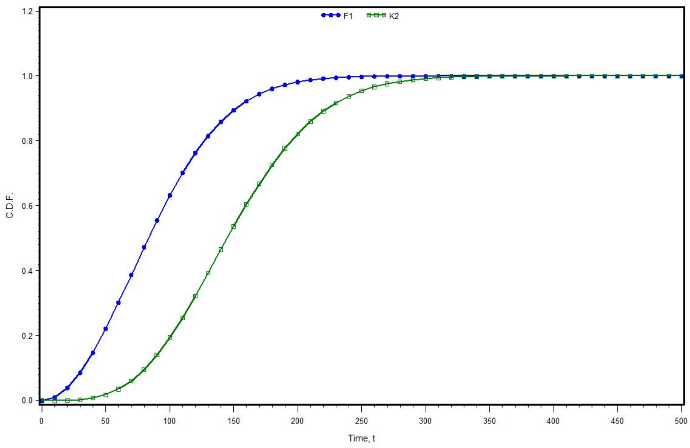
$$C_2(t;\hat{r},\hat{\mu},\hat{\nu}) = (C_{21}(t;\hat{r},\hat{\mu},\hat{\nu}), C_{22}(t;\hat{r},\hat{\mu},\hat{\nu}), C_{23}(t;\hat{r},\hat{\mu},\hat{\nu}))^T.$$

The asymptotic $1-\alpha$ confidence interval for $K_j(t)$ has the form (2.48) or, alternatively, (2.49) then $j = 2$.

We found by simulation finite sample confidence levels of the intervals obtained using asymptotic formulas with $1-\alpha = 0.9$. We simulating failure times T_{1j} and T_{2j} from Weibull distribution with following parameters:

$$T_{1j} \sim W(\alpha_1, \beta_1), \quad T_{2j} \sim\ W(\alpha_1, \beta_1),$$

$$\alpha_1 = \alpha_2 = 2, \quad \beta_1 = 100, \quad \beta_2 = 300.$$



Graph 2.8. Graphs of the trajectories of the parametric estimators $\hat{F}_1, \hat{K}_2$ (Weibull distribution)

The number of replications was 2000. For various values of t the proportions of confidence interval realizations covering the true value of the distributional function $K_2(t)$ are given below:

Table 2.16. Confidence level for finite samples ($n1 = n2 = 100$)

Time, t	50	100	200	300	400	500
$K_2(t)$	0.018	0.194	0.822	0.992	0.999	1.000
Confidence level (%)	89.4	89.2	89.2	89.5	89.5	90.2

Example 2.3.6. ***Loglogistic distribution*:** $S_1(t) = \frac{1}{1+(t/\mu)^\nu}$. Let us consider the case of complete samples. By (2.25) the loglikelihood function has the form

$$l(r;\theta) = \ln L(r,\theta) = \sum_{i=1}^{n_1} \ln f_1(T_{1i};\theta) + n_2 \ln r + \sum_{j=1}^{n_2} \ln f_1(rT_{2j};\theta).$$

In the case of loglogistic distribution

$$f_1(t;\mu,\nu) = \frac{\nu t^{\nu-1}}{\mu^\nu \left(1+\left(\frac{t}{\mu}\right)^\nu\right)^2};$$

$$\ln f_1(t;\mu,\nu) = \ln\nu + (\nu-1)\ln t - \nu\ln\mu - 2\ln\left(1+\left(\frac{t}{\mu}\right)^\nu\right),$$

$$\sum_{i=1}^{n_1} \ln f_1(t;\mu,\nu) = n_1 \ln\nu + (\nu-1)\sum_{i=1}^{n_1} \ln T_{1i} - \nu n_1 \ln\mu - 2\sum_{i=1}^{n_1} \ln\left(1+\left(\frac{T_{1i}}{\mu}\right)^{\nu}\right),$$

$$\ln f_1(rt;\mu,\nu) = \ln\nu + (\nu-1)(\ln r + \ln t) - \nu\ln\mu - 2\ln\left(1+\left(\frac{rt}{\mu}\right)^{\nu}\right).$$

So the loglikelihood function has the form

$$l(r;\mu,\nu) = n\ln\nu - \nu n\ln\mu + \nu n_2 \ln r + (\nu-1)\left(\sum_{i=1}^{n_1}\ln T_{1i} + \sum_{j=1}^{n_2}\ln T_{2j}\right) -$$

$$-2\sum_{i=1}^{n_1}\ln\left(1+\left(\frac{T_{1i}}{\mu}\right)^{\nu}\right) - 2\sum_{j=1}^{n_2}\ln\left(1+\left(\frac{rT_{2i}}{\mu}\right)^{\nu}\right).$$

Partial derivatives are

$$\dot{\ell}_r = \frac{\partial l}{\partial r} = \frac{\nu n_2}{r} - \frac{2\nu}{r}\sum_{j=1}^{n_2}\frac{\left(\frac{rT_{2j}}{\mu}\right)^{\nu}}{1+\left(\frac{rT_{2j}}{\mu}\right)^{\nu}} = 0;$$

$$\dot{\ell}_\mu = \frac{\partial l}{\partial \mu} = -\frac{\nu n}{\mu} + \frac{2\nu}{\mu}\sum_{i=1}^{n_1}\frac{\left(\frac{T_{1i}}{\mu}\right)^{\nu}}{1+\left(\frac{T_{1i}}{\mu}\right)^{\nu}} + \frac{2\nu}{\mu}\sum_{j=1}^{n_2}\frac{\left(\frac{rT_{2j}}{\mu}\right)^{\nu}}{1+\left(\frac{rT_{2j}}{\mu}\right)^{\nu}} = 0;$$

$$\dot{\ell}_\nu = \frac{\partial l}{\partial \nu} = \frac{n}{\nu} - n\ln\mu + n_2\ln r + \sum_{i=1}^{n_1}\ln T_{1i} + \sum_{j=1}^{n_2}\ln T_{2j} -$$

$$-2\sum_{i=1}^{n_1}\frac{\left(\frac{T_{1i}}{\mu}\right)^{\nu}\log\left(\frac{T_{1i}}{\mu}\right)}{1+\left(\frac{T_{1i}}{\mu}\right)^{\nu}} - 2\sum_{j=1}^{n_2}\frac{\left(\frac{rT_{2j}}{\mu}\right)^{\nu}\log\left(\frac{rT_{2j}}{\mu}\right)}{1+\left(\frac{rT_{2j}}{\mu}\right)^{\nu}} = 0.$$

Set

$$X_i = \left(\frac{T_{1i}}{\mu}\right)^{\nu} \quad Y_j = \left(,\frac{rT_{2i}}{\mu}\right)^{\nu}.$$

Second partial derivatives of the loglikelihood function are

$$\ddot{\ell}_{r^2} = -\frac{\nu n_2}{r^2} + \frac{2\nu}{r^2}\sum_{j=1}^{n_2}\frac{Y_j}{1+Y_j} - \frac{2\nu^2}{r^2}\sum_{i=1}^{n_2}\frac{Y_j}{(1+Y_j)^2}, \quad \ddot{\ell}_{r\mu} = \frac{2\nu^2}{r\mu}\sum_{j=1}^{n_2}\frac{Y_j}{(1+Y_j)^2},$$

$$\ddot{\ell}_{r\nu} = \frac{n_2}{r} - \frac{2}{r}\sum_{j=1}^{n_2}\frac{Y_j}{1+Y_j} - \frac{2\nu}{r}\frac{1}{\nu}\sum_{j=1}^{n_2}\frac{Y_j\ln Y_j}{(1+Y_j)^2},$$

$$\ddot{\ell}_{\mu^2} = \frac{n\nu}{\mu^2} - \frac{2\nu}{\mu^2}\left(\sum_{i=1}^{n_1}\frac{X_i}{1+X_i} + \sum_{j=1}^{n_2}\frac{Y_i}{1+Y_i}\right) - \frac{2\nu^2}{\mu^2}\left(\sum_{i=1}^{n_1}\frac{X_i}{(1+X_i)^2} + \sum_{j=1}^{n_2}\frac{Y_i}{(1+Y_i)^2};\right),$$

$$\ddot{\ell}_{\mu\nu} = -\frac{n}{\mu} + \frac{2}{\mu}\left(\sum_{i=1}^{n_1}\frac{X_i}{1+X_i} + \sum_{j=1}^{n_2}\frac{Y_j}{1+Y_j}\right) + \frac{2}{\mu}\left(\sum_{i=1}^{n_1}\frac{X_i\ln X_i}{(1+X_i)^2} + \sum_{j=1}^{n_2}\frac{Y_i\ln Y_i}{(1+Y_i)^2}\right),$$

$$\ddot{\ell}_{\nu^2} = -\frac{n}{\nu^2} - \frac{2}{\nu^2}\left(\sum_{i=1}^{n_1} \frac{X_i \ln^2 X_i}{(1+X_i)^2} + \sum_{j=1}^{n_2} \frac{Y_i \ln^2 Y_i}{(1+Y_i)^2}\right).$$

The random variables X_i and Y_j are identically distributed with the probability density function of the standard loglogistic distribution: $f_{X_i}(x) = f_{Y_j}(x) = 1/(1+x)^2$. It implies that for any $k > -2$ and $a \in (-1, k+1)$

$$g(a) = \mathbf{E}\frac{X_i^a}{(1+X_i)^k} = \int_0^\infty \frac{x^a}{(1+x)^{k+2}}dx = |y = \frac{1}{1+x}; x = \frac{1}{y} - 1; dx = -\frac{dy}{y^2}| =$$

$$= \int_0^1 \left(\frac{1}{y} - 1\right)^a y^k dy = \int_0^1 y^{k-a}(1-y)^a dy = \frac{\Gamma(k-a+1)\Gamma(a+1)}{\Gamma(k+2)}.$$

$$g'(a) = \mathbf{E}\frac{X_i^a \ln X_i}{(1+X_i)^k} = \int_0^\infty \frac{x^a}{(1+x)^{k+2}} \ln x dx =$$

$$= \frac{(\Gamma(k-a+1)\Gamma(a+1))'_a}{\Gamma(k+2)} = \frac{-\Gamma'(k-a+1)\Gamma(a+1) + \Gamma(k-a+1)\Gamma'(a+1)}{\Gamma(k+2)};$$

$$g''(a) = \mathbf{E}\frac{X_i^a \ln^2 X_i}{(1+X_i)^k} = \int_0^\infty \frac{x^a}{(1+x)^{k+2}} \ln^2 x dx =$$

$$= \frac{\Gamma''(k-a+1)\Gamma(a+1) - 2\Gamma'(k-a+1)\Gamma'(a+1) + \Gamma(k-a+1)\Gamma''(a+1)}{\Gamma(k+2)}.$$

If $a = 1$ and $k = 1$, then

$$\mathbf{E}\frac{X_i}{1+X_i} = \frac{\Gamma(1)\Gamma(2)}{\Gamma(3)} = \frac{1}{2}, \quad \mathbf{E}\frac{X_i \ln X_i}{1+X_i} = \frac{-\Gamma'(1) + \Gamma'(2)}{2};$$

if $a = 1$ and $k = 2$, then

$$\mathbf{E}\frac{X_i}{(1+X_i)^2} = \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} = \frac{1}{6},$$

$$\mathbf{E}\frac{X_i \ln X_i}{(1+X_i)^2} = 0, \quad \mathbf{E}\frac{X_i \ln^2 X_i}{(1+X_i)^2} = \frac{\Gamma''(2) - [\Gamma'(2)]^2}{3},$$

if $a = 2$ and $k = 2$, then

$$\mathbf{E}\frac{X_i^2}{(1+X_i)^2} = \frac{\Gamma(1)\Gamma(3)}{\Gamma(4)} = \frac{1}{3}, \quad \mathbf{E}\frac{X_i^2 \ln X_i}{(1+X_i)^2} = \frac{-2\Gamma'(1) + \Gamma'(3)}{6}.$$

Now we are able to compute the means of the second partial derivatives of the log-likelihood function:

$$\mathbf{E}\ddot{\ell}_{r^2} = \frac{-\nu n_2}{r^2} + \frac{2\nu}{r^2}\frac{n_2}{2} - \frac{2\nu^2}{r^2}\frac{n_2}{6} = -\frac{n_2 \nu^2}{3r^2},$$

$$\mathbf{E}\ddot{\ell}_{r\mu} = \frac{2\nu n_2}{r}\frac{\nu}{6\mu} = \frac{n_2\nu^2}{3r\mu}, \quad -\mathbf{E}\ddot{\ell}_{r\nu} = \frac{n_2}{r} - \frac{2n_2}{2r} - \frac{2}{r}0 = 0;$$

$$\mathbf{E}\ddot{\ell}_{\mu^2} = \frac{n\nu}{\mu^2} - \frac{2\nu}{\mu^2}\frac{n_1+n_2}{2} - \frac{2\nu^2}{\mu^2}\frac{n_1+n_2}{6} = \frac{n\nu}{\mu^2} - \frac{n\nu}{\mu^2} - \frac{n\nu^2}{3\mu^2} = -\frac{n\nu^2}{3\mu^2},$$

$$\mathbf{E}\ddot{\ell}_{\mu\nu} = -\frac{n}{\mu} + \frac{n}{\mu} + 0 = 0, -\mathbf{E}\ddot{\ell}_{\nu^2} = \frac{n}{3\nu^2}(3 + 2\Gamma''(2) - 2(\Gamma'(2))^2);$$

So the Fisher information matrix is

$$I(r,\lambda) = \begin{pmatrix} \frac{n_2\nu^2}{3r^2} & -\frac{n_2\nu^2}{3r\mu} & 0 \\ -\frac{n_2\nu^2}{3r\mu} & \frac{n\nu^2}{3\mu^2} & 0 \\ 0 & 0 & \frac{n\{3+2\Gamma''(2)-2[\Gamma'(2)]^2\}}{3\nu^2} \end{pmatrix}$$

and the inverse of the Fisher information matrix is

$$I^{-1} = \begin{pmatrix} \frac{3nr^2}{n_1n_2\nu^2} & \frac{3r\mu}{n_1\nu^2} & 0 \\ \frac{3r\mu}{n_1\nu^2} & \frac{3\mu^2}{n_1\nu^2} & 0 \\ 0 & 0 & \frac{3\nu^2}{n\{3+2\Gamma''(2)-2[\Gamma'(2)]^2\}} \end{pmatrix},$$

and the estimator of $\hat{\sigma}^2_{\hat{K}_j(t)}$ has the form

$$\hat{\sigma}^2_{\hat{K}_j(t)} = C_j^T(t;\hat{r},\hat{\mu},\hat{\nu})I^{-1}(\hat{r},\hat{\mu},\hat{\nu})C_j(t;\hat{r},\hat{\mu},\hat{\nu}). \tag{2.50}$$

The asymptotic $1-\alpha$ confidence interval for $K_j(t)$ has the form (2.48) or, alternatively, (2.49).

Taking into account that

$$S_1(t) = \frac{1}{1+\left(\frac{t}{\mu}\right)^{\nu}},$$

the ML estimator of the reliability function K_2 has the form

$$\hat{K}_2(t) = \hat{F}_1(t) - \int_0^t \hat{S}_1(t+\hat{r}y-y)d\hat{F}_1(y) =$$

$$= 1 - \frac{1}{1+\left(\frac{t}{\hat{\mu}}\right)^{\hat{\nu}}} - \frac{\hat{\nu}}{\hat{\mu}}\int_0^t \left(\frac{y}{\hat{\mu}}\right)^{\hat{\nu}-1}\frac{1}{1+\left(\frac{t+\hat{r}y-y}{\hat{\mu}}\right)^{\hat{\nu}}}\frac{1}{\left(1+\left(\frac{y}{\hat{\mu}}\right)^{\hat{\nu}}\right)^2}dy.$$

and the functions C_{2i} are

$$C_{21}(t) = \frac{\partial K_2(t)}{\partial r} =$$

$$= \nu^2\int_0^t \left(\frac{y}{\mu}\right)^{\nu}\left(\frac{t+ry-y}{\mu}\right)^{\nu-1}\left(1+\left(\frac{y}{\mu}\right)^{\nu}\right)^{-2}\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)^{-2}dy,$$

$$C_{22}(t)=\frac{\partial K_2(t)}{\partial \mu}=-\frac{\nu}{\mu}\left(\frac{t}{\mu}\right)^{\nu}\left(1+\left(\frac{t}{\mu}\right)^{\nu}\right)^{-2}+$$

$$+\int_0^t \frac{\nu\left(\frac{y}{\mu}\right)^{\nu-1}}{\mu^2\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)\left(1+\left(\frac{y}{\mu}\right)^{\nu}\right)^2}+\frac{\nu\left(\frac{y}{\mu}\right)^{(\nu-1)}(\nu-1)}{\mu^2\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)\left(1+\left(\frac{y}{\mu}\right)^{\nu}\right)^2}-$$

$$-\frac{\nu^2\left(\frac{y}{\mu}\right)^{(\nu-1)}\left(\frac{t+ry-y}{\mu}\right)^{\nu}}{\mu^2\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)^2\left(1+\left(\frac{y}{\mu}\right)^{\nu}\right)^2}-\frac{2\nu^2\left(\frac{y}{\mu}\right)^{(\nu-1)}\left(\frac{y}{\mu}\right)^{\nu}}{\mu^2\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)\left(1+\left(\frac{y}{\mu}\right)^{\nu}\right)^3}dy=$$

$$=-\frac{\nu}{\mu}\left(\frac{t}{\mu}\right)^{\nu}\left(1+\left(\frac{t}{\mu}\right)^{\nu}\right)^{-2}-\int_0^t \frac{\nu^2\left(\frac{y}{\mu}\right)^{\nu-1}}{\mu^2\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)\left(1+\left(\frac{y}{\mu}\right)^{\nu}\right)^2}\times$$

$$\times\frac{1-\left(\frac{y}{\mu}\right)^{\nu}-2\left(\frac{y}{\mu}\right)^{\nu}\left(\frac{t+ry-y}{\mu}\right)^{\nu}}{\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)\left(1+\left(\frac{y}{\mu}\right)^{\nu}\right)}dy,$$

$$C_{23}(t)=\frac{\partial K_2(t)}{\partial \nu}=\left(\frac{t}{\mu}\right)^{\nu}\left(1+\left(\frac{t}{\mu}\right)^{\nu}\right)^{-2}\ln\left(\frac{t}{\mu}\right)-$$

$$-\int_0^t \frac{\left(\frac{y}{\mu}\right)^{\nu-1}}{\mu\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)\left(1+\left(\frac{y}{\mu}\right)^{\nu}\right)^2}-\frac{\nu\left(\frac{y}{\mu}\right)^{\nu-1}\ln\left(\frac{y}{\mu}\right)}{\mu\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)\left(1+\left(\frac{y}{\mu}\right)^{\nu}\right)^2}+$$

$$+\frac{\nu\left(\frac{y}{\mu}\right)^{\nu-1}\left(\frac{t+ry-y}{\mu}\right)^{\nu}\ln\left(\frac{t+ry-y}{\mu}\right)}{\mu\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)^2\left(1+\left(\frac{y}{\mu}\right)^{\nu}\right)^2}+2\frac{\nu\left(\frac{y}{\mu}\right)^{\nu-1}\left(\frac{y}{\mu}\right)^{\nu}\ln\left(\frac{y}{\mu}\right)}{\mu\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)\left(1+\left(\frac{y}{\mu}\right)^{\nu}\right)^3}dy=$$

$$=\left(\frac{t}{\mu}\right)^{\nu}\left(1+\left(\frac{t}{\mu}\right)^{\nu}\right)^{-2}\ln\left(\frac{t}{\mu}\right)-$$

$$-\frac{1}{\mu}\int_0^t\left(\frac{y}{\mu}\right)^{\nu-1}\left(1+\left(\frac{y}{\mu}\right)^{\nu}\right)^{-2}\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)^{-1}\times$$

$$\times\left[\left(1+\left(\frac{y}{\mu}\right)^{\nu}-\nu\ln\left(\frac{y}{\mu}\right)\left(1-\left(\frac{y}{\mu}\right)^{\nu}\right)\right)\left(1+\left(\frac{y}{\mu}\right)\right)^{-1}-\right.$$

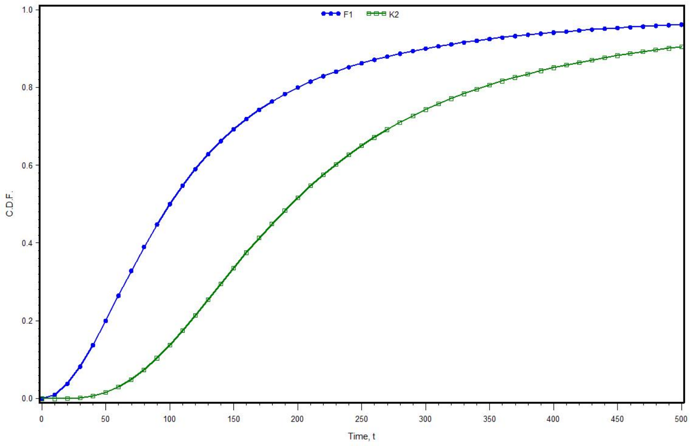
$$\left.-\nu\left(\frac{t+ry-y}{\mu}\right)^{\nu}\left(1+\left(\frac{t+ry-y}{\mu}\right)^{\nu}\right)^{-1}\ln\left(\frac{t+ry-y}{\mu}\right)\right]dy,$$

So the estimator $\hat{\sigma}^2_{\hat{K}_2(t)}$ has the form (2.50) and the asymptotic $1-\alpha$ confidence interval for $K_2(t)$ is of the form (2.48) or, alternatively, (2.49) taking $j=2$.

We found by simulation finite sample confidence levels of the intervals obtained using asymptotic formulas with $1-\alpha = 0.9$. We simulating failure times T_{1j} and T_{2j} from loglogistic distribution with following parameters:

$$T_{1j} \sim L(\alpha_1, \beta_1), \quad T_{2j} \sim \ L(\alpha_1, \beta_1),$$

$$\alpha_1 = \alpha_2 = 2, \quad \beta_1 = 100, \quad \beta_2 = 300.$$



Graph 2.9. Graphs of the trajectories of the parametric estimators $\hat{F}_1, \hat{K}_2$ (Loglogistic distribution)

The number of replications was 2000. For various values of t the proportions of confidence interval realizations covering the true value of the distributional function $K_2(t)$ are given below:

Table 2.16. Confidence level for finite samples ($n1 = n2 = 100$)

Time, t	50	100	200	300	400	500
$K_2(t)$	0.016	0.138	0.517	0.743	0.851	0.905
Confidence level (%)	89.0	88.8	90.4	89.6	89.5	90.5

Chapter 3

Failure-Time Degradation Models

3.1 Failure Degradation Model with covariates

For reliability characteristics estimation the following data we are going to analyze: failure times (possibly censored), explanatory variables (covariates, stresses) and the values of some observable quantity characterizing the degradation of units.

When we analyze the degradation data then the failure can occur non-traumatic or traumatic. A failure is called non-traumatic when the degradation attains a critical level z_0. Other failures are called traumatic. Traumatic failures may be of different types: related with production defects, caused by mechanical damages or by fatigue of components.

Suppose that under fixed constant covariate the degradation is stochastic process $Z(t)$, $t \geq 0$.

Suppose that the degradation process $Z(t)$ is non-decreasing with cadlag trajectories.

Denote by $T^{(k)}$ the moment of the traumatic failure of the kth mode, $k = 1, \cdots, s$.

We suppose that the random variables $T^{(1)}, \cdots, T^{(s)}$ are conditionally independent given the degradation Z.

Denote by $\tilde{\lambda}^{(k)}(t|Z) = \tilde{\lambda}^{(k)}(t|Z(s), 0 \leq s \leq t)$ the conditional failure rate of the traumatic failure of the kth mode given the degradation.

Suppose that this conditional failure rate has two additive components: one related to observed degradation values, other - to non-observable degradation (aging) and to possible shocks causing sudden traumatic failures. For example, observable degradation of tires is the wear of the protector. The failure rate of tire explosion depends on thickness of the protector, on non-measured degradation level of other tire components and on intensity of possible shocks (hitting a kerb, nail, etc.). So

$$\tilde{\lambda}^{(k)}(t|Z) = \lambda^{(k)}(Z(t)) + \mu^{(k)}(t). \tag{3.1}$$

The function $\lambda^{(k)}(z)$ characterizes the dependence of the rate of traumatic failures of the kth mode on degradation. The function $\mu^{(k)}$ characterizes the dependence of the rate of traumatic failures of the kth mode on other tire component.

Suppose that external covariates [25] influence degradation rate and traumatic event intensity.

Let $x(t) = \big(x_1(t), \ldots, x_s(t)\big)^T$ be a vector of possibly time dependent covariates. We assume in what follows that x_i are deterministic or realizations of bounded right continuous with finite left hand limits stochastic processes.

Denote by $Z(t|x)$ the degradation level at the moment t for units functioning under the covariate x.

We suppose that the covariates influence locally the scale of the traumatic failure time distribution component related to aging (non-observable degradation) and to possible shocks, i.e. the accelerated failure time (AFT) [1] model is true for this component. Let us explain it in detail. Denote by

$$S_1^{(k)}(t|Z) = \exp\{-\int_0^t \lambda^{(k)}[Z(u)]du\}, \quad S_2^{(k)}(t) = \exp\{-\int_0^t \mu^{(k)}(u)du\}$$

the survival functions corresponding to the failure rates $\lambda^{(k)}(Z(u))$ and $\mu^{(k)}(u)$. The first survival function is conditional given the degradation.

The AFT model defines the following relation of the second survival function and the covariates:

$$S_2^{(k)}(t|x) = S_2^{(k)}(\int_0^t e^{\beta_k^T x(s)}\, ds);$$

the parameters β_k have the same dimension as x. The covariate x may be replaced by some specified function $\varphi(x)$.

Set

$$f(t, x, \beta) = \int_0^t e^{\beta^T x(u)}\, du, \tag{3.2}$$

and denote by $g(t, x, \beta)$ the inverse of $f(t, x, \beta)$ with respect to the first argument. If $x =$const then

$$f(t, x, \beta) = e^{\beta^T x} t, \quad g(t, x, \beta) = e^{-\beta^T x} t.$$

The function $f(t, x, \beta)$ is time transformation in dependence on x. For units functioning under different covariates $x^{(1)}$ and $x^{(2)}$ two moments t_1 and t_2, respectively, are equivalent in the sense of degradation if they verify the equality $f(t_1, x^{(1)}, \beta) = f(t_2, x^{(2)}, \beta)$, i.e. we consider the following model for degradation process under covariates:

$$Z(t|x) = Z(f(t, x, \beta)). \tag{3.3}$$

The covariates have double influence on the distribution of the first traumatic failure component: via degradation and directly. So we combine the AFT and the proportional hazards models:

$$S_1^{(k)}(t|x, Z) = \exp\{-\int_0^t e^{\tilde\beta_k^T x(u)} \lambda^{(k)}(Z(u|x)du\}.$$

Denote by

$$S^{(k)}(t|x, Z) = \mathbf{P}(T^{(k)} > t|x(u), Z(u|x), 0 \le u \le t),$$

$$\tilde\lambda^{(k)}(t|x, Z) = -\frac{d}{dt} \ln S^{(k)}(t|x, Z)$$

the conditional distribution function and the failure rate of the traumatic failure of the kth mode given the covariates and the degradation. So we consider the following model:

$$\mathbf{P}(T^{(1)} > t, \dots, T^{(s)} > t | x(u), Z(u|x), 0 \le u \le t) = \prod_{k=1}^{s} S^{(k)}(t|x, Z), \tag{3.4}$$

$$S^{(k)}(t|x, Z) = \exp\left\{ -\int_0^t \tilde{\lambda}^{(k)}(t|x, Z)\, du \right\} =$$

$$= \exp\left\{ -\int_0^t e^{\tilde{\beta}_k^T x(u)} \lambda^{(k)}(Z(u|x))\, du - H^{(k)}(f(t, x, \beta_k)) \right\}, \tag{3.5}$$

where

$$\tilde{\lambda}^{(k)}(t|x, Z) = e^{\tilde{\beta}_k^T x(t)} \lambda^{(k)}(Z(t|x)) + e^{\beta_k^T x(t)} \mu^{(k)}(f(t, x, \beta_k)), \tag{3.6}$$

$$H^{(k)}(t) = \int_0^t \mu^{(k)}(u) du. \tag{3.7}$$

Denote by

$$T^{(0)} = \inf\{t : Z(t|x) \ge z_0\}. \tag{3.8}$$

and

$$S^{(0)}(t|x) = \mathbf{P}\big\{T^{(0)} > t \mid x(u), 0 \le u \le t\big\} =$$
$$= \mathbf{P}\big\{Z(t|x) < z_0 \mid x(u), 0 \le u \le t\big\} \tag{3.9}$$

the time to non-traumatic failure and its survival function under the covariate x, respectively.

The time of the unit failure

$$T = \min(T^{(0)}, T^{(1)}, \dots, T^{(s)}) \tag{3.10}$$

may be traumatic or non-traumatic.

Denote by

$$V = k \quad \text{if} \quad T = T^{(k)}, \quad k = 0, \dots, s, \tag{3.11}$$

the indicator of the failure mode. The failure mode 0 is non-traumatic. Others are traumatic.

Let us consider reliability characteristics which are interesting for applications. These are:

1) The survival function of the failure time under the covariate x:

$$S(t|x) = \mathbf{P}(T > t|x) = \mathbf{E}S(t|x, Z), \quad S(t|x, Z) = \mathbf{1}_{\{Z(t|x) < z_0\}} \prod_{k=1}^{s} S^{(k)}(t \mid x, Z). \tag{3.12}$$

2) Mean failure time under the covariate x:

$$e(x) = \mathbf{E}(T|x) = \mathbf{E}(\mathbf{E}(T|x, Z)), \quad \mathbf{E}(T|x, Z) = \int_0^{T^{(0)}} \prod_{k=1}^{s} S^{(k)}(t \mid x, Z) dt. \tag{3.13}$$

3) The probability that under the covariate x the non-traumatic failure is observed in the interval $[0,t]$:

$$P^{(0)}(t|x) = \mathbf{E}P^{(0)}(t|x,Z),$$

$$P^{(0)}(t|x,Z) = \mathbf{1}_{\{Z(t|x)\geq z_0\}} \prod_{k=1}^{s} S^{(k)}(T^{(0)} \mid x, Z). \tag{3.14}$$

In particular, the probability of observed non-traumatic failure under the covariate x in the interval $[0,\infty)$ is obtained.

4) The probability that under the covariate x a traumatic failure is observed in the interval $[0,t]$:

$$P^{(tr)}(t|x) = \mathbf{E}P^{(tr)}(t|x,Z),$$

$$P^{(tr)}(t|x,Z) = 1 - \prod_{k=1}^{s} S^{(k)}(t \wedge T^{(0)} \mid x, Z). \tag{3.15}$$

5) The probability that under the covariate x the traumatic failure of the kth mode, $k = 1,\ldots,s$, is observed in the interval $[0,t]$:

$$P^{(k)}(t|x) = \mathbf{E}P^{(k)}(t|x,Z),$$

$$P^{(k)}(t|x,Z) = \int_0^{t\wedge T^{(0)}} \prod_{l=1}^{s} S^{(l)}(s \mid x, Z)\, \lambda^{(k)}(s \mid x, Z)\, ds. \tag{3.16}$$

Suppose that the cause of some traumatic failure modes are eliminated. Note that elimination of a failure mode may increase the number of failures of other modes. Indeed, a failure of the lth mode is not observed if it is preceded by a failure of the kth mode but this failure might be observed if the kth failure mode would be eliminated.

If i_1th, ..., i_qth $(1 \leq i_1 < ... < i_q \leq s)$ traumatic failure modes are eliminated then the survival function $S(t|x)$, the mean $e(x)$ and the probabilities $P^{(0)}(t|x)$, $P^{(tr)}(t|x)$, and $P^{(k)}(t|x)$, $(k = 0,1,\ldots,s)$ are modified taking $\prod_{l\neq i_1,\ldots,i_q}$ instead of $\prod_{l=1}^{s}$ in the formulas (3.12)-(3.16). So an experiment using units with eliminated failure modes is not needed. The estimators of survival characteristics of units with eliminated failure modes is useful for planning possible ways of reliability improvement.

Suppose that a unit did not fail to the moment τ and we have some information about its covariable and degradation processes $(x(s), Z(s|x) \mid s \leq \tau)$.

Let $\mathcal{G}$ denote the σ-algebra generated by the possessed information about the degradation process and

$$\bar{\mathcal{G}}_\tau = \sigma(\mathcal{G} \cup \{T > \tau\}).$$

The conditional probabilities of the events considered in the previous section given the σ-algebra $\bar{\mathcal{G}}_\tau$ are: for $t > \tau$.

$$S(t \mid x, \tau, \mathcal{G}) = \frac{\mathbf{E}_{\mathcal{G}}\{S(t \mid x, Z)\}}{\mathbf{E}_{\mathcal{G}}\{S(\tau \mid x, Z)\}}, \tag{3.17}$$

$$P^{(k)}(t \mid x, \tau, \mathcal{G}) = \frac{\mathbf{E}_{\mathcal{G}}\{P^{(k)}(t \mid x, Z)\} - \mathbf{E}_{\mathcal{G}}\{P^{(k)}(\tau \mid x, Z)\}}{\mathbf{E}_{\mathcal{G}}\{S(\tau \mid x, Z)\}}, \tag{3.18}$$

$$P^{(tr)}(t \mid \tau, \mathcal{G}) = \frac{\mathbf{E}_{\mathcal{G}}\{P^{(tr)}(t \mid x, Z)\} - \mathbf{E}_{\mathcal{G}}\{P^{(tr)}(\tau \mid x, Z)\}}{\mathbf{E}_{\mathcal{G}}\{S(\tau \mid x, Z)\}}. \tag{3.19}$$

Moreover, the mean residual life of the unit is

$$e(x, \tau, \mathcal{G}) = \frac{\mathbf{E}_{\mathcal{G}}\{(T-\tau)\mathbf{1}_{\{T>\tau\}}|x\}}{\mathbf{E}_{\mathcal{G}}\{S(\tau \mid x, Z)\}}. \tag{3.20}$$

If $\mathcal{G} = \sigma(Z)$ then

$$\mathbf{E}_{\mathcal{G}}\{S(t \mid x, Z)\}) = \mathbf{E}\{S(t \mid x, Z)\}, \tag{3.21}$$

$$\mathbf{E}_{\mathcal{G}}\{(T-\tau)\mathbf{1}_{\{T>\tau\}}|x, Z\} = \int_{\tau}^{T^{(0)}} \prod_{l=1}^{s} S^{(l)}(t \mid x, Z)dt - \tau \mathbf{1}_{\{Z(\tau|x)<z_0\}} \prod_{l=1}^{s} S^{(l)}(\tau \mid x, Z). \tag{3.22}$$

3.2 Estimation of model parameters

3.2.1 The data

Suppose that n units are observed. The ith unit is tested under the vector of explanatory variables $x^{(i)}$, and at the moments

$$0 < t_{i1} < t_{i2} < .... < t_{im_i}$$

the values $Z_{ij} = Z_i(t_{ij}|x^{(i)})$ of the degradation level are supposed to be measured. The moments t_{ij} correspond to the scale of real functioning. For example, in the case of tire wear, t_{ij} mean kilometers done by the i-th tire until the j-th measurement. The values of covariates are supposed to be observed during the experiment. The most often they should be constant in time or step-functions.

Denote by $T_i = \min(T_i^{(0)}, \ldots, T_i^{(s)})$ the failure time and V_i - the failure mode indicator. The data may be right censored. Denote by C_i the censoring time of the ith unit, and set

$$\tilde{C}_i = C_i \wedge t_{im_i}, \quad X_i = T_i \wedge \tilde{C}_i, \quad \delta_i = \mathbf{1}_{\{T_i \le \tilde{C}_i\}}, \quad \tilde{\delta}_i = \mathbf{1}_{\{T_i \le \tilde{C}_i, V_i \ne 0\}}. \tag{3.23}$$

Denote by

$$\mu_i = \begin{cases} j, & \text{if} \quad X_i \in (t_{ij}, t_{i,j+1}], \quad j = 0, ..., m_i - 1, \\ m_i, & \text{if} \quad X_i = t_{im_i}, \end{cases} \tag{3.24}$$

the observed number of measurements of the i-th unit.

The data are the random vectors

$$(X_i, \delta_i, V_i, \mu_i, Z_{i1}, ..., Z_{i\mu_i}, x^{(i)}), \quad i = 1, ..., n. \tag{3.25}$$

If $\mu_i = 0$ then the degradation values are not observed.

3.2.2 Likelihood function construction

Suppose that the functions $\lambda^{(k)}(z)$ and $\mu^{(k)}(t)$ is from a class of functions

$$\lambda^{(k)}(z) = \lambda^{(k)}(z, \eta_k), \quad \mu^{(k)}(t) = \mu^{(k)}(t, \gamma_k), \tag{3.26}$$

where η_k, γ_k are possibly multi-dimensional parameters. For example, analysis of tire failure time and wear data shows that the intensities $\lambda^{(k)}(z)$ and $\mu^{(k)}(t)$ typically have the form $(z/\eta_{1k})^{\eta_{2k}}$ and $(t/\gamma_{1k})^{\gamma_{2k}}$.

Suppose at first that degradation processes $Z_i(t) = Z_i(t|x^{(i)})$ of all units are continuously observable. In this case conditional likelihood L and loglikelihood l functions given degradation for the parameters characterizing traumatic failures can be written as follows:

$$L = \prod_{i=1}^{n} \left\{ \sum_{k=1}^{s} \mathbf{1}_{\{V_i=k\}} \left[e^{\tilde{\beta}_k^T x^{(i)}(X_i)} \lambda^{(k)}(Z_i(X_i); \eta_k) + \right.\right.$$
$$\left.\left. + e^{\beta_k^T x^{(i)}(X_i)} \mu^{(k)}(f(X_i, x^{(i)}, \beta_k); \gamma_k) \right] \right\}^{\tilde{\delta}_i} \times$$
$$\times \exp \left\{ - \sum_{k=1}^{s} \left(\int_0^{X_i} e^{\tilde{\beta}_k^T x^{(i)}(u)} \lambda^{(k)}(Z_i(u); \eta_k)\, du - H^{(k)}(f(X_i, x^{(i)}, \beta_k); \gamma_k) \right) \right\},$$

$$l = \sum_{i=1}^{n} \sum_{k=1}^{s} \mathbf{1}_{\{V_i=k\}} \ln \left[e^{\tilde{\beta}_k^T x^{(i)}(X_i)} \lambda^{(k)}(Z_i(X_i); \eta_k) + \right.$$
$$\left. + e^{\beta_k^T x^{(i)}(X_i)} \mu^{(k)}(f(X_i, x^{(i)}, \beta_k); \gamma_k) \right] -$$
$$- \sum_{k=1}^{s} \left(\int_0^{X_i} e^{\tilde{\beta}_k^T x^{(i)}(u)} \lambda^{(k)}(Z_i(u); \eta_k)\, du - H^{(k)}(f(X_i, x^{(i)}, \beta_k); \gamma_k) \right). \tag{3.27}$$

If covariants are absent then

$$l = \sum_{i=1}^{n} \sum_{k=1}^{s} \mathbf{1}_{\{V_i=k\}} \ln \left[\lambda^{(k)}(Z_i(X_i); \eta_k) + \mu^{(k)}(X_i; , \gamma_k) \right] -$$
$$- \sum_{k=1}^{s} \left(\int_0^{X_i} \lambda^{(k)}(Z_i(u); \eta_k)\, du - H^{(k)}(X_i; \gamma_k) \right). \tag{3.28}$$

If the values of degradation processes are measured only at discrete times t_{ij} then the conditional likelihood function is modified replacing $Z_i(u)$ by their predictors $\hat{Z}_i(u)$ obtained from degradation data. The form of these predictors depends on the form of the degradation processes.

It was mentioned in introduction that the most applied stochastic processes describing degradation are general path models and time scaled stochastic processes with stationary and independent increments such as the gamma process, compound Poisson process and Wiener process with drift, the last not monotone.

Let us find the predictors for some specified degradation processes.

3.2.3 Example 1: Time scaled gamma process

Let m be a real time function. The degradation process Z is *time scaled* (by the scale function $m(t)$) *gamma process* if

1) it has independent increments, i.e. for any $0 < t_1 < \cdots < t_k$ the random variables $Z(t_1)$, $Z(t_2) - Z(t_1), \ldots, Z(t_k) - Z(t_{k-1})$ are independent;

2) for any $t > 0$ the random variable $Z(t)$ has the gamma distribution;

3) for any $t \geq 0$

$$\mathbf{E}\,(Z(t)) = m(t), \quad \mathbf{Var}\,(Z(t)) = \sigma^2\, m(t).$$

The definition implies that for any $x > 0$ the density of the r.v. $Z(t_j) - Z(t_{j-1})$, $j = 1, \ldots, k$, $t_0 = 0$, is

$$p_{Z(t_j)-Z(t_{j-1})}(x) = \frac{1}{\sigma^2 \Gamma\left(\frac{\Delta m_j}{\sigma^2}\right)} \left(\frac{x}{\sigma^2}\right)^{\frac{\Delta m_j}{\sigma^2}-1} e^{-\frac{x}{\sigma^2}}, \tag{3.29}$$

$$\Delta m_j = m(t_j) - m(t_{j-1}), \quad \Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx.$$

The density of $Z(t_j)$ is of the same form: Δm_j must be replaced by $m(t_j)$ in (3.29).

The degradation and its characteristics under covariate x are

$$Z(t|x) = Z(f(t, \beta, x)), \quad m(t|x) = \mathbf{E}(\, Z(t|x)) = m\,(f(t, \beta, x))\,, \tag{3.30}$$

$$\sigma^2(t|x) = \mathbf{Var}(\, Z(t|x) = \sigma^2 m(t|x).$$

$m(t|x)$ is the mean degradation under the covariate x.

a) Parametric form of the mean degradation

The form of the mean degradation $m(t)$ may be suggested by the form of observed degradation curves. In such a case $m(t)$ is chosen from some parametric class of functions (power or other time function depending on a finite-dimensional unknown parameter): $m(t) = m(t; \nu)$, $\nu = (\nu_1, ..., \nu_q)^T$. The data

$$(Z_{ij}, \mu_i), \quad i = \overline{1, n}, j = \overline{1, \mu_i}, \tag{3.31}$$

are used for estimation of the parameters $\theta = (\beta, \nu, \sigma^2)^T$.

For any $z > 0$ the density of the increment $\Delta Z_{ij} = Z_{ij} - Z_{i,j-1}$ has the form

$$p_{\Delta Z_{ij}}(z; \theta) = \frac{1}{\sigma^2 \Gamma\left(\frac{\Delta \mu_{ij}(\beta,\nu)}{\sigma^2}\right)} \left(\frac{z}{\sigma^2}\right)^{\frac{\Delta \mu_{ij}(\beta,\nu)}{\sigma^2}-1} e^{-\frac{z}{\sigma^2}},$$

where

$$\Delta \mu_{ij}(\beta, \nu) = m\left(f(t_{i,j}, \beta, x^{(i)}); \nu\right) - m\left(f(t_{i,j-1}, \beta, x^{(i)}); \nu\right). \tag{3.32}$$

The likelihood function of the degradation data (3.31) is

$$L_d(\theta) = \prod_{i=1}^{n}\prod_{j=1}^{\mu_i} p_{\Delta Z_{ij}}\Big(\Delta Z_{ij};\theta\Big), \tag{3.33}$$

where we set $p_{\Delta Z_{ij}}\Big(\Delta Z_{ij};\theta\Big) = 1$, if $X_i < t_{i1}$, i.e. when a traumatic event occurs earlier then the first measurement of degradation.

Denote by $\hat{\theta}$ the maximum likelihood estimator. Then for any x the estimator of the mean degradation $m(t|x)$ under the covariate x is

$$\hat{m}(t|x) = m\left(f(t,\hat{\beta},x);\hat{\nu}\right). \tag{3.34}$$

In the case of the data (3.25) degradation values are not measured continuously and the loglikelihood function (3.27) can not be used for estimation of the parameters $\beta_k, \tilde{\beta}_k$, η_k and γ_k. For modification of the loglikelihood (3.27) we need predictors of $Z_i(t)$.

Set

$$\tilde{Z}_i(t) = \mathbf{E}(Z_i(t)|Z_i(t_{i1}),\dots,Z_i(t_{i\mu_i})). \tag{3.35}$$

For $j = 1,\dots,\mu_i$ we have $\tilde{Z}_i(t_{ij}) = Z_{ij}$. For $t \in (t_{i,j-1}, t_{ij})$, $j = 1,\dots,\mu_i$, the conditional means (3.35) are

$$\tilde{Z}_i(t;\theta) = Z_{i,j-1} + \frac{\Delta m_{ij}(t;\beta,\nu)}{\Delta m_{ij}(\beta,\nu)}\Delta Z_{ij},$$

$$\Delta m_{ij}(t;\beta,\nu) = m\left(f(t,\beta,x^{(i)});\nu\right) - m\left(f(t_{i,j-1},\beta,x^{(i)});\nu\right). \tag{3.36}$$

For any $t > t_{i,\mu_i}$

$$\tilde{Z}_i(t;\theta) = Z_{i\mu_i}F_{\chi^2_{\kappa_i}}(2(z_0 - Z_{i\mu_i})) + \Delta m_{i,\mu_i+1}(t;\beta,\nu)F_{\chi^2_{\kappa_i+2}}(2(z_0 - Z_{i\mu_i})), \tag{3.37}$$

where $F_{\chi^2_n}(x)$ is the c.d.f. of the chi square distribution, $\kappa_i = 2\Delta m_{i,\mu_i+1}(t;\beta,\nu)$.

Note that for any $t > t_{i,\mu_i}$

$$\tilde{Z}_i(t;\theta) \to Z_{i\mu_i} + \Delta m_{i,\mu_i+1}(t;\beta,\nu), \quad \text{as} \quad z_0 \to \infty. \tag{3.38}$$

The predictors $\hat{Z}_i$ of Z_i are defined as

$$\hat{Z}_i(t) = \tilde{Z}_i(t;\hat{\theta}). \tag{3.39}$$

b) Unknown form of mean degradation

If the function m is completely unknown then non-parametric estimator of this function is used seeking predictors of the stochastic processes Z_i.

A piecewise-linear approximation of the process $Z_i(t) = Z(t|x^{(i)})$ on $[0, X_i]$ is

$$Z_i^*(t) = \sum_{j=1}^{\mu_i+1}\left[Z_i(t_{i,j-1}) + \frac{t - t_{i,j-1}}{t_{ij} - t_{i,j-1}}(Z_i(t_{ij}) - Z_i(t_{i,j-1}))\right]\mathbf{1}_{[t_{i,j-1},t_{ij}]}(t), \tag{3.40}$$

$t_{i0} = 0, \quad t_{i,\mu_i+1} = X_i$. We denoted by $g(t,x,\beta)$ the inverse of $f(t,x,\beta)$ with respect to the first argument. The distribution of the stochastic process

$$Z(t) = Z(g(t,x,\beta)|x), \tag{3.41}$$

does not depend on x, so the processes $Z_i^*(g(t,x^{(i)},\beta))$ are approximations of the process Z and can be used constructing an estimator of the mean

$$m(t) = \mathbf{E}Z(t) = \mathbf{E}Z(g(t,x,\beta)|x)). \tag{3.42}$$

These approximating processes are censored at the points $t_i^*(\beta) = g(X_i, x^{(i)}, \beta)$. Consider the ordered sequence of distinct moments

$$t^*_{(1)}(\beta) < \cdots < t^*_{(d)}(\beta), \quad d \le n.$$

Take the following pseudo-estimator (depending on β) of $m(t)$:

$$\tilde{m}(t,\beta) = \frac{1}{n}\sum_{i=1}^{n} Z_k^*(g(t,\beta,x^{(i)})), \quad t \in [0, t^*_{(1)}(\beta)],$$

$$\tilde{m}(t,\beta) = \tilde{m}(t^*_{(j-1)}(\beta),\beta) +$$

$$+\frac{\sum\limits_{i:t_i^*(\beta)>t^*_{(j-1)}(\beta)} \left(Z_i^*(g(t,\beta,x^{(i)})) - Z_i^*(g(t^*_{(j-1)}(\beta),\beta,x^{(i)}))\right)}{\sum\limits_{i:t_i^*(\beta)>t^*_{(j-1)}(\beta)} 1}, \tag{3.43}$$

$t \in (t^*_{(j-1)}(\beta), t^*_{(j)}(\beta)]$. The likelihood function from degradation data (3.31) is written in the form (3.33) putting

$$\theta = (\beta^T, \sigma^2)^T, \quad \Delta m_{ij}(\beta) = \tilde{m}\left(f(t_{i,j},\beta,x^{(i)}),\beta\right) - \tilde{m}\left(f(t_{i,j-1},\beta,x^{(i)}),\beta\right). \tag{3.44}$$

Denote by $\hat{\beta}$, $\hat{\sigma}^2$ the maximum likelihood estimators. The function $m(t)$ is estimated by the statistic $\hat{m}(t) = \tilde{m}(t,\hat{\beta})$.

Define $\tilde{Z}_i(t;\theta)$ by (3.36) and (3.37) replacing $\Delta m_{ij}(\beta,\nu)$ by $\Delta m_{ij}(\beta)$ given in (3.44).

The predictors of $Z_i(t)$ are $\hat{Z}_i(t) = \tilde{Z}_i(t;\hat{\theta})$.

3.2.4 Example 2: Shock processes

Assume that degradation results from shocks, each of them leading to an increment of degradation. Let $T_n, (n \geq 1)$ be the time of the nth shock and X_n the nth increment of the degradation level. Denote by $N(t)$ the number of shocks in the interval $[0, t]$. Set $X_0 = 0$. The degradation process is given by

$$Z(t) = \sum_{n=1}^{\infty} \mathbf{1}\{T_n \leq t\} X_n = \sum_{n=0}^{N(t)} X_n.$$

Kahle and Wendt [20] model T_n as the moments of transition of the doubly stochastic Poisson process, i.e. they suppose that the distribution of the number of shocks up to time t is given by

$$\mathbf{P}\{N(t) = k\} = \mathbf{E}\left\{ \frac{(Y\eta(t))^k}{k!} \exp\{-Y\eta(t)\} \right\},$$

where $\eta(t)$ is a deterministic function and Y is a nonnegative random variable with finite expectation. If Y is non-random, N is non-homogenous Poisson process, in particular, when $\eta(t) = \lambda t$, N is homogenous Poisson process. Other models for η may be used, for example, $\eta(t) = t^\alpha$, $\alpha > 0$.

Assume that $X_1, , X_2, \cdots$ are conditionally independent given $\{T_n\}$ and assume that the probability density functions of X_n given $\{T_n\}$ is g.

Let us consider the case when the number of shocks is modelled by non-homogenous Poisson process:

$$\mathbf{P}\{N(t_1) = i_1, N(t_2) - N(t_1) = i_2, \cdots, N(t_m) - N(t_{m-1}) = i_m\} =$$

$$\frac{\eta(t_1)^{i_1}}{i_1!} e^{-\eta(t_1)} \frac{[\eta(t_2) - \eta(t_1)]^{i_2}}{i_2!} e^{-[\eta(t_2)-\eta(t_1)]} \cdots \frac{[\eta(t_m) - \eta(t_{m-1})]^{i_m}}{i_m!} e^{-[\eta(t_m)-\eta(t_{m-1})]}. \tag{3.45}$$

The degradation process $Z(t)$ is a stochastic process with independent increments and for any $z \geq 0$ the density of the r.v. $Z(t) - Z(s)$, $0 \leq s, t$, is

$$p_{Z(t)-Z(s)}(z) = \sum_{k=1}^{\infty} g_k(z) \frac{[\eta(t) - \eta(s)]^k}{k!} e^{-[\eta(t)-\eta(s)]} \tag{3.46}$$

where g_k is the convolution of k densities g. For example, if the sizes of the shocks X_i have exponential distribution $\mathcal{E}(\xi)$: $g(u) = \xi e^{-\xi u}$, $u \geq 0$, then

$$g_k(u; \xi) = \frac{\xi^k u^{k-1}}{(k-1)!} e^{-\xi u}, \quad u \geq 0, \quad p_{Z(t)-Z(s)}(z) = \xi b e^{-\xi z - b} \sum_{k=0}^{\infty} \frac{(\xi b z)^k}{k!(k+1)!},$$

where $b = \eta(t) - \eta(s)$.

Denote by $a_1 = \mathbf{E}X_1$ and $a_2 = \mathbf{E}X_1^2$ the first two moments of the random variable X_1. The moments of $Z(t)$ are

$$\mathbf{E}(Z(t)) = a_1 \eta(t), \quad \mathbf{Var}(Z(t)) = a_2 \eta(t).$$

The degradation and its characteristics under covariate x are

$$Z(t|x) = Z(f(t,\beta,x)), \quad m(t|x) = \mathbf{E}(\,Z(t|x)) = \mu_1\,\eta\,(f(t,\beta,x))\,, \tag{3.47}$$

$$\sigma^2(t|x) = \mathbf{Var}(\,Z(t|x) = a_2\,\eta\,(f(t,\beta,x))\,.$$

a) Parametric form of the mean degradation

Suppose that g and η belong to some parametric classes $g(t) = g(t,\xi)$, $\xi = (\xi_1,...,\xi_p)^T$. and $\eta(t) = \eta(t;\nu)$, $\nu = (\nu_1,...,\nu_q)^T$. Set $\theta = (\beta^T,\nu^T,\xi^T)^T$. The likelihood function of the degradation data (3.31) is of the form (3.33), where for any $z > 0$, $0 \le s < t$, the density of the increment $Z(t) - Z(s)$ is

$$p_{Z(t)-Z(s)}(z;\theta) = \sum_{k=1}^{\infty} g_k(z;\xi)\frac{[\Delta\eta(s,t;\beta,\nu)]^k}{k!}e^{-\Delta\eta(s,t;\beta,\nu)}; \tag{3.48}$$

here

$$\Delta\eta(s,t;\beta,\nu) = \eta\left(f(t,\beta,x^{(i)});\nu\right) - \eta\left(f(s,\beta,x^{(i)});\nu\right). \tag{3.49}$$

Denote by $\hat\theta$ the maximum likelihood estimator. Then for any x the estimator of the mean degradation $m(t|x)$ under the covariate x has the form (3.34).

For $t \in (t_{i,j-1}, t_{ij})$, $j = 1,\ldots,\mu_i$, the conditional means (3.35) are

$$\tilde Z_i(t;\theta) = \frac{1}{p_{\Delta Z_{i,j}}(\Delta Z_{i,j};\theta)}\int_{Z_{i,j-1}}^{Z_{i,j}} z\; p_{Z(t)-Z_{i,j-1}}(z - Z_{i,j-1};\theta)p_{Z_{i,j}-Z(t)}(Z_{i,j} - z;\theta)dz. \tag{3.50}$$

For any $t > t_{i,\mu_i}$

$$\tilde Z_i(t;\theta) = \int_{Z_{i,\mu_i}}^{z_0} z\; p_{Z(t)-Z_{i,\mu_i}}(z - Z_{i,\mu_i};\theta)dz. \tag{3.51}$$

The predictors $\hat Z_i$ of Z_i are defined by the formula (3.39).

Note that as in the case of the gamma process (we set $\sigma^2 = a_2/a_1$)

$$\frac{\mathbf{Var}(Z(t))}{\mathbf{E}(Z(t))} = \sigma^2, \quad (Z(s),Z(t)) = \mathbf{Var}(Z(s\wedge t)),$$

so in terms of the first two moments the considered shock process and the gamma process are of identical structure.

b) Unknown form of mean degradation

The predictors of $Z_i(t)$ are defined as $\hat Z_i(t) = \tilde Z_i(t;\hat\theta)$, and $\tilde Z_i(t;\theta)$ are defined by (3.50) and (3.51) replacing $\Delta\eta(s,t;\beta,\nu)$ (given in (3.49)) by

$$\Delta\eta(s,t;\beta,\xi) = [\tilde m\left(f(t_{i,j},\beta,x^{(i)}),\beta\right) - \tilde m\left(f(t_{i,j-1},\beta,x^{(i)}),\beta\right)]/a_1(\xi),$$

where $\tilde m(t;\beta)$ is the pseudoestimator of the mean $m(t) = \mathbf{E}Z(t)$ given by (3.43).

3.2.5 Example 3: Path models

Suppose that the degradation process $Z(t)$ is of the following form:

$$Z(t) = \varphi(t, A, \nu), \tag{3.52}$$

where φ is a deterministic function and $A = (A_1, \dots, A_p)$ is a finite dimensional random vector and ν is a finite dimensional non-random parameter.

The form of the function φ may be suggested by the form of individual degradation curves. The degradation under the covariate x is modelled by

$$Z(t|x) = \varphi(f(t, x, \beta), A), \quad m(t|x) = E\varphi(f(t, x, \beta), A).$$

Let us consider the following typical example:

$$Z(t) = (t/A)^{\nu}; \tag{3.53}$$

here A is a positive random variable with unknown cumulative distribution function F, ν is a positive parameter. In particular case $\nu = 1$ this model fits well as the tire wear model [29].

The degradation process under covariate x is

$$Z(t|x) = Z(f(t, x, \beta)) = (f(t, x, \beta)/A)^{\nu}. \tag{3.54}$$

Even in the case $\nu = 1$ it is not necessary linear.

Suppose that n units are on test. The ith unit is tested under explanatory variable $x^{(i)}$. Denote by T_i, V_i the failure times and the failure modes, respectively. Suppose that the degradation values $Z^{(i)}$ at the moments T_i are observed. So the data has the form

$$(T_i, V_i, Z^{(i)}, x^{(i)}), \quad i = 1, ..., n. \tag{3.55}$$

The covariates $x^{(i)}$ are observed until the moments X_i.

Taking into account that the random variables

$$\ln A_i = \nu f(T_i, x^{(i)}, \beta) - \ln Z^{(i)}$$

are independent identically distributed with the mean, say m, which does not depend on β and ν, so these parameters are estimated by the method of least squares, minimizing the sum

$$\sum_{i=1}^{n} (\nu \ln f(T_i, x^{(i)}, \beta) - \ln Z^{(i)} - m)^2,$$

which gives the system of equations

$$n \sum_{i=1}^{n} \frac{\int_0^{T_i} x^{(i)} e^{\beta^T x^{(i)}(u)} du [\nu \ln f(T_i, x^{(i)}, \beta) - \ln Z^{(i)}]}{f(T_i, x^{(i)}, \beta)} -$$

$$-\sum_{i=1}^{n}\frac{\int_0^{T_i}x^{(i)}e^{\beta^T x^{(i)}(u)}du}{f(T_i,x^{(i)},\beta)}\sum_{j=1}^{n}[\nu\ln f(T_j,x^{(i)},\beta)-\ln Z^{(j)}]=0,$$

$$n\sum_{i=1}^{n}\ln f(T_i,x^{(i)},\beta)[\nu\ln f(T_i,x^{(i)},\beta)-\ln Z^{(i)}]-$$

$$-\sum_{i=1}^{n}\ln f(T_i,x^{(i)},\beta)\sum_{j=1}^{n}[\nu\ln f(T_j,x^{(i)},\beta)-\ln Z^{(j)}]=0.$$

If $x^{(i)}$ are constant then this system is:

$$n\sum_{i=1}^{n}\beta^T x^{(i)}[\nu\beta^T x^{(i)}+\nu\ln T_i-\ln Z^{(i)}]-\sum_{i=1}^{n}\beta^T x^{(i)}\sum_{j=1}^{n}[\nu\beta^T x^{(i)}+\nu\ln T_i-\ln Z^{(i)}]=0,$$

$$n\sum_{i=1}^{n}x^{(i)}[\nu\beta^T x^{(i)}+\nu\ln T_i-\ln Z^{(i)}]-\sum_{i=1}^{n}x^{(i)}\sum_{j=1}^{n}[\nu\beta^T x^{(i)}+\nu\ln T_i-\ln Z^{(i)}]=0.$$

Denote by $\hat\beta$ and $\hat\nu$ the obtained estimator.

The predictors of $Z_i(t)$ are defined as

$$\hat Z_i(t)=\left(\frac{f(t,x^{(i)},\hat\beta)}{f(T_i,x^{(i)},\hat\beta)}\right)^{\hat\nu}Z^{(i)}. \tag{3.56}$$

Note that $\hat Z_i(T_i)=Z^{(i)}$. If $x^{(i)}$ are constant over time then

$$\hat Z_i(t)=\left(\frac{t}{T_i}\right)^{\hat\nu}Z^{(i)}. \tag{3.57}$$

3.2.6 Modified loglikelihood

The modified loglikelihood function for the parameters $\beta_k, \tilde\beta_k$, η_k and γ_k from the data (3.25) (and (3.55)) is obtained modifying the loglikelihood function (3.27): the stochastic processes Z_i are replaced by their predictors $\hat Z_i$ in (3.27) (in the case of the data (3.55) take $X_i=T_i$):

$$\tilde l=\sum_{i=1}^{n}\sum_{k=1}^{s}\mathbf{1}_{\{V_i=k\}}\ln\Big[e^{\tilde\beta_k^T x^{(i)}(X_i)}\lambda^{(k)}(\hat Z_i(X_i);\eta_k)+$$

$$+e^{\beta_k^T x^{(i)}(X_i)}\mu^{(k)}(f(X_i,x^{(i)},\beta_k);\gamma_k)\Big]-$$

$$-\sum_{k=1}^{s}\left(\int_0^{X_i}e^{\tilde\beta_k^T x^{(i)}(u)}\lambda^{(k)}(\hat Z_i(u);\eta_k)\,du-H^{(k)}(f(X_i,x^{(i)},\beta_k);\gamma_k)\right). \tag{3.58}$$

If covariants are absent then

$$\tilde l=\sum_{i=1}^{n}\sum_{k=1}^{s}\mathbf{1}_{\{V_i=k\}}\ln\Big[\lambda^{(k)}(\hat Z_i(X_i);\eta_k)+\mu^{(k)}(X_i;,\gamma_k)\Big]-$$

$$-\sum_{k=1}^{s}\left(\int_0^{X_i}\lambda^{(k)}(\hat{Z}_i(u);\eta_k)\,du-H^{(k)}(X_i;\gamma_k)\right), \tag{3.59}$$

where the predictors $\hat{Z}_i$ are defined replacing $f(u,\beta,x^{(i)})$ by u in all formulas.

The loglikelihood (3.25) function can be modified and to the case when the two functions $\lambda^{(k)}$ (or the functions $H^{(k)}$, but not both) are completely unknown. In the case of linear path models such modifications and properties of estimators are given in Bagdonavičius *et al* ([8]).

Investigating the case of other degradation models is a subject for separate work.

3.3 Estimation of reliability characteristics

Let us consider estimation of reliability characteristics (3.12)-(3.22) when the mean degradation $m(t)$ is of parametric form. Set

$$\hat{Z}_i(t|x)=\hat{Z}_i\big(g(f(t,\hat{\beta},x),x^{(i)},\hat{\beta})\big), \tag{3.60}$$

where $\hat{Z}_i(t)$ are the predictors of the discreetly observed processes $Z_i(t|x^{(i)})$, $i=1,\ldots,n$. We considered construction of the predictors $\hat{Z}_i(t)$ in the previous section.

In particular case when $x, x^{(i)}$ are constant over time

$$\hat{Z}_i(t|x)=\hat{Z}_i(e^{\hat{\beta}(x-x^{(i)})}t). \tag{3.61}$$

The predictor of the non-traumatic failure of the ith unit under the covariate x is

$$\hat{T}_i^{(0)}(x)=\inf\{t:\hat{Z}_i(t|x)\geq z_0\}. \tag{3.62}$$

The formulas (3.12)-(3.16) imply the following estimators:

1) The estimator of the survival function of the failure time under the covariate x:

$$\hat{S}(t|x)=\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{\hat{Z}_i(t|x)<z_0\}}\prod_{k=1}^{s}\hat{S}^{(k)}(t\mid x,\hat{Z}_i), \tag{3.63}$$

where

$$\hat{S}^{(k)}(t\mid x,\hat{Z}_i)=\exp\left\{-\int_0^t e^{\hat{\beta}_k^T x(u)}\lambda^{(k)}(\hat{Z}_i(u|x),\hat{\eta}_k)\,du-H^{(k)}(f(t,x,\hat{\beta}_k),\hat{\gamma}_k)\right\}. \tag{3.64}$$

2) The estimator of the mean failure time under the covariate x:

$$\hat{e}(x)=\frac{1}{n}\sum_{i=1}^{n}\int_0^{\hat{T}_i^{(0)}(x)}\prod_{k=1}^{s}\hat{S}^{(k)}(t\mid x,\hat{Z}_i)dt. \tag{3.65}$$

3) The estimator of the probability that under the covariate x the non-traumatic failure is observed in the interval $[0,t]$:

$$\hat{P}^{(0)}(t|x)=\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{\hat{Z}_i(t|x)\geq z_0\}}\prod_{k=1}^{s}\hat{S}^{(k)}(\hat{T}_i^{(0)}(x)\mid x,\hat{Z}_i). \tag{3.66}$$

4) The estimator of the probability that under the covariate x a traumatic failure is observed in the interval $[0,t]$:

$$\hat{P}^{(tr)}(t|x) = 1 - \frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{s}\hat{S}^{(k)}(t \wedge \hat{T}_i^{(0)}(x) \mid x, \hat{Z}_i). \tag{3.67}$$

5) The estimator of the probability that under the covariate x the traumatic failure of the kth mode, $k = 1,\dots,s$, is observed in the interval $[0,t]$:

$$\hat{P}^{(k)}(t|x) = \frac{1}{n}\sum_{i=1}^{n}\int_0^{t\wedge\hat{T}_i^{(0)}(x)}\prod_{l=1}^{s}\hat{S}^{(l)}(s \mid x, \hat{Z}_i)\,\lambda^{(k)}(s \mid x, \hat{Z}_i)\,ds. \tag{3.68}$$

The estimators of survival characteristics of units with eliminated failure modes are obtained taking $\prod_{l\neq i_1,\dots,i_q}$ instead of $\prod_{l=1}^{s}$ in the formulas (3.63)-(3.68).

Conclusions

In the thesis, the following results which analysed formulated at the beginning aims are gained:
1. Mathematical definition of stand-by unit fluent switching from "warm" to "hot" conditions is formulated;
2. Tests for general "fluent switching hypothesis" formulated using Sedyakin's "reliability principle" and for particular fluent switching hypothesis formulated using accelerated failure time model are constructed; Asymptotic properties of the test statistics are investigated;
3. Parametric and nonparametric estimators of the cumulative distribution function of redundant system using reliability data of components tested under different stresses are constructed;
4. Asymptotic properties of the parametric and nonparametric estimators are investigated;
5. Asymptotic confidence intervals for cumulative distribution function of redundant system are constructed. Finite sample properties of the parametric and nonparametric estimators are investigated by simulation;
6. General simultaneous failure time and degradation regression data models are formulated. Maximum likelihood method for estimation of failure process and degradation process parameters using simultaneous degradation and multi-mode failure time regression data using predictors of degradation processes is modified;
7. The structure of modified likelihood function when the degradation process is modelled by time scaled gamma process, path processes, shock processes with the number of shocks modelled by non-homogenous Poisson process is investigated.

Appendix A

Delta method

Theorem A.0.1 *Let $\{a_n\}$ be a sequence of real numbers, $g = (g_1, ..., g_q) : \mathbf{R^p} \to \mathbf{R^q}$ be a differentiable vector-function, and*

$$J_g(x) = ||\frac{\partial g_i(x)}{\partial x_j}||_{q \times p}$$

be the Jacobi matrix of partial derivatives of coordinate functions g_i. If

$$a_n(X^{(n)} - x) \xrightarrow{\mathcal{D}} Z \quad as \quad a_n \to \infty \quad on \quad \mathbf{R^p},$$

then

$$a_n(g(X^{(n)}) - g(x)) \xrightarrow{\mathcal{D}} J_g(x)Z \quad as \quad a_n \to \infty. \tag{A.1}$$

Theorem A.0.2 *Suppose that*

1) $\{X_1^n \in D[0,\tau]\}$ and $\{X_2^n \in D[0,\tau]\}$ are sequences of catlag stochastic processes, the second being of bounded variation and bounded by a positive constant M;

2) $X_1, X_2 \in D[0,\tau]$ are cadlag stochastic processes of bounded variation the second being bounded by M such that

$$(a_n(X_1^n - X_1), a_n(X_2^n - X_2)) \xrightarrow{\mathcal{D}} (Z_1, Z_2),$$

on $D[0,\tau] \times D[0,\tau]$; here $Z_1, Z_2 \in D[0,\tau]$. Then

$$a_n\left(\int_0^{\cdot} X_1^n dX_2^n - \int_0^{\cdot} X_1 dX_2\right) \xrightarrow{\mathcal{D}} \int_0^{\cdot} Z_1 dX_2 + \int_0^{\cdot} X_1 dZ_2 \tag{A.2}$$

on $D[0,\tau]$. If Z_2 is not of bounded variation then the last integral is defined by

$$\int_0^t X_1(u)dZ_2(u) = X_1(t)Z_2(t) - X_1(0)Z_2(0) - \int_0^t Z_2(u)dX_1(u).$$

Theorem A.0.3 *Suppose that*

1) $x \in D[0,\tau]$ is a nondecreasing function, differentiable at the point

$$x^{-1}(p) = \inf\{t : x(t) \geq p\} \in (0,\tau),$$

where $p \in \mathbf{R}$ is a fixed number.

2) $\{X^{(n)} \in D[0,\tau]\}$ is a sequence of nondecreasing stochastic processes such that

$$a_n(X^n - x) \xrightarrow{\mathcal{D}} Z$$

on $D[0,\tau]$; here $Z \in D[0,\tau]$ is a nondecreasing process, continuous at the point $x^{-1}(p)$.

Then

$$a_n\left((X^n)^{-1}(p) - x^{-1}(p)\right) \xrightarrow{\mathcal{D}} -\frac{Z(x^{-1}(p))}{x'(x^{-1}(p))}. \tag{A.3}$$

Theorem A.0.4 *Suppose that*

1) x is a continuously differentiable function on $[0,\tau]$;

2) $\varphi = \varphi(t,\theta) : A \times B_\varepsilon(\theta_0) \to \mathbf{R}$, $\quad B_\varepsilon(\theta_0) \subset \mathbf{R^s}$, $A = [0,\tau_0]$ or $(0,\tau_0)$, is a continuous non-increasing in t function such that $0 < \varphi(t,\theta_0) < \tau$ for $t \in A$;

3) $\{X^{(n)} \in D[0,\tau]\}$ is a sequence of stochastic processes such that

$$\sqrt{n}(X^n - x) \xrightarrow{\mathcal{D}} Z$$

on $D[0,\tau]$, where Z is a continuous on $[0,\tau]$ stochastic process;

4) $\{\hat\theta^{(n)}\}$ is a sequence of random variables such that

$$\sqrt{n}(\hat\theta^{(n)} - \theta_0) \xrightarrow{\mathcal{D}} Y.$$

Then

$$\sqrt{n}(X^n(\varphi(\cdot,\hat\theta_0^{(n)})) - x(\varphi(\cdot,\theta_0)) \xrightarrow{\mathcal{D}} Z(\varphi(\cdot,\theta_0)) + x'(\varphi(\cdot,\theta_0))\varphi'_\theta(\cdot,\theta_0)Y$$

on $D(A)$.

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