VILNIUS UNIVERSITY

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DISCRETE MOMENTS OF THE RIEMANN ZETA FUNCTION AND DIRICHLET L-FUNCTIONS

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VILNIAUS UNIVERSITETAS

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RIEMANN'O DZETA FUNKCIJOS IR DIRICHLET L-FUNKCIJŲ DISKRETIEJI MOMENTAI

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This thesis is dedicated to my parents
for their love, endless support
and encouragement.

Contents

Introduction

History of the problem and main results

In mathematics, analytic number theory is a branch of number theory that uses methods from mathematical analysis to solve problems that concern the integers. It is often said to have begun with Dirichlet's introduction of Dirichlet L-functions. Let $s = \sigma + it$ be a complex variable, then Dirichlet L-functions are defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad (\sigma > 1),$$

where $\chi(n)$ is a Dirichlet character - a completely multiplicative function that obtains values from a unit circle. Dirichlet L-functions were used to give the first proof of Dirichlet's theorem on arithmetic progressions.

Theorem (Dirichlet's theorem, 1837)**.** *Let a and q be positive co-prime (i.e. (a,q)=1) integers. Then arithmetic progression*

$$a + qn, \quad n = 0, 1, 2, \dots$$

contains infinitely many prime numbers.

Analytic number theory can be divided into two main branches. The first branch is additive number theory that mainly uses Hardy-Littlewood circle and sieve methods. Best known conjecture from the field is the Goldbach conjecture that was proposed in the middle of the 18th century - *every even integer grater than two can be written as a sum of two prime numbers.*

The second branch of analytic number theory is multiplicative number theory. The key result in the field is the prime number theorem. Recall prime-counting function

$$\pi(x) = \#\{p | p - \text{prime},\ p \leq x\} = \sum_{p \leq x} 1.$$

Theorem (prime number theorem, 1896)**.**

$$\lim_{x \to \infty} \frac{\pi(x)}{x / \log x} = 1,$$

where $\log x = \ln x$*.*

In this thesis we denote $\ln x$ by $\log x$. The above result first time was published in 1896 (independently) by J. Hadamard and C. J. de la Vallee Poussin.

The prime number theorem is strongly connected with the Riemann zeta function. In 1859 B. Riemann published a paper that showed a connection between prime numbers and the zeros of the Riemann zeta function. The Riemann zeta function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\sigma > 1).$$

Moreover the Riemann zeta function can be expressed as the Euler product

$$\zeta(s) = \prod_{p-\text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (\sigma > 1). \tag{1}$$

The Euler product gives that $\zeta(s) \neq 0$, for $\sigma > 1$. The Riemann zeta function by analytic continuation can be extended elsewhere in the complex plane except for a simple pole at $s = 1$. It satisfies the functional equation

$$\zeta(s) = \Delta(s)\zeta(1-s), \qquad \text{where} \quad \Delta(s) := 2^s \pi^{s-1} \Gamma(1-s) \sin(\tfrac{\pi s}{2}). \tag{2}$$

The zeros that appear from the factor $\Delta(s)$ are called the trivial zeros ($\Delta(s) = 0$ if and only if $s = -2n$, for $n = 1, 2, \ldots$). From the Euler product (1) and the functional equation (2) we can deduce that the non trivial zeros (we denote them as $\rho = \beta + i\gamma$) must lie in the critical strip, $0 \leq \sigma \leq 1$. The line $\sigma = \frac{1}{2}$ is called the critical line and the Riemann hypothesis states that *all non-trivial zeros lie on the critical line.* If the Riemann hypothesis is true then the error term of the asymptotic formula of $\pi(x)$ is the best possible.

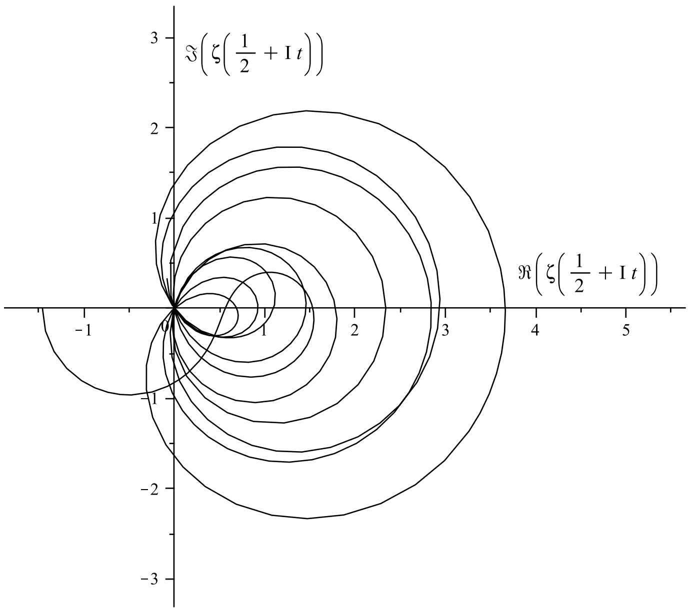


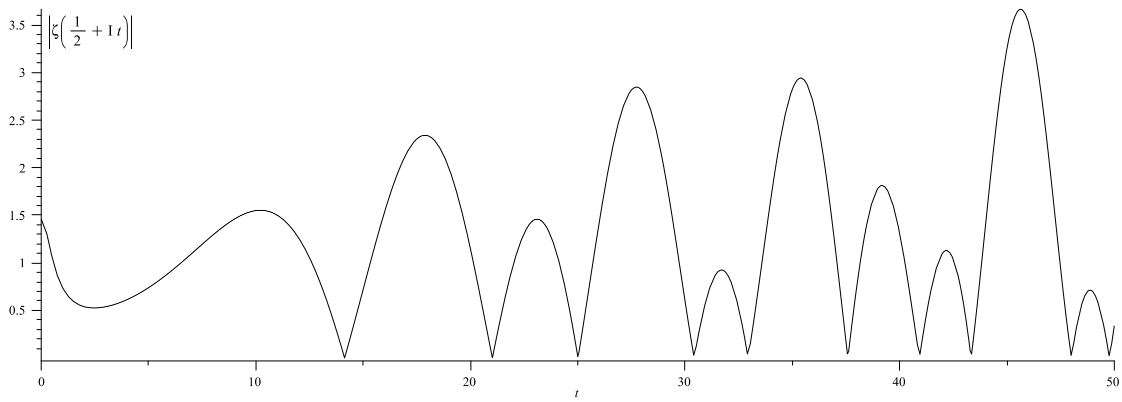
Figure 1: Curve $t \mapsto \zeta(\frac{1}{2} + it)$, where t varies from 0 to 50.

In Figure we see that the first non-trivial zeros of the Riemann zeta function lie on the critical line. At the moment it is known that the first $10^{22} + 10^4$ non-trivial zeros lie on the critical line.

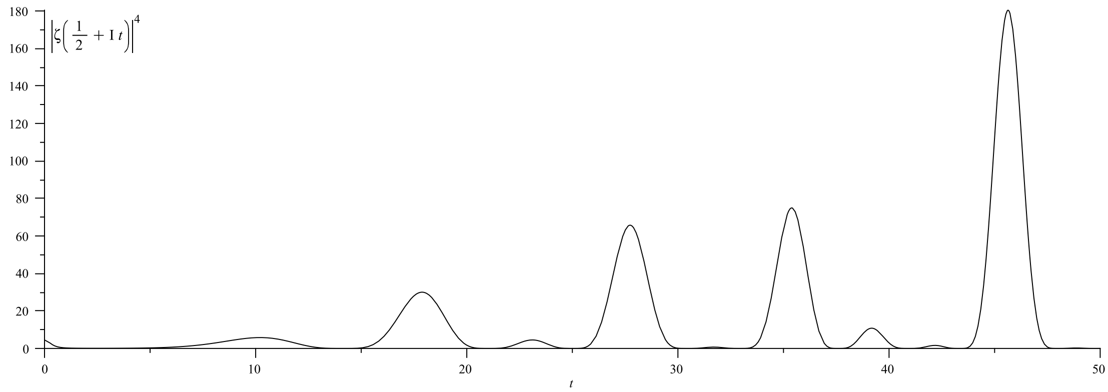
Next we will introduce some problems that concern the Riemann zeta function and Dirichlet L-functions and are related to the main results of the thesis.

The Riemann zeta function

Continuous moments



(a)



(b)

Figure 2: When power of the function increases we see where the Riemann zeta function obtains large values.

There are many unsolved problems related to the Riemann zeta function. One of them is the bounds for continuous moments

$$I_k(T) = \int_0^T \left|\zeta\Big(\frac{1}{2}+it\Big)\right|^{2k} dt.$$

(See Figure 2 (a) plot of $|\zeta(\frac{1}{2}+it)|$ and (b) plot of $|\zeta(\frac{1}{2}+it)|^4$) Currently only two asymptotic formulas for $I_k(T)$ are known, $k=1$ and $k=2$. In 1918 Hardy and Littlewood [23] proved the case $k=1$ and in 1928 Ingham [27] published the case $k=2$.

On the other hand some of expected order lower and upper bounds are known for $I_k(T)$. In 1978 Ramachandra [41] under the Riemann hypothesis for any non negative real k showed

$$I_k(T) \gg T(\log T)^{k^2}.$$

In 1981 Heath-Brown [25] proved the above result unconditionally for any non-negative rational k.

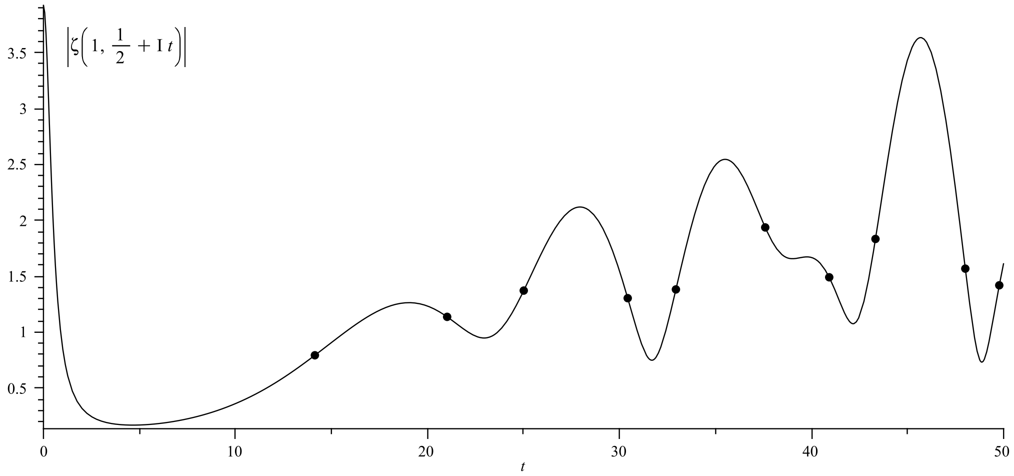
Conditional upper bounds were recently obtained by Soundararajan [44]. He showed that under the Riemann Hypothesis for any non negative real k holds

$$I_k(T) \ll T(\log T)^{k^2+\epsilon}, \tag{3}$$

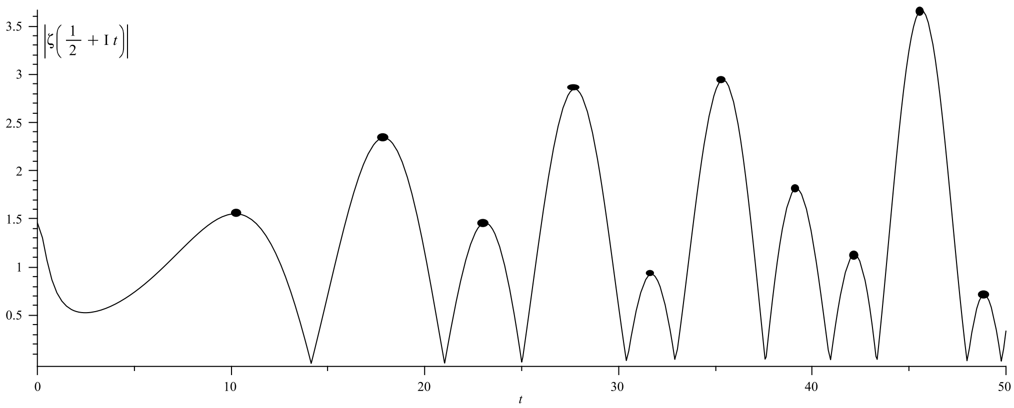
where ϵ is a fixed positive quantity. Further in this thesis by ϵ we denote a fixed positive quantity.

The upper bounds are connected with a well known hypothesis from the field that is called the Lindelöf hypothesis. *The Lindelöf hypothesis states that* $|\zeta(\frac{1}{2}+it)| \ll t^{\epsilon}$, $t > 1$. The hypothesis is equivalent to the statement $I_k(T) \ll T^{1+\epsilon}$. It is obviously that from the Riemann hypothesis we can deduce the Lindelöf hypothesis.

Discrete moments



(a)



(b)

Figure 3: (a) Plot of an absolute value of the derivative of the Riemann zeta function. Black dots indicate values obtained by the function at non-trivial zeros of the Riemann zeta function. In plot we denote $\zeta(1, \frac{1}{2}+It) = \zeta'(\frac{1}{2}+It)$. (b) Plot of an absolute value of the Riemann zeta function. Black dots indicate relative extremas.

Let $\rho = \beta + i\gamma$ be a non-trivial zero of the Riemann zeta function. The discrete moments

$$S_k(T) = \sum_{0<\gamma\le T} |\zeta'(\rho)|^{2k}$$

are considered as an important problem and only conditional (under the assumption of the

Riemann hypothesis) results are known (see Figure 3 (a)). In 1984 (I was born that year) Gonek [19] under the Riemann hypothesis proved an asymptotic formula for $S_k(T)$, when $k = 1$.

On the other hand Ng and Milinovich [34] under the Riemann Hypothesis for any positive integer k recently found the expected order lower bounds

$$S_k(T) \gg T(\log T)^{k^2+2k+1}.$$

Milinovich [35] under the Riemann hypothesis for any positive integer k found upper bounds

$$S_k(T) \ll T(\log T)^{k^2+2k+1+\epsilon},$$

Many authors investigated the following discrete moments. Let γ and γ^+ denote consecutive ordinates of the non-trivial zeros of the Riemann zeta function. Moments of the Riemann zeta function at its relative extrema on the critical line are defined by

$$M_k(T) = \sum_{0<\gamma\leq T} \max_{\gamma\leq\tau_\gamma\leq\gamma^+} \left|\zeta\left(\frac{1}{2} + i\tau_\gamma\right)\right|^{2k}$$

(see Figure 3 (b)). In 1985 Conrey and Ghosh [6] under the Riemann hypothesis for $k = 1$ obtained an asymptotic formula

$$M_1(T) \sim \frac{e^2 - 5}{2}\frac{T}{2\pi}(\log T)^2.$$

Recently Milinovich [36] under the Riemann hypothesis for all positive integers k found

$$T(\log T)^{k^2+1-\epsilon} \ll M_k(T) \ll T(\log T)^{k^2+1+\epsilon}.$$

Extreme values

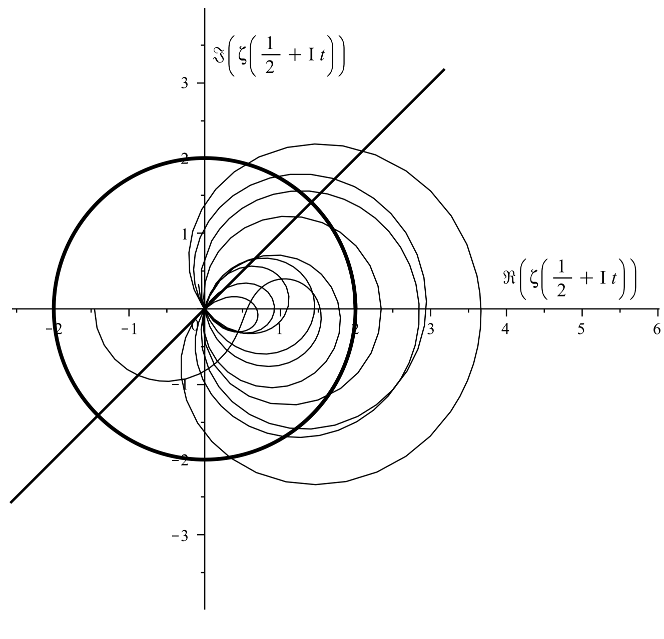


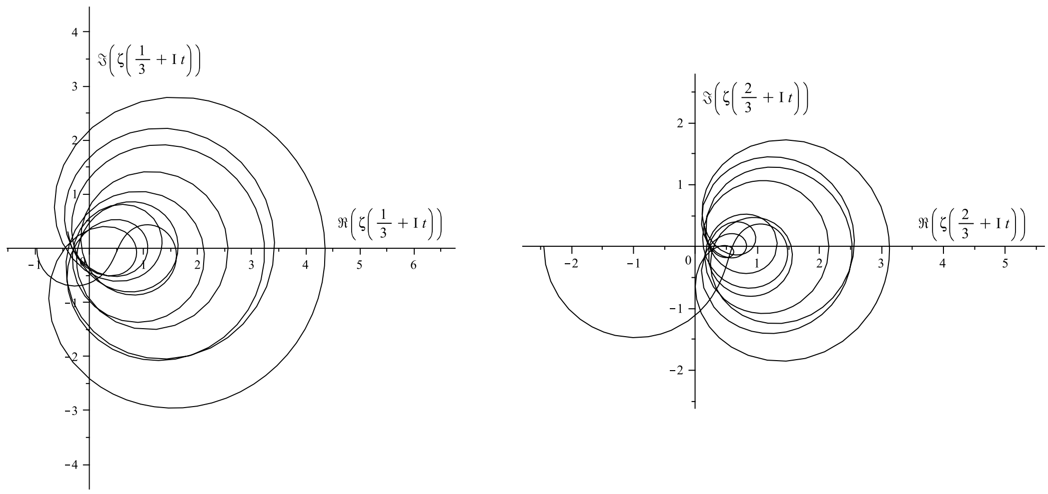
Figure 4: Curve $t \mapsto \zeta(\frac{1}{2} + it)$, where t varies from 0 to 50. The radius of the circle is 2 and the thick black line has 45° angle with the real axis.

Concerning the growth of the Riemann zeta function on the critical line recently Soundararajan [43] showed that

$$\max_{t\in[T,2T]} \left|\zeta(\tfrac{1}{2}+it)\right| \gg \exp\left((1+o(1))\sqrt{\frac{\log T}{\log\log T}}\right) \quad \text{as} \quad T\to\infty.$$

This result states that if we draw a circle with the center at the origin and the radius equal to $C\exp(\sqrt{\frac{\log T}{\log\log T}})$, here C is a fixed positive constant, we will find at least one point out of the circle and that point belongs to the curve (see Figure 4). However this result as well as all other Ω-estimates for the Riemann zeta function can not localize the extreme values.

Density



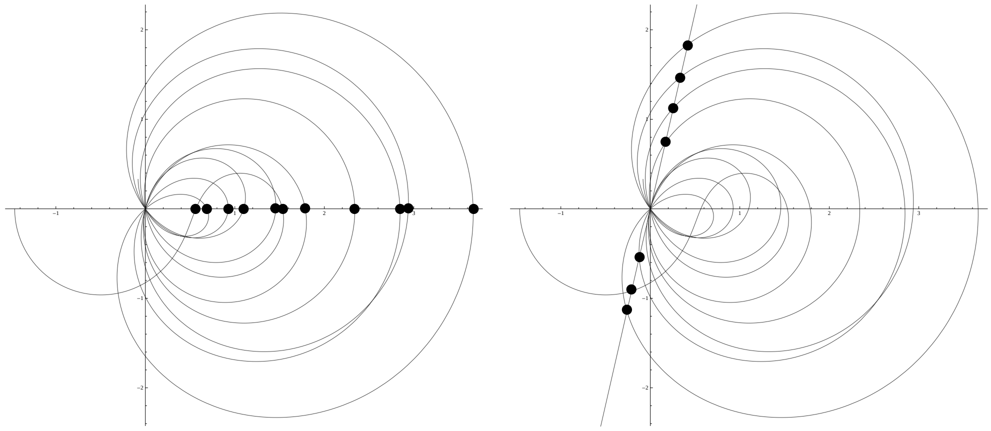
(a) (b)

Figure 5: Curves $t\mapsto\zeta(\frac{1}{3}+it)$ (a) and $t\mapsto\zeta(\frac{2}{3}+it)$ (b), where t varies from 0 to 50.

In 1914 Bohr and Courant [2] showed that values of the Riemann zeta function on the vertical line, which lie in the critical strip $\frac{1}{2}<\text{Re}s<1$, are dense in $\mathbb{C}$ (see Figure 5 (b)). Under the Riemann hypothesis Garunkštis and Steuding [17] showed that values of the Riemann zeta function on the vertical line which lie in the half-plane $\text{Re}s<\frac{1}{2}$ are not dense in $\mathbb{C}$ (see Figure 5 (a)). The question *whether the values of the Riemann zeta function on the critical line are dense* is open (see Figure 1).

Main results obtained for the Riemann zeta function

In Figure 6 we see that the real part of the curve $t\mapsto\zeta(\frac{1}{2}+it)$ has a tendency to be positive. That was noticed by H. M. Edwards. In his monograph [12] he writes *"...the real part of $\zeta(s)$ has a strong tendency to be positive."* (page 121). The critical line - that separates a curve $t\mapsto\zeta(\sigma+it)$, $\frac{1}{2}<\sigma<1$ that is dense in $\mathbb{C}$ from a curve $t\mapsto\zeta(\sigma+it)$, $\sigma<\frac{1}{2}$ that is not dense in $\mathbb{C}$ (under the Riemann hypothesis) - is very mysterious. It is not known whether the curve $t\mapsto\zeta(\frac{1}{2}+it)$ is dense in $\mathbb{C}$. Our main result (Corollary 1.5.1) states that the curve $t\mapsto\zeta(\frac{1}{2}+it)$ expand to all directions in the complex plane, i.e. if we draw a circle on the complex plane with the center at the origin and we draw a half-line that starts at the center then the curve $t\mapsto\zeta(\frac{1}{2}+it)$ intersect with the half-line outside the circle infinitely many times (see Figure 4).



(a) (b)

Figure 6: Curve $t \mapsto \zeta(\frac{1}{2}+it)$, where t varies from 0 to 50: (a) black thick dots are generalized Gram's points $t_n(0)$ and (b) black thick dots are generalized Gram's point $t_n(\frac{3}{7}\pi)$.

As a separate case of our result we can state *the Riemann zeta function obtains infinitely many negative values and they are unbounded.* Later we will formulate Corollary (1.5.1).

Recall the functional equation for the Riemann zeta function,

$$\zeta(s) = \Delta(s)\zeta(1-s), \qquad \text{where} \quad \Delta(s) := 2^s \pi^{s-1}\Gamma(1-s)\sin(\tfrac{\pi s}{2}). \tag{4}$$

It follows immediately that $\Delta(s)\Delta(1-s) = 1$, hence $\Delta(\frac{1}{2}+it)$ lies on the unit circle for real t. Given an angle $\phi \in [0, \pi)$, denote by $t_n(\phi)$ with $n \in \mathbb{N}$ the positive roots of the equation

$$e^{2i\phi} = \Delta(\tfrac{1}{2}+it)$$

in ascending order. These roots (we call them Generalized Gram's points, see Figure 6 (a) $t_n(0)$ and (b) $t_n(\frac{3}{7}\pi)$) correspond to intersections of the curve $t \mapsto \zeta(\frac{1}{2}+it)$ with straight lines $e^{i\phi}\mathbb{R}$ through the origin. Of special interest are intersections with the real line; in this case $\phi = 0$ and the roots are called Gram's points (see Figure 6 (a)), after Gram [22] who observed that the first of those roots separate consecutive zeta zeros on the critical line. The observation is called Gram's Law. It was shown by Titchmarsh [47] that Gram's law is violated infinitely often.

Now recall that $e^{-i\phi}\zeta(\frac{1}{2}+it_n(\phi))$ is real. Hence, we may write $t_n^+(\phi)$ in place of $t_n(\phi)$ if $e^{-i\phi}\zeta(\frac{1}{2}+it_n(\phi)) \geqslant 0$ and $t_n^-(\phi)$ if $e^{-i\phi}\zeta(\frac{1}{2}+it_n(\phi)) < 0$ (see Figure 7). Now we state the main result of the thesis.

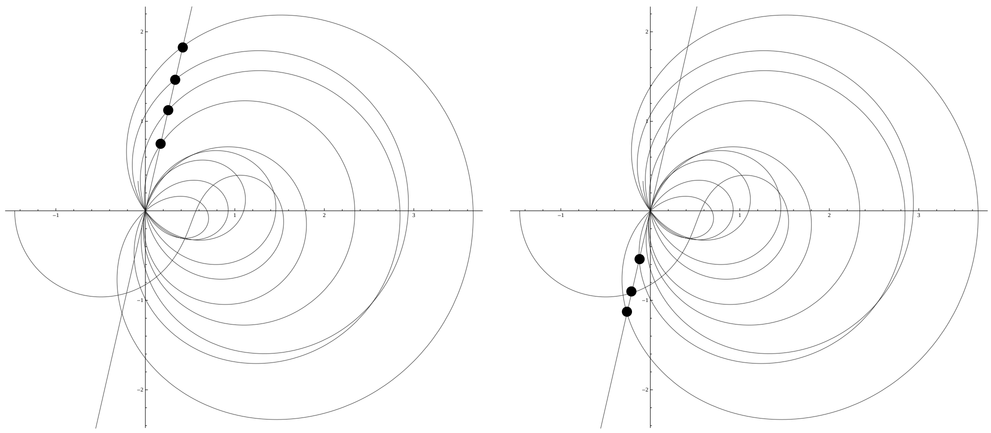
Corollary (1.5.1). *For any $\phi \in [0, \pi)$, there are arbitrary large positive and negative values of $e^{-i\phi}\zeta(\frac{1}{2}+it_n(\phi))$. More precisely,*

$$\max_{0<t_n^{\pm}(\phi)\leqslant T} \left|\zeta(\tfrac{1}{2}+it_n^{\pm}(\phi))\right| \gg (\log T)^{\frac{5}{4}}.$$

If the Riemann hypothesis is assumed then for any arbitrary small $\delta > 0$ we have

$$\max_{0<t_n^{\pm}(\phi)\leqslant T} \left|\zeta(\tfrac{1}{2}+it_n^{\pm}(\phi))\right| \gg (\log T)^{\frac{3}{2}-\delta}.$$

The corollary emerges from a combination of theorems concerning lower, upper bounds and asymptotic formula for the third moment. The approach was introduced by Kalpokas, Korolev



(a) (b)

Figure 7: Curve $t \mapsto \zeta(\frac{1}{2} + it)$, where t varies from 0 to 50: (a) black thick dots are values at generalized Gram's points $t_n^+(\frac{3}{7}\pi)$ and (b) black thick dots are values at generalized Gram's points $t_n^-(\frac{3}{7}\pi)$.

and Steuding [32].

Next we formulate a corollary that gives more information about extreme values (check Figure 4)

Corollary (1.5.2). *Let* $\phi \neq \frac{\pi}{2}$ *and* $\phi \in [0, \pi)$*, then*

$$\max_{0<t_n(\phi)\leq T} \left|\zeta(\tfrac{1}{2} + it_n(\phi))\right| \gg \exp\left(\left(\frac{1}{2} + o(1)\right)\sqrt{\frac{\log T}{\log\log T}}\right).$$

All the theorems that are stated in Chapter 1 are necessary to prove the Corollaries (1.5.1) and (1.5.2).

The first theorem considers the number of generalized Gram's points in the interval (0,T].

Theorem (1.2.1). *Uniformly for* $\phi \in [0, \pi)$*, as* $T \to \infty$*,*

$$\sum_{0<t_n(\phi)\leqslant T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

Next we prove asymptotic formulas for the first, the absolute second and the third moments. We use a contour integration with Cauchy's theorem along with a saddle point technique. To obtain simple poles with residues equal to one at generalized Gram's points we use an idea introduced by Kalpokas and Steuding [31].

Theorem (1.2.2). *Uniformly for* $\phi \in [0, \pi)$*, as* $T \to \infty$*,*

$$\sum_{0<t_n(\phi)\leqslant T} \zeta\left(\tfrac{1}{2} + it\right) = 2e^{i\phi}\cos\phi\,\frac{T}{2\pi}\log\frac{T}{2\pi e} + O\left(T^{\frac{1}{2}+\epsilon}\right), \tag{5}$$

and

$$\begin{aligned}\sum_{0<t_n(\phi)\leqslant T} \left|\zeta\left(\tfrac{1}{2} + it\right)\right|^2 &= \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)^2 + (2c + 2\cos(2\phi))\frac{T}{2\pi}\log\frac{T}{2\pi e} \\ &\quad + \frac{T}{2\pi} + O\left(T^{\frac{1}{2}+\epsilon}\right),\end{aligned} \tag{6}$$

where $c := \lim_{N\to\infty}(\frac{1}{N}\sum_{n=1}^{N}\frac{1}{n} - \log N) = 0.577\ldots$ *is the Euler-Mascheroni constant.*

Theorem (1.2.3). *Uniformly for* $\phi \in [0,\pi)$*, as* $T \to \infty$*,*

$$\sum_{0<t_n^{\phi}\leq T} \left(\zeta\left(\tfrac{1}{2}+it\right)\right)^3$$
$$= 2e^{i\phi}\cos\phi \frac{T}{2\pi} P_3\left(\log\frac{T}{2\pi}\right) + e^{2i\phi}\frac{T}{2\pi}\log\frac{T}{2\pi e} + O\left(T\right),$$

where $P_3(\log T)$ *is a suitable polynomial of degree 3.*

Further we investigate sums of the absolute values of the Riemann zeta function (and its derivatives) over the generalized Gram's points. The following theorem concerns the lower bounds. The idea is introduced by Rudnick and Soundararajan [42]. To adopt it we use the approach introduced by Kalpokas and Steuding [31] and later we use the generalized divisor function used by Heath-Brown [25].

Theorem (1.3.7). *For any rational* $k \geqslant 1$ *and any non-negative integer* l*, uniformly for* $\phi \in [0,\pi)$*, as* $T \to \infty$*,*

$$\sum_{0<t_n(\phi)\leqslant T} \left|\zeta^{(l)}\left(\tfrac{1}{2}+it_n(\phi)\right)\right|^{2k} \gg T(\log T)^{k^2+2kl+1}.$$

Next we investigate the upper bounds. The main approach is introduced by Soundararajan [44] and a discrete case is developed by Milinovich [35].

Theorem (1.4.1). *Assume the Riemann Hypothesis. For any non-negative integers* k *and* l*, uniformly for* $\phi \in [0,\pi)$*, as* $T \to \infty$

$$\sum_{0<t_n(\phi)\leqslant T} \left|\zeta^{(l)}\left(\tfrac{1}{2}+it_n(\phi)\right)\right|^{2k} \ll_{k,l,\varepsilon} T(\log T)^{k^2+2kl+1+\epsilon}.$$

Unconditionally for any non-negative real k*, uniformly for* $\phi \in [0,\pi)$*, as* $T \to \infty$

$$\sum_{0<t_n(\phi)\leqslant T} \left|\zeta\left(\tfrac{1}{2}+it_n(\phi)\right)\right|^{2k} \ll_{k,\varepsilon} T(\log T)^{k^2+1+\epsilon}.$$

The last three corollaries shows the connection between discrete and continuous moments. The idea was introduced by Christ and Kalpokas [4, 5].

Corollary (1.5.4). *As* $T \to \infty$

$$\int_0^T \zeta\left(\tfrac{1}{2}+it\right) d(\theta(t)) = \frac{T}{2}\log\frac{T}{2\pi e} + O\left(T^{\frac{1}{2}+\epsilon}\right),$$
$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 d(\theta(t)) = \frac{T}{2}\left(\log\frac{T}{2\pi e}\right)^2 + 2c\frac{T}{2}\log\frac{T}{2\pi e} + \frac{T}{2} + O\left(T^{\frac{1}{2}+\epsilon}\right),$$
$$\int_0^T \zeta\left(\tfrac{1}{2}+it\right)^3 d(\theta(t)) = \frac{T}{2}\log\frac{T}{2\pi e} + O\left(T^{\frac{1}{2}+\epsilon}\right),$$

where $\theta(t)$ *is defined in formula* (1.7)*.*

Using Corollary (1.5.4) we can obtain known lower and upper bounds for continuous moments (see Figure 2).

Corollary (1.5.5). *For any rational $k \geqslant 1$ and any non-negative integer l, as $T \to \infty$,*

$$\int_1^T \left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right|^{2k} \gg T(\log T)^{k^2+2kl}.$$

Corollary (1.5.6). *Assume the Riemann Hypothesis.*
For any non-negative real k, uniformly for $\phi \in [0,\pi)$, as $T \to \infty$,

$$\int_1^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} \ll T(\log T)^{k^2+\epsilon}.$$

For any non-negative integer k and any positive integer l, uniformly for $\phi \in [0,\pi)$, as $T \to \infty$,

$$\int_1^T \left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right|^{2k} \ll T(\log T)^{k^2+2kl+\epsilon}.$$

Dirichlet L-functions

The Dirichlet L-functions are the generalizations of the Riemann zeta function. It is conjectured that different primitive Dirichlet L-functions have no common nontrivial zeros (see Fujii [15, Conjecture 3], Perelli [38]). Towards this hypothesis Fujii [13] unconditionally obtained, that if χ_1 and χ_2 are different primitive characters to the same modulus q, then the positive proportion of zeros of $L(s,\chi_1)$ and $L(s,\chi_2)$ are non-coincident. Here a zero ρ is called a coincident zero of $L(s,\chi_1)$ and $L(s,\chi_2)$ if $L(\rho,\chi_1) = L(\rho,\chi_2) = 0$ with the same multiplicity. Assuming the Riemann Hypothesis for $\zeta(s)$ Conrey, Ghosh, and Gonek [8] proved that at most two-thirds of zeros of $\zeta(s)$ are also zeros of $L(s,\chi)$, where χ is a non-principle character. Conrey, Ghosh, and Gonek [8] also note that their method allows to show (under the Generalized Riemann Hypothesis (GRH)) that any two Dirichlet L-functions with non-equivalent characters have at most two-thirds of zeros in common. For related results see also Raghunathan [40].

Main results obtained for Dirichlet L-functions

Let $s = \sigma + it$ denote a complex variable. The Dirichlet L-function is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (\sigma > 1),$$

where $\chi(n)$ is a Dirichlet character modulo q. For χ mod 1 we get the Riemann zeta function $L(s,\chi) = \zeta(s)$. The Generalized Riemann Hypothesis (GRH) states that *inside the critical strip* $0 < \sigma < 1$ *every Dirichlet L-function has zeros only on the critical line* $\sigma = \frac{1}{2}$. Zeros in the critical strip are called non-trivial and we denote them by $\rho_\chi = \beta_\chi + i\gamma_\chi$. A Dirichlet character χ mod q is said to be primitive if it is not induced by any other character of modulus strictly less than q. The unique principal character modulo q is denoted by χ_0. The character χ_0 mod 1 is the only one principal and primitive character. For a Dirichlet character χ mod q the associated Gauss sum is defined by

$$G(n,\chi) = \sum_{a=1}^{q} \chi(a) \exp\left(2\pi i \frac{an}{q}\right).$$

If $n = 1$ we denote $\tau(\chi) = G(1,\chi)$. For a primitive character χ mod q we have $|\tau(\chi)| = \sqrt{q}$ and for the principal character χ_0 we have $\tau(\chi_0) = \mu(q)$, where $\mu(q)$ is the Möbius function.

To prove the following theorem we use a contour integration with Cauchy's theorem along with a saddle point technique.

Theorem (2.2.1). *Let A and B be positive constants. Let ψ mod Q and χ mod q be primitive Dirichlet characters and $\chi \neq \psi$. Then, uniformly for $Q \ll \log^A T$ and $q \ll \log^B T$, we have*

$$\sum_{0<\gamma_\chi\leq T} L(\rho_\chi,\psi)L(1-\rho_\chi,\overline{\psi}) = \frac{\phi(Q)}{Q}\frac{T}{2\pi}\log^2\frac{T}{2\pi} + a_1\frac{T}{2\pi}\log\frac{T}{2\pi} + a_2\frac{T}{2\pi} + O\left(T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon}T}}\right),$$

where real constants a_1, a_2 depend only on q, Q, and are defined by the formula (2.40) below.

If we assume GRH then the left-hand side of the last equality can be replaced by

$$\sum_{0<\gamma_\chi\leq T} |L(\tfrac{1}{2}+it_\chi,\psi)|^2$$

and the error term can be replaced by $O(q^{1+\varepsilon}Q^\varepsilon T^{\frac{1}{2}+\varepsilon} + qQ^{\frac{9}{2}+\varepsilon}T^\varepsilon + (QT)^{\frac{1}{2}+\varepsilon})$ uniformly for all Q and q.

Next we investigate the first moments.

Theorem (2.3.1). *Let A and B be positive constants. Let ψ mod Q and χ mod q be primitive Dirichlet characters and $\chi \neq \psi$. Then, uniformly for $Q \ll \log^A T$ and $q \ll \log^B T$, we have*

$$\sum_{0<\gamma_\chi\leq T} L(\rho_\chi,\psi) = \frac{T}{2\pi}\log\frac{Tq}{2\pi e} - \delta(q,Q)L(1,\chi\overline{\psi})\psi(-1)\tau(\psi)\frac{\tau(\overline{\chi}\psi_0)}{\phi(Q)}\frac{T}{2\pi} + \frac{L'}{L}(1,\psi\overline{\chi})\frac{T}{2\pi} + O\left(T\exp(-c\log^{\frac{1}{4}-\varepsilon}T)\right),$$

where $\delta(q,Q)=1$ if $q|Q$, $\delta(q,Q)=0$ otherwise, ψ_0 is the principal Dirichlet character mod$Q$ *and c is a positive absolute constant.*

Under GRH the error term can be replaced by $O\left((TQ)^{1/2+\varepsilon}q^\varepsilon\right)$, which is valid uniformly for all Q and q.

To deduce the following corollary we use Hölder's inequality.

Corollary (2.4.1). *Assume GRH. Let A be any positive real number. Let ψ mod Q and χ mod q be primitive Dirichlet characters and $\chi \neq \psi$. Then, uniformly for $q \ll (\log T)^A$ and $Q \ll (\log T)^{2-\varepsilon}$, we have*

$$\sum_{\substack{0<\gamma_\chi\leq T\\ L(1/2+\gamma_\chi,\psi)\neq 0}} 1 \gg \frac{Q}{\phi(Q)}T.$$

Actuality

In the thesis we present new results from the theory of the Riemann zeta and Dirichlet L-functions. We introduce a new kind of discrete moments for the Riemann zeta function, we show that the curve $\zeta(\frac{1}{2}+it)$, $t>0$ expands to all directions on the complex and we present a method how to localize extreme values on the critical line. For the moments of Dirichlet L-functions we find asymptotic formulas. Those formulas give an insight about the distribution of the zeros of the Dirichlet L-functions.

Aims and problems

Discrete moments of the Riemann zeta function on the critical line

The first aim of the thesis is to investigate value distribution of the Riemann zeta function on the critical line. The approach is to investigate discrete moments that emerges from the intersection points between a straight line crossing the origin and the curve of the Riemann zeta function on the critical line (see Figure 6)

$$S_k(T,\phi) = \sum_{0<t_n(\phi)\leq T} \zeta\left(\frac{1}{2}+it_n(\phi)\right)^k$$

and

$$S_{k,l}(T,\phi) = \sum_{0<t_n(\phi)\leq T} \left|\zeta^{(l)}\left(\frac{1}{2}+it_n(\phi)\right)\right|^{2k}.$$

We search solutions for the following problems

1. Asymptotic formulas for $S_0(T,\phi)$, $S_1(T,\phi)$, $S_2(T,\phi)$, $S_3(T,\phi)$ and $S_{1,0}(T,\phi)$.
2. Unconditional lower bounds for $S_{k,l}(T,\phi)$, where $k \geq 1$ is a rational number and l is a non negative integer.
3. Conditional upper bounds for $S_{k,0}(T,\phi)$, where k is a non negative real number.
4. Conditional upper bounds for $S_{k,l}(T,\phi)$, where k and l are positive integers.

Sums of Dirichlet L-function over non-trivial zeros of another Dirichlet L-function

The second aim of the thesis is to investigate sums of Dirichlet L-function over non-trivial zeros of another Dirichlet L-function. We search solutions for the problems that concerns asymptotic formulas for the following sums

$$\sum_{0<\gamma_\chi\leq T} L(\rho_\chi,\psi), \qquad \sum_{0<\gamma_\chi\leq T} L(\rho_\chi,\psi)L(1-\rho_\chi,\overline{\psi}).$$

Methods

We use recent methods introduced by Rudznik and Soundararajan [42], Soundararajan [43, 44] and well known methods introduced by Conrey, Gosh and Gonek [7, 8, 9, 10]. Also, elements of the theory of analytic functions (contour integration, residue theory, approximation theory, moment estimates) are used. Several new approaches are introduced by the author and his collaborators [31, 32].

Novelty

All results of the thesis are new.

Statements presented for the defense

Main statements

- The curve $\zeta(\frac{1}{2}+it)$, $t>0$ expands to all directions on the complex plane.
- We localize extreme values of the Riemann zeta function on the critical line

$$\max_{t\in[T,2T]} \left|\zeta(\tfrac{1}{2}+it)\right| \gg \exp\left((1+o(1))\sqrt{\frac{\log T}{\log\log T}}\right) \quad \text{as} \quad T\to\infty.$$

- We present a method that show how to transfer discrete moments to continuous moments.

  Recall the discrete moments

$$S_k(T,\phi)=\sum_{0<t_n(\phi)\le T}\zeta\left(\frac{1}{2}+it_n(\phi)\right)^k \quad \text{and} \quad S_{k,l}(T,\phi)=\sum_{0<t_n(\phi)\le T}\left|\zeta^{(l)}\left(\frac{1}{2}+it_n(\phi)\right)\right|^{2k}.$$

We present additional statements necessary to prove the main statements. We found:

1. asymptotic formulas for $S_0(T,\phi)$, $S_1(T,\phi)$, $S_2(T,\phi)$, $S_3(T,\phi)$ and $S_{1,0}(T,\phi)$,
2. unconditional lower bounds for $S_{k,l}(T,\phi)$, where $k\ge 1$ is a rational number and l is a non negative integer,
3. conditional upper bounds for $S_{k,0}(T,\phi)$, where k is a non negative real number,
4. conditional upper bounds for $S_{k,l}(T,\phi)$, where k and l are positive integers,
5. asymptotic formulas for

$$\sum_{0<\gamma_\chi\le T} L(\rho_\chi,\psi), \qquad \sum_{0<\gamma_\chi\le T} L(\rho_\chi,\psi)L(1-\rho_\chi,\overline{\psi}).$$

Approbation

Conferences

- 2011. 5th international conference in honor of Jonas Kubilius (September 4–10), Palanga (Lithuania).
- 2011. 27th Journees Arithmetiques (26 June–1 July 2011), Vilnius University (Lithuania).
- 2008. New Directions in the Theory of Universal Zeta and L Functions (6–10 October), Würzburg University (Germany).
- 2008. International Conference on Number Theory dedicated to the 60th birthday of Professor Antanas Laurinčikas (11–15 August), Šiauliai University (Lithuania).
- 2008. The 49th conference of Lithuanian Mathematical Society (25–26 June), Kaunas (Lithuania).

Visits

- 2011. Visiting prof. Jörn Steuding at Wuerzburg University (March 2011), Würzburg (Germany).
- 2010. Visiting prof. Jerzy Kaczorowski at Adam Mickiewicz University (February 2010), Poznan (Poland).
- 2009. Visiting prof. Jörn Steuding at Würzburg University (01 September 2009–31 January 2010), Würzburg (Germany).
- 2009. Visiting prof. Jörn Steuding at Würzburg University (24 April–24 June), Würzburg (Germany).
- 2008. Visiting prof. Alberto Perelli at Genova University (01–14 December), Genova (Italy).

Schools

- 2009. Winter school on explicit methods in Number Theory (26–30 January), Debrecen University (Hungary).

The results of the thesis were presented at the seminars on Number Theory of the Department of Mathematics and Informatics of Vilnius University.

Principal publications

The main results of the thesis are in the following papers:

Published papers

1. T. Christ, J. Kalpokas, Upper bounds of Discrete moments of the derivatives of the Riemann zeta function on the critical line, *Lithuanian math. Journal*, **52**, No. 3 (2012), 233 - 248.

2. J. Kalpokas, J. Steuding, On the value-distribution of the Riemann zeta function on the critical line, *Moscow Journal of Combinatorics and Number Theory*, **1**, No. 1 (2011), 23-38.

3. T. Christ, J. Kalpokas, J. Steuding, New results on the value distribution of the Riemann zeta function on the critical line. (Neue Resultate über die Wertverteilung der Riemannschen Zetafunktion auf der kritischen Geraden.) (German) *Math. Semesterber.* **57**, No. 2 (2010), 201-229.

4. R. Garunkštis, J. Kalpokas, J. Steuding, Sum of the Dirichlet L-function over nontrivial zeros of another Dirichlet L-function, *Acta Math. Hungar.* **128**, No. 3 (2010), 287-298.

5. J. Kalpokas, Sum of the periodic zeta function over the nontrivial zeros of the Riemann zeta function, R. Steuding (ed.) et al., *New directions in value-distribution theory of zeta and L-functions. Proceedings of the conference, Würzburg, Germany, October 6–10, 2008.* Shaker Verlag. (2009), 121-130.

6. R. Garunkštis, J. Kalpokas, Sum of the periodic zeta function over the nontrivial zeros of the Riemann zeta function, *Analysis, München*, **28**, No. 2 (2008), 209-217.

Submitted papers

1. T. Christ, J. Kalpokas, Lower bounds of Discrete moments of the derivatives of the Riemann zeta function on the critical line, submitted.

2. J. Kalpokas, M. Korolev, J. Steuding, Negative values of the Riemann zeta function on the critical line, submitted.

3. R. Garunkštis, J. Kalpokas, The discrete mean square of the Dirichlet L-function at nontrivial zeros of another Dirichlet L-function, submitted.

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Moreover, at the moment I hardly can imagine my life without - internet, Gmail, news from all over the world, flights, smartphone, macbook, free content (music, movies, tv shows), facebook, coffee to go, buying online nearly anything, freedom of speech, ArXiv and bunch of bars in Vilnius.[1]

[1] I truly believe that the last 8 lines will be extremely funny to read after 10 years.

Chapter 1

Discrete moments of the Riemann zeta function on the critical line

Let $s = \sigma + it$ denote a complex variable. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\sigma > 1).$$

We investigate the value-distribution of the Riemann zeta function $\zeta(s)$ on the critical line $s = \frac{1}{2} + i\mathbb{R}$. Recall the functional equation for the zeta function,

$$\zeta(s) = \Delta(s)\zeta(1-s), \qquad \text{where} \quad \Delta(s) := 2^s \pi^{s-1}\Gamma(1-s)\sin(\tfrac{\pi s}{2}). \tag{1.1}$$

It follows immediately that $\Delta(s)\Delta(1-s) = 1$, hence $\Delta(\frac{1}{2} + it)$ lies on the unit circle for real t. Given an angle $\phi \in [0, \pi)$, denote by $t_n(\phi)$ with $n \in \mathbb{N}$ the positive roots of the equation

$$e^{2i\phi} = \Delta(\tfrac{1}{2} + it)$$

in ascending order. These roots correspond to intersections of the curve $t \mapsto \zeta(\frac{1}{2} + it)$ with straight lines $e^{i\phi}\mathbb{R}$ through the origin (see Kalpokas and Steuding [31]).

Recall the function $\Delta(s)$ defined in (1.1). It is well-known that

$$\Delta(\sigma + it) = \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2} - \sigma - it} \exp(i(t + \tfrac{\pi}{4}))(1 + O(|t|^{-1})) \qquad \text{for} \quad |t| \geqslant 1 \tag{1.2}$$

uniformly for any σ from a bounded interval. Hence,

$$\frac{1}{\Delta(s) - e^{2i\phi}} = \frac{-e^{-2i\phi}}{1 - e^{-2i\phi}\Delta(s)} = -e^{-2i\phi}\left(1 + \sum_{k=1}^{\infty} e^{-2ki\phi}\Delta(s)^k\right) \tag{1.3}$$

for $\sigma > \frac{1}{2}$. Obviously, $\Delta(\frac{1}{2} + it)$ is a complex number from the unit circle for $t \in \mathbb{R}$. Moreover, $\Delta'(\frac{1}{2} + it)$ is non-vansishing for sufficiently large t as follows from the asymptotic formula

$$\frac{\Delta'}{\Delta}(\sigma + it) = -\log\frac{|t|}{2\pi} + O(|t|^{-1}) \qquad \text{for} \quad |t| \geqslant 1. \tag{1.4}$$

By (1.1) and (1.2) we can write

$$\Delta(\tfrac{1}{2}+it) = e^{-2\theta(t)}, \tag{1.5}$$

where

$$\theta(t) = \operatorname{Im}\log\left(\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)\right) - \frac{t}{2}\log\pi. \tag{1.6}$$

Using Stirling's formula for $t \geq 1$ we get

$$\theta(t) = \frac{t}{2}\log\frac{t}{2\pi e} - \frac{\pi}{8} + \sum_{k=1}^{\infty}\frac{a_k}{t^k}, \tag{1.7}$$

where the coefficients a_k can be computed explicitly[1]. The function $\theta(t)$ is differentiable and according to (1.7)

$$\theta'(t) = \tfrac{1}{2}\log\frac{t}{2\pi e} + \tfrac{1}{2} + O(t^{-2}) \tag{1.8}$$

holds for $t \geq 1$. Hence,

$$\tfrac{1}{2}\log\frac{t}{2\pi e} < \theta'(t) < \tfrac{1}{2}\log\frac{t}{2\pi e} + 1.$$

This implies that $\theta(t)$ is monotonously increasing for t large enough.

Due to (1.5), the solutions of $\Delta(\frac{1}{2}+it) - e^{i\phi} = 0$ correspond to the solutions of

$$\theta(t) \equiv \phi \mod \pi.$$

Next we introduce certain Dirichlet polynomials

$$X(s) = \sum_{n\leqslant X}\frac{x_n}{n^s}, \quad Y(s) = \sum_{m\leqslant Y}\frac{y_m}{m^s}, \tag{1.9}$$

where $X, Y \leqslant T$. Moreover, we define the following quantities

$$\mathcal{X}_0 = \max_{n\leqslant X}|x_n|, \quad \mathcal{Y}_0 = \max_{m\leqslant Y}|y_m|, \quad \mathcal{X}_1 = \sum_{n\leqslant X}\frac{|x_n|}{n}, \quad \mathcal{Y}_1 = \sum_{m\leqslant Y}\frac{|y_m|}{m}.$$

and we set

$$X_1(s) = \sum_{n\leqslant X}\frac{\overline{x}_n}{n^s}, \quad Y_1(s) = \sum_{m\leqslant Y}\frac{\overline{y}_m}{m^s}.$$

The following estimate will be used during the proofs

$$\zeta^{(l)}(\sigma+it) \ll \begin{cases} |t|^{\frac{1}{2}-\sigma+\epsilon}, & \text{if } \sigma \leq 0, \\ |t|^{\frac{1}{2}(1-\sigma)+\epsilon}, & \text{if } 0 < \sigma \leq 1, \\ |t|^{\epsilon}, & \text{if } \sigma > 1. \end{cases} \tag{1.10}$$

It can be derived from the case $l = 0$ (check [46]) by applying Cauchy's estimate for the derivatives of analytic functions to $\zeta(s)$ in a small disc centered at $s = \sigma + it$.

1.1 Lemmas

We shall use a variation of Lemma 5.1 from Ng [37]:

[1] e.g. $a_1 = 1/48$, $a_2 = 0$, $a_3 = 7/5760$,

Lemma 1.1.1. *Suppose the series* $f(s) = \sum_{n=1}^{\infty} \alpha_n n^{-s}$ *converges absolutely for* $\mathrm{Re}s > 1$ *and* $\sum_{n=1}^{\infty} |\alpha|_n n^{-\sigma} \ll (\sigma - 1)^{-\gamma}$ *for some* $\gamma \geq 0$ *as* $\sigma \to 1 + 0$. *Next, let* $X(s)$ *and* $Y(s)$ *be Dirichlet polynomials as defined in* (1.9). *Then, uniformly for* $a \in (1, 2]$ *and for any* $\alpha \in \mathbb{C}$ *with* $|\mathrm{Re}\alpha| < a - 1$,

$$J = \frac{1}{2\pi i} \int_{a+i}^{a+iT} f(s+\alpha) X(s+\alpha) Y(1 - s + \overline{\alpha}) \frac{\Delta'(s)}{\Delta(s)} ds$$
$$= -\frac{T}{2\pi} \left(\log \frac{T}{2\pi e} \right) \sum_{\substack{m \leqslant X \\ mn \leqslant Y}} \frac{\alpha_n x_m y_{mn}}{(mn)^{1+2\mathrm{Re}\alpha}} + O\left(\frac{Y^{a-\mathrm{Re}\alpha} (\log T)^2 \mathcal{X}_0 \mathcal{Y}_0}{(a + \mathrm{Re}\alpha - 1)^{\gamma+1}} \right),$$

where the implicit constant is absolute.

Proof. Changing the order of summation and the integration, we get

$$J = \sum_{n=1}^{\infty} \sum_{m \leqslant X,\, k \leqslant Y} \frac{k^{a-\overline{\alpha}}}{(mn)^{a+\alpha}} \frac{\alpha_n x_m y_k}{k} \frac{1}{2\pi} \int_1^T \frac{\Delta'}{\Delta}(a + it) \left(\frac{k}{mn} \right)^{it} dt.$$

Next, the contribution of O - term from (1.4) to J does not exceed in order

$$\mathcal{X}_0 \mathcal{Y}_0 (\log T) \sum_{n=1}^{\infty} \frac{|\alpha_n|}{n^{a+\mathrm{Re}\alpha}} \sum_{m \leqslant X} \frac{1}{m^{a+\mathrm{Re}\alpha}} \sum_{k \leqslant Y} k^{a-\mathrm{Re}\alpha-1} \ll \frac{Y^{a-\mathrm{Re}\alpha} (\log T)^2 \mathcal{X}_0 \mathcal{Y}_0}{(a + \mathrm{Re}\alpha - 1)^{\gamma+1}}.$$

Extracting the diagonal term (when $k = mn$) in the above expression for J, we get

$$J = \left(- \int_1^T \log \frac{t}{2\pi} \frac{dt}{2\pi} \right) \sum_{\substack{m \leqslant X \\ mn \leqslant Y}} \frac{\alpha_n x_m y_{mn}}{(mn)^{1+2\mathrm{Re}\alpha}} + O(J_1) + O\left(\frac{Y^{a-\mathrm{Re}\alpha} (\log T)^2 \mathcal{X}_0 \mathcal{Y}_0}{(a + \mathrm{Re}\alpha - 1)^{\gamma+1}} \right),$$

where

$$\begin{aligned} J_1 &= \sum_{n=1}^{\infty} \sum_{\substack{m \leqslant X,\, k \leqslant Y \\ mn \neq k}} \frac{k^{a-\mathrm{Re}\alpha}}{(mn)^{a+\mathrm{Re}\alpha}} \frac{|\alpha_n x_m y_k|}{k} |j_{k,mn}| \\ &\leq Y^{a-\mathrm{Re}\alpha-1} \mathcal{X}_0 \mathcal{Y}_0 \sum_{n=1}^{\infty} \sum_{\substack{m \leqslant X,\, k \leqslant Y \\ mn \neq k}} \frac{|\alpha_n|}{(mn)^{a+\mathrm{Re}\alpha}} |j_{k,mn}| \end{aligned}$$

with

$$j_{k,r} = \int_1^T \left(\log \frac{t}{2\pi} \right) \left(\frac{k}{r} \right)^{it} \frac{dt}{2\pi}.$$

Integrating by parts shows for $k \neq r$ that

$$|j_{k,r}| = \left| \int_1^T \frac{\log(t/(2\pi))}{2\pi \log (k/r)} d\left(\frac{k}{r} \right)^{it} \right| \leq \frac{2}{\pi} \log \frac{T}{2\pi} \left| \log \frac{k}{r} \right|^{-1}.$$

Setting $r = mn$, $\beta_r = \sum_{n|r} |\alpha_n|$ in the expression for J_1, we have

$$J_1 \ll Y^{a-\mathrm{Re}\alpha-1} (\log T) \mathcal{X}_0 \mathcal{Y}_0 \sum_{k \leqslant Y} \sum_{r=1}^{\infty} \frac{\beta_r}{r^{a+\mathrm{Re}\alpha}} \left| \log \frac{k}{r} \right|^{-1}.$$

Recall that $\zeta(s)$ has a simple pole at $s = 1$. Thus, the contribution of the terms with $r \leqslant k/2$

and $r > 3k/2$ does not exceed in order

$$\begin{aligned} & Y^{a-\mathrm{Re}\alpha-1}(\log T)\mathcal{X}_0\mathcal{Y}_0 \cdot Y \sum_{r=1}^{\infty} \frac{\beta_r}{r^{a+\mathrm{Re}\alpha}} \\ \ll \quad & Y^{a-\mathrm{Re}\alpha}\mathcal{X}_0\mathcal{Y}_0(\log T) \sum_{n=1}^{\infty} \frac{|\alpha_n|}{n^{a+\mathrm{Re}\alpha}} \sum_{m=1}^{\infty} \frac{1}{m^{a+\mathrm{Re}\alpha}} \\ \ll \quad & \frac{Y^{a-\mathrm{Re}\alpha}(\log T)\mathcal{X}_0\mathcal{Y}_0}{(a+\mathrm{Re}\alpha-1)^{\gamma+1}}. \end{aligned}$$

For $k/2 < r \leqslant 3k/2$, $r \neq k$ we set $r = k + \nu$; since $|\log(k/r)|^{-1} \ll k|\nu|^{-1}$, the corresponding part of J_1 can be estimated as follows:

$$\begin{aligned} & Y^{a-\mathrm{Re}\alpha 1}(\log T)\mathcal{X}_0\mathcal{Y}_0 \sum_{k\leqslant Y} \sum_{0<|\nu|\leqslant k/2} \frac{\beta_{k+\nu}}{(k+\nu)^{a+\mathrm{Re}\alpha}} \frac{k}{|\nu|} \\ \ll \quad & Y^{a-\mathrm{Re}\alpha}(\log T)\mathcal{X}_0\mathcal{Y}_0 \sum_{0<|\nu|\leqslant Y/2} \frac{1}{|\nu|} \sum_{2|\nu|\leqslant k\leqslant Y} \frac{\beta_{k+\nu}}{(k+\nu)^{a+\mathrm{Re}\alpha}} \\ \ll \quad & Y^{a-\mathrm{Re}\alpha}(\log T)^2\mathcal{X}_0\mathcal{Y}_0 \sum_{k=1}^{\infty} \frac{\beta_k}{k^{a+\mathrm{Re}\alpha}} \ll \frac{Y^{a-\mathrm{Re}\alpha}(\log T)^2\mathcal{X}_0\mathcal{Y}_0}{(a+\mathrm{Re}\alpha-1)^{\gamma+1}}. \end{aligned}$$

The lemma is proved.

□

Next lemma is a variation of Gonek's lemma:

Lemma 1.1.2. *Suppose the series* $f(s) = \sum_{n=1}^{\infty} \alpha_n n^{-s}$ *converges absolutely in the half-plane* $\mathrm{Re} s > 1$, $\sum_{n=1}^{\infty} |\alpha_n| n^{-\sigma} \ll (\sigma-1)^{-\gamma}$ *for some* $\gamma \geqslant 0$ *as* $\sigma \to 1+0$ *and* $\alpha_n \ll n^{\varepsilon}$ *for any* $\varepsilon > 0$. *Then we have, for any fixed integer* $m \geqslant 0$ *and* $c \geqslant 1$ *uniformly for* $a \in (1, 2]$,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{a+ic}^{a+iT} f(s)\Delta(1-s)\left(\frac{\Delta'}{\Delta}(s)\right)^m ds \\ = \quad & (-1)^m \sum_{n\leqslant \frac{T}{2\pi}} \alpha_n (\log n)^m + O\left(\frac{T^{a-\frac{1}{2}}}{(a-1)^{\gamma}}(\log T)^m + T^{\frac{1}{2}+\varepsilon}\right). \end{aligned}$$

For the proof we refer to Lemma 5 from [19] (in the original paper the remainder term is not uniform in $a > 1$).

We proceed with a modified version of Lemma 6 from [19].

Lemma 1.1.3. *Let* l *be a non-negative integer and* $1 \leq |t| \leq T$. *Then uniformly for* $\sigma \in [-\frac{1}{\log T}, 1 + \frac{1}{\log T}]$ *we have*

$$\zeta^{(l)}(1-s) = (-1)^l \sum_{k=0}^{l} \binom{l}{k} \zeta^{(k)}(s)\Delta(1-s)\left(\log\frac{t}{2\pi}\right)^{l-k} + O(t^{\frac{\sigma}{2}-1+\epsilon}).$$

Proof. Taking the l-th derivative of (1.1) according to Leibniz's rule, we find that

$$\zeta^{(l)}(1-s) = \sum_{k=0}^{l} \binom{l}{k} (-1)^k \zeta^{(k)}(s)\Delta^{(l-k)}(1-s). \tag{1.11}$$

Next we will show that

$$\Delta^{(\nu)}(1-s) = \Delta(1-s)\left(-\log\frac{t}{2\pi}\right)^{\nu} + O(t^{\sigma-\frac{3}{2}}(\log t)^{\nu-1}). \tag{1.12}$$

holds uniformly for σ from a bounded interval, $|t| \geq 1$ and ν a non-negative integer.

To prove (1.12) we use induction. The case $\nu = 0$ is true. Now suppose the equality (1.12) is proved for $\nu = 0, \ldots, \mu - 1$. We differentiate the identity

$$\Delta'(1-s) = \Delta(1-s)\frac{\Delta'}{\Delta}(1-s)$$

and obtain

$$\Delta^{(\mu)}(1-s) = \sum_{\nu=0}^{\mu-1} \binom{\mu-1}{\nu} \Delta^{(\nu)}(1-s) \left(\frac{\Delta'}{\Delta}\right)^{(\mu-\nu-1)} (1-s).$$

By (1.4) and Cauchy's estimate for the derivatives of an analytic function applied to a small disc centered at s, we find that

$$\left(\frac{\Delta'}{\Delta}\right)^{(\nu)} (1-s) \ll |t|^{-1}, \quad \text{for } \nu \geq 1.$$

Now this, (1.2) and (1.4) yield the proof of (1.12). The combination of (1.10), (1.11) and (1.12) finishes the proof of the Lemma. □

1.2 Asymptotic formulas

1.2.1 Number of intersection points

In this section we find the asymptotic formula for the number of intersection point between the straight line crossing the origin and the curve of the Riemann zeta function.

Theorem 1.2.1. *Uniformly for $\phi \in [0, \pi)$, as $T \to \infty$,*

$$\sum_{0 < t_n(\phi) \leqslant T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Proof. Recall that $\Delta(\frac{1}{2}+it)$ is a complex number from the unit circle whenever $t \in \mathbb{R}$. Moreover, $\Delta'(\frac{1}{2}+it)$ is non-vansishing, which follows from the asymptotic formula (1.4). Consequently, $\Delta(\frac{1}{2}+it)$ is spinning on the unit circle around the origin in clockwise direction with increasing speed as $t \to \infty$. Moreover, it follows that there exists no proper real interval $\mathcal{I}$ such that $\zeta(\frac{1}{2}+it)$ lies on a straight line $e^{i\phi}\mathbb{R}$ for all $t \in \mathcal{I}$. For the first, lets assume that

$$\Delta(\tfrac{1}{2}+iT) = \Delta(\tfrac{1}{2}) = 1. \tag{1.13}$$

Then the number of roots of the equation $\Delta(\frac{1}{2}+it) = e^{2i\phi}$ with $0 \leq t \leq T$ is up to the sign equal to the winding number of the curve

$$\eta : [0,1] \to \mathbb{C}, \quad \lambda \mapsto \eta(\lambda) := \Delta(\tfrac{1}{2}+i\lambda T).$$

This yields

$$-\sum_{0 < t_n(\phi) \leq T} 1 = \frac{1}{2\pi i}\int_\eta \frac{\mathrm{d}s}{s} = \frac{T}{2\pi}\int_0^1 \frac{\Delta'}{\Delta}(\tfrac{1}{2}+i\lambda T)\mathrm{d}\lambda.$$

In order to use (1.4) we divide the integration interval into two subintervals. Noting that there are only finitely many roots of $\Delta(\frac{1}{2}+it) - e^{2i\phi}$ for $0 < t \leq 1$, we find for the term with the integral on the right-hand side above

$$\frac{T}{2\pi}\left\{\int_0^{1/T}+\int_{1/T}^{1}\right\}\frac{\Delta'}{\Delta}(\tfrac{1}{2}+i\lambda T)\mathrm{d}\lambda$$
$$=O(1)+\frac{T}{2\pi}\int_{1/T}^{1}\left(-\log\frac{\lambda T}{2\pi}+O((\lambda T)^{-1})\right)\mathrm{d}\lambda.$$

Hence, the asymptotic formula of Theorem 1.2.1 follows by integration; however, to get rid of our assumption (1.13) on T, by (1.4) we may substitute this by any T at the expense of an error $O(\log T)$. This proves Theorem 1.2.1.

□

1.2.2 First and second moments

The following theorem gives asymptotic formulas for the associated first and second discrete moments

Theorem 1.2.2. *Uniformly for $\phi \in [0,\pi)$, as $T\to\infty$,*

$$\sum_{0<t_n(\phi)\leqslant T}\zeta\left(\tfrac{1}{2}+it_n(\phi)\right)=2e^{i\phi}\cos\phi\,\frac{T}{2\pi}\log\frac{T}{2\pi e}+O\left(T^{\frac{1}{2}+\epsilon}\right), \tag{1.14}$$

and

$$\sum_{0<t_n(\phi)\leqslant T}\left|\zeta\left(\tfrac{1}{2}+it_n(\phi)\right)\right|^2 = \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)^2+(2c+2\cos(2\phi))\frac{T}{2\pi}\log\frac{T}{2\pi e}+\frac{T}{2\pi}+O\left(T^{\frac{1}{2}+\epsilon}\right), \tag{1.15}$$

where $c:=\lim_{N\to\infty}(\frac{1}{N}\sum_{n=1}^{N}\frac{1}{n}-\log N)=0.577\ldots$ is the Euler-Mascheroni constant.

Proof of (1.14). We begin with the estimation

$$\left|\zeta(\tfrac{1}{2}+it)\right|\ll T^{1/6+\epsilon}.$$

It is a well-known bound from zeta function theory (see [28]). Note that it is sufficient to obtain (1.2.2) for the sum over the segment $c<t_n(\phi)\leqslant T$, where $c>32\pi$ is a large absolute constant (32π comes from the inequality $2(\frac{t}{2\pi})^{-\frac{1}{2}}<\frac{1}{2}$ that is used in the proof).

Next, without loss of generality we may set $T=\frac{1}{2}(t_\nu(\phi)+t_{\nu+1}(\phi))$. Since the points $s=\frac{1}{2}+it_n(\phi)$ are the roots of the function $\Delta(s)-e^{2i\phi}$ and

$$\sum_{c<t_n(\phi)\leqslant T}\zeta(\tfrac{1}{2}+it_n(\phi))=\sum_{c<t_n(\phi)\leqslant T}\overline{\zeta(\tfrac{1}{2}-it_n(\phi))}$$

the sum in question can be rewritten as a contour integral:

$$\sum_{c<t_n(\phi)\leqslant T}\zeta(\tfrac{1}{2}-it_n(\phi))=\frac{1}{2\pi i}\int_{\square}\zeta(1-s)\frac{\Delta'(s)}{\Delta(s)-e^{2i\phi}}ds;$$

here □ stands for the contour clockwise oriented rectangular contour with vertices $a+ic,\ a+iT,\ 1-a+iT,\ 1-a+ic$, where $a=1+(\log T)^{-1}$. Let $\mathcal{I}_1$ and $\mathcal{I}_3$ be integrals over right and left sides of contour, and $\mathcal{I}_2$ and $\mathcal{I}_4$ be the integrals over the top and bottom edges of the contour.

We may assume the constant c so large that the relations

$$|\Delta(a+it)| = \left(\frac{t}{2\pi}\right)^{1/2-a}\big(1+O(t^{-1})\big) \le 2\left(\frac{t}{2\pi}\right)^{-1/2} < \frac{1}{2}$$

hold for any $t > c$.

In view of (1.3) for $\mathcal{I}_1$ we have

$$\mathcal{I}_1 = -e^{-2i\phi}\int_{a+ic}^{a+iT}\zeta(s)\frac{\Delta'}{\Delta}(s)\left(1+\sum_{k=1}^{\infty}e^{-2ki\phi}\Delta(s)^k\right)ds = -e^{-2i\phi}(j_1+j_2),$$

where

$$j_1 = \int_{a+ic}^{a+iT}\zeta(s)\frac{\Delta'}{\Delta}(s)ds,$$

$$j_2 = \int_{a+ic}^{a+iT}\zeta(s)\frac{\Delta'}{\Delta}(s)\sum_{k=1}^{\infty}e^{-2ki\phi}\Delta(s)^k ds.$$

Lemma 1.1.1 with $X(s)\equiv 1$, $Y(s)\equiv 1$, applied to j_1, leads to

$$j_2 = -\frac{T}{2\pi}\log\frac{T}{2\pi e} + O\big((\log T)^6\big).$$

Next, by standard arguments we obtain

$$j_3 \ll \int_c^T \frac{\log t}{(a-1)}\frac{dt}{\sqrt{t}} \ll T^{\frac{1}{2}+\epsilon}.$$

Hence

$$\mathcal{I}_1 = e^{-2i\phi}\frac{T}{2\pi}\log\frac{T}{2\pi} + O\big(T^{\frac{1}{2}+\epsilon}\big).$$

Further, transforming the integral $\mathcal{I}_3$ via $s\mapsto 1-\overline{s}$ we find

$$\begin{aligned}\overline{\mathcal{I}_3} &= -\frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta(s)\frac{\Delta'(1-s)\,ds}{\Delta(1-s)-e^{-2i\phi}}\\ &= -\frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta(s)\frac{\Delta'}{\Delta}(s)\sum_{k=0}^{\infty}e^{-2i\phi}\Delta(s)^k ds.\end{aligned}$$

We notice that $\overline{I}_3 = e^{2i\phi}(j_2+j_3)$. Hence

$$\mathcal{I}_3 = \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) + O\big((\log T)^6\big).$$

In order to estimate the integral $\mathcal{I}_2$ over the top and bottom edges we write

$$\begin{aligned}F(s) &= \zeta(s)\Delta(1-s)\,\frac{\Delta'}{\Delta}(s)\,\frac{1}{\Delta(s)-e^{2i\phi}}\\ &= \zeta(1-s)\Delta(s)\frac{\Delta'}{\Delta}(s)\left(1+\frac{e^{2i\phi}}{\Delta(s)-e^{2i\phi}}\right).\end{aligned}$$

Since $T = \frac{1}{2}(t_\nu(\phi)+t_{\nu+1}(\phi))$ for some ν, the inequality $|\Delta(s)-e^{2i\phi}| > \frac{1}{3}$ from Paragraph 3.1.1 holds over the segment of integration. Using the bound $|\zeta(\sigma+it)| \ll t^{(1-\sigma)/3}\log t$, for

$s = \sigma + iT$, $\frac{1}{2} \leqslant \sigma \leqslant a$ (see [28]), we get

$$|F(s)| \ll (\log T)|\zeta(s)\Delta(1-s)| \ll (\log T)T^{\frac{1}{3}(1-\sigma)}T^{\sigma-\frac{1}{2}} \ll T^{\frac{1}{2}+\epsilon}.$$

In the case $1 - a \leqslant \sigma \leqslant \frac{1}{2}$ we have

$$|F(s)| \ll (\log T)|\zeta(1-s)\Delta(s)| \ll (\log T)T^{\frac{1}{3}(1-(1-\sigma))}\log T T^{\frac{1}{2}-\sigma} \ll T^{\frac{1}{2}+\epsilon}.$$

Thus, $\mathcal{I}_2 \ll T^{\frac{1}{2}+\epsilon}$. Finally, the bound $\mathcal{I}_4 = O(1)$ is obvious.

To loose the condition $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$ we note the sum over the intersection points over a bounded interval is bounded by $\ll T^{\frac{1}{6}+\epsilon}$ and is within agreement with the error term of the asymptotic formula.

We showed

$$\sum_{c<t_n(\phi)\leqslant T} \zeta(\tfrac{1}{2} + it_n(\phi)) = (1 + e^{-2i\phi})\frac{T}{2\pi}\log\frac{T}{2\pi} + O\big(T^{\frac{1}{2}+\epsilon}\big).$$

□

Proof of (1.15). We begin with the estimation

$$\big|\zeta\big(\tfrac{1}{2} + it\big)\big| \ll T^{1/6+\epsilon}.$$

It is a well-known bound from zeta function theory (see [28]). Note that it is sufficient to obtain (1.2.2) for the sum over the segment $c < t_n(\phi) \leqslant T$, where $c > 32\pi$ is a large absolute constant (32π comes from the inequality $2(\frac{t}{2\pi})^{-\frac{1}{2}} < \frac{1}{2}$ that is used in the proof).

Next, without loss of generality we may set $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$. Since the points $s = \frac{1}{2} + it_n(\phi)$ are the roots of the function $\Delta(s) - e^{2i\phi}$ and

$$\sum_{c<t_n(\phi)\leqslant T} |\zeta(\tfrac{1}{2} + it_n(\phi))|^2 = \sum_{c<t_n(\phi)\leqslant T} \overline{\zeta(\tfrac{1}{2} + it_n(\phi))}\zeta(\tfrac{1}{2} + it_n(\phi))$$

the sum in question can be rewritten as a contour integral:

$$\sum_{c<t_n(\phi)\leqslant T} |\zeta(\tfrac{1}{2} + it_n(\phi))|^2 = \frac{1}{2\pi i}\int_{\Box} \zeta(1-s)\zeta(s)\frac{\Delta'(s)}{\Delta(s) - e^{2i\phi}}ds;$$

here □ stands for the contour clockwise oriented rectangular contour with vertices $a + ic$, $a + iT$, $1 - a + iT$, $1 - a + ic$, where $a = 1 + (\log T)^{-1}$. Let $\mathcal{I}_1$ and $\mathcal{I}_3$ be integrals over right and left sides of contour, and $\mathcal{I}_2$ and $\mathcal{I}_4$ be the integrals over the top and bottom edges of the contour. We may assume the constant c so large that the relations

$$|\Delta(a + it)| = \left(\frac{t}{2\pi}\right)^{1/2-a}\big(1 + O(t^{-1})\big) \leq 2\left(\frac{t}{2\pi}\right)^{-1/2} < \frac{1}{2}$$

hold for any $t > c$.

In view of (1.3) for $\mathcal{I}_1$ we have

$$\mathcal{I}_1 = -e^{-2i\phi}\int_{a+ic}^{a+iT} \zeta^2(s)\frac{\Delta'}{\Delta}(s)\left(1 + \sum_{k=1}^{\infty} e^{-2ki\phi}\Delta(s)^k\right)ds = -e^{-2i\phi}(j_1 + j_2),$$

where

$$j_1 = \int_{a+ic}^{a+iT} \zeta^2(s) \frac{\Delta'}{\Delta}(s) ds,$$
$$j_2 = \int_{a+ic}^{a+iT} \zeta^2(s) \frac{\Delta'}{\Delta}(s) \sum_{k=1}^{\infty} e^{-2ki\phi} \Delta(s)^k ds.$$

Lemma 1.1.1 with $X(s) \equiv 1$, $Y(s) \equiv 1$, applied to j_1, leads to

$$j_1 = -\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) + O\big((\log T)^6\big).$$

Next, by standard arguments we obtain

$$j_3 \ll \int_c^T \frac{\log t}{(a-1)^2} \frac{dt}{\sqrt{t}} \ll T^{\frac{1}{2}+\epsilon}.$$

Hence

$$\mathcal{I}_1 = e^{-2i\phi}\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) + O\big(T^{\frac{1}{2}+\epsilon}\big).$$

Further, transforming the integral $\mathcal{I}_3$ via $s \mapsto 1-\overline{s}$ we find

$$\begin{aligned} \overline{\mathcal{I}_3} &= -\frac{1}{2\pi i}\int_{a+ic}^{a+iT} \zeta(s)^2 \Delta(1-s) \frac{\Delta'(1-s)\, ds}{\Delta(1-s) - e^{-2i\phi}} \\ &= -\frac{1}{2\pi i}\int_{a+ic}^{a+iT} \zeta(s)^2 \Delta(1-s) \frac{\Delta'}{\Delta}(s) \sum_{k=0}^{\infty} e^{-2i\phi} \Delta(s)^k ds \\ &= -j_1 - e^{-2i\phi}(j_2 + j_3), \end{aligned}$$

where

$$j_1 = \frac{1}{2\pi i}\int_{a+ic}^{a+iT} \zeta(s)^2 \Delta(1-s) \frac{\Delta'}{\Delta}(s) ds,$$
$$j_2 = \frac{1}{2\pi i}\int_{a+ic}^{a+iT} \zeta(s)^2 \frac{\Delta'}{\Delta}(s) ds,$$
$$j_3 = \frac{1}{2\pi i}\int_{a+ic}^{a+iT} \zeta(s)^2 \frac{\Delta'}{\Delta}(s) \sum_{k=1}^{\infty} e^{-2ki\phi} \Delta(s)^k ds.$$

We notice that $\mathcal{I}_1 = -e^{-2i\phi}(j_2 + j_3)$ and $\overline{\mathcal{I}}_3 = j_1 + \mathcal{I}_1$. By Lemma 1.1.2 we find

$$\mathcal{I}_3 = \sum_{n \le \frac{T}{2\pi}} d_2(n)\log n + e^{2i\phi}\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) + O\big(T^{\frac{1}{2}+\epsilon}\big).$$

The sum on the right hand side can be handled using summation by parts, since by standard bound for the Dirichlet divisor problem we have

$$\sum_{n \le x} d_2(n) = x\log x + (2c-1)x + O(x^{\frac{1}{2}+\epsilon}).$$

Hence

$$\mathcal{I}_3 = \frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^2 + (2c-2)\frac{T}{2\pi}\log\frac{T}{2\pi e} e^{2i\phi}\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) + O\big(T^{\frac{1}{2}+\epsilon}\big).$$

To loose the condition $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$ we note the sum over the intersection points over a bounded interval is bounded by $\ll T^{\frac{1}{6}+\epsilon}$ and is within agreement with the error term of the asymptotic formula.

In order to estimate the integral $\mathcal{I}_2$ over the top and bottom edges we write

$$\begin{aligned} F(s) &= \zeta^2(s)\Delta(1-s)\frac{\Delta'}{\Delta}(s)\frac{1}{\Delta(s)-e^{2i\phi}} \\ &= \zeta^2(1-s)\Delta(s)\frac{\Delta'}{\Delta}(s)\left(1+\frac{e^{2i\phi}}{\Delta(s)-e^{2i\phi}}\right). \end{aligned}$$

Since $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$ for some ν, the inequality $|\Delta(s) - e^{2i\phi}| > \frac{1}{3}$ from Paragraph 3.1.1 holds over the segment of integration. Using the bound $|\zeta(\sigma+it)| \ll t^{(1-\sigma)/3}\log t$, for $s = \sigma + iT$, $\frac{1}{2} \leqslant \sigma \leqslant a$ (see [28]), we get

$$|F(s)| \ll (\log T)|\zeta^2(s)\Delta(1-s)| \ll (\log T)\big(T^{\frac{1}{3}(1-\sigma)}\big)^2 T^{\sigma-\frac{1}{2}} \ll T^{\frac{1}{2}+\epsilon}.$$

In the case $1-a \leqslant \sigma \leqslant \frac{1}{2}$ we have

$$|F(s)| \ll (\log T)|\zeta^2(1-s)\Delta(s)| \ll (\log T)\big(T^{\frac{1}{3}(1-(1-\sigma))}\log T\big)^2 T^{\frac{1}{2}-\sigma} \ll T^{\frac{1}{2}+\epsilon}$$

Thus, $\mathcal{I}_2 \ll T^{\frac{1}{2}+\epsilon}$. Finally, the bound $\mathcal{I}_4 = O(1)$ is obvious.

To loose the condition $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$ we note the sum over the intersection points over a bounded interval is bounded by $\ll T^{\frac{1}{6}+\epsilon}$ and is within agreement with the error term of the asymptotic formula.

We showed

$$\begin{aligned} \sum_{0<t_n(\phi)\leqslant T} \left|\zeta\left(\tfrac{1}{2}+it_n(\phi)\right)\right|^2 &= \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)^2 + (2c+2\cos(2\phi))\frac{T}{2\pi}\log\frac{T}{2\pi e} \\ &\quad + \frac{T}{2\pi} + O\left(T^{\frac{1}{2}+\epsilon}\right). \end{aligned}$$

□

1.2.3 Third moment

The following theorem gives the asymptotic formula for the highest moment that is possible to handle.

Theorem 1.2.3. *Uniformly for $\phi \in [0,\pi)$, as $T \to \infty$,*

$$\begin{aligned} &\sum_{0<t_n^\phi\leq T} \zeta\left(\tfrac{1}{2}+it_n(\phi)\right)^3 \\ &= 2e^{i\phi}\cos\phi\frac{T}{2\pi}P_3\left(\log\frac{T}{2\pi}\right) + e^{2i\phi}\frac{T}{2\pi}\log\frac{T}{2\pi e} + O(T), \end{aligned}$$

where $P_3(\log T)$ is a suitable polynomial of degree 3.

Proof. The method of proof is along the lines of Kalpokas and Steuding [31]. It suffices to evaluate

$$S(T) = \sum_{c<t_n(\phi)\leqslant T} \zeta\big(\tfrac{1}{2}+it_n(\phi)\big)\zeta^2\big(\tfrac{1}{2}-it_n(\phi)\big),$$

where $c > 32\pi$ is an absolute constant and $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$ for some ν. Setting $a =$

$1+(\log T)^{-1}$, we find by Cauchy's theorem

$$\begin{aligned} S(T) &= \frac{1}{2\pi i}\left\{\int_{a+ic}^{a+iT}+\int_{a+iT}^{1-a+iT}+\int_{1-a+iT}^{1-a+ic}+\int_{1-a+ic}^{a+ic}\right\}\zeta(s)\zeta(1-s)^2\frac{\Delta'(s)}{\Delta(s)-e^{2i\phi}}\,ds \\ &= \sum_{k=1}^{4}\mathcal{I}_k, \end{aligned}$$

say.

First we consider $\mathcal{I}_1$. In view of (1.3) we obtain

$$\begin{aligned} \mathcal{I}_1 &= -\frac{e^{-2i\phi}}{2\pi i}\int_{a+ic}^{a+iT}\zeta^3(s)\Delta(1-s)\frac{\Delta'}{\Delta}(s)\left(1+\sum_{k=1}^{\infty}e^{-2ik\phi}\Delta^k(s)\right)ds \\ &= -e^{-2i\phi}\big(j_1+e^{-2i\phi}j_2+j_3\big), \end{aligned}$$

where

$$\begin{aligned} j_1 &= \frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta^3(s)\Delta(1-s)\frac{\Delta'}{\Delta}(s)ds, \\ j_2 &= \frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta^3(s)\frac{\Delta'}{\Delta}(s)\,ds, \\ j_3 &= \frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta^3(s)\frac{\Delta'}{\Delta}(s)\sum_{k=1}^{\infty}e^{-2i(k+1)\phi}\Delta^k(s)ds. \end{aligned}$$

By Gonek's Lemma 1.1.2 (with $m=1$) and Lemma 1.3.3 we have

$$j_1 = -\sum_{n\le\frac{T}{2\pi}}d_3(n)\log n\;+\;O\big(T^{\frac12+\varepsilon}\big)\;=\;-\frac{T}{2\pi}P_3\left(\log\frac{T}{2\pi}\right)\;+\;O\big(T^{\frac12+\varepsilon}\big),$$

where $P_3(x)$ is a computable polynomial of degree three.

Next, Lemma 1.1.1 with $X(s)\equiv 1$, $Y(s)\equiv 1$, applied to j_2, leads to

$$j_2 = -\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\;+\;O\big((\log T)^6\big).$$

Finally, by standard arguments we obtain

$$j_3 \ll \int_c^T\frac{\log t}{(a-1)^3}\frac{dt}{\sqrt{t}}\;\ll\;\sqrt{T}(\log T)^4.$$

Hence,

$$\mathcal{I}_1 = e^{-2i\phi}\frac{T}{2\pi}P_3\left(\log\frac{T}{2\pi}\right)\;+\;e^{-4i\phi}\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\;+\;O\big(T^{\frac12+\varepsilon}\big).$$

Further, transforming the integral $\mathcal{I}_3$ we find

$$\begin{aligned} \overline{\mathcal{I}_3} &= -\frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta(1-s)\zeta^2(s)\frac{\Delta'(1-s)\,ds}{\Delta(1-s)-e^{-2i\phi}} \\ &= -\frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta^3(s)\Delta(1-s)\frac{\Delta'}{\Delta}(s)\sum_{k=0}^{\infty}e^{-2i\phi}\Delta(s)^k ds. \end{aligned}$$

The latter expression equals $e^{2i\phi}\mathcal{I}_1$, hence we may deduce (do not forget to conjugate)

$$\mathcal{I}_3 = \frac{T}{2\pi}P_3\left(\log\frac{T}{2\pi}\right) + e^{2i\phi}\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) + O\big(T^{\frac{1}{2}+\varepsilon}\big).$$

In order to estimate the integral $\mathcal{I}_2$ over the top and bottom edges we write

$$\begin{aligned} F(s) &= \zeta^3(s)\Delta(1-s)\,\frac{\Delta'}{\Delta}(s)\,\frac{1}{\Delta(s)-e^{2i\phi}} \\ &= \zeta^3(1-s)\Delta(s)\frac{\Delta'}{\Delta}(s)\left(1+\frac{e^{2i\phi}}{\Delta(s)-e^{2i\phi}}\right). \end{aligned}$$

Since $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$ for some ν, the inequality $|\Delta(s) - e^{2i\phi}| > \frac{1}{3}$ from Paragraph 3.1.1 holds over the segment of integration. Using the bound $|\zeta(\sigma+it)| \ll t^{(1-\sigma)/3}\log t$, for $s = \sigma + iT$, $\frac{1}{2} \leqslant \sigma \leqslant a$ (see [28]), we get

$$|F(s)| \ll (\log T)|\zeta^3(s)\Delta(1-s)| \ll (\log T)\big(T^{\frac{1}{3}(1-\sigma)}\big)^3 T^{\sigma-\frac{1}{2}} \ll \sqrt{T}(\log T)^4.$$

In the case $1 - a \leqslant \sigma \leqslant \frac{1}{2}$ we have

$$|F(s)| \ll (\log T)|\zeta^3(1-s)\Delta(s)| \ll (\log T)\big(T^{\frac{1}{3}(1-(1-\sigma))}\log T\big)^3 T^{\frac{1}{2}-\sigma} \ll \sqrt{T}(\log T)^4.$$

Thus, $\mathcal{I}_2 \ll \sqrt{T}(\log T)^4$. Finally, the bound $\mathcal{I}_4 = O(1)$ is obvious. Collecting together the above results, we obtain

$$S(T) = (1+e^{-2i\phi})\frac{T}{2\pi}P_3\left(\log\frac{T}{2\pi}\right) + (e^{2i\phi}+e^{-4i\phi})\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) + O\big(T^{\frac{1}{2}+\varepsilon}\big)$$

Now it remains to note that we must multiply $S(T)$ by $e^{4i\phi}$ to obtain

$$\sum_{0<t_n(\phi)\leqslant T}\zeta^3\big(\tfrac{1}{2}+it_n(\phi)\big) = e^{4i\phi}S(T) + O(1).$$

The theorem is proved.

Remark. It is possible to compute the coefficients of the polynomial P_3 as follows. Define the polynomial $P_2(u) = A_2u^2 + A_1u + A_0$ by the relation

$$\sum_{n\leqslant x} d_3(n) = xP_2(\log x) + o(x),$$

which is a special case of the asymptotic from Lemma 1.3.3 we get

$$P_2(\log x) = \operatorname{res}_{s=1}\left(\frac{x^s\zeta^3(s)}{s}\right)$$

and hence $A_2 = \frac{1}{2}$, $A_1 = 3\gamma - 1$, $A_0 = 1 + 3(\gamma^2 - \gamma + \gamma_1)$, where $\gamma, \gamma_1, \ldots$ are the coefficients of Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + \gamma_1(s-1) + \ldots.$$

Thus, using the definition of $P_3(u)$ and Abel's summation formula, we find

$$P_3(u) = uP_2(u) - P_2(u) + P_2^{'}(u) - P_2^{''}(u) = \sum_{k=0}^{3} B_k u^k,$$

where $B_3 = A_2 = \frac{1}{2}$, $B_2 = A_1 - A_2 = 3\gamma - \frac{3}{2}$, $B_1 = A_0 - A_1 + 2A_2 = 3\big(\gamma_1 + (1-\gamma)^2\big)$ and $B_0 = -B_1 = -3\big(\gamma_1 + (1-\gamma)^2\big)$. For the values of the coefficients γ_j and P_2 we refer to [29].

□

1.3 Lower bounds

We start with the key proposition. Next we prove several statements about generalized divisor function and we finish the section with the proof of lower bounds for discrete moments.

1.3.1 The key proposition

We consider the discrete moments

$$S_1(T,\phi) = \sum_{0<t_n(\phi)\leqslant T} \zeta^{(l)}\left(\tfrac{1}{2} - it_n(\phi)\right) X\left(\tfrac{1}{2} + it_n(\phi)\right) Y\left(\tfrac{1}{2} - it_n(\phi)\right) \tag{1.16}$$

and

$$S_2(T,\phi) = \sum_{0<t_n(\phi)\leqslant T} \left|X\left(\tfrac{1}{2} + it_n(\phi) + \alpha\right)\right|^2 \tag{1.17}$$

with Dirichlet polynomials $X(s)$ and $Y(s)$ defined in (1.9). Our first aim is the following

Proposition 1.3.1. *Let $X(s)$ and $Y(s)$ be Dirichlet polynomials as defined in (1.9). Then uniformly for $\phi \in [0,\pi)$, as $T \to \infty$,*

$$\begin{aligned} S_1(T,\phi) =& e^{-2i\phi} \sum_{k=0}^{l} (-1)^{l+k} \binom{l}{k} \sum_{\substack{m\leqslant X \\ mn\leqslant Y}} \frac{(\log n)^k x_m y_{mn}}{mn} \frac{T}{2\pi} P_{l-k+1}\left(\log\frac{T}{2\pi}\right) \\ &+ (-1)^l \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) \sum_{\substack{m\leqslant Y \\ mn\leqslant X}} \frac{(\log m)^l\, y_m x_{mn}}{mn} + O(R_1), \end{aligned} \tag{1.18}$$

where $P_n(x)$ is a polynomial of degree n and

$$R_1 = (X+Y)(T^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1 + T^\epsilon \mathcal{X}_0\mathcal{Y}_0) + X^{\frac{1}{2}}Y^{\frac{1}{2}}T^{\frac{1}{6}+\epsilon}\mathcal{X}_0\mathcal{Y}_0 + T^\epsilon\mathcal{X}_0\mathcal{Y}_1;$$

moreover,

$$S_2(T,\phi) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) \sum_{n\leqslant X} \frac{|x_n|^2}{n^{1+2\mathrm{Re}\alpha}} + O(R_2), \tag{1.19}$$

where $\alpha \in \mathbb{C}$, $|\mathrm{Re}\alpha| < (\log T)^{-1}$ and

$$R_2 = X\sqrt{T}(\log T)^2 \sum_{n\leqslant X} \frac{|x_n|^2}{n} + X(\log T)^3 \mathcal{X}_0^2.$$

$S_2(T,\phi)$ is uniform with respect to α.

All implicit constants are absolute.

Proof of (1.18). We begin with the estimations

$$\begin{aligned}
&\left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right| \ll T^{1/6+\epsilon}, &(1.20)\\
&\left|X\left(\tfrac{1}{2}+it\right)\right| \leq \sum_{n\leqslant X}\frac{|x_n|}{\sqrt{n}} = \sum_{n\leqslant X}\sqrt{n}\frac{|x_n|}{n} \leq \sqrt{X}\mathcal{X}_1, \quad \left|Y\left(\tfrac{1}{2}+it\right)\right| \leq \sqrt{Y}\mathcal{Y}_1;
\end{aligned}$$

the first one is a well-known bound from zeta function theory (see [28]) and application of Cauchy's estimate for the derivatives of an analytic function applied to a small disc centered at s, whereas the second and third one are straightforward. Hence, it is sufficient to obtain (1.18) for the sum over the segment $c < t_n(\phi) \leqslant T$, where $c > 32\pi$ is a large absolute constant (32π comes from the inequality $2(\frac{t}{2\pi})^{-\frac{1}{2}} < \frac{1}{2}$ that is used in the proof).

Next, without loss of generality we may set $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$. Since the points $s = \frac{1}{2} + it_n(\phi)$ are the roots of the function $\Delta(s) - e^{2i\phi}$, the sum in question can be rewritten as a contour integral:

$$\begin{aligned}
&\sum_{c<t_n(\phi)\leqslant T} \zeta^{(l)}(\tfrac{1}{2}-it_n(\phi))X(\tfrac{1}{2}+it_n(\phi))Y(\tfrac{1}{2}-it_n(\phi))\\
&= \frac{1}{2\pi i}\int_{\square}\zeta^{(l)}(1-s)X(s)Y(1-s)\frac{\Delta'(s)}{\Delta(s)-e^{2i\phi}}ds;
\end{aligned}$$

here $\square$ stands for the contourclockwise oriented rectangular contour with vertices $a+ic$, $a+iT$, $1-a+iT$, $1-a+ic$, where $a = 1+(\log T)^{-1}$. Let $\mathcal{I}_1$ and $\mathcal{I}_3$ be integrals over right and left sides of contour, and $\mathcal{I}_2$ and $\mathcal{I}_4$ be the integrals over the top and bottom edges of the contour. We may assume the constant c so large that the relations

$$|\Delta(a+it)| = \left(\frac{t}{2\pi}\right)^{1/2-a}\left(1+O(t^{-1})\right) \leq 2\left(\frac{t}{2\pi}\right)^{-1/2} < \frac{1}{2}$$

hold for any $t > c$.

Moreover we will use the observations for $s = a+it$

$$\begin{aligned}
&|X(a+it)| \leq \sum_{n\leqslant X}\frac{|x_n|}{n^a} \leqslant \mathcal{X}_1, \quad |Y(1-a-it)| \leq \sum_{m\leqslant Y}\frac{m^a|y_m|}{m} \ll Y\mathcal{Y}_1, &(1.21)\\
&\left|\sum_{k=1}^{\infty}e^{-2ik\phi}\Delta(a+it)^k\right| \leq 2\left(\frac{t}{2\pi}\right)^{-1/2}\sum_{k=0}^{\infty}\frac{1}{2^k} \ll t^{-1/2}.
\end{aligned}$$

In view of (1.3) and Lemma 1.1.3 we have

$$\begin{aligned}
\mathcal{I}_1 =&e^{-2i\phi}(-1)^{l+1}\sum_{k=0}^{l}\binom{l}{k}\int_c^T\left(\log\frac{\tau}{2\pi}\right)^{l-k}\\
&\times d\left(\frac{1}{2\pi}\int_{a+ic}^{a+i\tau}\zeta^{(k)}(s)X(s)Y(1-s)\frac{\Delta'}{\Delta}(s)\left(1+\sum_{k=1}^{\infty}e^{-2ik\phi}\Delta(s)^k\right)ds\right)\\
&+O(YT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1),
\end{aligned}$$

where the error term comes from the application of (1.3), (1.21), and the error term of Lemma

1.1.3, i.e.

$$\frac{1}{2\pi}\int_{a+ic}^{a+iT} O(t^{-\frac{1}{2}+\epsilon})X(s)Y(1-s)\frac{\Delta'}{\Delta}(s)\left(1+\sum_{k=1}^{\infty}e^{-2ik\phi}\Delta(s)^k\right)ds$$
$$\ll YT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1.$$

Now we evaluate the measure function of $\mathcal{I}_1$. We write it as j_1+j_2, where

$$j_1=\frac{1}{2\pi i}\int_{a+ic}^{a+i\tau}\zeta^{(k)}(s)X(s)Y(1-s)\frac{\Delta'}{\Delta}(s)ds,$$
$$j_2=\frac{1}{2\pi i}\int_{a+ic}^{a+i\tau}\zeta^{(k)}(s)X(s)Y(1-s)\frac{\Delta'}{\Delta}(s)\sum_{k=1}^{\infty}e^{-2ik\phi}\Delta(s)^k ds.$$

By (1.21) we have

$$|j_2|\ll\zeta^{(k)}(a)Y\mathcal{X}_1\mathcal{Y}_1\int_c^{\tau}\frac{\log t dt}{\sqrt{t}}\ll Y\tau^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1.$$

Applying Lemma 1.1.1 to j_1 we get

$$j_1=(-1)^{k+1}\frac{\tau}{2\pi}\left(\log\frac{\tau}{2\pi e}\right)\sum_{\substack{m\leqslant X\\ mn\leqslant Y}}\frac{(\log n)^k x_m y_{mn}}{mn}+O\left(Y\tau^{\epsilon}\mathcal{X}_0\mathcal{Y}_0\right).$$

Hence

$$\mathcal{I}_1=e^{-2i\phi}\sum_{k=0}^{l}(-1)^{l+k}\binom{l}{k}\sum_{\substack{m\leqslant X\\ mn\leqslant Y}}\frac{(\log n)^k x_m y_{mn}}{mn}\frac{T}{2\pi}P_{l-k+1}\left(\log\frac{T}{2\pi}\right)$$
$$+O(YT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1+YT^{\epsilon}\mathcal{X}_0\mathcal{Y}_0+T^{\epsilon}\mathcal{X}_1\mathcal{Y}_0),$$

where

$$\frac{T}{2\pi}P_{l-k+1}\left(\log\frac{T}{2\pi}\right)+O(1)=\int_c^T\left(\log\frac{\tau}{2\pi}\right)^{l-k}d\left(\frac{\tau}{2\pi}\left(\log\frac{\tau}{2\pi e}\right)\right)$$

and $P_n(x)$ is a polynomial of degree n. The additional error term for $\mathcal{I}_1$ comes from the bound

$$\left|e^{-2i\phi}\sum_{k=0}^{l}(-1)^{l+k}\binom{l}{k}\sum_{\substack{m\leqslant X\\ mn\leqslant Y}}\frac{(\log n)^k x_m y_{mn}}{mn}\right|\ll T^{\epsilon}\mathcal{X}_1\mathcal{Y}_0.$$

In a similar way we may compute $\mathcal{I}_3$. We observe

$$\mathcal{I}_3=-\frac{1}{2\pi}\int_c^T\zeta^{(l)}(a-it)X(1-a+it)Y(a-it)\frac{\Delta'(1-a+it)}{\Delta(1-a+it)-e^{2i\phi}}dt.$$

This in combination with $\overline{X}(s)=X_1(\overline{s}),\overline{Y}(s)=Y_1(\overline{s})$ (check (1.9)) yield

$$\overline{\mathcal{I}}_3=-\frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta^{(l)}(s)X_1(1-s)Y_1(s)\frac{\Delta'(1-s)}{\Delta(1-s)-e^{-2i\phi}}ds.$$

In view of (1.3) we find

$$
\begin{aligned}
\overline{\mathcal{I}}_3 =& -\frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta^{(l)}(s)X_1(1-s)Y_1(s)\frac{\Delta'}{\Delta}(s)\bigg(1+\sum_{k=1}^{\infty}e^{-2ik\phi}\Delta(s)^k\bigg)ds\\
=& -(j_3+j_4),
\end{aligned}
$$

where

$$
\begin{aligned}
j_3 =& \frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta^{(l)}(s)X_1(1-s)Y_1(s)\frac{\Delta'}{\Delta}(s)ds\\
j_4 =& \frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta^{(l)}(s)X_1(1-s)Y_1(s)\frac{\Delta'}{\Delta}(s)\sum_{k=1}^{\infty}e^{-2ik\phi}\Delta(s)^k ds.
\end{aligned}
$$

By (1.21) we get

$$
|j_4| \ll XT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1
$$

and using Lemma 1.1.1 we find

$$
\overline{\mathcal{I}}_3 = (-1)^l\frac{T}{2\pi}\bigg(\log\frac{T}{2\pi e}\bigg)\sum_{\substack{m\leqslant Y\\ mn\leqslant X}}\frac{(\log m)^l\,\overline{y}_m\overline{x}_{mn}}{mn} + O\big(XT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1 + XT^{\epsilon}\mathcal{X}_0\mathcal{Y}_0\big).
$$

In order to estimate $\mathcal{I}_2$ we first note that the following inequalities hold along the line segment of the integration:

$$
\begin{aligned}
&|\zeta^{(l)}(1-s)| \ll T^{\frac{1}{2}+\epsilon}, \quad |X(s)| \le \sum_{n\leqslant X}\frac{|x_n|}{n}n^{1-\sigma} \ll X^{1-\sigma}\mathcal{X}_1,\\
&|Y(1-s)| \le \sum_{n\leqslant Y}\frac{|y_n|}{n}n^{\sigma} \ll Y^{\sigma}\mathcal{Y}_1,
\end{aligned}
$$

and, finally,

$$
\begin{aligned}
|\zeta^{(l)}(1-s)X(s)Y(1-s)| &\ll T^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1X\bigg(\frac{Y}{X}\bigg)^{\sigma}\\
&\ll XT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1\bigg\{\bigg(\frac{Y}{X}\bigg)^{1-a}+\bigg(\frac{Y}{X}\bigg)^{a}\bigg\}\\
&\ll (X+Y)T^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1.
\end{aligned}
$$

Next, by (1.4) we get

$$
\frac{\Delta'(s)}{\Delta(s)-e^{2i\phi}} = \frac{\Delta'(s)}{\Delta(s)}\bigg(1+\frac{e^{2i\phi}}{\Delta(s)-e^{2i\phi}}\bigg) \ll (\log T)\bigg(1+\frac{1}{|\Delta(s)-e^{2i\phi}|}\bigg).
$$

The second term in the brackets is bounded by an absolute constant. Indeed, in the case $\sigma \geqslant \frac{1}{2}+\frac{1}{3}\big(\log\frac{T}{2\pi}\big)^{-1}$ by (1.2) we have

$$
|\Delta(\sigma+iT)| = \bigg(\frac{T}{2\pi}\bigg)^{1/2-\sigma}\big(1+O(T^{-1})\big) \le e^{-1/3}\big(1+O(T^{-1})\big) < \frac{1}{2},
$$

and hence $|\Delta(s)-e^{2i\phi}|\geqslant 1-|\Delta(s)|>\frac{1}{2}$. Similarly, in the case $\sigma\leqslant\frac{1}{2}-\frac{1}{3}\big(\log\frac{T}{2\pi}\big)^{-1}$ we have

$$|\Delta(\sigma+iT)|\geq e^{1/3}\big(1+O(T^{-1})\big)>\frac{4}{3},\quad |\Delta(s)-e^{2i\phi}|>\frac{4}{3}-1=\frac{1}{3}.$$

Finally, let

$$\frac{1}{2}-\frac{1}{3}\left(\log\frac{T}{2\pi}\right)^{-1}<\sigma<\frac{1}{2}+\frac{1}{3}\left(\log\frac{T}{2\pi}\right)^{-1}.$$

Then, using the relations

$$\Delta\big(\tfrac{1}{2}+iT\big)=e^{-2i\vartheta(T)},\quad \Delta(\sigma+iT)=\tau e^{-2i\vartheta(T)}\big(1+O(T^{-1})\big),$$

where $\tau=\big(T/(2\pi)\big)^{1/2-\sigma}$ and $\vartheta=\vartheta(T)$ denotes the increment of any fixed continuous branch of the argument of $\pi^{-s/2}\Gamma\big(s/2\big)$ along the line segment with end-points $s=\frac{1}{2}$ and $s=\frac{1}{2}+iT$ (check [12]), we have $e^{-1/3}\leqslant\tau\leqslant e^{1/3}$ and

$$\begin{aligned}\Delta(\sigma+iT)-e^{2i\phi} &= \big(\Delta(\sigma+iT)-\Delta\big(\tfrac{1}{2}+iT\big)\big)+\big(\Delta\big(\tfrac{1}{2}+iT\big)-e^{2i\phi}\big)\\ &= (\tau-1)e^{-2i\vartheta}-2ie^{i(\phi-\vartheta)}\sin(\phi+\vartheta)+O(T^{-1})\\ &= e^{-i\vartheta}\big((\tau-1)\cos\vartheta+2\sin(\vartheta+\phi)\sin\phi-\\ &\quad -i((\tau-1)\sin\vartheta+2\sin(\vartheta+\phi)\cos\phi)\big)+O(T^{-1}).\end{aligned}$$

Thus we obtain

$$\begin{aligned}\big|\Delta(\sigma+iT)-e^{2i\phi}\big|^2 &= (\tau-1)^2+4\tau\sin^2(\vartheta+\phi)+O(T^{-1})\\ &\geqslant 4\tau\sin^2(\vartheta+\phi)+O(T^{-1}).\end{aligned}$$

Using the fact that $T=\frac{1}{2}(t_\nu(\phi)+t_{\nu+1}(\phi))$ for some ν, we finally get

$$\sin^2(\vartheta+\phi)=\sin^2\left(\pi\nu+\frac{\pi}{2}+O(T^{-1})\right)\geq\sin^2\frac{\pi}{3}=\frac{3}{4}$$

and hence, for sufficiently large T,

$$|\Delta(\sigma+iT)-e^{2i\phi}|^2\geq 4\cdot\frac{3}{4}e^{-1/3}+O(T^{-1})>2.$$

Thus, $|\Delta(s)-e^{2i\phi}|>\frac{1}{3}$ for any s under consideration, hence

$$\mathcal{I}_2\ll(X+Y)T^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1.$$

The integral $\mathcal{I}_4$ can be estimated in a similar way.

To loose the condition on T (i.e. $T=\frac{1}{2}(t_\nu(\phi)+t_{\nu+1}(\phi))$). We choose $T_0>T$ such that $T_0-T\ll 1$. First we notice that the contribution of the sum

$$\sum_{T<t_n(\phi)\leq T_0}\zeta^{(l)}(\tfrac{1}{2}-it_n(\phi))X(\tfrac{1}{2}+it_n(\phi))Y(\tfrac{1}{2}-it_n(\phi))$$

using (1.20) is bounded by

$$\ll X^{\frac{1}{2}}Y^{\frac{1}{2}}T^{\frac{1}{6}+\epsilon}\mathcal{X}_0\mathcal{Y}_0.$$

Next we check the error contribution from $\mathcal{I}_1$, i.e.

$$\mathcal{I}_1(T_0) = \mathcal{I}_1(T_0) - \mathcal{I}_1(T) + \mathcal{I}_1(T) = \mathcal{I}_1(T) + |\mathcal{I}_1(T_0) - \mathcal{I}_1(T)|.$$

We have

$$|\mathcal{I}_1(T_0) - \mathcal{I}_1(T)| \leq ||\mathcal{I}_1(T_0)| - |\mathcal{I}_1(T)|| \ll T^{\epsilon}\mathcal{X}_1\mathcal{Y}_0 + YT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1 + YT^{\epsilon}\mathcal{X}_0\mathcal{Y}_0,$$

since

$$\begin{aligned}||\mathcal{I}_1(T_0)| - |\mathcal{I}_1(T)|| \leq & \sum_{k=0}^{l}\binom{l}{k}\sum_{\substack{m\leqslant X\\ mn\leqslant Y}}\frac{(\log n)^k|x_m||y_{mn}|}{mn}\int_T^{T_0}\left(\log\frac{\tau}{2\pi}\right)^{l-k} d\left(\frac{\tau}{2\pi}\left(\log\frac{\tau}{2\pi e}\right)\right)\\ &+ O(YT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1 + YT^{\epsilon}\mathcal{X}_0\mathcal{Y}_0)\\ &\ll T^{\epsilon}\mathcal{X}_1\mathcal{Y}_0 + YT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1 + YT^{\epsilon}\mathcal{X}_0\mathcal{Y}_0.\end{aligned}$$

In the same way we show that the error term contribution from $\mathcal{I}_3$ is well controlled.

$$|\mathcal{I}_3(T_0) - \mathcal{I}_3(T)| \ll T^{\epsilon}\mathcal{Y}_0\mathcal{X}_1 + XT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1 + XT^{\epsilon}\mathcal{X}_0\mathcal{Y}_0.$$

The relation (1.18) is proved.

□

Proof of (1.19). In view of the inequalities

$$\left|X\left(\tfrac{1}{2} + it_n(\phi) + \alpha\right)\right|^2 \leq \left(\sum_{n\leqslant X}\frac{|x_n|}{n^{\frac{1}{2}+\mathrm{Re}\alpha}}\right)^2 \leq X\sum_{n\leqslant X}\frac{|x_n|^2}{n^{1+2\mathrm{Re}\alpha}}$$

it suffices to consider only the sum over the segment $c < t_n(\phi) \leqslant T$. Next, we may set $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$. Then we have

$$\begin{aligned}\sum_{c<t_n(\phi)\leqslant T}\left|X\left(\tfrac{1}{2} + it_n(\phi) + \alpha\right)\right|^2 &= \sum_{c<t_n(\phi)\leqslant T} X\left(\tfrac{1}{2} + it_n(\phi) + \alpha\right)X_1\left(\tfrac{1}{2} - it_n(\phi) + \overline{\alpha}\right)\\ &= \frac{1}{2\pi i}\int_{\square} X(s+\alpha)X_1(1-s+\overline{\alpha})\frac{\Delta'(s)}{\Delta(s) - e^{2i\phi}}\,ds,\end{aligned}$$

where □ stands for the rectangular contour defined in Section before. Denoting the integrals $\mathcal{I}_k$, $1 \leqslant k \leqslant 4$ as in Section 1.3.1, we get

$$\begin{aligned}\mathcal{I}_1 &= \frac{1}{2\pi i}\int_{a+ic}^{a+iT} X(s+\alpha)X_1(1-s+\overline{\alpha})\frac{\Delta'(s)}{\Delta(s) - e^{2i\phi}}ds\\ &= -\frac{1}{2\pi i}\int_{a+ic}^{a+iT} X(s+\alpha)X_1(1-s+\overline{\alpha})\frac{\Delta'(s)}{\Delta(s)}\sum_{k=1}^{\infty} e^{-2ik\phi}\Delta(s)^k\,ds.\end{aligned}$$

Estimating the integrand as in Section 1.3.1 we find

$$\mathcal{I}_1 \ll X\mathcal{X}_1^2\int_c^T\frac{\log t\,dt}{\sqrt{t}} \ll X\sqrt{T}(\log T)\mathcal{X}_1^2 \ll X\sqrt{T}(\log T)^2\sum_{n\leqslant X}\frac{|x_n|^2}{n}.$$

Furthermore,

$$\mathcal{I}_3 \;=\; -\frac{1}{2\pi}\int_c^T X(1-a+it+\alpha)X_1(a-it+\overline{\alpha})\frac{\Delta'(1-a+it)}{\Delta(1-a+it)-e^{2i\phi}}dt,$$

and the relations $\overline{X}(s) = X_1(\overline{s})$, $\overline{X}_1(s) = X(\overline{s})$ imply

$$\begin{aligned}\overline{\mathcal{I}}_3 \;&=\; -\frac{1}{2\pi i}\int_{a+ic}^{a+iT} X(s+\alpha)X_1(1-s+\overline{\alpha})\frac{\Delta'(1-s)}{\Delta(1-s)-e^{-2i\phi}}ds \\ &=\; -\frac{1}{2\pi i}\int_{a+ic}^{a+iT} X(s+\alpha)X_1(1-s+\overline{\alpha})\frac{\Delta'}{\Delta}(s)\,\frac{ds}{1-e^{-2i\phi}\Delta(s)} \;=\; -j_1+\mathcal{I}_1,\end{aligned}$$

where

$$j_1 \;=\; \frac{1}{2\pi i}\int_{a+ic}^{a+iT} X(s+\alpha)X_1(1-s+\overline{\alpha})\frac{\Delta'}{\Delta}(s)\,ds.$$

Lemma 1.1.1 with $f(s)\equiv 1$ (that is, $\alpha_1 = 1$, $\alpha_n = 0$ for $n > 1$), $\gamma = 0$ and $Y(s) = X_1(s)$, applied to j_1 yields

$$j_1 \;=\; -\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\sum_{n\leqslant X}\frac{|x_n|^2}{n^{1+2\mathrm{Re}\alpha}} \;+\; O\big(X(\log T)^3\mathcal{X}_0^2\big).$$

Using the above bound for $\mathcal{I}_1$, we derive

$$\mathcal{I}_3 \;=\; \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\sum_{n\leqslant X}\frac{|x_n|^2}{n^{1+2\mathrm{Re}\alpha}} \;+\; O\bigg(X\sqrt{T}(\log T)^2\sum_{n\leqslant X}\frac{|x_n|^2}{n}\bigg) \;+\; O\big(X(\log T)^3\mathcal{X}_0^2\big).$$

Estimating $\mathcal{I}_2$ and taking into account the bounds

$$X(s+\alpha) \;\ll\; X^{1-\sigma-\mathrm{Re}\alpha}\mathcal{X}_1, \quad X_1(1-s+\overline{\alpha}) \;\ll\; X^{\sigma-\mathrm{Re}\alpha}\mathcal{X}_1, \quad \frac{\Delta'(s)}{\Delta(s)-e^{2i\phi}} \;\ll\; \log T$$

for $s = \sigma+iT$, $1-a \le \sigma \leqslant a$, $T = \frac{1}{2}(t_\nu(\phi)+t_{\nu+1}(\phi))$, we get

$$\mathcal{I}_2 \;\ll\; X^{1-2\mathrm{Re}\alpha}(\log T)\mathcal{X}_1^2 \;\ll\; X(\log T)^2\sum_{n\leqslant X}\frac{|x_n|^2}{n}.$$

The integral $\mathcal{I}_4$ can be estimated in a similar way.

To loose the condition on T (i.e. $T = \frac{1}{2}(t_\nu(\phi)+t_{\nu+1}(\phi))$). We choose $T_0 > T$ such that $T_0 - T \ll 1$. First we notice that the contribution of the interval $(T, T_0]$ is bounded

$$\sum_{T<t_n(\phi)\leqslant T_0}\left|X\left(\tfrac{1}{2}+it_n(\phi)+\alpha\right)\right|^2 \ll X\log T\sum_{n\leqslant X}\frac{|x_n|^2}{n^{1+2\mathrm{Re}\alpha}}.$$

Next we notice that

$$\mathcal{I}_1(T_0) = \mathcal{I}_1(T) + ||\mathcal{I}_1(T_0)| - |\mathcal{I}_1(T)||$$

and

$$||\mathcal{I}_1(T_0)| - |\mathcal{I}_1(T)|| \ll X\log T\sum_{n\leqslant X}\frac{|x_n|^2}{n}.$$

For $\mathcal{I}_3$ we have

$$\mathcal{I}_3(T_0) = \mathcal{I}_3(T) + ||\mathcal{I}_3(T_0)| - |\mathcal{I}_3(T)||$$

and

$$||\mathcal{I}_3(T_0)| - |\mathcal{I}_3(T)|| \ll X \log T \sum_{n \leqslant X} \frac{|x_n|^2}{n} + X(\log T)^3 \mathcal{X}_0^2.$$

For $\mathcal{I}_2$ we have

$$||\mathcal{I}_2(T_0)| - |\mathcal{I}_2(T)|| \ll X(\log T)^2 \sum_{n \leqslant X} \frac{|x_n|^2}{n}.$$

Thus, formula (1.19) is proved.

□

1.3.2 Generalized divisor function

The properties of Generalized divisor function is used to prove the lower bounds for the discrete moments.

In the next four Lemmas, we will gather several properties of the generalized κ-th divisor function (check [25, Section 2]), which we will use in the sequel. Let κ be a positive real number. The generalized κ-th divisor function $d_\kappa : \mathbb{N} \to \mathbb{R}$ is given by the coefficients $d_\kappa(n)$ of

$$\zeta(s)^\kappa = \sum_{n=1}^{\infty} d_\kappa(n) n^{-s}, \qquad \sigma > 1,$$

where $d_\kappa(n)$ is multiplicative and on prime powers is defined by

$$d_\kappa(p^j) = \frac{\Gamma(\kappa + j)}{\Gamma(\kappa) j!}.$$

If κ is a positive integer the definition above coincides with the definition of the divisor function

$$d_\kappa(n) = \sum_{\substack{n_1, \ldots, n_\kappa \in \mathbb{N} \\ n_1 \cdots n_\kappa = n}} 1.$$

The generalized κ-th divisor function satiesfies the following properties:

Lemma 1.3.2. *Let κ be a positive real number and n a positive integer.*

1. *For $\kappa \geq 0$ and $n \geq 1$,we have $d_\kappa(n) \geq 0$.*
2. *For fixed n, $d_\kappa(n)$ increases with respect to κ.*
3. *For fixed $\kappa \geq 0$ and $\epsilon > 0$, we have $d_\kappa(n) \ll n^\epsilon$.*
4. *If j is an integer, then*

$$d_{\kappa j}(n) = \sum_{n = n_1 n_2 \ldots n_j} d_\kappa(n_1) d_\kappa(n_2) \ldots d_\kappa(n_j).$$

For a proof, we refer to [25, Lemma 1].

Lemma 1.3.3. *Let λ, μ be fixed positive real numbers. Then,*

$$\sum_{n \leq x} d_\lambda(n) d_\mu(n) \asymp_{\lambda,\mu} x(\log x)^{\lambda\mu - 1}$$

and, thus,

$$\sum_{n \leq x} d_\lambda(n) d_\mu(n) n^{-1} \asymp_{\lambda,\mu} (\log x)^{\lambda\mu}.$$

The assertions of Lemma 1.3.3 can be established by standard techniques based on Perron's formula, contour integration and Abel's summation.

Next we use Euler totient function that is defined by

$$\varphi(m) = \sum_{\substack{n \le m \\ (n,m)=1}} 1.$$

The following Lemma states

Lemma 1.3.4. *Let λ, μ be fixed positive real numbers. Then,*

$$\sum_{m \le x} d_\lambda(m) d_\mu(m) \left(\frac{\varphi(m)}{m} \right)^\mu \asymp_{\lambda,\mu} x(\log x)^{\lambda\mu - 1}$$

and, thus,

$$\sum_{m \le x} d_\lambda(m) d_\mu(m) \left(\frac{\varphi(m)}{m} \right)^\mu m^{-1} \asymp_{\lambda,\mu} (\log x)^{\lambda\mu}.$$

The assertions of Lemma 1.3.4 can be established by standard techniques based on Perron's formula, contour integration and Abel's summation.

Lemma 1.3.5. *For an arbitrary rational $k = \frac{p}{q} \ge 0$, $m \le x^{\frac{1}{2p}}$ and x sufficiently large, we have*

$$\sum_{\substack{n \le x \\ (m,n)=1}} \frac{d_k(n)}{n} \ge \left(\frac{1}{p} \frac{\phi(m)}{m} \log x \right)^k.$$

Proof. Let $k = \frac{p}{q}$ be a non-negative rational number. We consider the sum

$$W := \sum_{\substack{n \le \xi \\ (m,n)=1}} \frac{d_{\frac{1}{q}}(n)}{n}.$$

Taking q-th power, we get

$$W^q = \sum_{\substack{n \le \xi^q \\ (m,n)=1}} \frac{d_1(n,\xi)}{n},$$

where the coefficients $d_1(n,\xi)$ are given by

$$d_1(n,\xi) = \sum_{\substack{n_1 n_2 \cdots n_q = n \\ n_1, n_2, \ldots, n_q \le \xi}} d_{\frac{1}{q}}(n_1) d_{\frac{1}{q}}(n_2) \cdots d_{\frac{1}{q}}(n_q).$$

Note, that, as q is an integer, we have, according to property (4) of Lemma 1.3.2,

$$\sum_{n_1 n_2 \cdots n_q = n} d_{\frac{1}{q}}(n_1) d_{\frac{1}{q}}(n_2) \cdots d_{\frac{1}{q}}(n_q) = d_{\frac{1}{j} \cdot j}(n) = d_1(n) \equiv 1$$

for all positve integers n. Hence, we can easily deduce that

$$d_1(n,\xi) = d_1(n) = 1 \quad \text{if} \quad n \le \xi$$

and

$$d_1(n,\xi) \le d_1(n) = 1 \quad \text{if} \quad n > \xi.$$

Thus, we get

$$\sum_{\substack{n\le\xi\\(m,n)=1}} \frac{1}{n} \le W^q \le \sum_{\substack{n\le\xi^q\\(m,n)=1}} \frac{1}{n} \le 2q\frac{\phi(m)}{m}\log\xi$$

Using the inequality

$$\frac{\phi(m)}{m}\log\xi \le \sum_{\substack{n\le\xi\\(m,n)=1}} \frac{1}{n} \le 2\frac{\phi(m)}{m}\log\xi,$$

which is valid for $m \le \xi^{\frac{1}{2}}$ and ξ sufficiently large and which can be established by standard techniques. We get

$$\frac{\phi(m)}{m}\log\xi \le W^q \le 2q\frac{\phi(m)}{m}\log\xi$$

for $m \le \xi^{\frac{1}{2}}$. Therefore,

$$\left(\frac{\phi(m)}{m}\log\xi\right)^{\frac{1}{q}} \le W \le \left(2q\frac{\phi(m)}{m}\log\xi\right)^{\frac{1}{q}} \tag{1.22}$$

for $m \le \xi^{\frac{1}{2}}$. Taking the p-th power of W yields

$$W^p = \sum_{\substack{n\le\xi^p\\(m,n)=1}} \frac{d_{\frac{p}{q}}(n,\xi)}{n}$$

with coefficients

$$d_{\frac{p}{q}}(n,\xi) = \sum_{\substack{n_1n_2\cdots n_p=n\\n_1,n_2,\ldots,n_p\le\xi}} d_{\frac{1}{q}}(n_1)d_{\frac{1}{q}}(n_2)\cdots d_{\frac{1}{q}}(n_p).$$

By the same reasoning as above, we obtain

$$\sum_{\substack{n\le\xi\\(m,n)=1}} \frac{d_{\frac{p}{q}}(n)}{n} \le W^p \le \sum_{\substack{n\le\xi^p\\(m,n)=1}} \frac{d_{\frac{p}{q}}(n)}{n}$$

Using the upper bound for W^p from the above inequality and the lower bound for W from (1.22), we get

$$\sum_{\substack{n\le\xi^p\\(m,n)=1}} \frac{d_k(n)}{n} = \sum_{\substack{n\le\xi^p\\(m,n)=1}} \frac{d_{\frac{p}{q}}(n)}{n} \ge W^p \ge \left(\frac{\phi(m)}{m}\log\xi\right)^{\frac{p}{q}}.$$

for $m \le \xi^{\frac{1}{2}}$. Setting $x = \xi^p$, yields the assertion of the Lemma for $m \le x^{\frac{1}{2p}}$. □

Lemma 1.3.6. *Let l be a non-negative integer, r and k non-negative rational numbers. Then*

$$\sum_{\substack{m\leqslant x\\mn\leqslant x}} \frac{(\log m)^l d_r(m)d_k(mn)}{mn} \gg_{l,k,r} (\log x)^{l+kr+k}.$$

Proof. Let $k = \frac{p}{q} \ge 0$ be a rational number. We consider the sum

$$W := \sum_{\substack{m\leqslant x\\mn\leqslant x}} \frac{(\log m)^l d_r(m)d_k(mn)}{mn} = \sum_{m\le x} \frac{(\log m)^l d_r(m)}{m} \sum_{n\leqslant \frac{x}{m}} \frac{d_k(mn)}{n}$$

Certainly, the following estimates hold

$$W \geq \sum_{m\leq x} \frac{(\log m)^l d_r(m) d_k(m)}{m} \sum_{\substack{n\leqslant \frac{x}{m}\\ (m,n)=1}} \frac{d_k(n)}{n}$$
$$\geq \sum_{x^{\frac{1}{3p+1}} \leq m \leq x^{\frac{1}{2p+1}}} \frac{(\log m)^l d_r(m) d_k(m)}{m} \sum_{\substack{n\leqslant x^{\frac{2p}{2p+1}}\\ (m,n)=1}} \frac{d_k(n)}{n}.$$

Now, Lemma 1.3.5 yields

$$W \geq (3p+1)^{-l} \left(p+\tfrac{1}{2}\right)^{-k} (\log x)^{l+k} \sum_{x^{\frac{1}{3p+1}} \leq m \leq x^{\frac{1}{2+1}}} \frac{d_r(m) d_k(m)}{m} \left(\frac{\phi(m)}{m}\right)^k.$$

By Lemma 1.3.4, we get

$$W \gg_{k,l,r} (\log x)^{l+kr+k}$$

and the Lemma is proved.

□

1.3.3 Lower bounds

Theorem 1.3.7. *For any rational $k \geqslant 1$ and any non-negative integer l, uniformly for $\phi \in [0,\pi)$, as $T\to\infty$,*

$$\sum_{0<t_n(\phi)\leqslant T} \left|\zeta^{(l)}\left(\tfrac{1}{2}+it_n(\phi)\right)\right|^{2k} \gg T(\log T)^{k^2+2kl+1}.$$

Proof. Suppose that $k=\frac{p}{q}$ is a rational number with $p>q\geqslant 1$ and $(p,q)=1$. Let l be a non-negative integer. We set $r:=p-q$ and choose $\xi := T^{1/(4p)}$. First, we define fixed coefficients for the Dirichlet polynomials $X(s)$ and $Y(s)$ in (1.9) via

$$X(s) = \left(\sum_{n\leqslant\xi} \frac{d_{\frac{1}{q}}(n)}{n^s}\right)^p = \sum_{n\leqslant\xi^p} \frac{d_{\frac{p}{q}}(n;\xi)}{n^s}, \quad Y(s) = \left(\sum_{n\leqslant\xi} \frac{d_{\frac{1}{q}}(n)}{n^s}\right)^r = \sum_{n\leqslant\xi^r} \frac{d_{\frac{r}{q}}(n;\xi)}{n^s},$$

where $d_{\frac{m}{q}}(n;\xi)$ is given by

$$d_{\frac{m}{q}}(n;\xi) = \sum_{\substack{n=n_1\cdots n_m\\ n_1,\ldots,n_m\leqslant\xi}} d_{\frac{1}{q}}(n_1)\ldots d_{\frac{1}{q}}(n_m).$$

for $m=p,r$. From property (5) of Lemma 1.3.2 we can easily deduce that $d_{\frac{m}{q}}(n;\xi) = d_{\frac{m}{q}}(n)$ for $m\leqslant\xi$ and $0\leqslant d_{\frac{m}{q}}(n,\ \xi)\leqslant d_{\frac{m}{q}}(n)$ for $m>\xi$. Now, we consider the moment $S_1(T,\varphi)$ given by (1.16) with respect to the above chosen Dirichlet polynomials $X(s)$ and $Y(s)$. By statement (1.18) in Proposition 1.3.1, we have

$$S_1(T,\phi) = \sum_{j=0}^{l} (-1)^{l+j} \binom{l}{j} \frac{T}{2\pi} P_{l-j+1}\left(\log\frac{T}{2\pi}\right)\Sigma_1 + \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\Sigma_2 + O(R_1).$$

By Lemma 1.3.3,

$$\begin{aligned}\Sigma_1 &= \sum_{m\leqslant\xi^p, mn\leqslant\xi^r} \frac{(\log n)^j d_{\frac{p}{q}}(m;\xi) d_{\frac{r}{q}}(mn;\xi)}{mn} \\ &\leq (\log\xi^r)^j \sum_{n\leq\xi^r} \frac{d_{\frac{r}{q}}(n)}{n} \sum_{l|n} d_{\frac{p}{q}}(l) \\ &= (\log\xi^r)^j \sum_{n\leqslant\xi^r} \frac{d_{\frac{r}{q}}(n) d_{\frac{p}{q}+1}(n)}{n} \\ &\ll (\log T)^{(\frac{p}{q})^2-1+j},\end{aligned}$$

and by Lemma 1.3.6,

$$\begin{aligned}\Sigma_2 = \sum_{m\leqslant\xi^r, mn\leqslant\xi^p} \frac{(\log m)^l d_{\frac{r}{q}}(m;\xi) d_{\frac{p}{q}}(mn;\xi)}{mn} &\geq \sum_{m\leqslant\xi} \frac{(\log m)^l d_{\frac{r}{q}}(m) d_{\frac{p}{q}}(mn)}{mn} \\ &\gg (\log\xi)^{(\frac{p}{q})^2+l}.\end{aligned}$$

The error term of $S_1(T,\phi)$ is bounded by

$$\begin{aligned}R_1 &\ll (\xi^p+\xi^r) T^{\frac{1}{2}+\epsilon} \sum_{n\leqslant\xi^p} \frac{d_{\frac{p}{q}}(n;\xi)}{n} \sum_{m\leqslant\xi^r} \frac{d_{\frac{r}{q}}(m;\xi)}{m} + \xi^p\xi^r T^{\frac{1}{6}+\epsilon} \\ &\ll T^{3/4+\epsilon} \ll T^{4/5}.\end{aligned}$$

Thus, we obtain

$$|S_1(T,\phi)| \gg T(\log T)^{k^2+l+1}.$$

On the contrary, Hölder's inequality assures

$$\begin{aligned}|S_1(T,\phi)| &\leq \Bigg(\sum_{0<t_n(\phi)\leqslant T} \big|\zeta^{(l)}\big(\tfrac{1}{2}+it_n(\phi)\big)\big|^{2k}\Bigg)^{1/(2k)} \times \\ &\quad \times\Bigg(\sum_{0<t_n(\phi)\leqslant T} \big|X\big(\tfrac{1}{2}+it_n(\phi)\big)\big|^{2k/(2k-1)} \cdot \big|Y\big(\tfrac{1}{2}+it_n(\phi)\big)\big|^{2k/(2k-1)}\Bigg)^{1-1/(2k)} \\ &= \Bigg(\sum_{0<t_n(\phi)\leqslant T} \big|\zeta^{(l)}\big(\tfrac{1}{2}+it_n(\phi)\big)\big|^{2k}\Bigg)^{1/(2k)} \big(S_2(T,\phi)\big)^{1-1/(2k)},\end{aligned}$$

where $S_2(T,\phi)$ is given by (1.17). By statement (1.19) of Proposition 1.3.1 and Corollary 1.3.3, we find that

$$|S_2(T,\phi)| = \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right) \sum_{n\leqslant\xi^p} \frac{d^2_{\frac{p}{q}}(n;\xi)}{n} + O\big(\xi^p\sqrt{T}(\log T)^{k^2+1}\big) \ll T(\log T)^{k^2+1}.$$

Hence,

$$\sum_{0<t_n(\phi)\leqslant T} \big|\zeta\big(\tfrac{1}{2}+it_n(\phi)\big)\big|^{2k} \geq \frac{\big(S_1(T,\phi)\big)^{2k}}{\big(S_2(T,\phi)\big)^{2k-1}} \gg T(\log T)^{k^2+2kl+1}.$$

Theorem 1.3.7 is proved.

□

1.4 Upper bounds

The goal of this section is to prove the following theorem

Theorem 1.4.1. *Assume the Riemann Hypothesis.*
For $l = 0$ and any non-negative real k, uniformly for $\phi \in [0, \pi)$, as $T \to \infty$,

$$\sum_{0<t_n(\phi)\leqslant T} \left|\zeta^{(l)}\left(\tfrac{1}{2} + it_n(\phi)\right)\right|^{2k} \ll T(\log T)^{k^2+1+\epsilon}.$$

For any non-negative integer k and any positive integer l, uniformly for $\phi \in [0, \pi)$, as $T \to \infty$,

$$\sum_{0<t_n(\phi)\leqslant T} \left|\zeta^{(l)}\left(\tfrac{1}{2} + it_n(\phi)\right)\right|^{2k} \ll T(\log T)^{k^2+2kl+1+\epsilon}.$$

1.4.1 Outline of the proof

To prove the conditional upper bound for the discrete moments $S_{k,l}(T, \phi)$ we follow the ideas of Milinovich [35]. The latter established, under the assumption of the Riemann hypothesis, an upper bound for discrete moments connected to the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of the Riemann zeta function: *Assuming the Riemann hypothesis, for any positive integers k and l, as $T \to \infty$,*

$$\sum_{0<\gamma\leq T} |\zeta^{(l)}(\rho)|^{2k} \ll T(\log T)^{k^2+2kl+\varepsilon}. \tag{1.23}$$

As we will refer in big parts of the proof of Theorem 1.4.1 to Milinovich's work, we will briefly sketch his approach.

To proof (1.23), Milinovich relies on methods introduced by Soundararajan [44]: Assuming the Riemann hypothesis, Soundararajan gave upper bounds for meas$\{t \in [0, T] \mid \log|\zeta(\frac{1}{2} + it)| \geq V\}$ subject to $V \in (-\infty, \infty)$ and, thus, roughly speaking, extended a result due to Selberg (unpublished) which states that after a suitable normalization the values of $\log|\zeta(\frac{1}{2} + it)|$ are Gaussian normal distributed (the first published proof is due to Joyner [30]). These bounds allowed Soundararajan to deduce the conditional upper bound (3) for $I_k(T)$. Milinovich adjusted Soundararajan's approach to a discrete setting: under the assumption of the Riemann hypothesis, he approximated $\log^+|\zeta(s)|$ on and to the right of the critical line by rather short Dirichlet polynomials; here and in the sequel $\log^+|x|$ is defined by

$$\log^+|x| = \begin{cases} 0 & \text{if } |x| \leq 1, \\ \log|x| & \text{if } |x| > 1. \end{cases}$$

In particular, he proved the following

Lemma 1.4.2 (Milinovich [35], Lemma 3.1)**.** *Assume the Riemann Hypothesis. Let $\tau = |t| + e^{30}$ and $2 \leq x \leq \tau^2$. Let $\lambda_0 = 0.5671\ldots$ be the unique positive real number satisfying $e^{-\lambda_0} = \lambda_0$. Then, for $\sigma \geq \frac{1}{2}$, $2 \leq x \leq \tau^2$, and any $\lambda_0 \leq \lambda \leq \frac{\log x}{4}$ the estimate*

$$\log^+|\zeta(\sigma + it)| \leq \left| \sum_{\substack{p\leq x \\ p-prime}} \frac{1}{p^{\sigma+\frac{\lambda}{\log x}+it}} \frac{\log(x/p)}{\log x} \right| + \frac{(1+\lambda)}{2}\frac{\log\tau}{\log x} + O(\log\log\log\tau)$$

holds uniformly for $\frac{1}{2} \leq \sigma \leq \sigma_\lambda$, where $\sigma_\lambda = \frac{1}{2} + \frac{\lambda}{\log x}$.

By using a result of Gonek [20, 21], Milinovich could estimate high powers of the Dirichlet polynomial occuring in Lemma 1.4.2, if they were averaged over the non-trivial zeros. By means of a certain power technique, he was able to derive upper bounds for the cardinality of discrete sets

$$S_\rho(T,V) = \{\rho + \alpha \in [0,T] \mid \log|\zeta(\rho+\alpha)| \geq V\}$$

subject to $V \in (-\infty, \infty)$, for any fixed complex α with $|\alpha| < 1$ and $0 \leq \mathrm{Re}\alpha \leq (\log T)^{-1}$. These upper bounds for $\#S_\rho(T,V)$ allowed him to deduce the following estimate for discrete moments with respect to shifted zeros $\rho + \alpha$: *Assuming the Riemann hypothesis, for any positive real number k and any complex α with $|\alpha| < 1$ and $|\mathrm{Re}\alpha| \leq (\log T)^{-1}$, as $T \to \infty$,*

$$\sum_{0<\gamma\leq T} |\zeta(\rho+\alpha)|^{2k} \ll T(\log T)^{k^2+2k+\varepsilon}, \tag{1.24}$$

holds uniformly in α. Relying on Cauchy's integral formula, Milinovich [35] could immediatelly deduce (1.23) from (1.24).

We can adopt Milinovich's method to our case. This is essentially due to the fact that we get similar bounds for high powers of the Dirichlet polynomial approximating $\log^+|\zeta(s)|$ if we do not average over zeros ρ or shifted zeros $\rho + \alpha$, but if we average over generalized Gram points $\frac{1}{2} + it_n(\phi)$ or shifted generalized Gram points $\frac{1}{2} + \alpha + it_n(\phi)$. This is natural, in some sense, as, by comparing $N_\phi(T)$ with the Riemann-von Mangoldt formula, the points $\frac{1}{2} + it_n(\phi)$ are asymptotically similar distributed on the critical line as the non-trivial zeros $\frac{1}{2} + i\gamma$ under the assumption of the Riemann hypothesis. However, to obtain these bounds for high powers of Dirichlet polynomials averaged over the points $t_n(\phi)$, we will use a method different from the one Millinovich used in his case.

1.4.2 Frequency of large values

Relying on Proposition 1.3.1 formula (1.19), we are able to measure the frequency of large values of $\log|\zeta(\frac{1}{2} + it_n(\phi) + \alpha)|$. In the sequel we will use the notation $\log_3 x := \log\log\log x$.

Lemma 1.4.3. *Assume the Riemann Hypothesis. Let T be large, $V \geq 3$ a real number, $\Phi \in [0,\pi)$ and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $0 \leq \mathrm{Re}\alpha \leq (\log T)^{-1}$. Define the set*

$$S_\phi(T,V) := \{t_n(\phi) \in [0,T] : \log|\zeta(\tfrac{1}{2} + it_n(\phi) + \alpha)| \geq V\}.$$

Then, the following bounds for the cardinality $\#S_\phi(T,V)$ of the set $S_\phi(T,V)$ hold uniformly in Φ and α:

(i) If $\sqrt{\log\log T} \leq V \leq \log\log T$, then

$$\#S_\phi(T,V) \ll T(\log T)\frac{V}{\sqrt{\log\log T}} \exp\left(-\tfrac{V^2}{\log\log T}\left(1 - \tfrac{4}{\log_3 T}\right)\right).$$

(ii) If $\log\log T < V \leq \frac{1}{2}(\log\log T)\log_3 T$, then

$$\#S_\phi(T,V) \ll T(\log T)\frac{V}{\sqrt{\log\log T}} \exp\left(-\tfrac{V^2}{\log\log T}\left(1 - \tfrac{9V}{5\log\log T\log_3 T}\right)^2\right).$$

(iii) If $V \geq \frac{1}{2} \log\log T \log_3 T$*, then*

$$\#S_\phi(T,V) \ll T(\log T) \exp\left(-\tfrac{V}{201}\log V\right).$$

Note that in the cases (i) and (iii) the bounds for $\#S_\phi(T,V)$ are the same as the ones for $\#S_\rho(T,V)$ in Lemma 5.1 of [35]. In case (ii), we have

$$\begin{aligned}\#S_\phi(T,V) &\ll T(\log T)\frac{V}{\sqrt{\log\log T}} \exp\left(-\tfrac{V^2}{\log\log T}\left(1-\tfrac{9V}{5\log\log T\log_3 T}\right)^2\right) \\ &\ll T(\log T)\frac{V}{\sqrt{\log\log T}} \exp\left(-\tfrac{V^2}{\log\log T}\left(1-\tfrac{4V}{\log\log T\log_3 T}\right)\right).\end{aligned}$$

This second bound is the same as the bound for $\#S_\rho(T,V)$ in case (ii) of Lemma 5.1 of [35]. Our motivation to state Lemma 1.4.3 with a stronger bound for case (ii) is that it will help us to give a straightforward bound for Σ_2 in (1.25). Note that Milinovich's proof of his Lemma 5.1 directly implies this stronger bound.

Proof of Lemma 1.4.3. In the sequel p denotes always a prime number. We set $x := T^a$ with some $0 < a \leq \frac{1}{2}$ and put $z = x^b$ with $\frac{1}{\log\log T} \leq b \leq 1$. We define

$$\mathcal{S}(s) = \sum_{\substack{p \leq z \\ p-prime}} \frac{1}{p^s}\frac{\log(x/p)}{\log x}$$

Basically, with some smart technical refinements, the result can be derived from Lemma 1.4.2 and the inequality

$$\#N(T,V)V^{2k} \leq \sum_{0<t_n(\phi)\leq T} |\mathcal{S}(\tfrac{1}{2} + it_n(\phi) + \tfrac{\lambda_0}{\log x})|^{2k};$$

here $\#N(T,V)$ measures the cardinality of the set

$$N(T,V) = \{t_n(\phi) \in [T,2T] \bigm| |\mathcal{S}((\tfrac{1}{2} + it_n(\phi) + \tfrac{\lambda_0}{\log x})| \geq V\}.$$

For details we refer to [35].

We define the sequence $\alpha_k(n)$ by

$$\sum_{n \leq z^k} \frac{\alpha_k(n)}{n^s} = \left(\sum_{p\leq z}\frac{1}{p^s}\frac{\log(x/p)}{\log p}\right)^k.$$

It can be easily seen that $|\alpha_k(n)| \leq k!$. According to Proposition 1.3.1 we get for T large enough, for any positive integer k with $z^k \leq T$, uniformly for $\phi \in [0,\pi)$

$$\begin{aligned}\sum_{0<t_n(\phi)\leq T} |\mathcal{S}(\tfrac{1}{2}+it_n(\phi)+\tfrac{\lambda_0}{\log x})|^{2k} &\ll \frac{T}{2\pi}\log\frac{T}{2\pi e}\sum_{0<n\leq z^k}\frac{|\alpha_k(n)|^2}{n} \\ &\ll \frac{T}{2\pi}\log\frac{T}{2\pi e}k!\left(\sum_{0<p\leq z}\frac{1}{p}\frac{\log(x/p)}{\log x}\right)^k \\ &\ll \frac{T}{2\pi}\log\frac{T}{2\pi e}k!\left(\sum_{0<p\leq z}\frac{1}{p}\right)^k.\end{aligned}$$

Using Stirling's formula $k! \ll k^{\frac{1}{2}}(k/e)^k$ and the estimate $\sum_{p\leq z}p^{-1} \ll \log\log z$, we find that

$$\begin{aligned}\sum_{0<t_n(\phi)\leq T} |\mathcal{S}(\tfrac{1}{2}+it_n(\phi)+\tfrac{\lambda_0}{\log x})|^{2k} &\ll T(\log T)k^{\frac{1}{2}}\left(\frac{kb\log\log T}{e}\right)^k \\ &\ll T(\log T)\left(kb\log\log T\right)^k\end{aligned}$$

holds for T large enough, for any positive integer k with $z^k \leq T$, uniformly for $\phi \in [0,\pi)$. These are the same bounds which Milinovich [35] uses for $\mathcal{S}(\rho+\frac{\lambda_0}{\log x})$. Thus, the proof of Lemma 1.4.3 follows exactly the lines of the proof of Lemma 5.1 in [35] by just replacing ρ by $\frac{1}{2}+it_n(\phi)$. □

1.4.3 Upper bounds

Using Lemma 1.4.3 we are now able to prove the following Proposition.

Proposition 1.4.4. *Assume the Riemann Hypothesis. Let $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $|\mathrm{Re}\ \alpha| \leq (\log T)^{-1}$. For any positive real number k, uniformly for $\phi \in [0,\pi)$ and uniformly in α, as $T \to \infty$,*

$$\sum_{0<t_n(\phi)\leqslant T}\left|\zeta\left(\tfrac{1}{2}+it_n(\phi)+\alpha\right)\right|^{2k} \ll T(\log T)^{k^2+1+\epsilon}.$$

Proof. We follow the proof of Theorem 1.2 in [35]. First, we consider the case when $0 \leq \mathrm{Re}\ \alpha \leq (\log T)^{-1}$. Then, from this result, the case $-(\log T)^{-1} \leq \mathrm{Re}\ \alpha < 0$ can be derived via the functional equation of the Riemann zeta function.

Let $k \in \mathbb{R}$ be fixed. We partition the real axis into the intervals $I_1 = (-\infty, \sqrt{\log\log T}]$, $I_2 = (\sqrt{\log\log T}, 4k\log\log T]$ and $I_3 = (4k\log\log T, \infty)$. We set

$$\Sigma_i = \sum_{\nu\in I_i\cap\mathbb{Z}} e^{2k\nu}\#S_\phi(T,\nu) \qquad \text{for } i=1,2,3.$$

and observe that

$$\begin{aligned}\sum_{0<t_\phi(n)\leq T}\left|\zeta\left(\tfrac{1}{2}+it_n(\phi)+\alpha\right)\right|^{2k} &\leq \sum_{\nu\in\mathbb{Z}} e^{2k\nu}\left(\#S_\phi(T,\nu-1)-\#S_\phi(T,\nu)\right) \qquad (1.25)\\ &\leq \sum_{\nu\in\mathbb{Z}} e^{2k(\nu+1)}\#S_\phi(T,\nu) \\ &\ll \Sigma_1+\Sigma_2+\Sigma_3.\end{aligned}$$

Using the trivial bound $\#S_\phi(T,\nu) \leq N_\phi(T) \ll T\log T$ (see [31, Theorem 1]) that holds for every $\nu \in \mathbb{Z}$, we find that $\Sigma_1 \ll T(\log T)^{1+\epsilon}$. To estimate Σ_2 we use

$$\#S_\phi(T,\nu) \ll T(\log T)^{1+\epsilon} \exp\left(\frac{-\nu^2}{\log\log T}\right)$$

which follows from the first two cases of Lemma 1.4.3 when $\nu \in I_2 \cap \mathbb{Z}$. We get

$$\Sigma_2 \ll T(\log T)^{1+\epsilon} \int_3^{4k\log\log T} \exp\left(2ku - \frac{u^2}{\log\log T}\right) du \ll T(\log T)^{k^2+1+\epsilon}.$$

Finally, we bound Σ_3. If $\nu \in I_3 \cap \mathbb{Z}$, the three cases of Lemma 1.4.3 give

$$\#S(T,\nu) \ll T(\log T)^{1+\epsilon} e^{-4k\nu}.$$

Thus,

$$\Sigma_3 \ll T(\log T)^{1+\varepsilon} \int_{4k\log\log T}^{\infty} e^{-2ku} du \ll T(\log T)^{1-8k^2+\epsilon}.$$

Hence, collecting the estimates, we get with respect to (1.25)

$$\sum_{0<t_\phi(n)\le T} \left|\zeta\left(\tfrac{1}{2} + it_n(\phi) + \alpha\right)\right|^{2k} \ll T(\log T)^{k^2+1+\epsilon}$$

for every fixed positive real k when $|\alpha| \le 1$ and $0 \le \text{Re } \alpha \le (\log T)^{-1}$.

Now, using the functional equation (1.2) and the reflection principle $\overline{\zeta(s)} = \zeta(\overline{s})$, we get

$$\left|\zeta\left(\tfrac{1}{2} + it_n(\phi) + \alpha\right)\right| = \left|\Delta\left(\tfrac{1}{2} + it_n(\phi) + \alpha\right)\zeta\left(\tfrac{1}{2} + it_n(\phi) - \overline{\alpha}\right)\right| \le C\left|\zeta\left(\tfrac{1}{2} + it_n(\phi) - \overline{\alpha}\right)\right|$$

with some absolute constant $C > 0$, when $|\alpha| \le 1$ and $|\text{Re } \alpha| \le (\log T)^{-1}$. For $-(\log T)^{-1} \le \text{Re } \alpha \le 0$ we have

$$\sum_{0<t_\phi(n)\le T} \left|\zeta\left(\tfrac{1}{2} + it_n(\phi) + \alpha\right)\right|^{2k} \le C^{2k} \sum_{0<t_\phi(n)\le T} \left|\zeta\left(\tfrac{1}{2} + it_n(\phi) - \overline{\alpha}\right)\right|^{2k} \ll T(\log T)^{k^2+1+\epsilon}.$$

This finishes the proof. □

To deduce Theorem 1.4.1 from Proposition 1.4.4, we will use the following Lemma.

Lemma 1.4.5. *Let k and l be positive integers and let $R > 0$ be arbitrary. Then we have*

$$\sum_{0<t_\phi(n)\le T} \left|\zeta^{(l)}\left(\tfrac{1}{2} + it_n(\phi)\right)\right|^{2k} \le \left(\frac{l!}{R^l}\right)^{2k} \max_{|\alpha|\le R} \sum_{0<t_\phi(n)\le T} \left|\zeta\left(\tfrac{1}{2} + it_n(\phi) + \alpha\right)\right|^{2k}$$

Proof. The result follows by means of Cauchy's integral formula. The proof is the same as the proof of Lemma 7.1 in [35]; we just have to exchange ρ with $\frac{1}{2} + it_n(\phi)$. □

We are now ready to prove Theorem 1.4.1.

Proof of Theorem 1.4.1. The assertion of the Theorem for the case $l = 0$ follows directly from Proposition 1.4.4 (setting $\alpha = 0$). Let k and l be positive integers, then the assertion of the Theorem follows by Proposition 1.4.4 and Lemma 1.4.5 (setting $R = (\log T)^{-1}$). □

1.5 Corollaries

Combining the obtained results we are able to establish the following corollaries.

1.5.1 Expansion of the curve

Recall that $e^{-i\phi}\zeta(\frac{1}{2}+it_n(\phi))$ is real. Hence, we may write $t_n^+(\phi)$ in place of $t_n(\phi)$ if $e^{-i\phi}\zeta(\frac{1}{2}+it_n(\phi))\geqslant 0$ and $t_n^-(\phi)$ if $e^{-i\phi}\zeta(\frac{1}{2}+it_n(\phi))<0$ (Check Figures 6 and 7).

The corollary states that the curve of the Riemann zeta function on the critical line (see Figure 4) expands to all direction on the complex plane.

Corollary 1.5.1. *For any $\phi\in[0,\pi)$, there are arbitrary large positive and negative values of $e^{-i\phi}\zeta(\frac{1}{2}+it_n(\phi))$. More precisely,*

$$\max_{0<t_n^{\pm}(\phi)\leqslant T}\left|\zeta(\tfrac{1}{2}+it_n^{\pm}(\phi))\right|\gg(\log T)^{\frac{5}{4}}.$$

If the Riemann hypothesis is assumed for any arbitrary small $\delta>0$ we have

$$\max_{0<t_n^{\pm}(\phi)\leqslant T}\left|\zeta(\tfrac{1}{2}+it_n^{\pm}(\phi))\right|\gg(\log T)^{\frac{3}{2}-\delta}.$$

Proof. First we observe for any non-negative integer ℓ

$$\begin{aligned}&\sum_{t_n^{\pm}(\phi)\leqslant T}\left|\zeta(\tfrac{1}{2}+it_n^{\pm}(\phi))\right|^{2\ell+1}\\ &=\frac{1}{2}\sum_{t_n(\phi)\leqslant T}\left(\left|\zeta(\tfrac{1}{2}+it_n(\phi))\right|^{2\ell+1}\pm e^{-(2\ell+1)i\phi}\zeta\left(\tfrac{1}{2}+it_n(\phi)\right)^{2\ell+1}\right)\end{aligned}$$

(with the same choice of signs on either side of the equation). In view of Theorem 1.2.3 and Theorem 1.3.7 with $k=\frac{3}{2}$ we get

$$\sum_{t_n^{\pm}(\phi)\leqslant T}\left|\zeta(\tfrac{1}{2}+it_n^{\pm}(\phi))\right|^{3}\gg T(\log T)^{\frac{13}{4}}.$$

Since the number of intersection points $t_n(\phi)\leqslant T$ is bounded by $T\log T$ (see Theorem 1.2.1) and

$$\sum_{0<t_n^{\phi}\leqslant T}\left|\zeta\left(\tfrac{1}{2}+it_n(\phi)\right)\right|^2\ll T(\log T)^2$$

(see Theorem 1.15), we find

$$\begin{aligned}\sum_{t_n^{\pm}(\phi)\leqslant T}\left|\zeta(\tfrac{1}{2}+it_n^{\pm}(\phi))\right|^{3}&\ll\max_{t_n^{\pm}(\phi)\leqslant T}\left|\zeta(\tfrac{1}{2}+it_n^{\pm}(\phi))\right|\sum_{0<t_n^{\phi}\leqslant T}\left|\zeta\left(\tfrac{1}{2}+it_n(\phi)\right)\right|^2\\ &\ll\max_{t_n^{\pm}(\phi)\leqslant T}\left|\zeta(\tfrac{1}{2}+it_n^{\pm}(\phi))\right|T(\log T)^2.\end{aligned}$$

Comparing both estimates we arrive at

$$\max_{t_n^{\pm}(\phi)\leqslant T}\left|\zeta(\tfrac{1}{2}+it_n^{\pm}(\phi))\right|\gg(\log T)^{\frac{5}{4}},$$

If we assume Riemann Hypothesis we can use the following estimate (see Theorem 1.4.1) that holds for any non-negative real k, uniformly for $\phi \in [0, \pi)$, as $T \to \infty$,

$$\sum_{0<t_n^\phi \leqslant T} \left|\zeta\left(\tfrac{1}{2} + it_n(\phi)\right)\right|^{2k} \ll T(\log T)^{k^2+1+\epsilon}.$$

We have

$$T(\log T)^{\frac{13}{4}} \ll \sum_{t_n^\pm(\phi)\leqslant T} \left|\zeta(\tfrac{1}{2} + it_n^\pm(\phi))\right|^3 \ll \max_{t_n^\pm(\phi)\leqslant T} \left|\zeta(\tfrac{1}{2} + it_n^\pm(\phi))\right|^\alpha \sum_{0<t_n^\phi \leqslant T} \left|\zeta\left(\tfrac{1}{2} + it_n(\phi)\right)\right|^{3-\alpha}$$
$$\ll \max_{t_n^\pm(\phi)\leqslant T} \left|\zeta(\tfrac{1}{2} + it_n^\pm(\phi))\right|^\alpha T(\log T)^{(\frac{3}{2}-\frac{\alpha}{2})^2+1+\epsilon}.$$

Comparing both sides we arrive at

$$\max_{t_n^\pm(\phi)\leqslant T} \left|\zeta(\tfrac{1}{2} + it_n^\pm(\phi))\right| \gg (\log T)^{\frac{3}{2}-\frac{\alpha}{4}-\frac{\epsilon}{\alpha}}.$$

After choosing $\alpha = 2\delta$ and $\epsilon = \delta^2$, where δ is positive and arbitrary small we get

$$\max_{t_n^\pm(\phi)\leqslant T} \left|\zeta(\tfrac{1}{2} + it_n^\pm(\phi))\right| \gg (\log T)^{\frac{3}{2}-\delta}.$$

which proves the corollary.

□

1.5.2 Extreme values

Corollary 1.5.2. *Let $\phi \neq \frac{\pi}{2}$ and $\phi \in [0, \pi)$, then*

$$\max_{0<t_n(\phi)\leq T} \left|\zeta(\tfrac{1}{2} + it_n(\phi))\right| \gg \exp\left(\left(\frac{1}{2} + o(1)\right)\sqrt{\frac{\log T}{\log\log T}}\right).$$

Proof. Our argument follows Soundararajan [43]. Taking $X = Y$, resp. $x_n = y_n$ in (1.9), we get

$$S_1(T) = \sum_{0<t_n(\phi)\leqslant T} \zeta\left(\tfrac{1}{2} - it_n(\phi)\right) |X\left(\tfrac{1}{2} + it_n(\phi)\right)|^2$$

for (1.16). Comparing with (1.17) we find

$$|S_1(T)| \leq S_2(T) \max_{0<t_n\leq T} |\zeta(\tfrac{1}{2} + it_n(\phi))|. \tag{1.26}$$

Now let $L = \exp(\sqrt{\log X \log\log X})$ where X is a sufficiently large parameter which will be chosen later. Following Soundararajan [43], we define $x_n = n^{\frac{1}{2}} f(n)$, where f is the multiplicative function such that $f(p^k) = 0$ for all primes p and positive integers $k \geq 2$,

$$f(p) = \frac{L}{\sqrt{p}\log p}$$

for all primes p satisfying $L^2 \leq p \leq \exp((\log L)^2)$, and $f(p) = 0$ for all other primes. We observe that

$$\mathcal{X}_0 = \max_{n\leq X} \sqrt{n} f(n) \leq L^m \prod_{j=1}^m \frac{1}{\log p_j},$$

where $p_1, \ldots, p_m$ are the least distinct m prime numbers in $[L^2, \exp((\log L)^2]$ for which $n =$

$p_1 \cdot \ldots \cdot p_m \leq X$. Since $X \geq n \geq L^{2m}$ then $L^m \leq X^{\frac{1}{2}}$ and $\mathcal{X}_0 < L^m \leq X^{\frac{1}{2}}$. Moreover, since $f(n) \leq 1$ for any n, we find

$$\mathcal{X}_1 = \sum_{n\leq X} \frac{f(n)}{\sqrt{n}} \leq \sum_{n\leq X} \frac{1}{\sqrt{n}} \ll X^{\frac{1}{2}}$$

as well as

$$\mathcal{X}_2 = \sum_{n\leq X} \frac{|x_n|^2}{n} = \sum_{n\leq X} f^2(n) = \sum_{\substack{n=p_1\ldots p_m\leq X \\ L^2<p_1\ldots p_m\leq e^{L^2}}} \frac{L^{2m}}{(p_1\log p_1\ldots p_m\log p_m)^2}$$
$$\leq \prod_{L^2<p\leq e^{L^2}} \left(1+\frac{L^2}{p^2\log^2 p}\right) < \exp\left(L^2\sum_{p>L^2}\frac{1}{p^2\log^2 p}\right) < e.$$

Inserting these bounds in the asymptotic formulas of Proposition 1.3.1 yields

$$S_1(T) = (1+e^{-2i\phi})\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\sum_{mn\leqslant X}\frac{f(m)f(mn)}{\sqrt{n}} + O(X^2T^{\frac{1}{2}}(\log T)^2)$$

and

$$S_2(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\sum_{n\leqslant X}|f(n)|^2 + O(XT^{\frac{1}{2}}(\log T)^2 + X^2(\log T)^3).$$

Let $X = T^{\frac{1}{4}-\epsilon}$, then the main terms in the latter formulas dominate the error terms and we may deduce from (1.26) that

$$\max_{0<t_n\leq T}|\zeta(\tfrac{1}{2}+it_n(\phi))| \geq \frac{|S_1(T)|}{S_2(T)} \gg \left(\sum_{mn\leqslant X}\frac{f(m)f(mn)}{\sqrt{n}}\right)\left(\sum_{n\leqslant X}|f(n)|^2\right)^{-1}.$$

Soundararajan [43] proved that the right hand-side is $\geq \exp\left((1+o(1))\sqrt{\frac{\log X}{\log\log X}}\right)$ which gives the desired estimate by letting $\epsilon \to 0$.

□

1.5.3 Continuous moments

Proposition 1.5.3. *Let k be any non-negative real number and l any non-negative integer. Then, as $T \to \infty$,*

$$\int_T^{2T}\left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right|^{2k}\theta'(t)dt = \int_0^{\pi}\sum_{T\leq t_n(\phi)\leq 2T}\left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right|^{2k}d\phi.$$

Proof. Let $T_i := g_{M+i}$ with $i = 0, ..., N$ denote the generalized Gram's points that lie in the interval $[T, 2T]$. We define a smooth function $[c_0/\pi, \infty) \ni x \mapsto t_x$ via

$$\theta(t_x) = \pi \cdot x.$$

Then, $t_n = g_n$ and $t_{n+\phi/\pi} = t_n(\phi)$ for every positive integer n and every $\phi \in [0, \pi)$. Hence, we get

$$\int_{t_n}^{t_{n+1}} g(t)d\theta(t) = \int_0^1 g(t_{n+u})d\theta(t_{n+u}) = \int_0^1 g(t_{n+u})d(\pi(n+u)) = \qquad (1.27)$$

$$= \int_0^1 g(t_{n+u})\pi du = \int_0^\pi g(t_{n+\phi/\pi})\pi d(\phi/\pi) = \int_0^\pi g(t_n(\phi))d\phi.$$

Therefore,

$$\int_{T_1}^{T_N} g(t)d\theta(t) = \sum_{M\le n\le M+N} \int_{t_n}^{t_{n+1}} g(t)d\theta(t) =$$

$$= \int_0^\pi \left(\sum_{M\le n\le M+N} g(t_n(\phi)) \right) d\phi = \int_0^\pi \left(\sum_{T_1\le t_n(\phi)\le T_N} g(t) \right) d\phi.$$

Noting that the segements $[T, T_1]$ and $[T_N, 2T]$ can be treated in a way analogue to (1.27), the assertion of the Proposition follows. □

Corollary 1.5.4. *As $T\to\infty$,*

$$\begin{aligned}\int_T^{2T} \zeta\left(\tfrac12+it\right) d(\theta(t)) =& \frac{T}{2}\log\frac{T}{2\pi e} + O\left(T^{\frac12+\epsilon}\right),\\ \int_T^{2T} \left|\zeta\left(\tfrac12+it\right)\right|^2 d(\theta(t)) =& \frac{T}{2}\left(\log\frac{T}{2\pi e}\right)^2 + cT\log\frac{T}{2\pi e} + \frac{T}{2} + O\left(T^{\frac12+\epsilon}\right),\\ \int_T^{2T} \zeta\left(\tfrac12+it\right)^3 d(\theta(t)) =& \frac{T}{2}\log\frac{T}{2\pi e} + O\left(T^{\frac12+\epsilon}\right),\end{aligned}$$

where c is the Euler-Mascheroni constant.

Proof of Corollary 1.5.4. By Theorem 1.2.2 we have for any $\varepsilon > 0$, any $\phi\in[0,\pi)$, as $T\to\infty$,

$$\begin{aligned}\sum_{0<t_n(\phi)\leqslant T} \zeta\left(\tfrac12+it\right) =& 2e^{i\phi}\cos(\phi)\frac{T}{2\pi}\log\frac{T}{2\pi e} + O\left(T^{\frac12+\epsilon}\right),\\ \sum_{0<t_n(\phi)\leqslant T} \left|\zeta\left(\tfrac12+it\right)\right|^2 =& \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)^2 + (2c+2\cos(2\phi))\frac{T}{2\pi}\log\frac{T}{2\pi e}\\ &+ \frac{T}{2\pi} + O\left(T^{\frac12+\epsilon}\right),\end{aligned}$$

where c is the Euler-Mascheroni constant. By Theorem 1.2.3 we have for any $\varepsilon > 0$, any $\phi\in[0,\pi)$, as $T\to\infty$,

$$\begin{aligned}\sum_{0<t_n(\phi)\leqslant T} \zeta\left(\tfrac12+it_n(\phi)\right)^3 =& 2e^{3i\phi}\cos(\phi)\frac{T}{2\pi}P_3\left(\log\frac{T}{2\pi}\right) + 2e^{3i\phi}\cos(3\phi)\frac{T}{2\pi}\log\frac{T}{2\pi e}\\ &+ O\left(T^{\frac12+\varepsilon}\right),\end{aligned}$$

where $P_3(x)$ is a computable polynomial of degree three. Using the asymptotic formulas above, we can easily deduce the assertions of Corollary 1.5.4 via Proposition 1.5.3. Note that

$$\int_0^\pi e^{i\phi}\cos(\phi)d\phi = \int_0^\pi e^{i3\phi}\cos(3\phi)d\phi = \frac{\pi}{2},$$

$$\int_0^\pi \cos(2\phi)d\phi = \int_0^\pi e^{i3\phi}\cos(\phi)d\phi = 0.$$

□

Corollary 1.5.5. *For any rational $k \geqslant 1$ and any non-negative integer l, as $T \to \infty$,*

$$\int_1^T \left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right|^{2k} \gg T(\log T)^{k^2+2kl}.$$

Proof of Corollary 1.5.5. Using the asymptotic extension (1.8) for $\theta'(t)$, Proposition 1.5.3 yields for any rational $k \geq 1$ and any non-negative integer l

$$\int_T^{2T} \left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \asymp \frac{1}{\log T}\int_0^{\pi} \sum_{T\leq t_n(\phi)\leq 2T} \left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right|^{2k} d\phi.$$

Combining this with Theorem 1.3.7, we get for any rational $k \geq 1$ and any non-negative integer l

$$\begin{aligned}\int_1^T |\zeta^{(l)}(\tfrac{1}{2}+it)|^{2k}dt &\geq \sum_{j=0}^{\infty}\int_{T/2^{j+1}}^{T/2^j} |\zeta^{(l)}(\tfrac{1}{2}+it)|^{2k}dt\\ &\gg \sum_{j=0}^{\infty}\frac{1}{\log T}\int_0^{\pi}\sum_{\frac{T}{2^{j+1}}\leq t_n(\phi)\leq\frac{T}{2^j}} |\zeta^{(l)}(\tfrac{1}{2}+it)|^{2k}d\phi\\ &\gg T(\log T)^{k^2+2kl}.\end{aligned}$$

Thus, Corollary 1.5.5 follows. □

Corollary 1.5.6. *Assume the Riemann Hypothesis.*
For any non-negative real k, uniformly for $\phi \in [0,\pi)$, as $T \to \infty$,

$$\int_1^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} \ll T(\log T)^{k^2+\epsilon}.$$

For any non-negative integer k and any positive integer l, uniformly for $\phi \in [0,\pi)$, as $T \to \infty$,

$$\int_1^T \left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right|^{2k} \ll T(\log T)^{k^2+2kl+\epsilon}.$$

Proof of Corollary 1.5.6. Using the asymptotic extension (1.8) for $\theta'(t)$, Proposition 1.5.3 yields for any non-negative real k and any non-negative integer l

$$\int_T^{2T} \left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \asymp \frac{1}{\log T}\int_0^{\pi} \sum_{T\leq t_n(\phi)\leq 2T} \left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right|^{2k} d\phi.$$

Combining this with Theorem 1.4.1, we prove Corollary 1.5.6. □

Chapter 2

Sums of Dirichlet L-function over non-trivial zeros of another Dirichlet L-function

Let $s = \sigma + it$ denote a complex variable. The Dirichlet L-function is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (\sigma > 1),$$

where $\chi(n)$ is a Dirichlet character modulo q. For χ mod 1 we get the Riemann zeta function $L(s,\chi) = \zeta(s)$. The Generalized Riemann Hypothesis (GRH) states that inside the critical strip $0 < \sigma < 1$ every Dirichlet L-function has zeros only on the critical line $\sigma = \frac{1}{2}$. Zeros in the critical strip are called non-trivial and we denote them by $\rho_\chi = \beta_\chi + i\gamma_\chi$. A Dirichlet character χ mod q is said to be primitive if it is not induced by any other character of modulus strictly less than q. The unique principal character modulo q is denoted by χ_0. The character χ_0 mod 1 is the only one principal and primitive character. For a Dirichlet character χ mod q the associated Gauss sum is defined by

$$G(n,\chi) = \sum_{a=1}^{q} \chi(a) \exp\left(2\pi i \frac{an}{q}\right).$$

If $n = 1$ we denote $\tau(\chi) = G(1,\chi)$. For a primitive character χ mod q we have $|\tau(\chi)| = \sqrt{q}$ and for the principal character χ_0 we have $\tau(\chi_0) = \mu(q)$, where $\mu(q)$ is the Möbius function.

Next we recall several facts about Dirichlet L-functions. Dirichlet L-function to a primitive character ψ mod Q satisfies the functional equation (Apostol [1, Theorem 12.11])

$$L(1-s,\psi) = \tau(\psi)\frac{1}{Q}\left(\frac{Q}{2\pi}\right)^s \Gamma(s)\left(\exp\left(-\frac{\pi i s}{2}\right) + \psi(-1)\exp\left(\frac{\pi i s}{2}\right)\right) L(s,\overline{\psi}). \tag{2.1}$$

Thus Theorem from Heath-Brown [24] and an application of the Phragmen-Lindelöf principle yield the estimates

$$L(s,\psi) \ll |QT|^{\frac{3}{16}+\varepsilon} \quad \text{for} \quad \frac{1}{2} \le \sigma \le 1 + \frac{1}{\log QT},\ |t| \ge 1, \tag{2.2}$$

$$L(s,\psi) \ll |QT|^{\frac{1}{2}+\varepsilon} \quad \text{for} \quad -\frac{1}{\log QT} \le \sigma < \frac{1}{2},\ |t| \ge 1 \tag{2.3}$$

uniformly in $|t| \ll T$. Under GRH the bound (2.2) can be replaced by

$$L(s,\psi) \ll |QT|^{\varepsilon} \quad \text{for} \quad \frac{1}{2} \le \sigma \le 1 + \frac{1}{\log QT}, \ |t| \ge 1 \tag{2.4}$$

uniformly in $|t| \ll T$. The bound (2.4) can be obtained similarly as in Titchmarsh [47, Theorem 13.5], see also Garunkštis [16, Theorem 4].

We rewrite the functional equation (2.1) as

$$L(1-s,\psi) = \Delta(1-s,\psi) L(s,\overline{\psi}), \tag{2.5}$$

where $\Delta(s,\psi)$ is a meromorphic function with only real zeros and poles satisfying the functional equation

$$\Delta(s,\psi)\Delta(1-s,\overline{\psi}) = 1. \tag{2.6}$$

Stirling's formula gives, for $t > 1$,

$$\Delta(\sigma+it,\psi) = \left(\frac{tQ}{2\pi}\right)^{\sigma-\frac{1}{2}-it} \exp\left(i\left(t+\frac{\pi}{4}\right)\right)\left(1+O\left(\frac{1}{t}\right)\right) \tag{2.7}$$

and

$$\frac{\Delta'}{\Delta}(s,\psi) = \frac{\Delta'}{\Delta}(1-s,\overline{\psi}) = -\log\frac{tQ}{2\pi} + O\left(\frac{1}{t}\right). \tag{2.8}$$

Logarithmic differentiation of (2.5) leads to

$$\frac{L'}{L}(1-s,\chi) = \frac{\Delta'}{\Delta}(1-s,\chi) - \frac{L'}{L}(s,\overline{\chi}). \tag{2.9}$$

For the logarithmic derivative we have the partial fraction decomposition (see Prachar [39, Chapter 7, Theorem 4.1])

$$\frac{L'}{L}(s,\chi) = \sum_{|t-\gamma_\chi|\le 1} \frac{1}{s-\rho_\chi} + O(\log q(|t|+2)) \quad \text{for} \quad -1 \le \sigma \le 2, t \ge 1. \tag{2.10}$$

For $q \ge 1$, $\chi \bmod q$ and $t \ge 0$ we have (see Prachar [39, Chapter 7, Theorem 3.3])

$$N_\chi(t+1) - N_\chi(t) := \#\{\rho_\chi = \beta_\chi + i\gamma_\chi : t < \gamma_\chi \le t+1\} \ll \log q(t+2). \tag{2.11}$$

Thus the zeros ρ_χ cannot lie too dense: for any given $t_0 \ge 2$ there exists a $t = t(\chi)$, $t \in (t_0, t_0+1]$, such that

$$\min_{\gamma_\chi} |t-\gamma_\chi| \gg \frac{1}{\log qt}. \tag{2.12}$$

In view of the expression (2.10) we get, for t satisfying (2.12),

$$\frac{L'}{L}(\sigma+it,\chi) \ll (\log qt)^2 \quad \text{for} \quad -1 \le \sigma \le 2, \ t \ge 2. \tag{2.13}$$

2.1 Lemmas

In the proofs of Theorems 2.2.1 and 2.3.1 the following modified Gonek lemma (c.f. Gonek [19, Lemma 5]) will be important.

Lemma 2.1.1. *Assume that* $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ *converges for* $\sigma > 1$ *and* $a(n) = O(n^{\varepsilon})$. *Let* $a = 1 + \frac{1}{\log(QT)}$. *Then*

$$\frac{1}{2\pi i}\int_{a+i}^{a+iT}\left(\frac{m}{2\pi}\right)^s \Gamma(s)\exp\left(\delta\frac{\pi i s}{2}\right)\sum_{n=1}^{\infty}\frac{a(n)}{n^s}ds$$
$$= \begin{cases} \sum_{n\le \frac{Tm}{2\pi}} a(n)\exp\left(-2\pi i\frac{n}{m}\right) + O(m^a T^{1/2+\varepsilon}) & \text{if } \delta = -1, \\ O(m^a) & \text{if } \delta = +1. \end{cases}$$

Proof. Because of the absolute convergence we may interchange the order of summation and integration. For the integral we use Lemma 1 from [10] and for the sum Lemma 4 from [19]. □

Next three technical lemmas will be useful in the proof of Theorem 2.2.1.

Lemma 2.1.2. *Let* $\chi \bmod q$ *and* $\psi \bmod Q$ *be primitive Dirichlet characters and* $\chi \neq \psi$ *with* $Q \ll \log^A T$ *and* $q \ll \log^B T$, *where* A *and* B *are positive constants. We have*

$$\sum_{mnl\le \frac{TQ}{2\pi}} \Lambda(m)\chi(m)\psi(n)\psi(l)\exp\left(-2\pi i\frac{mnl}{Q}\right)$$
$$= -\psi(-1)\tau(\psi)\frac{\phi(Q)}{Q}\frac{L'}{L}(1,(\chi\overline{\psi}))\frac{T}{2\pi}\left(\log\frac{TQ}{2\pi e} + 2\gamma + 2\sum_{p|Q}\frac{\log p}{p-1} + \left(\frac{L''}{L'} - \frac{L'}{L}\right)(1,\chi\overline{\psi})\right)$$
$$+ \delta(q,Q)\chi(-1)\tau(\chi\psi_0)\frac{1}{\phi(Q)}\frac{TQ}{2\pi}L(1,(\chi\overline{\psi}))^2 + O\left(T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon} T}}\right).$$

Under GRH the error term can be replaced by $O\left(q\,Q(\log q)^2(TQ)^{\frac{1}{2}+\varepsilon} + q\,Q^5\log(TQ)\right)$ *uniformly for all* Q *and* q.

Proof. By the orthogonality of Dirichlet characters the left-hand side of the formula in Lemma 2.1.2 can be written in the following way.

$$\sum_{mnl\le \frac{TQ}{2\pi}} \Lambda(m)\chi(m)\psi(n)\psi(l)\exp\left(-2\pi i\frac{mnl}{Q}\right)$$
$$= \sum_{a=1}^{[q,Q]}\chi(a)\sum_{b=1}^{Q}\psi(b)\sum_{c=1}^{Q}\psi(c)\exp\left(-2\pi i\frac{abc}{Q}\right)\sum_{\substack{mnl\le\frac{TQ}{2\pi}\\ m\equiv a \bmod [q,Q]\\ n\equiv b \bmod Q\\ l\equiv c \bmod Q}}\Lambda(m) \tag{2.14}$$
$$= \frac{1}{\phi^2(Q)}\sum_{\substack{\omega' \bmod Q\\ \omega'' \bmod Q}}\sum_{a=1}^{[q,Q]}\chi(a)\sum_{b=1}^{Q}\overline{\omega'}(b)\psi(b)\sum_{c=1}^{Q}\overline{\omega''}(c)\psi(c)\exp\left(-2\pi i\frac{abc}{Q}\right)$$
$$\times\sum_{\substack{mnl\le\frac{TQ}{2\pi}\\ m\equiv a \bmod [q,Q]}}\Lambda(m)\omega'(n)\omega''(l) = S_1 + S_2 + S_3 + S_4,$$

where S_j, $j = 1,2,3,4$ correspond to the following cases:

- $S_1 : \omega' = \omega'_0$ and $\omega'' = \omega''_0$,
- $S_2 : \omega' = \omega'_0$ and $\omega'' \neq \omega''_0$,
- $S_3 : \omega' \neq \omega'_0$ and $\omega'' = \omega''_0$,
- $S_4 : \omega' \neq \omega'_0$ and $\omega'' \neq \omega''_0$,

here $\omega_0' = \omega_0''$ is the principle character modulo Q.

First we consider S_1. Since ψ is a primitive Dirichlet character, the Gauss sum is separable, i.e. $G(-ab,\psi) = \overline{\psi}(-ab)\tau(\psi)$ (see Davenport [11, Section 9]). By this and by the orthogonality of Dirichlet characters we have

$$\begin{aligned} S_1 =& \psi(-1)\tau(\psi)\frac{1}{\phi(Q)}\sum_{a=1}^{[q,Q]}\chi(a)\overline{\psi}(a)\sum_{\substack{mnl\le \frac{TQ}{2\pi}\\ m\equiv a \bmod [q,Q]}}\Lambda(m)\omega_0'(n)\omega_0''(l)\\ =&\psi(-1)\tau(\psi)\frac{1}{\phi([q,Q])}\frac{1}{\phi(Q)}\sum_{\eta \bmod [q,Q]}\sum_{a=1}^{[q,Q]}\overline{\eta}(a)\chi(a)\overline{\psi}(a)\sum_{mnl\le\frac{TQ}{2\pi}}\Lambda(m)\eta(m)\omega_0'(n)\omega_0''(l). \end{aligned}$$

In the last formula the sum over a is not equal to zero if and only if $\eta = \chi\overline{\psi}$. Thus

$$S_1 = \psi(-1)\tau(\psi)\frac{1}{\phi(Q)}\sum_{mnl\le\frac{TQ}{2\pi}}\Lambda(m)(\chi\overline{\psi})(m)\omega_0'(n)\omega_0''(l). \tag{2.15}$$

Second we consider S_2. In formula (2.1.2) interchanging the summation (over b and c) and using the equality $G(-ac,\psi) = \overline{\psi}(-ac)\tau(\psi)$ we obtain

$$\begin{aligned} S_2 =& \frac{1}{\phi^2(Q)}\sum_{\substack{\omega'' \bmod Q\\ \omega''\ne\omega_0''}}\sum_{a=1}^{[q,Q]}\chi(a)\sum_{c=1}^{Q}\overline{\omega''}(c)\psi(c)\sum_{b=1}^{Q}\psi(b)\exp\left(-2\pi i\frac{abc}{Q}\right)\\ &\times\sum_{\substack{mnl\le\frac{TQ}{2\pi}\\ m\equiv a \bmod [q,Q]}}\Lambda(m)\omega_0'(n)\omega''(l)\\ =&\psi(-1)\tau(\psi)\frac{1}{\phi^2(Q)}\sum_{\substack{\omega'' \bmod Q\\ \omega''\ne\omega_0''}}\sum_{a=1}^{[q,Q]}\chi(a)\overline{\psi}(a)\sum_{c=1}^{Q}\overline{\omega''}(c)\sum_{\substack{mnl\le\frac{TQ}{2\pi}\\ m\equiv a \bmod [q,Q]}}\Lambda(m)\omega_0'(n)\omega''(l). \end{aligned}$$

Hence $S_2 = 0$, since the sum over c is equal to zero.

In the same way we obtain $S_3 = 0$.

Finally we consider S_4. First we note that, for $\omega'\ne\omega_0'$, $\omega''\ne\omega_0''$ and $(a,[q,Q])>1$,

$$\left|\sum_{\substack{m\le\frac{TQ}{2\pi}\\ m\equiv a \bmod [q,Q]}}\Lambda(m)\sum_{nl\le\frac{TQ}{2\pi m}}\omega'(n)\omega''(l)\right| \le \phi^2(Q)\sum_{\substack{m\le\frac{TQ}{2\pi}\\ m\equiv a \bmod [q,Q]}}\Lambda(m) = O(\phi^2(Q)\log(TQ)).$$

This yields (note that the formula below is split into two lines)

$$\begin{aligned} S_4 =& \frac{1}{\phi^2(Q)}\sum_{\substack{\omega' \bmod Q\\ \omega'\ne\omega_0'}}\sum_{\substack{\omega'' \bmod Q\\ \omega''\ne\omega_0''}}\sum_{a=1}^{[q,Q]}\chi(a)\sum_{b=1}^{Q}\overline{\omega'}(b)\psi(b)\sum_{c=1}^{Q}\overline{\omega''}(c)\psi(c)\exp\left(-2\pi i\frac{abc}{Q}\right)\\ &\sum_{\substack{mnl\le\frac{TQ}{2\pi}\\ m\equiv a \bmod [q,Q]\\ (a,[q,Q])=1}}\Lambda(m)\omega'(n)\omega''(l) + O\left(\phi^4(Q)\phi([q,Q])\log(TQ)\right). \end{aligned}$$

If $(ab,Q)=1$, then $G(-ab,\overline{\omega''}\psi) = (\omega''\overline{\psi})(-ab)\tau(\overline{\omega''}\psi)$ (see Davenport [11, Section 9]). Then by the orthogonality of Dirichlet characters we have

$$S_4 = \frac{1}{\phi([q,Q])} \frac{1}{\phi^2(Q)} \sum_{\eta \bmod [q,Q]} \sum_{\substack{\omega' \bmod Q \\ \omega' \neq \omega'_0}} \sum_{\substack{\omega'' \bmod Q \\ \omega'' \neq \omega''_0}} (\omega''\overline{\psi})(-1)\tau(\overline{\omega''}\psi) \sum_{a=1}^{[q,Q]} \overline{\eta}(a)\chi(a)\omega''(a)\overline{\psi}(a)$$
$$\sum_{b=1}^{Q} \overline{\omega'}(b)\omega''(b) \sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)\eta(m)\omega'(n)\omega''(l) + O\left(\phi^4(Q)\phi([q,Q])\log(TQ)\right).$$

In the last formula the sum over b does not vanish if and only if $\omega' = \omega''$. Further we write $S_4 = T_1 + T_2$, where T_1 and T_2 correspond to the following cases:

- $T_1 : \eta = \eta_0$,
- $T_2 : \eta \neq \eta_0$.

Let $\delta(q,Q) = 1$ if $q|Q$ and $\delta(q,Q) = 0$ otherwise. Then

$$\begin{aligned} T_1 =& \frac{1}{\phi([q,Q])} \frac{1}{\phi(Q)} \\ &\times \sum_{\substack{\omega' \bmod Q \\ \omega' \neq \omega'_0}} (\omega'\overline{\psi})(-1)\tau(\overline{\omega'}\psi) \sum_{a=1}^{[q,Q]} \eta_0(a)\chi(a)\omega'(a)\overline{\psi}(a) \sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)\eta_0(m)\omega'(n)\omega''(l) \quad (2.16) \\ =& \delta(q,Q)(\chi\psi_0)(-1)\tau(\chi\psi_0)\frac{1}{\phi(Q)} \sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)\eta_0(m)(\overline{\chi}\psi)(n)(\overline{\chi}\psi)(l), \end{aligned}$$

since the sum over a does not vanish if and only if $\omega' = \overline{\chi}\psi$; and this equality happens if and only if $q|Q$. By definition of T_2 we have

$$\begin{aligned} T_2 =& \frac{1}{\phi([q,Q])} \frac{1}{\phi(Q)} \sum_{\substack{\eta \bmod [q,Q] \\ \eta \neq \eta_0}} \sum_{\substack{\omega' \bmod Q \\ \omega' \neq \omega'_0}} (\omega'\overline{\psi})(-1)\tau(\overline{\omega'}\psi) \\ & \sum_{a=1}^{[q,Q]} \overline{\eta}(a)\chi(a)\omega'(a)\overline{\psi}(a) \sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)\eta(m)\omega'(n)\omega''(l), \end{aligned} \quad (2.17)$$

where $\omega''(l)$ is a non-principle character mod Q.

In view of the above we can write

$$\sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)\chi(m)\psi(n)\psi(l) \exp\left(-2\pi i \frac{mnl}{Q}\right) = S_1 + T_1 + T_2 + O\left(\phi^4(Q)\phi([q,Q])\log(TQ)\right),$$

where S_1, T_1, and T_2 are given by formulas (2.15), (2.16), and (2.17). We will see that sums S_1 and T_1 produce the main term, and the sum T_2 contributes to the error term.

Next we consider S_1. Note that by conditions of the lemma the character $\chi\overline{\psi}$ is not a principal character. By Perron's formula (see Titchmarsh [47, Lemma 3.12])

$$\begin{aligned} -\frac{\phi(Q)}{\psi(-1)\tau(\psi)} S_1 =& -\sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)(\chi\overline{\psi})(m)\omega'_0(n)\omega''_0(l) \\ =& \frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'}{L}(s,(\chi\overline{\psi}))L(s,\omega'_0)L(s,\omega''_0)\left(\frac{TQ}{2\pi}\right)^s \frac{ds}{s} + O\left(\frac{TQ\log^2(TQ)}{U}\right), \end{aligned} \quad (2.18)$$

where $a = 1 + 1/\log(TQ)$.

In Prachar [39, Chapter 8, Theorem 6.2] considering $q \ll \log^B T$ with B being any positive constant we find that

$$L(s,\chi) \neq 0 \quad \text{for} \quad \sigma > 1 - \frac{c}{\log^{\frac{3}{4}+\varepsilon} T},$$

where c is an absolute positive constant. With regard to this zero-free region for $L(s,\chi)$ let $b_1 = 1 - c/\log^{\frac{3}{4}+\varepsilon} T$. Shifting the line of integration and noting that $L(s,\omega_0') = L(s,\omega_0'')$ we get

$$\begin{aligned} -\sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)(\chi\overline{\psi})(m)\omega_0'(n)\omega_0''(l) &= \operatorname{Res}_{s=1} \frac{L'}{L}(s,\chi\overline{\psi})L^2(s,\omega_0')\left(\frac{TQ}{2\pi}\right)^s \frac{1}{s} \\ &- \frac{1}{2\pi i}\left\{\int_{a+iU}^{b_1+iU} + \int_{b_1+iU}^{b_1-iU} + \int_{b_1-iU}^{a-iU}\right\} \frac{L'}{L}(s,\chi\overline{\psi})L^2(s,\omega_0')\left(\frac{TQ}{2\pi}\right)^s \frac{ds}{s} \\ &+ O\left(\frac{TQ\log^2(TQ)}{U}\right). \end{aligned} \tag{2.19}$$

Note that

$$\operatorname{Res}_{s=1} \frac{L'}{L}(s,\chi\overline{\psi})L^2(s,\omega_0')\left(\frac{TQ}{2\pi}\right)^s \frac{1}{s} = \lim_{s\to 1} \frac{d}{ds}\left((s-1)^2 \frac{L'}{L}(s,\chi\overline{\psi})L^2(s,\omega_0')\left(\frac{TQ}{2\pi}\right)^s \frac{1}{s}\right).$$

To calculate this residue we use the following expansion (see Davenport [11, Section 4, formula (6)] and Titchmarsh [47, formula (2.1.16)])

$$\begin{aligned} L^2(s,\omega_0') = \left(\zeta(s)\prod_{p|Q}\left(1-p^{-s}\right)\right)^2 &= \left(\frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} \gamma_n (s-1)^n\right)^2 \prod_{p|Q}\left(1-p^{-s}\right)^2 \\ &= \left(\frac{1}{(s-1)^2} + \frac{2\gamma}{(s-1)} + \sum_{m=0}^{\infty} a_m (s-1)^m\right)\prod_{p|Q}\left(1-p^{-s}\right)^2, \end{aligned}$$

where γ is the Euler constant, γ_n, $n = 1, 2, \ldots$, and a_m, $m = 0, 1, \ldots$ are absolute real constants. Then by (see Apostol [1, Theorem 2.4])

$$\prod_{p|Q}\left(1-p^{-1}\right) = \frac{\phi(Q)}{Q}, \tag{2.20}$$

$$\frac{d}{ds}\left(\prod_{p|Q}\left(1-p^{-s}\right)\right)^2 = 2\prod_{p|Q}\left(1-p^{-s}\right)^2 \sum_{p|Q} \frac{\log p}{p^s - 1} \tag{2.21}$$

and

$$\frac{d}{ds}\frac{L'}{L}(s,\chi\overline{\psi}) = \left(\frac{L''L - (L')^2}{L^2}\right)(s,\chi\overline{\psi})$$

we get

$$\begin{aligned}
&\operatorname{Res}_{s=1} \frac{L'}{L}(s,\chi\overline{\psi})L^2(s,\omega_0')\left(\frac{TQ}{2\pi}\right)^s\frac{1}{s}\\
&= \lim_{s\to 1}\frac{d}{ds}\left((1+2\gamma(s-1))\frac{L'}{L}(s,\chi\overline{\psi})\left(\frac{TQ}{2\pi}\right)^s\frac{1}{s}\prod_{p|Q}\left(1-p^{-s}\right)^2\right)\\
&= (2\gamma-1)\frac{L'}{L}(1,\chi\overline{\psi})\frac{TQ}{2\pi}\prod_{p|Q}\left(1-p^{-1}\right)^2+\frac{L''L-(L')^2}{L^2}(1,\chi\overline{\psi})\frac{TQ}{2\pi}\prod_{p|Q}\left(1-p^{-1}\right)^2\\
&\quad+\prod_{p|Q}\left(1-p^{-1}\right)^2\frac{L'}{L}(1,\chi\overline{\psi})\frac{TQ}{2\pi}\log\frac{TQ}{2\pi}+2\frac{L'}{L}(1,\chi\overline{\psi})\frac{TQ}{2\pi}\prod_{p|Q}\left(1-p^{-1}\right)^2\sum_{p|Q}\frac{\log p}{p-1}\\
&= \left(\frac{\phi(Q)}{Q}\right)^2\frac{L'}{L}(1,\chi\overline{\psi})\frac{TQ}{2\pi}\left(\log\frac{TQ}{2\pi e}+2\gamma+2\sum_{p|Q}\frac{\log p}{p-1}+\left(\frac{L''}{L'}-\frac{L'}{L}\right)(1,\chi\overline{\psi})\right).
\end{aligned}$$

This and formulas (2.19), (2.18) yield

$$\begin{aligned}
S_1 =& -\psi(-1)\tau(\psi)\frac{\phi(Q)}{Q}\frac{L'}{L}(1,(\chi\overline{\psi}))\frac{T}{2\pi}\left(\log\frac{TQ}{2\pi e}+2\gamma+2\sum_{p|Q}\frac{\log p}{p-1}+\left(\frac{L''}{L'}-\frac{L'}{L}\right)(1,\chi\overline{\psi})\right)\\
&+\mathcal{E},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{E} =& -\frac{\psi(-1)\tau(\psi)}{2\pi i\phi(Q)}\left\{\int_{a+iU}^{b_1+iU}+\int_{b_1+iU}^{b_1-iU}+\int_{b_1-iU}^{a-iU}\right\}\frac{L'}{L}(s,(\chi\overline{\psi}))L(s,\omega_0')L(s,\omega_0'')\left(\frac{TQ}{2\pi}\right)^s\frac{ds}{s}\\
&+O\left(\frac{|\tau(\psi)|}{\phi(Q)}\frac{TQ\log^2(TQ)}{U}\right).
\end{aligned}$$

Next we evaluate the integrals in $\mathcal{E}$. In view of inequality $\phi(Q)\gg Q^{1-\varepsilon}$ we obtain

$$\frac{|\tau(\psi)|}{\phi(Q)}\ll Q^{-\frac{1}{2}+\varepsilon}. \tag{2.22}$$

By this and by formulas (2.2), (2.13) we have that the first and the third integrals in $\mathcal{E}$ are bounded by

$$\ll Q^{-\frac{1}{2}+\varepsilon}(\log([q,Q]U))^2(QU)^{\frac{3}{8}+\varepsilon}TQ/U.$$

Next we consider the second integral in $\mathcal{E}$. We brake it into three parts

$$-\frac{1}{2\pi i}\left\{\int_{b_1-iU}^{b_1-i}+\int_{b_1-i}^{b_1+i}+\int_{b_1+i}^{b_1+iU}\right\}\frac{L'}{L}(s,(\chi\overline{\psi}))L(s,\omega_0')L(s,\omega_0'')\left(\frac{TQ}{2\pi}\right)^s\frac{ds}{s}.$$

Again, using formulas (2.2) and (2.13) we see that the first and the third integrals in the last formula are bounded by

$$\ll Q^{-\frac{1}{2}+\varepsilon}(\log([q,Q]U))^2(QU)^{\frac{3}{8}+\varepsilon}(TQ)^{b_1}$$

and the second integral is bounded by

$$\ll Q^{-\frac{1}{2}+\varepsilon}(TQ)^{b_1}(\log([q,Q]))^2(Q)^{\frac{3}{8}+\varepsilon}.$$

Now we choose $U = T^{1-b_1}$. By $Q \ll \log^A T$ and $q \ll \log^B T$ we obtain that

$$\mathcal{E} \ll T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon} T}}.$$

In the same way we get

$$T_1 = \delta(q,Q)\chi(-1)\tau(\chi\psi_0)\frac{1}{\phi(Q)}\frac{TQ}{2\pi}L(1,(\chi\overline{\psi}))^2 + O\left(T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon} T}}\right)$$

and

$$T_2 \ll T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon} T}}.$$

Under GRH we choose $b_1 = 1/2 + 1/\log QT$, $U = QT$ and make use of the bound (2.4). This finishes the proof of Lemma 2.1.2.

□

Lemma 2.1.3. *Let ψ mod Q be a primitive Dirichlet character. Then, for $x \to \infty$,*

$$\sum_{mn\le x} \psi(m)\psi(n)\exp\left(-2\pi i\frac{mn}{Q}\right)$$
$$= -\psi(-1)\tau(\psi)\frac{\phi(Q)}{Q^2}x\left(\log\frac{x}{e} + 2\gamma + 2\sum_{p|Q}\frac{\log p}{p-1}\right) + O(Q^{-\frac{1}{8}+\varepsilon}x^{\frac{11}{16}+\varepsilon} + Q^4)$$

uniformly in Q. Under GRH the error term can be replaced by $O(Q^{-1/2+\varepsilon}x^{1/2+\varepsilon} + Q^4)$ uniformly in Q.

Proof. By the orthogonality of Dirichlet characters the sum in Lemma 2.1.3 can be written as

$$\begin{aligned}
S &= \sum_{mn\le x} \psi(m)\psi(n)\exp\left(-2\pi i\frac{mn}{Q}\right)\\
&= \frac{1}{\phi^2(Q)}\sum_{\eta \bmod Q}\sum_{\omega \bmod Q}\sum_{a=1}^{Q}\overline{\eta}(a)\psi(a)\sum_{b=1}^{Q}\overline{\omega}(b)\psi(b)\exp\left(-2\pi i\frac{ab}{Q}\right)\sum_{mn\le x}\eta(m)\omega(n)\\
&= \psi(-1)\tau(\psi)\frac{1}{\phi(Q)}\sum_{mn\le x}\eta_0(m)\omega_0(n) + O(\phi(Q)^4).
\end{aligned}$$

Perron's formula yields

$$\sum_{mn\le x}\eta_0(n)\omega_0(l) = -\frac{1}{2\pi i}\int_{a-iU}^{a+iU} L(s,\eta_0)L(s,\omega_0)x^s\frac{ds}{s} + O\left(\frac{x\log^2 x}{U}\right),$$

where $a = 1/\log x$. Let $b_1 = 1/2 + 1/\log x$. Shifting the line of integration we get

$$\begin{aligned}
\sum_{mn\le x}\eta_0(n)\omega_0(l) &= -\operatorname{Res}_{s=1} L(s,\eta_0)L(s,\omega_0)x^s\frac{1}{s}\\
&\quad + \frac{1}{2\pi i}\left\{\int_{a+iU}^{b_1+iU} + \int_{b_1+iU}^{b_1-iU} + \int_{b_1-iU}^{a-iU}\right\} L(s,\eta_0)L(s,\omega_0)x^s\frac{ds}{s} + O\left(\frac{x\log^2 x}{U}\right),
\end{aligned}$$

The definition of residue, formulas (2.20) and (2.21) lead to

$$\mathrm{Res}_{s=1} L(s,\eta_0)L(s,\omega_0)x^s\frac{1}{s} = \frac{\phi(Q)^2}{Q^2}x\left(\log\frac{x}{e}+2\gamma+2\sum_{p|Q}\frac{\log p}{p-1}\right).$$

Hence, we have

$$S=-\,\psi(-1)\tau(\psi)\frac{\phi(Q)}{Q^2}x\left(\log(x/e)+2\left(\gamma+\sum_{p|Q}\frac{\log p}{p-1}\right)\right)+O(\phi(Q)^4)+\mathcal{E},$$

where

$$\begin{aligned}\mathcal{E}=&\psi(-1)\tau(\psi)\frac{1}{2\pi i\phi(Q)}\left\{\int_{a+iU}^{b_1+iU}+\int_{b_1+iU}^{b_1-iU}+\int_{b_1-iU}^{a-iU}\right\}L(s,\eta_0)L(s,\omega_0)x^s\frac{ds}{s}\\&+O\left(\frac{|\tau(\psi)|}{\phi(Q)}\frac{x\log^2 x}{U}\right).\end{aligned}$$

It remains to bound $\mathcal{E}$. By formulas (2.2) and (2.22) we have that the first and the last integrals in $\mathcal{E}$ are bounded by $\ll Q^{-1/2+\varepsilon}(QU)^{3/8+\varepsilon}x/U$. The second integral in $\mathcal{E}$ we brake into three parts

$$\left\{\int_{b_1-iU}^{b_1-i}+\int_{b_1-i}^{b_1+i}+\int_{b_1+i}^{b_1+iU}\right\}L(s,\eta_0)L(s,\omega_0)x^s\frac{ds}{s}.$$

In the last formula the first and the third integrals are both bounded by $\ll Q^{-1/2+\varepsilon}(QU)^{3/8+\varepsilon}x^{1/2}$ and the second integral is bounded by $\ll Q^{-1/2+\varepsilon}x^{1/2}$. We choose $U=x^{1/2}$. Then

$$\mathcal{E}\ll Q^{-\frac{1}{8}+\varepsilon}x^{\frac{11}{16}+\varepsilon}.$$

If we assume GRH then the bound (2.4) gives $\mathcal{E}\ll Q^{-1/2+\varepsilon}x^{1/2+\varepsilon}$. Lemma 2.1.3 is proved.

□

The last Lemma will be useful in the proof of Theorem 2.3.1. It is a weaker case of Lemma 2.1.2.

Lemma 2.1.4. *Let $\chi \bmod q$ and $\psi \bmod Q$ be primitive Dirichlet characters and $\chi\neq\psi$ with $Q\ll\log^A T$ and $q\ll\log^B T$, where A and B are positive constants. We have*

$$\begin{aligned}&\sum_{mn\le\frac{TQ}{2\pi}}\Lambda(m)\chi(m)\psi(n)\exp\left(-2\pi i\frac{mn}{Q}\right)\\&\quad=-\psi(-1)\tau(\psi)\frac{L'}{L}(1,\chi\overline{\psi})\frac{T}{2\pi}+\delta(q,Q)\frac{1}{\phi(Q)}G(-1,\chi\psi_0)L(1,\overline{\chi}\psi)\frac{TQ}{2\pi}+O\left(T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon}T}}\right).\end{aligned}$$

Under GRH the error term can be replaced by $O\left((TQ)^{1/2+\varepsilon}\right)$ uniformly for all $Q$and q.

Proof. By the orthogonality of Dirichlet characters the left-hand side of the formula in Lemma

2.1.4 can be written in the following way.

$$
\begin{aligned}
&\sum_{mn\le\frac{TQ}{2\pi}}\Lambda(m)\chi(m)\psi(n)\exp\left(-2\pi i\frac{mn}{Q}\right)\\
&=\frac{1}{\phi(Q)}\sum_{\omega \mod Q}\sum_{a=1}^{Q}\psi(a)\overline{\omega}(a)\sum_{b=1}^{[q,Q]}\chi(b)\exp\left(-2\pi i\frac{ab}{Q}\right)\sum_{\substack{mn\le\frac{TQ}{2\pi}\\ m\equiv b \mod [q,Q]}}\Lambda(m)\omega(n)\\
&=\frac{1}{\phi(Q)}\sum_{b=1}^{[q,Q]}\chi(b)\sum_{a=1}^{Q}\psi(a)\omega_0(a)\exp\left(-2\pi i\frac{ab}{Q}\right)\sum_{\substack{mn\le\frac{TQ}{2\pi}\\ m\equiv b \mod [q,Q]}}\Lambda(m)\omega_0(n)\\
&\quad+\frac{1}{\phi(Q)}\sum_{\substack{\omega \mod Q\\ \omega\ne\omega_0}}\sum_{a=1}^{Q}\psi(a)\overline{\omega}(a)\sum_{b=1}^{[q,Q]}\chi(b)\exp\left(-2\pi i\frac{ab}{Q}\right)\sum_{\substack{mn\le\frac{TQ}{2\pi}\\ m\equiv b \mod [q,Q]}}\Lambda(m)\omega(n)\\
&=S_1+S_2.
\end{aligned}
$$

By ω_0 we denote the principal Dirichlet character $\mod Q$.

First we deal with S_1. We note that

$$
\sum_{a=1}^{Q}\psi(a)\omega_0(a)\exp\left(-2\pi i\frac{ab}{Q}\right)=G(-b,\psi)=\overline{\psi}(b)G(-1,\psi).
$$

In view of [1, Thm. 8.10], we obtain $S_1=0$ whenever $(b,[q,Q])>1$. We use the orthogonality of Dirichlet characters one more time and we get

$$
\begin{aligned}
S_1=&G(-1,\psi)\frac{1}{\phi(Q)}\sum_{b=1}^{[q,Q]}\chi(b)\overline{\psi}(b)\sum_{\substack{mn\le\frac{TQ}{2\pi}\\ m\equiv b \mod [q,Q]}}\Lambda(m)\omega_0(n) \qquad (2.23)\\
=&G(-1,\psi)\frac{1}{\phi(Q)}\frac{1}{\phi([q,Q])}\sum_{\eta \mod [q,Q]}\left(-\sum_{mn\le\frac{TQ}{2\pi}}\Lambda(m)\eta(m)\omega_0(n)\right)\\
&\times\sum_{b=1}^{[q,Q]}\chi(b)\overline{\psi}(b)\omega_0(b)\overline{\eta}(b).
\end{aligned}
$$

Now we consider S_2. If $\omega\ne\omega_0$ and $(b,[q,Q])=d$, $d>1$, then we have

$$
\sum_{\substack{mn\le\frac{TQ}{2\pi}\\ m\equiv b \mod [q,Q]}}\Lambda(m)\omega(n)=O\Bigg(\sum_{\substack{m\le\frac{TQ}{2\pi}\\ m\equiv b \mod [q,Q]}}\Lambda(m)\Bigg)=O(\log(TQ)).
$$

We note

$$
\begin{aligned}
&\sum_{\substack{\omega \mod Q\\ \omega\ne\omega_0}}\sum_{b=1}^{[q,Q]}\chi(b)\sum_{a=1}^{Q}\psi(a)\overline{\omega}(a)\exp\left(-2\pi i\frac{ab}{Q}\right)\\
&=\sum_{\substack{\omega \mod Q\\ \omega\ne\omega_0}}G(-1,\psi\overline{\omega})\sum_{b=1}^{[q,Q]}\chi(b)\overline{\psi}(b)\omega(b)+O(Q).
\end{aligned}
$$

Hence after using the orthogonality of Dirichlet characters we get

$$S_2 = \frac{1}{\phi(Q)} \sum_{\substack{\omega \mod Q \\ \omega \neq \omega_0}} \sum_{a=1}^{Q} \psi(a)\overline{\omega}(a) \sum_{b=1}^{[q,Q]} \chi(b) \exp\left(-2\pi i \frac{ab}{Q}\right) \sum_{\substack{mn \leq \frac{TQ}{2\pi} \\ m \equiv b \mod [q,Q]}} \Lambda(m)\omega(n) \tag{2.24}$$

$$= \frac{1}{\phi(Q)} \frac{1}{\phi([q,Q])} \sum_{\substack{\omega \mod Q \\ \omega \neq \omega_0}} G(-1, \psi\overline{\omega}) \sum_{\eta \mod [q,Q]} \left(- \sum_{mn \leq \frac{TQ}{2\pi}} \Lambda(m)\eta(m)\omega(n) \right)$$

$$\times \sum_{b=1}^{[q,Q]} \chi(b)\overline{\psi}(b)\omega(b)\overline{\eta}(b) + O(Q^{\varepsilon} \log(TQ)).$$

Now combining (2.23) and (2.24) the left-hand side of the formula in Lemma 2.1.4 can be written in the following way.

$$\frac{1}{\phi(Q)} \frac{1}{\phi([q,Q])} \sum_{\omega \mod Q} G(-1, \psi\overline{\omega}) \sum_{\eta \mod [q,Q]} \left(- \sum_{mn \leq \frac{TQ}{2\pi}} \Lambda(m)\eta(m)\omega(n) \right) \tag{2.25}$$

$$\times \sum_{b=1}^{[q,Q]} \chi(b)\overline{\psi}(b)\omega(b)\overline{\eta}(b) + O(Q^{\varepsilon} \log(TQ)).$$

The sum over b is not equal to zero in the following three cases

$$\sum_{b=1}^{[q,Q]} \chi(b)\overline{\psi}(b)\omega(b)\overline{\eta}(b) = \phi([q,Q]) \qquad \text{for} \quad \begin{cases} \omega = \omega_0, \eta = \overline{\psi}\chi, \\ q|Q, \omega = \psi\overline{\chi}, \eta = \eta_0, \\ \omega \neq \omega_0, \eta = \chi\overline{\psi}\omega. \end{cases} \tag{2.26}$$

By Perron's formula we obtain

$$- \sum_{mn \leq \frac{TQ}{2\pi}} \Lambda(m)\eta(m)\omega(n) = \frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'}{L}(s,\eta)L(s,\omega) \left(\frac{TQ}{2\pi}\right)^s \frac{ds}{s} + O\left(\frac{T \log^2 T}{U}\right).$$

In [39, Chapter 8, Thm. 6.2] considering $q \ll \log^B T$ with B being any positive constant we find

$$L(s,\chi) \neq 0 \quad \text{for} \quad \sigma > 1 - \frac{c}{\log^{\frac{3}{4}+\epsilon} T},$$

where c is an absolute positive constant. With regard to this zero-free region for $L(s,\chi)$ let $b_1 = 1 - c/\log^{(\frac{3}{4}+\epsilon)} T$. Shifting the line of integration we get

$$- \sum_{mn \leq \frac{TQ}{2\pi}} \Lambda(m)\eta(m)\omega(n)$$

$$= \text{Res}_{s=1} \frac{L'}{L}(s,\eta)L(s,\omega) \left(\frac{TQ}{2\pi}\right)^s \frac{1}{s}$$

$$- \frac{1}{2\pi i} \left\{ \int_{a+iU}^{b_1+iU} + \int_{b_1+iU}^{b_1-iU} + \int_{b_1-iU}^{a-iU} \right\} \frac{L'}{L}(s,\eta)L(s,\omega) \left(\frac{TQ}{2\pi}\right)^s \frac{ds}{s} + O\left(\frac{T \log^2 T}{U}\right). \tag{2.27}$$

According to (2.26) we calculate the residues. In the first case we have

$$\lim_{s\to 1}\frac{d}{ds}\left((s-1)\frac{L'}{L}(s,\chi\overline{\psi})L(s,\omega_0)\left(\frac{TQ}{2\pi}\right)^s\frac{1}{s}\right)=\frac{\phi(Q)}{Q}\frac{TQ}{2\pi}\frac{L'}{L}(1,\chi\overline{\psi}).$$

In the second case we find

$$\lim_{s\to 1}\frac{d}{ds}\left((s-1)\frac{L'}{L}(s,\eta_0)L(s,\overline{\chi}\psi)\left(\frac{TQ}{2\pi}\right)^s\frac{1}{s}\right)=-\frac{TQ}{2\pi}L(1,\overline{\chi}\psi).$$

The residue in the third case is equal to zero.

Now combining (2.25), (2.26) and the values of the residues we get that the main terms of the left-hand side of the formula in Lemma 2.1.4

$$-\psi(-1)\tau(\psi)\frac{L'}{L}(1,\chi\overline{\psi})\frac{T}{2\pi}+\delta(q,Q)\frac{1}{\phi(Q)}G(-1,\chi\psi_0)L(1,\overline{\chi}\psi)\frac{TQ}{2\pi},$$

where $\delta(q,Q)=1$ if $q|Q$ and $\delta(q,Q)=0$ otherwise.

In a standard way we evaluate the integrals in (2.27). We consider

$$-\frac{1}{2\pi i}\left\{\int_{a+iU}^{b_1+iU}+\int_{b_1+iU}^{b_1-iU}+\int_{b_1-iU}^{a-iU}\right\}\frac{L'}{L}(s,\overline{\chi}\psi)L(s,\omega_0)\left(\frac{TQ}{2\pi}\right)^s\frac{ds}{s}+O\left(\frac{T\log^2 T}{U}\right).$$

According to (2.2) and (2.13) the first integral can be estimated as

$$\begin{aligned}\int_{a+iU}^{b_1+iU}\frac{L'}{L}(s,\overline{\chi}\psi)L(s,\omega_0)\left(\frac{TQ}{2\pi}\right)^s\frac{ds}{s}&=O\left((TQ)^a\frac{(UQ)^{\frac12}}{U}\log^3([q,Q]U)\right)\\&=O\left(TU^{-\frac12}Q^{\frac32}\log^3([q,Q]U)\right).\end{aligned}$$

We get the same bound for the third integral.

For the second integral we find

$$\begin{aligned}&\left\{\int_{b_1+iU}^{b_1+i}+\int_{b_1+i}^{b_1-i}+\int_{b_1-i}^{b_1-iU}\right\}\frac{L'}{L}(s,\overline{\chi}\psi)L(s,\omega_0)\left(\frac{TQ}{2\pi}\right)^s\frac{ds}{s}\\&=O\left(T^{b_1}U^{\frac12}Q^{\frac12+b_1}\log^3([q,Q]U)\right)+O\left((TQ)^{b_1}\log^{\frac32}T\right),\end{aligned}$$

where the second error term corresponds to the integral on the path $[b_1-i,b_1+i]$.

Now we choose $U=T^{1-b_1}$, $Q\ll\log^A T$ and $q\ll\log^B T$, where A and B are positive constants. Then we obtain an error

$$O\left(T^{1-\frac{c}{\log^{\frac34+\epsilon}T}}\right).$$

The other cases in (2.25) give the same error term.

Under GRH we choose $b_1=1/2+1/\log QT$, $U=QT$ and make use of the bound (2.4). This finishes the proof of Lemma 2.1.4.

□

2.2 Second moment

Theorem 2.2.1. *Let A and B be positive constants. Let $\psi \bmod Q$ and $\chi \bmod q$ be primitive Dirichlet characters and $\chi \neq \psi$. Then, uniformly for $Q \ll \log^A T$ and $q \ll \log^B T$, we have*

$$\sum_{0<\gamma_\chi \le T} L(\rho_\chi, \psi) L(1-\rho_\chi, \overline{\psi}) = \frac{\phi(Q)}{Q} \frac{T}{2\pi} \log^2 \frac{T}{2\pi} + a_1 \frac{T}{2\pi} \log \frac{T}{2\pi} + a_2 \frac{T}{2\pi} + a_3 \frac{T}{2\pi} + O\left(T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon} T}} \right),$$

where real constants a_1, a_2, a_3 depend only on q, Q, and are defined by the formula (2.40) below.

If we assume GRH then the left-hand side of the last equality can be replaced by

$$\sum_{0<\gamma_\chi \le T} |L(\tfrac{1}{2} + it_\chi, \psi)|^2$$

and the error term can be replaced by $O(q^{1+\varepsilon} Q^\varepsilon T^{\frac{1}{2}+\varepsilon} + q Q^{\frac{9}{2}+\varepsilon} T^\varepsilon + (QT)^{\frac{1}{2}+\varepsilon})$ uniformly for all Q and q.

Proof. Let $\chi \bmod q$ and $\psi \bmod Q$ be primitive Dirichlet characters such that $\chi \neq \psi$. The proof of the theorem relies on the method of Conrey, Ghosh, and Gonek [9]. The idea is to interpret the sum in question as a sum of residues, resp. a contour integral

$$\sum_{0<\gamma_\chi \le T} L(\rho_\chi, \psi) L(1-\rho_\chi, \overline{\psi}) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{L'}{L}(s,\chi) L(s,\psi) L(1-s, \overline{\psi}) ds, \tag{2.28}$$

which can be evaluated by the modified Gonek lemma (Lemma 2.1.1). We choose an appropriate path of integration $\mathfrak{C}$. In view of the bound for zeros (2.11) we can choose $\frac{1}{\log q} \ll b \le 1$ and $T \ge 2$ such that

$$\min_{\gamma_\chi} |b - \gamma_\chi| \gg \frac{1}{\log q} \quad \text{and} \quad \min_{\gamma_\chi} |T - \gamma_\chi| \gg \frac{1}{\log qT}. \tag{2.29}$$

Let $a = 1 + 1/\log(QT)$ and define the contour $\mathfrak{C}$ to be the rectangle with vertices $a+ib$, $a+iT$, $1-a+iT$, $1-a+ib$. Then

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{L'}{L}(s,\chi) L(s,\psi) L(1-s,\overline{\psi}) ds \qquad (2.30) \\ &= \frac{1}{2\pi i} \left\{ \int_{a+ib}^{a+iT} + \int_{a+iT}^{1-a+iT} + \int_{1-a+iT}^{1-a+ib} + \int_{1-a+ib}^{a+ib} \right\} \frac{L'}{L}(s,\chi) L(s,\psi) L(1-s,\overline{\psi}) ds \\ &=: \sum_{j=1}^{4} \mathcal{J}_j. \end{aligned}$$

First we consider

$$\mathcal{J}_1 = \frac{1}{2\pi i} \int_{a+i}^{a+iT} \frac{L'}{L}(s,\chi) L(s,\psi) L(1-s,\overline{\psi}) ds + O(1).$$

By applying the functional equation (2.1) we get

$$\begin{aligned}\mathcal{J}_1 =&\tau(\overline{\psi})\frac{1}{Q}\frac{1}{2\pi i}\int_{a+ib}^{a+iT}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(-\frac{\pi i s}{2}\right)\frac{L'}{L}(s,\chi)L(s,\psi)L(s,\psi)ds\\ &+\tau(\overline{\psi})\psi(-1)\frac{1}{Q}\frac{1}{2\pi i}\int_{a+ib}^{a+iT}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(\frac{\pi i s}{2}\right)\frac{L'}{L}(s,\chi)L(s,\psi)L(s,\psi)ds\\ =&\mathcal{J}_{11}+\mathcal{J}_{12}.\end{aligned}$$

In view of the modified Gonek lemma (Lemma 2.1.1) we have

$$\mathcal{J}_{11} = -\tau(\overline{\psi})\frac{1}{Q}\sum_{mnl\le\frac{TQ}{2\pi}}\Lambda(m)\chi(m)\psi(n)\psi(l)\exp\left(-2\pi i\frac{mnl}{Q}\right)+O((QT)^{\frac{1}{2}+\varepsilon})$$

and $\mathcal{J}_{12} = O(Q^{\frac{1}{2}})$. By this, Lemma 2.1.2, and the equality

$$\tau(\overline{\psi})\psi(-1)\tau(\psi) = Q \tag{2.31}$$

we obtain

$$\begin{aligned}\mathcal{J}_1 =&\frac{\phi(Q)}{Q}\frac{L'}{L}(1,(\chi\overline{\psi}))\frac{T}{2\pi}\left(\log\frac{TQ}{2\pi e}+2\gamma+2\sum_{p|Q}\frac{\log p}{p-1}+\left(\frac{L''}{L'}-\frac{L'}{L}\right)(1,\chi\overline{\psi})\right)\\ &-\delta(q,Q)(\chi\psi_0)(-1)\tau(\chi\psi_0)\frac{\tau(\overline{\psi})}{\phi(Q)}\frac{T}{2\pi}L(1,(\chi\overline{\psi}))^2+O\left(T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon}T}}\right).\end{aligned} \tag{2.32}$$

Under GRH the error term in (2.32) can be replaced by $O\left(q^{1+\varepsilon}\,Q^{\varepsilon}T^{\frac{1}{2}+\varepsilon}+q\,Q^{9/2+\varepsilon}T^{\varepsilon}+(QT)^{1/2+\varepsilon}\right)$.

Second we consider $\mathcal{J}_2$ and $\mathcal{J}_4$. Using bounds (2.2), (2.3), and (2.13) we get

$$\mathcal{J}_2 = \frac{1}{2\pi i}\int_{a+iT}^{1-a+iT}\frac{L'}{L}(s,\chi)L(s,\psi)L(1-s,\overline{\psi})ds = O\left((QT)^{11/16+\varepsilon}\log^2(qT)\right). \tag{2.33}$$

Similarly,

$$\mathcal{J}_4 \ll (Q)^{11/16+\varepsilon}\log^2 q. \tag{2.34}$$

Under GRH we use (2.4) instead of (2.2) and we obtain $\mathcal{J}_2,\mathcal{J}_4 \ll (QT)^{1/2+\varepsilon}\log^2(qT)$.

Next we consider $\mathcal{J}_3$. A change of variables $s\mapsto 1-\overline{s}$ gives

$$\mathcal{J}_3 = -\frac{1}{2\pi i}\int_{a+ib}^{a+iT}\frac{L'}{L}(1-\overline{s},\chi)L(1-\overline{s},\psi)L(\overline{s},\overline{\psi})ds.$$

By complex conjugation we get

$$\overline{\mathcal{J}_3} = -\frac{1}{2\pi i}\int_{a+ib}^{a+iT}\frac{L'}{L}(1-s,\overline{\chi})L(1-s,\overline{\psi})L(s,\psi)ds.$$

The functional equation (2.1) and its logarithmic derivative (2.9) together with the property (2.8) lead to

$$\begin{aligned}&\frac{L'}{L}(1-s,\overline{\chi})L(1-s,\overline{\psi}) = \left(\frac{\Delta'}{\Delta}(s,\chi)-\frac{L'}{L}(s,\chi)\right)\\ &\times\tau(\overline{\psi})\frac{1}{Q}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\left(\exp\left(-\frac{\pi i s}{2}\right)+\psi(-1)\exp\left(+\frac{\pi i s}{2}\right)\right)L(s,\psi).\end{aligned}$$

Then

$$
\begin{aligned}
\overline{\mathcal{J}_3} =& -\tau(\overline{\psi})\frac{1}{2\pi i Q}\int_{a+ib}^{a+iT}\frac{\Delta'}{\Delta}(s,\chi)\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(-\frac{\pi is}{2}\right)L^2(s,\psi)ds \\
& -\psi(-1)\tau(\overline{\psi})\frac{1}{2\pi iQ}\int_{a+ib}^{a+iT}\frac{\Delta'}{\Delta}(s,\chi)\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(\frac{\pi is}{2}\right)L^2(s,\psi)ds \qquad (2.35)\\
& +\tau(\overline{\psi})\frac{1}{2\pi iQ}\int_{a+ib}^{a+iT}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(-\frac{\pi is}{2}\right)\frac{L'}{L}(s,\chi)L^2(s,\psi)ds \\
& +\psi(-1)\tau(\overline{\psi})\frac{1}{2\pi iQ}\int_{a+ib}^{a+iT}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(\frac{\pi is}{2}\right)\frac{L'}{L}(s,\chi)L^2(s,\psi)ds \\
=& \sum_{j=1}^{4}\mathcal{F}_j,
\end{aligned}
$$

say.

First we consider $\mathcal{F}_1$. We rewrite $\mathcal{F}_1$ in the following way.

$$
\mathcal{F}_1 = -\frac{1}{Q}\tau(\overline{\psi})\int_b^T\frac{\Delta'}{\Delta}(a+i\tau,\chi)d\left(\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(-\frac{\pi is}{2}\right)L^2(s,\psi)ds\right).
$$

By Lemmas 2.1.1 and 2.1.3 we get

$$
\begin{aligned}
& \frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(-\frac{\pi is}{2}\right)L^2(s,\psi)ds \\
& = \sum_{mn\le\frac{\tau Q}{2\pi}}\psi(m)\psi(n)\exp\left(-2\pi i\frac{mn}{Q}\right)+O\left((Q\tau)^{\frac12+\varepsilon}\right) \\
& = \psi(-1)\tau(\psi)\frac{\phi(Q)}{Q}\frac{\tau}{2\pi}\left(\log\frac{\tau Q}{2\pi e}+2\gamma+2\sum_{p|Q}\frac{\log p}{p-1}\right)+O\left(Q^{-\frac18+\varepsilon}\tau^{\frac{11}{16}+\varepsilon}+Q^4\right).
\end{aligned}
$$

Hence, in view of the asymptotic formula (2.8) for the logarithmic derivative of the delta function and the property (2.31) of the Gauss sum we obtain

$$
\begin{aligned}
\mathcal{F}_1 =& -\frac{\phi(Q)}{Q}\int_b^T\left(-\log\frac{\tau q}{2\pi}+O\left(\frac{1}{\tau}\right)\right) \\
& \times d\left(\frac{\tau}{2\pi}\left(\log\frac{\tau Q}{2\pi e}+2\gamma+2\sum_{p|Q}\frac{\log p}{p-1}\right)+O\left(Q^{-\frac18+\varepsilon}\tau^{\frac{11}{16}+\varepsilon}+Q^4\right)\right) \qquad (2.36)\\
=& \frac{\phi(Q)}{Q}\frac{T}{2\pi}\left(\log\frac{Tq}{2\pi e}\log\frac{TQ}{2\pi e}+2\log\frac{Tq}{2\pi e}\left(\gamma+\sum_{p|Q}\frac{\log p}{p-1}\right)+1\right) \\
& +O\left(Q^{-\frac18+\varepsilon}T^{\frac{11}{16}+\varepsilon}\log q+Q^4\log(qT)\right).
\end{aligned}
$$

Under GRH the error term for $\mathcal{F}_1$ can be replaced by $O\left(Q^{-1/2+\varepsilon}T^{1/2+\varepsilon}\log q+Q^4\log(qT)\right)$.

Reasoning similarly to $\mathcal{F}_1$ we obtain

$$
\begin{aligned}
\mathcal{F}_2 =& -\frac{1}{Q}\psi(-1)\tau(\overline{\psi})\int_b^T\frac{\Delta'}{\Delta}(a+i\tau,\chi) \\
& \times d\left(\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(\frac{\pi is}{2}\right)L(s,\psi)ds\right) \qquad (2.37)\\
=& O(Q^{\frac12}\log(Tq)).
\end{aligned}
$$

Under GRH for $\mathcal{F}_2$ we use the same bound as in (2.37).

We turn to the integral $\mathcal{F}_3$. Using Lemmas 2.1.1 and 2.1.2 we get

$$\begin{aligned}\mathcal{F}_3 =& -\frac{1}{Q}\tau(\overline{\psi})\sum_{mnl\le\frac{TQ}{2\pi}}\Lambda(m)\chi(m)\psi(n)\psi(l)\exp\left(-2\pi i\frac{mnl}{Q}\right)+O\left(Q^{\frac{1}{2}}T^{\frac{1}{2}+\varepsilon}\right)\\ =&\frac{\phi(Q)}{Q}\frac{L'}{L}(1,(\chi\overline{\psi}))\frac{T}{2\pi}\left(\log\frac{TQ}{2\pi e}+2\gamma+2\sum_{p|Q}\frac{\log p}{p-1}+\left(\frac{L''}{L'}-\frac{L'}{L}\right)(1,\chi\overline{\psi})\right)\\ &-\delta(q,Q)(\chi\psi_0)(-1)\tau(\chi\psi_0)\frac{\tau(\overline{\psi})}{\phi(Q)}\frac{T}{2\pi}L(1,(\chi\overline{\psi}))^2+O\left(T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon}T}}\right).\end{aligned} \tag{2.38}$$

Under GRH the error term can be replaced by $O\left(q^{1+\varepsilon}\,Q^{\varepsilon}T^{\frac{1}{2}+\varepsilon}+q\,Q^{9/2+\varepsilon}T^{\varepsilon}+(QT)^{1/2+\varepsilon}\right)$.

In a similar way as above we have

$$\mathcal{F}_4 \ll Q^{\frac{1}{2}}\log(Tq). \tag{2.39}$$

The last bound we use also under GRH.

Now in view of (2.28), (2.30), (2.32)–(2.39) we obtain

$$\begin{aligned}&\sum_{0<\gamma_\chi\le T}L(\rho_\chi,\psi)L(1-\rho_\chi,\overline{\psi})\\ &=\frac{\phi(Q)}{Q}\frac{T}{2\pi}\left(\log\frac{Tq}{2\pi e}\log\frac{TQ}{2\pi e}+2\log\frac{Tq}{2\pi e}\left(\gamma+\sum_{p|Q}\frac{\log p}{p-1}\right)+1\right)\\ &+\frac{\phi(Q)}{Q}\frac{T}{2\pi}\left[\left(\log\frac{TQ}{2\pi e}+2\left(\gamma+\sum_{p|Q}\frac{\log p}{p-1}\right)\right)\left(\frac{L'}{L}(1,(\chi\overline{\psi}))+\frac{L'}{L}(1,(\overline{\chi}\psi))\right)\right.\\ &\left.+\frac{L'}{L}\cdot\left(\frac{L''}{L'}-\frac{L'}{L}\right)((1,\chi\overline{\psi})+(1,\overline{\chi}\psi))\right]\\ &-\delta(q,Q)\left(\tau(\chi\psi_0)\tau(\overline{\psi})L(1,(\chi\overline{\psi}))^2+\tau(\overline{\chi}\psi_0)\tau(\psi)L(1,(\overline{\chi}\psi))^2\right)\frac{(\chi\psi_0)(-1)}{\phi(Q)}\frac{T}{2\pi}\\ &+O\left(T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon}T}}\right).\end{aligned} \tag{2.40}$$

Accordingly, the notes after each formula (2.28), (2.30), (2.32)–(2.39) give the error term under GRH. By this Theorem 2.2.1 is proved.

□

2.3 First moment

Theorem 2.2.1 extends the following theorem

Theorem 2.3.1. *Let A and B be positive constants. Let $\psi \bmod Q$ and $\chi \bmod q$ be primitive Dirichlet characters and $\chi\neq\psi$. Then, uniformly for $Q\ll\log^A T$ and $q\ll\log^B T$, we have*

$$\begin{aligned}\sum_{0<\gamma_\chi\le T}L(\rho_\chi,\psi)=&\frac{T}{2\pi}\log\frac{Tq}{2\pi e}-\delta(q,Q)L(1,\chi\overline{\psi})\psi(-1)\tau(\psi)\frac{\tau(\overline{\chi}\psi_0)}{\phi(Q)}\frac{T}{2\pi}\\ &+\frac{L'}{L}(1,\psi\overline{\chi})\frac{T}{2\pi}+O\left(T\exp(-c\log^{\frac{1}{4}-\varepsilon}T)\right),\end{aligned}$$

where $\delta(q,Q) = 1$ *if* $q|Q$*,* $\delta(q,Q) = 0$ *otherwise,* ψ_0 *is the principal Dirichlet character* $\bmod Q$ *and* c *is a positive absolute constant.*

Under GRH the error term can be replaced by $O\left((TQ)^{1/2+\varepsilon}q^{\varepsilon}\right)$*, which is valid uniformly for all* Q *and* q*.*

Proof. The proof of the theorem relies on the same idea and the same method which are used to prove Theorem 2.2.1.

Let $\chi \bmod q$ and $\psi \bmod Q$ be primitive Dirichlet characters such that $\chi \neq \psi$. Now let $a = 1 + 1/\log(QT)$ and define the contour $\mathfrak{C}$ to be the rectangle with vertices $a+ib$, $a+iT$, $1-a+iT$, $1-a+ib$ satisfying the same conditions as in Theorem 2.2.1. Then we sum in question is equal to a contour integral

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{L'}{L}(s,\chi)L(s,\psi)ds \\
&= \frac{1}{2\pi i}\left\{\int_{a+ib}^{a+iT} + \int_{a+iT}^{1-a+iT} + \int_{1-a+iT}^{1-a+ib} + \int_{1-a+ib}^{a+ib}\right\} \frac{L'}{L}(s,\chi)L(s,\psi)ds \\
&= \sum_{j=1}^{4} \mathcal{J}_j.
\end{aligned}$$

First we consider

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{2\pi i}\int_{a+ib}^{a+iT} \frac{L'}{L}(s,\chi)L(s,\psi)ds \\
&= \frac{1}{2\pi}\int_{b}^{T} \frac{L'}{L}(a+it,\chi)L(a+it,\psi)dt \\
&= -\sum_{m=2}^{\infty}\sum_{n=1}^{\infty} \frac{\chi(m)\Lambda(m)\psi(n)}{(mn)^a}\frac{1}{2\pi}\int_b^T \frac{1}{(mn)^{it}}dt \\
&= -\sum_{m=2}^{\infty}\sum_{n=1}^{\infty} \frac{\chi(m)\Lambda(m)\psi(n)}{(mn)^a}\frac{1}{2\pi}\left(\frac{1}{-i\log(mn)(mn)^{it}}\Bigg|_b^T\right) \\
&= O\left(\frac{L'}{L}(a,\chi)L(a,\psi)\right) = O\left(\frac{\zeta'}{\zeta}(a)\zeta(a)\right).
\end{aligned}$$

By the Laurent expansions at $s=1$,

$$\begin{aligned}
\zeta(s) &= \frac{1}{s-1} + \gamma + O(s-1), \\
\frac{\zeta'}{\zeta}(s) &= \frac{-1}{s-1} + \gamma + O(s-1),
\end{aligned}$$

where $\gamma := \lim_{N\to\infty}(\frac{1}{N}\sum_{n=1}^{N}\frac{1}{n} - \log N) = 0.577\ldots$ is the Euler-Mascheroni constant, we get

$$\mathcal{J}_1 = O(\log^2 QT).$$

Second we consider $\mathcal{J}_2$. Using (2.2), (2.3) and (2.13) we get

$$\mathcal{J}_2 = \frac{1}{2\pi i}\int_{a+iT}^{1-a+iT} \frac{L'}{L}(s,\chi)L(s,\psi)ds = O\left((QT)^{\frac{1}{2}+\varepsilon}\log^2(qT)\right).$$

Next we consider $\mathcal{J}_4$. Similarly as before we get

$$\mathcal{J}_4 = \frac{1}{2\pi i}\int_{1-a+ib}^{a+ib} \frac{L'}{L}(s,\chi)L(s,\psi)ds = O\left(Q^{\frac{1}{2}+\varepsilon}\log^2(q+1)\right).$$

Hence, $\mathcal{J}_1,\ \mathcal{J}_2,\ \mathcal{J}_4 = O\left((QT)^{\frac{1}{2}+\varepsilon}\log^2(qT)\right)$. Under GRH we use the same bound.

Now we consider $\mathcal{J}_3$. A change of variables $s \mapsto 1-\overline{s}$ gives

$$\mathcal{J}_3 \quad = \quad -\frac{1}{2\pi i}\int_{a+ib}^{a+iT} \frac{L'}{L}(1-\overline{s},\chi)L(1-\overline{s},\psi)ds.$$

After conjunction we get

$$\overline{\mathcal{J}_3} \quad = \quad -\frac{1}{2\pi i}\int_{a+ib}^{a+iT} \frac{L'}{L}(1-s,\overline{\chi})L(1-s,\overline{\psi})ds.$$

By the functional equation (2.5) and its logarithmic derivative the integrand of $\overline{\mathcal{J}_3}$ can be rewritten as

$$\begin{aligned}\frac{L'}{L}(1-s,\overline{\chi})L(1-s,\overline{\psi}) \quad &= \quad \left(\frac{\Delta'}{\Delta}(s,\chi) - \frac{L'}{L}(s,\chi)\right)\frac{1}{Q}G(1,\overline{\psi})\left(\frac{Q}{2\pi}\right)^s\Gamma(s) \\ &\quad \times\left(\exp\left(-\frac{\pi is}{2}\right) + \psi(-1)\exp\left(+\frac{\pi is}{2}\right)\right)L(s,\psi);\end{aligned}$$

here $G(n,\chi)$ denotes the Gauss sum associated with $\chi \mod q$, given by

$$G(n,\chi) = \sum_{a=1}^{q}\chi(a)\exp\left(2\pi i\frac{an}{q}\right).$$

Then

$$\begin{aligned}\overline{\mathcal{J}_3} \quad = \quad & -\frac{1}{Q}G(1,\overline{\psi})\frac{1}{2\pi i}\int_{a+ib}^{a+iT}\frac{\Delta'}{\Delta}(s,\chi)\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(-\frac{\pi is}{2}\right)L(s,\psi)ds \\ & -\frac{1}{Q}G(-1,\overline{\psi})\frac{1}{2\pi i}\int_{a+ib}^{a+iT}\frac{\Delta'}{\Delta}(s,\chi)\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(+\frac{\pi is}{2}\right)L(s,\psi)ds \\ & +\frac{1}{Q}G(1,\overline{\psi})\frac{1}{2\pi i}\int_{a+ib}^{a+iT}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(-\frac{\pi is}{2}\right)\frac{L'}{L}(s,\chi)L(s,\psi)ds \\ & +\frac{1}{Q}G(-1,\overline{\psi})\frac{1}{2\pi i}\int_{a+ib}^{a+iT}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(+\frac{\pi is}{2}\right)\frac{L'}{L}(s,\chi)L(s,\psi)ds \\ = \quad & \sum_{j=1}^{4}\mathcal{F}_j,\end{aligned}$$

say.

First we consider $\mathcal{F}_1$. We rewrite it in the following way

$$\mathcal{F}_1 = -\frac{1}{Q}G(1,\overline{\psi})\int_b^T\frac{\Delta'}{\Delta}(a+i\tau,\chi)\,d\left(\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(-\frac{\pi is}{2}\right)L(s,\psi)ds\right).$$

By Lemma 2.1.1 we get

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(-\frac{\pi i s}{2}\right)L(s,\psi)ds\\
&\quad=\sum_{n\le\frac{\tau Q}{2\pi}}\psi(n)\exp\left(-2\pi i\frac{n}{Q}\right)+O\left(\tau^{\frac{1}{2}+\epsilon}\right)\\
&\quad=\sum_{a=1}^{Q}\psi(a)\exp\left(-2\pi i\frac{a}{Q}\right)\sum_{\substack{n\le\frac{\tau Q}{2\pi}\\ n\equiv a \mod Q}}1+O\left(Q\tau^{\frac{1}{2}+\epsilon}\right)\\
&\quad=G(-1,\psi)\frac{\tau}{2\pi}+O\left(Q\tau^{\frac{1}{2}+\epsilon}\right).
\end{aligned}$$

According to (2.8) we have

$$\begin{aligned}
\mathcal{F}_1 &= -\frac{1}{Q}G(1,\overline{\psi})\int_b^T\left(-\log\frac{\tau q}{2\pi}+O\left(\frac{1}{\tau}\right)\right)d\left(G(-1,\psi)\frac{\tau}{2\pi}+O\left(Q\tau^{\frac{1}{2}+\epsilon}\right)\right)\\
&= \frac{T}{2\pi}\log\frac{Tq}{2\pi e}+O\left((QT)^{\frac{1}{2}+\epsilon}\log(Tq)\right),
\end{aligned}$$

where we have used that

$$G(1,\overline{\psi})G(-1,\psi)=Q.$$

Second, we consider

$$\mathcal{F}_2=-\frac{1}{Q}G(-1,\overline{\psi})\int_b^T\frac{\Delta'}{\Delta}(a+i\tau,\chi)\,d\left(\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(+\frac{\pi i s}{2}\right)L(s,\psi)ds\right).$$

By (2.8) and Lemma 2.1.1 we have

$$\mathcal{F}_2=\frac{1}{Q}G(-1,\overline{\psi})\int_b^T\left(\log\frac{tq}{2\pi}+O\left(\frac{1}{t}\right)\right)dO(Q)=O(Q^{\frac{1}{2}}\log(Tq)).$$

Now we calculate

$$\mathcal{F}_3=\frac{1}{Q}G(1,\overline{\psi})\frac{1}{2\pi i}\int_{a+i}^{a+iT}\left(\frac{Q}{2\pi}\right)^s\Gamma(s)\exp\left(-\frac{\pi i s}{2}\right)\frac{L'}{L}(s,\chi)L(s,\psi)ds+O(1).$$

Using Lemma 2.1.1 and Lemma 2.1.4 we get

$$\mathcal{F}_3=\frac{L'}{L}(1,\chi\overline{\psi})\frac{T}{2\pi}-\delta(q,Q)\frac{1}{\phi(Q)}G(1,\overline{\psi})G(-1,\chi\psi_0)L(1,\overline{\chi}\psi)\frac{T}{2\pi}+O\left(T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon}T}}\right).$$

Finaly we consider $\mathcal{F}_4$. By Lemma 2.1.1 we have $\mathcal{F}_4=O(Q^{\frac{1}{2}+\varepsilon})$.
Gathering everything together, after conjugation we have

$$\begin{aligned}
\sum_{0<\gamma_\chi\le T}L(\rho_\chi,\psi)=&\frac{T}{2\pi}\log\frac{Tq}{2\pi e}-\delta(q,Q)L(1,\chi\overline{\psi})\psi(-1)\tau(\psi)\frac{\tau(\overline{\chi}\psi_0)}{\phi(Q)}\frac{T}{2\pi}+\frac{L'}{L}(1,\psi\overline{\chi})\frac{T}{2\pi}\\
&+O\left(T^{1-\frac{c}{\log^{\frac{3}{4}+\epsilon}T}}\right).
\end{aligned}$$

Assuming GRH we get an asymptotic formula with the error term

$$O\left((TQ)^{\frac{1}{2}+\epsilon}q^{\epsilon}\right),$$

uniformly for $q,\ Q \ll T^{1-\epsilon}$.

This proves the theorem.

□

2.4 Corollary

Theorems 2.2.1 and 2.3.1 lead to the following corollary

Corollary 2.4.1. *Assume GRH. Let A be any positive real number. Let $\psi \bmod Q$ and $\chi \bmod q$ be primitive Dirichlet characters and $\chi \neq \psi$. Then, uniformly for $q \ll (\log T)^A$ and $Q \ll (\log T)^{2-\varepsilon}$, we have*

$$\sum_{\substack{0<\gamma_\chi\leq T\\ L(1/2+\gamma_\chi,\psi)\neq 0}} 1 \gg \frac{Q}{\phi(Q)}T.$$

Proof. Let

$$S_1 := \sum_{0<\gamma_\chi\leq T} L(\rho_\chi,\psi) \quad \text{and} \quad S_2 := \sum_{0<\gamma_\chi\leq T} |L(\rho_\chi,\psi)|^2.$$

By Hoelder's inequality we get

$$\sum_{\substack{0<\gamma_\chi\leq T\\ L(1/2+\gamma_\chi,\psi)\neq 0}} 1 \geq \frac{|S_1|^2}{S_2}. \tag{2.41}$$

To evaluate sums S_1 and S_2 we will use Theorems 2.3.1 and 2.2.1. First we state several helpful inequalities. From Davenport [11, formulas (11) and (13) of Section 14], for a non-principal Dirichlet character $\chi \bmod q$, we have

$$|L(1,\chi)| \ll \log q, \qquad |L'(1,\chi)| \ll (\log q)^2, \qquad |L''(1,\chi)| \ll (\log q)^3.$$

By Siegel's Theorem (see Davenport [11, Section 21]), for a real primitive Dirichlet character $\chi \bmod q$, we have

$$|L(1,\chi)| \gg q^{-\varepsilon}. \tag{2.42}$$

If $\chi \bmod q$ is a complex primitive character we use the zero free region of $L(s,\chi)$ (see Davenport [11, formula (6) of Section 14]) and follow the proof of Li [33, Corollary 7] to obtain the lower bound

$$|L(1,\chi)| \gg q^{-\varepsilon}. \tag{2.43}$$

We will show that the lower bounds (2.42) and (2.43) are valid also for an imprimitive character χ. Indeed, if χ is the imprimitive character induced by a primitive character χ_1 then

$$L(s,\chi) = L(s,\chi_1)\prod_{p|q}\left(1-\frac{\chi_1(p)}{p^s}\right).$$

Thus for imprimitive character the bounds (2.42) and (2.43) follow by

$$\left|\prod_{p|q}\left(1-\frac{\chi_1(p)}{p}\right)\right| \geq \prod_{p|q}\left(1-\frac{1}{p}\right) = \frac{\phi(q)}{q} \gg q^{-\varepsilon}.$$

By above we have that

$$\delta(q,Q)L(1,\chi\overline{\psi})\psi(-1)\tau(\psi)\frac{\tau(\overline{\chi}\psi_0)}{\phi(Q)}\frac{T}{2\pi} \ll Q^{\frac{1}{2}}\log(Q)T \ll T(\log T)^{1-\varepsilon}$$

and

$$\frac{L'}{L}(1,\psi\overline{\chi})\frac{T}{2\pi} \ll (qQ)^{\varepsilon}T \ll T(\log T)^{\varepsilon}.$$

This and Theorem 2.3.1 give that

$$S_1 \gg T\log T.$$

We will find an upper bound for $|S_2|$. We have

$$\sum_{p|Q}\frac{\log p}{p-1} \ll \log Q,$$

$$\begin{aligned}&\delta(q,Q)\left(\tau(\chi\psi_0)\tau(\overline{\psi})L(1,(\chi\overline{\psi}))^2+\tau(\overline{\chi}\psi_0)\tau(\psi)L(1,(\overline{\chi}\psi))^2\right)\frac{(\chi\psi_0)(-1)}{\phi(Q)}\frac{T}{2\pi}\\ &\ll \phi(Q)Q^{\frac{1}{2}}(\log Q)^2\frac{T}{\phi(Q)} \ll T\log T,\end{aligned}$$

and

$$\frac{L''}{L}(1,\psi\overline{\chi})\frac{T}{2\pi} \ll (qQ)^{\varepsilon}T \ll T\log T.$$

Then Theorem 2.2.1 and the formula (2.40) lead to

$$S_2 \ll \frac{\phi(Q)}{Q}T(\log T)^2.$$

Now the corollary folows by formula (2.41).

□

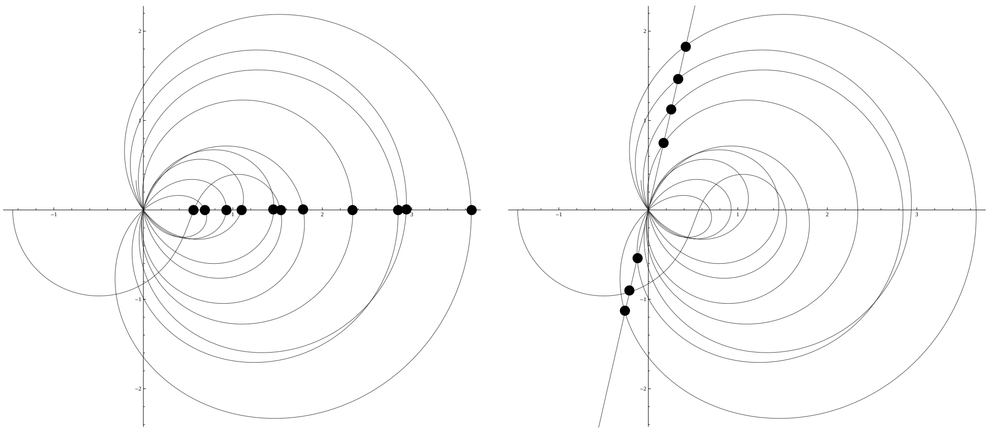
Conclusions

In the thesis the following results for the Riemann zeta function and Dirichlet L-functions are established:

1. The curve $t \mapsto \zeta(\frac{1}{2} + it)$, $t > 0$ expands to all directions on the complex plane.
2. We can localize extreme values of the Riemann zeta function on the critical line.
3. We can transfer discrete moments to continuous moments.
4. Asymptotic formulas a sum of a Dirichlet L-function over the zeros of another Dirichlet L-function give us continuous lower bound for the number of non-coincide non-trivial zeros.

Santrauka

Analizinė skaičių teorija yra skaičių teorijos dalis, kurioje naudojantis matematinės analizės ir kompleksinio kintamojo funkcijų tyrimo metodais, sprendžiami uždaviniai susiję su sveikaisiais skaičiais. Manoma, kad analizinės skaičių teorijos pradžią žymi Dirichlet eilučių ir Dirichlet L-funkcijų taikymai.



(a) (b)

Figure 2.1: Pavaizduota kreivė $t \mapsto \zeta(\frac{1}{2} + it)$, kai t kinta nuo 0 iki 50: (a) juodi taškai yra Gram'o taškai $t_n(0)$, (b) juodi taškai yra apibendrintieji Gram'o taškai $t_n(\frac{3}{7}\pi)$.

Iš paveikslo matyti, kad kreivės $t \mapsto \zeta(\frac{1}{2}+it)$ realioji dalis linkusi būti teigiama. Tai pastebėjo Edwards'as ir savo monografijoje [12] rašė *"...the real part of $\zeta(s)$ has a strong tendency to be positive"* [1] (p. 121). Kritinė tiesė yra riba, skirianti kreives $t \mapsto \zeta(\sigma + it)$, $\frac{1}{2} < \sigma < 1$, kurios yra visur tirštos aibėje $\mathbb{C}$, nuo kreivių $t \mapsto \zeta(\sigma + it)$, $\sigma < \frac{1}{2}$, kurios nėra visur tirštos aibėje $\mathbb{C}$ (jeigu galioja Riemann'o hipotezė). Klausimas, ar kreivė $t \mapsto \zeta(\frac{1}{2} + it)$ yra visur tiršta aibėje $\mathbb{C}$, yra sunkus ir neišspręstas uždavinys. Pagrindinis disertacijos rezultatas yra išvada (1.5.1), kuri teigia, kad kreivė $t \mapsto \zeta(\frac{1}{2} + it)$ plečiasi į visas puses, t. y. jeigu mes nubrėšime apskritimą, kurio centras yra koordinačių pradžioje ir spindulys lygus T, bei nubrėšime spindulį einantį iš apskritimo centro, tai kreivė $t \mapsto \zeta(\frac{1}{2}+it)$ kirs nubrėžtą spindulį be galo daug kartų už apskritimo ribų (žr. 4 pav.). Atskiras išvados (1.5.1) atvejis gali būti formuluojamas taip: *Riemann'o dzeta funkcija ant kritinės tiesės įgyja be galo daug neigiamų reikšmių ir jos yra neaprėžtos.*

[1] "...realioji $\zeta(s)$ dalis linkusi būti teigiama".

Notation

$s = \sigma + it$	complex variable, where $\sigma, t \in \mathbb{R}$ and $i = \sqrt{-1}$
ϕ	real number from the interval $[0, \pi)$
$t_n(\phi)$	generalized Gram's points
$X(s),\ Y(s),\ X_1(s),\ Y_1(s)$	Dirichlet polynomials
$d_\kappa(n)$	generalized divisor function for $n \in \mathbb{N}$, where $\kappa > 0$ and $\kappa \in \mathbb{R}$
$\Gamma(s)$	Euler gamma-function defined, for $\sigma > 0$, by $\Gamma(s) = \int\limits_0^\infty e^{-x} x^{s-1} dx$, and by analytic continuation elsewhere
$\zeta(s)$	Riemann zeta function defined, for $\sigma > 1$, by $\zeta(s) = \sum\limits_{m=1}^\infty m^{-s}$, and by analytic continuation elsewhere
$\chi \mod q, \psi \mod Q$	Dirichlet characters, where $q, Q \in \mathbb{N}$
χ_0, ψ_0	principal Dirichlet characters
$G(n, \chi)$	Gauss sum
$\psi(n)$	Euler totient function
$L(s, \chi)$	Dirichlet L-function defined, for $\sigma > 1$, by $L(s, \chi) = \sum\limits_{m=1}^\infty \chi(m) m^{-s}$, and by analytic continuation elsewhere

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