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ASYMPTOTIC ANALYSIS OF THE SUMS OF HEAVY-TAILED RANDOM VARIABLES

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VILNIAUS UNIVERSITETAS

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Lina Dindienė,
Vilnius.

Contents

Notation

$S_n = \sum_{i=1}^n X_i$.
$S_{(n)} = \max\{S_1, \dots, S_n\}$.
$S_n^{\Theta +} = \sum_{i=1}^n \Theta_i X_i^+$.
$S_n^{\Theta} = \sum_{i=1}^n \Theta_i X_i$.
$X_{(n)} = \max\{X_1, \dots, X_n\}$.
$G_n(x) = \mathrm{P}(X_{(n)} < x)$.
$S_n^{(+)} = X_1^+ + \dots + X_n^+$.
$x^+ = \max(x, 0)$.
$Z_\tau = \Theta_1 + \dots + \Theta_\tau$.
$H_n(x) = n^{-1}(F_1(x) + \dots + F_n(x))$.
$S_{(\tau)}^{\Theta} = \max_{k \le \tau} S_k^{\Theta}$.
For positive functions $a(x)$ and $b(x)$:
$a(x) \sim b(x)$ if $\lim\limits_{x\to\infty} a(x)/b(x) = 1$;
$a(x) \lesssim b(x)$ if $\limsup\limits_{x\to\infty} a(x)/b(x) \le 1$;
$a(x) \gtrsim b(x)$ if $\liminf\limits_{x\to\infty} a(x)/b(x) \ge 1$;
$a(x) \asymp b(x)$ if $0 < \liminf\limits_{x\to\infty} \frac{a(x)}{b(x)} \le \limsup\limits_{x\to\infty} \frac{a(x)}{b(x)} < \infty$ (also are called weakly equivalent);
$a(x) = o(b(x))$ if $\lim\limits_{x\to\infty} a(x)/b(x) = 0$.
$\lfloor x \rfloor$ denotes the integer part of a real number x.
$x \vee y$ denotes the maximal value between real numbers x and y.
$x \wedge y$ denotes the minimal value between real numbers x and y.
$\mathbb{1}_A$ - the indicator function of an event A.
All limit relationships hold for x tending to ∞, unless stated otherwise.

Chapter 1

Introduction

The notion of heavy-tailed distribution function (d. f.) naturally appears in the analysis of the sum of random variables (r.v.s). Nowadays such functions are widely applicable in stochastic systems and their importance is obvious: modeling large claim size in insurance and finance, extremal events and other risk processes. Various other popular samples follow heavy-tailed d. f. (distribution of wealth, file sizes in computer systems, connection durations, web pages sizes and others).

The main characteristic of heavy-tailed distribution is that there are a few large values compared to the other values of the given sample. Besides that, not all moments exist and other statistics are used for heavy-tailed d. f. So the classical central limit theorem or confidence interval formulas can not be applicable for such distributions. Hence, some special approaches are needed to handle it.

Since the heavy-tailed r.v.s have the property that the small observations are asymptotically negligible compared to the largest one, many researchers try to compare asymptotically the tail probability of the sum with the tail probability of the maximal element. These and similar asymptotics are widely described and discussed in many papers and monographs, which deal with the sum of heavy-tailed r.v.s. We will also discuss a few problems (arised from the already solved one) of such sums.

In this doctoral disertation we consider the sequence of real-valued r.v.s $X_1, \dots, X_n$ with heavy-tailed d. f.s.

Our first problem is to investigate the asymptotic tail of sum $S_n := X_1 + \cdots + X_n$ for dependent nonidentically distributed summands. Dependence among primary r.v.s is important for practical situations: variables are often related to each other. In Chapter 3, motivated by the paper of

[42] (see also [15]), we restrict some conditions to the (heavy-tailed) distribution of $X_{(n)} := \max\{X_1, \dots, X_n\}$ and prove the weak max-sum (see (3.1)) equivalence among quantities $\mathrm{P}(S_n > x)$, $\mathrm{P}(X_{(n)} > x)$ and $\sum_{i=1}^{n} \mathrm{P}(X_i > x)$ for nonidentically distributed r.v.s. We give some copula-based examples of dependence structures.

The analysis of the sums S_n led us to the discussion about the randomly weighted sums $S_n^{\Theta} := \sum_{i=1}^{n} \Theta_i X_i$, where $X_1, \dots, X_n$ are real-valued r.v.s with some dependence structure and distributions $F_1, \dots, F_n$, respectively; $\Theta_1, \dots, \Theta_n$ are arbitrarily dependent positive r.v.s., independent of $X_1, \dots, X_n$ (Chapter 4).

We consider two questions. The first of them is about the closure property of the sum S_n^{Θ} in the case of long-tailed primary variables $X_1, \dots, X_n$. More precisely, we investigate when, given that distributions $F_1, \dots, F_n$ are from the long-tailed distribution class (see Section 2.1 for definition), the distribution function (d. f.) of sum S_n^{Θ} is long-tailed too.

The second question is the asymptotic equivalence of the tail probabilities $\mathrm{P}(S_n^{\Theta} > x)$ and $\mathrm{P}(S_n^{\Theta+} > x)$, where $S_n^{\Theta+} := \Theta_1 X_1^+ + \dots + \Theta_n X_n^+$, that is, for a given dependence structure among the heavy-tailed r.v.s $X_1, \dots, X_n$, whether it holds that

$$\mathrm{P}(S_n^{\Theta} > x) \;\sim\; \mathrm{P}(S_n^{\Theta+} > x) \tag{1.1}$$

for $x \to \infty$?

In Chapter 4 we extend the result on the closure property and tail asymptotics of randomly weighted sums S_n^{Θ} under similar dependence structure as in [72] for any $n \geq 2$. Also, we study the case where the distribution of random vector $(X_1, \dots, X_n)$ is generated by an absolutely continuous copula. In particular, we show that, if the distribution of $(X_1, \dots, X_n)$ is generated by the FGM copula, marginal distributions are from a certain class of heavy-tailed d. f. and random weights are bounded, then the probabilities $\mathrm{P}(S_n^{\Theta} > x)$ and $\mathrm{P}(S_n^{\Theta+} > x)$ are asymptotically equivalent to $\sum_{k=1}^{n} \mathrm{P}(\Theta_k X_k > x)$.

The last subject we investigate (Chapter 5) is randomly weighted and randomly stoped sums $S_k^{\Theta} := \sum_{i=1}^{k} \Theta_i X_i$, $k \geq 1$, where $\{X_1, X_2, \dots\}$ is a sequence of identically distributed r.v.s, having a certain dependence structure, with heavy-tailed d. f. F_X; $\Theta_1, \Theta_2, \dots$ are some nonnegative r.v.s. We consider the random maximum of these sums,

$$S_{(\tau)}^{\Theta} \;=\; \max_{k \leq \tau} S_k^{\Theta}$$

with nonnegative integer-valued r.v. τ. We assume that $\{X_1, X_2, \dots\}$, $\{\Theta_1, \Theta_2, \dots\}$ and τ are mutually independent. We are interested in the asymptotics of tail probability $\mathrm{P}(S_{(\tau)}^{\Theta} > x)$ and $\mathrm{P}(Z_\tau > x)$ as $x \to \infty$, where $Z_\tau := \Theta_1 + \cdots + \Theta_\tau$.

In Chapter 5 we specify the conditions, under which relation $\mathrm{P}(Z_\tau > x) = o(\overline{F_X}(x))$ holds for a wide class of heavy tailed distribution functions and dependence structures. Together, we extend the main result in [73] to a wider class of dependence structure.

All results presented in this dissertation are achieved by the author of the thesis together with the co-authors. The theorems and propositions proved in the dissertation are original and can be considered as new. The main result of Chapter 3 is based on the papers [21] and [69]. The closure property and tail probability for randomly weighted sums of dependent r.v.s are proved in Chapter 4 and submited to the journal. The theorems presented in Chapter 5 are published in paper [20]. In the last Chapter 6 we make the conclusions of our results.

Chapter 2

Background

In this chapter we introduce the concepts and notations we use in the dissertation.

2.1 Heavy-tailed distributions

A distribution of r.v. X, supported on $[0, \infty)$, is said to be heavy-tailed if $\mathrm{E}e^{\delta X} = \infty$ for all $\delta > 0$ and light-tailed otherwise. We recall the definitions of some classes of heavy-tailed d. f. s. Let $\overline{F}(x) := 1 - F(x)$ for all real x. A d. f. F supported on $[0, \infty)$ belongs to the regularly varying-tailed class ($F \in \mathscr{R}$) if there exists a constant $a > 0$ such that

$$\lim_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-a},$$

holds for any fixed positive y,
belongs to the consistently varying-tailed class ($F \in \mathscr{C}$) if

$$\lim_{y \nearrow 1} \limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1,$$

belongs to the dominatedly varying-tailed class ($F \in \mathscr{D}$) if for any fixed $y \in (0, 1)$

$$\limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty,$$

is long-tailed ($F \in \mathscr{L}$) if, for every fixed y,

$$\lim_{x \to \infty} \frac{\overline{F}(x + y)}{\overline{F}(x)} = 1,$$

is subexponential ($F \in \mathscr{S}$) if

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2,$$

where F^{*2} denotes convolution of $F(x)$ with itself, and belongs to the class $\mathscr{S}^*$ (is strongly subexponential) if $m := \int_{[0,\infty)} x\mathrm{d}F(x) < \infty$ and

$$\int_0^x \overline{F}(x-y)\overline{F}(y)\mathrm{d}y \sim 2m\overline{F}(x), \quad x \to \infty.$$

If a d. f. F is supported on $\mathbb{R}$, then F belongs to any of these classes, if the d. f. $F(x)\mathbf{1}_{\{x\geq 0\}}$ belongs to the corresponding class. In the case of finite mean, it holds that

$$\mathscr{R} \subset \mathscr{C} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{S}^* \subset \mathscr{S} \subset \mathscr{L}$$

(see [25], [36]). For the example of d. f. which is subexponential but does not belong to $\mathscr{S}^*$, see [18]; for d. f. which is dominatedly varying-tailed but not long-tailed (hence, not in $\mathscr{S}$ and $\mathscr{S}^*$), see [17] and [24] (Example 1.4.2). For more details on heavy-tailed distributions, see [24].

Denote

$$\overline{F}_*(y) := \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \quad \overline{F}^*(y) := \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \quad y > 1,$$

and define the upper and lower Matuszewska indices of d. f. F, respectively:

$$J_F^+ := -\lim_{y\to\infty} \frac{\log \overline{F}_*(y)}{\log y}, \quad J_F^- := -\lim_{y\to\infty} \frac{\log \overline{F}^*(y)}{\log y}.$$

Additionally, let

$$L_F \ := \ \lim_{y\searrow 1} \overline{F}_*(y).$$

Parameter L_F and the Matuszewska indices are important quantities for the characterization of the classes of heavy-tailed d. f. s. In particular (see, e.g., [8]), the following four statements are equivalent:

(i) $F \in \mathscr{D}$, (ii) $\overline{F}_*(y) > 0$ for some $y > 1$, (iii) $L_F > 0$, (iv) $J_F^+ < \infty$.

Also, $F \in \mathscr{C}$ if and only if $L_F = 1$.

2.2 Dependence structures

Recall some concepts of negative dependence.

DEFINITION 2.2.1 ([44], Definition 1.1). *R.v.s $X_1, \dots, X_n$ are said to be upper extended negatively dependent (UEND), if there exists a positive constant M_1, such that, for each real $x_1, \dots, x_n$,*

$$\mathrm{P}(X_1 > x_1, \dots, X_n > x_n) \le M_1 \prod_{i=1}^{n} \mathrm{P}(X_i > x_i); \tag{2.2.1}$$

they are said to be lower extended negatively dependent (LEND), if there exists some positive constant M_2, such that for each real $x_1, \dots, x_n$

$$\mathrm{P}(X_1 \le x_1, \dots, X_n \le x_n) \le M_2 \prod_{i=1}^{n} \mathrm{P}(X_i \le x_i); \tag{2.2.2}$$

and they are said to be extended negatively dependent (END), if they are both UEND and LEND.

When $M_1 = 1$ and $M_2 = 1$ in (2.2.1) and (2.2.2), the r.v.s $X_1, \dots, X_n$ are said to be upper negatively dependent (UND) and lower negatively dependent (LND), respectively, and they are said to be negatively dependent (ND) if (2.2.1) and (2.2.2) both hold with $M_1 = 1$ and $M_2 = 1$, see [22], [9], [58].

For negatively dependent r.v.s one subset of them is "high" and other disjoint subsets are "low". Such property is rather natural and appears in life insurance and financial mathematics.

DEFINITION 2.2.2. *Random variables $X_1, \dots, X_n$ are called pairwise upper extended negatively dependent (pUEND), if*

$$\mathrm{P}\big(X_i > x_i, X_j > x_j\big) \;\le\; M_3 \mathrm{P}(X_i > x_i)\mathrm{P}(X_j > x_j) \tag{2.2.3}$$

for all $x_i, x_j \in \mathbb{R}$, $i \ne j$, $i, j \in \{1, \dots, n\}$, and some $M_3 > 0$.

DEFINITION 2.2.3. *$X_1, \dots, X_n$ are pairwise negatively (pND) dependent (or negatively quadrant dependent, according to [38]), if* (2.2.3) *holds with $M_3 = 1$:*

$$\mathrm{P}\big(X_i > x_i, X_j > x_j\big) \;\le\; \mathrm{P}(X_i > x_i)\mathrm{P}(X_j > x_j) \tag{2.2.4}$$

for all $x_i, x_j \in \mathbb{R}$, $i \ne j$, $i, j \in \{1, \dots, n\}$.

Inequality (2.2.4) is equivalent to

$$\mathrm{P}\big(X_i \le x_i, X_j \le x_j\big) \;\le\; \mathrm{P}(X_i \le x_i)\mathrm{P}(X_j \le x_j) \tag{2.2.5}$$

for all $x_i, x_j \in \mathbb{R}$, $i \neq j$, $i, j \in \{1, \ldots, n\}$. Indeed, if inequality (2.2.4) holds, then

$$\begin{aligned} \mathrm{P}(X_i > x_i)\mathrm{P}(X_j > x_j) &= (1 - \mathrm{P}(X_i \leq x_i))(1 - \mathrm{P}(X_j \leq x_j)) \\ &= 1 - \mathrm{P}(X_i \leq x_i) - \mathrm{P}(X_j \leq x_j) + \mathrm{P}(X_i \leq x_i)\mathrm{P}(X_j \leq x_j) \\ &\geq \mathrm{P}\big(X_i > x_i, X_j > x_j\big). \end{aligned}$$

It follows, by the formula $\mathrm{P}(\overline{A}\ \overline{B}) = 1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(AB)$, that

$$\begin{aligned} &\mathrm{P}(X_i \leq x_i)\mathrm{P}(X_j \leq x_j) \\ &\quad\geq \mathrm{P}\big(X_i > x_i, X_j > x_j\big) - 1 + \mathrm{P}(X_i \leq x_i) + \mathrm{P}(X_j \leq x_j) \\ &\quad= 1 - \mathrm{P}(X_i \leq x_i) - \mathrm{P}(X_j \leq x_j) + \mathrm{P}\big(X_i \leq x_i, X_j \leq x_j\big) \\ &\qquad -1 + \mathrm{P}(X_i \leq x_i) + \mathrm{P}(X_j \leq x_j), \end{aligned}$$

which is the same as (2.2.5). Hence, if r.v.s are pND, they are pairwise UND and pairwise LND at the same time. Note, that pND does not imply mutual ND ([22]). Also, if r.v.s $X_1, \ldots, X_n$ are UND (LND) the any subset of size $2 \leq k \leq n$ is UND (LND) too.

Acording to Definition 2.2.1, the UND/LND/ND r.v.s have the following useful transformation properties.

Lemma 2.2.1 ([58], Lemma 1.1)**.** 1) *If r.v.s* $\{X_k, k = 1, 2\ldots\}$ *are LND (UND) and* $\{f_k(\cdot), k = 1, 2, \ldots\}$ *are all monotone increasing real functions, then* $\{f_k(X_k), k = 1, 2\ldots\}$ *are also LND (UND) ;*

2) *if r.v.s* $\{X_k, k = 1, 2\ldots\}$ *are LND (UND) and* $\{f_k(\cdot), k = 1, 2, \ldots\}$ *are all monotone decreasing real functions, then* $\{f_k(X_k), k = 1, 2\ldots\}$ *are also UND (LND);*

3) *if r.v.s* $\{X_k, k = 1, 2\ldots\}$ *are ND and* $\{f_k(\cdot), k = 1, 2, \ldots\}$ *are either all monotone increasing or all monotone decreasing real functions, then* $\{f_k(X_k), k = 1, 2\ldots\}$ *are also ND;*

4) *if r.v.s* $\{X_k, k = 1, 2\ldots\}$ *are nonnegative and UND, then for each* $n = 1, 2, \ldots,$

$$\mathrm{E}\Big(\prod_{k=1}^{n} X_k\Big) \leq \prod_{k=1}^{n} \mathrm{E}X_k.$$

Recall one more dependence structure related to the UEND structure.

Definition 2.2.4 ([73], (2.1)). *Identically distributed r.v.s* $X_1, \ldots, X_n$ *are said to be bivariate upper tail independent (BUTI), if* $\mathrm{P}(X_i > x) > 0$ *for all* $x \in (-\infty, \infty)$, $i = 1, \ldots, n$, *and*

$$\lim_{x\to\infty} \mathrm{P}(X_i > x | X_j > x) = 0$$

for all $1 \leq i \neq j < n$.

Note that the BUTI is strictly larger than the UEND structure. To see this, consider two positive r.v.s ξ_1 and ξ_2 with the joint tail probability

$$\mathrm{P}(\xi_1 > x, \xi_2 > y) = \frac{1}{(x \vee 1)(y \vee 1)(1 + x + y)}, \quad x \geq 0, \ y \geq 0.$$

The marginal distributions are

$$\mathrm{P}(\xi_1 > x, \xi_2 > 0) = \mathrm{P}(\xi_1 > x) = \frac{1}{(x \vee 1)(1 + x)}, \quad x \geq 0$$

and

$$\mathrm{P}(\xi_2 > y, \xi_1 > 0) = \mathrm{P}(\xi_2 > y) = \frac{1}{(y \vee 1)(1 + y)}, \quad y \geq 0.$$

For $x \geq 1$ r.v.s (ξ_1, ξ_2) have the BUTI structure:

$$\mathrm{P}(\xi_1 > x | \xi_2 > x) = \frac{1 + x}{x} \frac{1}{1 + 2x} \to 0, \quad x \to \infty.$$

Such a pair (ξ_1, ξ_2) is bivariate upper tail independent, but not UEND (see [46], Example 3.1):

$$\sup_{x,y \geq 1} \frac{\mathrm{P}(\xi_1 > x, \xi_2 > y)}{\mathrm{P}(\xi_1 > x)\mathrm{P}(\xi_2 > y)} = \sup_{x,y \geq 1} \left(1 + \frac{xy}{1 + x + y}\right) = \infty,$$

that is, the fraction not bounded from above by some positive constant M.

Definition 2.2.5 ([41]). *Real-valued r.v.s $X_1, \ldots, X_n$ with d. f. $F_1, \ldots, F_n$ are said to be pairwise quasi-asymptotically independent (pQAI) if* $\mathrm{P}(X_i > x) > 0$ *for all x and i, and*

$$\lim_{x \to \infty} \mathrm{P}(|X_i| \wedge X_j > x | X_i \vee X_j > x) = 0, \ i \neq j,$$

or equivalently,

$$\lim_{x \to \infty} \frac{\mathrm{P}(|X_i| > x, X_j > x) + \mathrm{P}(X_i < -x, X_j > x)}{\mathrm{P}(X_i > x) + \mathrm{P}(X_j > x)} = 0.$$

Definition 2.2.6 ([31]). *R.v.s. $X_1, \ldots, X_n$ are pairwise strong quasi-asymptotically indepent (pSQAI) if, for any $i \neq j$,*

$$\lim_{x \wedge y \to \infty} \mathrm{P}(|X_i| > x | X_j > y) = 0. \tag{2.2.6}$$

The property of asymptotic tail independence (Definitions 2.2.4–2.2.6) means that the probability of two nonnegative random variables to be large is negligible comparing with the probability of one variable being large.

Bellow we present an implication of dependence structures mentioned in this chapter. The arrow in Figure 2.1 means "follows", for example, if r.v.s are ND, it follows, that they are pND.

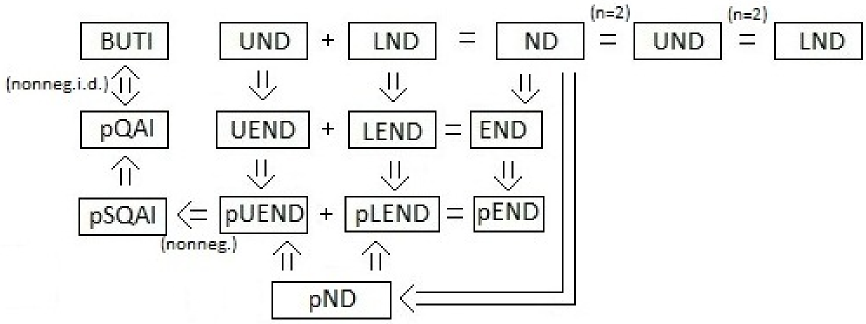

Figure 2.1: Implication of dependence structures

2.3 Copula

In this section we introduce the notion of a copula, which we use later to construct the dependence between random variables.

By the Sklar theorem (see [51], Theorem 2.10.9), any joint distribution function $F(x_1,\dots,x_n) = \mathrm{P}(X_1 \le x_1,\dots,X_n \le x_n)$ of a random vector $(X_1,\dots,X_n)$ with the marginal distribution functions $F_i(x) = \mathrm{P}(X_i \le x)$, $i = 1,\dots,n$, can be written as

$$F(x_1,\dots,x_n) = C(F_1(x_1),\dots,F_n(x_n)) \tag{2.3.1}$$

for all $x_i \in \mathbb{R}$, $i = 1,\dots,n$, where C is a copula. Moreover, if marginals $F_1,\dots,F_n$ are continuous, then the copula C satisfying (2.3.1) is unique and is given by

$$C(u_1,\dots,u_n) = F(F_1^{-1}(u_1),\dots,F_n^{-1}(u_1)),$$

where $F_i^{-1}(u) = \inf\{x : F_i(x) \ge u\}$, $i = 1,\dots,n$. Conversely, if C is a copula and $F_1,\dots,F_n$ are distribution functions, then (2.3.1) defines the n-dimensional joint distribution function with marginals $F_1,\dots,F_n$.

Definition 2.3.1 ([51], Definition 2.10.6). *For any $n \ge 2$, a function $C\colon [0,1]^n \to [0,1]$ is called a n-dimensional copula (shortly, copula) if*

(1) $C(u_1,\dots,u_{i-1},0,u_i,\dots,u_n) = 0$ for any $i \in \{1,\dots,n\}$;

(2) $C(1,\dots,1,u_i,1,\dots,1) = u_i$ for any $i \in \{1,\dots,n\}$;

(3) C is n-increasing, i.e. $\forall(x_1,\dots,x_n) \in [0,1]^n$, $\forall(y_1,\dots,y_n) \in [0,1]^n$, $x_i \le y_i$, $i = 1,\dots,n$, it holds

$$\sum_{J\subset\{1,\dots,n\}} (-1)^{|J|} C(u_1^J,\dots,u_n^J) \ge 0, \text{ where } u_i^J = \begin{cases} x_i, & \text{if } i \in J, \\ y_i, & \text{if } i \notin J. \end{cases}$$

In the bivariate case the last property can be simplified. For every u_1, u_2, v_1, v_2 in $[0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$, $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$.

Another important property: for every copula $C(u_1, \ldots, u_n)$ there exists Fréchet-Hoeffding lower and upper bounds ([51], Section 2.5). That is

$$W(u_1, \ldots, u_n) \leq C(u_1, \ldots, u_n) \leq M(u_1, \ldots, u_n),$$

where $W(u_1, \ldots, u_n) := \max\left\{\sum_{i=1}^n u_i - n + 1, 0\right\}$ and $M(u_1, \ldots, u_n) := \min\{u_1, \ldots, u_n\}$. The function M is always the copula, while the function W is the copula in the bivariate case, and it can be the copula for $n > 2$ with some aditional conditions ([64], Section 2.1.2).

Copula is very convenient tool of modeling the dependence between random variables. There are numbers of various forms of copulas and their constructions. In [47] we can find the main classes of the copulas: Archimedean, Marshall-Olkin and Elliptical. The construction of the pair copulas is described in [1] and [47]. Below we write the copulas which we will use in our examples.

1. Independence copula:

$$C^{\mathrm{I}}(u_1, \ldots, u_n) = \prod_{i=1}^n u_i. \tag{2.3.2}$$

2. Generalized FGM copula:

$$C^{\mathrm{GFGM}}(u_1, \ldots, u_n) = \prod_{l=1}^n u_l \left(1 + \sum_{1 \leq i < j \leq n} \theta_{ij}(1 - u_i^\alpha)(1 - u_j^\alpha)\right)^m; \tag{2.3.3}$$

with $\alpha > 0$, $m \in \{0, 1, 2, \ldots\}$ and the parameters θ_{ij} which are real numbers such that $C^{\mathrm{GFGM}}(u_1, \ldots, u_n)$ is a proper n-dimensional copula. Obviously, if the θ_{ij} all are nonpositive and take values from a corresponding admissible region, then

$$C^{\mathrm{GFGM}}(u_1, \ldots, u_n) \quad \leq \quad u_1 \ldots u_n,$$

i.e. we obtain the LND structure.

The special cases of (2.3.3) are well-known:

– If $m = 0$, we get the independence copula.

– If $m = 1$ and $\alpha = 1$, we get the classical multivariate FGM copula

$$C^{\mathrm{FGM}}(u_1, \ldots, u_n) = \prod_{l=1}^n u_l \left(1 + \sum_{1 \leq i < j \leq n} \theta_{ij}(1 - u_i)(1 - u_j)\right),$$

which was introduced by Farlie [26], Gumbel [33] and Morgenstern [50] in the case $n = 2$. This copula was widely investigated and used in practice. The well-known limitation of FGM copula is that it does not allow the modeling of high dependencies. For example, if $n = 2$ then the admissible region for the parameter θ_{12} is $[-1, 1]$ and correlation ρ between corresponding uniformly distributed random variables is $\rho = \theta_{12}/3$, thus the range for correlation ρ is $[-1/3, 1/3]$. If $n = 3$, the conditions for parameters can be summarized as follows: $\theta_{12}+\theta_{13}+\theta_{23} \geq -1$, $\theta_{13} + \theta_{23} - \theta_{12} \leq 1$, $\theta_{12} + \theta_{23} - \theta_{13} \leq 1$, $\theta_{12} + \theta_{13} - \theta_{23} \leq 1$.

– If $m = 1$, $n = 2$ and $\alpha > 0$ we get the copula introduced by Huang and Kotz [34]. It was shown that the admissible range of θ_{12} is $-\min\{1, \alpha^{-2}\} \leq \theta_{12} \leq \alpha^{-1}$ and correlation ρ between the corresponding uniformly distributed random variables is $\rho = 3\theta_{12}\alpha^2(\alpha+2)^{-2}$, thus the range for correlation ρ is $-3(\alpha + 2)^{-2} \min\{1, \alpha^2\} \leq \rho \leq 3\alpha(\alpha + 2)^{-2}$.

– If $m \geq 1$, $n = 2$ and $\alpha > 0$ we get the copula introduced by Bekrizadeh et al. [7]. They have shown that the admissible range of θ_{12} is $-\min\{1, (m\alpha^2)^{-1}\} \leq \theta_{12} \leq (m\alpha)^{-1}$ and correlation between corresponding uniformly distributed random variables is given by formula

$$\begin{aligned}
\rho &= 12\int_0^1 \int_0^1 C^{\mathrm{GFGM}}(u, v)\mathrm{d}u\mathrm{d}v - 3 \\
&= 12\sum_{k=1}^{m} \binom{m}{k} \theta_{12}^k \left(\frac{\Gamma(k+1)\Gamma(2/\alpha)}{\alpha\Gamma(k+1+2/\alpha)}\right)^2.
\end{aligned}$$

Because of the weak dependence generated by the FGM family, many authors considered the modifications of this class. Examples of modified FGM copula can be found in [6], [3], among others.

The finding of the admissible region for parameters θ_{ij} in (2.3.3) is technical, although straightforward, task. Essentially, it requires the verification that the corresponding copula density (if exists) $c^{\mathrm{GFGM}}(u_1, \ldots, u_n) = \partial^n C^{\mathrm{GFGM}}(u_1, \ldots, u_n)/\partial u_1 \ldots \partial u_n$ is nonnegative for all $u_1, \ldots, u_n$. In the case of copula (2.3.3) with $m = 1$,

$$c^{\mathrm{FGM}}(u_1, \ldots, u_n) = 1 + \sum_{1\leq i<j\leq n} \theta_{ij}(1 - (1 + \alpha)u_i^\alpha)(1 - (1 + \alpha)u_j^\alpha)$$

and these conditions can be obtained by considering the 2^n cases for $u_k = 0$ or 1, $k = 1, \ldots, n$, and verifying that $c^{\mathrm{FGM}}(u_1, \ldots, u_n) \geq 0$. For example, if $m = 1$ and $n = 3$, then these conditions are the following:

$$1 + \alpha^2\theta \geq 0, \quad \theta_{ij} \geq \begin{cases} \frac{\alpha\theta - 1}{1+\alpha} & \text{if} \quad \alpha\theta > 1, \\ \frac{1}{\alpha}\frac{\alpha\theta-1}{1+\alpha} & \text{if} \quad \alpha\theta \leq 1, \end{cases} \quad 1 \leq i < j \leq 3, \quad \text{when} \quad \alpha > 1,$$

and

$$1 + \alpha\theta \geq 0, \quad \theta_{ij} \geq \begin{cases} \frac{1}{\alpha}\frac{\alpha\theta-1}{1+\alpha} & \text{if} \quad \alpha\theta > 1, \\ \frac{\alpha\theta - 1}{1+\alpha} & \text{if} \quad \alpha\theta \leq 1, \end{cases} \quad 1 \leq i < j \leq 3,$$

when $0 < \alpha \leq 1$, with $\theta := \theta_{12} + \theta_{13} + \theta_{23}$.

3. Ali-Mikhail-Haq copula:

$$C^{\mathrm{AMH}}(u_1, \ldots, u_n) = \frac{u_1 \ldots u_n}{1 - \theta(1-u_1)\ldots(1-u_n)}, \quad -1 \leq \theta < 1. \tag{2.3.4}$$

4. Frank copula:

$$C^{\mathrm{F}}(u_1, \ldots, u_n) = -\frac{1}{\theta} \log\left(1 + \frac{(\mathrm{e}^{-\theta u_1} - 1)\ldots(\mathrm{e}^{-\theta u_n} - 1)}{(\mathrm{e}^{-\theta} - 1)^{n-1}}\right), \quad \theta > 0. \tag{2.3.5}$$

5. Clayton copula:

$$C^{\mathrm{Cl}}(u_1, \ldots, u_n) = \left(u_1^{-\theta} + \cdots + u_n^{-\theta} - n + 1\right)^{-1/\theta}, \quad \theta > 0. \tag{2.3.6}$$

Chapter 3

The max-sum equivalence

In this chapter we investigate the (weak) equivalence relations among the tail probabilities of the sums $S_n := \sum_{k=1}^{n} X_k$, $S_n^{(+)} := \sum_{k=1}^{n} X_k^+$, $S_{(n)} := \max\{S_1, \dots, S_n\}$ and $\sum_{k=1}^{n} \overline{F_k}(x)$. The analysis of the so-called max-sum equivalence

$$\mathrm{P}(S_{(n)} > x) \sim \mathrm{P}(S_n > x) \ \sim \ \mathrm{P}(X_{(n)} > x) \sim \sum_{i=1}^{n} \mathrm{P}(X_i > x), \qquad (3.1)$$

where $X_{(n)} := \max\{X_1, \dots, X_n\}$, has essential applications in ruin theory, where probability $\mathrm{P}(S_{(n)} > x)$ stands as the ruin probability of an insurance company (see Section 3.3). The quantities $S_{(n)}$ and S_n are the main elements of modeling the risk management (see [52]). Besides, the asymptotic relation (3.1) allows us to reduce the calculation of $\mathrm{P}(S_{(n)} > x)$ to the calculation of $\mathrm{P}(X_{(n)} > x)$ and posseses the principle of big jump: for large x, one of n summands $X_1, \dots, X_n$ is large, while others are relatively small.

Such the sums were investigated earlier in a number of papers. One of the first studies of sums (for independent identically distributed (i.i. d.) positive r.v.s) was done in [16]. Geluk and De Vries [32] showed that for i.i. d. subexponential r.v.s the asymptotic $\mathrm{P}(S_n > x) \sim \sum_{i=1}^{n} \mathrm{P}(X_i > x)$ holds under the proper condition for X_i, $i = 1, \dots, n$. Later, Geluk and Tang [31] obtained this relation for dependent subexponential r.v.s with nonidentical distributions. Geluk and Ng [30] proved the asymptotic $\mathrm{P}(S_{(n)} > x) \sim \mathrm{P}(S_n > x)$ for independent r.v.s with long-tailed distributions $F_1, \dots, F_n$. In case of dependent r.v.s, relation (3.1) was discussed in [59], [37], [74] among others. Li and Tang [42] showed asymptotic (3.1) for independent r.v.s. under the condition that their maximum belongs to the specific class of heavy-tailed distributions. Below, motivated by the main result of [42], we prove the weak max-sum equivalence

$$C_1 \sum_{i=1}^{n} \mathrm{P}(X_i > x) \lesssim \mathrm{P}(S_{(n)} > x) \lesssim C_2 \sum_{i=1}^{n} \mathrm{P}(X_i > x)$$

with some positive constants C_1 and C_2 for dependent r.v.s.

3.1 Main result

The two following propositions (see [69]) present our first results on the quantities $\mathrm{P}(S_{(n)} > x)$, $\mathrm{P}(S_n^{(+)} > x)$ and $\overline{G}_n(x) = \mathrm{P}(\max\{X_1, \ldots, X_n\} > x)$ when r.v.s are pairwise negatively dependent.

Proposition 3.1.1. *Let $X_1, \ldots, X_n$ be pND real-valued r.v.s with corresponding distributions $F_1, \ldots, F_n$. If $G_n \in \mathscr{D}$, then*

$$\mathrm{P}(S_{(n)} > x) \ \leq \ \mathrm{P}(S_n^{(+)} > x) \ \lesssim \ \frac{1}{L_{G_n}} \overline{G}_n(x). \tag{3.1.1}$$

Furthermore, if $G_n \in \mathscr{L} \cap \mathscr{D}$, then

$$\mathrm{P}\left(S_{(n)} > x\right) \ \leq \ \mathrm{P}\left(S_n^{(+)} > x\right) \ \lesssim \ \overline{G}_n(x). \tag{3.1.2}$$

Proposition 3.1.2. *Let $X_1, \ldots, X_n$ be pND r.v.s.*
(i) *If $G_n \in \mathscr{D}$ and $F_i(-x) = o(\overline{F}_i(x))$ for $i = 1, \ldots, n$, then*

$$\mathrm{P}(S_{(n)} > x) \ \geq \ \mathrm{P}(S_n > x) \ \gtrsim \ L_{G_n} \overline{G}_n(x). \tag{3.1.3}$$

(ii) *If $G_n \in \mathscr{C}$ and $F_i(-x) = o(\overline{F}_i(x))$ for $i = 1, \ldots, n$, then*

$$\mathrm{P}(S_{(n)} > x) \ \geq \ \mathrm{P}(S_n > x) \ \gtrsim \ \overline{G}_n(x). \tag{3.1.4}$$

(iii) *If $G_n \in \mathscr{L} \cap \mathscr{D}$ and $F_i(A) = 0$ for some finite $A < 0$, $i = 1, \ldots, n$, then relations in* (3.1.4) *hold.*

Using inequality (3.1.2) from Proposition 3.1.1 and Proposition 3.1.2 (iii), we obtain:

Corollary 3.1.1. *Let $X_1, \ldots, X_n$ be nonnegative pND r.v.s. If $G_n \in \mathscr{L} \cap \mathscr{D}$, then*

$$\mathrm{P}(S_{(n)} > x) \ = \ \mathrm{P}(S_n > x) \ \sim \ \overline{G}_n(x).$$

Remark 3.1.1. Note that class $\mathscr{D}$ is closed under max operation, i.e. if $F_k \in \mathscr{D}$ for all $k = 1, \ldots, n$, then $G_n \in \mathscr{D}$ (the inverse statement obviously does not hold). Moreover, the constant L_{G_n} appearing in Propositions 3.1.1 and 3.1.2 can be estimated from below as follows:

$$L_{G_n} \geq \left(\sum_{k=1}^{n} \frac{1}{L_{F_k}} \right)^{-1} > 0, \tag{3.1.5}$$

where $L_{F_k} := \lim_{y \searrow 1} \liminf \frac{\overline{F_k}(xy)}{\overline{F_k}(x)}$. To show this, for any $y > 0$ write

$$\begin{aligned} \frac{\overline{G}_n(xy)}{\overline{G}_n(x)} &= \frac{\mathrm{P}\Big(\bigcup_{k=1}^{n} \{X_k > xy\} \Big)}{\mathrm{P}\Big(\bigcup_{k=1}^{n} \{X_k > x\} \Big)} \leq \frac{\sum_{k=1}^{n} \mathrm{P}(X_k > xy)}{\mathrm{P}\Big(\bigcup_{i=1}^{n} \{X_i > x\} \Big)} \\ &= \sum_{k=1}^{n} \frac{\mathrm{P}(X_k > xy)}{\mathrm{P}\Big(\bigcup_{i=1}^{n} \{X_i > x\} \Big)} \leq \sum_{k=1}^{n} \frac{\mathrm{P}(X_k > xy)}{\mathrm{P}(X_k > x)}, \end{aligned}$$

which implies

$$\frac{1}{L_{G_n}} = \lim_{y \nearrow 1} \limsup \frac{\overline{G}_n(xy)}{\overline{G}_n(x)} \leq \sum_{k=1}^{n} \lim_{y \nearrow 1} \limsup \frac{\overline{F_k}(xy)}{\overline{F_k}(x)} = \sum_{k=1}^{n} \frac{1}{L_{F_k}} < \infty,$$

or (3.1.5). Hence, $L_{G_n} > 0$, which is equivalent to $G_n \in \mathscr{D}$.

Remark 3.1.2. The statement of Corollary 3.1.1 holds if $F_k \in \mathscr{C}$ for $k = 1, \ldots, n$ and r.v.s $X_1, \ldots, X_n$ are nonnegative pND. To see that $G_n \in \mathscr{C}$, note that for any x, y it holds

$$\begin{aligned} \frac{\overline{G_n}(xy)}{\overline{G_n}(x)} &= \frac{\mathrm{P}\big(\bigcup_{k=1}^{n} \{X_k > xy\} \big)}{\mathrm{P}\big(\bigcup_{k=1}^{n} \{X_k > x\} \big)} \\ &\geq \frac{\sum_{k=1}^{n} \overline{F_k}(xy) - \sum_{1 \leq i < j \leq n} \mathrm{P}(X_i > xy, X_j > xy)}{\sum_{k=1}^{n} \overline{F_k}(x)} \\ &\geq \min_{1 \leq k \leq n} \left\{ \frac{\overline{F_k}(xy)}{\overline{F_k}(x)} \right\} - \frac{\sum_{1 \leq i < j \leq n} \overline{F_i}(xy) \overline{F_j}(xy)}{\sum_{k=1}^{n} \overline{F_k}(x)} \end{aligned}$$

by pND property. Hence,

$$\begin{aligned} 1 &\geq \lim_{y \searrow 1} \liminf \frac{\overline{G_n}(xy)}{\overline{G_n}(x)} \\ &\geq \lim_{y \searrow 1} \liminf \min_{1 \leq k \leq n} \left\{ \frac{\overline{F_k}(xy)}{\overline{F_k}(x)} \right\} - \lim_{y \searrow 1} \limsup \sum_{j=1}^{n} \overline{F_j}(xy) \\ &\geq \min_{1 \leq k \leq n} \left\{ \lim_{y \searrow 1} \liminf \frac{\overline{F_k}(xy)}{\overline{F_k}(x)} \right\} = 1, \end{aligned}$$

that is, $L_{G_n} = 1$ and we know that this holds for consistently varying tail distributions ($G_n \in \mathscr{C}$).

Later we generalized these two propositions. The improved results with the wider dependence structure are the main results of this chapter.

Denote the d. f. $H_n(x) := n^{-1}(F_1(x) + \cdots + F_n(x))$ and assume that $\overline{H_n}(x) > 0$ for all x. Introduce the following condition:

$$\sum_{1\le k<l\le n} \mathrm{P}(X_k > x, X_l > x) \ = \ o(1)\overline{H}_n(x), \quad x \to \infty, \tag{3.1.6}$$

or, equivalently,

$$\mathrm{P}(X_k > x, X_l > x) \ = \ o(1)\overline{H}_n(x) \ \text{ for all } \ k, l = 1, \dots, n, \ k < l. \tag{3.1.7}$$

The random variables satisfying (3.1.6) allow a wide range of dependence structures. In particular, they cover the pND r.v.s and even some positive dependence structures (see Section 3.5). They also include the pQAI structure (Definition 2.2.5), if $X_1, \dots, X_n$ are all nonnegative. Note that under some stronger dependence conditions, related equivalence results for subexponential r.v.s were established by Geluk and Tang [31], Jiang et al. [35].

When $X_1, \dots, X_n$ are real-valued and identically distributed r.v.s, the dependence structure in (3.1.7) coincides with the BUTI structure (Definition 2.2.4):

$$\lim_{x\to\infty} \mathrm{P}(X_i > x | X_j > x) = \lim_{x\to\infty} \frac{\mathrm{P}(X_i > x, X_j > x)}{\mathrm{P}(X_j > x)} = 0.$$

The main result of the chapter is the following theorem, which generalizes Propositions 3.1.1–3.1.2.

Theorem 3.1.1. *Let r.v.s $X_1, \dots, X_n$ satisfy condition* (3.1.6). *If $H_n \in \mathscr{D}$ (or, equivalently, $G_n \in \mathscr{D}$). Then*

$$\mathrm{P}(S_{(n)} > x) \ \le \ \mathrm{P}(S_n^{(+)} > x) \ \lesssim \ L_{H_n}^{-1} n\overline{H_n}(x). \tag{3.1.8}$$

If, in addition, $H_n(-x) = o(\overline{H_n}(x))$, then

$$\mathrm{P}(S_{(n)} > x) \ \ge \ \mathrm{P}(S_n > x) \ \gtrsim \ L_{H_n} n\overline{H_n}(x). \tag{3.1.9}$$

Here,

$$L_{H_n} \ = \ L_{G_n} \quad \text{and} \quad n\overline{H_n}(x) \ \sim \ \overline{G_n}(x). \tag{3.1.10}$$

Remark 3.1.3. Since pND r.v.s satisfy condition (3.1.6), Theorem 3.1.1 generalizes the result of Propositions 3.1.1–3.1.2 and, moreover, the constant L_{G_n} in (3.1.1), (3.1.3) can be replaced by L_{H_n}.

REMARK 3.1.4. In the case where $F_k \in \mathscr{C} \subset \mathscr{D}$, $k = 1, \ldots, n$, we have $H_n \in \mathscr{C}$ and thus $L_{H_n} = 1$ in Theorem 3.1.1.

In the case of identically distributed random variables we obtain the following corollary:

Corollary 3.1.2. *Let assumptions of Theorem 3.1.1 hold and let $X_1, \ldots, X_n$ be identically distributed with common distribution F. Then relations* (3.1.8) *and* (3.1.9) *hold with $L_{H_n} = L_{G_n} = L_F$ and $\overline{H_n}(x) = \overline{F}(x)$.*

3.2 Proof of main result

We start this section with the following useful proposition.

Proposition 3.2.1. *Assume that condition* (3.1.6) *holds. Then $\overline{G_n}(x) \sim n\overline{H_n}(x)$, and therefore $L_{G_n} = L_{H_n}$.*

PROOF. We have

$$\overline{G_n}(x) = \mathrm{P}\left(\bigcup_{k=1}^{n}\{X_k > x\}\right) \leq \sum_{k=1}^{n} \mathrm{P}(X_k > x). \tag{3.2.1}$$

On the other hand,

$$\overline{G_n}(x) \geq \sum_{k=1}^{n} \mathrm{P}(X_k > x) - \sum_{1\leq k<l\leq n} \mathrm{P}(X_k > x, X_l > x). \tag{3.2.2}$$

(3.1.6) and (3.2.1), (3.2.2) imply that $\overline{H_n}(x)$ is positive if and only if $\overline{G_n}(x) > 0$ is positive for $x \to \infty$. Then

$$\limsup \frac{\overline{G_n}(x)}{n\overline{H_n}(x)} \leq \frac{\sum_{k=1}^{n} \mathrm{P}(X_k > x)}{n\overline{H_n}(x)} = 1$$

and

$$\begin{aligned}\liminf \frac{\overline{G_n}(x)}{n\overline{H_n}(x)} &\geq \frac{\sum_{k=1}^{n} \mathrm{P}(X_k > x) - \sum_{1\leq k<l\leq n} \mathrm{P}(X_k > x, X_l > x)}{n\overline{H_n}(x)} \\ &\geq 1 - \limsup \frac{\sum_{1\leq k<l\leq n} \mathrm{P}(X_k > x, X_l > x)}{n\overline{H_n}(x)} = 1,\end{aligned}$$

implying $\overline{G_n}(x) \sim n\overline{H_n}(x)$ and, thus,

$$L_{G_n} = \lim_{y\searrow 1} \limsup_{x\to\infty} \frac{\overline{G}_n(yx)}{\overline{G}_n(x)} = \lim_{y\searrow 1} \limsup_{x\to\infty} \frac{n\overline{H_n}(yx)}{n\overline{H_n}(x)} = L_{H_n}.$$

□

First we prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Relations (3.1.10) hold by Proposition 3.2.1, implying the equivalence of $H_n \in \mathscr{D}$ and $G_n \in \mathscr{D}$.

We first show the upper bound (3.1.8). For any $0 < v < 1$ and $x > 0$ write

$$\begin{aligned}
&\mathrm{P}(S_n^{(+)} > x) \\
&\le \mathrm{P}\Big(\bigcup_{k=1}^{n}\{X_k^+ > (1-v)x\}\Big) + \mathrm{P}\Big(S_n^{(+)} > x, \bigcap_{k=1}^{n}\{X_k^+ \le (1-v)x\}\Big) \\
&\le n\overline{H}_n((1-v)x) + \mathrm{P}\Big(S_n^{(+)} > x, \bigcup_{i=1}^{n}\Big\{X_i^+ > \frac{x}{n}\Big\}, \bigcap_{k=1}^{n}\{X_k^+ \le (1-v)x\}\Big) \\
&=: I_1(v,x) + I_2(v,x). \qquad (3.2.3)
\end{aligned}$$

We have by $H_n \in \mathscr{D}$ that

$$\lim_{v\searrow 0}\limsup_{x\to\infty}\frac{I_1(v,x)}{L_{H_n}^{-1}n\overline{H_n}(x)} = L_{H_n}\lim_{v\searrow 0}\limsup_{x\to\infty}\frac{\overline{H_n}((1-v)x)}{\overline{H_n}(x)} = L_{H_n}L_{H_n}^{-1} = 1.$$

As for $I_2(v,x)$, we have

$$\begin{aligned}
I_2(v,x) &\le \sum_{i=1}^{n}\mathrm{P}\Big(S_n^{(+)} > x, X_i^+ > \frac{x}{n}, \bigcap_{k=1}^{n}\{X_k^+ \le (1-v)x\}\Big) \\
&\le \sum_{i=1}^{n}\mathrm{P}\Big(S_n^{(+)} - X_i^+ > vx, X_i^+ > \frac{x}{n}\Big) \\
&\le \sum_{i=1}^{n}\mathrm{P}\Big(\bigcup_{\substack{j=1\\ j\neq i}}^{n}\Big\{X_j^+ > \frac{vx}{n-1}\Big\}, X_i^+ > \frac{x}{n}\Big) \\
&\le \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mathrm{P}\Big(X_j^+ > \frac{vx}{n}, X_i^+ > \frac{vx}{n}\Big).
\end{aligned}$$

Hence, by (3.1.6) and assumption $H_n \in \mathscr{D}$, we obtain

$$\begin{aligned}
&\limsup\frac{I_2(v,x)}{L_{H_n}^{-1}n\overline{H_n}(x)} \\
&\le L_{H_n}\limsup\frac{\sum_{i\neq j}\mathrm{P}(X_i > vx/n, X_j > vx/n)}{n\overline{H_n}(vx/n)}\limsup\frac{\overline{H_n}(vx/n)}{\overline{H_n}(x)} \\
&= 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\limsup_{x\to\infty}\frac{\mathrm{P}(S_n^{(+)} > x)}{L_{H_n}^{-1}n\overline{H_n}(x)} &\le \lim_{v\searrow 0}\limsup_{x\to\infty}\frac{I_1(v,x)}{L_{H_n}^{-1}n\overline{H_n}(x)} + \lim_{v\searrow 0}\limsup_{x\to\infty}\frac{I_2(v,x)}{L_{H_n}^{-1}n\overline{H_n}(x)} \\
&= 1.
\end{aligned}$$

To obtain the lower bound, note that for any $v > 0$ and $x > 0$

$$\begin{aligned} P(S_n > x) &\geq P\Big(S_n > x, \bigcup_{k=1}^{n}\{X_k > (1+v)x\}\Big) \\ &\geq \sum_{k=1}^{n} P(S_n > x, X_k > (1+v)x) \\ &\quad - \sum_{1\leq i<j\leq n} P(S_n > x, X_i > (1+v)x, X_j > (1+v)x) \\ &=: I_3(v,x) - I_4(v,x). \end{aligned} \tag{3.2.4}$$

Here,

$$I_4(v,x) \leq \sum_{1\leq i<j\leq n} P(X_i > x, X_j > x) = o(\overline{H_n}(x)) \tag{3.2.5}$$

according to (3.1.6).

For $I_3(v,x)$ we have

$$\begin{aligned} I_3(v,x) &\geq \sum_{k=1}^{n} P(S_n - X_k > -vx, X_k > (1+v)x) \\ &\geq \sum_{k=1}^{n} \big(P(S_n - X_k > -vx) + \overline{F_k}((1+v)x) - 1\big) \\ &= n\overline{H_n}((1+v)x) - \sum_{k=1}^{n} P(S_n - X_k \leq -vx) \\ &=: I_{31}(v,x) - I_{32}(v,x). \end{aligned} \tag{3.2.6}$$

Here,

$$\lim_{v\searrow 0}\liminf_{x\to\infty} \frac{I_{31}(v,x)}{L_{H_n} n\overline{H_n}(x)} = 1. \tag{3.2.7}$$

For term $I_{32}(v,x)$ we have

$$\begin{aligned} I_{32}(v,x) &= \sum_{k=1}^{n} P\Bigg(\sum_{\substack{i=1\\ i\neq k}}^{n}(-X_i) \geq vx\Bigg) \leq \sum_{k=1}^{n} P\Bigg(\bigcup_{\substack{i=1\\ i\neq k}}^{n}\Big\{-X_i \geq \frac{v}{n-1}x\Big\}\Bigg) \\ &\leq n^2 H_n\Big(-\frac{v}{n-1}x\Big) = o(1)\overline{H_n}\Big(\frac{v}{n-1}x\Big) = o(\overline{H_n}(x)) \end{aligned} \tag{3.2.8}$$

by the assumption of theorem and by $H_n \in \mathscr{D}$. Hence, by (3.2.4)–(3.2.8),

$$\begin{aligned} \liminf_{x\to\infty} \frac{P(S_n > x)}{L_{H_n} n\overline{H_n}(x)} &\geq \lim_{v\searrow 0}\liminf_{x\to\infty} \frac{I_{31}(v,x)}{L_{H_n} n\overline{H_n}(x)} - \lim_{v\searrow 0}\limsup_{x\to\infty} \frac{I_{32}(v,x)}{L_{H_n} n\overline{H_n}(x)} \\ &\quad - \lim_{v\searrow 0}\limsup_{x\to\infty} \frac{I_4(v,x)}{L_{H_n} n\overline{H_n}(x)} = 1. \end{aligned}$$

This completes the proof. □

Proof of Proposition 3.1.1. We prove only the second part of Proposition 3.1.1, while the first part is analogous to the proof of the first part of Theorem 3.1.1 .

If $G_n \in \mathscr{L} \cap \mathscr{D}$, then substitute vx in the above proof of Theorem 3.1.1 with $\ell(x)$, where $\ell(x)$ is a positive function satisfying $\ell(x) \to \infty$, $\ell(x) = o(x)$, and

$$\overline{G}_n(x - \ell(x)) \ \sim \ \overline{G}_n(x), \tag{3.2.9}$$

by $G_n \in \mathscr{L}$ (see [31], [28]). Rewrite (3.2.3) as follows

$$\mathrm{P}(S_n^{(+)} > x) \leq I_1(\ell(x)) + I_2(\ell(x))$$

In this case, the estimate for $I_2(\ell(x))$ remains the same, i.e. $I_2(\ell(x)) = o(\overline{G}_n(x))$:

$$\begin{aligned} I_2(\ell(x)) &= \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \mathrm{P}\Big(X_j^+ > \frac{\ell(x)}{n}, X_i^+ > \frac{\ell(x)}{n}\Big) \\ &\leq \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \overline{F}_j\Big(\frac{\ell(x)}{n}\Big)\overline{F}_i\Big(\frac{\ell(x)}{n}\Big) \lesssim \overline{G}_n\Big(\frac{\ell(x)}{n}\Big)\overline{G}_n\Big(\frac{\ell(x)}{n}\Big) \ = \ o(\overline{G}_n(x)), \end{aligned}$$

by the pND property and Proposition 3.2.1. Whereas for $I_1(\ell(x))$, due to (3.2.9) and Proposition 3.2.1, it holds, that $I_1(\ell(x)) \sim \overline{G}_n(x)$:

$$\limsup \frac{I_1(\ell(x))}{\overline{G}_n(x)} \ = \ \limsup \frac{\sum_{i=1}^n \overline{F}_i(x - \ell(x))}{\overline{G}_n(x)} = \limsup \frac{\overline{G}_n(x - \ell(x))}{\overline{G}_n(x)} = 1.$$

□

Proof of Proposition 3.1.2.

(i) The proof is identical to that of the second part of Theorem 3.1.1.

(ii) See Remark 3.1.4.

(iii) Again, replacing vx in (3.2.4) in the proof of Theorem 3.1.1 by the function $\ell(x)$ given in (3.2.9), for $I_{31}(\ell(x))$ we have

$$I_{31}(\ell(x)) \ = \ \sum_{i=1}^{n} \overline{F}_i(x + \ell(x)) \sim \overline{G}_n(x + \ell(x)) \sim \overline{G}_n(x),$$

by Proposition 3.2.1 and (3.2.9). The term $I_4(\ell(x)) = o(\overline{G_n}(x))$ remains the same. Finally, $I_{32}(\ell(x))$:

$$\begin{aligned} I_{32}(\ell(x)) &= \sum_{k=1}^{n} \mathrm{P}\left(\sum_{\substack{i=1\\ i\neq k}}^{n}(-X_i) \geq \ell(x)\right) \leq \sum_{k=1}^{n} \mathrm{P}\left(\bigcup_{\substack{i=1\\ i\neq k}}^{n}\Big\{-X_i \geq \frac{\ell(x)}{n-1}\Big\}\right) \\ &\leq n\sum_{i=1}^{n} F_i\Big(-\frac{\ell(x)}{n-1}\Big) \ = \ o(1)\sum_{i=1}^{n}\overline{F}_i(x) \ \sim \ o(\overline{G}_n(x)) \end{aligned}$$

for large x by the assumption of proposition. This ends the proof. □

3.3 Aplication to ruin theory

The assumption that the r.v.s $X_1, \ldots, X_n$ are nonidentically distributed is important for insurance mathematics, because the result can be applied to some discrete-time risk models with insurance and financial risks, proposed by Nyrhinen [53], [54]. Namely, set $X_k = \Theta_k \xi_k$, where ξ_k, $k = 1, \ldots, n$, are real-valued r.v.s, which represent the successive net losses for an insurance company, or can be understood as the total claim amount minus the total premium income within year k, and Θ_k, $1 \le k \le n$, are nonnegative r.v.s which stand for the discount factor from year k to year 0. In such a model, the r.v.s ξ_k and Θ_k are called the insurance risk and financial risk, respectively, and $\mathrm{P}(S_{(n)} > x) =: \psi(x, n)$ represents the finite-time ruin probability by year n with initial capital $x > 0$. The obtained asymptotic relations in Section 3.1 are important not only from the theoretical point of view, but also they can be used in practice as a numerical tool allowing to approximate the ruin probability $\psi(x, n)$ by the tail distribution of the maximal random variable $X_{(n)}$.

Firstly we study the question when the conditions of the Propositions 3.1.1 and 3.1.2 are satisfied for the $X_k = \Theta_k \xi_k$. The same contitions are required in Theorem 3.1.1.

Lemma 3.3.1 below gives a simple condition for $X_1, \ldots, X_n$ to be upper, lower or pairwise negatively dependent.

Lemma 3.3.1. *Assume that $\xi_1, \ldots, \xi_n$ are independent, almost surely positive r.v.s, $\Theta_1, \ldots, \Theta_n$ are UND (LND, pND) r.v.s, independent of $\{\xi_1, \ldots, \xi_n\}$. Then $\Theta_1\xi_1, \ldots, \Theta_n\xi_n$ are UND (LND, pND, respectively).*

Proof. Assume that $\Theta_1, \ldots, \Theta_n$ are UND r.v.s. Then

$$
\begin{aligned}
&\mathrm{P}(\Theta_1\xi_1 > x_1, \ldots, \Theta_n\xi_n > x_n)\\
&\quad= \int_{(0,\infty)} \cdots \int_{(0,\infty)} \mathrm{P}\Big(\Theta_1 > \frac{x_1}{y_1}, \ldots, \Theta_n > \frac{x_n}{y_n}\Big) \mathrm{d}F_{\xi_1}(y_1) \ldots \mathrm{d}F_{\xi_n}(y_n)\\
&\quad\le \int_{(0,\infty)} \cdots \int_{(0,\infty)} \mathrm{P}\Big(\Theta_1 > \frac{x_1}{y_1}\Big) \ldots \mathrm{P}\Big(\Theta_n > \frac{x_n}{y_n}\Big) \mathrm{d}F_{\xi_1}(y_1) \ldots \mathrm{d}F_{\xi_n}(y_n)\\
&\quad= \mathrm{P}(\Theta_1\xi_1 > x_1) \ldots \mathrm{P}(\Theta_n\xi_n > x_n).
\end{aligned}
$$

The cases of LND and pND are analogous. □

We obtain the following proposition.

Proposition 3.3.1. *Assume that* $\xi_1, \ldots, \xi_n$ *are independent, almost surely positive r.v.s from* $\mathscr{D}$*. Assume also that* $\Theta_1, \ldots, \Theta_n$ *are pND r.v.s, independent of* $\xi_1, \ldots, \xi_n$*, such that* $\mathrm{P}(\Theta_i \in [a, b]) = 1$ *for all* $i = 1, \ldots, n$ *and some* $0 < a \leq b < \infty$*. Then relations* (3.1.1) *and* (3.1.3) *hold.*

Remark 3.3.1. Since pND r.v.s satisfy condition (3.1.6), conditions of Proposition 3.3.1 imply that more general relations (3.1.8) and (3.1.9) hold.

Proof. Note that the conditions of the proposition imply

$$G_n(x) = \mathrm{P}(\max\{\Theta_1\xi_1, \ldots, \Theta_n\xi_n\} \leq x) \in \mathscr{D}, \tag{3.3.1}$$

since, by Remark 3.1.1, $\mathrm{P}(\max\{\xi_1, \ldots, \xi_n\} \leq x) \in \mathscr{D}$ and hence, for any $0 < y < 1$,

$$\begin{aligned}\limsup \frac{\mathrm{P}(\max\{\Theta_1\xi_1, \ldots, \Theta_n\xi_n\} > xy)}{\mathrm{P}(\max\{\Theta_1\xi_1, \ldots, \Theta_n\xi_n\} > x)} &\leq \limsup \frac{\mathrm{P}(b\max\{\xi_1, \ldots, \xi_n\} > xy)}{\mathrm{P}(a\max\{\xi_1, \ldots, \xi_n\} > x)} \\ &= \limsup \frac{\mathrm{P}(\max\{\xi_1, \ldots, \xi_n\} > xya/b)}{\mathrm{P}(\max\{\xi_1, \ldots, \xi_n\} > x)},\end{aligned}$$

which is finite because tail of dominatedly varying distribution never turns into zero. It remains to apply Lemma 3.3.1, which says that r.v.s $\Theta_1\xi_1, \ldots, \Theta_n\xi_n$ are ND too. Hence, the needed conditions of Propositions 3.1.1–3.1.2 (and Theorem 3.1.1) are satisfied and required equations hold. □

Finally note that, in the case $F_{\xi_k}(x) := \mathrm{P}(\xi_k \leq x) \in \mathscr{D}$ and $\mathrm{P}(\Theta_k \in [a, b]) = 1$, the constant L_{F_k} ($F_k(x)$ is the distribution of $\Theta_k\xi_k$) appearing in (3.1.5) can be estimated by the constants defined through the function $\overline{F_{\xi_k}}_*(y) = \liminf \frac{\mathrm{P}(\xi_k > xy)}{\mathrm{P}(\xi_k > x)}$, $y \geq 1$. It is easy to see that

$$L_{F_k} \;\geq\; \lim_{y \searrow 1} \overline{F_{\xi_k}}_*(y)\, \overline{F_{\xi_k}}_*\left(\frac{b}{a}\right).$$

Indeed,

$$\begin{aligned}L_{F_k}^{-1} &= \lim_{y \nearrow 1} \limsup \frac{\mathrm{P}(\Theta_k\xi_k > xy)}{\mathrm{P}(\Theta_k\xi_k > x)} = \lim_{y \nearrow 1} \limsup \frac{1}{\frac{\mathrm{P}(\Theta_k\xi_k > x)}{\mathrm{P}(\Theta_k\xi_k > xy)}} \\ &= \frac{1}{\lim_{y \nearrow 1} \liminf \frac{\mathrm{P}(\Theta_k\xi_k > x)}{\mathrm{P}(\Theta_k\xi_k > xy)}} \leq \frac{1}{\lim_{y \searrow 1} \liminf \frac{\mathrm{P}(a\xi_k > xy)}{\mathrm{P}(b\xi_k > x)}} \\ &= \frac{1}{\lim_{y \searrow 1} \liminf \frac{\mathrm{P}(\xi_k > \frac{xy}{a})}{\mathrm{P}(\xi_k > \frac{x}{a})} \frac{\mathrm{P}(\xi_k > \frac{x}{a})}{\mathrm{P}(\xi_k > \frac{x}{b})}} = \frac{1}{\lim_{y \searrow 1} \liminf \frac{\mathrm{P}(\xi_k > xy)}{\mathrm{P}(\xi_k > x)} \frac{\mathrm{P}(\xi_k > x\frac{b}{a})}{\mathrm{P}(\xi_k > x)}}.\end{aligned}$$

3.4 Numerical simulations

In this section we perform some numerical simulations in order to check the accuracy of the asymptotic relations obtained in Corollary 3.1.1. We compare the tail probabilities $\mathrm{P}(S_n > x)$ and $\overline{G}_n(x)$ for several values of x, assuming that r.v.s X_k are distributed according to the common Pareto law with parameters $\kappa, \beta > 0$:

$$F(x;\kappa,\beta) = 1 - \left(\frac{\kappa}{\kappa + x}\right)^{\beta}, \quad x \geq 0, \tag{3.4.1}$$

which belongs to the class $\mathscr{C} \subset \mathscr{L} \cap \mathscr{D}$. We assume that $\{(X_{2k-1}, X_{2k}), k \geq 1\}$ are independent replications of (X_1, X_2) with the joint distribution

$$F_{X_1,X_2}(x,y) = \max\Big\{\alpha F(x)F(y) + (1-\alpha)(F(x) + F(y) - 1),\ 0\Big\}, \tag{3.4.2}$$

with parameter $\alpha \in (0,1)$ (see eq. (4.2.7) in [51]). Since $\mathrm{P}(X_1 > x, X_2 > y) \leq \alpha\overline{F}(x)\overline{F}(y)$ for all x, y, X_1 and X_2 are ND r.v.s. Hence, by construction, $X_1, \ldots, X_n$ (n – even) are nonnegative pND r.v.s. Moreover, according to Remark 3.1.2, $G_n \in \mathscr{C}$. For our simulations we choose parameters:

1. $\kappa = 1$, $\beta = 2$ and $\alpha = 0.5$ (I case);
2. $\kappa = 2$, $\beta = 2$ and $\alpha = 0.5$ (II case);
3. $\kappa = 5$, $\beta = 2$ and $\alpha = 0.2$ (III case);
4. $\kappa = 5$, $\beta = 2$ and $\alpha = 0.7$ (IV case);
5. $\kappa = 5$, $\beta = 3$ and $\alpha = 0.8$ (V case).

We set $n = 10, 20, 50$ and $x = 100, 500, 1000, 2000$. The procedure of the computation of $\mathrm{P}(S_n > x)$ and $\overline{G_n}(x)$ in Corollary 3.1.1 consists of the following steps:

- Step 1. Assign a value for the variable x and set $m = k = 0$;
- Step 2. Generate the dependent r.v.s $X_1, \ldots, X_n$ from (3.4.1) and (3.4.2);
- Step 3. Calculate the sum value and the maximal value of $X_1, \ldots, X_n$:

$$S_n = \sum_{i=1}^{n} X_i \quad \text{and} \quad X_{(n)} = \max\{X_1, \ldots, X_n\};$$

- Step 4. Compare the two values S_n and $X_{(n)}$ with x: if $S_n > x$, then $m = m + 1$, and if $X_{(n)} > x$, then $k = k + 1$;

- Step 5. Repeat step 2 through step 4, $N = 2 \times 10^6$ times;
- Step 6. Calculate the estimates of the two tail probabilities $\mathrm{P}(S_n > x)$ and $\overline{G}_n(x)$ as, respectively, m/N and k/N.

For specific values of x, the simulated values of $\mathrm{P}(S_n > x)$ and $\overline{G_n}(x)$ are presented in Table 3.1 (I–V cases, respectively). It can be found from the table, that, the larger x becomes, the smaller the difference between the simulated values of $\mathrm{P}(S_n > x)$ and $\overline{G}_n(x)$ is. Therefore, the approximate relationship in Corollary 3.1.1 is reasonable.

I case: $\kappa = 1$, $\beta = 2$ and $\alpha = 0.5$.						
	n=10		n=20		n=50	
x	$P(S_n > x)$	$\overline{G_n}(x)$	$P(S_n > x)$	$\overline{G_n}(x)$	$P(S_n > x)$	$\overline{G_n}(x)$
100	0.002060	0.001524	0.005871	0.002942	0.080627	0.007374
500	0.000125	0.000118	0.000120	0.000106	0.000394	0.000285
1000	0.000013	0.000013	0.000037	0.000036	0.000101	0.000088
2000	0.000004	0.000004	0.000007	0.000007	0.000017	0.000015
II case: $\kappa = 2$, $\beta = 2$ and $\alpha = 0.5$.						
	n=10		n=20		n=50	
x	$P(S_n > x)$	$\overline{G_n}(x)$	$P(S_n > x)$	$\overline{G_n}(x)$	$P(S_n > x)$	$\overline{G_n}(x)$
100	0.011146	0.005762	0.0560811	0.011466	0.864862	0.028541
500	0.000258	0.000226	0.000603	0.000469	0.002375	0.001171
1000	0.000072	0.000067	0.000141	0.000124	0.000392	0.000300
2000	0.000012	0.000012	0.000026	0.000025	0.000078	0.000069
III case: $\kappa = 5$, $\beta = 2$ and $\alpha = 0.2$.						
	n=10		n=20		n=50	
x	$P(S_n > x)$	$\overline{G_n}(x)$	$P(S_n > x)$	$\overline{G_n}(x)$	$P(S_n > x)$	$\overline{G_n}(x)$
100	0.066922	0.026984	0.294649	0.053017	0.963822	0.127339
500	0.001436	0.001181	0.002333	0.003541	0.020995	0.005916
1000	0.000322	0.000295	0.000744	0.000619	0.002516	0.001488
2000	0.000067	0.000063	0.000170	0.000152	0.000463	0.000354
IV case: $\kappa = 5$, $\beta = 2$ and $\alpha = 0.7$						
	n=10		n=20		n=50	
x	$P(S_n > x)$	$\overline{G_n}(x)$	$P(S_n > x)$	$\overline{G_n}(x)$	$P(S_n > x)$	$\overline{G_n}(x)$
100	0.229526	0.037845	0.895448	0.074061	1.000000	0.175933
500	0.002414	0.001694	0.007813	0.003377	0.170259	0.008353
1000	0.000488	0.000412	0.001241	0.000859	0.006647	0.002074
2000	0.000119	0.000111	0.000251	0.000212	0.000847	0.000515
V case: $\kappa = 5$, $\beta = 3$ and $\alpha = 0.8$						
	n=10		n=20		n=50	
x	$P(S_n > x)$	$\overline{G_n}(x)$	$P(S_n > x)$	$\overline{G_n}(x)$	$P(S_n > x)$	$\overline{G_n}(x)$
100	0.012617	0.001952	0.271284	0.003838	0.999997	0.009773
500	0.000023	0.000018	0.000039	0.000069	0.000559	0.000081
1000	0.000002	0.000002	0.000007	0.000006	0.000020	0.000011
2000	0.000001	0.000001	0.000002	0.000004	0.000002	0.000004

Table 3.1: The empirical values of $P(S_n > x)$ and $\overline{G_n}(x)$

3.5 Modelling negative dependence structures with copulas

In this section we discuss some copula-based examples of dependence structures, satisfying (3.1.6). It is clear that any pND or pUEND r.v.s $X_1, \dots, X_n$ satisfy (3.1.6).

3.5.1 Generalized FGM copula

Consider the generalized Farlie-Gumbel-Morgenstern (FGM) copula (2.3.3).

Note that any pair of variables $X_1, \dots, X_n$ linked by copula (2.3.3) satisfy $\mathrm{P}(X_k \le x, X_l \le y) = C_{kl}^{\mathrm{GFGM}}(F_k(x), F_l(y))$, $k \ne l$, where

$$C_{kl}^{\mathrm{GFGM}}(u,v) \;=\; uv(1+\theta_{kl}(1-u^\alpha)(1-v^\alpha))^m. \tag{3.5.1}$$

Obviously, (3.5.1) implies $C_{kl}^{\mathrm{GFGM}}(u,v) \le uv$, $k<l$, whenever all θ_{kl} are nonpositive. Hence, the generalized FGM copula (2.3.3) provides the pND structure if $\theta_{kl} \le 0$, $1 \le k < l \le n$. The following proposition shows that this copula also captures the pUEND structure.

Proposition 3.5.1. *Let the distribution of* $(X_1, \dots, X_n)$ *be generated by copula in* (2.3.3). *Then*

$$\mathrm{P}(X_k > x, X_l > y) \;\le\; C_{kl}\overline{F_k}(x)\overline{F_l}(y), \tag{3.5.2}$$

where $C_{kl} := 1 + \max\{\alpha, 1\}((|\theta_{kl}|+1)^m - 1)$.

Proof. For every $k<l$, by (3.5.1), we have that

$$\mathrm{P}(X_k \le x, X_l \le y) = F_k(x)F_l(y)[1+\theta_{kl}(1-F_k^\alpha(x))(1-F_l^\alpha(y))]^m.$$

Hence,

$$\begin{aligned}
&\mathrm{P}(X_k > x, X_l > y)\\
&\quad= 1 - F_k(x) - F_l(y) + \mathrm{P}(X_k \le x, X_l \le y)\\
&\quad= 1 - F_k(x) - F_l(y) + F_k(x)F_l(y)\big(1+\theta_{kl}(1-F_k^\alpha(x))(1-F_l^\alpha(y))\big)^m\\
&\quad= \overline{F_k}(x) + \overline{F_l}(y) - 1 + (1-\overline{F_k}(x) - \overline{F_l}(y) + \overline{F_k}(x)\overline{F_l}(y))\\
&\qquad\times\left(1+\sum_{i=1}^m \binom{m}{i}\theta_{kl}^i(\overline{F_k^\alpha}(x))^i(\overline{F_l^\alpha}(y))^i\right)\\
&\quad= \overline{F_k}(x)\overline{F_l}(y) + (1-\overline{F_k}(x) - \overline{F_l}(y) + \overline{F_k}(x)\overline{F_l}(y))\\
&\qquad\times \sum_{i=1}^m \binom{m}{i}\theta_{kl}^i(\overline{F_k^\alpha}(x))^i(\overline{F_l^\alpha}(y))^i,
\end{aligned}$$

where $\overline{F_k^\alpha}(x) := 1 - F_k^\alpha(x)$. Using inequality $1 - u^\alpha \le \max\{\alpha, 1\}(1-u)$, $u \in [0,1]$, we get

$$\begin{aligned} \mathrm{P}(X_k > x, X_l > y) &\le \overline{F_k}(x)\overline{F_l}(y) + \overline{F_k^\alpha}(x)\overline{F_l^\alpha}(y) \times \sum_{i=1}^{m} \binom{m}{i} |\theta_{kl}|^i \\ &\le \Big(1 + \max\{\alpha, 1\}((|\theta_{kl}|+1)^m - 1)\Big)\overline{F_k}(x)\overline{F_l}(y). \end{aligned}$$

□

By (3.5.2), FGM copula generates a pUEND structure.

3.5.2 Ali–Mikhail–Haq copula

Consider the copula (2.3.4) and let $\mathrm{P}(X_1 \le x_1, \ldots, X_n \le x_n) = C^{\mathrm{AMH}}(F_1(x_1), \ldots, F_n(x_n))$. Then, for $k \ne l$,

$$\mathrm{P}(X_k \le x, X_l \le y) = \frac{F_k(x)F_l(y)}{1 - \theta\overline{F_k}(x)\overline{F_l}(y)}$$

and hence

$$\begin{aligned} &\mathrm{P}(X_k > x, X_l > y) \\ &= 1 - F_k(x) - F_l(y) + \frac{F_k(x)F_l(y)}{1 - \theta\overline{F_k}(x)\overline{F_l}(y)} \le \overline{F_k}(x)\overline{F_l}(y) \qquad (3.5.3) \end{aligned}$$

if $-1 \le \theta \le 0$. In the case $0 < \theta < 1$, we have

$$\mathrm{P}(X_k \le x, X_l \le y) \le \frac{1}{1-\theta} F_k(x)F_l(y), \qquad (3.5.4)$$

$$\mathrm{P}(X_k > x, X_l > y) \le \frac{1}{1-\theta} \overline{F_k}(x)\overline{F_l}(y). \qquad (3.5.5)$$

(3.5.4) is obvious. In order to show (3.5.5) it suffices to verify that

$$1 - u - v + \frac{uv}{1 - \theta(1-u)(1-v)} \le \frac{(1-u)(1-v)}{1-\theta}, \quad 0 \le u, v \le 1,\ 0 < \theta < 1.$$

The proof is straightforward, we omit it.

By (3.5.3)–(3.5.5), the copula in (2.3.4) generates the pND structure if $-1 \le \theta \le 0$ and the pEND structure if $0 < \theta < 1$.

3.5.3 Frank copula

Consider the copula (2.3.5) and assume that $\mathrm{P}(X_1 \le x_1, \ldots, X_n \le x_n) = C^{\mathrm{F}}(F_1(x_1), \ldots, F_n(x_n))$. Then $\mathrm{P}(X_k \le x, X_l \le y) = Q^{\mathrm{F}}(F_k(x), F_l(y))$, $k \ne l$, where

$$C^{\mathrm{F}}(u, v) := -\frac{1}{\theta} \log\Big(1 + \frac{(\mathrm{e}^{-\theta u} - 1)(\mathrm{e}^{-\theta v} - 1)}{\mathrm{e}^{-\theta} - 1}\Big).$$

In this case the copula density is bounded:

$$\begin{aligned} c^{\mathrm{F}}(u,v) &= \frac{-\theta(\mathrm{e}^{-\theta}-1)\mathrm{e}^{-\theta(u+v)}}{((\mathrm{e}^{-\theta}-1)+(\mathrm{e}^{-\theta u}-1)(\mathrm{e}^{-\theta v}-1))^2} \\ &\le \frac{\theta}{(1-\mathrm{e}^{-\theta})\mathrm{e}^{-2\theta}} =: c_\theta. \end{aligned}$$

Thus, denoting the corresponding marginal densities $f_k(x)$, we have

$$\begin{aligned} \mathrm{P}(X_k>x, X_l>y) &= \int_{w>x,z>y} c^{\mathrm{F}}(F_k(w),F_l(z))f_k(w)f_l(z)\mathrm{d}w\,\mathrm{d}z \\ &\le c_\theta \overline{F_k}(x)\overline{F}_l(y), \quad k\neq l, \end{aligned}$$

i.e. the Frank copula generates the pUEND structure.

3.5.4 Clayton copula

Consider the copula (2.3.6) and assume $\mathrm{P}(X_1\le x_1,\dots,X_n\le x_n) = C^{\mathrm{Cl}}(F_1(x_1),\dots,F_n(x_n))$. Then $\mathrm{P}(X_k\le x, X_l\le y)=C^{\mathrm{Cl}}(F_k(x),F_l(y))$, where

$$C^{\mathrm{Cl}}(u,v) = \left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}.$$

Note that if $\theta\to 0$ then $C^{\mathrm{Cl}}(u,v)$ tends to uv, i.e. we obtain the independence copula, whereas if $\theta\to\infty$ then $C^{\mathrm{Cl}}(u,v)$ tends to $\min(u,v)$, i.e. comonotonicity copula.

We will show that for any $k\neq l$ and $x,y\in\mathbb{R}$ it holds

$$\mathrm{P}(X_k>x,X_l>y)\le(1+\theta)\overline{F_k}(x)\overline{F_l}(y).$$

This implies the pUEND property and, hence, relation (3.1.6). The proof of this inequality follows from identity $\mathrm{P}(X_k>x,X_l>y)=1-F_k(x)-F_l(y)+\mathrm{P}(X_k\le x,X_l\le y)$ and the following lemma.

Lemma 3.5.1. *For any $(u,v)\in[0,1]^2$ and $\theta>0$ it holds*

$$(u^{-\theta}+v^{-\theta}-1)^{-1/\theta} \le uv+\theta(1-u)(1-v).$$

Proof. Denote, for convenience, $C_\theta(u,v):=(u^{-\theta}+v^{-\theta}-1)^{-1/\theta}$. Take any small $\epsilon>0$ and write

$$\begin{aligned} &C_\theta(u,v)-C_\epsilon(u,v) \\ &= \int_\epsilon^\theta \frac{\partial C_t(u,v)}{\partial t}\,\mathrm{d}t \\ &= \int_\epsilon^\theta \frac{(u^{-t}+v^{-t}-1)\log(u^{-t}+v^{-t}-1)-u^{-t}\log u^{-t}-v^{-t}\log v^{-t}}{t^2(u^{-t}+v^{-t}-1)^{1+1/t}}\,\mathrm{d}t \\ &= \int_\epsilon^\theta C_t(u,v)\frac{(u^{-t}+v^{-t}-1)\log(u^{-t}+v^{-t}-1)-u^{-t}\log u^{-t}-v^{-t}\log v^{-t}}{t^2(u^{-t}+v^{-t}-1)}\,\mathrm{d}t. \end{aligned}$$

For all $(u,v) \in [0,1]^2$ and $t > 0$ we have

$$C_t(u,v) \;\leq\; \sqrt{uv} \tag{3.5.6}$$

and

$$\frac{(u^{-t}+v^{-t}-1)\log(u^{-t}+v^{-t}-1) - u^{-t}\log u^{-t} - v^{-t}\log v^{-t}}{t^2(u^{-t}+v^{-t}-1)} \;\leq\; \frac{(1-u)(1-v)}{\sqrt{uv}}. \tag{3.5.7}$$

Bound (3.5.6) is due to inequality $(u^{-t/2} - v^{-t/2})^2 + u^{-t/2}v^{-t/2} \geq 1$. In order to proof (3.5.7) we use the following inequality

$$(x+y-1)\log(x+y-1) - x\log x - y\log y \;\leq\; (x+y-1)\log x \log y, \tag{3.5.8}$$

for any $x \geq 1$, $y \geq 1$. Denote

$$f(x,y) \;:=\; (x+y-1)\log(x+y-1) - x\log x - y\log y - (x+y-1)\log x\log y.$$

Then (3.5.8) follows by noting that $f(1,y) = 0$ for any $y \geq 1$ and

$$\frac{\partial f(x,y)}{\partial x} \;=\; -\Big(\log x \log y + \frac{y-1}{x}\log y + \log\frac{xy}{x+y-1}\Big) \leq 0, \quad x, y \geq 1.$$

By (3.5.8),

$$\frac{(u^{-t}+v^{-t}-1)\log(u^{-t}+v^{-t}-1) - u^{-t}\log u^{-t} - v^{-t}\log v^{-t}}{t^2(u^{-t}+v^{-t}-1)} \;\leq\; \log u \log v,$$

where, by inequality $\log x \leq (x-1)/\sqrt{x}$, $x \geq 1$ (see [49], p. 272),

$$-\log u = \log(1/u) \leq \frac{1/u - 1}{1/\sqrt{u}} = \frac{1-u}{\sqrt{u}}.$$

Inequalities (3.5.6), (3.5.7) imply

$$C_\theta(u,v) \;\leq\; C_\epsilon(u,v) + (\theta - \epsilon)(1-u)(1-v).$$

Taking $\epsilon \to 0$, we obtain the desired inequality. □

Summarizing, we have the following corollary.

Corollary 3.5.1. *Let r.v.s* $X_1, \ldots, X_n$ *have corresponding univariate distributions* $F_1, \ldots, F_n$*, such that* $H_n \in \mathscr{D}$*, and let the dependence structure be generated by either of the copulas in* (2.3.3), (2.3.4), (2.3.5) *or* (2.3.6)*. Then asymptotic relation* (3.1.8) *holds. If, in addition,* $H_n(-x) = o(\overline{H_n}(x))$*, then* (3.1.9) *holds too.*

Chapter 4

Randomly weighted sums and their closure property

In this chapter we study the closure property and probability tail asymptotics for randomly weighted sums $S_n^\Theta = \Theta_1 X_1 + \cdots + \Theta_n X_n$ of heavy-tailed dependent random variables $X_1, \ldots, X_n$ and positive random weights $\Theta_1, \ldots, \Theta_n$.

Together we prove the asymptotic equivalence between the tail probabilities of $S_{(n)}^\Theta := \max\{S_1^\Theta, \ldots, S_n^\Theta\}$, S_n^Θ and $S_n^{\Theta+} := \sum_{i=1}^n \Theta_i X_i^+$. Such relation is not only of theoretical interest but also has practical implications as it allows, for large x, to replace the sum of real-valued r.v.s by much easier to handle sum of r.v.s concentrated on $[0, \infty)$. Also it shows that in the context of the model with the insurance and financial risk, the tail probabilities of the stochastic present value of the aggregate net losses, S_n^Θ, and the maximal net loss, $S_{(n)}^\Theta$, asymptotically are the same.

4.1 Literature review

In the case $\Theta_1 = \cdots = \Theta_n = 1$ the convolution closure of class $\mathscr{L}$ was proved in [23] (Theorem 3(b)) when $n = 2$ (in fact, they proved the closure for more general class $\mathscr{L}_\gamma$) and in [52]. The closure property for some other heavy-tailed classes was studied by Leslie [40], Tang and Tsitsiashvili [61], Cai and Tang [10], Geluk and Ng [30], Foss et al. [28], Watanabe and Yamamuro [68].

The closure property of randomly weighted sums S_n^Θ was studied in [12] and [72].

The probability tail asymptotics for sums S_n^Θ of independent heavy tailed r.v.s $X_1, \dots, X_n$ with $\Theta_1, \dots, \Theta_n$ being nonnegative bounded r.v.s were investigated in [61], [62], [63], [12], [71] among others.

Weak equivalence between the quantities $\mathrm{P}(S_n^\Theta > x)$ and $\sum_{i=1}^n \mathrm{P}(\Theta_i X_i > x)$ with r.v.s having a certain dependence structure was proved in [29]. For pQAI r.v.s Chen and Yuen [13] showed that $\mathrm{P}(S_n^\Theta > x) \sim \sum_{i=1}^n \mathrm{P}(\Theta_i X_i > x)$. The same asymptotics, with some dependence among $X_1, \dots, X_n$, was considered in [66].

We note that both mentioned questions are closely related: the proof asymptotic equivalence (1.1) is based on the uniform closure property.

Recently, Yang et al. [72] considered the randomly weighted sum S_2^Θ under the following dependence structure between real-valued r.v.s X_1 and X_2:

$$\begin{aligned} \mathrm{P}(X_2 > x | X_1 = y) &\sim h_1(y)\overline{F_2}(x), \\ \mathrm{P}(X_1 > x | X_2 = y) &\sim h_2(y)\overline{F_1}(x), \end{aligned} \tag{4.1.1}$$

uniformly in $y \in \mathbb{R}$, where $h_k : \mathbb{R} \mapsto (0, \infty)$, $k = 1, 2$, are measurable functions. Such a dependence structure, proposed by Asimit and Badescu [4], can be easily checked for some well-known bivariate copulas, allowing both positive and negative dependence, see, e.g., [4], [43], [72]. The main result of Yang et al. [72] is the following:

Theorem 4.1 ([72]). *Assume that X_1, X_2 are real-valued r.v.s with distributions $F_k \in \mathscr{L}$, satisfying relation (4.1.1); Θ_1, Θ_2 are arbitrarily dependent, but independent of X_1, X_2, and such that $\mathrm{P}(a \le \Theta_k \le b) = 1$, $k = 1, 2$, with some constants $0 < a \le b < \infty$. Then the distribution of S_2^Θ is in $\mathscr{L}$ and*

$$\mathrm{P}(S_{(2)}^\Theta > x) \sim \mathrm{P}(S_2^\Theta > x) \sim \mathrm{P}(S_2^{\Theta+} > x),$$

where $S_{(2)}^\Theta = \max\{S_1^\Theta, S_2^\Theta\}$.

Our goal is to is to extend the result on the closure property and tail asymptotics of randomly weighted sums S_n^Θ under similar dependence structure to (4.1.1) for any $n \ge 2$.

4.2 Main results

Let $n \ge 2$ be an integer. Consider the real-valued r.v.s $X_1, \dots, X_n$ with corresponding distributions $F_1, \dots, F_n$, such that $\overline{F_k}(x) > 0$ for $k = 1, \dots, n$, and assume the following dependence structures.

Assumption A. For each $k = 2, \dots, n$ relation

$$P(X_k > x | X_1 = y_1, \dots, X_{k-1} = y_{k-1}) \sim g_k(y_1, \dots, y_{k-1}) \overline{F_k}(x) \tag{4.2.1}$$

holds uniformly for all $(y_1, \dots, y_{k-1}) \in \mathbb{R}^{k-1}$, i.e.

$$\lim_{x\to\infty} \sup_{(y_1,\dots,y_{k-1})\in\mathbb{R}^{k-1}} \left| \frac{P(X_k > x | X_1 = y_1, \dots, X_{k-1} = y_{k-1})}{g_k(y_1, \dots, y_{k-1}) \overline{F_k}(x)} - 1 \right| = 0,$$

where $g_k \colon \mathbb{R}^{k-1} \mapsto \mathbb{R}_+ := (0, \infty)$, $k = 2, \dots, n$, are measurable functions.

Assumption B. For each $k = 2, \dots, n$, relation

$$P\left(\sum_{i=1}^{k-1} w_i X_i > x | X_k = y \right) \sim h_k^{(w)}(y) P\left(\sum_{i=1}^{k-1} w_i X_i > x \right) \tag{4.2.2}$$

holds uniformly for all $y \in \mathbb{R}$ and $\overline{w}_{k-1} := (w_1, \dots, w_{k-1}) \in [a, b]^{k-1}$, with some positive constants $0 < a \le b < \infty$, i.e.

$$\lim_{x\to\infty} \sup_{y\in\mathbb{R}} \sup_{\overline{w}_{k-1}\in[a,b]^{k-1}} \left| \frac{P\left(\sum_{i=1}^{k-1} w_i X_i > x | X_k = y \right)}{h_k^{(w)}(y) P\left(\sum_{i=1}^{k-1} w_i X_i > x \right)} - 1 \right| = 0,$$

where $h_k^{(w)} \colon \mathbb{R} \mapsto \mathbb{R}_+$, $k = 1, \dots, n$, are measurable functions, maybe dependent on $\overline{w}_{k-1}$.

If, for some $i \in \{1, \dots, k-1\}$, $y_i = y_i^*$ in (4.2.1) is not attainable value of X_i the conditional probability in there is treated as unconditional and therefore $g_k(y_1, \dots, y_i^*, \dots, y_{k-1}) = 1$ for such y_i^*. The same agreement holds for (4.2.2).

Clearly, the uniformity in (4.2.1) and (4.2.2) imply that $\mathrm{E} g_k(X_1, \dots, X_{k-1}) = \mathrm{E} h_k^{(w)}(X_k) = 1$ for $k = 2, \dots, n$.

Our first main result is the following theorem.

Theorem 4.2.1. *Let $X_1, \dots, X_n$ be real-valued r.v.s satisfying Assumptions A, B, and let $\Theta_1, \dots, \Theta_n$ be random weights, independent of $X_1, \dots, X_n$, such that $P(a \le \Theta_k \le b) = 1$, $k = 1, \dots, n$, $0 < a \le b < \infty$. If $F_k \in \mathscr{L}$, for all $k = 1, \dots, n$, then d. f. $P(S_n^{\Theta} \le x)$ belongs to $\mathscr{L}$.*

In order to obtain our second main result we have to strengthen the assumption of dependence from assumptions A, B to the following:

Assumption C. For arbitrary nonempty sets of indices $I = \{k_1, \dots, k_m\} \subset \{1, 2, \dots, n\}$ and $J = \{r_1, \dots, r_p\} \subset \{1, 2, \dots, n\} \backslash I$, relation

$$P\left(\sum_{k\in I} w_k X_k > x \Big| X_r = y_r \text{ with } r \in J \right) \sim h_{I,J}^{(w)}(y_{r_1}, \dots, y_{r_p}) P\left(\sum_{k\in I} w_k X_k > x \right)$$

holds uniformly for all $(y_{r_1},\dots,y_{r_p}) \in \mathbb{R}^p$ and $(w_{k_1},\dots,w_{k_m}) \in [a,b]^m$, $0 < a \le b < \infty$, with some measurable function $h^{(w)}_{I,J}\colon \mathbb{R}^p \mapsto \mathbb{R}_+$, such that $h^{(w)}_{I,J}(y_{r_1},\dots,y_{r_p})$ is bounded uniformly in $w_k \in [a,b], k \in I$ and $(y_{r_1},\dots,y_{r_p}) \in \mathbb{R}^p$.

Clearly, Assumption C implies both Assumptions A and B with $g_k(y_1,\dots,y_{k-1}) \equiv h^{(w)}_{\{k\},\{1,\dots,k-1\}}(y_1,\dots,y_{k-1})$ and $h^{(w)}_k(y) \equiv h^{(w)}_{\{1,\dots,k-1\},\{k\}}(y)$, $k = 2,\dots,n$.

Theorem 4.2.2. *Let $X_1,\dots,X_n$ be real-valued r.v.s satisfying Assumption C and let $\Theta_1,\dots,\Theta_n$ be random weights, independent of $X_1,\dots,X_n$, such that $\mathrm{P}(a \le \Theta_k \le b) = 1$, $k = 1,\dots,n$, $0 < a \le b < \infty$. If $F_k \in \mathscr{L}$ for all $k = 1,\dots,n$, then*

$$\mathrm{P}(S^{\Theta}_n > x) \;\sim\; \mathrm{P}(S^{\Theta+}_n > x) \;\sim\; \mathrm{P}(S^{\Theta}_{(n)} > x). \tag{4.2.3}$$

Remark 4.2.1. In the case $n = 2$, conjunction of assumptions A and B coincides with Assumption C, which is the same as condition (4.1.1). Thus, Theorems 4.2.1–4.2.2 generalize the result in Theorem 4.1.

Remark 4.2.2. If conditions of Theorem 4.2.2 are satisfied and $X_1,\dots,X_n$ are independent, then relations (4.2.3) were proved in [66] (Lemma 4) and [12] (Theorem 2.1); moreover, the interval $[a,b]$ can be generalized to $(0,b]$ if, additionally, Θ_k's are associated (see Theorem 2.2 in [12]).

4.3 Proofs of theorems

4.3.1 Proof of Theorem 4.2.1

The proof of Theorem 4.2.1 is essentially based on the uniform closure property of the sum $S^w_n := w_1X_1 + \cdots + w_nX_n$: if assumptions A and B are satisfied and each $F_k \in \mathscr{L}$, then the distribution of sum S^w_n is uniformly in $\mathscr{L}$ too.

Lemma 4.3.1. *Let $X_1,\dots,X_n$ be the real-valued r.v.s with corresponding distributions $F_1,\dots,F_n$ and let Assumptions A, B hold. If $F_k \in \mathscr{L}$, $k = 1,\dots,n$, then for any $K > 0$ the relation*

$$\mathrm{P}\big(S^w_n > x - K\big) \;\sim\; \mathrm{P}\big(S^w_n > x\big) \tag{4.3.1}$$

holds uniformly for all $\overline{w}_n \in [a,b]^n$.

Proof. It is sufficient to prove that

$$\limsup_{x\to\infty}\sup_{\overline{w}_n\in[a,b]^n}\frac{\mathrm{P}(S_n^w>x-K)}{\mathrm{P}(S_n^w>x)}\leq 1. \tag{4.3.2}$$

By Remark 4.2.1, relation (4.3.1) holds for $n=2$ (see Lemma 3.1 in [72]). Suppose that relation (4.3.2) holds for some $n=N\geq 2$, i.e.

$$\mathrm{P}(S_N^w>x-K)\ \sim\ \mathrm{P}(S_N^w>x) \tag{4.3.3}$$

with above uniformity. We will prove that (4.3.2) holds for $n=N+1$. This will prove the statement of the lemma.

Let $\epsilon\in(0,1)$ be an arbitrary constant. By $F_{N+1}\in\mathscr{L}$, we have that

$$\begin{aligned}
&\limsup\sup_{w_{N+1}\in[a,b]}\frac{\mathrm{P}\left(w_{N+1}X_{N+1}>x-K\right)}{\mathrm{P}\left(w_{N+1}X_{N+1}>x\right)}\\
&\leq\ \limsup\sup_{w_{N+1}\in[a,b]}\frac{\overline{F}_{N+1}\left(\frac{x}{w_{N+1}}-\frac{K}{a}\right)}{\overline{F}_{N+1}\left(\frac{x}{w_{N+1}}\right)}\\
&\leq\ \limsup\sup_{z\geq x/b}\frac{\overline{F}_{N+1}\left(z-\frac{K}{a}\right)}{\overline{F}_{N+1}(z)}=1.
\end{aligned}$$

Let $\epsilon\in(0,1)$ be an arbitrary constant. Therefore, for any $\epsilon>0$ there exists such $x_1>K$ that for all $x>x_1$

$$\begin{aligned}
1\ &\leq\ \sup_{x>x_1}\sup_{w_{N+1}\in[a,b]}\frac{\mathrm{P}\left(w_{N+1}X_{N+1}>x-K\right)}{\mathrm{P}\left(w_{N+1}X_{N+1}>x\right)}\\
&\leq\ \sup_{x>x_1}\sup_{z\geq x/b}\frac{\mathrm{P}(X_{N+1}>z-\frac{K}{a})}{\mathrm{P}(X_{N+1}>z)}\leq 1+\epsilon.
\end{aligned}\tag{4.3.4}$$

Also, condition (4.2.1) implies that

$$\begin{aligned}
(1-\epsilon)\overline{F_{N+1}}(x)g_{N+1}(y_1,\ldots,y_N)\ &\leq\ \mathrm{P}(X_{N+1}>x|X_1=y_1,\ldots,X_N=y_N)\\
&\leq\ (1+\epsilon)\overline{F_{N+1}}(x)g_{N+1}(y_1,\ldots,y_N)
\end{aligned}\tag{4.3.5}$$

for all $y_i\in\mathbb{R}$, $i=1,\ldots,N$ and $x\geq x_2\geq x_1$.

If $x\geq\max\{bx_2,x_2\}$, then

$$\begin{aligned}
&\frac{\mathrm{P}(S_{N+1}^w>x-K)}{\mathrm{P}(S_{N+1}^w>x)}\\
&=\ \frac{(\int_{\mathcal{D}_1}+\int_{\mathcal{D}_2})\mathrm{P}(w_{N+1}X_{N+1}>x-K-\sum_{i=1}^N w_iy_i|X_1=y_1,\ldots,X_N=y_N)\mathrm{d}F_{\mathbf{X}}(\mathbf{y})}{(\int_{\mathcal{D}_3}+\int_{\mathcal{D}_4})\mathrm{P}(w_{N+1}X_{N+1}>x-\sum_{i=1}^N w_iy_i|X_1=y_1,\ldots,X_N=y_N)\mathrm{d}F_{\mathbf{X}}(\mathbf{y})}\\
&=:\ \frac{I_{11}(x)+I_{12}(x)}{I_{21}(x)+I_{22}(x)}\ \leq\ \max\left\{\frac{I_{11}(x)}{I_{21}(x)},\frac{I_{12}(x)}{I_{22}(x)}\right\},
\end{aligned}\tag{4.3.6}$$

where

$$\begin{aligned}
\mathcal{D}_1 &:= \{(y_1,\dots,y_N) : \sum_{i=1}^N w_i y_i \le x - bx_2 - K\},\\
\mathcal{D}_2 &:= \{(y_1,\dots,y_N) : \sum_{i=1}^N w_i y_i > x - bx_2 - K\},\\
\mathcal{D}_3 &:= \{(y_1,\dots,y_N) : \sum_{i=1}^N w_i y_i \le x - bx_2\},\\
\mathcal{D}_4 &:= \{(y_1,\dots,y_N) : \sum_{i=1}^N w_i y_i > x - bx_2\},
\end{aligned}$$

and $F_{\boldsymbol{X}}(\boldsymbol{x}) := F_{X_1,\dots,X_N}(x_1,\dots,x_N)$. Since $x \ge bx_2$, $x \ge x_2 \ge x_1$, relations (4.3.4), (4.3.5) imply that

$$\begin{aligned}
&\sup_{\overline{w}_{N+1}\in[a,b]^{N+1}} \frac{I_{11}(x)}{I_{21}(x)}\\
&\le \frac{1+\epsilon}{1-\epsilon} \sup_{\overline{w}_{N+1}\in[a,b]^{N+1}} \frac{\int_{\mathcal{D}_1} \mathrm{P}(w_{N+1}X_{N+1} > x - K - \sum_{i=1}^N w_i y_i) g_{N+1}(\boldsymbol{y}) \mathrm{d}F_{\boldsymbol{X}}(\boldsymbol{y})}{\int_{\mathcal{D}_1} \mathrm{P}(w_{N+1}X_{N+1} > x - \sum_{i=1}^N w_i y_i) g_{N+1}(\boldsymbol{y}) \mathrm{d}F_{\boldsymbol{X}}(\boldsymbol{y})}\\
&\le \frac{1+\epsilon}{1-\epsilon} \sup_{\overline{w}_{N+1}\in[a,b]^{N+1}} \sup_{\overline{y}_N\in\mathcal{D}_1} \frac{\mathrm{P}(w_{N+1}X_{N+1} > x - K - \sum_{i=1}^N w_i y_i)}{\mathrm{P}(w_{N+1}X_{N+1} > x - \sum_{i=1}^N w_i y_i)}\\
&\le \frac{1+\epsilon}{1-\epsilon} \sup_{z\ge x_2} \frac{\mathrm{P}(X_{N+1} > z - K)}{\mathrm{P}(X_{N+1} > z)} \le \frac{(1+\epsilon)^2}{1-\epsilon}, \qquad (4.3.7)
\end{aligned}$$

where $g_{N+1}(\boldsymbol{y}) := g_{N+1}(y_1,\dots,y_N)$.

On the other hand, condition (4.2.2) implies that

$$\begin{aligned}
(1-\epsilon) h^{(w)}_{N+1}(y_{N+1}) \mathrm{P}(S^w_N > x) &\le \mathrm{P}(S^w_N > x | X_{N+1} = y_{N+1})\\
&\le (1+\epsilon) h^{(w)}_{N+1}(y_{N+1}) \mathrm{P}(S^w_N > x) \quad (4.3.8)
\end{aligned}$$

for all $y_{N+1} \in \mathbb{R}$, $\overline{w}_N \in [a,b]^N$ and $x \ge x_3 \ge x_2$. Hence,

$$
\begin{aligned}
I_{22}(x) = \;& \mathrm{P}\Big(S_N^w > x - bx_2, S_{N+1}^w > x\Big)\\
\geq \;& \mathrm{P}\Big(S_N^w > x, S_{N+1}^w > x\Big)\\
= \;& \mathrm{P}(S_N^w > x, X_{N+1} \geq 0) + \mathrm{P}(S_N^w + w_{N+1}X_{N+1} > x, X_{N+1} < 0)\\
= \;& \int_{[0,\infty)} \mathrm{P}(S_N^w > x | X_{N+1} = y_{N+1})\mathrm{d}F_{N+1}(y_{N+1})\\
& + \int_{(-\infty,0)} \mathrm{P}(S_N^w > x - w_{N+1}y_{N+1} | X_{N+1} = y_{N+1})\mathrm{d}F_{N+1}(y_{N+1})\\
\geq \;& (1-\epsilon)\int_{[0,\infty)} \mathrm{P}(S_N^w > x)h_{N+1}^{(w)}(y_{N+1})\mathrm{d}F_{N+1}(y_{N+1})\\
& + (1-\epsilon)\int_{(-\infty,0)} \mathrm{P}(S_N^w > x - w_{N+1}y_{N+1})h_{N+1}^{(w)}(y_{N+1})\mathrm{d}F_{N+1}(y_{N+1})\\
= \;& (1-\epsilon)\mathrm{P}(S_N^w > x)\mathrm{E}h_{N+1}^{(w)}(X_{N+1})\mathbb{1}_{\{X_{N+1}\geq 0\}}\\
& + (1-\epsilon)\int\limits_{(-\infty,0)} \mathrm{P}(S_N^w > x - w_{N+1}y_{N+1})h_{N+1}^{(w)}(y_{N+1})\mathrm{d}F_{N+1}(y_{N+1})
\end{aligned}
\tag{4.3.9}
$$

for all $\overline{w}_{N+1} \in [a,b]^{N+1}$ and $x \geq x_3$. Here, $\mathrm{E}h_{N+1}^{(w)}(X_{N+1})\mathbb{1}_{\{X_{N+1}\geq 0\}} > 0$ because of heavy-tailedness of F_{N+1}. Similarly, under (4.3.8),

$$
\begin{aligned}
I_{12}(x) \;=\;& \mathrm{P}(S_{N+1}^w > x - K, S_N^w > x - bx_2 - K)\\
\leq\;& \mathrm{P}(S_{N+1}^w > x - K, S_N^w > x - K) + \mathrm{P}(x - bx_2 - K < S_N^w \leq x - K)\\
=\;& \mathrm{P}(S_N^w > x - K, X_{N+1} \geq 0) + \mathrm{P}(S_N^w + w_{N+1}X_{N+1} > x - K, X_{N+1} < 0)\\
& + \mathrm{P}(x - bx_2 - K < S_N^w \leq x - K)\\
\leq\;& (1+\epsilon)\mathrm{P}(S_N^w > x - K)\mathrm{E}h_{N+1}^{(w)}(X_{N+1})\mathbb{1}_{\{X_{N+1}\geq 0\}}\\
& + (1+\epsilon)\int_{(-\infty,0)} \mathrm{P}(S_N^w > x - K - w_{N+1}y_{N+1})h_{N+1}^{(w)}(y_{N+1})\mathrm{d}F_{N+1}(y_{N+1})\\
& + \mathrm{P}(S_N^w > x - bx_2 - K) - \mathrm{P}(S_N^w > x - K)
\end{aligned}
\tag{4.3.10}
$$

for $x \geq x_3$ and all $\overline{w}_{N+1} \in [a,b]^{N+1}$.

Relations (4.3.9), (4.3.10) imply that

$$\begin{aligned}
&\limsup_{x\to\infty} \sup_{\overline{w}_{N+1}\in[a,b]^{N+1}} \frac{I_{12}(x)}{I_{22}(x)} \\
&\le \frac{1}{1-\epsilon} \limsup_{x\to\infty} \sup_{\overline{w}_N\in[a,b]^N} \left(\frac{\mathrm{P}(S_N^w > x - bx_2 - K)}{\mathrm{P}(S_N^w > x)} - \frac{\mathrm{P}(S_N^w > x - K)}{\mathrm{P}(S_N^w > x)} \right) \\
&\quad + \frac{1+\epsilon}{1-\epsilon} \max\left\{ \limsup_{x\to\infty} \sup_{\overline{w}_N\in[a,b]^N} \frac{\mathrm{P}(S_N^w > x - K)}{\mathrm{P}(S_N^w > x)}, \right. \\
&\quad\quad \left. \limsup_{x\to\infty} \sup_{\overline{w}_N\in[a,b]^N} \sup_{y_{N+1}<0} \frac{\mathrm{P}(S_N^w > x - w_{N+1}y_{N+1} - K)}{\mathrm{P}(S_N^w > x - w_{N+1}y_{N+1})} \right\}.
\end{aligned}$$

From hypothesis (4.3.3) we obtain that

$$\limsup_{x\to\infty} \sup_{\overline{w}_{N+1}\in[a,b]^{N+1}} \frac{I_{12}(x)}{I_{22}(x)} \le \frac{1+\epsilon}{1-\epsilon}. \tag{4.3.11}$$

Hence, by (4.3.6), (4.3.7), (4.3.11) we get

$$\limsup_{x\to\infty} \sup_{\overline{w}_{N+1}\in[a,b]^{N+1}} \frac{\mathrm{P}(S_{N+1}^w > x - K)}{\mathrm{P}(S_{N+1}^w > x)} \le \frac{(1+\epsilon)^2}{1-\epsilon}.$$

The arbitrariness of $\epsilon > 0$ implies inequality (4.3.2) for $n = N+1$. □

It is easy to see that the result in Lemma 4.3.1 can be reformulated replacing the constant K in (4.3.1) by some infinitely increasing function $K(x)$ (see the arguments in [75]), which does not depend on w. If Lemma 4.3.1 holds, then

$$\lim_{x\to\infty} \sup_{\overline{w}_n\in[a,b]^n} \left| \frac{\mathrm{P}(S_n^w > x - K)}{\mathrm{P}(S_n^w > x)} - 1 \right| = 0$$

holds uniformly for all $\overline{w}_n \in [a,b]^n$. We can choose an increasing sequence of positive numbers $\{q_n, n \ge 1\}$ such that for all $x \ge q_n$,

$$\sup_{\overline{w}_n\in[a,b]^n} \left| \frac{\mathrm{P}(S_n^w > x + n)}{\mathrm{P}(S_n^w > x)} - 1 \right| + \sup_{\overline{w}_n\in[a,b]^n} \left| \frac{\mathrm{P}(S_n^w > x - n)}{\mathrm{P}(S_n^w > x)} - 1 \right| \le \frac{1}{n}.$$

If we set $K(x) = n$, $q_{n-1} \le x < q_n$ then we see, that $K(x) \nearrow \infty$ and

$$\lim_{x\to\infty} \sup_{\overline{w}_n\in[a,b]^n} \left| \frac{\mathrm{P}(S_n^w > x \pm K(x))}{\mathrm{P}(S_n^w > x)} - 1 \right| = 0.$$

Thus we have:

Corollary 4.3.1. *Assume the conditions in Lemma 4.3.1. Then there exists a positive nondecreasing function $K(x)$, satisfying $K(x) \nearrow \infty$, such that the relation*

$$\mathrm{P}\big(S_n^w > x \pm K(x)\big) \sim \mathrm{P}\big(S_n^w > x\big) \tag{4.3.12}$$

holds uniformly for $\overline{w}_n \in [a,b]^n$.

Proof of Theorem 4.2.1. Using Lemma 4.3.1, we obtain that for any $K>0$

$$\begin{aligned}
\mathrm{P}(S_n^\Theta > x-K) &= \int\cdots\int_{[a,b]^n} \mathrm{P}(S_n^w > x-K)\mathrm{P}(\Theta_1\in \mathrm{d}w_1,\dots,\Theta_n\in\mathrm{d}w_n)\\
&\sim \int\cdots\int_{[a,b]^n} \mathrm{P}(S_n^w > x)\mathrm{P}(\Theta_1\in \mathrm{d}w_1,\dots,\Theta_n\in\mathrm{d}w_n)\\
&= \mathrm{P}(S_n^\Theta > x).
\end{aligned}$$

□

4.3.2 Proof of Theorem 4.2.2

The proof of Theorem 4.2.2 is based on the following lemma. Set $S_n^{w+} := \sum_{k=1}^n w_k X_k^+$, $S_{(n)}^w := \max\{S_1^w,\dots,S_n^w\}$.

Lemma 4.3.2. *Let $X_1,\dots,X_n$ $(n\ge 2)$ be real-valued r.v.s with corresponding distributions $F_1,\dots,F_n$, such that each $F_k\in\mathscr{L}$. Then, under Assumption C,*

$$\mathrm{P}(S_n^w > x) \;\sim\; \mathrm{P}(S_n^{w+} > x) \;\sim\; \mathrm{P}(S_{(n)}^w > x)$$

uniformly for all $\overline{w}_n\in[a,b]^n$.

Proof. Since $S_n^w \le S_{(n)}^w \le S_n^{w+}$, we only need to prove that

$$\mathrm{P}(S_n^{w+} > x) \;\lesssim\; \mathrm{P}(S_n^w > x). \tag{4.3.13}$$

Obviously, for positive x, it holds

$$\begin{aligned}
\mathrm{P}(S_n^{w+} > x) &= \mathrm{P}(S_n^w > x) + \mathrm{P}(S_n^{w+} > x, S_n^w \le x)\\
&= \mathrm{P}(S_n^w > x) + \sum_I \mathrm{P}(S_n^{w+} > x, S_n^w \le x, \mathcal{A}_I(X))\\
&= \mathrm{P}(S_n^w > x) + \sum_I p_I, \qquad (4.3.14)
\end{aligned}$$

where the sum $\sum_I$ is taken over all nonempty subsets $I\subset\{1,2,\dots,n\}$ and

$$\mathcal{A}_I(X) \;:=\; \Big(\bigcap_{k\in I}\{X_k\ge 0\}\Big)\bigcap\Big(\bigcap_{k\in I^c}\{X_k<0\}\Big).$$

Let $I = \{k_1, \ldots, k_m\}$ be a fixed subset of indices with nonempty $I^c = \{r_1, \ldots, r_{n-m}\}$. Set $l := n - m$, $F_{\boldsymbol{X}_r}(\boldsymbol{x}_r) := F_{X_{r_1},\ldots,X_{r_l}}(x_{r_1}, \ldots, x_{r_l})$ and write

$$\begin{aligned}
& p_I \\
= \;& \mathrm{P}\Big(\sum_{k\in I} w_k X_k > x, \sum_{k\in I} w_k X_k + \sum_{r\in I^c} w_r X_r \le x, X_k \ge 0, k \in I; X_r < 0, r \in I^c \Big) \\
\le \;& \mathrm{P}\Big(\sum_{k\in I} w_k X_k > x, \sum_{k\in I} w_k X_k + \sum_{r\in I^c} w_r X_r \le x, X_r < 0, r \in I^c \Big) \\
= \;& \mathrm{P}\Big(\sum_{k\in I} w_k X_k > x, X_r < 0, r \in I^c \Big) \\
& - \mathrm{P}\Big(\sum_{k\in I} w_k X_k + \sum_{r\in I^c} w_r X_r > x, X_r < 0, r \in I^c \Big) \\
\le \;& \int_{(-\infty,0)} \!\!\cdots\! \int \mathrm{P}\Big(\sum_{k\in I} w_k X_k > x \Big| X_r = y_r, r \in I^c \Big) \mathrm{d}F_{\boldsymbol{X}_r}(\boldsymbol{y}_r) \\
& - \int_{(-\infty,0)} \!\!\cdots\! \int \mathrm{P}\Big(\sum_{k\in I} w_k X_k > x - b \sum_{r\in I^c} y_r \Big| X_r = y_r, r \in I^c \Big) \mathrm{d}F_{\boldsymbol{X}_r}(\boldsymbol{y}_r) \\
\le \;& C \Bigg(\int_{(-\infty,0)} \!\!\cdots\! \int \pi'_I(x, y_r, r \in I^c) \mathrm{d}F_{X_{r_1},\ldots,X_{r_l}}(y_{r_1}, \ldots, y_{r_l}) \\
& - \int_{(-\infty,0)} \!\!\cdots\! \int \pi''_I(x, y_r, r \in I^c) \mathrm{d}F_{X_{r_1},\ldots,X_{r_l}}(y_{r_1}, \ldots, y_{r_l}) \Bigg) \\
=: \;& C p'_I,
\end{aligned}$$

where

$$\begin{aligned}
\pi'_I(x, y_r, r \in I^c) \;&:=\; \frac{\mathrm{P}\Big(\sum_{k\in I} w_k X_k > x \Big| X_r = y_r, r \in I^c \Big)}{h^{(w)}_{I,I^c}(y_{r_1}, \ldots, y_{r_l})}, \\
\pi''_I(x, y_r, r \in I^c) \;&:=\; \frac{\mathrm{P}\Big(\sum_{k\in I} w_k X_k > x - b \sum_{r\in I^c} y_r \Big| X_r = y_r, r \in I^c \Big)}{h^{(w)}_{I,I^c}(y_{r_1}, \ldots, y_{r_l})}
\end{aligned}$$

and where we have used that, by Assumption C,

$$\sup_{w_k\in[a,b],k\in I} \; \sup_{(y_{r_1},\ldots,y_{r_l})\in\mathbb{R}^l} h^{(w)}_{I,I^c}(y_{r_1}, \ldots, y_{r_l}) \;\le\; C < \infty.$$

According to the Fatou lemma, Assumption C and Lemma 4.3.1,

$$\begin{aligned}
&\limsup_{x\to\infty} \sup_{w_k\in[a,b],k\in I} \frac{p'_I}{\mathrm{P}\big(\sum_{k\in I} w_k X_k > x\big)} \\
\le & \int\limits_{(-\infty,0)} \cdots \int\limits_{(-\infty,0)} \limsup_{x\to\infty} \sup_{w_k\in[a,b],k\in I} \frac{\pi'_I(x, y_r, r\in I^c)}{\mathrm{P}(\sum_{k\in I} w_k X_k > x)} \mathrm{d}F_{X_{r_1},\dots,X_{r_l}}(y_{r_1},\dots,y_{r_l}) \\
- & \int\limits_{(-\infty,0)} \cdots \int\limits_{(-\infty,0)} \liminf_{x\to\infty} \inf_{w_k\in[a,b],k\in I} \frac{\pi''_I(x, y_r, r\in I^c)}{\mathrm{P}(\sum_{k\in I} w_k X_k > x)} \mathrm{d}F_{X_{r_1},\dots,X_{r_l}}(y_{r_1},\dots,y_{r_l}) \\
= & \quad 0.
\end{aligned}$$

Since $p_I \le C p'_I$, for each subset I in (4.3.14) we obtain that

$$\limsup_{x\to\infty} \sup_{\overline{w}_n\in[a,b]^n} \frac{p_I}{\mathrm{P}\big(\sum_{k\in I} w_k X_k > x\big)} = 0.$$

This, together with (4.3.14), imply

$$\begin{aligned}
\liminf_{x\to\infty} \inf_{\overline{w}_n\in[a,b]^n} \frac{\mathrm{P}(S_n^w > x)}{\mathrm{P}(S_n^{w+} > x)} &\ge 1 - \sum_I \limsup_{x\to\infty} \sup_{\overline{w}_n\in[a,b]^n} \frac{p_I}{\mathrm{P}(S_n^{w+} > x)} \\
&= 1 - \sum_I \limsup_{x\to\infty} \sup_{\overline{w}_n\in[a,b]^n} \frac{p_I}{\mathrm{P}\big(\sum_{k\in I} w_k X_k > x\big)} \\
&= 1.
\end{aligned}$$

Thus, relation (4.3.13) holds and lemma is proved. □

Proof of Theorem 4.2.2. Similarly, as in the case of Theorem 4.2.1, the proof follows immediately from Lemma 4.3.2:

$$\begin{aligned}
\mathrm{P}(S_n^{\Theta+} > x) &= \int\limits_{[a,b]^n} \cdots \int \mathrm{P}(S_n^{w+} > x)\mathrm{P}(\Theta_1 \in \mathrm{d}w_1,\dots,\Theta_n \in \mathrm{d}w_n) \\
&\sim \int\limits_{[a,b]^n} \cdots \int \mathrm{P}(S_n^{w} > x)\mathrm{P}(\Theta_1 \in \mathrm{d}w_1,\dots,\Theta_n \in \mathrm{d}w_n) \\
&= \mathrm{P}(S_n^{\Theta} > x).
\end{aligned}$$

□

4.4 The case of copula-based dependence

In this section we demonstrate how the functions g_k, $h_k^{(w)}$ and $h_{I,J}^{(w)}$, appearing in Assumptions A, B and C, can be found when the dependence structure among $X_1,\dots,X_n$ is generated by n-dimensional absolutely continuous copula $C(u_1,\dots,u_n)$.

4.4.1 General copula dependence

Assume that the distribution of vector $(X_1, \dots, X_n)$ is given by

$$\mathrm{P}(X_1 \le x_1, \dots, X_n \le x_n) = C(F_1(x_1), \dots, F_n(x_n)), \quad (x_1, \dots, x_n) \in [-\infty, \infty]^n, \tag{4.4.1}$$

where $C(u_1, \dots, u_n)$ is some absolutely continuous copula function with corresponding positive copula density $c(u_1, \dots, u_n)$. Assume that $F_1, \dots, F_n$ are absolutely continuous with corresponding positive densities $f_1, \dots, f_n$.

Consider first the case of assumptions A and B.

Let $C_k(u_1, \dots, u_k) := C(u_1, \dots, u_k, 1, \dots, 1)$, where $k = 2, \dots, n$, be the k-dimensional marginal copulas. Also write $C_1(u_1) = u_1$. Let the corresponding copula densities be $c_k(u_1, \dots, u_k)$, where $k = 1, \dots, n$. Denote $\widetilde{C}_k(u_1, \dots, u_k) := C_{k-1}(u_1, \dots, u_{k-1}) - C_k(u_1, \dots, u_k)$ and let

$$\widetilde{c}_k(u_1, \dots, u_k) \;\; := \;\; \frac{\partial^{k-1} \widetilde{C}_k(u_1, \dots, u_k)}{\partial u_1 \dots \partial u_{k-1}}. \tag{4.4.2}$$

Further, we introduce the following assumption: for any $k = 2, \dots, n$, there exists positive limit

$$\bar{c}_k(u_1, \dots, u_{k-1}, 1-) \;\; := \;\; \lim_{u \searrow 0} \frac{\widetilde{c}_k(u_1, \dots, u_{k-1}, 1-u)}{u} \tag{4.4.3}$$

uniformly for $(u_1, \dots, u_{k-1}) \in [0,1]^{k-1}$.

Denote $X_1^*, \dots, X_n^*$ the corresponding independent copies of r.v.s $X_1, \dots, X_n$ and set $S_k^{w*} := w_1 X_1^* + \dots + w_k X_k^*$, $k = 1, \dots, n$.

Proposition 4.4.1. *Assume that the distribution of random vector* $(X_1, \dots, X_n)$ *is given by* (4.4.1) *with some absolutely continuous copula* $C(u_1, \dots, u_n)$ *and absolutely continuous marginal distributions* $F_1, \dots, F_n$. *Then Assumption A is equivalent to* (4.4.3) *and in this case functions* g_k, $k = 2, \dots, n$ *are given by*

$$g_k(y_1, \dots, y_{k-1}) \;\; = \;\; \frac{\bar{c}_k(F_1(y_1), \dots, F_{k-1}(y_{k-1}), 1-)}{c_{k-1}(F_1(y_1), \dots, F_{k-1}(y_{k-1}))}. \tag{4.4.4}$$

Furthermore, Assumption B is equivalent to the existence of positive limits

$$h_k^{(w)}(y) \;\; := \;\; \lim_{x \to \infty} \frac{\mathrm{E} c_k(F_1(X_1^*), \dots, F_{k-1}(X_{k-1}^*), F_k(y)) \mathbb{1}_{\{S_{k-1}^{w*} > x\}}}{\mathrm{E} c_{k-1}(F_1(X_1^*), \dots, F_{k-1}(X_{k-1}^*)) \mathbb{1}_{\{S_{k-1}^{w*} > x\}}}, \tag{4.4.5}$$

uniformly for $\overline{w}_{k-1} \in [a,b]^{k-1}$, $y \in \mathbb{R}$ *and* $k = 2, \dots, n$.

Proof. Denote the k-dimensional density function of vector $(X_1, \dots, X_k)$ by $f_{X_1, \dots, X_k}(y_1, \dots, y_k)$. Clearly,

$$f_{X_1, \dots, X_k}(y_1, \dots, y_k) \;\; = \;\; c_k(F_1(y_1), \dots, F_k(y_k)) f_1(y_1) \cdots f_k(y_k), \tag{4.4.6}$$

which is positive for all $k = 1, \dots, n$ by the positivity of copula density c and marginal densities $f_1, \dots, f_n$. Hence,

$$
\begin{aligned}
&\mathrm{P}(X_k > x | X_1 = y_1, \dots, X_{k-1} = y_{k-1}) \\
&= \frac{\partial^{k-1} \mathrm{P}(X_k > x, X_1 \le y_1, \dots, X_{k-1} \le y_{k-1})}{\partial y_1 \dots \partial y_{k-1}} \frac{1}{f_{X_1,\dots,X_{k-1}}(y_1, \dots, y_{k-1})} \\
&= \frac{\widetilde{c}_k(F_1(y_1), \dots, F_{k-1}(y_{k-1}), F_k(x))}{c_{k-1}(F_1(y_1), \dots, F_{k-1}(y_{k-1}))}, \qquad (4.4.7)
\end{aligned}
$$

which follows from (4.4.6) and equality

$$
\begin{aligned}
&\frac{\partial^{k-1} \mathrm{P}(X_k > x, X_1 \le y_1, \dots, X_{k-1} \le y_{k-1})}{\partial y_1 \dots \partial y_{k-1}} \\
&= \widetilde{c}_k(F_1(y_1), \dots, F_{k-1}(y_{k-1}), F_k(x)) f_1(y_1) \dots f_{k-1}(y_{k-1}).
\end{aligned}
$$

The last equality holds by (4.4.2).

By (4.4.7), Assumption A is equivalent to

$$
\begin{aligned}
&\lim_{x \to \infty} \frac{\widetilde{c}_k(F_1(y_1), \dots, F_{k-1}(y_{k-1}), F_k(x))}{\overline{F_k}(x)} \\
&= g_k(y_1, \dots, y_{k-1}) c_{k-1}(F_1(y_1), \dots, F_{k-1}(y_{k-1}))
\end{aligned}
$$

for some positive functions g_k, uniformly for $(y_1, \dots, y_{k-1}) \in \mathbb{R}^{k-1}$, $k = 2, \dots, n$. Clearly, the last relation is equivalent to (4.4.3), and (4.4.4) holds.

Lets deal with Assumption B. Since $F_k(x)$ is absolutely continuous, we have

$$
\mathrm{P}\big(S_{k-1}^w > x | X_k = y\big) = \frac{\partial \mathrm{P}(S_{k-1}^w > x, X_k \le y)}{\partial y} \frac{1}{f_k(y)}. \qquad (4.4.8)
$$

It is easy to see that

$$
\begin{aligned}
&\frac{\partial \mathrm{P}(S_{k-1}^w > x, X_k \le y)}{\partial y} \\
&= f_k(y) \int_{\mathcal{R}} c_k(F_1(u_1), \dots, F_{k-1}(u_{k-1}), F_k(y)) f_1(u_1) \dots f_{k-1}(u_{k-1}) \mathrm{d}u_1 \dots \mathrm{d}u_{k-1} \\
&= f_k(y) \mathrm{E} c_k(F_1(X_1^*), \dots, F_{k-1}(X_{k-1}^*), F_k(y)) \mathbb{1}_{\{w_1 X_1^* + \dots + w_{k-1} X_{k-1}^* > x\}},
\end{aligned}
$$

where $\mathcal{R} := \{(u_1, \dots, u_{k-1}) : \sum_{i=1}^{k-1} w_i u_i > x\}$. Hence, by (4.4.8) and equality $\mathrm{P}(S_{k-1}^w > x) = \mathrm{E} c_{k-1}(F_1(X_1^*), \dots, F_{k-1}(X_{k-1}^*)) \mathbb{1}_{\{S_{k-1}^{w*} > x\}}$, we obtain

$$
\begin{aligned}
&\mathrm{P}\big(S_{k-1}^w > x | X_k = y\big) \\
&= \frac{\mathrm{E} c_k(F_1(X_1^*), \dots, F_{k-1}(X_{k-1}^*), F_k(y)) \mathbb{1}_{\{S_{k-1}^{w*} > x\}}}{\mathrm{E} c_{k-1}(F_1(X_1^*), \dots, F_{k-1}(X_{k-1}^*)) \mathbb{1}_{\{S_{k-1}^{w*} > x\}}} \mathrm{P}\big(S_{k-1}^w > x\big).
\end{aligned}
$$

This implies the second statement of proposition. □

Next we formulate the similar result in the case of Assumption C. For any (not necessarily nonempty) subsets $I = \{k_1, \dots, k_m\}$, $J = \{r_1, \dots, r_p\} \subset \{1, \dots, n\} \backslash I$ denote by $c_{I,J}(u_k, k \in I, u_r, r \in J)$ the copula density corresponding to random vector $(X_{k_1}, \dots, X_{k_m}, X_{r_1}, \dots, X_{r_p})$, i.e.

$$\begin{aligned} & f_{X_{k_1},\dots,X_{k_m},X_{r_1},\dots,X_{r_p}}(y_{k_1}, \dots, y_{k_m}, y_{r_1}, \dots, y_{r_p}) \\ &= c_{I,J}\big(F_k(y_k), k \in I, F_r(y_r), r \in J\big) \prod_{k \in I} f_k(y_k) \prod_{r \in J} f_r(y_r), \end{aligned}$$

and let $c_I(u_{k_1}, \dots, u_{k_m}) := c_{I,\varnothing}(u_{k_1}, \dots, u_{k_m})$, $c_J(u_{r_1}, \dots, u_{r_p}) := c_{\varnothing,J}(u_{r_1}, \dots, u_{r_p})$.

Proposition 4.4.2. *Assume that the distribution of random vector $(X_1, \dots, X_n)$ is given by (4.4.1) with some absolutely continuous copula $C(u_1, \dots, u_n)$ and absolutely continuous marginal distributions $F_1, \dots, F_n$. Then Assumption C is equivalent to the existence of positive, uniformly bounded limits*

$$\begin{aligned} & h_{I,J}^{(w)}(y_{r_1}, \dots, y_{r_p}) \\ &:= \frac{1}{c_J(F_r(y_r), r \in J)} \lim_{x \to \infty} \frac{\mathrm{E} c_{I,J}(F_k(X_k^*), k \in I, F_r(X_r^*), r \in J) \mathbb{1}_{\{\sum_{k \in I} w_k X_k^* > x\}}}{\mathrm{E} c_I(F_k(X_k^*), k \in I) \mathbb{1}_{\{\sum_{k \in I} w_k X_k^* > x\}}} \end{aligned}$$

uniformly for $w_k \in [a, b]$, $k \in I$, $y_r \in \mathbb{R}$, $r \in J$ and all nonempty sets of indices $I = \{k_1, \dots, k_m\} \subset \{1, 2, \dots, n\}$ and $J = \{r_1, \dots, r_p\} \subset \{1, 2, \dots, n\} \backslash I$.

Proof. The proof is similar to that of Proposition 4.4.1. We have

$$\begin{aligned} & \mathrm{P}\Big(\sum_{k \in I} w_k X_k > x \big| X_r = y_r, r \in J\Big) \\ &= \frac{\partial^p \mathrm{P}(\sum_{k \in I} w_k X_k > x, X_r \le y_r, r \in J)}{\partial y_{r_1} \dots \partial y_{r_p}} \frac{1}{f_{X_{r_1},\dots,X_{r_p}}(y_{r_1}, \dots, y_{r_p})}, \end{aligned}$$

where

$$\begin{aligned} & \frac{\partial^p \mathrm{P}(\sum_{k \in I} w_k X_k > x, X_r \le y_r, r \in J)}{\partial y_{r_1} \dots \partial y_{r_p}} \\ &= \prod_{r \in J} f_r(y_r) \int\limits_{\sum_{k \in I} w_k u_k > x} c_{I,J}(F_{k_1}(u_{k_1}), \dots, F_{k_m}(u_{k_m}), F_{r_1}(y_{r_1}), \dots, F_{r_p}(y_{r_p})) \\ &\qquad \times \prod_{k \in I} f_k(u_k) \mathrm{d}u_{k_1} \dots \mathrm{d}u_{k_m} \end{aligned}$$

and $f_{X_{r_1},\dots,X_{r_p}}(y_{r_1}, \dots, y_{r_p}) = c_J(F_{r_1}(y_{r_1}), \dots, F_{r_p}(y_{r_p})) \prod_{r \in J} f_r(y_r)$. Now the proof follows observing that

$$\mathrm{P}\Big(\sum_{k \in I} w_k X_k > x) = \mathrm{E} c_I(F_{k_1}(X_{k_1}^*), \dots, F_{k_m}(X_{k_m}^*)) \mathbb{1}_{\{\sum_{k \in I} w_k X_k^* > x\}}.$$

□

4.4.2 The case of FGM copula

In this subsection, we assume that $C(u_1,\ldots,u_n) = C^{\mathrm{GFGM}}(u_1,\ldots,u_n)$, given by (2.3.3) with $m = 1$. In this case,

$$C_k(u_1,\ldots,u_k) \;=\; \prod_{l=1}^{k} u_l \left(1 + \sum_{1\le i<j\le k} \theta_{ij}(1-u_i)(1-u_j)\right), \qquad (4.4.9)$$

and the corresponding copula densities are given by

$$c_k(u_1,\ldots,u_k) \;=\; 1 + \sum_{1\le i<j\le k} \theta_{ij}(1-2u_i)(1-2u_j), \qquad (4.4.10)$$

$k = 2,\ldots,n$. Everywhere below we assume that the parameters θ_{ij} are such that $c_n(u_1,\ldots,u_n) > 0$ for all $(u_1,\ldots,u_n) \in [0,1]^n$. Obviously, this implies that $c_k(u_1,\ldots,u_k) > 0$ for all $(u_1,\ldots,u_k) \in [0,1]^k$, $k = 2,\ldots,n$.

Next, we make the following assumption:

Assumption D. For each $k = 1,\ldots,n-1$ there exists limit

$$\lim_{x\to\infty} \frac{\overline{F}_k(x/w_k)}{\overline{F}_1(x/w_1)+\cdots+\overline{F}_{n-1}(x/w_{n-1})} \;=:\; a_k^{(w)} \in (0,1]$$

uniformly for $\overline{w}_{n-1} \in [a,b]^{n-1}$.

To illustrate Assumption D, suppose that $F_1,\ldots,F_n$ are such that $\overline{F_i}(x) \sim c_i\overline{G}(x)$ with some positive constants c_i, $i = 1,\ldots,n$, and a d. f. $G(x)$ with $\overline{G}(x) > 0$ for all x. Then Assumption D is satisfied if, e.g., $G(x)$ is some regularly varying function, i.e. $\overline{G}(x) = L(x)x^{-\alpha}$, $x > 0$, $\alpha \ge 0$ ($L(x)$ is a slowly varying function). In this case,

$$a_k^{(w)} \;=\; \frac{c_k}{c_1(w_1/w_k)^\alpha + \cdots + c_{n-1}(w_{n-1}/w_k)^\alpha}.$$

On the other hand, if $a = b$ and $G(x)$ is any d. f. with $\overline{G}(x) > 0$ for all x, then

$$a_k^{(w)} \;=\; \frac{c_k}{c_1 + \cdots + c_{n-1}}.$$

Next we derive the expressions for functions g_k and $h_k^{(w)}$, omitting the case of function $h_{I,J}^{(w)}$, for which the corresponding expression is complicated and does not carry much interest.

For a distribution F, denote $\widetilde{F} := 1 - 2F = 2\overline{F} - 1$.

Proposition 4.4.3. *Assume that $n \geq 2$ and $X_1, \ldots, X_n$ are real-valued r.v.s whose distribution is generated by FGM copula in (4.4.9), marginal distributions $F_1, \ldots, F_n$ are absolutely continuous and $F_i \in \mathscr{L} \cap \mathscr{D}$, $i = 1, \ldots, n$. Then*

$$g_k(y_1, \ldots, y_{k-1}) \;=\; 1 - \frac{\sum_{1 \leq l \leq k-1} \theta_{lk} \widetilde{F}_l(y_l)}{c_{k-1}(F_1(y_1), \ldots, F_{k-1}(y_{k-1}))}, \quad k = 2, \ldots, n.$$

If $n \geq 3$ and Assumption D holds, then

$$h_k^{(w)}(y) \;=\; 1 - \widetilde{F}_k(y) \sum_{1 \leq l \leq k-1} \theta_{lk} a_{l,k-1}^{(w)}, \quad k = 3, \ldots, n,$$

where $a_{l,k-1}^{(w)} := a_l^{(w)} / (a_1^{(w)} + \cdots + a_{k-1}^{(w)})$.

Proof. We apply Proposition 4.4.1. Obviously,

$$\begin{aligned} &\widetilde{C}_k(u_1, \ldots, u_k) \\ &\quad = (1 - u_k) C_{k-1}(u_1, \ldots, u_{k-1}) - u_1 \cdots u_k (1 - u_k) \sum_{1 \leq l \leq k-1} \theta_{lk} (1 - u_l), \end{aligned}$$

implying that $\tilde{c}_k(u_1, \ldots, u_k)$ in (4.4.2) is

$$\tilde{c}_k(u_1, \ldots, u_k) \;=\; (1 - u_k) c_{k-1}(u_1, \ldots, u_{k-1}) - u_k (1 - u_k) \sum_{1 \leq l \leq k-1} \theta_{lk} (1 - 2u_l).$$

Hence, condition (4.4.3) is satisfied (uniformly in $(u_1, \ldots, u_{k-1}) \in [0,1]^{k-1}$) and

$$\begin{aligned} \bar{c}_k(u_1, \ldots, u_{k-1}, 1-) &= \lim_{u \searrow 0} \left(c_{k-1}(u_1, \ldots, u_{k-1}) - (1 - u) \sum_{1 \leq l \leq k-1} \theta_{lk} (1 - 2u_l) \right) \\ &= c_{k-1}(u_1, \ldots, u_{k-1}) - \sum_{1 \leq l \leq k-1} \theta_{lk} (1 - 2u_l). \end{aligned}$$

Hence, by (4.4.4),

$$g_k(y_1, \ldots, y_{k-1}) \;=\; 1 - \frac{\sum_{1 \leq l \leq k-1} \theta_{lk} (1 - 2F_l(y_l))}{c_{k-1}(F_1(y_1), \ldots, F_{k-1}(y_{k-1}))}.$$

Consider now function $h_k^{(w)}(y)$. For $k = 2, \ldots, n$ we have

$$h_k^{(w)}(y) \;=\; \lim_{x \to \infty} \frac{\varphi_k^{(w)}(x, y)}{\varphi_{k-1}^{(w)}(x)},$$

where, by (4.4.5) and (4.4.10),

$$\begin{aligned}
\varphi_k^{(w)}(x,y) &:= \mathrm{E}c_k(F_1(X_1^*),\dots,F_{k-1}(X_{k-1}^*),F_k(y))\mathbb{I}_{\{S_{k-1}^{w*}>x\}} \\
&= \mathrm{P}(S_{k-1}^{w*}>x)+\sum_{1\le l<m\le k-1}\theta_{lm}\mathrm{E}\widetilde{F}_l(X_l^*)\widetilde{F}_m(X_m^*)\mathbb{I}_{\{S_{k-1}^{w*}>x\}} \\
&\qquad + \widetilde{F}_k(y)\sum_{1\le l\le k-1}\theta_{lk}\mathrm{E}\widetilde{F}_l(X_l^*)\mathbb{I}_{\{S_{k-1}^{w*}>x\}}, \\
\varphi_{k-1}^{(w)}(x) &:= \mathrm{E}c_{k-1}(F_1(X_1^*),\dots,F_{k-1}(X_{k-1}^*))\mathbb{I}_{\{S_{k-1}^{w*}>x\}} \\
&= \mathrm{P}(S_{k-1}^{w*}>x)+\sum_{1\le l<m\le k-1}\theta_{lm}\mathrm{E}\widetilde{F}_l(X_l^*)\widetilde{F}_m(X_m^*)\mathbb{I}_{\{S_{k-1}^{w*}>x\}}.
\end{aligned}$$

Rewrite now

$$\frac{\varphi_k^{(w)}(x,y)}{\varphi_{k-1}^{(w)}(x)} = 1+\widetilde{F}_k(y)b_k^{(w)}(x),$$

where

$$b_k^{(w)}(x) := \frac{\sum\limits_{1\le l\le k-1}\theta_{lk}\mathrm{E}\widetilde{F}_l(X_l^*)\mathbb{I}_{\{S_{k-1}^{w*}>x\}}}{\mathrm{P}(S_{k-1}^{w*}>x)+\sum\limits_{1\le l<m\le k-1}\theta_{lm}\mathrm{E}\widetilde{F}_l(X_l^*)\widetilde{F}_m(X_m^*)\mathbb{I}_{\{S_{k-1}^{w*}>x\}}}.$$

It remains to prove that, uniformly in $\overline{w}_{k-1}\in[a,b]^{k-1}$,

$$b_k^{(w)}(x) \to -\sum_{1\le l\le k-1}\theta_{lk}a_{l,k-1}^{(w)} =: b_k^{(w)}, \quad k=3,\dots,n. \tag{4.4.11}$$

Rewrite

$$\begin{aligned}
&b_k^{(w)}(x) \\
&= \frac{2\sum\limits_{1\le l\le k-1}\theta_{lk}\mathrm{E}\overline{F_l}(X_l^*)\mathbb{I}_{\{S_{k-1}^{w*}>x\}}-\mathrm{P}(S_{k-1}^{w*}>x)\sum\limits_{1\le l\le k-1}\theta_{lk}}{2\sum\limits_{1\le l<m\le k-1}\theta_{lm}\mathrm{E}Y_{lm}^*\mathbb{I}_{\{S_{k-1}^{w*}>x\}}+\mathrm{P}(S_{k-1}^{w*}>x)+\mathrm{P}(S_{k-1}^{w*}>x)\sum\limits_{1\le l<m\le k-1}\theta_{lm}} \\
&= \frac{2\sum_{1\le l\le k-1}\theta_{lk}\frac{\mathrm{E}\overline{F_l}(X_l^*)\mathbb{I}_{\{S_{k-1}^{w*}>x\}}}{\mathrm{P}(S_{k-1}^{w*}>x)}-\sum_{1\le l\le k-1}\theta_{lk}}{2\sum_{1\le l<m\le k-1}\theta_{lm}\frac{\mathrm{E}Y_{lm}^*\mathbb{I}_{\{S_{k-1}^{w*}>x\}}}{\mathrm{P}(S_{k-1}^{w*}>x)}+1+\sum_{1\le l<m\le k-1}\theta_{lm}},
\end{aligned}$$

where $Y_{lm}^* := 2\overline{F_l}(X_l^*)\overline{F_m}(X_m^*)-\overline{F_l}(X_l^*)-\overline{F_m}(X_m^*)$.

The desired convergence (4.4.11) will follow if we show that

$$\mathrm{E}\overline{F_l}(X_l^*)\mathbb{I}_{\{S_{k-1}^{w*}>x\}} \sim \frac{1}{2}(1-a_{l,k-1}^{(w)})\mathrm{P}(S_{k-1}^{w*}>x), \quad l=1,\dots,k-1, \tag{4.4.12}$$

$$\mathrm{E}Y_{lm}^*\mathbb{I}_{\{S_{k-1}^{w*}>x\}} \sim -\frac{1}{2}\mathrm{P}(S_{k-1}^{w*}>x), \quad 1\le l<m\le k-1, \tag{4.4.13}$$

uniformly in $\overline{w}_{k-1} \in [a,b]^{k-1}$.

To show (4.4.12), take $Y_i = X_i^*$, $a_i(x) \equiv \overline{F_i}(x)$ in Corollary 4.5.1 below and note that condition (4.5.16) is satisfied:

$$\mathrm{E}\overline{F_i}(X_i^*)1\!\!\mathrm{I}_{\{X_i^*>x\}} \;=\; \overline{F_j}(x)\int_x^\infty \frac{\overline{F_i}(y)}{\overline{F_j}(x)}\,\mathrm{d}F_i(y) \;=\; o(\overline{F_j}(x)),\; j\neq i,$$

because, by Assumption D, $\overline{F_i}(x) \sim c_{ij}\overline{F_j}(x)$ with some positive constant c_{ij}. Combining Corollary 4.5.1, Proposition 4.5.1 (i) and using that $\mathrm{E}\overline{F_l}(X_l^*) = 1/2$ for all $l = 1,\dots,n$ (since distribution F_l has positive density), we get

$$\begin{aligned}\lim_{x\to\infty}\frac{\mathrm{E}\overline{F_l}(X_l^*)1\!\!\mathrm{I}_{\{S_{k-1}^{w*}>x\}}}{\mathrm{P}(S_{k-1}^{w*}>x)} &= \mathrm{E}\overline{F_l}(X_l^*)\lim_{x\to\infty}\frac{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)-\overline{F}_l(x/w_l)}{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)}\\ &= \frac{1}{2}(1-a_{l,k-1}^{(w)}),\quad l=1,\dots,k-1,\end{aligned}$$

uniformly in $\overline{w}_{k-1} \in [a,b]^{k-1}$ (note that $0 < a_{l,k-1} < 1$ because $\sum_{l=1}^{k-1} a_{l,k-1}^{(w)} = 1$ and $a_{l,k-1}^{(w)} > 0$, $k \geq 3$). Thus, we get (4.4.12).

The proof of relation (4.4.13) is similar. If $k > 3$, then, by Corollary 4.5.1,

$$\begin{aligned}&\lim_{x\to\infty}\frac{\mathrm{E}Y_{lm}^*1\!\!\mathrm{I}_{\{S_{k-1}^{w*}>x\}}}{\mathrm{P}(S_{k-1}^{w*}>x)}\\ &= \lim_{x\to\infty}\frac{\mathrm{E}(2\overline{F_l}(X_l^*)\overline{F_m}(X_m^*)-\overline{F_l}(X_l^*)-\overline{F_m}(X_m^*))1\!\!\mathrm{I}_{\{S_{k-1}^{w*}>x\}}}{\mathrm{P}(S_{k-1}^{w*}>x)}\\ &= 2\mathrm{E}\overline{F_l}(X_l^*)\mathrm{E}\overline{F_m}(X_m^*)\lim_{x\to\infty}\frac{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)-\overline{F_l}(x/w_l)-\overline{F_m}(x/w_m)}{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)}\\ &\quad-\mathrm{E}\overline{F_l}(X_l^*)\lim_{x\to\infty}\frac{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)-\overline{F_l}(x/w_l)}{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)}\\ &\quad-\mathrm{E}\overline{F_m}(X_m^*)\lim_{x\to\infty}\frac{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)-\overline{F_m}(x/w_m)}{\sum_{i=1}^{k-1}\overline{F}_i(x/w_i)} \;=\; -\frac{1}{2}\end{aligned}$$

uniformly in $\overline{w}_{k-1} \in [a,b]^{k-1}$. The case $k = 3$ in (4.4.13) easily follows from arguments above and (4.5.18). The proof is complete. □

Consider now the tail asymptotics of the sum $S_n^\Theta = \Theta_1 X_1 + \cdots + \Theta_n X_n$ in the case when the distribution of vector $(X_1,\dots,X_n)$ is given by (4.4.9). The next proposition shows that in this case the probabilities $\mathrm{P}(S_n^\Theta > x)$ and $\mathrm{P}(S_n^{\Theta+} > x)$ asymptotically are the same and are both asymptotically equivalent to $\mathrm{P}(\Theta_1 X_1 > x) + \cdots + \mathrm{P}(\Theta_n X_n > x)$. This result follows from Theorem 1 in [66] proved in the case pSQAI structure (Definition 2.2.6). It easy to see that the FGM distribution given by (4.4.9) satisfies (2.2.6) (see, e.g., [31]).

Proposition 4.4.4. *Suppose that $n \geq 2$ and $X_1, \ldots, X_n$ are real-valued r.v.s with corresponding distributions $F_1, \ldots, F_n$, such that $F_i \in \mathscr{L} \cap \mathscr{D}$, $i = 1, \ldots, n$. Let the distribution of vector $(X_1, \ldots, X_n)$ is generated by the FGM copula in* (4.4.9). *If* $\mathrm{P}(0 < \Theta_k \leq b) = 1$, $k = 1, \ldots, n$, *then*

$$\begin{aligned} \mathrm{P}(S_n^{\Theta} > x) \;&\sim\; \mathrm{P}(S_n^{\Theta +} > x) \;\sim\; \mathrm{P}(S_{(n)}^{\Theta} > x) \\ &\sim\; \mathrm{P}\Big(\max_{k=1,\ldots,n} \Theta_k X_k > x\Big) \;\sim\; \sum_{k=1}^{n} \mathrm{P}(\Theta_k X_k > x). \quad (4.4.14) \end{aligned}$$

Remark 4.4.1. The proof of relations in (4.4.14) is based essentially on two facts: first, the fact that the distribution of the product ΘX, where Θ and X are independent r.v.s with $0 < \Theta \leq b$ a.s. and $F_X \in \mathscr{L} \cap \mathscr{D}$, is again in $\mathscr{L} \cap \mathscr{D}$ (see Lemmas 3.9 and 3.10 in [61]); second, the result as in (4.4.14) but with products $\Theta_k X_k$ replaced by the (dependent) r.v.s Y_k, such that $F_{Y_k} \in \mathscr{L} \cap \mathscr{D}$, $k = 1, \ldots, n$. Alternatively, the relation in (4.4.14) can be derived replacing the Θ_k's by w_k's and then proving the corresponding relations *uniformly* with respect to $\overline{w}_n = (w_1, \ldots, w_n)$. For instance, using Proposition 4.5.1 (ii) and representation

$$\mathrm{P}(S_n^{w} > x) \;=\; \mathrm{P}(S_n^{w*} > x) + \sum_{1 \leq i < j \leq n} \Theta_{ij} \int\limits_{w_1 y_1 + \cdots + w_n y_n > x} \mathrm{d}H_{ij}(y_1, \ldots, y_n),$$

where $H_{ij}(y_1, \ldots, y_n) := F_1(y_1) \ldots F_n(y_n) \overline{F_i}(y_i) \overline{F_j}(y_j)$, or directly applying (4.5.1) below for the pSQAI r.v.s, we have that for the FGM copula case it holds

$$\mathrm{P}(S_n^{w} > x) \;\sim\; \mathrm{P}(S_n^{w*} > x) \;\sim\; \sum_{k=1}^{n} \overline{F}_k(x/w_k)$$

uniformly for $\overline{w}_n \in [a, b]^n$. Hence

$$\begin{aligned} &\mathrm{P}(S_n^{\Theta} > x) \\ &\sim \int \cdots \int\limits_{[a,b]^n} \big(\mathrm{P}(w_1 X_1 > x) + \cdots + \mathrm{P}(w_n X_n > x)\big) \mathrm{P}(\Theta_1 \in \mathrm{d}w_1, \ldots, \Theta_n \in \mathrm{d}w_n) \\ &= \mathrm{P}(\Theta_1 X_1 > x) + \cdots + \mathrm{P}(\Theta_n X_n > x). \end{aligned}$$

Obviously, the last approach leads to a weaker result as it requires the restriction $\Theta_k \in [a, b] \subset (0, b]$, $k = 1, \ldots, n$, unless the d. f. s $F_1, \ldots, F_n$ are in the class $\mathscr{C}$, see Proposition 4.5.1 (ii) below.

4.5 Auxiliary results

In this section we present some useful statements, which are used proving the corresponding results in the case of FGM copula.

Proposition 4.5.1. *Suppose that $Y_1, \dots, Y_n$ are real-valued independent r.v.s with corresponding distributions $F_{Y_1}, \dots, F_{Y_n}$.*
(i) If $F_{Y_i} \in \mathscr{L} \cap \mathscr{D}$, $i = 1, \dots, n$, then

$$\mathrm{P}(w_1 Y_1 + \cdots + w_n Y_n > x) \ \sim \ \sum_{i=1}^{n} \overline{F}_{Y_i}(x/w_i) \tag{4.5.1}$$

uniformly for $\overline{w}_n \in [a, b]^n$, where $0 < a \leq b < \infty$.
(ii) If $F_{Y_i} \in \mathscr{C}$, $i = 1, \dots, n$, then relation (4.5.1) holds uniformly for $\overline{w}_n \in (0, b]^n$.

Proof. (i) The proof of this fact follows from Theorem 2.1 in [41] (note that Li's result also holds for more general, pSQAI, dependence structure).

(ii) Denote $S^w_{Y,n} := w_1 Y_1 + \cdots + w_n Y_n$ and write for any $\delta \in (0, 1)$ and $x > 0$

$$\begin{aligned}
&\mathrm{P}(S^w_{Y,n} > x) \\
&\geq \sum_{i=1}^{n} \mathrm{P}(S^w_{Y,n} > x, w_i Y_i > x + \delta x) - \sum_{1 \leq i < j \leq n} \mathrm{P}(w_i Y_i > x + \delta x, w_j Y_j > x + \delta x) \\
&=: \ p^w_1(x) - p^w_2(x).
\end{aligned}$$

Obviously,

$$p^w_2(x) \ \leq \ \Big(\sum_{i=1}^{n} \overline{F}_{Y_i}(x/w_i) \Big)^2 \ = \ o\Big(\sum_{i=1}^{n} \overline{F}_{Y_i}(x/w_i) \Big) \tag{4.5.2}$$

uniformly in $\overline{w}_n \in (0, b]^n$. For $p^w_1(x)$ we have

$$\begin{aligned}
p^w_1(x) \ &\geq \ \sum_{i=1}^{n} \mathrm{P}(S^w_{Y,n} - w_i Y_i > -\delta x, w_i Y_i > x + \delta x) \\
&= \ \sum_{i=1}^{n} \mathrm{P}(w_i Y_i > x + \delta x) - \sum_{i=1}^{n} \mathrm{P}(S^w_{Y,n} - w_i Y_i \leq -\delta x, w_i Y_i > x + \delta x) \\
&=: \ p^w_{11}(x) - p^w_{12}(x).
\end{aligned}$$

Here,

$$\liminf_{x \to \infty} \inf_{\overline{w}_n \in (0,b]^n} \frac{p^w_{11}(x)}{\sum_{i=1}^{n} \overline{F}_{Y_i}(x/w_i)} \geq \liminf_{x \to \infty} \inf_{\overline{w}_n \in (0,b]^n} \min_{1 \leq i \leq n} \frac{\overline{F}_{Y_i}((1+\delta)x/w_i)}{\overline{F}_{Y_i}(x/w_i)}, \tag{4.5.3}$$

where, for any $i = 1, \ldots, n$,

$$\begin{aligned}
&\liminf_{x\to\infty} \inf_{w_i\in(0,b]} \frac{\overline{F}_{Y_i}((1+\delta)x/w_i)}{\overline{F}_{Y_i}(x/w_i)} \\
&\geq \lim_{x\to\infty} \inf_{z\geq x/b} \frac{\overline{F}_{Y_i}((1+\delta)z)}{\overline{F}_{Y_i}(z)} \\
&= \liminf_{x\to\infty} \frac{\overline{F}_{Y_i}((1+\delta)x)}{\overline{F}_{Y_i}(x)} \longrightarrow 1 \quad \text{if} \quad \delta \searrow 0
\end{aligned} \tag{4.5.4}$$

by the definition of class $\mathscr{C}$. We get from (4.5.3)–(4.5.4) that

$$\lim_{\delta\searrow 0} \liminf_{x\to\infty} \inf_{\overline{w}_n\in(0,b]^n} \frac{p_{11}^w(x)}{\sum_{i=1}^n \overline{F}_{Y_i}(x/w_i)} \geq 1. \tag{4.5.5}$$

For the term $p_{12}^w(x)$ we get

$$\begin{aligned}
p_{12}^w(x) &\leq \sum_{i=1}^n \mathrm{P}(S_{Y,n}^w - w_iY_i \leq -\delta x)\mathrm{P}(w_iY_i > x) \\
&\leq \mathrm{P}(b(Y_1^- + \cdots + Y_n^-) \leq -\delta x) \sum_{i=1}^n \overline{F}_{Y_i}(x/w_i) \\
&= o(1) \sum_{i=1}^n \overline{F}_{Y_i}(x/w_i)
\end{aligned} \tag{4.5.6}$$

uniformly in $\overline{w}_n \in (0,b]^n$. (4.5.2), (4.5.5) and (4.5.6) imply

$$\liminf_{x\to\infty} \inf_{\overline{w}_n\in(0,b]^n} \frac{\mathrm{P}(S_{Y,n}^w > x)}{\sum_{i=1}^n \overline{F}_{Y_i}(x/w_i)} \geq \liminf_{x\to\infty} \inf_{\overline{w}_n\in(0,b]^n} \frac{p_1^w(x)}{\sum_{i=1}^n \overline{F}_{Y_i}(x/w_i)} \geq 1.$$

In order to show the upper asymptotic bound in (4.5.1), write

$$\begin{aligned}
&\mathrm{P}(S_{Y,n}^w > x) \\
&= \mathrm{P}\Big(S_{Y,n}^w > x, \bigcup_{i<j}\{w_iY_i > \delta x/(n-1), w_jY_j > \delta x/(n-1)\}\Big) \\
&+ \mathrm{P}\Big(S_{Y,n}^w > x, \bigcap_{i<j}\big\{\{w_iY_i \leq \delta x/(n-1)\} \cup \{w_jY_j \leq \delta x/(n-1)\}\big\}\Big) \\
&\leq \sum_{i<j} \mathrm{P}(w_iY_i > \delta x/(n-1))\mathrm{P}(w_jY_j > \delta x/(n-1)) + \mathrm{P}\Big(\bigcup_{i=1}^n\{w_iY_i > (1-\delta)x\}\Big) \\
&\leq \Big(\sum_{i=1}^n \mathrm{P}(w_iY_i > \delta x/(n-1))\Big)^2 + \sum_{i=1}^n \mathrm{P}(w_iY_i > (1-\delta)x) =: r_1^w(x) + r_2^w(x),
\end{aligned} \tag{4.5.7}$$

where we have used that for any sets $A_1,\dots,A_n$ it holds $\bigcap_{1\le i<j\le n}\{A_i\cup A_j\}\subset\bigcup_{i=1}^n\bigcap_{j\ne i}A_j$. It is easy to see that $r_1^w(x)=o(1)\sum_{i=1}^n\overline{F}_{Y_i}(x/w_i)$ and, by the definition of class $\mathscr{C}$,

$$\lim_{\delta\searrow 0}\limsup_{x\to\infty}\sup_{\overline{w}_n\in(0,b]^n}\frac{r_2^w(x)}{\sum_{i=1}^n\overline{F}_{Y_i}(x/w_i)}\ \le\ 1.$$

This and (4.5.7) complete the proof of proposition. □

REMARK 4.5.1. Uniform asymptotic relation (4.5.1) was investigated earlier in a number of papers. Tang and Tsitsiashvili [62] obtained this relation for independent r.v.s with common subexponential d. f. and weights $\overline{w}_n\in[a,b]^n$, $0<a\le b<\infty$. Subexponential r.v.s (independent or dependent) were also investigated by Zhu and Gao [76], Wang [66]. Liu et al. [46] and Wang at al. [67] proved relation (4.5.1) for identically distributed r.v.s from class $\mathscr{L}\cap\mathscr{D}$ allowing some dependence among primary variables with weights $\overline{w}_n\in[a,b]^n$, $0<a\le b<\infty$. Li [41] showed that this uniform asymptotics holds for nonidentically distributed (with some dependence) r.v.s from the class $\mathscr{C}$ or $\mathscr{L}\cap\mathscr{D}$ and $\overline{w}_n\in[a,b]^n$, $0<a\le b<\infty$.

Proposition 4.5.2. *Suppose that $Y_1,Y_2,\dots$ are real-valued independent r.v.s with corresponding distributions $F_{Y_1},F_{Y_2},\dots$ and $a_i\colon(-\infty,\infty)\to[0,\infty)$, $i=1,2$, are measurable functions.*

(i) If $0<\mathrm{E}a_1(Y_1)<\infty$, $F_{Y_i}\in\mathscr{L}\cap\mathscr{D}$, $i=2,\dots,k$, where $k\ge 2$ is an arbitrary integer, and

$$\mathrm{E}a_1(Y_1)1\!\!1_{\{Y_1>x\}}\ =\ o(\overline{F_{Y_2}}(x)+\cdots+\overline{F_{Y_k}}(x)),\tag{4.5.8}$$

then, uniformly for $\overline{w}_k\in[a,b]^k$, $0<a\le b<\infty$, it holds

$$\begin{aligned}\mathrm{E}a_1(Y_1)1\!\!1_{\{w_1Y_1+\cdots+w_kY_k>x\}}\ &\sim\ \mathrm{E}a_1(Y_1)\mathrm{P}(w_2Y_2+\cdots+w_kY_k>x)\\ &\sim\ \mathrm{E}a_1(Y_1)\big(\overline{F_{Y_2}}(x/w_2)+\cdots+\overline{F_{Y_k}}(x/w_k)\big);\end{aligned}\tag{4.5.9}$$

(ii) if $0<\mathrm{E}a_i(Y_i)<\infty$, $F_{Y_i}\in\mathscr{D}$, $i=1,2$, and

$$\mathrm{E}a_i(Y_i)1\!\!1_{\{Y_i>x\}}\ =\ o(\overline{F_{Y_j}}(x)),\ i,j=1,2,\ i\ne j,\tag{4.5.10}$$

then

$$\mathrm{E}a_1(Y_1)a_2(Y_2)1\!\!1_{\{w_1Y_1+w_2Y_2>x\}}\ =\ o(\overline{F_{Y_1}}(x/w_1)+\overline{F_{Y_2}}(x/w_2))\tag{4.5.11}$$

uniformly for $\overline{w}_2\in(0,b]^2$.

(iii) if $0 < \mathrm{E}a_i(Y_i) < \infty$*,* $i = 1, 2$*,* $F_{Y_i} \in \mathscr{L} \cap \mathscr{D}$*,* $i = 3, \dots, k$*, where* $k \geq 3$ *is an arbitrary integer, and*

$$\mathrm{E}a_i(Y_i)1\!\!\mathrm{I}_{\{Y_i>x\}} \quad = \quad o(\overline{F_{Y_3}}(x) + \cdots + \overline{F_{Y_k}}(x)),\ i = 1, 2, \tag{4.5.12}$$

then, uniformly for $\overline{w}_k \in [a, b]^k$*,* $0 < a \leq b < \infty$*, it holds*

$$\mathrm{E}a_1(Y_1)a_2(Y_2)1\!\!\mathrm{I}_{\{w_1Y_1+\cdots+w_kY_k>x\}} \sim \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)(\overline{F_{Y_3}}(x/w_3)+\cdots+\overline{F_{Y_k}}(x/w_k)). \tag{4.5.13}$$

Proof. (i) By Corollary 4.3.1 we can choose some positive function $K_1(x)$, $K_1(x) \leq x$ such that $K_1(x) \nearrow \infty$ and

$$\mathrm{P}(w_2Y_2 + \cdots + w_kY_k > x \pm K_1(x)) \sim \mathrm{P}(w_2Y_2 + \cdots + w_kY_k > x) \tag{4.5.14}$$

uniformly for $w_2, \dots, w_k \in [a, b]$. Next, write

$$\begin{aligned} &\mathrm{E}a_1(Y_1)1\!\!\mathrm{I}_{\{w_1Y_1+\cdots+w_kY_k>x\}} \\ &= \quad \mathrm{E}a_1(Y_1)1\!\!\mathrm{I}_{\{w_1Y_1+\cdots+w_kY_k>x\}}(1\!\!\mathrm{I}_{\{w_1|Y_1|\leq K_1(x)\}} + 1\!\!\mathrm{I}_{\{w_1|Y_1|>K_1(x)\}}) \\ &=: \quad i_1(x) + i_2(x). \end{aligned}$$

By (4.5.14) we have

$$\begin{aligned} &\limsup_{x\to\infty} \sup_{\overline{w}_k\in[a,b]^k} \frac{i_1(x)}{\mathrm{E}a_1(Y_1)\mathrm{P}(w_2Y_2 + \cdots + w_kY_k > x)} \\ &\leq \quad \limsup_{x\to\infty} \sup_{\overline{w}_k\in[a,b]^k} \frac{\mathrm{P}(w_2Y_2 + \cdots + w_kY_k > x - K_1(x))}{\mathrm{P}(w_2Y_2 + \cdots + w_kY_k > x)} = 1. \end{aligned}$$

This, together with Proposition 4.5.1 (i), yields

$$i_1(x) \lesssim \mathrm{E}a_1(Y_1)(\overline{F_{Y_2}}(x/w_2) + \cdots + \overline{F_{Y_k}}(x/w_k))$$

uniformly in $\overline{w}_k \in [a, b]^k$.

For the lower bound, by (4.5.14) and Proposition 4.5.1 (i), we can write

$$\begin{aligned} i_1(x) \quad &\geq \quad \mathrm{E}a_1(Y_1)1\!\!\mathrm{I}_{\{w_2Y_2+\cdots+w_kY_k>x+K_1(x), w_1|Y_1|\leq K_1(x)\}} \\ &= \quad \mathrm{E}a_1(Y_1)1\!\!\mathrm{I}_{\{w_1|Y_1|\leq K_1(x)\}}\mathrm{P}(w_2Y_2 + \cdots + w_kY_k > x + K_1(x)) \\ &\sim \quad \mathrm{E}a_1(Y_1)\mathrm{P}(w_2Y_2 + \cdots + w_kY_k > x) \\ &\sim \quad \mathrm{E}a_1(Y_1)(\overline{F_{Y_2}}(x/w_2) + \cdots + \overline{F_{Y_k}}(x/w_k)) \end{aligned}$$

uniformly in $\overline{w}_k \in [a, b]^k$.

It remains to show that $i_2(x) = o(\overline{F_{Y_2}}(x/w_2) + \cdots + \overline{F_{Y_k}}(x/w_k))$. Write

$$\begin{aligned} i_2(x) \quad &\leq \quad \mathrm{E}a_1(Y_1)(1\!\!\mathrm{I}_{\{w_1Y_1>x/2\}} + 1\!\!\mathrm{I}_{\{w_2Y_2+\cdots+w_kY_k>x/2\}})1\!\!\mathrm{I}_{\{w_1|Y_1|>K_1(x)\}} \\ &\leq \quad \mathrm{E}a_1(Y_1)1\!\!\mathrm{I}_{\{Y_1>x/(2b)\}} + \mathrm{E}a_1(Y_1)1\!\!\mathrm{I}_{\{|Y_1|>K_1(x)/b\}}\mathrm{P}(w_2Y_2 + \cdots + w_kY_k > x/2). \end{aligned}$$

Hence, by assumption (4.5.8), Proposition 4.5.1 (i) and the definition of class $\mathscr{D}$ we get

$$\begin{aligned} i_2(x) &\lesssim o\big(\overline{F_{Y_2}}(x/(2b)) + \cdots + \overline{F_{Y_k}}(x/(2b))\big) \\ &\quad + o(1)\big(\overline{F_{Y_2}}(x/(2w_2)) + \cdots + \overline{F_{Y_k}}(x/(2w_k))\big) \\ &= o\big(\overline{F_{Y_2}}(x/w_2) + \cdots + \overline{F_{Y_k}}(x/w_k)\big) \end{aligned}$$

uniformly in $\overline{w}_k \in [a,b]^k$.

(ii) We have by (4.5.10) and $F_{Y_i} \in \mathscr{D}$, $i = 1,2$, that

$$\begin{aligned} &\mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{w_1Y_1+w_2Y_2>x\}} \\ &\quad\le \mathrm{E}a_2(Y_2)\mathrm{E}a_1(Y_1)\mathbb{1}_{\{Y_1>x/(2w_1)\}} + \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)\mathbb{1}_{\{Y_2>x/(2w_2)\}} \\ &\quad= \mathrm{E}a_2(Y_2)o(\overline{F_{Y_2}}(x/(2w_1))) + \mathrm{E}a_1(Y_1)o(\overline{F_{Y_1}}(x/(2w_2))) \\ &\quad= o(\overline{F_{Y_1}}(x/w_1) + \overline{F_{Y_2}}(x/w_2)) \end{aligned}$$

uniformly for $\overline{w}_2 \in (0,b]^2$.

(iii) Choose $K_2(x) > 0$ such that $K_2(x) \le x$, $K_2(x) \nearrow \infty$ and

$$\mathrm{P}(w_3Y_3 + \cdots + w_kY_k > x \pm K_2(x)) \sim \mathrm{P}(w_3Y_3 + \cdots + w_kY_k > x) \qquad (4.5.15)$$

uniformly for $w_3, \ldots, w_k \in [a,b]$. Now, split

$$\begin{aligned} &\mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{w_1Y_1+\cdots+w_kY_k>x\}} \\ &\quad= \mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{w_1Y_1+\cdots+w_kY_k>x\}}(\mathbb{1}_{\{|w_1Y_1+w_2Y_2|\le K_2(x)\}} \\ &\qquad + \mathbb{1}_{\{|w_1Y_1+w_2Y_2|>K_2(x)\}}) =: k_1(x) + k_2(x). \end{aligned}$$

Similarly as in case (i), we have

$$\begin{aligned} k_1(x) &\sim \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)(\overline{F_{Y_3}}(x/w_3) + \cdots + \overline{F_{Y_k}}(x/w_k)), \\ k_2(x) &= o(\overline{F_{Y_3}}(x/w_3) + \cdots + \overline{F_{Y_k}}(x/w_k)). \end{aligned}$$

Indeed, by (4.5.15) and Proposition 4.5.1 (i),

$$\begin{aligned} k_1(x) &\le \mathrm{E}a_1(Y_1)a_2(Y_2)\mathrm{P}(w_3Y_3 + \cdots + w_kY_k > x - K_2(x)) \\ &\sim \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)\mathrm{P}(w_3Y_3 + \cdots + w_kY_k > x) \\ &\sim \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)(\overline{F_{Y_3}}(x/w_3) + \cdots + \overline{F_{Y_k}}(x/w_k)), \\ k_1(x) &\ge \mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{|w_1Y_1+w_2Y_2|\le K_2(x)\}}\mathrm{P}(w_3Y_3 + \cdots + w_kY_k > x + K_2(x)) \\ &\sim \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)\mathrm{P}(w_3Y_3 + \cdots + w_kY_k > x) \\ &\sim \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)(\overline{F_{Y_3}}(x/w_3) + \cdots + \overline{F_{Y_k}}(x/w_k)) \end{aligned}$$

uniformly for $\overline{w}_k \in [a,b]^k$, where we have used that

$$\begin{aligned} \mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{|w_1Y_1+w_2Y_2|>K_2(x)\}} &\leq \mathrm{E}a_1(Y_1)\mathbb{1}_{\{b|Y_1|>K_2(x)/2\}}\mathrm{E}a_2(Y_2) \\ &\quad + \mathrm{E}a_2(Y_2)\mathbb{1}_{b|Y_2|>K_2(x)/2\}}\mathrm{E}a_1(Y_1) \ \to \ 0. \end{aligned}$$

For $k_2(x)$ we have

$$\begin{aligned} k_2(x) &\leq \mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{w_1Y_1+w_2Y_2>x/2\}} \\ &\quad +\mathrm{E}a_1(Y_1)a_2(Y_2)\mathbb{1}_{\{|w_1Y_1+w_2Y_2|>K_2(x)\}}\mathrm{P}(w_3Y_3+\cdots+w_kY_k>x/2) \\ &=: k_{21}(x)+k_{22}(x), \end{aligned}$$

where, by assumption (4.5.12), Proposition 4.5.1 (i) and the definition of class $\mathscr{D}$,

$$\begin{aligned} k_{21}(x) &\leq \mathrm{E}a_2(Y_2)\mathrm{E}a_1(Y_1)\mathbb{1}_{\{w_1Y_1>x/4\}} + \mathrm{E}a_1(Y_1)\mathrm{E}a_2(Y_2)\mathbb{1}_{\{w_2Y_2>x/4\}} \\ &= \mathrm{E}a_2(Y_2)o\Big(\sum_{i=3}^{k}\overline{F_{Y_i}}(x/(4w_1))\Big) + \mathrm{E}a_1(Y_1)o\Big(\sum_{i=3}^{k}\overline{F_{Y_i}}(x/(4w_2))\Big) \\ &= o\Big(\sum_{i=3}^{k}\overline{F_{Y_i}}(x/w_i)\Big) \end{aligned}$$

and

$$k_{22}(x) = o(1)\sum_{i=3}^{k}\overline{F_{Y_i}}(x/(2w_i))$$

uniformly for $\overline{w}_k \in [a,b]^k$. □

Corollary 4.5.1. *Assume that $k \geq 2$ and $Y_1,\ldots,Y_k$ are real-valued independent r.v.s, such that $F_{Y_i} \in \mathscr{L}\cap\mathscr{D}$, $i=1,\ldots,k$. Let $a_i\colon(-\infty,\infty)\to[0,\infty)$, $i=1,\ldots,k$, be measurable functions such that $0<\mathrm{E}a_i(Y_i)<\infty$ for each i and let*

$$\mathrm{E}a_i(Y_i)\mathbb{1}_{\{Y_i>x\}} = o(\overline{F_{Y_j}}(x)),\ i,j=1,\ldots,k,\ i\neq j. \tag{4.5.16}$$

Then, uniformly for $\overline{w}_k\in[a,b]^k$, for all $l=1,\ldots,k$ it holds

$$\mathrm{E}a_l(Y_l)\mathbb{1}_{\{w_1Y_1+\cdots+w_kY_k>x\}} \sim \mathrm{E}a_l(Y_l)\sum_{j=1,j\neq l}^{k}\overline{F_{Y_j}}(x/w_j), \tag{4.5.17}$$

and for all l,m, $1\leq l<m\leq k$, it holds

$$\begin{aligned} &\mathrm{E}a_l(Y_l)a_m(Y_m)\mathbb{1}_{\{w_1Y_1+\cdots+w_kY_k>x\}} \\ &= \begin{cases} o\Big(\overline{F_{Y_1}}(x/w_1)+\overline{F_{Y_2}}(x/w_2)\Big), & k=2, \\ \mathrm{E}a_l(Y_l)\mathrm{E}a_m(Y_m)\sum\limits_{\substack{j=1\\ j\neq l,j\neq m}}^{k}\overline{F_{Y_j}}(x/w_j)(1+o(1)), & k\geq 3. \end{cases} \end{aligned} \tag{4.5.18}$$

PROOF. Observe that (4.5.16) with $i = 1$ implies all three conditions (4.5.8), (4.5.10), (4.5.12) with $i = 1$. Then the statement follows straightforwardly from Proposition 4.5.2. $\square$

Chapter 5

Randomly weighted and stopped dependent sums

In this chapter we deal with the tail behavior of the random sums $S_\tau^\Theta := \sum_{k=1}^{\tau} \Theta_k X_k$ and its maximum $S_{(\tau)}^\Theta := \max_{k\le\tau} S_k^\Theta$ with identically distributed (i. d.) dependent heavy-tailed r.v.s $X_1, X_2, \dots$, nonnegative random weights $\Theta_1, \Theta_2, \dots$ and nonnegative counting random variable τ. These three quantities are mutually independent.

Also we study the tail distribution of randomly stopped sum

$$Z_\tau \ := \ \Theta_1 + \cdots + \Theta_\tau,$$

because the asymptotic behavior of $\mathrm{P}(Z_\tau > x)$ has an influence for the behavior of the tail distribution of random maximum $S_{(\tau)}^\Theta$. Such randomly stopped sums apear in the analysis of collective risk model (for example [48]), compound renewal model (see [60]), the model of teletrafic arrivals ([27]), the context of weighted branching processes, fixed point equations of smoothing transforms ([5], [45]), etc.

5.1 Preliminaries

Recently, Olvera-Cravioto [55] studied the asymptotic tail behavior of random sum S_τ^Θ and random maximum $S_{(\tau)}^\Theta$, when $X_1, X_2, \dots$ are independent i. d. random variables with consistently varying common d. f. F_X. Yang et al. [73] generalized the results of Olvera-Cravioto [55] to a certain extent. The main results in both papers state that under assumption $\mathrm{P}(Z_\tau > x) = o(\overline{F_X}(x))$ and some other conditions on the distributions of r.v.s

$\{\Theta_k, k \geq 1\}$, X and τ, probability $\mathrm{P}(S^{\Theta}_{(\tau)} > x)$ is weakly tail-equivalent to $\mathrm{E}\sum_{k=1}^{\tau} \mathrm{P}(\Theta_k X_k > x)$, that is

$$0 < \liminf_{x\to\infty} \frac{\mathrm{P}(S^{\Theta}_{(\tau)} > x)}{\mathrm{E}\sum_{k=1}^{\tau} \mathrm{P}(\Theta_k X_k > x)} \leq \limsup_{x\to\infty} \frac{\mathrm{P}(S^{\Theta}_{(\tau)} > x)}{\mathrm{E}\sum_{k=1}^{\tau} \mathrm{P}(\Theta_k X_k > x)} < \infty. \tag{5.1.1}$$

The asymptotics of the probability $\mathrm{P}(Z_\tau > x)$ with i. i. d. heavy-tailed r.v.s $\Theta_i, i \geq 1$ was studied extensively in the literature. In particular, a well-known result (see [24], Theorem A3.20)) states that if $F_\Theta \in \mathscr{S}$ and τ is light-tailed, then

$$\mathrm{P}(Z_\tau > x) \sim \mathrm{E}\tau \overline{F}_\Theta(x). \tag{5.1.2}$$

If $F_\Theta \in \mathscr{L} \cap \mathscr{D}$ and $\overline{F_\tau}(x) = o(\overline{F_\Theta}(x))$, then relation (5.1.2) was obtained in [52] and [2]. If $F_\Theta \in \mathscr{S}^*$, Denisov et al. [19] proved that

$$P(Z_\tau > x) \sim \mathrm{E}\tau\overline{F_\Theta}(x) + \overline{F}_\tau\Big(\frac{x}{\mathrm{E}\Theta}\Big).$$

In case of some dependence structures within r.v.s $\Theta_1, \Theta_2, \ldots$, similar asymptotics as (5.1.2) were obtained in [65] (for class $\mathscr{L} \cap \mathscr{D}$), [14], [44] (both for class $\mathscr{C}$) under some additional conditions.

We now introduce the following assumption.

Assumption E. Let $X, X_1, X_2, \ldots$ be a sequence of UEND (with dominating constant $\kappa > 0$) real-valued r.v.s with common d. f. $F_X \in \mathscr{D}$, such that $J^-_{F_X} > 0$ and $F_X(-x) = o(\overline{F_X}(x))$; let $\Theta_1, \Theta_2, \ldots$ be a sequence of nonnegative r.v.s (not necessarily independent and identically distributed) and let τ be a nondegenerate at zero nonnegative integer-valued r.v. with distribution function F_τ. $\{X, X_1, X_2, \ldots\}$, $\{\Theta_1, \Theta_2, \ldots\}$ and τ are mutually independent.

In addition, assume that there exists $\epsilon \in (0, J^-_{F_X})$ such that

$$\mathrm{E}(X^+)^{1+\epsilon} < \infty \tag{5.1.3}$$

and

$$\mathrm{E}\sum_{i=1}^{\tau} \Theta_i^{J^-_{F_X}-\epsilon} < \infty, \qquad \mathrm{E}\sum_{i=1}^{\tau} \Theta_i^{J^+_{F_X}+\epsilon} < \infty. \tag{5.1.4}$$

The following theorem was proved in [73].

Theorem 5.1.1. ([73]) *Let Assumption E and conditions* (5.1.3), (5.1.4) *be satisfied. If*

$$\mathrm{P}(Z_\tau > x) \;=\; o(\overline{F_X}(x)), \tag{5.1.5}$$

then

$$L_{F_X}\mathrm{E}\sum_{i=1}^{\tau}\mathrm{P}(\Theta_i X_i>x)\ \lesssim\ \mathrm{P}(S_{(\tau)}^{\Theta}>x)\ \lesssim\ L_{F_X}^{-1}\mathrm{E}\sum_{i=1}^{\tau}\mathrm{P}(\Theta_i X_i>x). \quad (5.1.6)$$

Remark 5.1.1. Since

$$\begin{aligned}\mathrm{E}\sum_{i=1}^{\tau}\Theta_i^{J_{F_X}^{-}-\epsilon} &= \mathrm{E}\Big(\sum_{i=1}^{\tau}\Theta_i^{J_{F_X}^{-}-\epsilon}\mathbf{1}_{\{\Theta_i\le 1\}}\Big)+\mathrm{E}\Big(\sum_{i=1}^{\tau}\Theta_i^{J_{F_X}^{-}-\epsilon}\mathbf{1}_{\{\Theta_i>1\}}\Big)\\ &\le \mathrm{E}\tau+\mathrm{E}\sum_{i=1}^{\tau}\Theta_i^{J_{F_X}^{+}+\epsilon},\end{aligned}$$

the first restriction in (5.1.4) can be dropped as $\mathrm{E}\tau<\infty$. Besides, if $\Theta,\Theta_1,\Theta_2,\ldots$ are identically distributed, then

$$\begin{aligned}\mathrm{E}\sum_{i=1}^{\tau}\Theta_i^{J_{F_X}^{+}+\epsilon}=\mathrm{E}\sum_{n=0}^{\infty}\sum_{i=1}^{n}\Theta_i^{J_{F_X}^{+}+\epsilon}\mathrm{P}(\tau=n) &= \mathrm{E}\Theta^{J_{F_X}^{+}+\epsilon}\sum_{n=0}^{\infty}n\mathrm{P}(\tau=n)\\ &= \mathrm{E}\Theta^{J_{F_X}^{+}+\epsilon}\mathrm{E}\tau.\end{aligned}$$

Clearly, if the random series $Z_\infty:=\Theta_1+\Theta_2+\ldots$ converges almost surely (it is typical in insurance mathematics, where X_i denotes the net loss over period i and Θ_i represents the stochastic discount from time i to 0), then condition

$$\mathrm{P}(Z_\infty>x)\ =\ o(\overline{F_X}(x)) \quad (5.1.7)$$

is sufficient for relation (5.1.5) to hold. So that, the statement of Theorem 5.1.1 is valid if (5.1.5) is replaced by (5.1.7).

Corollary 5.1.1. *If Assumption E, conditions* (5.1.3), (5.1.4) *and* (5.1.7) *are satisfied, then relation* (5.1.6) *holds.*

Consider now the case $\mathrm{P}(Z_\infty=\infty)>0$. For example, if $\Theta_1,\Theta_2,\ldots$ are nonnegative independent r.v.s, then, according to the Kolmogorov's three series theorem, the inequality $\frac{x}{1+x}\le\min(x,1)$ for $x\ge0$ and Problem 2 in [56] (p. 388), $\mathrm{P}(Z_\infty=\infty)=1$ if and only if $\sum_{k=1}^{\infty}\mathrm{E}\min\{\Theta_k,1\}=\infty$. This fact can be extended for arbitrarily dependent nonnegative r.v.s as well, see [57]. If, additionally, r.v.s $\Theta,\Theta_1,\Theta_2,\ldots$ are identically distributed, then the last condition is equivalent to $\mathrm{E}\Theta>0$. Identically distributed weights are rather natural when studying the present value of investment portfolio of n risky assets with X_i, denoting the potential loss of ith asset over a period, and Θ_i being the stochastic discount factor over the period. Clearly, in such a case relation (5.1.7) does not hold and some other approaches must be used in order to obtain the asymptotics of $\mathrm{P}(Z_\tau>x)$.

5.2 Asymptotics of $\mathrm{P}(Z_\tau > x)$

In this section we study the asymptotics of $\mathrm{P}(Z_\tau > x)$, when $\Theta_1, \Theta_2, \ldots$ are identically distributed r.v.s. The next proposition is a modification of Theorem 1 in [70]. In this case, more general dependence structure of r.v.s $\Theta_1, \Theta_2, \ldots$ is considered.

Proposition 5.2.1. *Let $\Theta, \Theta_1, \Theta_2, \ldots$ be nonnegative END r.v.s with common marginal d. f. F_Θ and finite positive mean $\mathrm{E}\Theta$. Let τ be a nonnegative integer-valued r.v., independent of the sequence $\Theta, \Theta_1, \Theta_2, \ldots$.*
(i) *If $F_\Theta \in \mathscr{D}$ and $\overline{F_\Theta}(x) \asymp \overline{F_\tau}(x)$, then $F_\tau \in \mathscr{D}$, $\mathrm{E}\tau < \infty$ and*

$$L_{F_\Theta}\mathrm{E}\tau\overline{F_\Theta}(x) + L_{F_\tau}\overline{F_\tau}\Big(\frac{x}{\mathrm{E}\Theta}\Big) \;\lesssim\; \mathrm{P}(Z_\tau > x) \;\lesssim\; L_{F_\Theta}^{-1}\mathrm{E}\tau\overline{F_\Theta}(x) + L_{F_\tau}^{-1}\overline{F_\tau}\Big(\frac{x}{\mathrm{E}\Theta}\Big); \tag{5.2.1}$$

(ii) *if $F_\Theta \in \mathscr{D}$, $\overline{F_\tau}(x) = o(\overline{F_\Theta}(x))$, then $\mathrm{E}\tau < \infty$ and*

$$L_{F_\Theta}\mathrm{E}\tau\overline{F_\Theta}(x) \;\lesssim\; \mathrm{P}(Z_\tau > x) \;\lesssim\; L_{F_\Theta}^{-1}\mathrm{E}\tau\overline{F_\Theta}(x); \tag{5.2.2}$$

(iii) *if $F_\tau \in \mathscr{D}$, $\mathrm{E}\tau < \infty$ and $\overline{F_\Theta}(x) = o(\overline{F_\tau}(x))$, then*

$$L_{F_\tau}\overline{F_\tau}\Big(\frac{x}{\mathrm{E}\Theta}\Big) \;\lesssim\; \mathrm{P}(Z_\tau > x) \;\lesssim\; L_{F_\tau}^{-1}\overline{F_\tau}\Big(\frac{x}{\mathrm{E}\Theta}\Big). \tag{5.2.3}$$

For the upper asymptotic relations in (5.2.1), (5.2.2), (5.2.3), *the assumption that $\Theta_1, \Theta_2, \ldots$ are END can be replaced by weaker assumption that $\Theta_1, \Theta_2, \ldots$ are UEND.*

Proof. The proof follows similarly as in [70].

(i) As in the proof of Theorem 1 of [70], split

$$\begin{aligned} P(Z_\tau > x) &= \Bigg(\sum_{n=1}^{M} + \sum_{n=M+1}^{\lfloor(1-\epsilon)x(\mathrm{E}\Theta)^{-1}\rfloor} + \sum_{n=\lfloor(1-\epsilon)x(\mathrm{E}\Theta)^{-1}\rfloor+1}^{\infty}\Bigg)\mathrm{P}(Z_n > x)\mathrm{P}(\tau = n) \\ &=: Q_1 + Q_2 + Q_3, \end{aligned} \tag{5.2.4}$$

for each triplet $\epsilon \in (0,1)$, $M \in \mathbb{N}$, $x > 0$ such that $\lfloor(1-\epsilon)x(\mathrm{E}\Theta)^{-1}\rfloor \geq M + 1$. Clearly, by conditions of the proposition, $F_\tau \in \mathscr{D}$, because

$$\limsup_{x\to\infty}\frac{\overline{F_\tau}(xy)}{\overline{F_\tau}(x)} \;\leq\; \limsup_{x\to\infty}\frac{\overline{F_\tau}(xy)}{\overline{F_\Theta}(xy)}\,\limsup_{x\to\infty}\frac{\overline{F_\Theta}(xy)}{\overline{F_\Theta}(x)}\,\limsup_{x\to\infty}\frac{\overline{F_\Theta}(x)}{\overline{F_\tau}(x)} \;<\; \infty.$$

Moreover, conditions of the proposition imply $\mathrm{E}\tau < \infty$. Indeed, since $\limsup_{x\to\infty}\overline{F_\tau}(x)/\overline{F_\Theta}(x) \leq c_1$ for some $c_1 > 0$, $\forall\epsilon \in (0,1)\ \exists x^* : \sup_{x > x^*}\overline{F_\tau}(x) \leq (1+\epsilon)c_1\overline{F_\Theta}(x) \leq 2c_1\mathrm{P}(\Theta > x)$ and we obtain that $\mathrm{P}(\tau > x) \leq 2c_1\mathrm{P}(\Theta > x)$,

$x \geq x^*$. Hence

$$\begin{aligned} \mathrm{E}\tau &= \int_{[0,\infty)} \mathrm{P}(\tau > x)\mathrm{d}x = \int_{[0,x^*)} \mathrm{P}(\tau > x)\mathrm{d}x + \int_{[x^*,\infty)} \mathrm{P}(\tau > x)\mathrm{d}x \\ &\leq x^* + 2c_1 \int_{[x^*,\infty)} \mathrm{P}(\Theta > x)\mathrm{d}x \leq x^* + 2c_1\mathrm{E}\Theta < \infty. \end{aligned}$$

Using Lemma 5.4.2 below, for each fixed M it holds

$$Q_1 \lesssim \overline{F_\Theta}(x) L^{-1}_{F_\Theta} \sum_{n=1}^{M} n\mathrm{P}(\tau = n). \tag{5.2.5}$$

For the term Q_2 write

$$\begin{aligned} Q_2 &= \sum_{n=M+1}^{\lfloor (1-\epsilon)x(\mathrm{E}\Theta)^{-1} \rfloor} \mathrm{P}(Z_n - n\mathrm{E}\Theta > x - n\mathrm{E}\Theta)\mathrm{P}(\tau = n) \\ &\leq \sum_{n=M+1}^{\lfloor (1-\epsilon)x(\mathrm{E}\Theta)^{-1} \rfloor} \mathrm{P}(Z_n - n\mathrm{E}\Theta > \epsilon x)\mathrm{P}(\tau = n), \end{aligned}$$

where, by Lemma 5.4.3, $\mathrm{P}(Z_n - n\mathrm{E}\Theta > \epsilon x) \leq c_2 n \overline{F_\Theta}(\epsilon x)$ for some $c_2 = c_2(\epsilon, \kappa, \mathrm{E}\Theta)$. Since $F_\Theta \in \mathscr{D}$, $\overline{F_\Theta}(\epsilon x)/\overline{F_\Theta}(x) \lesssim c_3$ for some finite constant $c_3 = c_3(\epsilon)$. Hence, similarly to (3.3) in [70], it follows that

$$\begin{aligned} Q_2 &\lesssim c_2 \sum_{n=M+1}^{\lfloor (1-\epsilon)x(\mathrm{E}\Theta)^{-1} \rfloor} \overline{F_\Theta}(\epsilon x) n\mathrm{P}(\tau = n) \\ &\lesssim c_2 c_3 \overline{F_\Theta}(x) \sum_{n=M+1}^{\lfloor (1-\epsilon)x(\mathrm{E}\Theta)^{-1} \rfloor} n\mathrm{P}(\tau = n) \\ &\lesssim c_4 \overline{F_\Theta}(x) \sum_{n=M+1}^{\infty} n\mathrm{P}(\tau = n) \end{aligned} \tag{5.2.6}$$

with some $c_4 = c_4(\epsilon, \kappa, \mathrm{E}\Theta)$. Finally,

$$Q_3 \leq \sum_{n=\lfloor (1-\epsilon)x(\mathrm{E}\Theta)^{-1} \rfloor + 1}^{\infty} \mathrm{P}(\tau = n) = \overline{F_\tau}((1-\epsilon)x(\mathrm{E}\Theta)^{-1}). \tag{5.2.7}$$

Relations (5.2.5)–(5.2.7) and (5.2.4) imply that, for all $\epsilon \in (0, 1)$, $M \in \mathbb{N}$ and sufficiently large x,

$$\begin{aligned} &\frac{\mathrm{P}(Z_\tau > x)}{L^{-1}_{F_\Theta}\mathrm{E}\tau\overline{F_\Theta}(x) + L^{-1}_{F_\tau}\overline{F_\tau}(x(\mathrm{E}\Theta)^{-1})} \\ &\quad \leq \frac{Q_2}{L^{-1}_{F_\Theta}\mathrm{E}\tau\overline{F_\Theta}(x)} + \max\left\{ \frac{Q_1}{L^{-1}_{F_\Theta}\mathrm{E}\tau\overline{F_\Theta}(x)}, \frac{Q_3}{L^{-1}_{F_\tau}\overline{F_\tau}(x(\mathrm{E}\Theta)^{-1})} \right\}. \end{aligned}$$

Hence,

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\mathrm{P}(Z_\tau > x)}{L_{F_\Theta}^{-1}\mathrm{E}\tau\overline{F_\Theta}(x) + L_{F_\tau}^{-1}\overline{F_\tau}(x(\mathrm{E}\Theta)^{-1})} \\
&\le \; c_4 L_{F_\Theta}\frac{\sum_{n=M+1}^{\infty} n\mathrm{P}(\tau = n)}{\mathrm{E}\tau} + \max\left\{\frac{\sum_{n=1}^{M} n\mathrm{P}(\tau = n)}{\mathrm{E}\tau}, L_{F_\tau}\limsup_{x\to\infty}\frac{\overline{F_\tau}((1-\epsilon)x)}{\overline{F_\tau}(x)}\right\}.
\end{aligned}$$

Letting $M \to \infty$ and $\epsilon \searrow 0$, we obtain the statement in case (i).

The proof of asymptotic lower estimate of (5.2.1) is similar to the proof in [70]. We present it here for convenience. For any $\epsilon \in (0,1)$, positive integer M and sufficiently large x (e.g., $\lfloor(1+\epsilon)x(\mathrm{E}\Theta)^{-1}\rfloor) \ge M$) we have

$$P(Z_\tau > x) \;\; \ge \;\; Q_1 + Q_4, \tag{5.2.8}$$

where Q_1 is the same as earlier and

$$Q_4 \;:=\; \sum_{n=\lfloor(1+\epsilon)x(\mathrm{E}\Theta)^{-1}\rfloor+1}^{\infty} \mathrm{P}(Z_n > x)\mathrm{P}(\tau = n).$$

Conditions of the proposition and (5.4.2) imply that

$$\begin{aligned}
\liminf \frac{Q_1}{\overline{F}_\Theta(x)} \;&\ge\; \sum_{n=1}^{M} \liminf \frac{\mathrm{P}(Z_n > x)}{\overline{F_\Theta}(x)}\,\mathrm{P}(\tau = n) \\
&\ge\; L_{F_\Theta}\sum_{n=1}^{M} n\mathrm{P}(\tau = n).
\end{aligned}$$

Therefore,

$$\liminf \frac{Q_1}{L_{F_\Theta}\mathrm{E}\tau\overline{F}_\Theta(x)} \ge 1. \tag{5.2.9}$$

For term Q_4 we have

$$\begin{aligned}
Q_4 \;&\ge\; \sum_{n>(1+\epsilon)x(\mathrm{E}\Theta)^{-1}} \mathrm{P}\left(\frac{Z_n}{n} - \mathrm{E}\Theta > -\frac{\epsilon\mathrm{E}\Theta}{1+\epsilon}\right)\mathrm{P}(\tau = n) \\
&\ge\; \inf_{n>(1+\epsilon)x(\mathrm{E}\Theta)^{-1}} \mathrm{P}\left(\frac{Z_n}{n} - \mathrm{E}\Theta > -\frac{\epsilon\mathrm{E}\Theta}{1+\epsilon}\right)\overline{F}_\tau\big((1+\epsilon)x(\mathrm{E}\Theta)^{-1}\big).
\end{aligned}$$

By Lemma 5.4.4, we have that

$$\lim_{n\to\infty} \mathrm{P}\left(\frac{Z_n}{n} - \mathrm{E}\Theta > -\frac{\epsilon\mathrm{E}\Theta}{1+\epsilon}\right) \;=\; 1,$$

so that

$$\liminf \frac{Q_4}{L_{F_\tau}\overline{F_\tau}((1+\epsilon)x(\mathrm{E}\Theta)^{-1})} \ge 1. \tag{5.2.10}$$

The lower estimate of (5.2.1) follows from relations (5.2.8)–(5.2.10), because for sufficiently large x, any $\epsilon \in (0,1)$ and positive integer M

$$\begin{aligned} & \frac{\mathrm{P}\left(Z_\tau > x\right)}{L_{F_\Theta}\mathrm{E}\tau\overline{F}_\Theta(x) + L_{F_\tau}\overline{F}_\tau(x(\mathrm{E}\Theta)^{-1})} \\ & \qquad \geq \min\left\{\frac{Q_1}{L_{F_\Theta}\mathrm{E}\tau\overline{F}_\Theta(x)}, \frac{Q_4}{L_{F_\tau}\overline{F_\tau}(x(\mathrm{E}\Theta)^{-1})}\right\}. \end{aligned}$$

(ii) The proof of this part follows the proof of the same part in Theorem 1 of [70]. Similarly as in part (i), for the upper estimate $Q_3 \leq \overline{F_\tau}((1-\epsilon)x(\mathrm{E}\Theta)^{-1}) = o(\overline{F_\Theta}(x))$ for every fixed $\epsilon \in (0,1)$. For the lower estimate we have that $\mathrm{P}(Z_\tau > x) \geq Q_1$, since $Q_4 = o(\overline{F_\Theta}(x))$.

(iii) For the lower estimate, by (5.2.8)–(5.2.10), $Q_1 = o(\overline{F_\tau}(x))$ then $\mathrm{P}(Z_\tau > x) \geq \overline{F}_\tau\big((1+\epsilon)x(\mathrm{E}\Theta)^{-1}\big)$ for any fixed $\epsilon \in (0,1)$.

The proof of the upper estimate is analogous to the proof of the same part in Theorem 1 of Yang et al. [70]. For completeness of the proof we write it here. For every $\epsilon \in (0,1)$ we have

$$\begin{aligned} \mathrm{P}\left(Z_\tau > x\right) &= \sum_{n \leq (1-\epsilon)x(\mathrm{E}\Theta)^{-1}} \mathrm{P}(Z_n > x)\mathrm{P}(\tau = n) \\ &\quad + \sum_{n > (1-\epsilon)x(\mathrm{E}\Theta)^{-1}} \mathrm{P}(Z_n > x)\mathrm{P}(\tau = n) \\ &=: J_1 + J_2. \end{aligned} \tag{5.2.11}$$

Because $J_2 \leq \overline{F_\tau}((1-\epsilon)x(\mathrm{E}\Theta)^{-1})$,

$$\lim_{\epsilon \searrow 0} \limsup_{x\to\infty} \frac{J_2}{\overline{F_\tau}(x(\mathrm{E}\Theta)^{-1})} \leq L_{F_\tau}^{-1}. \tag{5.2.12}$$

According to Lemma 5.4.5, there exists a sequence of UEND r.v.s $Y_1, Y_2, \ldots$ such that, almost surely, $\Theta_n \leq Y_n$, $n = 1, 2, \ldots$, $F_Y \in \mathscr{D}$, $\overline{F_Y}(x) = o(\overline{F_\tau}(x))$. Therefore,

$$\begin{aligned} J_1 &\leq \sum_{n \leq (1-\epsilon)x\mu_Y^{-1}} \mathrm{P}(Z_n^Y > x)\mathrm{P}(\tau = n) \\ &\quad + \sum_{(1-\epsilon)x\mu_Y^{-1} < n \leq (1-\epsilon)x(\mathrm{E}\Theta)^{-1}} \mathrm{P}(Z_n > x)\mathrm{P}(\tau = n) \\ &=: J_{11} + J_{12} \end{aligned} \tag{5.2.13}$$

with finite $\mu_Y := \mathrm{E}Y \geq \mathrm{E}\Theta$ and $Z_n^Y := \sum_{k=1}^n Y_k$, $n \geq 0$.

Using Lemma 5.4.3 we obtain for sufficiently large x and some positive constants $c_5 = c_5(\epsilon)$, $c_6 = c_6(\epsilon)$

$$\begin{aligned} J_{11} &\le \sum_{n\le(1-\epsilon)x\mu_Y^{-1}} \mathrm{P}\left(Z_n^Y - n\mu_Z > \epsilon x\right)\mathrm{P}(\tau = n) \\ &\le c_5 \sum_{n\le(1-\epsilon)x\mu_Y^{-1}} n\overline{F_Y}(\epsilon x)\mathrm{P}(\tau = n) \\ &\le c_6 \mathrm{E}\tau\overline{F_Y}(x). \end{aligned}$$

Hence, using $\overline{F_Y}(x) = o(\overline{F_\tau}(x))$ and $F_\tau \in \mathscr{D}$, we have that for every fixed $\epsilon \in (0,1)$

$$\limsup \frac{J_{11}}{\overline{F_\tau}(x(\mathrm{E}\Theta)^{-1})} = 0. \tag{5.2.14}$$

Finally, we deal with J_{12}. Clearly,

$$\begin{aligned} J_{12} &\le \sum_{(1-\epsilon)x\mu_Y^{-1}<n\le(1-\epsilon)x(\mathrm{E}\Theta)^{-1}} \mathrm{P}\left(\frac{Z_n}{n} - \mathrm{E}\Theta > \frac{\mathrm{E}\Theta\epsilon}{1-\epsilon}\right)\mathrm{P}(\tau = n) \\ &\le \sup_{n>(1-\epsilon)x\mu_Y^{-1}} \mathrm{P}\left(\frac{Z_n}{n} - \mathrm{E}\Theta > \frac{(\mathrm{E}\Theta)\epsilon}{1-\epsilon}\right)\overline{F_\tau}\big((1-\epsilon)x\mu_Y^{-1}\big). \end{aligned}$$

By Lemma 5.4.4, the first term in the last expression vanishes as $x \to \infty$ for every fixed $\epsilon \in (0,1)$. This and assumption $F_\tau \in \mathscr{D}$ imply that (with the same ϵ)

$$\limsup_{x\to\infty} \frac{J_{12}}{\overline{F_\tau}(x(\mathrm{E}\Theta)^{-1})} = 0. \tag{5.2.15}$$

The upper estimate in (5.2.3) follows from (5.2.11)–(5.2.15). □

The following result for strongly subexponential r.v.s is proved by Denisov [19] (Theorem 1 (ii)).

Proposition 5.2.2. *([19]) Let $\Theta, \Theta_1, \Theta_2, \ldots$ be a sequence of nonnegative independent r.v.s with common d. f. $F_\Theta \in \mathscr{S}^*$ and finite positive mean $\mathrm{E}\Theta$. Let τ be a nondegenerate nonnegative integer-valued r.v., independent of $\Theta, \Theta_1, \Theta_2, \ldots$. If there exists $c > \mathrm{E}\Theta$ such that $\overline{F_\tau}(x) = o(\overline{F_\Theta}(cx))$, then $\mathrm{E}\tau < \infty$ and*

$$\mathrm{P}(Z_\tau > x) \sim \mathrm{E}\tau\overline{F_\Theta}(x). \tag{5.2.16}$$

Remark 5.2.1. Note that in more restrictive cases, the assumption of Proposition 5.2.2 can be simplified. For example, if the same main conditions of the proposition hold, $F_\Theta \in \mathscr{L} \cap \mathscr{D}$ and $\overline{F_\tau}(x) = o(\overline{F_\Theta}(x))$, then relation (5.2.16) holds (see [52] (Theorem 2.3) and [19] (Theorem 8)).

Remark 5.2.2. It is easy to see that, under the conditions of Proposition 5.2.2, the closure of the class $\mathscr{S}^*$ holds, i.e. $F_{Z_\tau} \in \mathscr{S}^*$ (see [39]).

5.3 Main results

Applying results in Section 5.2, which deal with case of identically distributed r.v.s. $\Theta_1, \Theta_2, \ldots$, we obtain the following theorems, which constitute the main results of this chapter.

Theorem 5.3.1. *Let r.v.s* $\Theta, \Theta_1, \Theta_2, \ldots$ *be identically distributed and let Assumption E be satisfied. Assume that* (5.1.3) *and* $\mathrm{E}\Theta^{J^+_{F_X}+\epsilon} < \infty$ *hold.*

(i) *If* $F_\Theta \in \mathscr{D}$ *and either* $\overline{F_\Theta}(x) \sim c^* \overline{F_\tau}(x)$ *for some* $c^* > 0$ *or* $\overline{F_\tau}(x) = o(\overline{F_\Theta}(x))$, *then relation* (5.1.6) *holds;*

(ii) *if* $F_\tau \in \mathscr{D}$, $\mathrm{E}\tau < \infty$ *and* $\overline{F_\Theta}(x) = o(\overline{F_\tau}(x))$, $\overline{F_\tau}(x) = o(\overline{F_X}(x))$, *then* (5.1.6) *holds.*

Proof. First note that condition $\mathrm{E}(X^+)^{1+\epsilon} < \infty$ implies $J^+_{F_X} \geq 1$ and, thus, $\mathrm{E}\Theta < \infty$. Observe that, by Markov's inequality and Lemma 5.4.1,

$$\overline{F_\Theta}(x) \leq x^{-(J^+_{F_X}+\epsilon)} \mathrm{E}\Theta^{J^+_{F_X}+\epsilon} = o(\overline{F_X}(x)). \tag{5.3.1}$$

(i) From the (i) part of Proposition 5.2.1 we have that $F_\tau \in \mathscr{D}$, if $\overline{F_\Theta}(x) \sim c^* \overline{F_\tau}(x)$. Then we note that $\overline{F_\tau}(x/\mathrm{E}\Theta) = o(\overline{F_X}(x))$ is equivalent to $\overline{F_\tau}(x) = o(\overline{F_X}(x))$ if $F_\tau \in \mathscr{D}$, $F_X \in \mathscr{D}$. Combining this and (5.3.1), from (5.2.1) we get that condition (5.1.5) is fulfilled.

Similarly, if $\overline{F_\tau}(x) = o(\overline{F_\Theta}(x))$, then (5.2.2) holds, for $F_\Theta \in \mathscr{D}$. Hence, under (5.3.1), condition (5.1.5) is satisfied.

(ii) Conditions imply that relation (5.2.3) holds and (5.1.5) is satisfied.
□

The next theorem presents the case of the strongly subexponential class $\mathscr{S}^*$.

Theorem 5.3.2. *Let* $\Theta, \Theta_1, \Theta_2, \ldots$ *be i.i. d. r.v.s. and let Assumption E be satisfied. Assume that* (5.1.3) *and* $\mathrm{E}\Theta^{J^+_{F_X}+\epsilon} < \infty$ *hold. If* $F_\Theta \in \mathscr{S}^*$ *and there exists* $c > \mathrm{E}\Theta$ *such that* $\overline{F_\tau}(x) = o(\overline{F_\Theta}(cx))$, *then* (5.1.6) *holds.*

Proof. Proposition 5.2.2 and relation (5.3.1) imply the main condition (5.1.5). Hence, relation (5.1.6) holds. □

5.4 Auxiliary lemmas

The first lemma is a well-known property of class $\mathscr{D}$ (see [61], Lemma 3.5).

Lemma 5.4.1. *For a d. f.* $F \in \mathscr{D}$ *with its upper Matuszewska index* J_F^+ *it holds that*

$$x^{-p} \;=\; o(\overline{F}(x)) \quad \textit{for any} \quad p > J_F^+.$$

Next two lemmas are used in proving Proposition 5.2.1.

Lemma 5.4.2. *Let* $\Theta_1, \Theta_2, \ldots$ *be pUEND r.v.s with common d. f.* $F_\Theta \in \mathscr{D}$. *Then, for any fixed* $n \geq 1$,

$$\mathrm{P}(Z_n > x) \lesssim L_{F_\Theta}^{-1} n \overline{F_\Theta}(x). \tag{5.4.1}$$

If, in addition, $F_\Theta(-x) = o(\overline{F}_\Theta(x))$, *then for any fixed* $n \geq 1$

$$\mathrm{P}(Z_n > x) \;\gtrsim\; L_{F_\Theta} n \overline{F}_\Theta(x). \tag{5.4.2}$$

Proof. It is obvious that inequality (5.4.1) holds for $n = 1$. If $n \geq 2$, then for any fixed $\epsilon \in (0, 1)$,

$$\begin{aligned}
&\mathrm{P}(Z_n > x)\\
&\leq \; \mathrm{P}\Big(\Theta_i > \frac{\epsilon x}{n}, \Theta_j > \frac{\epsilon x}{n} \quad \text{for some } 1 \leq i < j \leq n\Big)\\
&\qquad + \mathrm{P}\Big(Z_n > x \text{ and } \Big\{\Theta_i \leq \frac{\epsilon x}{n} \text{ or } \Theta_j \leq \frac{\epsilon x}{n}\Big\} \text{ for every pair } 1 \leq i, j \leq n\Big)\\
&=: \; \mathrm{P}(A) + \mathrm{P}(B).
\end{aligned}$$

Clearly, $B \subset \Big\{Z_n > x, \Theta_j > (1-\epsilon)x \text{ for some } j \text{ and } \Theta_i \leq \frac{\epsilon x}{n} \text{ for all } i \neq j\Big\}$. Using this and the definition of pUEND,

$$\begin{aligned}
\mathrm{P}(Z_n > x) \;&\leq\; \sum_{1 \leq i < j \leq n} \mathrm{P}\Big(\Theta_i > \frac{\epsilon x}{n}, \Theta_j > \frac{\epsilon x}{n}\Big) + \sum_{j=1}^{n} \mathrm{P}(\Theta_j > (1-\epsilon)x)\\
&\leq\; \kappa n^2 \Big(\overline{F_\Theta}\Big(\frac{\epsilon x}{n}\Big)\Big)^2 + n\overline{F_\Theta}((1-\epsilon)x).
\end{aligned}$$

Here, since $F_\Theta \in \mathscr{D}$, for any $n \geq 2$ and $\epsilon \in (0, 1)$, it holds that $(\overline{F_\Theta}(\epsilon x/n))^2 = o(\overline{F_\Theta}(x))$. Hence,

$$\begin{aligned}
\limsup_{x \to \infty} \frac{\mathrm{P}(Z_n > x)}{\overline{F_\Theta}(x)} \;&\leq\; n \lim_{\epsilon \searrow 0} \limsup_{x \to \infty} \frac{\overline{F_\Theta}((1-\epsilon)x)}{\overline{F_\Theta}(x)}\\
&=\; n L_{F_\Theta}^{-1}.
\end{aligned}$$

Consider now the lower estimate. For $n = 1$ relation (5.4.2) is evident. Suppose that $n \geq 2$. For $\epsilon \in (0, 1)$ and $x > 0$ we have

$$\begin{aligned}
\mathrm{P}(Z_n > x) \;&\geq\; \mathrm{P}\big(Z_n > x, \max_{1 \leq k \leq n} \Theta_k > (1+\epsilon)x\big)\\
&\geq\; \sum_{k=1}^{n} \mathrm{P}\big(Z_n > x, \Theta_k > (1+\epsilon)x\big)\\
&\qquad - \sum_{1 \leq i < j \leq n} \mathrm{P}\big(Z_n > x, \Theta_i > (1+\epsilon)x, \Theta_j > (1+\epsilon)x\big)\\
&=:\; P_1 - P_2.
\end{aligned} \tag{5.4.3}$$

According to the conditions of lemma, r.v.s Θ_i, Θ_j are pUEND for all $i \neq j$. Thus,

$$P_2 \leq \sum_{1\leq i<j\leq n} \mathrm{P}\Big(\Theta_i > (1+\epsilon)x,\ \Theta_j > (1+\epsilon)x\Big) \leq \kappa\left(n\overline{F}_\Theta((1+\epsilon)x)\right)^2. \quad (5.4.4)$$

Since $\Theta_1, \Theta_2, \dots$ are identically distributed, we have for P_1

$$\begin{aligned} P_1 &\geq \sum_{k=1}^{n} \mathrm{P}(Z_n - \Theta_k \geq -\epsilon x,\ \Theta_k > (1+\epsilon)x) \\ &\geq \sum_{k=1}^{n} \Big(\overline{F}_\Theta((1+\epsilon)x) + \mathrm{P}(Z_n - \Theta_k \geq -\epsilon x) - 1\Big) \\ &= n\overline{F}_\Theta((1+\epsilon)x) - \sum_{k=1}^{n} \mathrm{P}\left(\sum_{\substack{l=1\\ l\neq k}}^{n} \Theta_l < -\epsilon x\right). \end{aligned} \quad (5.4.5)$$

For fixed k

$$\begin{aligned} \mathrm{P}\left(\sum_{\substack{l=1\\ l\neq k}}^{n} \Theta_l < -\epsilon x\right) &\leq \mathrm{P}\Big(\Theta_l < -\frac{\epsilon x}{n} \text{ for some } 1\leq l\leq n,\ l\neq k\Big) \\ &\leq nF\Big(-\frac{\epsilon x}{n}\Big). \end{aligned}$$

Hence, conditions of the lemma imply that

$$\limsup_{x\to\infty} \frac{\sum_{k=1}^{n} \mathrm{P}\left(\sum_{\substack{l=1\\ l\neq k}}^{n} \Theta_l < -\epsilon x\right)}{n\overline{F}((1+\epsilon)x)} \leq n \limsup_{x\to\infty} \frac{F(-\frac{\epsilon x}{n})}{\overline{F}((1+\epsilon)x)} = 0$$

for each fixed $\epsilon \in (0,1)$. The last relation and (5.4.5) yield

$$\liminf_{x\to\infty} \frac{P_1}{n\overline{F}((1+\epsilon)x)} \geq 1 \quad (5.4.6)$$

for fixed $\epsilon \in (0,1)$ and $n \geq 1$.

Relation (5.4.2) follows now from (5.4.3), (5.4.4), (5.4.6) and the definition of L_F. Lemma 5.4.2 is proved. □

The next lemma is a generalization of Corollary 3.1 in [58], where the structure UND has been used. The proof is almost identical to the proof of Corollary 3.1 in [58] and, thus, is omitted.

Lemma 5.4.3. *If $\Theta_1, \Theta_2, \dots$ are UEND r.v.s with common d. f. $F_\Theta \in \mathscr{D}$ and mean $\mathrm{E}\Theta = 0$, then, for each $\gamma > 0$, there exists a constant $c = c(\kappa, \gamma)$, independent of x and n, such that*

$$\mathrm{P}(Z_n > x) \leq cn\overline{F_\Theta}(x)$$

for all $x \geq \gamma n$ and $n \geq 1$.

The following auxiliary result is the law of large numbers for END r.v.s. The proof of lemma can be found in [11].

Lemma 5.4.4. *Let* $\xi_1, \xi_2, \ldots$ *be a sequence of identically distributed END r.v.s. If* $\mathrm{E}|\xi_1|$ *exists then, almost surely,*

$$\frac{\xi_1 + \cdots + \xi_n}{n} \to \mathrm{E}\xi_1 \tag{5.4.7}$$

as $n \to \infty$.

The last lemma is the generalization of Lemma 4 in [70] with UND r.v.s. Here we use the UEND structure, but it does not change the proof.

Lemma 5.4.5. *Let* $\Theta_1, \Theta_2, \ldots$ *be a sequence of UEND r.v.s with common d. f.* F_Θ *satisfying* $\overline{F_\Theta}(0-) > 0$ *and* $\overline{F_\Theta}(x) = o\big(\overline{F_\tau}(x)\big)$ *for some d. f.* $F_\tau \in \mathscr{D}$*. Then there exists a sequence of UEND r.v.s* $\eta_1, \eta_2, \ldots$ *with common d. f.* $F_\eta \in \mathscr{D}$ *such that, a.s.,* $\Theta_n \leq \eta_n$, $n = 1, 2, \ldots$ *and* $\overline{F_\eta}(x) = o\big(\overline{F_\tau}(x)\big)$.

Chapter 6

Conclusions

Here we make the conclusions of the main results obtained in this dissertation.

1. Tail distributions of $S_n^{(+)}$, $S_{(n)}$ and the sum $\sum_{i=1}^n \mathrm{P}(X_i > x)$ are weakly equivalent, if primary random variables $X_1, \ldots, X_n$ are dependent according to a certain structure and the distribution of maximal element is dominatedly varying-tailed.

2. The sum S_n^Θ of dependent (under the given structure) random variables $X_1, \ldots, X_n$ belongs to the class $\mathscr{L}$, if the marginal distributions $F_1, \ldots, F_n$ are from the long-tailed distribution class. Besides that, the tail distributions of S_n^Θ, $S_n^{\Theta+}$ and $S_{(n)}^\Theta$ are equivalent if random weights $\Theta_1, \ldots, \Theta_n$ are bounded and independent of random variables $X_1, \ldots, X_n$. For example, this result holds if dependence of random variables $X_1, \ldots, X_n$ is generated by the well-known FGM copula.

3. With the assumption that identically distributed UEND random variables $X_1, X_2 \ldots$, bounded random weights $\Theta_1, \Theta_2, \ldots$ and the stopping moment τ are heavy-tailed, the asymptotic lower and upper bounds for the tail distribution of $S_{(\tau)}^\Theta$ (maximum of randomly stopped sums) are derived. The conditions for this result are shown for the wide class of heavy tailed distribution functions and dependence structures. With some additional requirements the tail distribution of the sum of random weights is asymptotically negligible compared to the tail distribution of r.v.s $X_1, X_2 \ldots$.

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