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# On Value Distribution of Certain Beurling Zeta-Functions

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**Abstract:** In this paper, the approximation of analytic functions by shifts $\zeta_{\mathcal{P}}(s+i\tau)$ of Beurling zeta-functions $\zeta_{\mathcal{P}}(s)$ of certain systems $\mathcal{P}$ of generalized prime numbers is discussed. It is required that the system of generalized integers $\mathcal{N}_{\mathcal{P}}$ generated by $\mathcal{P}$ satisfies $\sum_{m\leqslant x,\, m\in\mathcal{N}} 1 = ax + O(x^\delta)$, $a>0$, $0\leqslant\delta<1$, and the function $\zeta_{\mathcal{P}}(s)$ in some strip lying in $\widehat{\sigma}<\sigma<1$, $\widehat{\sigma}>\delta$, which has a bounded mean square. Proofs are based on the convergence of probability measures in some spaces.

**Keywords:** Beurling zeta-function; generalized integers; generalized prime numbers; weak convergence of probability measures

**MSC:** 11M41

## 1. Introduction

The Riemann zeta-function $\zeta(s)$, $s=\sigma+it$, is defined, for $\sigma>1$ by

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}=\prod_{q}\left(1-\frac{1}{q^s}\right)^{-1},$$

where the product is taken over prime numbers $q$, has a meromorphic continuation to the complex plane with the unique simple pole $s=1$, $\text{Res}_{s=1}\zeta(s)=1$ (see, for example, [1]), and has several generalizations. One of them is Beurling zeta-functions.

The system $\mathcal{P}$ of real numbers $1<p_1\leqslant p_2\leqslant\cdots\leqslant p_n\leqslant\cdots$, $p_n\to\infty$ as $n\to\infty$, is called generalized prime numbers. From numbers of system $\mathcal{P}$, the system $\mathcal{N}_{\mathcal{P}}$ of generalized integers

$$p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}\cdots,\quad \alpha_j\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ j=1,\ldots,r,\ldots,$$

is obtained. As in the theory of rational primes $q$, the main attention is devoted to asymptotics of the function

$$\pi_{\mathcal{P}}(x)=\sum_{\substack{p\leqslant x\\ p\in\mathcal{P}}}1,\quad x\to\infty.$$

Together with $\pi_{\mathcal{P}}(x)$, the number of generalized integers $m$

$$\mathcal{N}_{\mathcal{P}}(x)=\sum_{\substack{m\leqslant x\\ m\in\mathcal{N}_{\mathcal{P}}}}1,\quad x\to\infty,$$

is considered. The above sums are taken by counting multiplicities of $p$ and $m$, respectively. By the Landau result [2], it is known that the estimate

$$\mathcal{N}_{\mathcal{P}}(x)=ax+O\left(x^\delta\right),\quad 0\leqslant\delta<1,\ a>0, \tag{1}$$

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implies

$$\pi_{\mathcal{P}}(x) = \int_{2}^{x} \frac{\mathrm{d}u}{\log u} + O\Big(x\mathrm{e}^{-c\sqrt{\log x}}\Big), \quad c > 0.$$

The distribution of generalized numbers was studied by Beurling [3], Borel [4], Diamond [5–7], Mallavin [8], Nyman [9], Ryavec [10], Stankus [11], Zhang [12], Hilberdink and Lapidus [13], Schlage-Puhta and Vindas [14], Debruyne, Schlage-Puhta and Vindas [15], and others. Among other problems studied in the above works, the central place is occupied by the relation between

$$\mathcal{N}_{\mathcal{P}}(x) = ax + O\bigg(\frac{x}{(\log x)^{\alpha}}\bigg), \quad \alpha > 0, \tag{2}$$

and

$$\pi_{\mathcal{P}}(x) = \int_{2}^{x} \frac{\mathrm{d}u}{\log u} + O\bigg(\frac{x}{(\log x)^{\beta}}\bigg), \quad \beta > 0.$$

For example, in [9], it was obtained that the above estimates with arbitrary $\alpha$ and $\beta$ are equivalent. The papers [6,8,16] are devoted to formulae for $\pi_{\mathcal{P}}(x)$, with the remainder term of order $O(x\mathrm{e}^{-c_1(\log x)^{\beta}})$ implied by $\mathcal{N}_{\mathcal{P}}(x)$ with the remainder term $O(x\mathrm{e}^{-c_2(\log x)^{\alpha}})$. Beurling proved [3] that the asymptotics

$$\pi_{\mathcal{P}}(x) \sim \frac{x}{\log x}, \quad x \to \infty, \tag{3}$$

follows from (2) with $\alpha > 3/2$, and this is not true with $\alpha = 3/2$ for all systems of generalized primes. Moreover, for the investigation of $\pi_{\mathcal{P}}(x)$, he introduced the zeta-functions $\zeta_{\mathcal{P}}(s)$ defined in some half-planes by the Euler product

$$\zeta_{\mathcal{P}}(s) = \prod_{p \in \mathcal{P}} \bigg(1 - \frac{1}{p^s}\bigg)^{-1},$$

or by the Dirichlet series

$$\zeta_{\mathcal{P}}(s) = \sum_{m \in \mathcal{N}_{\mathcal{P}}}^{\infty} \frac{1}{m^s}.$$

The convergence of the latter objects depends on the system $\mathcal{P}$ of generalized primes.

It is easily seen that in case (1), the series for $\zeta_{\mathcal{P}}(s)$ is absolutely convergent for $\sigma > 1$. Actually, the partial summation formula shows that

$$\sum_{\substack{m \leqslant x \\ m \in \mathcal{N}_{\mathcal{P}}}} \frac{1}{m^s} = \frac{1}{x^s}\mathcal{N}_{\mathcal{P}}(x) + s\int_{1}^{x} \frac{\mathcal{N}_{\mathcal{P}}(x)}{x^{s+1}}\,\mathrm{d}x. \tag{4}$$

Since, for $\sigma > 1$, the integral

$$\int_{1}^{\infty} \frac{\mathcal{N}_{\mathcal{P}}(x)}{x^{s+1}}\,\mathrm{d}x$$

is absolutely and uniformly convergent for $\sigma \geqslant 1 + \varepsilon$, $\forall \varepsilon > 0$, and $x^{-s}\mathcal{N}_{\mathcal{P}}(x) = o(1)$, so from (4) we have

$$\zeta_{\mathcal{P}}(s) = s\int_{1}^{\infty} \frac{\mathcal{N}_{\mathcal{P}}(x)}{x^{s+1}}\,\mathrm{d}x. \tag{5}$$

Thus, $\zeta_{\mathcal{P}}(s)$ is analytic in the half-plane $\sigma > 1$. Moreover, in this half-plane,

$$\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^s}\right)^{-1} = \sum_{m\in\mathcal{N}_{\mathcal{P}}}\frac{1}{m^s}.$$

Now, the functions $\zeta_{\mathcal{P}}(s)$ are called Beurling zeta-functions.

As it was observed by Beurling [3], it suffices to consider $\mathcal{N}_{\mathcal{P}}(x)$ in place of $\mathcal{N}_{\mathcal{P}}(x^\delta)$, $\delta \neq 1$, because the latter case reduces after normalization to $\mathcal{N}_{\mathcal{P}}(x)$.

An important problem is the analytic continuation of the function $\zeta_{\mathcal{P}}(s)$. Suppose that (1) is true. Then, (5) implies

$$\zeta_{\mathcal{P}}(s) = \frac{as}{s-1} + s\int_1^\infty \frac{r(x)}{x^{s+1}}\,\mathrm{d}x, \quad r(x) = O(x^\delta),\ \delta < 1,$$

the latter integral being absolutely and uniformly convergent for $\sigma \geqslant \delta + \varepsilon$, $\forall\varepsilon > 0$. Therefore, the function $\zeta_{\mathcal{P}}(s)$ has analytic continuation to the half-plane $\sigma > \delta$, except for a simple pole at the point $s = 1$ with residue $a$.

Much attention is devoted to analytic continuation for the function $\zeta_{\mathcal{P}}(s)$ in [13]. For this, the generalized von Mongoldt function

$$\Lambda_{\mathcal{P}}(m) = \begin{cases} \log p & \text{if } m = p^k,\ p\in\mathcal{P},\ k\in\mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\psi_{\mathcal{P}}(x) = \sum_{\substack{m\leqslant x\\ m\in\mathcal{N}_{\mathcal{P}}}} \Lambda_{\mathcal{P}}(m)$$

are used. Let

$$\psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\varepsilon}), \quad \alpha\in[0,1),\ \forall\varepsilon > 0.$$

Then, in [13], it is proved that $\zeta_{\mathcal{P}}(s)$ has an analytic continuation to the half-plane $\sigma > \alpha$, except for a simple pole at the point $s = 1$. Under certain additional conditions, the latter estimate is necessary as well.

There is another method for the analytic continuation of $\zeta_{\mathcal{P}}(s)$ cultivated in [13]. However, for our aims, we limit ourselves by the analytic continuation to the half-plane $\sigma > \delta$ because, throughout the paper, we suppose the validity of the axiom (1).

The paper [17] is devoted to zero-distribution of $\zeta_{\mathcal{P}}(s)$, where various zero-density results corresponding to those of $\zeta(s)$ are given. We stress that in [17], the Beurling prime number theorem [3] was strengthened, and it was proved that asymptotics (3) is implied by the estimate of Cesàro type

$$\int_1^x \frac{\mathcal{N}_{\mathcal{P}}(t) - at}{t}\left(1-\frac{t}{x}\right)^m \mathrm{d}t = O\left(\frac{x}{(\log x)^\alpha}\right), \quad \alpha > \frac{3}{2},\ x\to\infty,$$

with some $m \in \mathbb{N}$.

In the present paper, differently from the cited above works, including [14,17], that are devoted to prime number theorem, analytic continuation and zeros of $\zeta_{\mathcal{P}}(s)$, we focus on the approximation properties of the Beurling zeta-functions. More precisely, we consider the approximation of a set of analytic functions $f(s)$ by shifts $\zeta_{\mathcal{P}}(s+i\tau)$, $\tau\in\mathbb{R}$, i.e., such $\tau$ that, for some compact sets $K$ and $\varepsilon > 0$,

$$\sup_{s\in K}|\zeta_{\mathcal{P}}(s+i\tau) - f(s)| < \varepsilon.$$

The case of the Riemann zeta-function shows that the results of such a type have serious theoretical (functional independence, zero-distribution, moment problem, ...) and practical (approximation theory, quantum mechanics) applications, see [18]. Moreover, investigations of the approximation of analytic functions by zeta-functions have an impact on the Linnik–Ibragimov conjecture on the universality of the Dirichlet series; see Section 1.6 of [19].

For our aims, the mean square estimate for $\zeta_{\mathcal{P}}(s)$ is needed. Let

$$M(\sigma, T) \stackrel{\text{def}}{=} \int_0^T |\zeta_{\mathcal{P}}(\sigma + it)|^2 \, dt,$$

and $\widehat{\sigma} = \inf\{\sigma : M(\sigma, T) \ll_\sigma T, \sigma > \delta\}$. Suppose that $\widehat{\sigma} < 1$, and define $D_{\mathcal{P}} = \{s \in \mathbb{C} : \widehat{\sigma} < \sigma < 1\}$. Here, and in what follows, the notation $z \ll_\varepsilon y$, $z \in \mathbb{C}$, $y > 0$ is a synonym of $z = O(y)$ with implied constant depending on $\varepsilon$. Denote by $\mathcal{H}(D_{\mathcal{P}})$ the space of analytic on $D_{\mathcal{P}}$ functions endowed with the topology of uniform convergence on compacta.

It is well-known that the Riemann zeta-function $\zeta(s)$ and some other zeta-functions are universal, i.e., their shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$ are approximately defined in certain strip analytic functions; see [18–25] for results and problems. We believe that the function $\zeta_{\mathcal{P}}(s)$ for some systems of generalized prime numbers $\mathcal{P}$ also has similar approximation properties. However, every case of system $\mathcal{P}$ requires a separate investigation. In the paper, we propose the following result for the approximation of analytic functions by shifts $\zeta_{\mathcal{P}}(s + i\tau)$. In what follows, $m_L A$ denotes the Lebesgue measure of $A \subset \mathbb{R}$. The main result of the paper is the following theorem.

**Theorem 1.** *Assume that the system $\mathcal{P}$ satisfies the axiom (1). Then, there exists a non-empty closed subset $F_{\mathcal{P}} \subset \mathcal{H}(D_{\mathcal{P}})$, such that, for all compact sets $K \subset D_{\mathcal{P}}$, $f(s) \in F_{\mathcal{P}}$ and $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} m_L \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta_{\mathcal{P}}(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

*In addition, the limit*

$$\lim_{T \to \infty} \frac{1}{T} m_L \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta_{\mathcal{P}}(s + i\tau) - f(s)| < \varepsilon \right\}$$

*exists and is positive for all, but at most countably many, $\varepsilon > 0$.*

Theorem 1 will be proved in Section 5.

Let $\mathcal{B}(\mathbb{X})$ stand for the Borelean $\sigma$-field of the topological space $\mathbb{X}$, and, for $A \in \mathcal{B}(\mathcal{H}(D_{\mathcal{P}}))$,

$$P_{T,\mathcal{P}}(A) = \frac{1}{T} m_L \{\tau \in [0, T] : \zeta_{\mathcal{P}}(s + i\tau) \in A\}.$$

Theorem 1 will be derived from the next theorem on weak convergence of $P_{T,\mathcal{P}}$ as $T \to \infty$.

**Theorem 2.** *Suppose that the system $\mathcal{P}$ satisfies the axiom (1). Then $P_{T,\mathcal{P}}$, as $T \to \infty$, weakly converges to a certain measure $P_{\mathcal{P}}$ on $(\mathcal{H}(D_{\mathcal{P}}), \mathcal{B}(\mathcal{H}(D_{\mathcal{P}})))$.*

Theorem 2 will be proved in Section 4.

We recall some examples connected to the hypotheses of Theorems 1 and 2.

A problem of the validity of axiom (1) is not easy. The following interesting example is known; see [13]. Let the system of generalized integers $\mathcal{N}_\mathcal{P}$ be generated by the system

$$\mathcal{P} = (2, \sqrt{3}, 5, 5, \sqrt{7}, \sqrt{11}, 13, 13, \dots),$$

i.e., $\mathcal{P}$ includes 2, rational primes $q \equiv 1(\mathrm{mod}\, 4)$ with multiplicity 2, and $\sqrt{q}$ with rational primes $q \equiv 3(\mathrm{mod}\, 4)$. Then, it is known that

$$\mathcal{N}_\mathcal{P}(x) = \frac{\pi}{4}x + O\Big(x^{23/73}\Big).$$

In [11], the system $\mathcal{P}$ of shifted rational primes $q = \pi(r) + 1$ with $r > 0$, $\pi(r) = \sum_{q \leqslant r} 1$, was considered, and it was obtained that

$$\mathcal{N}_\mathcal{P}(x) = ax + O\left(x \exp\left\{-\left(1 - \frac{c \log_3 x}{\log_2 x}\right)\sqrt{\frac{1}{2}\log x \log_2 x}\right\}\right),$$

where $\log_n x = \underbrace{\log \dots \log}_{n} x$, $a > 0$, $c > 0$. This shows that the estimate (1), even for a comparatively simple system $\mathcal{P}$, is difficult to reach.

Write generalized numbers in another form

$$1 = \nu_1 < \nu_2 < \cdots$$

with corresponding multiplicities $1 = a_1, a_2, \dots$. Then, we have

$$\mathcal{N}_\mathcal{P}(x) = \sum_{\nu_m \leqslant x} a_m,$$

and

$$\zeta_\mathcal{P}(s) = \sum_{m=1}^{\infty} \frac{a_m}{\nu_m^s}.$$

In [26], the following result has been obtained. Suppose that (1) is true, and $\nu_{m+1} - \nu_m \gg \exp\{-\nu_m^\kappa\}$ with every $\kappa > 0$. Then, for $\sigma > (1+\delta)/2$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\zeta_\mathcal{P}(\sigma + it)|^2 \, \mathrm{d}t = \sum_{m=1}^{\infty} \frac{a_m^2}{\nu_m^{2\sigma}}.$$

This implies that $\widehat{\sigma} = (1+\delta)/2 < 1$ in this case.

We divide the proof of Theorem 2 into parts. We start with weak convergence of probability measures in comparatively simple spaces and finish in the space $\mathcal{H}(D_\mathcal{P})$.

## 2. Case of Compact Group

Define the set

$$\Omega = \prod_{p \in \mathcal{P}} \{s \in \mathbb{C} : |s| = 1\}.$$

The elements of $\Omega$ are all functions $\omega : \mathcal{P} \to \{s \in \mathbb{C} : |s| = 1\}$. We equipped $\Omega$ with the product topology and operation of pointwise multiplication. Since the unit circle is a compact set, by the Tikhonov theorem [27], $\Omega$ is a compact topological group. For $A \in \mathcal{B}(\Omega)$, set

$$P_{T,\mathcal{P}}^{\Omega}(A) = \frac{1}{T} m_L\Big\{\tau \in [0, T] : \Big(p^{-i\tau} : p \in \mathcal{P}\Big) \in A\Big\}.$$

**Lemma 1.** *$P_{T,\mathcal{P}}^{\Omega}$ weakly converges to a certain measure $P_\mathcal{P}^{\Omega}$ on $(\Omega, \mathcal{B}(\Omega))$ as $T \to \infty$.*

**Proof.** It suffices to show that the Fourier transform of $P^{\Omega}_{T,\mathcal{P}}$ converges to a certain continuous function. Characters of $\Omega$ have the form

$$\prod_{p\in\mathcal{P}} \omega^{k_p}(p),$$

where $\omega(p)$ denotes the $p$th component of $\omega \in \Omega$, and $k_p$ are integer rational numbers, where only a finite number of them are not zero. Therefore,

$$\mathcal{F}_{T,\mathcal{P}}(\Bbbk) = \frac{1}{T}\int_0^T \left( {\prod_{p\in\mathcal{P}}}^{*} p^{-i\tau k_p} \right) \mathrm{d}\tau,$$

where $\Bbbk = (k_p : p \in \mathcal{P})$, and the star $*$ shows that $k_p \neq 0$ for a finite set of generalized primes $p$, is the Fourier transform of the measure $P^{\Omega}_{T,\mathcal{P}}$. Define two sets of $\Bbbk$:

$$K_1 = \left\{ \Bbbk : {\sum_{p\in\mathcal{P}}}^{*} k_p \log p = 0 \right\}, \qquad K_2 = \left\{ \Bbbk : {\sum_{p\in\mathcal{P}}}^{*} k_p \log p \neq 0 \right\}.$$

Then, we have

$$\mathcal{F}_{T,\mathcal{P}}(\Bbbk) = \begin{cases} 1 & \text{if } \Bbbk \in K_1, \\ \frac{1-\exp\left\{-iT\sum^{*}_{p\in\mathcal{P}} k_p \log p\right\}}{iT\left(1-\exp\left\{-i\sum^{*}_{p\in\mathcal{P}} k_p \log p\right\}\right)} & \text{if } \Bbbk \in K_2. \end{cases}$$

Thus,

$$\lim_{T\to\infty} \mathcal{F}_{T,\mathcal{P}}(\Bbbk) = \begin{cases} 1 & \text{if } \Bbbk \in K_1, \\ 0 & \text{if } \Bbbk \in K_2. \end{cases}$$

The limit function is continuous in the discrete topology; therefore, this implies that $P^{\Omega}_{T,\mathcal{P}}$ weakly converges to the measure $P^{\Omega}_{\mathcal{P}}$ on $(\Omega, \mathcal{B}(\Omega))$ given by the Fourier transform $\mathcal{F}_{\mathcal{P}}(\Bbbk)$,

$$\mathcal{F}_{\mathcal{P}}(\Bbbk) = \begin{cases} 1 & \text{if } \Bbbk \in K_1, \\ 0 & \text{if } \Bbbk \in K_2. \end{cases}$$

□

**Remark 1.** *If the system $\mathcal{P}$ is linearly independent over the field of rational numbers, then*

$$\mathcal{F}_{\mathcal{P}}(\Bbbk) = \begin{cases} 1 & \textit{if } \Bbbk = (\mathbf{0}), \\ 0 & \textit{if } \Bbbk \neq (\mathbf{0}). \end{cases}$$

*In this case, the limit measure $P^{\Omega}_{\mathcal{P}}$ is the Haar measure $P_H$, which is invariant with respect to translations by elements $\omega \in \Omega$, i.e., for every $\omega \in \Omega$ and $A \in \mathcal{B}(\Omega)$,*

$$P_H(A) = P_H(\omega A) = P_H(A\omega).$$

*Obviously, in this case, the numbers of $\mathcal{P}$ must be different.*

Lemma 1 is a starting point to consider limit distributions in space $\mathcal{H}(D_{\mathcal{P}})$. The simplest case is of an absolutely convergent Dirichlet series. Let $\eta > 1 - \widehat{\sigma}$ be fixed. For $m \in \mathcal{N}_{\mathcal{P}}$ and $n \in \mathbb{N}$, set

$$a_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\eta} \right\},$$

and

$$\zeta_{n,\mathcal{P}}(s) = \sum_{m\in\mathcal{N}_{\mathcal{P}}} \frac{a_n(m)}{m^s}.$$

It is not difficult to see that the series for $\zeta_{n,\mathcal{P}}(s)$ is absolutely convergent, say, for $\sigma > 0$. Thus, $\zeta_{n,\mathcal{P}}(s)$ is an element of $\mathcal{H}(D_\mathcal{P})$. For $A \in \mathcal{B}(\mathcal{H}(D_\mathcal{P}))$, define

$$P_{T,n,\mathcal{P}}(A) = \frac{1}{T} m_L\{\tau \in [0,T] : \zeta_{\mathcal{P},n}(s+i\tau) \in A\}.$$

**Lemma 2.** *Assume that the system $\mathcal{P}$ satisfies the axiom (1). Then, $P_{T,n,\mathcal{P}}$ weakly converges to a certain measure $P_{n,\mathcal{P}}$ on $(\mathcal{H}(D_\mathcal{P}), \mathcal{B}(\mathcal{H}(D_\mathcal{P})))$ as $T \to \infty$.*

**Proof.** Extend the function $\omega(p)$ to the set $\mathcal{N}_\mathcal{P}$ by using the equality

$$\omega(m) = \omega^{a_1}(p_1) \cdots \omega^{a_r}(p_r)$$

for $m = p_1^{a_1} \cdots p_r^{a_r}$. Consider the mapping $h_{n,\mathcal{P}} : \Omega \to \mathcal{H}(D_\mathcal{P})$ given by

$$h_{n,\mathcal{P}}(\omega) = \sum_{m \in \mathcal{N}_\mathcal{P}} \frac{\omega(m) a_n(m)}{m^s}, \quad \omega \in \Omega.$$

The latter definition implies that

$$h_{n,\mathcal{P}}\left(p^{-i\tau} : p \in \mathcal{P}\right) = \sum_{m \in \mathcal{N}_\mathcal{P}} \frac{a_n(m)}{m^{s+i\tau}} = \zeta_{n,\mathcal{P}}(s+i\tau). \tag{6}$$

Moreover, the absolute convergence of the series

$$\sum_{m \in \mathcal{N}_\mathcal{P}} \frac{\omega(m) a_n(m)}{m^s}$$

for $\sigma > 0$ ensures the continuity of the mapping $h_{n,\mathcal{P}}$. In view of (6), we have

$$P_{T,n,\mathcal{P}}(A) = \frac{1}{T} m_L\left\{\tau \in [0,T] : \left(p^{-i\tau} : p \in \mathcal{P}\right) \in h_{n,\mathcal{P}}^{-1} A\right\} = P_{T,\mathcal{P}}^{\Omega}\left(h_{n,\mathcal{P}}^{-1} A\right)$$

for all $A \in \mathcal{B}(\mathcal{H}(D_\mathcal{P}))$. This shows that $P_{T,n,\mathcal{P}} = P_{T,\mathcal{P}}^{\Omega} h_{n,\mathcal{P}}^{-1}$, where

$$P_{T,\mathcal{P}}^{\Omega} h_{n,\mathcal{P}}^{-1}(A) = P_{T,\mathcal{P}}^{\Omega}\left(h_{n,\mathcal{P}}^{-1} A\right), \quad A \in \mathcal{B}(\mathcal{H}(D_\mathcal{P})),$$

and $h_{n,\mathcal{P}}^{-1} A$ denotes the preimage of the set $A$. These remarks, Lemma 1, and the preservation of weak convergence under continuous mappings (see, for example, [28], Chapter 5) prove that $P_{T,n,\mathcal{P}}$, as $T \to \infty$ weakly converges to the measure $P_{n,\mathcal{P}} = h_{n,\mathcal{P}}^{-1} P_\mathcal{P}^{\Omega}$, where $P_\mathcal{P}^{\Omega}$ is from Lemma 1. □

### 3. Some Estimates

To pass from the function $\zeta_{n,\mathcal{P}}(s)$ to $\zeta_\mathcal{P}(s)$, we need some estimates between these functions. We start with an integral representation for $\zeta_{n,\mathcal{P}}(s)$. As usual, let $\Gamma(s)$ stand for the Euler gamma-function, and, for $n \in \mathbb{N}$, define

$$l_n(s) = \eta^{-1} \Gamma\left(\eta^{-1} s\right) n^s,$$

where the number $\eta$ is from the definition of $a_n(m)$.

**Lemma 3.** *Suppose that axiom (1) is valid. Then, for $s \in D$, the representation*

$$\zeta_{n,\mathcal{P}}(s) = \frac{1}{2\pi i} \int_{\eta - i\infty}^{\eta + i\infty} \zeta_\mathcal{P}(s+z) l_n(z) \, \mathrm{d}z \tag{7}$$

*holds.*

**Proof.** Let $a$ and $b$ be positive numbers. Then, the classical Mellin formula

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(z)b^{-z}\,\mathrm{d}z = \mathrm{e}^{-b}$$

is valid. Therefore, for $m \in \mathcal{N}_{\mathcal{P}}$,

$$\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} m^{-z} l_n(z)\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\Gamma\left(\frac{z}{\eta}\right)\left(\frac{m}{n}\right)^{(-z/\eta)\eta}\mathrm{d}\left(\frac{z}{\eta}\right) = a_n(m).$$

Hence,

$$\begin{aligned}\zeta_{n,\mathcal{P}}(s) &= \sum_{m\in\mathcal{N}_{\mathcal{P}}}\frac{a_n(m)}{m^s} = \frac{1}{2\pi i}\sum_{m\in\mathcal{N}_{\mathcal{P}}}\int_{\eta-i\infty}^{\eta+i\infty}\frac{1}{m^{s+z}}l_n(z)\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\left(\sum_{m\in\mathcal{N}_{\mathcal{P}}}\frac{1}{m^{s+z}}\right)l_n(z)\,\mathrm{d}z. \end{aligned} \tag{8}$$

Since $\eta > 1-\widehat{\sigma}$, we have $\mathrm{Re}(s+z) > 1$. Moreover, the properties of the function $\Gamma(s)$ ensure the change in order integration and summation. Thus, (8) implies the representation of the lemma. □

There is a sequence of compact embedded sets $\{K_l : l \in \mathbb{N}\} \subset D_{\mathcal{P}}$, $D_{\mathcal{P}} = \bigcup_{l=1}^{\infty} K_l$, such that every compact set $K \subset D_{\mathcal{P}}$ lies in some $K_l$. Then,

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|}, \quad g_1, g_2 \in \mathcal{H}(D_{\mathcal{P}}),$$

is a metric in $\mathcal{H}(D_{\mathcal{P}})$ inducing its topology of uniform convergence on compacta.

**Lemma 4.** *Suppose that axiom* (1) *is valid. Then,*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_1^T \rho(\zeta_{\mathcal{P}}(s+i\tau), \zeta_{n,\mathcal{P}}(s+i\tau)) = 0.$$

**Proof.** By the formula for $\rho$, it is sufficient to prove that, for every compact set $K \subset D_{\mathcal{P}}$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_1^T \sup_{s\in K}|\zeta_{\mathcal{P}}(s+i\tau) - \zeta_{n,\mathcal{P}}(s+i\tau)| = 0. \tag{9}$$

Thus, fix a compact set $K \subset D_{\mathcal{P}}$. Then, there is $\varepsilon > 0$ satisfying $\widehat{\sigma}+\varepsilon \leqslant \sigma \leqslant 1-\varepsilon/2$ for $\sigma+it \in K$. We apply Lemma 3. Let $\eta = 1$, and $\eta_1 = \widehat{\sigma}+\varepsilon/2-\sigma$ with above $\sigma$. Then $\eta_1 < 0$. The integrand in (7) possesses a simple pole at $z = 0$ (a pole of $\Gamma(s)$), and a simple pole at $z = 1-s$ (a pole of $\zeta_{\mathcal{P}}(s+z)$). Actually, it is obvious that $0 \in (\eta_1, \eta)$ and $1-\sigma \in (\eta_1, \eta)$. Moreover, since $\eta_1 \geqslant \widehat{\sigma}+\varepsilon/2-1+\varepsilon/2$, $\widehat{\sigma}-1+\varepsilon > -1$, the pole $z = -1$ of $\Gamma(s)$ does not lie in the strip $\eta_1 < \mathrm{Re} z < \eta$.

Now, the residue theorem and Lemma 3 yields, for $s \in K$,

$$\zeta_{n,\mathcal{P}}(s) - \zeta_{\mathcal{P}}(s) = \frac{1}{2\pi i} \int\limits_{\eta_1 - i\infty}^{\eta_1 + i\infty} \zeta_{\mathcal{P}}(s) l_n(z)\, \mathrm{d}z + \operatorname*{Res}_{z=1-s} \zeta_{\mathcal{P}}(s+z) l_n(z).$$

Hence, for $s \in K$,

$$\begin{aligned}
&\zeta_{n,\mathcal{P}}(s+i\tau) - \zeta_{\mathcal{P}}(s+i\tau) \\
&= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau + it + iu\Big) l_n\Big(\widehat{\sigma} + \frac{\varepsilon}{2} - \sigma + iu\Big)\, \mathrm{d}u + a l_n(1 - s - i\tau) \\
&= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau + iu\Big) l_n\Big(\widehat{\sigma} + \frac{\varepsilon}{2} - s + iu\Big)\, \mathrm{d}u + a l_n(1 - s - i\tau) \\
&\ll \int\limits_{-\infty}^{\infty} \Big|\zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau + iu\Big)\Big| \sup_{s \in K}\Big|l_n\Big(\widehat{\sigma} + \frac{\varepsilon}{2} - s + iu\Big)\Big|\, \mathrm{d}u + \sup_{s \in K}|l_n(1 - s - i\tau)|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{T} \int\limits_{0}^{T} \sup_{s \in K}|\zeta_{\mathcal{P}}(s+i\tau) - \zeta_{n,\mathcal{P}}(s+i\tau)|\, \mathrm{d}\tau \\
&\quad \ll \int\limits_{-\infty}^{\infty} \left(\frac{1}{T}\int\limits_{0}^{T} \Big|\zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau + iu\Big)\Big|\, \mathrm{d}\tau\right) \sup_{s \in K}|l_n(1 - s + iu)|\, \mathrm{d}u \\
&\qquad + \frac{1}{T}\int\limits_{0}^{T} \sup_{s \in K}|l_n(1 - s - i\tau)|\, \mathrm{d}\tau \\
&\quad \stackrel{\text{def}}{=} J_1 + J_2.
\end{aligned} \tag{10}$$

By the definition of $\widehat{\sigma}$,

$$\int\limits_{0}^{T} \Big|\zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau\Big)\Big|^2 \mathrm{d}\tau \ll_\varepsilon T.$$

Therefore, in view of the Cauchy–Schwarz inequality,

$$\begin{aligned}
\int\limits_{0}^{T} \Big|\zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau + iu\Big)\Big|\, \mathrm{d}\tau &\leqslant \sqrt{T}\left(\int\limits_{0}^{T} \Big|\zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau + iu\Big)\Big|^2 \mathrm{d}\tau\right)^{1/2} \\
&\leqslant \sqrt{T}\left(\int\limits_{-|u|}^{T+|u|} \Big|\zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau\Big)\Big|^2 \mathrm{d}\tau\right)^{1/2} \\
&\ll_\varepsilon \sqrt{T}(T + |u|)^{1/2} \ll_\varepsilon \sqrt{T}\Big(\sqrt{T} + \sqrt{u}\Big) \\
&\ll_\varepsilon T\big(1 + \sqrt{u}\big).
\end{aligned} \tag{11}$$

The most important ingredient of the function $l_n(s)$ is $\Gamma(s)$ and is estimated as

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0.$$

Therefore, for $s \in K$,

$$l_n\left(\widehat{\sigma} + \frac{\varepsilon}{2} + 1 - s + iu\right) \ll n^{\widehat{\sigma}+\varepsilon/2-\sigma} \exp\{-c|u-t|\} \ll_K n^{-\varepsilon/2} \exp\{-c_1|u|\}, \quad c_1 > 0.$$

This, together with (11), yields

$$J_1 \ll_{K,\varepsilon} n^{-\varepsilon/2} \int_{-\infty}^{+\infty} (1+\sqrt{u}) \exp\{-c_1|u|\}\, \mathrm{d}u \ll_{\varepsilon,K} n^{-\varepsilon/2}. \tag{12}$$

Similarly, as above, we obtain that, for $s \in K$,

$$l_n(1-s-i\tau) \ll n^{1-\sigma} \exp\{-c|t+\tau|\} \ll_K n^{1-\widehat{\sigma}-\varepsilon} \exp\{-c_2|\tau|\}, \quad c_2 > 0.$$

Therefore,

$$J_2 \ll_K n^{1-\widehat{\sigma}-\varepsilon} \frac{1}{T} \int_0^T \exp\{-c_2|\tau|\}\, \mathrm{d}\tau \ll_K n^{1-\widehat{\sigma}-\varepsilon} T^{-1}.$$

The latter bound, (12) and (10), prove (9). The lemma is proved. □

## 4. Proof of Theorem 2

We derive Theorem 2 from Lemmas 2 and 4 and the following statement (see, for example, [28], Theorem 4.2) is applied to the case $\mathcal{H}(D_{\mathcal{P}})$.

**Lemma 5.** *Assume that $\xi_{nk}$ and $\widehat{\xi}_n$, $n,k \in \mathbb{N}$, are $\mathcal{H}(D_{\mathcal{P}})$-valued random elements given on a space $(\mathbb{X}, \mathcal{B}(\mathbb{X}), \nu)$. Let*

$$\xi_{nk} \xrightarrow[n\to\infty]{\mathcal{D}} \xi_k, \qquad \xi_k \xrightarrow[k\to\infty]{\mathcal{D}} \xi,$$

*and for $\varepsilon > 0$,*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \nu\left\{\rho\left(\widehat{\xi}_n, \xi_{nk}\right) \geqslant \varepsilon\right\} = 0,$$

*where $\xrightarrow{\mathcal{D}}$ stands for the convergence in distribution. Then $\widehat{\xi}_n \xrightarrow[n\to\infty]{\mathcal{D}} \xi$.*

We remind the reader that $P_{n,\mathcal{P}}$ is from Lemma 2. Using Lemma 5 requires some convergence properties for $P_{n,\mathcal{P}}$. Recall that the sequence $\{P_{n,\mathcal{P}} : n \in \mathbb{N}\}$ is tight if, for every $\varepsilon > 0$, there is a compact set $K \subset \mathcal{H}(D_{\mathcal{P}})$ such that

$$P_{n,\mathcal{P}}(K) > 1 - \varepsilon$$

with all $n \in \mathbb{N}$.

**Lemma 6.** *Suppose that the system $\mathcal{P}$ satisfies the axiom (1). Then, the sequence $\{P_{n,\mathcal{P}} : n \in \mathbb{N}\}$ is tight.*

**Proof.** Let $K_l$ be a fixed compact set in the definition of $\rho$. Then, the Cauchy integral theorem, for $s \in K_l$, implies

$$\zeta_{\mathcal{P}}(s+i\tau) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\zeta_{\mathcal{P}}(z+i\tau)}{z-s}\, \mathrm{d}z,$$

where $\mathcal{L}$ is a closed simple curve lying in $D$ and enclosing the set $K_l$. Hence,

$$\sup_{s\in K_l}|\zeta_{\mathcal{P}}(s+i\tau)|^2 \ll \int_{\mathcal{L}} \frac{|\mathrm{d}z|}{|z-s|^2} \int_{\mathcal{L}} |\zeta_{\mathcal{P}}(z+i\tau)|^2|\mathrm{d}z| \ll_{K_l} \int_{\mathcal{L}} |\zeta_{\mathcal{P}}(\mathrm{Re}z+i\mathrm{Im}z+i\tau)|^2|\mathrm{d}z|.$$

Therefore,

$$\frac{1}{T}\int_0^T \sup_{s\in K_l}|\zeta_{\mathcal{P}}(s+i\tau)|^2\,\mathrm{d}\tau \ll_{K_l} \int_{\mathcal{L}} \left(\frac{1}{T}\int_0^T |\zeta_{\mathcal{P}}(\mathrm{Re}z+i\mathrm{Im}z+i\tau)|^2\,\mathrm{d}\tau\right)|\mathrm{d}z| \ll_{K_l} 1 \leqslant B_l < \infty.$$

From this, we have

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_l}|\zeta_{\mathcal{P}}(s+i\tau)|\,\mathrm{d}\tau \leqslant \sqrt{B_l}.$$

Then, in view of (9),

$$\begin{aligned}\sup_{n\in\mathbb{N}}\limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_l}|\zeta_{n,\mathcal{P}}(s+i\tau)|\,\mathrm{d}\tau \leqslant &\sup_{n\in\mathbb{N}}\limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_l}|\zeta_{\mathcal{P}}(s+i\tau)-\zeta_{n,\mathcal{P}}(s+i\tau)|\,\mathrm{d}\tau \\ &+ \limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_l}|\zeta_{\mathcal{P}}(s+i\tau)|\,\mathrm{d}\tau \\ \leqslant &C_l < \infty. \end{aligned} \tag{13}$$

Let $\beta_T$ be the random variable on the space $(\widehat{\Omega}, \mathcal{A}, \nu)$ and uniformly distributed in $[0, T]$. Define $\mathcal{H}(D_{\mathcal{P}})$-valued random elements

$$\xi_{T,n} = \xi_{T,n}(s) = \zeta_{n,\mathcal{P}}(s + i\beta_T)$$

and $\xi_n = \xi_n(s)$ having the distribution $P_{n,\mathcal{P}}$. We fix $\varepsilon > 0$, and set $V = V_l = 2^{-l}\varepsilon^{-1}C_l$. Then, in virtue of (13) and Lemma 2,

$$\begin{aligned}\nu\left\{\sup_{s\in K_l}|\xi_n(s)| \geqslant V_l\right\} &\leqslant \limsup_{T\to\infty} \nu\left\{\sup_{s\in K_l}|\xi_{T,n}(s)| \geqslant V_l\right\} \\ &\leqslant \sup_{n\in\mathbb{N}}\limsup_{T\to\infty} \frac{1}{V_l T}\int_0^T \sup_{s\in K_l}|\zeta_{n,\mathcal{P}}(s+i\tau)|\,\mathrm{d}\tau = \frac{\varepsilon}{2^l}\end{aligned} \tag{14}$$

for all $n \in \mathbb{N}$. Let $K = \left\{h \in \mathcal{H}(D_{\mathcal{P}}) : \sup_{s\in K_l} |h(s)| \leqslant V_l, l \in \mathbb{N}\right\}$. Then, $K$ is a compact set in $\mathcal{H}(D_{\mathcal{P}})$, and, by (14),

$$\begin{aligned}P_{n,\mathcal{P}}(K) &= 1 - P_{n,\mathcal{P}}(\mathcal{H}(D_{\mathcal{P}}) \setminus K) = 1 - P_{n,\mathcal{P}}\left(g(s) \in \mathcal{H}(D_{\mathcal{P}}) : \exists l : \sup_{s\in K_l} |g(s)| \geqslant V_l\right) \\ &= 1 - P_{n,\mathcal{P}}\left(\bigcup_{l=1}^{\infty}\left\{g(s) \in \mathcal{H}(D_{\mathcal{P}}) : \sup_{s\in K_l} |g(s)| \geqslant V_l\right\}\right) \\ &\geqslant 1 - \sum_{l=1}^{\infty} P_{n,\mathcal{P}}\left(g(s) \in \mathcal{H}(D_{\mathcal{P}}) : \sup_{s\in K_l} |g(s) \geqslant V_l\right) \\ &= 1 - \sum_{l=1}^{\infty} \nu\left\{\sup_{s\in K_l} |\xi_n(s)| \geqslant V_l\right\} \geqslant 1 - \varepsilon\sum_{l=1}^{\infty} 2^{-l} = 1 - \varepsilon\end{aligned}$$

for all $n \in \mathbb{N}$. This proves the lemma. $\square$

**Proof of Theorem 2.** We will apply Lemma 5. Since by Lemma 6, the sequence $\{P_{n,\mathcal{P}} : n \in \mathbb{N}\}$ is tight, it is relatively compact in virtue of the classical Prokhorov theorem; see, for example, [28], Theorem 6.1. This means that every subsequence of $\{P_{n,\mathcal{P}}\}$ possesses a subsequent weak convergent to a probability measure on $(\mathcal{H}(D_{\mathcal{P}}), \mathcal{B}(\mathcal{H}(D_{\mathcal{P}})))$. Thus, there is $\{P_{n_r,\mathcal{P}}\} \subset \{P_{n,\mathcal{P}}\}$ and a probability measure $P_{\mathcal{P}}$ on $(\mathcal{H}(D_{\mathcal{P}}), \mathcal{B}(\mathcal{H}(D_{\mathcal{P}})))$ such that $P_{n_r,\mathcal{P}}$ converges weakly to $P_{\mathcal{P}}$ as $r \to \infty$. Using the notation of the proof of Lemma 6, we have

$$\xi_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} P_{\mathcal{P}}. \tag{15}$$

Moreover, in view of Lemma 2,

$$\xi_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} \xi_n. \tag{16}$$

Define one more $\mathcal{H}(D_{\mathcal{P}})$-valued random element

$$\widehat{\xi}_T = \widehat{\xi}_T(s) = \zeta_{\mathcal{P}}(s + i\beta_T).$$

Then Lemma 4 implies that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{r\to\infty} \limsup_{T\to\infty} \nu\left\{\rho\left(\widehat{\xi}_T, \xi_{T,n}\right) \geqslant \varepsilon\right\} \\
&\quad = \lim_{r\to\infty} \limsup_{T\to\infty} \frac{1}{T} m_L\{\tau \in [0,T] : \rho(\zeta_{\mathcal{P}}(s+i\tau), \zeta_{n_r,\mathcal{P}}(s+i\tau)) \geqslant \varepsilon\} \\
&\quad \leqslant \lim_{r\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_0^T \rho(\zeta_{\mathcal{P}}(s+i\tau), \zeta_{n_r,\mathcal{P}}(s+i\tau))\, d\tau = 0.
\end{aligned}$$

This equality, together with (15) and (16), shows that for $\xi_{n_r}$, $\xi_{T,n}$ and $\widehat{\xi}_T$, the conditions of Lemma 5 are fulfilled. Therefore, the relation

$$\widehat{\xi}_T \xrightarrow[T\to\infty]{\mathcal{D}} P_{\mathcal{P}}$$

holds, and this implies the weak convergence of $P_{T,\mathcal{P}}$ to $P_{\mathcal{P}}$ as $T \to \infty$. The proof is completed. $\square$

## 5. Proof of Theorem 1

Theorem 1 is a consequence of Theorem 2 and the equivalents of weak convergence.

We remind the reader that the support of the measure $P_{\mathcal{P}}$ is a closed minimal set $S_{\mathcal{P}} \subset \mathcal{H}(D_{\mathcal{P}})$ satisfying $P_{\mathcal{P}}(S_{\mathcal{P}}) = 1$. The set $S_{\mathcal{P}}$ contains all $g \in \mathcal{H}(D_{\mathcal{P}})$ such that for any open neighborhood $\mathcal{G}$ of $g$, the inequality $P_{\mathcal{P}}(\mathcal{G}) > 0$ holds.

**Proof of Theorem 1.** Let $F_{\mathcal{P}}$ be the support of the limit measure $P_{\mathcal{P}}$ in Theorem 2. Then, $F_{\mathcal{P}}$ is a closed set, and $F_{\mathcal{P}} \neq \varnothing$ because $P_{\mathcal{P}}(F_{\mathcal{P}}) = 1$. We will prove that the set $F_{\mathcal{P}}$ has approximation properties of the theorem.

Suppose that $f(s) \in F_{\mathcal{P}}$, and

$$\mathcal{G}_\varepsilon = \left\{h \in \mathcal{H}(D_{\mathcal{P}}) : \sup_{s\in K} |h(s) - f(s)| < \varepsilon\right\},$$

i.e., $\mathcal{G}_\varepsilon$ is an open neighborhood of an element $f(s)$ of the support $F_{\mathcal{P}}$. Hence, by the support property,

$$P_{\mathcal{P}}(\mathcal{G}_\varepsilon) > 0. \tag{17}$$

Moreover, using Theorem 2 and Theorem 2.1 of [28] with open sets implies the inequality

$$\liminf_{T\to\infty} P_{T,\mathcal{P}}(\mathcal{G}_\varepsilon) \geqslant P_{\mathcal{P}}(\mathcal{G}_\varepsilon).$$

Thus, the notations for $P_{T,\mathcal{P}}$ and $\mathcal{G}_\varepsilon$ lead to

$$\liminf_{T\to\infty} \frac{1}{T} m_L \left\{ \tau \in [0,T] : \sup_{s\in K} |\zeta_{\mathcal{P}}(s+i\tau) - f(s)| < \varepsilon \right\} > 0.$$

To prove the second statement of the theorem, we deal with continuity sets. We remind the reader that a set $A \in \mathcal{B}(\mathbb{X})$ is a continuity set of a measure $P$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ if $P(\partial A) = 0$, where $\partial A$ is the boundary of $A$.

The set $\partial\mathcal{G}_\varepsilon$ of the set $\mathcal{G}_\varepsilon$ belongs to the set

$$\left\{ h \in \mathcal{H}(D_{\mathcal{P}}) : \sup_{s\in K} |h(s) - f(s)| = \varepsilon \right\}.$$

Hence, the sets $\partial\mathcal{G}_{\varepsilon_1}$ and $\partial\mathcal{G}_{\varepsilon_2}$ for different $\varepsilon_1$ and $\varepsilon_2$ have no common elements. From this remark, it follows that $P_{\mathcal{P}}(\partial\mathcal{G}_\varepsilon) > 0$ for at most countably many values of $\varepsilon$, or, in the above terminology, the set $\mathcal{G}_\varepsilon$ is a continuity set of the measure $P_{\mathcal{P}}$ for all but at most countably many $\varepsilon > 0$. Thus, Theorem 2 and Theorem 2.1 of [28] with continuity sets show that the limit

$$\lim_{T\to\infty} P_{T,\mathcal{P}}(\mathcal{G}_\varepsilon) = P_{\mathcal{P}}(\mathcal{G}_\varepsilon)$$

exists, and in view of (17), is positive for all but at most countably many $\varepsilon > 0$. This and the notations for $P_{T,\mathcal{P}}$ and $\mathcal{G}_\varepsilon$ give the second assertion of the theorem. The theorem is proved. □

## 6. Conclusions

Every system $\mathcal{P}$ of real numbers $1 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_r \leqslant \cdots$, $\lim_{n\to\infty} p_n = \infty$ is called generalized prime numbers. We consider the zeta-function $\zeta_{\mathcal{P}}(s)$, $s = \sigma + it$ associated with the system $\mathcal{P}$. We assume that the system of generalized integers $\mathcal{N}_{\mathcal{P}}$ obtained from $\mathcal{P}$ satisfies the axiom

$$\sum_{\substack{m\leqslant x \\ m\in\mathcal{N}}} 1 = ax + O(x^\delta), \quad a > 0,\ 0 \leqslant \delta < 1.$$

Then, for $\sigma > 1$, the function $\zeta_{\mathcal{P}}(s)$ is defined by

$$\zeta_{\mathcal{P}}(s) = \sum_{m\in\mathcal{N}} \frac{1}{m^s} = \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

and has analytic continuation to the region $\delta < \sigma < 1$. Additionally, we suppose that $\zeta_{\mathcal{P}}(s)$ has the bounded mean square

$$\int_0^T |\zeta_{\mathcal{P}}(\sigma+it)|^2 \, dt \ll_\sigma T, \quad T \to \infty,$$

for some $\sigma > \widehat{\sigma}$ with some $\delta < \widehat{\sigma} < 1$.

We consider probabilistic and approximation properties of the function $\zeta_{\mathcal{P}}(s)$. We prove a limit theorem for $\zeta_{\mathcal{P}}(s)$ in the space of analytic functions $\mathcal{H}(D_{\mathcal{P}})$, $D_{\mathcal{P}} = \{s \in \mathbb{C} : \widehat{\sigma} < \sigma < 1\}$, i.e., that

$$\frac{1}{T} m_L \{\tau \in [0,T] : \zeta_{\mathcal{P}}(s+i\tau) \in A\}, \quad A \in \mathcal{B}(H(D_{\mathcal{P}})),$$

converges weakly to a certain probability measure $P_{\mathcal{P}}$ as $T \to \infty$. From this, we deduce that the shifts $\zeta_{\mathcal{P}}(s + i\tau)$ approximate a certain closed subset of $\mathcal{H}(D_{\mathcal{P}})$.

For identification of the limit measure $P_{\mathcal{P}}$ and universality of the function $\zeta_{\mathcal{P}}(s)$, some stronger restrictions for the system $\mathcal{P}$ are needed. We are planning to apply this in the future.

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