Lietuvos matematikos rinkinys ISSN 0132-2818
Proc. of the Lithuanian Mathematical Society, Ser. A Vol. **57**, 2016
DOI: 10.15388/LMR.A.2016.05 pages 24–28

Nonlinear thermal conductivity in gases

Arvydas Juozapas Janavičius[1], **Sigita Turskienė**[2]

[1] *Faculty of Technology, Physical and Biomedical Sciences, Šiauliai University*
Vilniaus str. 141, LT-76353 Šiauliai, Lithuania

[2] *Faculty of Technology, Physical and Biomedical Sciences, Šiauliai University*
P. Višinskio str. 19, LT-77156 Šiauliai, Lithuania

E-mail: AYanavy@gmail.com, turskienes@gmail.com

Abstract. The nonlinear diffusion equation corresponds to the diffusion processes which can occur with a finite velocity. A.J. Janavičius proposed nonlinear equation which describes more exactly the diffusion of impurities in Si crystals in many interesting practical applications. The heat transfer in gases is also based on diffusion of gas molecules from hot regions to the coldest ones with a finite velocity by random Brownian motions. In this case the heat transfer can be considered using similar nonlinear thermal diffusivity equation. The approximate analytical solution of this nonlinear equation can be used for the experimental analysis of thermal conductivity coefficients using temperature profiles dependence on different temperatures and pressures in gases.

Keywords: nonlinear thermal diffusivity equation, approximate analytical solution, temperature profiles.

Introduction

We discussed the thermodiffusion in semiconductors [5] excited by ultraviolet or X-rays [2] and metals heated by lasers [8]. The mathematical methods derived in papers [5, 2, 8, 4] for the formulation and solution of the nonlinear heat transfer equations in gases have been used. These results can be important for engineering applications. We assume that the process of heat spreading is similar to other diffusion processes, and in the nonlinear case, can be described by nonlinear flow density [4]. In this case the frequency of the jumps [7] depends upon the molecule coordinates, concentration and temperature.

The coefficient of thermal conductivity of gases can be expressed in the following way [9]

$$K = \frac{1}{2} nk\lambda\overline{v} = \frac{k\overline{v}}{2\sqrt{2}\pi d^2}, \qquad \lambda = \frac{1}{\sqrt{2}\pi d^2 n}, \qquad \overline{v} = \sqrt{\frac{8RT}{\pi\mu}}. \tag{1}$$

Here λ is mean values of a free path, $\overline{v}$ – mean velocities of molecular movement, n – number of molecules per unit volume, k – Boltzmann constant, T – temperature of gases, μ – molar mass, R - gas constant, d – diameter of a molecule.

The equation of thermal conductivity of gases [9] for one-dimensional case

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(D_m(T)\frac{\partial T}{\partial x}\right), \qquad D_m(T) = \frac{K}{\rho c_p} = \frac{k\overline{v}}{2\sqrt{2}\pi d^2 p} T = D_{m0} T \tag{2}$$

can be obtained from the heat flow continuity equation [9]

$$\frac{\partial T}{\partial t} = -\mathrm{div}\big(-D_m(T)\mathrm{grad}(T)\big) \tag{3}$$

where $p = nkT$, c_p is the specific heat capacity of gases at constant pressure p, ρ is density of gases, D_m – coefficient of thermal diffusion.

The equation (2) can be rewritten in more a convenient form

$$\frac{\partial T}{\partial t} = D_{m0}\left[\frac{\partial}{\partial x}\left(T\frac{\partial T}{\partial x}\right)\right]. \tag{4}$$

The nonlinear heat conduction equation [10] can be rewritten by introducing nonlinear equation

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial x}\left(k_e(E)\frac{\partial E}{\partial x}\right) \tag{5}$$

for energy density E.

The complicated approximate analytical solution [10] $E(x,t)$ of the equation (5) cannot be experimentally measured. In our case (4) temperatures $T(x,t)$ can be measured directly and compared to the theoretical calculations.

1 Nonlinear heat diffusion equation in one-dimensional case

The solution of (4) can be obtained by introducing similarity variable [4] ξ and function $f(\xi)$

$$\xi = \frac{x}{\sqrt{(D_{m0}T_e t)}}, \qquad T = T_e f(\xi), \quad 0 \leqslant \xi \leqslant \xi_0,\ 0 \leqslant x \leqslant x_0,\ x_0 = \xi_0\sqrt{(D_{m0}T_e t)} \tag{6}$$

which depends on environment temperature T_e and constant D_{m0}.

By substituting (6) into (4) we obtain nonlinear differential equation

$$2\frac{\partial}{\partial \xi}\left(f\frac{\partial f}{\partial \xi}\right) + \xi\frac{\partial f}{\partial \xi} = 0. \tag{7}$$

The solution of this nonlinear equation can be expanded by power series including boundary condition at $\xi = \xi_0$

$$f(\xi) = \sum_{n=1}^{\infty} a_n(\xi-\xi_0)^n, \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z = \xi - \xi_0,\ -\xi_0 \leqslant z \leqslant 0. \tag{8}$$

The equation (7) can be applied to $f(z)$ for the modified variable z

$$2\frac{\partial}{\partial z}\left(f\frac{\partial f}{\partial z}\right) + z\frac{\partial f}{\partial z} + \xi_0\frac{\partial f}{\partial z} = 0. \tag{9}$$

By requiring that solution $f(z)$ (8) of nonlinear equation (9) can be expressed by power series, we obtain recurrences relations [3] between coefficients a_n

$$2(n+1)\sum_{m=0}^{n+1}(n+2-m)a_{n+2-m}a_m + na_n + (n+1)\xi_0 a_{n+1} = 0, \quad n = 0,1,2,\dots \tag{10}$$

Table 1. Dependence of parameters a_1, a_2, ξ_0 for the approximate solution $f(z)$ (16) on relative temperature differences $\frac{\Delta T}{T_e}$ at heat sources.

$\frac{\Delta T}{T_e}$	a_1	a_2	ξ_0
0.05	−0.446843	−0.087049	0.114448
0.1	−0.481917	−0.090069	0.216244
0.2	−0.544857	−0.094755	0.394076

2 The approximate analytical solution

We restrict the approximation (8) by polynomials. From the formula (10) at $n = 0, 1, 2$ we obtain the following system of equations:

$$4a_2a_0 + 2{a_1}^2 + \xi_0 a_1 = 0, \quad a_0 = 1, \tag{11}$$

$$12a_3a_0 + 12a_1a_2 + a_1 + 2\xi_0 a_2 = 0, \tag{12}$$

$$24a_4a_0 + 24a_3a_1 + 12a_2^2 + 2a_2 + 3\xi_0 a_3 = 0, \quad a_4 = 0, \tag{13}$$

where in (13) was used approximation $a_4 = 0$ and $a_0 = 1$ for boundary condition $f(0) = 1$.

The solution $f(z)$ (8) must satisfy equation (9) and boundary conditions

$$T(z) = T_e f(0) = T_e, \quad a_0 = 1 \tag{14}$$

at maximum penetration point of heat when $T(\xi = \xi_0) = T_e$. The approximate expression of temperature T_s at heat source is

$$T = T_e f(-\xi_0) = T_s, \qquad \sum_{m=1}^{n+1} (-1)^m a_m {\xi_0}^m = \frac{\Delta T}{T_e}, \qquad \Delta T = T_s - T_e. \tag{15}$$

By using the approximate solution

$$f(z) = a_0 + a_1 z + a_2 z^2, \quad a_0 = 1, \tag{16}$$

we obtain the following system of equations

$$4a_2a_0 + 2{a_1}^2 + \xi_0 a_1 = 0, \quad a_0 = 1, \tag{17}$$

$$12a_3a_0 + 12a_1a_2 + a_1 + 2\xi_0 a_2 = 0, \tag{18}$$

$$a_2\xi_0^2 - a_1\xi_0 = \frac{T_s}{T_e} - a_0, \qquad \frac{T_s}{T_e} - 1 = \frac{T_s - T_e}{T_e} = \frac{\Delta T}{T_e}. \tag{19}$$

Taking in the care that values of $\frac{\Delta T}{T_e}$ are sufficiently small, we can find from (17), (18), (19) the following approximate solutions (9) with constants presented in Table 1.

In Table 1 the approximate solution (8) of the equation (9) for temperature differences $\Delta T_1 = T_s - T_e = 0.05T_e$, $\Delta T_2 = 0.1T_e$ and $\Delta T_3 = 0.2T_e$ are presented for source temperature T_s and environmental temperature T_e.

The constants a_1, a_2 practically do not change significantly at different $\frac{\Delta T}{T_e}$ what means that approximate solution is sufficiently exact for practical calculations of temperatures. The constant ξ_0 defining the maximum heat penetration depth x_0

$$x_0 = \xi_0 \sqrt{(}D_{m0}T_e t). \tag{20}$$

Table 2. Dependence of parameters a_1, a_2, a_3, ξ_0 for the approximate solution $F(z)$ of (8) on relative temperature difference at heat source $\frac{\Delta T}{T_e}$.

$\frac{\Delta T}{T_e}$	a_1	a_2	a_3	ξ_0
0.05	−0.478967	−0.101916	−0.007086	0.106800
0.1	−0.509157	−0.103549	−0.006758	0.204821
0.2	−0.565578	−0.106188	−0.006198	0.380151

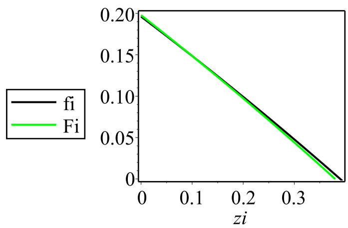

Fig. 1. Profiles f_i and F_i for approximate solutions $(f(z)-1)$ and $(F(z)-1)$.

which is directly proportional to square root from temperature of environment. The obtained solution of the equation (9) presented in Table 1 and experimental heat penetration depths can be used for defining the heat thermal diffusion coefficients $D_{m0}T_e$ $[m^2 s^{-1}]$. In this way the dependence of D_{m0} on temperature at constant pressure p (2) can be obtained.

We can find a more exact solution by the system of equations (11), (12) and (13), when $a_4 = 0$ and the boundary condition (15) is as follows

$$-a_3{\xi_0}^3 + a_2{\xi_0}^2 - a_1\xi_0 = \frac{\Delta T}{T_e}, \qquad \Delta T = T_s - T_e. \tag{21}$$

The results for more exact solutions $F(z)$ are presented in Table 2. For graphically representation in Fig. 1 we introduced the new functions f_i and F_i instead $(f-1)$ and $(F-1)$. We obtained that profiles f_i and F_i for obtained approximate solutions $f(z)$ and $F(z)$ presented in Table 1 and Table 2 practically coincide, when at heat source we used $(f-1) \approx (F-1) = (T_s - T_e)/T_e = 0.2$.

3 Results and conclusions

A similar task and approach was considered for nonlinear diffusion [4, 1, 6] in gases. In this case the definition of diffusion coefficients, which can be used at average values of frequencies of molecule jumps in the frontier region of diffusion profiles, was improved. For practical calculation of temperature profiles, the coefficients a_1, a_2, a_3 at $\Delta T/T_e = 0.1$ sufficient exact (21) and average values of $\overline{v}$ for approximate D_{m0} (2), the evaluation can be used. For definition of D_{m0} values dependencies on temperatures and pressures, the values $_1, a_2, a_3$ presented in Table 2 at $\Delta T/T_e = 0.05$ can be used. The results presented in Table 1 and Table 2 show that heat penetration depths (19) are approximately proportional to $\Delta T/T_e$ values.

References

[1] B.F. Apostol. On a non-linear describing clouds and wreaths of smoke. *Phys. Lett. A*, **235**:363–366, 1997.

[2] S. Balakauskas, A.J. Janavičius, V.Kazlauskienė, A. Mekys, R. Purlys and J. Storasta. Superdiffusion in si crystal lattice irradiated by soft x-rays. *Acta Phys. Pol. A*, **114**:779–790, 2008.

[3] P.F. Filchakov. *Handbook of High Mathematics*. Scientific Thought, Kiev, 1973.

[4] A.J. Janavičius. Method for solving the nonlinear diffusion equation. *Phys. Lett. A*, **224**:159–162, 1997.

[5] A.J. Janavičius. The nonlinear diffusion in the nonisothermical case. *Acta Phys. Pol. A*, **93**:505–512, 1998.

[6] A.J. Janavičius, G. Luza and D. Jurgaitis. The nonlinear diffusion equation desribing spread of impurities of high density. *Acta Phys. Pol. A*, **107**:475–483, 2004.

[7] A.J. Janavičius and A. Poškus. Nonlinear diffusion equation with diffusion coefficient directly proportional to concentration of impurities. *Acta Phys. Pol. A*, **107**:519–521, 2005.

[8] A.J. Janavičius and S. Turskienė. Modeling of thermodiffusion inertia in metal films heated with ultrashort laser pulses. *Acta Phys. Pol. A*, **110**:511–521, 2006.

[9] G. Joos and I.M. Freeman. *Theoretical Physics*. Dover Publications, Inc., New York, 1986.

[10] J. Yu, Y. Yang and A. Campo. Approximate solution of the nonlinear heat conduction equation in a semi-infinite domain. *Math. Probl. Eng.*, **2010**:24–26, 2010.

REZIUMĖ

Netiesinis šilumos laidumas dujose

A. J. Janavičius, S. Turskienė

Panašus netiesinės priemaišų difuzijos kietuose kūnuose uždavinys anksčiau išnagrinėtas įvedus difuzijos koeficientą proporcingą difunduojančių priemašų koncentracijai. Tai patikslina difuzijos proceso modelį ir teorinius priemaišų pasiskirstymo profilius gaunant baigtinį difuzijos greitį. Šiame darbe gautas termodifuzijos koeficientas proporcingas absoliutinei temperatūrai, naudojant klasikinę šilumos laidumo teoriją dujose. Taip pat gaunamas baigtinis šilumos sklidimo greitis (20). Gauti apytiksliai netiesinės termodifuzijos lygties (4) sprendiniai leidžia nustatyti termodifuzijos koeficientus, nagrinėjant eksperimentinius temperatūrų profilius intervale (6) ir jų priklausomybę nuo temperatūros ir slėgio.

Raktiniai žodžiai: netiesinė termodifuzijos lygtis, apytikslis analizinis sprendinys, temperatūros profilis.