

A Joint Elliott Type Theorem for Twists of L -Functions of Elliptic Curves

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Abstract. We consider a collection of L -functions of elliptic curves twisted by a Dirichlet character modulo q (q is a prime number), and prove for this collection a joint limit theorem for weakly convergent probability measures in the space of analytic functions as $q \rightarrow \infty$. The limit measure is given explicitly.

Keywords: Dirichlet character, elliptic curve, L -function of elliptic curve, probability measure, weak convergence.

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1 Introduction

Let E be an elliptic curve over the field of rational numbers given by the Weierstrass equation

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

with discriminant $\Delta = -16(4a^3 + 27b^2) \neq 0$. For each prime number p , denote by E_p the reduction modulo p of the curve E which is a curve over the finite field \mathbb{F}_p , and define the integer $\lambda(p)$ by the equality

$$|E(\mathbb{F}_p)| = p + 1 - \lambda(p),$$

where $|E(\mathbb{F}_p)|$ is the number of points of curve E_p . The L -function $L_E(s)$, $s = \sigma + it$, of the curve E is defined by the Euler product

$$L_E(s) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1},$$

which, in virtue of the estimate

$$|\lambda(p)| \leq 2\sqrt{p}$$

is absolutely convergent for $\sigma > 3/2$. Moreover, the function $L_E(s)$ is analytically continued to an entire function, see, for example, [12].

Now let χ be a Dirichlet character modulo q . Then the twist $L_E(s, \chi)$ of the function $L_E(s)$ is defined, for $\sigma > \frac{3}{2}$, by the Euler product

$$L_E(s, \chi) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s-1}}\right)^{-1}$$

and can be expanded in the Dirichlet series

$$\sum_{m=1}^{\infty} \frac{\lambda(m)\chi(m)}{m^s}.$$

In the sequel, we assume that q is a prime number. Then it was observed in [12] that the function $L_E(s, \chi)$, as $L_E(s)$, is also entire one.

Limit theorems for $L_E(s, \chi)$ with increasing q were began to study in [6], [7], [8], [9], however, only in the half-plane of absolute convergence $\sigma > \frac{3}{2}$. In [12], a limit theorem in the space of analytic functions $H(D)$, $D = \{s \in \mathbb{C} : \sigma > 1\}$, for the function $L_E(s, \chi)$ has been obtained. For its statement, we need some notation and definitions. For $Q \geq 2$, let

$$M_Q = \sum_{q \leq Q} \sum_{\substack{\chi = \chi(\text{mod } q) \\ \chi \neq \chi_0}} 1,$$

where, as usual, χ_0 denotes the principal character modulo q . It is well known that

$$M_Q = \frac{Q^2}{2 \log Q} + \mathcal{O}\left(\frac{Q^2}{\log^2 Q}\right).$$

Denote by γ the unite circle $\{s \in \mathbb{C} : |s| = 1\}$, and define $\Omega = \prod_p \gamma_p$, where $\gamma_p = \gamma$ for all primes p . The infinite-dimensional torus Ω with the product topology and operation of pointwise multiplication is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(X)$ is the Borel σ -field of the space X , the probability Haar measure m_H exists, and this gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of an element $\omega \in \Omega$ to the coordinate space γ_p , and, on $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$ -valued random element $L_E(s, \omega)$ by the formula

$$L_E(s, \omega) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\omega(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta} \left(1 - \frac{\lambda(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1}.$$

Let P_{L_E} be the distribution of $L_E(s, \omega)$, i. e.,

$$P_{L_E}(A) = m_H\{\omega \in \Omega : L_E(s, \omega) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Then the main result of [12] is the following limit theorem.

Theorem 1. *Suppose that $Q \rightarrow \infty$. Then*

$$\frac{1}{M_Q} \#\{\chi(\text{mod } q), q \leq Q, \chi \neq \chi_0 : L_E(s, \chi) \in A\}, \quad A \in \mathcal{B}(H(D))$$

converges weakly to P_{L_E} .

Here $\#A$ stands for the cardinality of the set A .

In [8], a joint limit theorem of type of Theorem 1 has been obtained for a collection of moduli of twists of L -functions of elliptic curves, however, in the region $\sigma > \frac{3}{2}$, only. We note that P. D. T. A. Elliott was the first who began to study limit theorems with increasing modulus for Dirichlet L -functions [4], [5].

The aim of this paper is a multidimensional analogue of Theorem 1. For $j = 1, \dots, r$, let E_j be an elliptic curve over the field of rational numbers given by the equation

$$y^2 = x^3 + a_j x + b_j, \quad a_j, b_j \in \mathbb{Z}$$

with discriminant $\Delta_j = -16(4a_j^3 + 27b_j^2) \neq 0$. Consider the corresponding L -function

$$L_{E_j}(s) = \prod_{p|\Delta_j} \left(1 - \frac{\lambda_j(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta_j} \left(1 - \frac{\lambda_j(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}.$$

Suppose that N_j is the conductor of the curve E_j . Then, by the Weil-Shimura-Taniyama conjecture proved in [3], see also Theorem 14.6 of [10], the function $L_{E_j}(s)$ coincides with L -function of a new cusp form of weight 2 and level N_j . This shows that $L_{E_j}(s)$ is an entire function.

By Theorem 14.20 of [10], the twist $L_{E_j}(s, \chi)$ of $L_{E_j}(s)$ with a character χ modulo q is again a new cusp form of weight 2 and level $N_j q^2$. Therefore, the function $L_{E_j}(s, \chi)$ is also an entire function.

Let, for brevity, $\underline{E} = (E_1, \dots, E_r)$, $\underline{L}_{\underline{E}}(s, \chi) = (L_{E_1}(s, \chi), \dots, L_{E_r}(s, \chi))$ and $A_Q = \{\chi(\text{mod } q) : q \leq Q, \chi \neq \chi_0\}$. Moreover, define

$$\underline{L}_{\underline{E}}(s, \omega) = (L_{E_1}(s, \omega), \dots, L_{E_r}(s, \omega)),$$

where, for $j = 1, \dots, r$,

$$L_{E_j}(s, \omega) = \prod_{p|\Delta_j} \left(1 - \frac{\lambda_j(p)\omega(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta_j} \left(1 - \frac{\lambda_j(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1}.$$

Denote by $P_{\underline{L}_{\underline{E}}}$ the distribution of the $H^r(D)$ -valued random element $\underline{L}_{\underline{E}}(s, \omega)$, i. e.,

$$P_{\underline{L}_{\underline{E}}}(A) = m_H \left\{ \omega \in \Omega : \underline{L}_{\underline{E}}(s, \omega) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)).$$

Then we have the following statement.

Theorem 2. *Suppose that $Q \rightarrow \infty$. Then*

$$P_{Q, \underline{E}}(A) \stackrel{\text{def}}{=} \frac{1}{M_Q} \#\{\chi \in A_Q : \underline{L}_{\underline{E}}(s, \chi) \in A\}, \quad A \in \mathcal{B}(H^r(D))$$

converges weakly to $P_{\underline{L}_{\underline{E}}}$.

2 Auxiliary results

We start with a joint limit theorem which is a generalization of Lemma 1 from [12]. We extend the function $\omega(p)$ to the set \mathbb{N} by the formula

$$\omega(m) = \sum_{p^l | m, p^{l+1} \nmid m} \omega^l(p), \quad m \in \mathbb{N}.$$

Lemma 1. For $j = 1, \dots, r$, let $\{a_{m,j} : m \in \mathbb{N}\}$ be a sequence of complex numbers such that

$$\sum_{m \leq n} |a_{m,j}|^2 = \mathcal{O}(n^{2\alpha}), \quad \alpha > 0$$

as $n \rightarrow \infty$, and, for $\omega \in \Omega$ and $\sigma > \alpha + \frac{1}{2}$, let $X_j(s, \omega) = \sum_{m=1}^{\infty} \frac{a_{m,j}\omega(m)}{m^s}$.

Suppose that $\{A_m : m \in \mathbb{N}\}$ is a sequence of finite subsets of the torus Ω such that, for each $\omega \in \bigcup_{m=1}^{\infty} A_m$, $X_j(s, \omega)$ has an analytic continuation to the half plane $D_\alpha = \{s \in \mathbb{C} : \sigma > \alpha\}$ satisfying the following conditions:

$$1^0 \text{ As } |t| \rightarrow \infty,$$

$$\frac{1}{\#A_m} \sum_{\omega \in A_m} |X_j(\sigma + it, \omega)|^2 = \mathcal{O}(|t|^A), \quad A > 0$$

uniformly for $m \in \mathbb{N}$ and σ in compact subsets of the interval (α, ∞) ;

$$\sum_{\omega \in A_m} |X_j(\sigma, \omega)|^2 = \mathcal{O}(\#A_m)$$

as $m \rightarrow \infty$, uniformly for s on compact subsets of D_α ;

Moreover, suppose that

$$\frac{\#\{A \cap A_m\}}{\#A_m}, \quad A \in \mathcal{B}(\Omega)$$

converges weakly to the Haar measure m_H . Then

$$\frac{1}{\#A_m} \#\{\omega \in A_m : (X_1(s, \omega), \dots, X_r(s, \omega)) \in A\},$$

where $A \in H^r(D_\alpha)$, converges weakly to the distribution of the random element $(X_1(s, \omega), \dots, X_r(s, \omega))$ as $m \rightarrow \infty$.

Proof. A way of the proof is completely analogical to that in one-dimensional case presented in [2], Proposition 4.4.1. In our case, the metric in $H^r(D_\alpha)$ inducing its topology of uniform convergence on compacta is applied, and the joint case is reduced to the one-dimensional case. \square

The next lemma is devoted to checking the hypotheses of Lemma 1, and contains an approximate functional equation of L -functions of cusp forms of weight 2 and level N . Let $F(z)$ be a new form of weight 2 and level N with

Fourier coefficients $c(m)$. Moreover, let $\Gamma(s)$, as usual, denote the gamma-function, and let $\Gamma(s, z)$ be the incomplete gamma-function,

$$\Gamma(s, z) = \int_z^\infty e^{-t} t^{s-1} dt, \quad \sigma > 0, \quad z \in \mathbb{R}.$$

Lemma 2. [1]. *Suppose that $L(s, F)$, $s = \sigma + it$, is the L-function associated to the form F , $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, $M > \frac{t\sqrt{N}}{4}$, $r = e^{i(\frac{\pi}{2} - \delta(t))}$ with $0 < \delta(t) \leq \frac{\pi}{2}$. Then*

$$\begin{aligned} L(s, F) &= \frac{1}{\Gamma(s)} \sum_{m \leq M} \frac{c(m)}{m^s} \Gamma\left(s, \frac{2\pi mr}{\sqrt{N}}\right) \\ &\quad - \frac{\mu N^{1-s} (2\pi)^{2(s-1)}}{\Gamma(s)} \sum_{m \leq M} \frac{c(m)}{m^{2-s}} \Gamma\left(2-s, \frac{2\pi m}{\sqrt{N}r}\right) + \frac{(2\pi)^s}{\Gamma(s)} R, \end{aligned}$$

where

$$\begin{aligned} |R| &< e^{-\frac{\pi t}{2}} e^{\delta(t)\left(t - \frac{4M}{\sqrt{N}}\right)} N^{\frac{1-\sigma}{2}} \sqrt{M} \delta^{-1}(t) \\ &\quad \times \left(1 + \frac{\log M + \sigma + 1}{2t\delta(t)} + \frac{(\sigma - 1)(\log M + 2)}{4(t\delta(t))^2}\right). \end{aligned}$$

Let G be a compact Abelian group. Then, on $(G, \mathcal{B}(G))$, the probability Haar measure μ can be defined. We recall that a sequence $\{x_m : m \in \mathbb{N}\} \subset G$ is said to be uniformly distributed if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n f(x_m) = \int_G f d\mu$$

for any real bounded Borel measurable function f .

The next lemma is a criterion of uniform distribution for sequences in G .

Lemma 3. *The sequence $\{x_m : m \in \mathbb{N}\} \subset G$ is uniformly distributed in G if and only if, for any nontrivial character χ_G , the equality*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n \chi_G(x_m) = 0$$

holds.

The proof of the lemma can be found in [11], Chapter 4, Corollary 1.2.

3 Proof of Theorem 2

In the notation of Lemma 1, we have that $a_{mj} = \lambda_j(m)\chi(m)$. Since $\lambda_j(m)$ coincides with Fourier coefficients of a new cusp form, we have, by the estimate (14.53) from [10], that

$$\sum_{m \leq n} |a_{mj}|^2 = O(n^2).$$

Therefore, $\alpha = 1$ in Lemma 1.

Denote by \mathbb{P} the set of all prime numbers. Let χ be a Dirichlet character modulo $q \in \mathbb{P}$, and

$$\hat{\chi}(p) = \begin{cases} \chi(p), & \text{if } p \in \mathbb{P} \setminus \{q\}, \\ 1, & \text{if } p = q. \end{cases}$$

Then $\{\hat{\chi}(p) : p \in \mathbb{P}\}$ is an element of the torus Ω . Putting

$$l_{E_j, q}(s) = \begin{cases} 1 - \frac{\lambda_j(q)}{q^s}, & \text{if } q \mid \Delta, \\ 1 - \frac{\lambda_j(q)}{q^s} + \frac{1}{q^{2s-1}}, & \text{if } q \nmid \Delta, \end{cases}$$

we have that

$$L_{E_j}(s, \chi) = l_{E_j, q}(s)L_{E_j}(s, \hat{\chi}). \tag{3.1}$$

Denote by p_m the m th prime number, and define $A_m = \{\chi(\text{mod } p_m) : \chi \neq \chi_0\}$. Obviously, $\#A_m = p_m - 2$. Define one more set $\hat{A}_m = \{\hat{\chi} : \chi \in A_m\}$. On a certain probability space $(\hat{\Omega}, \mathcal{A}, P)$, define the $H^r(D)$ -valued random elements $\underline{X}_m(s)$ and $\hat{\underline{X}}_m(s)$ by the formulae

$$P(\underline{X}_m(s) = L_{\underline{E}}(s, \chi)) = \frac{1}{p_m - 2}, \quad \chi \in A_m,$$

$$P(\hat{\underline{X}}_m(s) = L_{\underline{E}}(s, \hat{\chi})) = \frac{1}{p_m - 2}, \quad \hat{\chi} \in \hat{A}_m.$$

By the definition of $l_{E_j, p_m}(s)$, we see that

$$l_{E_j, p_m}(s) \rightarrow 1 \tag{3.2}$$

in the space $H(D)$ as $m \rightarrow \infty$. Hence, $\underline{X}_m(s)$ converges in distribution to $\hat{\underline{X}}_m$ as $m \rightarrow \infty$. Therefore, for the proof that

$$Q_q(A) \stackrel{\text{def}}{=} \frac{1}{q-2} \#\{\chi(\text{mod } q), \chi \neq \chi_0 : \underline{L}(s, \chi) \in A\}, \quad A \in \mathcal{B}(H^r(D))$$

converges weakly to $P_{\underline{L}}$ as $q \rightarrow \infty$ it suffices to obtain that the random element $\hat{\underline{X}}_m$ converges in distribution to $P_{\underline{L}_{\underline{E}}}$. Thus, the sequence $\{\hat{A}_m : m \in \mathbb{N}\}$ corresponds the sequence $\{A_m : m \in \mathbb{N}\}$ in Lemma 1, and $\underline{X}(s, \omega) = \underline{L}_{\underline{E}}(s, \hat{\chi})$. Moreover, in virtue of (3.1) and (3.2), $\underline{L}_{\underline{E}}(s, \hat{\chi})$ can be replaced by $\underline{L}_{\underline{E}}(s, \chi)$.

It remains to check other hypotheses of Lemma 1. Let K be a compact subset of $(1, \infty)$. We continue with estimate for

$$D_q(\sigma, t) \stackrel{\text{def}}{=} \frac{1}{q-2} \sum_{\chi(\text{mod } q)} |L_{E_j}(\sigma + it, \chi)|^2, \quad t > 0,$$

when q runs prime numbers. For this, we apply Lemma 2 with $M = ct\sqrt{N_j}$ $\times q \log^2 q$, $c > 0$, and $\delta = t^{-1}$. We have

$$D_q(\sigma, t) \ll_K D_{1,q}(\sigma, t) + D_{2,q}(\sigma, t) + D_{3,q}(\sigma, t), \tag{3.3}$$

where

$$\begin{aligned}
 D_{1,q}(\sigma, t) &= \frac{1}{q} \sum_{\chi(\bmod q)} \frac{1}{|\Gamma(s)|^2} \left| \sum_{m \leq M} \frac{\lambda(m)\chi(m)}{m^{\sigma+it}} \Gamma\left(\sigma + it, \frac{2\pi mr}{\sqrt{N_j q}}\right) \right|^2 \\
 &\ll \frac{1}{|\Gamma(s)|^2} \sum_{m \leq M} \frac{|\lambda(m)|^2}{m^{2\sigma}} \left| \Gamma\left(\sigma + it, \frac{2\pi mr}{\sqrt{N_j q}}\right) \right|^2 \frac{1}{q} \sum_{\chi(\bmod q)} |\chi(m)|^2 \\
 &+ \frac{1}{|\Gamma(s)|^2} \sum_{\substack{m \leq M \\ n \leq M \\ m \neq n}} \frac{|\lambda(m)||\lambda(n)|}{m^\sigma n^\sigma} \left| \Gamma\left(\sigma + it, \frac{2\pi mr}{\sqrt{N_j q}}\right) \right| \\
 &\quad \times \left| \Gamma\left(\sigma + it, \frac{2\pi nr}{\sqrt{N_j q}}\right) \right| \left| \frac{1}{q} \sum_{\chi(\bmod q)} \chi(m)\overline{\chi(n)} \right|, \tag{3.4}
 \end{aligned}$$

$$\begin{aligned}
 D_{2,q}(\sigma, t) &= \frac{1}{q} \sum_{\chi(\bmod q)} \frac{N_j^{2-2\sigma} q^{4-4\sigma}}{|\Gamma(s)|^2} \left| \sum_{m \leq M} \frac{\lambda(m)\chi(m)}{m^{2-\sigma-it}} \Gamma\left(2 - \sigma - it, \frac{2\pi m}{\sqrt{N_j qr}}\right) \right|^2 \\
 &= \frac{N_j^{2-2\sigma} q^{4-4\sigma}}{|\Gamma(s)|^2} \sum_{m \leq M} \frac{|\lambda(m)|^2}{m^{4-2\sigma}} \left| \Gamma\left(2 - \sigma - it, \frac{2\pi m}{\sqrt{N_j qr}}\right) \right|^2 \frac{1}{q} \sum_{\chi(\bmod q)} |\chi(m)|^2 \\
 &+ \frac{N_j^{2-2\sigma} q^{4-4\sigma}}{|\Gamma(s)|^2} \sum_{\substack{m \leq M \\ n \leq M \\ m \neq n}} \frac{|\lambda(m)||\lambda(n)|}{m^{2-\sigma} n^{2-\sigma}} \left| \Gamma\left(2 - \sigma - it, \frac{2\pi m}{\sqrt{N_j qr}}\right) \right| \\
 &\quad \times \left| \Gamma\left(2 - \sigma - it, \frac{2\pi n}{\sqrt{N_j qr}}\right) \right| \left| \frac{1}{q} \sum_{\chi(\bmod q)} \chi(m)\overline{\chi(n)} \right|, \tag{3.5}
 \end{aligned}$$

$$D_{3,q}(\sigma, t) = \frac{1}{|\Gamma(s)|^2} R^2. \tag{3.6}$$

Let $d(m) = \sum_{k|m} 1$ be the divisor function. Using the well - known bounds

$$|\lambda(m)| \leq \sqrt{m}d(m), \quad \sum_{m \leq x} d^2(m) \ll x \log^4 x$$

as well as [1]

$$|\Gamma(\sigma + it, Ar)| \ll A^\sigma e^{-(\frac{\pi}{2} - \delta(t))t},$$

and the properties of the gamma-function, we find that the first term in the right-hand side of (3.4) is estimated as

$$\ll_K q^{2-2\sigma} (\log q)^{c_1} t^{A_1} \ll_K t^{A_1}, \quad c_1 > 0, \quad A_1 > 0 \tag{3.7}$$

uniformly in $\sigma \in K$ and q . Moreover, in view of the equalities

$$\sum_{\chi(\bmod q)} \chi(m)\overline{\chi(n)} = \begin{cases} q - 1, & \text{if } m \equiv n(\bmod q), \\ 0, & \text{if } m \not\equiv n(\bmod q) \end{cases} \tag{3.8}$$

and the bound $d(m) \ll m^\varepsilon$, $\varepsilon > 0$, we obtain that the second term in the right-hand side of (3.4) has a bound

$$\begin{aligned} &\ll \sum_{\substack{m \leq M \\ m \equiv n \pmod{q}}} \sum_{n \leq M} \frac{|\lambda(m)||\lambda(n)|}{m^\sigma n^\sigma} \left(\frac{m}{q}\right)^\sigma \left(\frac{n}{q}\right)^\sigma \\ &\ll \frac{1}{q^{2\sigma}} \sum_{k \leq M/q} \sum_{m \leq M} \sqrt{md(m)}\sqrt{m+kq} d(m+kq) \\ &\ll q^{2-2\sigma+\varepsilon} t^{A_2} \ll_K t^{A_2}, \quad A_2 > 0 \end{aligned}$$

uniformly in $\sigma \in K$ and q . This together with (3.7) shows that

$$D_{1,q}(\sigma, t) \ll_K t^{A_3}, \quad A_3 > 0 \tag{3.9}$$

uniformly in $\sigma \in K$ and q .

In a similar way, we obtain that

$$D_{2,q}(\sigma, t) \ll_K t^{A_4}, \quad A_4 > 0 \tag{3.10}$$

uniformly in $\sigma \in K$ and q . The definition of R and the choice of M and $\delta(t)$ imply the estimate

$$D_{3,q}(\sigma, t) \ll_K e^{8c \log^2 q} q^{1-\sigma} \sqrt{q} (\log q)^{c_3} t^{A_5} \ll_K t^{A_5}, \quad A_5 > 0$$

uniformly in $\sigma \in K$ and q . From this, (3.9), (3.10), and (3.3) we have that

$$D_q(\sigma, t) \ll_K t^A, \quad A > 0$$

uniformly in $\sigma \in K$ and q . Obviously, then

$$\frac{1}{q-2} \sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} |L_{E_j}(\sigma + it, \chi)|^2 \ll_K |t|^A$$

uniformly in $\sigma \in K$ and q .

In a similar manner, we obtain that

$$\sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} |L_{E_j}(\sigma, \chi)|^2 = \mathcal{O}(q)$$

as $q \rightarrow \infty$, uniformly for s on compact subsets of the half-plane D .

Next we will consider the sequence $\{\hat{A}_m : m \in \mathbb{N}\}$, and will prove that it is uniformly distributed. Let χ_Ω be a character of the group Ω . Then it is well known that

$$\chi_\Omega(\omega) = \prod_p \omega^{k_p}(p), \quad \omega \in \Omega,$$

where only a finite number of integers k_p are distinct from zero. Hence,

$$\chi_\Omega(\omega) = \omega(m_1) \overline{\omega(m_2)}$$

with $(m_1, m_2) = 1$. Then we have

$$\begin{aligned} \frac{1}{\#\hat{A}_m} \sum_{\omega \in \hat{A}_m} \chi_\Omega(\omega) &= \frac{1}{p_m - 2} \sum_{\omega \in \hat{A}_m} \omega(m_1) \overline{\omega(m_2)} \\ &= \frac{1}{p_m - 2} \sum_{\chi \pmod{p_m}, \chi \neq \chi_0} \hat{\chi}(m_1) \overline{\hat{\chi}(m_2)}. \end{aligned} \tag{3.11}$$

The numbers $m_1, m_2 \in \mathbb{N}$ are fixed. Therefore, for sufficiently large m ,

$$p_m \nmid m_1, p_m \nmid m_2, \text{ and } p_m \nmid (m_1 - m_2).$$

Thus, taking into account (3.11) and (3.8), we find that

$$\begin{aligned} \frac{1}{\#\hat{A}_m} \sum_{\omega \in \hat{A}_m} \chi_\Omega(\omega) &= -\frac{1}{p_m - 2} + \frac{1}{p_m - 2} \sum_{\chi \pmod{p_m}} \chi(m_1) \overline{\chi(m_2)} \\ &= -1/(p_m - 2) \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$. Therefore, by Lemma 3, the sequence $\{\hat{A}_m : m \in \mathbb{N}\}$ is uniformly distributed. Thus, all hypotheses of Lemma 1 are fulfilled, and we have that

$$\frac{1}{q - 2} \#\{\chi \pmod{q}, \chi \neq \chi_0 : \underline{L}(s, \hat{\chi}) \in A\}, \quad A \in \mathcal{B}(H^r(D))$$

converges weakly to $P_{\underline{L}_E}$ as $q \rightarrow \infty$, and this is true for Q_q as well.

From the weak convergence of Q_q to $P_{\underline{L}_E}$ as $q \rightarrow \infty$, it follows that of $P_{Q, \underline{E}}$ as $Q \rightarrow \infty$, see [12].

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