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Asymptotic results on nearly nonstationary processes

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Beveik nestacionarių procesų asimptotiniai rezultatai

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Contents

List of Figures

List of Tables

Notation and abbreviations

Notations	Descriptions
i.i.d.	Independent identically distributed.
LSE	Least squares estimator.
MLE	Maximum likelihood estimate.
NNS	Nearly nonstationary.
CDF	Cumulative distribution function.
$AR(d)$	Autoregressive process of order d.
$y_{n,k}$	The first order nearly nonstationary process $(y_{n,k}, k \leq n, n \geq 1)$.
ε_k	i.i.d.random variables, innovations of nearly nonstationary process.
$\widehat{\phi}_n$	The least squares estimate in first order nearly nonstationary process.
$\widehat{\varepsilon}_k$	The estimated residuals from first order nearly nonstationary process.
$\mathbb{R}$	The real numbers set.
$\mathbb{N}$	The natural numbers set.
$\mathrm{D}[0,1]$	Skorohod (CÀDLÀG functions) space on $[0,1]$.
$\mathrm{C}[0,1]$	Continuous functions space on $[0,1]$.
$\mathrm{C}^1[0,1]$	Space of fonctions on $[0,1]$ with continuous derivative.
$\mathrm{H}^o_\alpha[0,1]$	Separable Hölder space with index α on $[0,1]$.
$\xrightarrow[n\to\infty]{\mathbb{R}}$	Convergence in distribution in $\mathbb{R}$.
$\xrightarrow[n\to\infty]{\mathrm{E}}$	convergence in distribution in a metric space E.
$\xrightarrow[n\to\infty]{\mathrm{P}}$	Convergence in probability.
$\stackrel{\mathcal{D}}{=}$	Equality in distribution.

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Table 0.1 – Continued

Notations	Descriptions
W	A standard Wiener process $(W(t), t \in [0,1])$.
U_γ	An Ornstein-Uhlenbeck process $(U_\gamma(t), t \in [0,1])$.
$\mathfrak{N}(\mu, \sigma^2)$	Gaussian distribution with mean μ and variance σ^2.
W_n^{pl}	The polygonal line process $(W_n^{\mathrm{pl}}(t), t \in [0,1])$ build on i.i.d. random variables.
S_n^{pl}	The polygonal line process $(S_n^{\mathrm{pl}}(t), t \in [0,1])$ build on $y_{n,k}$'s.
$\widehat{W}_n^{\mathrm{pl}}$	The polygonal line process $(\widehat{W}_n^{\mathrm{pl}}(t), t \in [0,1])$ build on residuals.
$\lVert f \rVert_\infty$	Uniform norm of function f in the Skorohod and continuous functions space.
$\lVert f \rVert_\alpha$	Norm of the function f in the Hölder space H_α^o.
$\log(n)$	Natural logarithm.

Asymptotic results on nearly nonstationary processes

Abstract

We study some Hölderian functional central limit theorems for the polygonal partial sum processes built on a first order nearly nonstationary autoregressive process $y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k$ and its least squares residuals $\widehat{\varepsilon}_k$ with ϕ_n converging to 1 and i.i.d. centered square-integrable innovations. In the case where $\phi_n = \mathrm{e}^{\gamma/n}$ with a negative constant γ, we prove that the limiting process depends on Ornstein – Uhlenbeck one. In the case where $\phi_n = 1 - \gamma_n/n$, with γ_n tending to infinity slower than n, the convergence to Brownian motion is established in Hölder space in terms of the rate of γ_n and the integrability of the ε_k's.

We also investigate some epidemic change in the innovations of the first order nearly nonstationary autoregressive process $AR(1)$. Two types of models are considered. For $0 \leq \alpha < 1$, we build the α-Hölderian uniform increments statistics based on the observations and on the least squares residuals to detect the short epidemic change in the process under consideration. Under the assumptions for innovations we find the limit of the statistics under null hypothesis, some conditions of consistency and we perform a test power analysis. We also discuss the interplay between the various parameters to detect the shortest epidemics.

Résultats asymptotiques sur des processus quasi non stationnaires

Résumé

Nous étudions certains théorèmes limite centraux fonctionnels hölderiens pour des processus autorégressifs d'ordre un quasi non stationnaires $y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k$ et leurs résidus au sens des moindres carrés avec $\phi_n \to 1$ et des innovations i.i.d. centrées, de carré intégrable. Dans le cas $\phi_n = \mathrm{e}^{\gamma/n}$ avec $\gamma < 0$, la limite en loi est une fonction d'un processus d'Ornstein-Uhlenbeck intégré. Dans le cas $\phi_n = 1 - \gamma_n/n$ avec $\gamma_n \to \infty$, $\gamma_n/n \to 0$, la convergence vers le mouvement brownien est établie dans l'espace de Hölder en termes de vitesse de divergence γ_n et d'intégrabilité des innovations ε_k.

Nous considérons également une rupture épidémique dans les innovations de pro- cessus autorégressifs d'ordre un quasi non stationnaires $AR(1)$. Deux types de modèles sont considérés. Pour $0 \leq \alpha < 1$ nous construisons une statistique α-hölderienne basée sur les accroissements uniformes des observations ou des résidus pour détecter une courte rupture épidémique dans les processus considérés. Sous certaines hypothèses pour les innovations, nous trouvons la loi limite de la statistique sous l'hypothèse nulle, les conditions de consistance et nous effectuons une analyse de la puissance du test statistique. Nous discutons également l'interaction entre les différents paramètres pour la détectabilité des plus courtes épidémies.

Beveik nestacionarių procesų asimptotiniai rezultatai

Santrauka

Disertacijoje nagrinėjami dalinių sumų laužčių procesai sudaryti iš pirmos eilės beveik nestacionaraus proceso $y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k$ bei jo mažiausių kvadratų liekanų $\widehat{\varepsilon}_k$, kai $\phi_n \to 1$ ir inovacijos yra nepriklausomi, vienodai pasiskirstę ir bent kvadratu integruojami atsitiktiniai dydžiai su nuliniu vidurkiu. Įrodomos funkcinės ribinės teoremos šiems laužčių procesams Hiolderio erdvėje. Kai $\phi_n = e^{\gamma/n}$, $\gamma < 0$, įrodoma, kad ribinis procesas priklauso nuo Ornsteino–Uhlenbecko proceso. Kitu atveju, kai $\phi_n = 1 - \gamma_n/n$ ir γ_n artėja į begalybę lėčiau nei n, įrodomas konvergavimas į Brauno judesį Hiolderio erdvėje, atsižvelgiant į γ_n divergavimo greitį bei inovacijų integruojamumą.

Toliau nagrinėjamas epideminio pasikeitimo modelis beveik nestacionaraus pirmos eilės autoregresinio proceso inovacijoms. Nagrinėjami du modeliai. Iš stebėjimų bei liekanų konstruojama tolydžiųjų prieaugių α-Hiolderio statistika, kai $0 \le \alpha < 1$. Remiantis prielaidomis inovacijoms, randama statistikos ribinis procesas prie nulinės hipotezės, suderinamumo sąlygos, atliekama galios analizė. Taip pat aptariamas parametrų sąryšis siekiant aptikti kuo trumpesnį epideminį pasikeitimą.

Introduction

Research topic. The thesis is devoted to an asymptotic analysis of the first order nearly nonstationary autoregressive processes. We consider a sample $y_{n,1}, \dots, y_{n,n}$, where $y_{n,k}$ is generated by first order nearly nonstationary process

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k, \quad k \le n, \quad n \ge 1,$$

$\phi_n \to 1$ as $n \to \infty$, innovations $(\varepsilon_k, k = 0, \dots n)$ are centered, at least square integrable random variables.

We investigate functional limit theorems for the process $(y_{n,k})$ in the space of continuous function and in the Hölder spaces. Also, we prove the Hölderian functional limit theorems for least square residuals $(\widehat{\varepsilon}_k, k = 0, \dots n)$ of the process under investigation. We use the two type of parameterizations of the coefficient ϕ_n: the first is $\phi_n = \mathrm{e}^{\gamma/n}$ and the second one $\phi_n = 1 - \gamma_n/n$ with $\gamma_n \to \infty$ and $\gamma_n/n \to 0$ as $n \to \infty$. These two parameterizations give different limiting distribution in the functional limit theorems. The limit in case one is a functional of an integrated Ornstein-Uhlenbeck process, while in case two the limit is functional depending on the Wiener process.

In this thesis we apply functional limit theorems to the epidemic change detection in the mean of innovations, i.e., we discuss the model

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k + a_{n,k}, \quad k \le n, \quad n \ge 1,$$

where

$$a_{n,k} = a_n \mathbf{1}_{\mathbb{I}_n^*}(k).$$

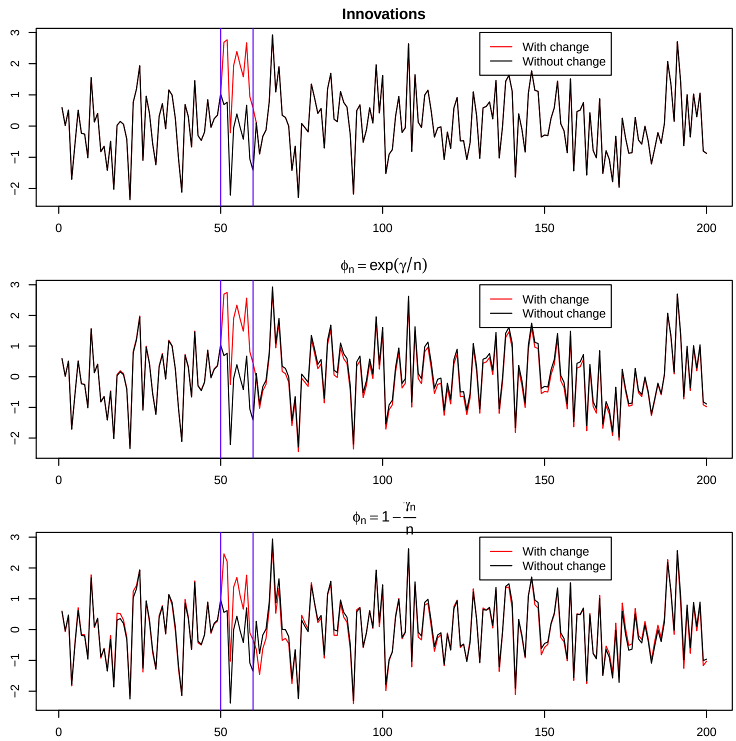

Figure 0.2: Trajectory of the innovations and the NNS $AR(1)$ residuals with and without the epidemic change. Blue vertical lines denote the beginning and the end of the epidemic change.

Here $\mathbf{1}_{\mathbb{I}_n^*}(k)$ is the indicator function of the index set

$$\mathbb{I}_n^* = \{k^* + 1, \dots, m^*\}$$

that denotes the epidemic change with the unknown beginning k^* and end m^*.

Such epidemic change is reflected in trajectories of $y_{n,k}$ and $\widehat{\varepsilon}_k$ (see figures 0.2 and 0.3). Thus we deal with uniform increments statistics build both on $y_{n,k}$'s and $\widehat{\varepsilon}_k$'s. This leads to different results.

For the test statistics under investigation, we find the limit of the statistics under the null hypothesis of no change. Also we investigate the consistency of statistics, power analysis and we discuss the interplay between various parameters to detect the shorter epidemics.

Actuality. Nearly nonstationary autoregressive processes are important in statistics and particularly in econometrics. One important feature of such processes is their behaviour in the neighbourhood of an unit root. This question have been investigated by a number of authors: P.C.B. Phillips, L. Giraitis, N.H. Chan,

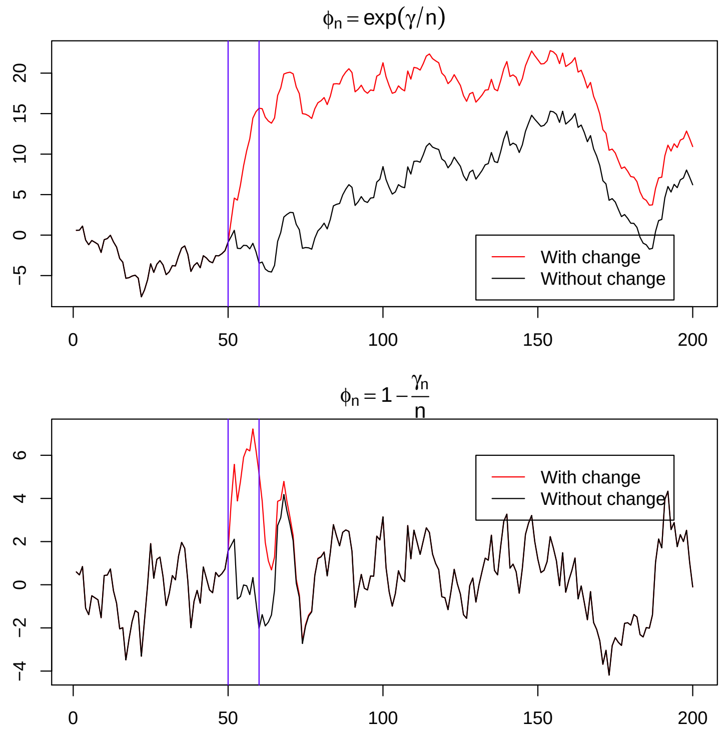

Figure 0.3: Trajectory of the NNS $AR(1)$ with and without the epidemic change. Blue vertical lines denote the beginning and the end of the epidemic change.

etc. For more references and details, see section "State of the art".

Aim and tasks. The aim of the thesis is to prove the functional limit theorems for the nearly nonstationary processes and to apply them to the epidemic change detection in the mean of innovations. The tasks of the thesis are:

– to analyse the functional convergence of polygonal line processes build on the $y_{n,k}$'s and residuals $\widehat{\varepsilon}_k$;
– to build and study test statistics for the epidemic change detection;
– to make numerical experiments for the epidemic change detection.

Novelty. In the thesis we prove various the Hölderian functional central limit theorems for the first order nearly nonstationary autoregressive processes. New results on the epidemic change detection by Hölderian type statistics in nearly nonstationary first order autoregressive process are established.

Main results. Functional limit theorems for the first order nearly nonstationary autoregressive process in continuous function and Hölder spaces are proved (theorems 4.1.3, 4.1.8, 4.1.9); Hölderian functional limit theorems for residuals are given (theorems 4.2.2, 4.2.8); Levin and Kline type statistics build on $y_{n,k}$'s for epidemic detection under null hypothesis of no change is investigated (theorems 5.1.1, 5.1.2); Hölderian type statistics is studied (theorems 5.1.3, 5.1.4); consistency of Levin and Kline and Hölderian type statistics is given (theorem 5.2.1); Hölderian type statistics build on residuals for epidemic detection under null hypothesis of no change is investigated (theorems 6.1.1, 6.1.2); consistency of such type statistics in special case is considered (theorem 6.2.1).

Methods. Methods and results of probability theory, statistics and functional analysis are used. Numerical experiments are performed with a free software environment for statistical computing and graphics **R**.

Publications.

1. J. Markevičiūtė, A. Račkauskas, Ch. Suquet. Functional central limit theorems for sums of nearly nonstationary processes. *Lithuanian mathematical journal*, 52(3): 282-296, 2012.
2. J. Markevičiūtė, A. Račkauskas, Ch. Suquet. Testing the epidemic change in nearly nonstationary processes. *Nonlinear Analysis: Modelling and Control*, *To appear*, 2013.
3. J. Markevičiūtė, A. Račkauskas, Ch. Suquet. Epidemic change test based on residuals for nearly nonstationary process. (**preprint**)

Conferences.

1. The First German-Polish Joint Conference on Probability Theory and Mathematical Statistics, Torun, Poland, 2013 06 06 - 2013 06 09. Talk "Testing the epidemic change in nearly nonstationary processes".
2. Conference "Non-stationarity in Statistics and Risk Management", Luminy, France, 2013 01 21 – 2013 01 25. Talk "Functional limit theorems for residuals of nearly nonstationary processes".
3. 53rd conference of Lithuanian Mathematical Society, Klaipėda, Lithuania, 2012 06 11 - 2012 06 12. Talk "Functional central limit theorems for nearly nonstationary processes and applications for testing epidemic change".
4. 2nd conference of young scientists by Lithuanian Academy of Sciences "Interdisciplinary research of physical and technological sciences", Vilnius, 2012 02 14. Poster "Weak law of large numbers for the first order nearly nonstationary autoregressive processes in the functional spaces".

5. 52nd conference of Lithuanian Mathematical Society, Vilnius, 2011 06 16 – 2011 06 17. Talk "Functional limit theorems for residuals of nearly non stationary processes".
6. 1st conference of young scientists by Lithuanian Academy of Sciences "Interdisciplinary research of physical and technological sciences", Vilnius, 2011 02 08. Poster "The choice of dimension of high frequency data smoothing".

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I also place on record, my sense of gratitude to one and all who, directly or indirectly, have lent their helping hand in this venture.

Structure of the thesis. The chapter 2 of the thesis is devoted to the state of the art. We give some necessary background and tools in the chapter 3. Functional limit theorems and some supplementary results are proved in the chapter 4. Chapter 5 contains the analysis of the epidemic change with the statistics build on the process $y_{n,k}$. We investigate the statistics build on residuals in the chapter 6. Finally we give conclusions and the list of bibliography.

2
State of the Art

In this chapter we give the definition of the first order autoregressive process. We review the main results related with these processes and we motivate the choice to investigate first order nearly nonstationary autoregressive process. Also we give some information on the change point and epidemic change problems.

2.1 First order autoregressive process

The first order autoregressive process $AR(1)$ is a very important process in applications of statistics and economics. The autoregressive model is a time series model and it is one of linear prediction formulas that predicts an output based on the previous outputs. The $AR(1)$ equation is a standard linear difference equation

$$y_k = \phi y_{k-1} + \varepsilon_k, \quad k = 0, \pm 1, \pm 2, \pm 3, \dots \tag{2.1}$$

where (ε_k) are innovations and give the variability in the time series. It is well known (see for example Tsay [2002]) in the case $|\phi| < 1$, the system (2.1) is said to be stable, i.e., the effect of the changes in the past reduces as the time goes on.

Besides, for $|\phi| < 1$, the solution of (2.1) is a function of the error terms from the past. For $|\phi| > 1$, the system (2.1) blows up. It means that the change in the past has an increasing influence for the future. For the practical reasons it is natural to have a system that is less affected by the past, thus the values of $|\phi|$ is typically assumed to be less than one. Stationary autoregressive model has a mean reverting property, i.e., the trajectory of the process moves towards the long-term mean. When the coefficient ϕ is equal to 1 the process defined by (2.1) is nonstationary, i.e., it has a unit root or 1 is a root of the process's characteristic equation. Nonstationary process fail to have mean reverting property. The trajectory of such process moves up and down without the tendency of tending to the any particular point.

In practice, the coefficient ϕ is unknown, so it has to be estimated. Usually one uses the least squares estimator (LSE):

$$\widehat{\phi} = \frac{\sum_{k=1}^{n} y_k y_{k-1}}{\sum_{k=1}^{n} y_{k-1}^2}. \tag{2.2}$$

Other possible estimation methods are Yule-Walker equations (method of moments) or maximum likelihood estimate. Note, that if (ε_k)'s are normally distributed, the least squares estimate $\widehat{\phi}$ is also a maximum likelihood estimate of ϕ. When $|\phi| < 1$ it is well known (see, for example, Mann and Wald [1943] and Anderson [1959]) that the standardized LSE is asymptotically normal:

$$\left(\sum_{k=1}^{n} y_{k-1}^2\right)^{1/2} (\widehat{\phi} - \phi) \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,1). \tag{2.3}$$

It is worth to mention that with another normalization the latter result becomes:

$$\sqrt{n}(\widehat{\phi} - \phi) \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0, 1-\phi^2).$$

However when $\phi = 1$, the limit distribution of the properly standardized sequence of the least-squares estimators is non-normal. It has been shown by White [1958], see also Rao [1978], that

$$\left(\sum_{k=1}^{n} y_{k-1}^2\right)^{1/2} (\widehat{\phi} - 1) \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\frac{1}{2}(W^2(1) - 1)}{\left(\int_0^1 W^2(t)\,\mathrm{d}t\right)^{1/2}}. \tag{2.4}$$

Putting another normalization, the following convergence is true:

$$n(\widehat{\phi}-1)\xrightarrow[n\to\infty]{\mathbb{R}}\frac{\int_0^1 W(t)\,\mathrm{d}W(t)}{\int_0^1 W^2(t)\,\mathrm{d}t}.$$

Evans and Savin ([1981], [1984]) have found in extensive simulation experiment that the statistical properties of the coefficient estimator and associated t test in a stationary $AR(1)$ with a root near unity are close to those of a random walk. This is observed even in sample size of size 100. Similar results have been found when the $AR(1)$ is mildly explosive. Thus, according to Evans and Savin ([1981], [1984]), (2.4) can be used to approximate the distribution of standardized estimate of ϕ, when it is close to one. However, Chan and Wei [1987] have observed that neither (2.3) nor (2.4) seems to be intuitive approximations because of nonsmooth transition from normal distribution to the distribution of $\left(\frac{1}{2}(W^2(1)-1)\right) \big/ \left(\left(\int_0^1 W^2(t)\,\mathrm{d}t\right)^{1/2}\right)$. Also Ahtola and Tiao [1984] investigating the score function with respect to the ϕ, i.e.,

$$\sigma^2\left(\sum_{k=1}^{n} y_{k-1}^2\right)(\widehat{\phi}-\phi),$$

have established that normal approximation of its distribution becomes poor in finite samples when ϕ approaches unity and eventually fails even as an asymptotic distribution when $\phi=1$. These results lead to an interest to investigate the so called nearly nonstationary or nearly integrated processes.

2.2 Nearly nonstationary first order autoregressive process

2.2.1 Definition and parameterization

The *nearly nonstationary* first order autoregressive process $(y_{n,k} : k = 0, 1, \ldots, n; n = 1, 2, \ldots)$ is generated by the triangular array scheme

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k, \tag{2.5}$$

where $\phi_n \to 1$, as $n \to \infty$, (ε_k) is a sequence of innovations usually with $\mathbb{E}\varepsilon_k = 0$ and finite variance σ^2. The initialization $(y_{n,0}, n \geq 0)$ plays an important role and will be precised later in discussion of every case.

In all the literature related with the nearly nonstationary processes, the model (2.5) is reparameterized in terms of closeness of ϕ_n to unity. Phillips [1987] uses the parameterization $\phi_n = \mathrm{e}^{\gamma/n}$, where γ is a constant. In fact, Phillips treated parameter γ as noncentrality parameter. When $\gamma = 0$, the model has a unit root. When $\gamma < 0$ and n is fixed, then $0 < \phi_n < 1$ and obviously the model is stationary. Similarly, when $\gamma > 0$ and n is fixed, then $\phi_n > 1$ and the model has properties of the explosive one in finite data sample. When the ratio γ/n is close to zero and $\gamma < 0$, the coefficient ϕ_n is close to one and the model can be thought of as having a root in the neighbourhood of unity. Similar parameterization, for example, $\phi_n = 1 - \gamma/n$ with $\gamma > 0$ have been used by Chan and Wei [1987], Cox and Llatas [1991], Park [2003], Dzhaparidze et al. [1994] etc.

The paper by Andrews and Guggenberger [2008] investigates the parameterization $\phi_n = 1 - \gamma_n/n$, where $\gamma_n \to \gamma \in [0, \infty)$. In this case the parameter ϕ_n is also very near unit root in the sense that $1 - \phi_n = O(n^{-1})$. Phillips and Magdalinos [2007] have defined the parameter ϕ_n in the form $\phi_n = 1 + \gamma/k_n$, $\gamma \in \mathbb{R}$, which represents moderate deviations from unity when (k_n) is a deterministic sequence increasing to infinity at a rate slower than n, so that $k_n = o(n)$, as $n \to \infty$. Putting $\gamma < 0$ the model defined by (2.5) is considered as nearly nonstationary.

Moreover, Giraitis and Phillips [2006] investigate the first order AR model without intercept when the autoregressive parameter ϕ_n deviates from unity by more than $O(n^{-1})$, i.e., $n(1-\phi_n) \to \infty$. Thus, for nearly nonstationary first order autoregressive process one can parametrize $\phi_n = 1 - \gamma_n/n$, where $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$ and their results still applies.

2.2.2 Limit of the standardized LSE of ϕ_n

In the section 2.1 we have observed that the standardized LSE estimate has a different limit in the stationary and nonstationary models and that there is nonsmooth transition between them. Here we recall the main results of the limit distributions of the standardized LSE estimate in the first order nearly nonstationary autoregressive model under various parametrizations.

Phillips [1987] have found the limit of the standardized LSE of the coefficient ϕ_n, which depends on the Wiener and Ornstein-Uhlenbeck processes, when the innovations are strong mixing:

$$n(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\int_0^1 U_\gamma(t)\,\mathrm{d}W(t) + \frac{1}{2}\left(1 - \frac{\sigma}{\sigma'}\right)}{\int_0^1 U_\gamma^2(t)\,\mathrm{d}t}, \tag{2.6}$$

where $\gamma < 0$ and $\sigma' = \lim_{n\to\infty} \mathbb{E}\left(n^{-1}\left(\sum_{k=1}^n \varepsilon_k\right)^2\right)$. If innovations are i.i.d., the latter result reduces to

$$n(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\int_0^1 U_\gamma(t)\,\mathrm{d}W(t)}{\int_0^1 U_\gamma^2(t)\,\mathrm{d}t}.$$

Chan and Wei [1987] have shown that the limiting distribution of $\left(\sum_{k=1}^n y_{n,k-1}^2\right)^{1/2}$ $(\widehat{\phi}_n - \phi_n)$ is $\mathcal{L}(\gamma)$ $(\gamma > 0)$ which is a quotient of stochastic integrals of standard Wiener process:

$$\mathcal{L}(\gamma) = \frac{\int_0^1 (1+bt)^{-1} W(t)\,\mathrm{d}W(t)}{\left(\int_0^1 (1+bt)^{-2} W^2(t)\,\mathrm{d}t\right)^{1/2}},$$

where $b = \mathrm{e}^{2\gamma} - 1$. They have assumed that initialization is $y_{n,0} = 0$ and that innovations are martingale difference sequence with respect to an increasing sequence of σ-fields. Later Chan [1988] under the same assumptions have established that $\mathcal{L}(\gamma)$ can be rewritten in terms of Ornstein-Uhlenbeck process:

$$\mathcal{L}(\gamma) \stackrel{\mathcal{D}}{=} \frac{\int_0^1 U_\gamma(t)\,\mathrm{d}W(t)}{(\int_0^1 U_\gamma^2(t)\,\mathrm{d}t)^{1/2}}.$$

So, essentially the result of Chan and Wei [1987] is the same as Phillips [1987]. Furthermore, Chan [1990] have investigated innovations in the domain of attraction of stable law with index $\alpha \in [0, 2]$. He have found the following result for the LSE of nearly nonstationary $AR(1)$ model

$$n(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\int_0^1 X_\alpha(t)\,\mathrm{d}U_\alpha(t)}{\int_0^1 X_\alpha^2(t)\,\mathrm{d}t},$$

where γ is a real number. Here $X_\alpha(t)$ satisfies the differential equation

$$\mathrm{d}X_\alpha(t) = -\gamma X_\alpha(t)\,\mathrm{d}t + \mathrm{d}U_\alpha(t),$$

where $X_\alpha(0) = 0$ and $U_\alpha = (U_\alpha(t), t \in [0, 1])$ is a Lévy process defined on the Skorohod space $\mathrm{D}[0, 1]$.

To get more information on the properties like (2.6) one usually studies the rate of convergence. In Kubilius and Račkauskas [1996] the rate of convergence in (2.6) is estimated with respect to Lévy-Prohorov metric π. Further Račkauskas [1996] investigate the convergence (2.6) with respect to a smooth functions topology using an approach based on the convergence rate results in the central limit theorem in Banach spaces.

Under the assumptions that (ε_k) are i.i.d., initialization $y_{n,0} = \sum_{j=0}^{\infty} \phi_n^j \varepsilon_{-j}$ and $\phi_n = 1 - \gamma_n/n$, $\gamma_n \to 0$, as $n \to \infty$, Andrews and Guggenberger [2008] derived that

$$(2\gamma_n)^{-1/2} n(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} C,$$

where C is a Cauchy random variable. In fact, when $\gamma_n \to 0$, as $n \to \infty$ the AR parameter ϕ_n is so close to the unity that the initial condition $y_{n,0}$ dominates the behavior of $y_{n,k}$ for all $k = 1, 2, \ldots, n$. While changing the parameterization of the coefficient these authors obtained different results. By defining $\phi_n = 1 - \gamma_n/n$, $\gamma_n \to \gamma \in (0, \infty]$, as $n \to \infty$ Andrews and Guggenberger [2008] have derived

- for $\gamma \in (0, \infty)$

$$n(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\int_0^1 U_\gamma^*(t)\,\mathrm{d}W(t)}{\int_0^1 (U_\gamma^*(t))^2\,\mathrm{d}t},$$

  where the process $U_\gamma^* = (U_\gamma^*, t \in [0,1])$ is defined from a standard normal random variable Z and an Ornstein-Uhlenbeck process $U_\gamma = (U_\gamma(t), t \in [0,1])$ by:

$$U_\gamma^*(t) = U_\gamma(t) + (2\gamma)^{-1/2} \mathrm{e}^{-\gamma t} Z. \tag{2.7}$$

- for $\gamma = \infty$

$$(1 - \phi_n^2)^{-1/2} n^{1/2} (\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,1).$$

The latter result provides for the coefficient that deviates from unity more than $O(n^{-1})$ the usual Gaussian limit theory still applies. In fact, this result is obtained due to the results of Giraitis and Phillips [2006] where the authors have assumed that (ε_k) are stationary and ergodic martingale difference sequence with respect to the natural filtration, initialization satisfies $\mathbb{E}y_0^2 = o(n^{1/2})$ and $n(1 - \phi_n) \to \infty$ holds. Note that the convergence rate in this case depends on how close ϕ_n is to unity.

Similar cases have been investigated by Chan and Zhang [2009]. Authors assume that the innovations are heavy tailed and have infinite variance. In particular, they show that when $\lim_{n\to\infty} n(1-\phi_n) = \gamma$, where γ is a constant, then under some regularity conditions the limit distribution of the least squares estimator of ϕ_n is a functional of fractional Ornstein-Uhlenbeck stable processes.

Investigating the coefficient defined by $\phi_n = 1 + \gamma/k_n$, $\gamma < 0$, in the nearly nonstationary case Phillips and Magdalinos [2007] obtain

$$\sqrt{nk_n}(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0, -2\gamma). \tag{2.8}$$

In this case the authors assume that innovations are centered i.i.d. random variables with finite variance and the process $(y_{n,k})$ is initialized at some $y_{n,0} = o_P(\sqrt{k_n})$. Phillips and Magdalinos [2007] note that, putting $k_n = n^{\varrho}$ yields a convergence rate $n^{1/2+\varrho/2}$ for the serial correlation coefficient $(\widehat{\phi}_n - \phi_n)$, which for $\varrho \in (0, 1)$ covers the interval $(n^{1/2}, n)$ providing a link between the $\sqrt{n}$ and n asymptotics of stationary and nearly nonstationary autoregressions. Though the parametrization $\phi_n = 1 + \gamma/n^{\varrho}$ is very intuitive, the (2.8) result is more general. It allows arbitrarily large neighborhoods of unity, with ϕ_n approaching 1 slower than any polynomial rate, such as $k_n = \log(n)$.

To sum up, the limit distribution of properly standardized LSE depend on the parametrization of the model. In particular, it depends on how close the coefficient ϕ_n is to 1. If the coefficient is further removed from the unity (for example $n(1-\phi_n) \to \infty$, as $n \to \infty$) the standard Gaussian limit theory still holds, while for the coefficients "very" close to the 1 (like $\liminf_{n\to\infty} n(1 - \phi_n) > 0$) the limit distribution is the one of a functional depending on the Ornstein-Uhlenbeck process.

Dzhaparidze et al. [1994] also consider the parameter estimation problem in the nearly nonstationary first order autoregression. They describe the sequential procedure for estimating the parameter γ. For fixed $t \in [0, 1]$, the estimator for γ is defined by

$$\widehat{\gamma}_{n,[nt]} = \begin{cases} \dfrac{-\int_0^{[nt]/n} n^{-1/2} y_{n,[nt]}(s_-)\, \mathrm{d}(n^{-1/2} y_{n,[nt]}(s))}{\int_0^{[nt]/n} n^{-1} y^2_{n,[nt]}(s)\, \mathrm{d}s}, & \int_0^{[nt]/n} n^{-1} y^2_{n,[nt]}(s)\, \mathrm{d}s > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Note that the LSE of γ based on only $[nt]$ observations is given by $\widehat{\gamma}_{n,[nt]}$, while the LSE of γ based on all observations is

$$\widehat{\gamma}_{n,n} = \frac{n\left(\sum_{k=1}^{n} y_{n,k-1}^2 - \sum_{k=1}^{n} y_{n,k} y_{n,k-1}\right)}{\sum_{k=1}^{n} y_{n,k-1}^2}.$$

Then under some regularity conditions Dzhaparidze et al. [1994] obtain

$$\left(\int_0^{[nt]/n} n^{-1} y_{n,[nt]}^2(s)\,\mathrm{d}s\right)(\gamma - \widehat{\gamma}_{n,[nt]}) \xrightarrow[n\to\infty]{\mathrm{D}[0,1]} \left(\int_0^t Y^2(s)\,\mathrm{d}s\right)(\gamma - \widehat{\gamma}_t)$$

with

$$\widehat{\gamma}_t = \begin{cases} \dfrac{-\int_0^t Y(s)\,\mathrm{d}Y(s)}{\int_0^t Y^2(s)\,\mathrm{d}s}, & \int_0^t Y^2(s)\,\mathrm{d}s > 0 \\ 0, & \text{elsewhere,} \end{cases}$$

where $Y(t) = \int_0^t \mathrm{e}^{\gamma(s-t)}\,\mathrm{d}M(s)$ and M is a continuous semimartingale on $[0,1]$.

2.2.3 Other coefficient estimation methods

Cox and Llatas [1991] study asymptotic properties of a class of estimators of the first order nearly nonstationary autoregressive model coefficient ϕ_n. The class of estimators considered are those obtained by solving nonlinear equations:

$$\Psi_n(\widehat{\phi_n}) = \sum_{k=0}^{n-1} y_{n,k}\psi(y_{n,k+1} - \widehat{\phi}_n y_{n,k}) = 0. \tag{2.9}$$

Here ψ is a continuously differentiable and satisfies the second order Lipschitz condition. Then Cox and Llatas [1991] obtain that there exists a sequence $(\widehat{\phi}_n)$ of solutions (2.9) such that $(\widehat{\phi}_n - \phi_n) = O_P(n^{-1})$ and for such sequence

$$n(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\int_0^1 U_\gamma(t)\,\mathrm{d}\widetilde{W}(t)}{\int_0^1 U_\gamma^2(t)\,\mathrm{d}t}$$

where $U_\gamma(t)$ is Ornstein-Uhlenbeck process defined by the stochastic differential equation

$$\mathrm{d}U_\gamma(t) = -\gamma U_\gamma(t)\,\mathrm{d}t + \mathrm{d}W(t), \quad U_\gamma(0) \stackrel{\mathcal{D}}{=} \mathfrak{N}(0, \sigma^2/2\gamma)$$

and $(W(t), \widetilde{W}(t))$ is a two dimensional Brownian motion with

$$\mathbb{E}(W^2(t)) = t\mathbb{E}(\varepsilon_1^2), \quad \mathbb{E}(\widetilde{W}^2(t)) = t\mathbb{E}(\psi^2(\varepsilon_1)),$$
$$\mathbb{E}(W(t)\widetilde{W}(t)) = t\mathbb{E}(\varepsilon_1\psi(\varepsilon_1)), \quad t \in [0,1].$$

In addition, Cox [1991] consider a three parameter first order nearly nonstationary autoregressive model, where the parameters are the mean, autoregressive coefficient and variance of the innovations. Three different estimators are considered: the exact Gaussian MLE, the conditional maximum likelihood or LSE and some "naive" estimators. It is shown that the estimators converge in distribution to analogous estimators for a continuous-time Ornstein-Uhlenbeck process.

2.2.4 Limit theorems for the partial sums of the process $(y_{n,k})$ and residuals

Phillips [1987] independently with Cumberland and Sykes [1982] have found that the sequence of normalized processes $(n^{-1/2}y_{n,[nt]}, t \in [0,1])$ converges weakly to an Ornstein-Uhlenbeck process in the classical Skorohod space D$[0,1]$ in the case where $\phi_n = \mathrm{e}^{\gamma/n}$. The same result has been obtained by Andrews and Guggenberger [2008] with $\phi_n = 1 - \gamma_n/n$, $\gamma_n \to \gamma \in [0,\infty)$, as $n \to \infty$. In the case where $\gamma_n \to \gamma \in (0,\infty)$, as $n \to \infty$ and initialization satisfies condition $y_{n,0} = \sum_{j=0}^{\infty} \phi_n^j \varepsilon_{-j}$ Andrews and Guggenberger [2008] have established the convergence

$$n^{-1/2}(y_{[nt]}, t \in [0,1]) \xrightarrow[n\to\infty]{\mathrm{D}[0,1]} \sigma U_\gamma^*,$$

where U_γ^* is defined by (2.7). Moreover putting $\gamma_n \to 0$, as $n \to \infty$ they have shown

$$\sigma^{-1}(2\gamma_n)^{1/2} n^{-1/2} y_{n,[nt]} \xrightarrow[n\to\infty]{\mathbb{R}} Z \sim \mathfrak{N}(0,1),$$

for each $t \in [0,1]$ and Z does not depend on t. In contrast, with the initial condition $y_{n,0} = o_P(n)$, the result is the D$[0,1]$ weak convergence of $n^{-1/2}(y_{n,[nt]}, t \in [0,1])$ to $\sigma W(t)$. Again, one can notice that the limit distribution differs depending on the closeness of the coefficient ϕ_n to 1 and the initial condition.

Further Phillips and Magdalinos [2007] found

$$n^{-1} y_{[nt]}^2 \xrightarrow[n\to\infty]{\mathrm{P}} 0, \quad \text{for each} \quad t \in [0,1],$$

when $\phi_n = 1 + \gamma/k_n$, $\gamma < 0$, with initialization $y_0 = o_P(\sqrt{k_n})$.

The central limit theorem for the sums $\sum_{k=1}^{n} y_{n,k}$, $n \geq 1$ is proved by various authors in different cases. Phillips [1987] investigates the case where $\phi_n = \mathrm{e}^{\gamma/n}$.

Under normalization $n^{-3/2}$ the limit is some integral of an Ornstein-Uhlenbeck process

$$n^{-3/2} \sum_{k=1}^{n} y_{n,k} \xrightarrow[n\to\infty]{\mathbb{R}} \sigma \int_0^1 U_\gamma(t)\, \mathrm{d}t.$$

The same result is obtained by Andrews and Guggenberger [2008] with $\phi_n = 1 - \gamma_n/n$, $\gamma_n \to \gamma \in [0,\infty)$, as $n \to \infty$. Moreover putting the parametrization $\phi_n = 1 - \gamma_n/n$, $\gamma_n \to \gamma \in (0,\infty)$ and the initial condition $y_{n,0} = \sum_{j=0}^{\infty} \phi_n^j \varepsilon_{-j}$ the central limit theorem now is

$$n^{-3/2} \sum_{k=1}^{n} y_{n,k} \xrightarrow[n\to\infty]{\mathbb{R}} \sigma \int_0^1 U_\gamma^*(t)\, \mathrm{d}t,$$

where U_γ^* is defined by (2.7). While in the case $\gamma_n \to 0$, as $n \to \infty$ and initialization is $y_{n,0} = \sum_{j=0}^{\infty} \phi_n^j \varepsilon_{-j}$ they have shown that the limit is :

$$(2\gamma_n)^{1/2} n^{-3/2} \sum_{k=1}^{n} y_{n,k-1} \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,\sigma^2).$$

Next Giraitis and Phillips [2006] in case $\phi_n = 1-\gamma_n/n$, $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$, have established that

$$n^{-1/2}(1-\phi_n) \sum_{k=1}^{n} y_{n,k} \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,\sigma^2).$$

One can see, that under such parametrization, the asymptotic distribution of the sample mean of $y_{n,k}$ is normal random variable with a convergence rate that depends on ϕ_n.

Further Phillips and Magdalinos [2007] have proved the following weak law of large numbers in case $\phi_n = 1 + \gamma/k_n$ and $\gamma < 0$:

$$(nk_n)^{-1} \sum_{k=1}^{n} y_{n,k}^2 \xrightarrow[n\to\infty]{\mathrm{P}} \frac{\sigma^2}{-2\gamma}.$$

The convergence rates of this result provide a bridge between the results for nonstationary (or nearly nonstationary) and stationary processes. According to Phillips and Magdalinos [2007] this easy to explain by putting $k_n = n^{\varrho}$ for some $\varrho \in (0,1)$. Using this parametrization, ϕ_n approaches the boundary with the stationary region when $\varrho \to 0$ and the boundary of nearly nonstationary region when $\varrho \to 1$.

The partial sums of the residuals in various type models are frequently utilized in many areas such as detecting parameter changes or probability density estimation. Several authors have investigated the limiting distributions of the partial sums for nearly nonstationary first order autoregression under various error structures. For example, Shin [1998] investigates the same parametrization as Phillips [1987] $\phi_n = \mathrm{e}^{\gamma/n}$ and under zero-mean i.i.d. assumption for innovations with variance σ^2 and $\sup_k \mathbb{E}\,|\varepsilon_k|^{2+\delta}$, $\delta > 0$ he has established

$$n^{-1/2}\sum_{k=1}^{[nt]}\widehat{\varepsilon}_k \xrightarrow[n\to\infty]{\mathrm{D}[0,1]} W(t) - A^{-1}BJ(t), \quad t \in [0,1],$$

where $A = \int_0^1 U_\gamma^2(r)\,\mathrm{d}r$, $B = \int_0^1 U_\gamma(r)\,\mathrm{d}W(r)$ and $J(t) = \int_0^t U_\gamma(r)\,\mathrm{d}r$. Also Chan and Liu [2010] study the goodness-of-fit test of the residual empirical process of a nearly unstable long-memory time series.

2.3 Change points and epidemic change detection

Change point problems have a variety of applications in economics, medicine, biology, engineering, etc. Studies concern detecting one change point as well as multiple change points. A special case of multiple change point problem is the epidemic change. To describe the epidemic change, suppose we are given a sequence $X_1, \ldots, X_n$. The standard null hypothesis is

$$H_0: \quad X_1, \ldots, X_n \quad \text{all have the same parameter} \quad \theta_0$$

(e.g. mean, median, variance, etc.) against the alternative

$$H_A: \quad \text{there exists such integers} \quad 1 < k^* < m^* < n \quad \text{that}$$
$$\theta_1 = \ldots = \theta_{k^*} = \theta_{m^*+1} = \ldots = \theta_n = \theta_0 \quad \text{and} \quad \theta_{k^*+1} = \ldots = \theta_{m^*} = \theta_A.$$

Here k^* denotes the (unknown) time or location at which the epidemics starts, m^* is the end and we denote $\ell^* = m^* - k^*$ as the length of the epidemic change. That is, at first the parameter θ is in one state, then at some point a change occurs (the value θ_0 changes to θ_A) and after a certain period the state comes back to the initial one.

There is a lot of literature related with the testing for change points, estimation of them and forecasting the models with the structural breaks. According to the method the data are obtained, there exist two different formulations of the problem. Off-line (or a posteriori) change-points problem arises when the series of observations is complete, i.e., the sample is finite. The sequential change-points problem is formulated when the detection is performed in real time (or on-line). The commonly used methods for detecting the change point is cumulative sums (CUSUM), maximum liklihood, Bayesian methods. More on the change point problem one can find in the books by Brodsky and Darkhovsky [1993], Csörgő and Horváth [1997], Hackl and Westlund [1991], Chen and Gupta [2000]. Hackl and Westlund [1989] give a lot of references concentrated on two topics: detection of non-constancy of parameters in regression and time-series models and statistical analysis of models with time-varying parameters. Peron [2006] wrote a review on the methodological issues related to estimation, testing and computation of the linear models with the structural changes. A central theme in this review is the interplay between structural change and unit root and on methods to distinguish between them two. Among many others, the surveys by Bhattacharya [1994], Khodadadi and Asgharian [2008] concentrate on testing the hypothesis of "no change", estimating the change point by a point estimator or a confidence set.

One way to construct test statistics for detecting the epidemic change of mean is to construct the uniform increments statistics:

$$T_{0,n}(X_1, \ldots, X_n) = \max_{1\le k,\ell\le n} \left| \sum_{j=k+1}^{k+\ell} X_j - \frac{\ell}{n} \sum_{j=1}^{n} X_j \right|. \tag{2.10}$$

To the best of our knowledge, the changed segment in mean problem for i.i.d. random variables have been formulated for the first time by Levin and Kline [1985] (we also refer to Csörgő and Horváth [1997] section 1.4). Other statistics are offered also. For example, Gombay [1994] investigates rank and sign statistics. Siegmund [1986] considers parametric framework for detecting the changed segment, while Lombard [1987] suggests nonparametric tests. Yao [1993] have studied various parametric test statistics in order to detect an epidemic change in the mean value of a sequence of independent normally distributed random variables. Ramanayake and Gupta [2003] build the likelihood ratio statistic and a likelihood ratio type statistic to detect the epidemic change in mean in a sequence of independent exponential random variables. Further Ramanayake and Gupta [2004] investigated the epidemic change of the natural parameter of the independent sequence given from the exponential family. The likelihood ratio statistic for such hypothesis testing is derived. Gut and Steinebach [2005] propose a two-step sequential procedure to detect the epidemic change. Fellouris et al. [2010] have used the CUSUM procedure for this problem in the framework of counting process.

We study statistics of the type (2.10). Račkauskas and Suquet [2004b] observe, that this statistics can detect only epidemics whose the length ℓ^* is such that $n^{1/2} = o_P(\ell^*)$. For shorter epidemics, Račkauskas and Suquet [2004b] have proposed to improve the statistics by weighting. Let $\alpha \in [0, 1/2)$ and $X_1, \ldots, X_n$ be any sample and define statistics by by

$$T_{\alpha,n} = T_{\alpha,n}(X_1, \ldots, X_n) = \max_{1 \le \ell \le n} \ell^{-\alpha} \max_{1 \le k \le n-\ell} \left| \sum_{j=k+1}^{k+\ell} X_j - \frac{\ell}{n} \sum_{j=1}^{n} X_j \right|. \tag{2.11}$$

Račkauskas and Suquet [2004b] have shown that for any $0 < \alpha < 1/2$ statistics $T_{\alpha,n}(X_1, \ldots, X_n)$ detects epidemics with $n^\delta = o_P(\ell^*)$, where $\delta = (1-2\alpha)/(2-2\alpha)$ ranges in $(0, 1/2)$. Further, Mikosch and Račkauskas [2010] have studied the limit behavior of $T_{\alpha,n}$ with regularly varying random variables and $\alpha > 1/2$. Graiche et al. [2011] propose Hölderian type statistics based on independent not identically distributed or α-mixing random variables to test the epidemic change. From the statistical point of view it is interesting to relax the assumption of independence. For example, Rastenė [2011] has investigated the change segment problem in the coefficient of the first order autoregressive process.

3
Preliminaries

In this chapter we give all the necessary background for the further chapters. We provide the main results related with the Hölder space that will be used further. Also, we describe the invariance principle in function spaces and we present the main tools that are necessary for the reading convenience of this thesis.

Throughout the thesis $W = (W(t), t \in [0,1])$ is a standard Brownian motion. Also, the following process plays an important role in all the thesis:

$$U_\gamma(t) = \int_0^t \mathrm{e}^{(t-s)\gamma}\,\mathrm{d}W(s) = W(t) + \gamma \int_0^t \mathrm{e}^{(t-s)\gamma} W(s)\,\mathrm{d}s, \quad t \in [0,1]. \tag{3.1}$$

Actually, $U_\gamma = (U_\gamma(t), t \in [0,1])$ is an Ornstein-Uhlenbeck process, generated by the stochastic differential equation

$$\mathrm{d}U_\gamma(t) = \gamma U_\gamma(t)\,\mathrm{d}t + \mathrm{d}W(t), \quad t \in [0,1].$$

with the initial condition $U_\gamma(0) = 0$ and parameter $\gamma < 0$.

3.1 Hölder space

We focus in this thesis on the functional convergence in the space of continuous functions and Hölder spaces. We denote by $\mathrm{C}[0,1]$ the space of continuous functions $f:[0,1]\mapsto\mathbb{R}$. Equipped with the supremum norm

$$\|f\| = \sup_{0\le t\le 1} |f(t)|,$$

$\mathrm{C}[0,1]$ is a complete, separable Banach space.

For $\alpha\in[0,1)$ the Hölder space

$$\mathrm{H}^o_\alpha[0,1] := \left\{ f \in C[0,1] : \lim_{\delta\to 0}\omega_\alpha(f,\delta) = 0 \right\},$$

endowed with the norm $\|f\|_\alpha := |f(0)| + \omega_\alpha(f,1)$, where

$$\omega_\alpha(f,\delta) := \sup_{\substack{s,t\in[0,1]\\ 0<t-s<\delta}} \frac{|f(t)-f(s)|}{|t-s|^\alpha},$$

is a separable Banach space. In the special case where $\alpha = 0$, the set $\mathrm{H}^o_0[0,1]$ coincides with $\mathrm{C}[0,1]$ and the norms $\|f\|_0$ and $\|f\|_\infty$ are equivalent.

The functional framework of Hölder space is interesting in the theory of stochastic processes since very often the continuous stochastic process under study has a better regularity than the bare continuity. Also, the weak convergence of a sequence of stochastic processes in some functions space E provides results about the asymptotic distribution of functionals of the paths which are continuous with respect to the topology of E. Since the Hölder spaces are topologically embedded in $\mathrm{C}[0,1]$ and $\mathrm{D}[0,1]$, they support more continuous functionals. From this point of view, the alternative framework of Hölder spaces gives functional limit theorems of a broader scope (see more in Juodis et al. [2009]).

Throughout the thesis we work with random polygonal lines and study their asymptotic behaviour in Hölder topology. As a polygonal line is characterized by its vertices, it is useful to know how its Hölderian asymptotic behaviour depends on the control of its vertices. To explain this, it is convenient here to represent a polygonal line π_n with vertices $(l/n, V_l)$, $0\le l\le n$, $V_0 = 0$, under the form:

$$\pi_n(t) = (1-\{nt\})V_{[nt]} + \{nt\}\,V_{[nt]+1}, \quad 0\le t\le 1, \tag{3.2}$$

where $\{nt\} = nt - [nt]$ is the fractional part of nt. We claim that the Hölder norm of such a line is reached at two vertices, that is

$$\|\pi_n\|_\alpha = \max_{0\le j<k\le n} \frac{|V_k - V_j|}{(\frac{k}{n} - \frac{j}{n})^\alpha}. \tag{3.3}$$

From (3.3) we immediately deduce that

$$\|\pi_n\|_\alpha \le 2n^\alpha \max_{1\le l\le n} |V_l|. \tag{3.4}$$

To prove that the Hölder norm of a polygonal line is reached at two vertices (equality (3.3)), it is convenient to generalize a bit by considering more general weight functions than $h \mapsto h^\alpha$.

Lemma 3.1.1. *Let $\rho : [0,1] \to \mathbb{R}$ be a weight function satisfying the following properties.*

i) ρ is concave.

ii) $\rho(0) = 0$ and ρ is positive on $(0,1]$.

iii) ρ is non decreasing on $[0,1]$.

Let $t_0 = 0 < t_1 < \cdots < t_n = 1$ be a partition of $[0,1]$ and f be a real valued polygonal line function on $[0,1]$ with vertices at the t_i's, i.e. f is continuous on $[0,1]$ and its restriction to each interval $[t_i, t_{i+1}]$ is an affine function. Define

$$R(s,t) := \frac{|f(t) - f(s)|}{\rho(t-s)}, \quad 0 \le s < t \le 1.$$

Then

$$\sup_{0\le s<t\le 1} R(s,t) = \max_{0\le i<j\le n} R(t_i, t_j). \tag{3.5}$$

Proof. Obviously (3.5) will be established if we prove that

$$R(s,t) \le \max_{0\le i<j\le n} R(t_i, t_j), \tag{3.6}$$

for every pair of real numbers s,t such that $0 \le s < t \le 1$. This in turn, is easily deduced from the following estimates where in each configuration considered, f is supposed to be affine on $[a,b]$.

$$R(s,t) \le \begin{cases} R(a,b) & \text{if } a \le s < t \le b, \\ \max\Big(R(s,a), R(s,b)\Big) & \text{if } s < a \le t \le b, \\ \max\Big(R(a,t), R(b,t)\Big) & \text{if } a \le s \le b < t. \end{cases}$$

In the first configuration,

$$f(t) - f(s) = \frac{f(b) - f(a)}{b - a}(t - s),$$

whence

$$R(s,t) = R(a,b)\frac{t-s}{\rho(t-s)}\frac{\rho(b-a)}{b-a}. \tag{3.7}$$

By concavity of ρ, the function $h \mapsto \rho(h)/h$ is non increasing on $(0,1]$, as the slope of the chord between 0 and h. So $\rho(t-s)/(t-s) \geq \rho(b-a)/(b-a)$, whence $\frac{t-s}{\rho(t-s)}\frac{\rho(b-a)}{b-a} \leq 1$ and (3.7) gives $R(s,t) \leq R(a,b)$.

In the second configuration, let us parameterize the segment $[a,b]$ by putting $t = (1-u)a + ub$, $u \in [0,1]$. Then $t - s = (1-u)(a-s) + u(b-s)$ and as $t \mapsto f(t) - f(s)$ is affine on $[a,b]$, $f(t)-f(s) = (1-u)(f(a)-f(s)) + u(f(b)-f(s))$. Now to estimate $R(s,t)$, using triangular inequality for the numerator and the concavity of ρ for the denominator gives:

$$R(s,t) \leq \frac{(1-u)\,|f(a)-f(s)| + u\,|f(b)-f(s)|}{(1-u)\rho(a-s) + u\rho(b-s)} = \frac{Au+B}{Cu+D} = A' + \frac{B'}{Cu+D},$$

where the constants $A, A', \dots, D$ depend on f, ρ, a, b and s (which is fixed here). As ρ is non decreasing, $(1-u)\rho(a-s) + u\rho(b-s) \geq \rho(a-s) > 0$, so $Cu + D$ remains positive when u varies between 0 and 1. It follows that the homographic function $A' + B'/(Cu+D)$ is monotonic on $[0,1]$ and hence reaches its maximum at $u = 0$ or at $u = 1$. This gives $R(s,t) \leq \max\big(R(s,a), R(s,b)\big)$.

The bound for $R(s,t)$ in the third configuration is obtained in a completely similar way, so we omit the details. □

Remark 3.1.2. In the case of vector valued polygonal lines, the result and the proof are still valid, replacing $|f(t) - f(s)|$ by $\|f(t) - f(s)\|$ in the definition of $R(s,t)$.

The next theorem gives a characterization of the tightness of sequences of random elements in a Hölder space (see Suquet [1999] Theorem 13 for the case $0 < \alpha < 1$ and Proposition 1 for $\alpha = 0$).

Theorem 3.1.3. *The sequence (ξ_n) of random elements in $\mathrm{H}^o_\alpha[0,1]$, $0 \leq \alpha < 1$, is tight if and only if*

(a) $\lim_{A\to\infty} \sup_{n\geq 1} \mathbb{P}\left(\|\xi_n\|_\infty > A\right) = 0$;

(b) $\forall \epsilon > 0$, $\lim_{\delta\to 0} \sup_{n\geq 1} \mathbb{P}\left(\omega_\alpha(\xi_n,\delta) \geq \epsilon\right) = 0$.

3.2 Invariance principle

Consider the polygonal line process constructed on i.i.d. random variables (ε_j)

$$W_n^{\mathrm{pl}}(t) = \sum_{j=1}^{[nt]} \varepsilon_j + (nr - [nt])\varepsilon_{[nt]+1}, \quad t \in [0,1]. \tag{3.8}$$

This process lies in the continuous function space $\mathrm{C}[0,1]$ and in each Hölder space $\mathrm{H}_\alpha^o[0,1]$, for $0 < \alpha < 1$. The limiting behaviour of such processes is well known. The classical Donsker-Prohorov invariance principle states that, if $\mathbb{E}\varepsilon_1 = 0$ and $0 < \sigma^2 := \mathrm{Var}(\varepsilon_1) = \mathbb{E}\varepsilon_1^2 < \infty$, then

$$n^{-1/2}\sigma^{-1}W_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{C}[0,1]} W. \tag{3.9}$$

This result has a lot of applications, especially in statistics, and continues to receive many extensions.

Hölderian invariance principle is also established. By the classical Levy's result on the modulus of continuity of W, $W \in \mathrm{H}_\alpha^o[0,1]$ with probability one for every $0 \leq \alpha < 1/2$. Lamperti [1962] proved that if $0 < \alpha < 1/2$ and $\mathbb{E}\,|\varepsilon_0|^p < \infty$, where $p > 1/(1/2 - \alpha)$, then

$$n^{-1/2}\sigma^{-1}W_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\alpha^0[0,1]} W \tag{3.10}$$

holds. This result was derived again by Kerkyacharian and Roynette [1991] by another method using analysis given by Ciesielski [1960] of Hölder spaces by triangular functions. Further generalizations were given by Hamadouche [2000] and Račkauskas and Suquet [1999]. The result (3.10) have been completed and extended by Račkauskas and Suquet [2004a]. They have proved that for $p > 2$ with $\alpha = 1/2 - 1/p$ (i.e. $0 < \alpha < 1/2$) the convergence (3.10) holds if and only if

$$\lim_{t\to\infty} t^p P(|\varepsilon_1| \geq t) = 0. \tag{3.11}$$

Note that condition (3.11) can be rewritten as

$$\lim_{t\to\infty} t^{1/(1/2-\alpha)} P(|\varepsilon_1| \geq t) = 0.$$

Condition (3.11) provides precise relation between the strength of the convergence (3.10) and the integrability of summands. Compared with the classical Donsker invariance principle, it shows the price to be paid for functional convergence in a stronger topology. When $\alpha > 0$, condition (3.11) implies that $\mathbb{E}\,|\varepsilon_1|^p < \infty$ for $p < (1/2-\alpha)^{-1}$ and in particular $\mathbb{E}\varepsilon_1^2 < \infty$. We note also that condition (3.11) with $p = 2$, so $\alpha = 0$, does not imply the convergence (3.9).

3.3 Tools

The first two results in this section help us to reduce the proof of functional limit theorems to zero initial condition. The central point is the fact, that all α-Hölder norms of a function f are equivalent if $f \in \mathrm{C}^1[0,1]$.

Lemma 3.3.1. *If $f \in \mathrm{C}^1[0,1]$ and f is non constant, then all its α-Hölder norms are equivalent in the sense that there exists positive constants b and c such that $b \le \omega_\alpha(f,1) \le c$, where b and c do not depend on α. If f is constant $\|f\|_\alpha = |f(0)|$ for every $0 < \alpha < 1$.*

Proof. Recall, that

$$\omega_\alpha(f,1) = \sup_{0\le s<t\le 1} \frac{|f(t)-f(s)|}{|t-s|^\alpha}.$$

Since f' is continuous on $[0,1]$, for any $0 \le s < t \le 1$, there is a $\theta \in (0,1)$ such that $f(t)-f(s) = (t-s)f'(s+\theta(t-s))$. From this we immediately deduce that

$$|f(t)-f(s)| \le \frac{|f(t)-f(s)|}{|t-s|^\alpha} \le \|f'\|_\infty\,,$$

whence

$$\operatorname{osc}(f) := \sup_{0\le s<t\le 1} |f(t)-f(s)| \le \omega_\alpha(f,1) \le \|f'\|_\infty\,.$$

If $\operatorname{osc}(f) = 0$, then f is constant (and conversely), so $\omega_\alpha(f,1) = 0$ for every $\alpha \in (0,1)$. Else we can put $b = \operatorname{osc}(f) > 0$ and $c = \|f'\|_\infty$ to conclude. □

Here we give a more precise result of this type for the exponential function $t \mapsto a^t$, where $0 < a < 1$.

Lemma 3.3.2. *For $0 < a < 1$, let f be the map $[0,1] \to \mathbb{R}_+$, $t \mapsto a^t$. Then for every $\alpha \in [0,1]$,*

$$1-a \le \sup_{0\le s<t\le 1} \frac{a^s - a^t}{(t-s)^\alpha} \le -\ln a. \tag{3.12}$$

Moreover, if p_n is the polygonal line of linear interpolation of f between the points

k/n, $0 \le k \le n$, then

$$1 - a \le \omega_\alpha(p_n, 1) \le \omega_\alpha(f, 1) \le -\ln a. \tag{3.13}$$

Proof. Putting $t - s = h$, we deduce immediately from the factorisation

$$\frac{a^s - a^t}{(t-s)^\alpha} = a^s \frac{1 - a^h}{h^\alpha}$$

that

$$\sup_{0 \le s < t \le 1} \frac{a^s - a^t}{(t-s)^\alpha} = \sup_{0 < h \le 1} \frac{1 - a^h}{h^\alpha} \tag{3.14}$$

The function $h \mapsto 1 - a^h$ being concave on $[0, 1]$, its graphic representation is above its chord between the points with abscissas 0 and 1 and below its tangent at the origin. This provides the inequalities:

$$(1 - a)h \le 1 - a^h \le (-\ln a)h, \quad h \in [0, 1].$$

Hence for every $h \in (0, 1]$,

$$(1 - a)h^{1-\alpha} \le \frac{1 - a^h}{h^\alpha} \le (-\ln a)h^{1-\alpha}.$$

Taking the supremum over $h \in (0, 1]$ and accounting (3.14), we obtain (3.12). From Lemma 3.1.1 it is clear that $\omega_\alpha(p_n, 1) \le \omega_\alpha(f, 1)$. Together with the obvious inequality $1 - a = p_n(0) - p_n(1) \le \omega_\alpha(p_n, 1)$, this gives (3.13). □

The next results are useful tools to investigate the limiting behaviour of the test statistics.

Lemma 3.3.3. *Suppose $\alpha \in [0, 1)$. Consider the functionals g_n and g defined on the Hölder space $\mathrm{H}_\alpha^o[0, 1]$ by*

$$g_n(x) := \max_{1 \le i < j \le n} I_\alpha(x, i/n, j/n), \quad g(x) := \sup_{0 \le s < t \le 1} I_\alpha(x, s, t), \tag{3.15}$$

where

$$I_\alpha(x, s, t) := \frac{|x(t) - x(s) - (t - s)x(1)|}{|t - s|^\alpha}, \quad 0 < t - s \le 1. \tag{3.16}$$

Then g_n and g are Lipschitz on

$$G_\alpha = \{x \in \mathrm{H}_\alpha^o[0, 1] : x(0) = 0\}$$

with the same constant $C = 2$, if $\alpha \in (0,1)$. Also, g_n and g are Lipschitz on $G_0 = \{x \in \mathrm{C}[0,1] : x(0) = 0\}$ with the same constant $C = 2$, if $\alpha = 0$.

Further, for any tight sequence of random elements $(\eta_n)_{n\geq 0}$ in $\mathrm{C}[0,1]$ or $\mathrm{H}^o_\alpha[0,1]$, it holds

$$g_n(\eta_n) = g(\eta_n) + o_{\mathrm{P}}(1). \tag{3.17}$$

To prove Lemma 3.3.3 it is convenient to use the two following lemmas which one can find in Račkauskas and Suquet [2004b].

Lemma 3.3.4. *Let (η_n) be a tight sequence of random elements in separable Banach space B and g_n, g be continuous functionals $B \to \mathbb{R}$. Assume that g_n converges pointwise to g on B and that (g_n) is equicontinuous. Then*

$$g_n(\eta_n) = g(\eta_n) + o_{\mathrm{P}}(1).$$

Lemma 3.3.5. *Let $(\mathcal{B}, \|\|)$ be a vector normed space and $q : \mathcal{B} \to \mathbb{R}$ such that*

a) q is subadditive: $q(x+y) \leq q(x) + q(y)$, $x, y \in \mathcal{B}$;

b) q is symmetric: $q(-x) = q(x)$, $x \in \mathcal{B}$;

c) for some constant C, $q(x) \leq C\,\|x\|$, $x \in \mathcal{B}$.

Then q satisfies the Lipschitz condition

$$|q(x+y) - q(x)| \leq C\,\|y\|\,, \quad x, y \in \mathcal{B}. \tag{3.18}$$

If $\mathcal{F}$ is any set of functionals q fulfilling a), b) and c) with the same constant C, then a), b) and c) are inherited by $g(x) := \sup\{q(x) : q \in \mathcal{F}\}$ which therefore satisfies (3.18).

Proof of Lemma 3.3.3. Here we shall give an unified proof for the cases $\alpha = 0$ and $\alpha \in (0,1)$. Since the spaces $(\mathrm{C}, ||\cdot||_\infty)$ and $(H^o_0, ||\cdot||_0)$ are isomorphic, thus putting $\alpha = 0$ in the proof gives the special case of g_n and g being Lipschitz on $\mathrm{C}[0,1]$. To show that $q = I_\alpha(\cdot, s, t)$ is Lipschitz, we shall use the Lemma 3.3.5 whose conditions a) and b) are obviously satisfied while condition c) follows from

$$q(x) = I_\alpha(x, s, t) \leq \frac{x(t) - x(s)}{|t-s|^\alpha} + |t-s|^{1-\alpha}\,|x(1)| \leq 2\,\|x\|_\alpha\,. \tag{3.19}$$

Define the closed subspace $G_\alpha = \{x \in \mathrm{H}^o_\alpha[0,1] : x(0) = 0\}$. From (3.19) we see that for any $0 \leq s < t \leq 1$, the functional $q = I_\alpha(\cdot, s, t)$ satisfies on G_α the

Condition c) of Lemma 3.3.5 with the constant $C = 2$. It follows by Lemma 3.3.5 that g_n as well as g are Lipschitz on G_α with this same constant $C = 2$. As a result, the sequence $(g_n)_{n\geq 2}$ is equicontinuous on G_α.

Now by Lemma 3.3.4, the proof of (3.17) is reduced to check the pointwise convergence of g_n to g on G_α. Let us fix an arbitrary function $x \in G_\alpha$. By the first inequality in (3.19) and the definition of the space H_α^o, the function $I_\alpha(x,\cdot,\cdot)$ receives a continuous extension $\tilde{I}_\alpha(x,\cdot,\cdot)$ to the diagonal by putting $\tilde{I}_\alpha(x,s,s) := 0$ for every $s \in [0,1]$. Since I_α is non negative and $\tilde{I}_\alpha(x,\cdot,\cdot)$ is null along the diagonal, the functionals g_n and g defined by (3.15) satisfy

$$g_n(x) := \max_{1\leq i\leq j\leq n} \tilde{I}_\alpha(x, i/n, j/n), \quad g(x) := \sup_{0\leq s\leq t\leq 1} \tilde{I}_\alpha(x,s,t).$$

Next we observe that the value of the functional $g(x)$ appears as the supremum of the continuous bivariate function $\tilde{I}_\alpha(x,\cdot,\cdot)$ on the closed triangular domain $K := \{(s,t) \in [0,1]^2 : 0 \leq s \leq t \leq 1\}$. By compactness of K, this supremum is reached at some point $(s_0,t_0) \in K$. For $n \geq 1$, let us define the integer

$$i_n := \begin{cases} [ns_0] & \text{if } s_0 \geq 1/n, \\ 1 & \text{if } 0 < s_0 < 1/n, \end{cases} \qquad j_n := \begin{cases} [nt_0] & \text{if } t_0 \geq 1/n, \\ 1 & \text{if } 0 < t_0 < 1/n. \end{cases}$$

Noting that $1 \leq i_n \leq j_n \leq n$, we have

$$I_\alpha(x, i_n/n, j_n/n) \leq g_n(x) \leq g(x) = \tilde{I}_\alpha(x, s_0, t_0).$$

Clearly $(i_n/n, j_n/n)$ converges in K to (s_0,t_0), so letting n tend to infinity in the above inequalities gives the convergence of $g_n(x)$ to $g(x)$ by continuity of $\tilde{I}_\alpha(x,\cdot,\cdot)$. As x was arbitrary in G_α, the pointwise onvergence of g_n to g is established. □

In the last chapter we build test statistics on residuals to test the hypothesis about epidemic change in mean of innovations. The following two results are useful in the proofs of this chapter.

First suppose that we have a sample $X_1,\dots,X_n$ and assume that

$$H_0' : X_1,\dots,X_n \quad \text{are independent identically distributed random variables with mean denoted by } \mu_0.$$

Then Theorem 3 in Račkauskas and Suquet [2004b] finds the limit of test statistics under null hypothesis:

Theorem 3.3.6. *Let $0 < \alpha < 1/2$. Under H_0', assume that*

$$\lim_{t\to\infty} t^{1/(1/2-\alpha)} P(|\varepsilon_1| \geq t) = 0.$$

Then

$$\sigma^{-1} n^{-1/2} UI_{\alpha,n} \xrightarrow[n\to\infty]{\mathbb{R}} UI_{\alpha,\infty},$$

where

$$UI_{\alpha,n} = UI_{\alpha,n}(X_1,\dots,X_n) = \max_{1\leq \ell\leq n} \left(\frac{\ell}{n}\left(1-\frac{\ell}{n}\right)\right)^{-\alpha} \max_{1\leq k\leq n-\ell} \left|\sum_{j=k+1}^{k+\ell} X_j - \frac{\ell}{n}\sum_{j=1}^{n} X_j\right|$$

and

$$UI_{\alpha,\infty} = \sup_{0<t-s<1} \frac{|W(t)-W(s)-(t-s)W(1)|}{|(t-s)(1-(t-s))|^{\alpha}}.$$

Note, that we use the weight ℓ/n and not $\ell/n\cdot(1-\ell/n)$, but in view of Lemma 3.3.3 and Hölderian invariance principle the Theorem 3.3.6 can be adapted as follows when we replace the statistics $UI_{\alpha,n}$ by the statistics $T_{\alpha,n}$.

Corollary 3.3.7. *Let $0 < \alpha < 1/2$. Under H_0', assume that*

$$\lim_{t\to\infty} t^{1/(1/2-\alpha)} P(|\varepsilon_1| \geq t) = 0.$$

Then

$$\sigma^{-1} n^{-1/2+\alpha} T_{\alpha,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_{\alpha,\infty},$$

where

$$T_{\alpha,n} = T_{\alpha,n}(X_1,\dots,X_n) = \max_{1\leq \ell\leq n} \ell^{-\alpha} \max_{1\leq k\leq n-\ell} \left|\sum_{j=k+1}^{k+\ell} X_j - \frac{\ell}{n}\sum_{j=1}^{n} X_j\right|$$

and

$$T_{\alpha,\infty} = \sup_{0<t-s<1} \frac{|W(t)-W(s)-(t-s)W(1)|}{|t-s|^{\alpha}}.$$

Next, assume that $X_1,\dots,X_n$ are regularly varying random variables (the precise definition of regularly varying random variables is given by definition 6.0.1, page 84). Define two statistics:

$$\mathcal{M}_{\alpha,n} = \max_{1\le \ell\le n} \ell^{-\alpha} \max_{1\le k\le n-\ell} \left| \sum_{j=k+1}^{k+\ell} X_j \right|, \quad n\ge 0$$

and

$$\mathcal{T}_{\alpha,n} = \max_{1\le \ell\le n} (\ell(1-\ell/n))^{-\alpha} \max_{1\le k\le n-\ell} \left| \sum_{j=k+1}^{k+\ell} X_j - \frac{\ell}{n}\sum_{j=1}^{n} X_j \right|, \quad n\ge 0.$$

Then the following result of Mikosch and Račkauskas [2010] holds under the null hypothesis H_0':

Theorem 3.3.8. *Consider an i.i.d.sequence* (X_i) *of random variables which are regularly varying with index* $p > 2$ *and have mean zero if it exists. Then, for* $\alpha \in (1/2 - 1/p, 1]$,

$$\lim_{n\to\infty} \mathbb{P}\left(b_n^{-1}\mathcal{M}_{\alpha,n} \le x\right) = \Phi_\alpha(x) = \mathrm{e}^{-x^{-\alpha}}, \quad x > 0,$$

where the normalizing sequence is given by

$$b_n = \inf\left\{x \in \mathbb{R} : \mathbb{P}(|X| \le x) \ge 1 - 1/n\right\}.$$

Moreover

$$\lim_{n\to\infty} \mathbb{P}\left(b_n^{-1}\mathcal{T}_{\alpha,n} \le x\right) = \Phi_\alpha(x), \quad x > 0.$$

The next corollary shows that the behavior of statistics $\mathcal{M}_{\alpha,n}(X_1,\dots,X_n)$ and $T_{\alpha,n}(X_1,\dots,X_n)$ coincides.

Corollary 3.3.9. *Under the assumptions of Theorem 3.3.8, the sequence* $(b_n^{-1}\mathcal{M}_{\alpha,n})$ *has the same limit distribution as the sequence*

$$b_n^{-1}T_{\alpha,n} = b_n^{-1} \max_{1\le \ell\le n} \ell^{-\alpha} \max_{1\le k\le n-\ell} \left| \sum_{j=k+1}^{k+\ell} X_j - \frac{\ell}{n}\sum_{j=1}^{n} X_j \right|.$$

For the proof of this corollary, see Remark 2.6 in Mikosch and Račkauskas [2010].

4

Functional limit theorems

In this chapter we prove the functional limit theorems for the partial sums of the first order nearly nonstationary autoregressive processes in the space of continuous functions and in the Hölder spaces. Further we prove the functional limit theorems for the partial sums of the residuals of the process under investigation in Hölder space. Also we introduce some supplementary results that might be of independent interest. As noticed in the chapter 2, finite dimensional of functional weak limit theorems for nearly non stationnary processes depend on the convergence rate of ϕ_n to 1. In this chapter our aim is to investigate functional central limit theorems in the two situations where ϕ_n tends to 1 at the rate $1/n$ or slower, that is $n(1-\phi_n)$ tends to infinity. More precisely, we restrict our study to the two following parameterizations introduced respectively in Phillips [1987] and Giraitis and Phillips [2006].

- *Case 1*: $\phi_n = \mathrm{e}^{\gamma/n}$ (γ is a negative constant);
- *Case 2*: $\phi_n = 1 - \frac{\gamma_n}{n}$, $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$.

4.1 Functional central limit theorems for sums of nearly nonstationary processes

Recall that we investigate the asymptotic behavior of the first-order autoregressive process $(y_{n,k} : k = 1, \ldots, n; n = 1, 2, \ldots)$ given by

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k, \tag{4.1}$$

where $0 < \phi_n < 1$ for fixed n, $\phi_n \to 1$, as $n \to \infty$, (ε_k) is a sequence of independent identically distributed random variables with $\mathbb{E}\varepsilon_k = 0$ and $y_{n,0}$ is a random variable. Despite the fact that $(y_{n,k})$ is a triangular array, for simplicity, we omit the index n in this chapter and we write $y_k = \phi_n y_{k-1} + \varepsilon_k$, $k = 1, 2, \ldots, n$.

In this section we focus on polygonal line partial sums processes built on the y_k's:

$$S_n^{\mathrm{pl}}(t) := \sum_{k=1}^{[nt]} y_{k-1} + (nt - [nt])y_{[nt]}, \quad t \in [0, 1], \quad n \geq 0, \quad S_n^{\mathrm{pl}}(0) = 0. \tag{4.2}$$

Note that the definition of S_n^{pl} is quite unusual with a general term y_{k-1} where one would expect y_k. This definition is more convenient from the technical point of view. However, asymptotic results proved remain true with y_{k-1} replaced by y_k as well.

The estimate of the Hölder norm (3.4), page 21, enables us to reduce the investigation of the asymptotic behaviour of the random polygonal line S_n^{pl} (properly normalized) to the case where the initialization in (4.1) is given by $y_{n,0} = 0$. Indeed let us associate to each autoregressive process $(y_{n,k})$ satisfying (4.1), the process $(y'_{n,k})$ defined by

$$y'_{n,k} = y_{n,k} - \phi_n^k y_{n,0}. \tag{4.3}$$

Then $(y'_{n,k})$ satisfies (4.1) with initialization $y'_{n,0} = 0$ and the same ε_k's and the above mentioned reduction may be formulated as follows.

Proposition 4.1.1. *Let $S_n^{\mathrm{pl}\prime}$ be the polygonal line process obtained by substituting in (4.2) the $y_{n,j}$'s by the $y'_{n,j}$'s. Assume that $c_n S_n^{\mathrm{pl}\prime}$ converges in distribution in $\mathrm{H}_\alpha^o[0,1]$, where the c_n's are some positive normalizing constants. Then $c_n S_n^{\mathrm{pl}}$*

converges in distribution in $\mathrm{H}_\alpha^o[0,1]$ *to the same limit provided that*

$$nc_n y_{n,0} = o_P(1). \tag{4.4}$$

Proof. The stochastic process $c_n S_n^{\mathrm{pl}} - c_n S_n^{\mathrm{pl}\prime}$ is a random polygonal line π_n which according to the representation (3.2) is determined by its vertices $(l/n, V_l)$, $0 \le l \le n$, $V_0 = 0$, where

$$V_l = \sum_{j=0}^{l-1} c_n \phi_n^j y_{n,0} = \frac{1-\phi_n^l}{1-\phi_n} c_n y_{n,0}.$$

As $\pi_n(0) = 0$, accounting Lemma 3.1.1 we have

$$\|\pi_n\|_\alpha = \omega_\alpha(\pi_n, 1) = \frac{c_n |y_{n,0}|}{1-\phi_n} \max_{1\le l<k\le n} \frac{\left|(\phi_n^n)^{k/n} - (\phi_n^n)^{l/n}\right|}{|k/n - l/n|^\alpha}.$$

Applying inequalities (3.13) in Lemma 3.3.2 with the function f_n defined on $[0,1]$ by $t \mapsto \phi_n^t$, we obtain

$$c_n |y_{n,0}| \frac{1-\phi_n^n}{1-\phi_n} \le \|\pi_n\|_\alpha \le c_n |y_{n,0}| \frac{(-n \ln \phi_n)}{1-\phi_n}.$$

As ϕ_n tends to 1, for the two models under consideration, this gives

$$\|\pi_n\|_\alpha \sim nc_n |y_{n,0}|, \quad n \to \infty.$$

Thus assuming that $c_n S_n^{\mathrm{pl}\prime}$ has a limiting distribution in $\mathrm{H}_\alpha^o[0,1]$, we deduce of this estimate that, if $nc_n y_{n,0} = o_P(1)$, then $c_n S_n^{\mathrm{pl}}$ converges in $\mathrm{H}_\alpha^o[0,1]$ to the same limiting distribution. □

Remark 4.1.2. Assume that $c_n S_n^{\mathrm{pl}\prime}$ has a limiting distribution in $\mathrm{H}_\alpha^o[0,1]$ and $nc_n |y_{n,0}|$ is not stochastically bounded in $\mathbb{R}$. Since

$$\|\pi_n\|_\alpha \le \left\|c_n S_n^{\mathrm{pl}}\right\|_\alpha + \left\|c_n S_n^{\mathrm{pl}\prime}\right\|_\alpha,$$

we have

$$\left\|c_n S_n^{\mathrm{pl}}\right\|_\alpha \ge \|\pi_n\|_\alpha - \left\|c_n S_n^{\mathrm{pl}\prime}\right\|_\alpha.$$

As $nc_n |y_{n,0}|$ is not stochastically bounded in $\mathbb{R}$, so $\|\pi_n\|_\alpha \to \infty$ as $n \to \infty$ and together with boundedness of $c_n S_n^{\mathrm{pl}\prime}$ we obtain that $c_n S_n^{\mathrm{pl}}$, for any α, is not stochastically bounded in $\mathrm{H}_\alpha^o[0,1]$ and cannot converge in this space.

4.1.1 First type model

In this section we study the process (4.1) in the case where $\phi_n = \mathrm{e}^{\gamma/n}$ with a constant $\gamma < 0$. Note that for Theorem 4.1.3 and only here, instead of putting any direct assumption on the ε_j's, we assume rather some functional weak convergence of W_n^{pl} to W. This extends the scope of the result far beyond the case where the ε_j's are i.i.d. (for some Hölderian invariance principles, in the case of weakly dependent random variables, see Hamadouche [2000]).

Theorem 4.1.3. *In the case 1 where* (y_k) *is generated by* (4.1) *with* $\phi_n = \mathrm{e}^{\gamma/n}$, $\gamma < 0$, *suppose that the sequence of polygonal lines* $(n^{-1/2}W_n^{\mathrm{pl}})$ *converges weakly to the standard Brownian motion* W *either in* $\mathrm{C}[0,1]$ *or in* $\mathrm{H}_\alpha^o[0,1]$ *for some* $0 < \alpha < 1/2$. *Suppose moreover that* $y_{n,0} = o_P(n^{1/2})$. *Then* $n^{-3/2}S_n^{\mathrm{pl}}$ *converges weakly, as* $n \to \infty$, *in the space under consideration to the integrated Ornstein-Uhlenbeck process* J *defined by:*

$$J(t) := \int_0^t U_\gamma(s)\,\mathrm{d}s, \quad 0 \le t \le 1, \tag{4.5}$$

where $U_\gamma(s) = \int_0^s \mathrm{e}^{\gamma(s-r)}\,\mathrm{d}W(r)$.

Remark 4.1.4. The result in Theorem 4.1.3 is formulated for the variance equal to 1. If variance is known and equal to σ^2, then under the conditions of Theorem 4.1.3 the following result holds:

$$n^{-3/2}\sigma^{-1}S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{E}} J, \tag{4.6}$$

where E denotes either $\mathrm{C}[0,1]$ or $\mathrm{H}_\alpha^o[0,1]$ for $0 < \alpha < 1/2$.

Remark 4.1.5. If variance is unknown by Slutsky's Theorem it can be replaced in (4.6) by it's estimator

$$\widehat{\sigma}^2 := \frac{1}{n}\sum_{k=1}^n \widehat{\varepsilon}_k^2, \tag{4.7}$$

since Phillips [1987] established, that

$$\widehat{\sigma}^2 \xrightarrow[n\to\infty]{\mathrm{P}} \sigma^2. \tag{4.8}$$

Proof of theorem 4.1.3. Since the Banach spaces $(C[0,1], \|\ \|_\infty)$ and $(\mathrm{H}_0^o, \|\ \|_0)$ are isomorphic, the unified proof proposed here for the spaces $\mathrm{H}_\alpha^o[0,1]$, $0 \le \alpha < 1/2$, includes the special case of the space C[0, 1]. By Proposition 4.1.1 and assumption $y_{n,0} = o_P(n^{1/2})$, it is enough to give the proof in the case where $y_{n,0} = 0$.

The idea of the proof is to approximate the polygonal line $n^{-3/2} S_n^{\mathrm{pl}}$ by some linear interpolation of a smooth process J_n which is a functional of $n^{-1/2} W_n^{\mathrm{pl}}$, continuous in Hölder topology, with $\left\| n^{-3/2} S_n^{\mathrm{pl}} - J_n \right\|_\alpha = o_P(1)$.

Successive polygonal approximations of $n^{-3/2} S_n^{\mathrm{pl}}$.

We detail first the successive approximations of $\pi_{n,1} := n^{-3/2} S_n^{\mathrm{pl}}$ by the polygonal lines $\pi_{n,2}$, $\pi_{n,3}$, $\pi_{n,4}$ where the later has vertices $(l/n, V_{l,4})$ given by

$$V_{l,4} = \int_0^{l/n} n^{-1/2} W_n^{\mathrm{pl}}(s) \, \mathrm{d}s + \gamma \int_0^{l/n} \int_0^s \mathrm{e}^{\gamma(s-r)} n^{-1/2} W_n^{\mathrm{pl}}(r) \, \mathrm{d}r \, \mathrm{d}s, \tag{4.9}$$

and satisfies

$$\left\| n^{-3/2} S_n^{\mathrm{pl}} - \pi_{n,4} \right\|_\alpha = o_P(1). \tag{4.10}$$

To control the distance in Hölder norm between polygonal lines, we use the following property. Let π_n be a polygonal line with representation (3.2), page 20. As a consequence of (3.4), page 21, if we approximate each V_l by some $\widetilde{V}_l$ in such a way that $|V_l - \widetilde{V}_l| = o_P(n^{-\alpha})$, *uniformly* in $1 \leq l \leq n$, then the corresponding polygonal line $\widetilde{\pi_n}$ satisfies $\|\pi_n - \widetilde{\pi_n}\|_\alpha = o_P(1)$.

In what follows, we denote the successive polygonal lines approximating $n^{-3/2} S_n^{\mathrm{pl}}$ by $\pi_{n,i}$ and their vertices by $(l/n, V_{l,i})$, $i = 1, 2, 3, 4$. At each step we will use the following facts

$$\left\| n^{-1/2} W_n^{\mathrm{pl}} \right\|_\infty \quad \text{is stochastically bounded} \tag{4.11}$$

and

$$\omega_\alpha\left(n^{-1/2} W_n^{\mathrm{pl}}, \frac{1}{n} \right) \xrightarrow[n \to \infty]{\mathrm{P}} 0, \tag{4.12}$$

by tightness in $\mathrm{H}_\alpha^o[0, 1]$, $0 \leq \alpha < 1/2$, see Theorem 3.1.3 (page 22).

We start with $\pi_{n,1} = n^{-3/2} S_n^{\mathrm{pl}}$ for which

$$V_{l,1} = Y_l = n^{-3/2} \sum_{k=1}^{l} y_{k-1}.$$

We express y_k in terms of innovations

$$y_k = \sum_{j=1}^{k} \mathrm{e}^{(k-j)\gamma/n} \varepsilon_j.$$

Noting that $\varepsilon_j = W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) - W_n^{\mathrm{pl}}\left(\frac{j-1}{n}\right)$, we obtain

$$\begin{aligned} y_k &= \sum_{j=1}^{k} \mathrm{e}^{(k-j)\gamma/n} \left(W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) - W_n^{\mathrm{pl}}\left(\frac{j-1}{n}\right)\right) \\ &= W_n^{\mathrm{pl}}\left(\frac{k}{n}\right) + \sum_{j=1}^{k-1} \mathrm{e}^{(k-j)\gamma/n}(1-\mathrm{e}^{-\gamma/n}) W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) \\ &= W_n^{\mathrm{pl}}\left(\frac{k}{n}\right) + \frac{\gamma}{n}\sum_{j=1}^{k-1} \mathrm{e}^{(k-j)\gamma/n} W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) + \frac{\gamma^2 u_n}{2n^2}\sum_{j=1}^{k-1} \mathrm{e}^{(k-j)\gamma/n} W_n^{\mathrm{pl}}\left(\frac{j}{n}\right), \end{aligned}$$

where $u_n = 2n^2\gamma_n^{-2}\left(1-\mathrm{e}^{-\gamma/n} - \gamma n^{-1}\right)$. As

$$\mathrm{e}^{-\gamma/n} = 1 - \frac{\gamma}{n} + \frac{\gamma^2}{2n^2} + o\left(\frac{1}{n^2}\right),$$

it follows

$$u_n = -1 + \frac{2n^2}{\gamma^2} o\left(\frac{1}{n^2}\right) \to -1, \quad \text{as} \quad n \to \infty. \tag{4.13}$$

Now our first approximation consist in neglecting the last term in the sum above, which gives the polygonal line $\pi_{n,2}$ with

$$V_{l,2} = \frac{1}{n}\sum_{k=1}^{l} W_n\left(\frac{k-1}{n}\right) + \frac{\gamma}{n^2}\sum_{k=1}^{l}\sum_{j=1}^{k-2} \mathrm{e}^{(k-j-1)\gamma/n} W_n\left(\frac{j}{n}\right), \tag{4.14}$$

where $W_n := n^{-1/2} W_n^{\mathrm{pl}}$ for writing simplicity. For the approximation error, we have the following bound valid for $n \geq |\gamma|$:

$$|V_{l,2} - V_{l,1}| \leq \frac{\gamma^2 \mathrm{e}^{\gamma}}{2n} \left\| W_n \right\|_\infty .$$

Next, approximating Riemann sums by integrals in (4.14), we obtain the polygonal line $\pi_{n,3}$ with

$$V_{l,3} = \int_0^{l/n} W_n(s)\,\mathrm{d}s + \frac{\gamma}{n}\sum_{k=1}^{l} \mathrm{e}^{\gamma k/n}\int_0^{k/n} \mathrm{e}^{-\gamma r} W_n(r)\,\mathrm{d}r. \tag{4.15}$$

Let us estimate the error of approximation. For any $f \in \mathrm{C}[0,1]$,

$$\frac{1}{n}\sum_{j=1}^{k-k_0} f\Big(\frac{j+j_0}{n}\Big) - \int_0^{k/n} f(s)\,\mathrm{d}s = \sum_{j=1}^{k-k_0}\int_{(j-1)/n}^{j/n}\Big(f\Big(\frac{j+j_0}{n}\Big)-f(s)\Big)\,\mathrm{d}s - \int_{(k-k_0)/n}^{k/n} f(s)\,\mathrm{d}s,$$

whence

$$\left|\frac{1}{n}\sum_{j=1}^{k-k_0} f\Big(\frac{j+j_0}{n}\Big) - \int_0^{k/n} f(s)\,\mathrm{d}s\right| \le \omega_0\Big(f,\frac{1+j_0}{n}\Big) + \|f\|_\infty \frac{k_0}{n}. \tag{4.16}$$

Moreover,

$$\text{if } f \in \mathrm{H}_\alpha^o[0,1], \quad \omega_0(f,\delta) \le \omega_\alpha(f,\delta)\delta^\alpha. \tag{4.17}$$

If $f(t) = g(t)h(t)$ with g of class $\mathrm{C}^1[0,1]$ and $h \in \mathrm{C}[0,1]$,

$$\omega(gh,\delta) \le \|g\|_\infty \omega(h,\delta) + \|g'\|_\infty \|h\|_\infty \delta. \tag{4.18}$$

Using (4.16)–(4.18), we obtain the uniform bound

$$|V_{l,3} - V_{l,2}| \le \frac{1+\gamma\mathrm{e}^\gamma}{n^\alpha}\omega_\alpha\Big(W_n,\frac{1}{n}\Big) + \frac{\gamma\mathrm{e}^\gamma(2+\gamma\mathrm{e}^\gamma)}{n}\|W_n\|_\infty .$$

Finally, we replace the last sum remaining in (4.15) by an integral of $f_n(s) := \mathrm{e}^{\gamma s}\int_0^s \mathrm{e}^{-\gamma r}W_n(r)\,\mathrm{d}r$, $s \in [0,1]$, noting that $|f_n'(s)| \le (1+\gamma\mathrm{e}^\gamma)\|W_n\|_\infty$ for each $s \in [0,1]$. This gives the polygonal line $\pi_{n,4}$ with vertices

$$V_{l,4} = \int_0^{l/n} W_n(s)\,\mathrm{d}s + \gamma\int_0^{l/n}\mathrm{e}^{\gamma s}\int_0^s \mathrm{e}^{-\gamma r}W_n(r)\,\mathrm{d}r\,\mathrm{d}s. \tag{4.19}$$

The approximation error is given by the uniform bound

$$|V_{l,4} - V_{l,3}| \le \frac{1+\gamma\mathrm{e}^\gamma}{n}\|W_n\|_\infty .$$

Noting that $\pi_{n,4}$ is exactly the polygonal line defined by (4.9) (page 34), gathering all the estimate of errors above, recalling (3.4) (page 21), we obtain finally with some positive constants C_γ and C'_γ:

$$\Big\|n^{-3/2}S_n^{\mathrm{pl}} - \pi_{n,4}\Big\|_\alpha \le C_\gamma\omega_\alpha\Big(W_n,\frac{1}{n}\Big) + C'_\gamma\|W_n\|_\infty n^{\alpha-1}. \tag{4.20}$$

Recalling (4.11) and (4.12), it follows that

$$\left\| n^{-3/2} S_n^{\mathrm{pl}} - \pi_{n,4} \right\|_\alpha \xrightarrow[n\to\infty]{\mathrm{P}} 0,$$

so (4.10) (page 34) is proved.

Convergence of J_n.

Next we note that $\pi_{n,4}$ is exactly the linear interpolation at the points $t_{n,l} = l/n$ of the random function:

$$J_n(t) := \int_0^t n^{-1/2} W_n^{\mathrm{pl}}(s) \, \mathrm{d}s + \gamma \int_0^t \int_0^s \mathrm{e}^{\gamma(s-r)} n^{-1/2} W_n^{\mathrm{pl}}(r) \, \mathrm{d}r \, \mathrm{d}s.$$

By an elementary chaining argument, the interpolation error is controlled by

$$\| J_n - \pi_{n,4} \|_\alpha \leq 4\omega_\alpha \Big(J_n, \frac{1}{n} \Big),$$

which converges in probability to zero, provided that J_n converges weakly in $\mathrm{H}_\alpha^o[0,1]$. Indeed, if J_n converge weakly in $\mathrm{H}_\alpha^o[0,1]$, then it is tight in $\mathrm{H}_\alpha^o[0,1]$, thus, according to Theorem 3.1.3 (page 22), we obtain

$$\forall \epsilon > 0, \quad \lim_{n\to\infty} \mathbb{P} \left(\omega_\alpha \left(J_n, \frac{1}{n} \right) \geq \epsilon \right) = 0.$$

Now, it only remains to check that J_n converges weakly to J in $\mathrm{H}_\alpha^o[0,1]$. As the operator

$$\mathrm{H}_\alpha^o[0,1] \to \mathrm{H}_\alpha^o[0,1] \quad : \quad x \longmapsto \int_0^\bullet x(s) \, \mathrm{d}s + \gamma \int_0^\bullet \int_0^s \mathrm{e}^{\gamma(s-r)} x(r) \, \mathrm{d}r \, \mathrm{d}s$$

is continuous on $\mathrm{H}_\alpha^o[0,1]$, this last convergence follows from the convergence of $n^{-1/2} W_n^{\mathrm{pl}}$ to W (see (3.10), page 23). □

Taking into account the classical Donsker-Prohorov invariance principle (3.9), page 23, and the Hölderian invariance principle (3.10), page 23, we have the following corollary of Theorem 4.1.3 in the classical case of i.i.d. innovations.

Corollary 4.1.6. *Assume that (y_k) is generated by (4.1) with $\phi_n = \mathrm{e}^{\gamma/n}$, $\gamma < 0$ and that the (ε_k)'s are i.i.d. and centered. Then the weak convergence of $\sigma^{-1} n^{-3/2} S_n^{\mathrm{pl}}$ to J holds*

- *in $\mathrm{C}[0,1]$ provided that $\mathbb{E}\varepsilon_1^2 = \sigma^2 < \infty$ and $y_{n,0} = o_P(n^{1/2})$;*
- *in $\mathrm{H}_\alpha^o[0,1]$ for $0 < \alpha < 1/2$ under condition (3.11) (page 23) and $y_{n,0} = o_P(n^{1/2})$.*

4.1.2 Second type model

In this section we investigate the polygonal line process S_n^{pl} built on the y_k's, as defined by (4.2), where $\phi_n = 1 - \gamma_n/n$ and $\gamma_n \to \infty$ slower than n. Also the innovations (ε_k) are supposed i.i.d. with zero mean and known variance.

A key point in all the following limit theorems is to keep a good control on the asymptotic behavior of $\max_{1\le k\le n}|y_k|$. This is provided by the following Lemma which may be of independent interest.

Lemma 4.1.7. *Suppose the process (y_k) is generated by (4.1) and $\phi_n = 1-\gamma_n/n$, where (γ_n) is a sequence of non negative numbers such that $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$. Suppose moreover that $y_{n,0} = 0$. Let $p \ge 2$. Assume that the innovations (ε_k) are i.i.d. and satisfy*

$$\begin{aligned} &\lim_{t\to\infty} t^p P(|\varepsilon_0| > t) = 0, && \text{if } p > 2; \\ &\mathbb{E}\varepsilon_0^2 < \infty, && \text{if } p = 2. \end{aligned} \tag{4.21}$$

For $p \ge 2$, put $\alpha = 1/2 - 1/p$. Then

$$n^{-1/2}\gamma_n^{\alpha} \max_{1\le k\le n} |y_k| \xrightarrow[n\to\infty]{\text{P}} 0. \tag{4.22}$$

The proof of this Lemma is given in section 4.3.2, page 54.

We start with asymptotic behavior of S_n^{pl} in the space C[0, 1].

Theorem 4.1.8. *Suppose the process (y_k) is generated by (4.1) and $\phi_n = 1-\gamma_n/n$, where (γ_n) is a sequence of non negative numbers such that $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$. Assume also that the innovations (ε_k) are i.i.d. with $\mathbb{E}\varepsilon_k = 0$, $\mathbb{E}\varepsilon_k^2 = 1$ and that $y_{n,0} = o_P(n^{-1/2}(1-\phi_n)^{-1})$. Then the following convergence holds.*

$$n^{-1/2}(1-\phi_n)S_n^{\text{pl}} \xrightarrow[n\to\infty]{\text{C[0,1]}} W. \tag{4.23}$$

Proof. Using Proposition 4.1.1 and the assumption $y_{n,0} = o_P(n^{-1/2}(1-\phi_n)^{-1})$ it suffices to prove the result when $y_{n,0} = 0$. To prove (4.23), in view of the Donsker-Prohorov invariance principle (see Billingsley [1986]), it is enough to show that

$$\Delta_n = \|\xi_n\|_\infty \xrightarrow[n\to\infty]{\text{P}} 0, \tag{4.24}$$

where

$$\xi_n = \frac{1-\phi_n}{n^{1/2}} S_n^{\text{pl}} - n^{-1/2} W_n^{\text{pl}}.$$

We observe that ξ_n is a polygonal line with vertices at the points $t_{n,k} = k/n$, $0 \le k \le n$. Its supremum norm is reached at one of its vertices. Hence

$$\Delta_n = \sup_{0\le t\le 1} \left| \frac{1-\phi_n}{n^{1/2}} S_n^{\mathrm{pl}}(t) - n^{-1/2} W_n^{\mathrm{pl}}(t) \right| = n^{-1/2} \max_{1\le k\le n} \left| (1-\phi_n) \sum_{j=1}^{k} y_{j-1} - \sum_{j=1}^{k} \varepsilon_j \right|.$$

For every $k \ge 1$, it follows from (4.1) that $\sum_{j=1}^k y_j = \phi_n \sum_{j=1}^k y_{j-1} + \sum_{j=1}^k \varepsilon_j$, whence

$$(1-\phi_n) \sum_{j=1}^{k} y_{j-1} = -y_k + \sum_{j=1}^{k} \varepsilon_j, \tag{4.25}$$

so Δ_n reduces to

$$\Delta_n = n^{-1/2} \max_{1\le k\le n} |y_k|.$$

By the particular case where $p = 2$ in Lemma 4.1.7, the convergence (4.22) holds true with $\alpha = 0$. Hence $n^{-1/2} \max_{1\le k\le n} |y_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0$ and (4.24) follows. The proof of the theorem is complete. □

Next we extend Theorem 4.1.8 by proving convergence of S_n^{pl} in the Hölder space $\mathrm{H}_\beta^o[0,1]$, $0 < \beta < \alpha$, of course under stronger condition on (ε_k) than finiteness of the second moment. The necessity of an extra restriction on the divergence of γ_n like (4.27) below and the optimality of this later remain an open question.

Theorem 4.1.9. *Suppose* (y_k) *is generated by* (4.1) *and* $\phi_n = 1 - \gamma_n/n$, *where* (γ_n) *is a sequence of non negative numbers such that* $\gamma_n \to \infty$ *and* $\gamma_n/n \to 0$, *as* $n \to \infty$. *Assume also that the innovations* (ε_k) *are i.i.d. and satisfy condition* (3.11) *(page 23) for some* $p > 2$. *Put* $\alpha = \frac{1}{2} - \frac{1}{p}$. *Then for* $0 < \beta < \alpha$,

$$n^{-1/2}(1-\phi_n) S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\beta^o[0,1]} W, \tag{4.26}$$

provided that $y_{n,0} = o_P(n^{-1/2}(1-\phi_n)^{-1})$ *and*

$$\liminf_{n\to\infty} \gamma_n n^{-\frac{\beta}{\alpha}} > 0. \tag{4.27}$$

Proof. By Račkauskas and Suquet [2004a], condition (4.21) gives the weak convergence of $n^{-1/2} W_n^{\mathrm{pl}}$, defined by (3.8), page 23, to the standard Brownian motion in the space $\mathrm{H}_\alpha^o[0,1]$. By continuous embedding of Hölder spaces, the same convergence remains true in $\mathrm{H}_\beta^o[0,1]$ for $0 < \beta < \alpha$. Therefore it is enough to show that

$$D_{n,\beta} := \|\zeta_n\|_\beta \xrightarrow[n\to\infty]{\mathrm{P}} 0, \tag{4.28}$$

where

$$\zeta_n := n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} - n^{-1/2}W_n^{\mathrm{pl}}.$$

Note that ζ_n is a polygonal line with vertices at the points $t_{n,k} = k/n$, $0 \le k \le n$. According to Lemma 3.1.1, page 21, the Hölderian norm of such a polygonal line is reached at two vertices, so

$$\left\| n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} - n^{-1/2}W_n^{\mathrm{pl}} \right\|_\beta \le \max_{1\le j<k\le n} \frac{\left|n^{-1/2}(y_k - y_j)\right|}{|k/n - j/n|^\beta} \le 2n^{\beta-\frac{1}{2}} \max_{1\le k\le n} |y_k| \,.$$

Using Proposition 4.1.1 and the assumption $y_{n,0} = o_P(n^{-1/2}(1-\phi_n)^{-1})$ it suffices to prove (4.28) when $y_{n,0} = 0$. Then, by Lemma 4.1.7, $\max_{1\le k\le n} |y_k| = o_P(n^{1/2}\gamma_n^{-\alpha})$, so the convergence (4.28) is satisfied provided that

$$\limsup_{n\to\infty} \frac{n^\beta}{\gamma_n^\alpha} < \infty,$$

which is equivalent to our assumption (4.27). □

Remark 4.1.10. If variance of innovations is equal to σ^2, then under conditions of Theorem 4.1.8 we have

$$n^{-1/2}(1-\phi_n)\sigma^{-1}S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{C}[0,1]} W \tag{4.29}$$

and under conditions of Theorem 4.1.9 we obtain

$$n^{-1/2}(1-\phi_n)\sigma^{-1}S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\beta^o[0,1]} W. \tag{4.30}$$

Remark 4.1.11. If variance σ^2 is unknown by Slutsky's Theorem it can be replaced in (4.29) and (4.30) by its estimator defined by (4.7) if

$$\widehat{\sigma}^2 \xrightarrow[n\to\infty]{\mathrm{P}} \sigma^2.$$

And the latter result is true by Lemma 4.3.1.

4.2 Functional central limit theorems for residuals of nearly nonstationary processes

In this section we establish the convergence in Hölder spaces of the polygonal line processes $\widehat{W}_n^{\mathrm{pl}} = (\widehat{W}_n^{\mathrm{pl}}(t), t \in [0,1])$ build on the residuals $(\widehat{\varepsilon}_k)$

$$\widehat{W}_n^{\mathrm{pl}}(t) := \sum_{k=1}^{[nt]} \widehat{\varepsilon}_k + (nt - [nt])\widehat{\varepsilon}_{[nt]+1}, \quad t \in [0,1], \quad n \geq 0, \quad \widehat{W}_n^{\mathrm{pl}}(0) = 0. \tag{4.31}$$

We investigate the same two type of parameterizations as in previous section. The residuals of the model (4.1) are defined by

$$\widehat{\varepsilon}_k = y_k - \widehat{\phi}_n y_{k-1} = \varepsilon_k - (\widehat{\phi}_n - \phi_n) y_{k-1} \tag{4.32}$$

where $\widehat{\phi}_n$ is the LSE of the coefficient ϕ_n as defined by (2.2), page 7. We assume that innovations (ε_k) are centered and $\mathbb{E}\varepsilon_1^2 = 1$. The condition that variance is equal to 1 is just for the technical simplicity, but all the proofs holds also for $\mathbb{E}\varepsilon_1^2 = \sigma^2$.

The estimate of the Hölder norm (3.4), page 21, helps us to reduce the investigation of the asymptotic behaviour of the properly normalized random polygonal line $\widehat{W}_n^{\mathrm{pl}}$ to the case where the initialization in (4.1) is given by $y_{n,0} = 0$. Indeed let us associate to each autoregressive process $(y_{n,k})$ satisfying (4.1), the process $(y'_{n,k})$ defined by

$$y'_{n,k} = y_{n,k} - \phi_n^k y_{n,0}. \tag{4.33}$$

Then $(y'_{n,k})$ satisfies (4.1) with initialization $y'_{n,0} = 0$. Then we obtain

$$\widehat{\varepsilon}_j = \varepsilon_j - (\widehat{\phi}_n - \phi_n) y_{n,j-1}$$

and

$$\widehat{\varepsilon}'_j = \widehat{\varepsilon}_j + (\widehat{\phi}_n - \phi_n)\phi_n^{j-1} y_{n,0}.$$

So the above mentioned reduction may be formulated as follows.

Proposition 4.2.1. *Let $\widehat{W}_n^{\mathrm{pl}\prime}$ be the polygonal line process obtained by substituting in (4.31) the $\widehat{\varepsilon}_j$'s by the $\widehat{\varepsilon}'_j$'s. Assume that $n^{-1/2}\widehat{W}_n^{\mathrm{pl}\prime}$ converges in distribution in $\mathrm{H}_\alpha^o[0,1]$. Then $n^{-1/2}\widehat{W}_n^{\mathrm{pl}}$ converges in distribution in $\mathrm{H}_\alpha^o[0,1]$ to the same limit* *provided that* $c_n(\widehat{\phi}_n - \phi_n) = O_P(1)$ *and*

$$n^{1/2} c_n^{-1} y_{n,0} = o_P(1). \tag{4.34}$$

Proof. The stochastic process $n^{-1/2}(\widehat{W}_n^{\mathrm{pl}\prime} - \widehat{W}_n^{\mathrm{pl}})$ is a random polygonal line π_n. According to representation (3.2), π_n is determined by its vertices $(l/n, V_l)$, $0 \le l \le n$, $V_0 = 0$, where

$$V_l = \sum_{j=0}^{l-1} n^{-1/2}(\widehat{\phi}_n - \phi_n)\phi_n^j y_{n,0} = \frac{1-\phi_n^l}{1-\phi_n} n^{-1/2}(\widehat{\phi}_n - \phi_n) y_{n,0}.$$

Applying Lemma 3.3.2, page 24, from

$$\begin{aligned}\|\pi_n\|_\alpha &= \frac{n^{-1/2}\,|y_{n,0}|\,\left|\widehat{\phi}_n - \phi_n\right|}{1-\phi_n} \max_{1\le l<k\le n} \frac{\left|\phi_n^k - \phi_n^l\right|}{|k/n - l/n|^\alpha} \\ &= \frac{n^{-1/2}\,|y_{n,0}|\,\left|\widehat{\phi}_n - \phi_n\right|}{1-\phi_n} \max_{1\le l<k\le n} \frac{\left|(\phi_n^n)^{k/n} - (\phi_n^n)^{l/n}\right|}{|k/n - l/n|^\alpha}\end{aligned}$$

we obtain

$$\|\pi_n\|_\alpha \le n^{-1/2}\,|y_{n,0}|\,\left|\widehat{\phi}_n - \phi_n\right| \frac{(-n\ln\phi_n)}{1-\phi_n}.$$

Assuming that $c_n(\widehat{\phi}_n - \phi_n) = O_P(1)$, we obtain

$$\|\pi_n\|_\alpha = n^{1/2} c_n^{-1}\,|y_{n,0}|\,\frac{(-\ln\phi_n)}{1-\phi_n} O_P(1).$$

Finally, as ϕ_n tends to 1, for the two models under consideration, this gives

$$\|\pi_n\|_\alpha = O_P(n^{1/2} c_n^{-1}\,|y_{n,0}|), \quad n\to\infty.$$

Since $n^{-1/2}\widehat{W}_n^{\mathrm{pl}\prime}$ converges in distribution in $\mathrm{H}_\alpha^o[0,1]$, we deduce $n^{-1/2}\widehat{W}_n^{\mathrm{pl}}$ converges in $\mathrm{H}_\alpha^o[0,1]$ to the same limit distribution provided that $n^{1/2}c_n^{-1}y_{n,0} = o_P(1)$. □

4.2.1 First type model

For the process $\widehat{W}_n^{\mathrm{pl}}$ defined by (4.31) we prove invariance principle and we find necessary and sufficient condition for it.

Theorem 4.2.2. *Let* $\alpha \in (0, 1/2)$. *Suppose that* (y_k) *is generated by* (4.1), $\phi_n = \mathrm{e}^{\gamma/n}$ *and* $\gamma < 0$ *is a constant. Moreover assume that* (ε_k) *are independent, identically distributed random variables with* $\mathbb{E}\varepsilon_0 = 0$ *and* $\mathbb{E}\varepsilon_0^2 = 1$ *and*

$y_{n,0} = o_P(n^{1/2})$. *Then*

$$n^{-1/2}\widehat{W}_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}^o_\alpha[0,1]} W - A^{-1}BJ \tag{4.35}$$

if and only if condition (3.11) *(page 23) holds. Here* $A = \int_0^1 U_\gamma^2(t)\,\mathrm{d}t$, $B = \int_0^1 U_\gamma(t)\,\mathrm{d}W(t)$ *and* $J(t)$ *is an integrated Ornstein-Uhlenbeck process defined by* (4.5).

Remark 4.2.3. If variance σ^2 is known then under conditions of Theorem 4.2.2, we obtain

$$n^{-1/2}\sigma^{-1}\widehat{W}_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}^o_\alpha[0,1]} W - A^{-1}BJ \tag{4.36}$$

if and only if condition (3.11) holds.

Remark 4.2.4. If variance is unknown by Slutsky's Theorem it can be replaced in (4.36) by its estimator defined by (4.7) via Phillips [1987] result (4.8).

For the proof of the Theorem 4.2.2 we need the following technical lemmas whose proofs are deferred to subsection 4.3.3.

Lemma 4.2.5. *Let* N_n, D_n, N, D *be real valued random variables with* D_n *and* D *non negative. Assume that* $P(D = 0) = 0$, $P(D_n = 0)$ *tends to* 0 *and that* (N_n, D_n) *converges in distribution on* $\mathbb{R}^2$ *to* (N, D). *Define*

$$\Phi_n := \begin{cases} \frac{N_n}{D_n} & on\ \{D_n \neq 0\} \\ 0 & on\ \{D_n = 0\} \end{cases}$$

Then Φ_n *converges in distribution to* N/D.

Lemma 4.2.6. *Suppose that the process* (y_k) *is defined by* (4.1) *with* $\phi_n = \mathrm{e}^{\gamma/n}$, $\gamma < 0$ *and* $y_0 = 0$. *Let* (ε_k) *are i.i.d. random variables with mean* 0 *and satisfies condition* (3.11) *(page 23). Define*

$$A_{n,0} := n^{-2}\sum_{k=1}^{n} y_{k-1}^2,$$

$$D(n^{-1/2}W_n^{\mathrm{pl}}) := \int_0^1 \left(n^{-1/2}W_n^{\mathrm{pl}} + \gamma\int_0^r \mathrm{e}^{(r-s)\gamma}n^{-1/2}W_n^{\mathrm{pl}}\,\mathrm{d}s\right)^2\mathrm{d}r.$$

Then

$$\left|D(n^{-1/2}W_n^{\mathrm{pl}}) - A_{n,0}\right| = o_{\mathrm{P}}(n^{-\alpha}). \tag{4.37}$$

Lemma 4.2.7. *Suppose that the process* (y_k) *is defined by* (4.1) *with* $\phi_n = \mathrm{e}^{\gamma/n}$*,* $\gamma < 0$ *and* $y_0 = 0$*. Let* (ε_k) *are i.i.d. random variables with mean* 0 *and satisfies condition* (3.11) *(page 23). Define*

$$\begin{aligned} B_{n,0} &:= n^{-1}\sum_{k=1}^{n}\varepsilon_k y_{k-1}, \\ N(n^{-1/2}W_n^{\mathrm{pl}}) &:= \frac{1}{2}\left(n^{-1/2}W_n^{\mathrm{pl}}(1) + \gamma\int_0^1 \mathrm{e}^{(1-s)\gamma}n^{-1/2}W_n^{\mathrm{pl}}(s)\,\mathrm{d}s\right)^2 \\ &\quad - \gamma\int_0^1\left(n^{-1/2}W_n^{\mathrm{pl}}(r) + \gamma\int_0^r \mathrm{e}^{(r-s)\gamma}n^{-1/2}W_n^{\mathrm{pl}}(s)\,\mathrm{d}s\right)^2\mathrm{d}r - \frac{1}{2}. \end{aligned}$$

Then

$$|N(W_n) - B_{n,0}| = o_{\mathrm{P}}(n^{-\alpha}). \tag{4.38}$$

Proof of theorem 4.2.2. Proposition 4.2.1 and assumption $y_{n,0} = o_P(n^{1/2})$ enables us to reduce the proof to the case where $y_{n,0} = 0$.

To prove sufficiency, at first, we express $\widehat{W}_n^{\mathrm{pl}}$ in terms of W_n^{pl} and S_n^{pl}:

$$\begin{aligned} n^{-1/2}\widehat{W}_n^{\mathrm{pl}} &= n^{-1/2}W_n^{\mathrm{pl}} - n^{-1/2}(\widehat{\phi}_n - \phi_n)S_n^{\mathrm{pl}} \\ &= n^{-1/2}W_n^{\mathrm{pl}} - \frac{n^{-1}\sum_{k=1}^n \varepsilon_k y_{k-1}}{n^{-2}\sum_{k=1}^n y_{k-1}^2}\cdot n^{-3/2}S_n^{\mathrm{pl}}. \end{aligned} \tag{4.39}$$

Note that according to (2.2), page 7,

$$\widehat{\phi}_n - \phi_n = \frac{\sum_{j=1}^n y_j y_{j-1}}{\sum_{j=1}^n y_{j-1}^2} - \phi_n = \frac{n^{-1}\sum_{k=1}^n \varepsilon_k y_{k-1}}{n^{-2}\sum_{k=1}^n y_{k-1}^2}$$

Next, using U_γ definition (see (3.1), page 19) one obtains

$$\int_0^1 U_\gamma(r)\,\mathrm{d}W(r) = \frac{1}{2}\left(U_\gamma^2(1) - 1 - 2\gamma\int_0^1 U_\gamma^2(r)\,\mathrm{d}r\right)$$

(see for example Phillips [1987]), so we notice that

$$T(W) = W - A^{-1}BJ$$

where T is the following operator

$$T : \mathrm{H}_\alpha^o[0,1] \longrightarrow \mathrm{H}_\alpha^o[0,1] : \quad x \longmapsto T(x) := x - \frac{N(x)}{D(x)}F(x)$$

here

$$\begin{aligned} N(x) &:= \frac{1}{2}\left(x(1)+\gamma\int_0^1 \mathrm{e}^{(1-s)\gamma}x(s)\,\mathrm{d}s\right)^2 \\ &\quad -\gamma\int_0^1\left(x(r)+\gamma\int_0^r \mathrm{e}^{(r-s)\gamma}x(s)\,\mathrm{d}s\right)^2\mathrm{d}r-\frac{1}{2} \\ D(x) &:= \int_0^1\left(x(r)+\gamma\int_0^r \mathrm{e}^{(r-s)\gamma}x(s)\,\mathrm{d}s\right)^2\mathrm{d}r \\ F(x)(t) &:= \int_0^t\left(x(r)+\gamma\int_0^r \mathrm{e}^{(r-s)\gamma}x(s)\,\mathrm{d}s\right)\mathrm{d}r, \quad t\in[0,1]. \end{aligned}$$

for $x \in \mathrm{H}_\alpha^o[0,1]$. It is obvious, that the domain of operator T is

$$H_T := \left\{x \in \mathrm{H}_\alpha^o[0,1] : D(x) \neq 0\right\}.$$

Further note, that H_T is the Hölder space deprived of the zero functions. Indeed, from the equations $D(x)=0$, recalling that x is a continuous function on $[0,1]$, we obtain for every $r \in [0,1]$

$$x(r)+\gamma\int_0^r \mathrm{e}^{(r-s)\gamma}x(s)\,\mathrm{d}s = 0. \tag{4.40}$$

Thus any continuous solution x of $D(x)=0$ satisfies

$$x(r) = -\gamma \mathrm{e}^{r\gamma}\int_0^r \mathrm{e}^{-s\gamma}x(s)\,\mathrm{d}s. \tag{4.41}$$

Further from the continuity of x follows, that the right hand side of (4.41) is obviously derivable, consequently x is itself derivable and for all $r \in (0,1)$ we obtain $x'(r)=0$. This implies that x is a constant on $[0,1]$ (it is continuous at 0 and at 1). Conversely, let r tend to 0 in (4.41). Then by continuity of x we obtain $x(0)=0$ and since x is a constant, $x(r)=0$ for every $r \in [0,1]$. Thus we obtain

$$\mathbb{P}(W \in \mathrm{H}_\alpha^o[0,1] \setminus H_T) = \mathbb{P}(W=0).$$

Next, we observe, that if $W=0$ it follows that $W(1)=0$, i.e., event $\{W=0\}$ is included in the event $\{W(1)=0\}$. Recall, that $W(t) \sim N(0,t)$, so $W(1)$ is a standard Gaussian random variable, then $\mathbb{P}(W(1)=0)=0$ and this gives

$$\mathbb{P}(W=0)=0. \tag{4.42}$$

We obtain the convergence (4.35) by proving that

(a) T is continuous operator on H_T and $\mathbb{P}(W \in \mathrm{H}_\alpha^o[0,1] \setminus H_T)=0$,

(b) $\left\| n^{1/2}\widehat{W}_n^{\mathrm{pl}} - T(n^{-1/2}W_n^{\mathrm{pl}}) \right\|_\alpha \xrightarrow[n\to\infty]{\mathrm{P}} 0.$

We start with the continuity of T. Operator T is the difference of two operators. The first one is the identity on $\mathrm{H}_\alpha^o[0,1]$, obviously continuous. The second one is

$$\widetilde{T}(x) := \frac{N(x)}{D(x)} \cdot F(x), \quad x \in H_T.$$

First we show that $N : \mathrm{H}_\alpha^o[0,1] \to \mathbb{R}$ and $D : \mathrm{H}_\alpha^o[0,1] \to \mathbb{R}$ are continuous. Let us check first the continuity of D. By triangular inequality of L_2 norm applied to the function $f(x)(r) = x(r) + \gamma \int_0^r \mathrm{e}^{(r-s)\gamma} x(s)\,\mathrm{d}s$,

$$\begin{aligned}
\left| D^{1/2}(x) - D^{1/2}(y) \right| &= \left| \left(\int_0^1 (f(x)(r))^2 \,\mathrm{d}r \right)^{1/2} - \left(\int_0^1 (f(y)(r))^2 \,\mathrm{d}r \right)^{1/2} \right| \\
&\le \left| \left(\int_0^1 \left((x(r) - y(r)) + \gamma \int_0^r \mathrm{e}^{(r-s)\gamma} (x(s) - y(s))\,\mathrm{d}s \right)^2 \mathrm{d}r \right)^{1/2} \right| \\
&\le \left| \left(\int_0^1 \left(|x(r) - y(r)| + \gamma \int_0^r \mathrm{e}^{(r-s)\gamma} \,|x(s) - y(s)|\, \mathrm{d}s \right)^2 \mathrm{d}r \right)^{1/2} \right| \\
&\le \|x - y\|_\infty \left(\frac{1}{2\gamma} (\mathrm{e}^{2\gamma} - 1) \right)^{1/2}.
\end{aligned}$$

Here we remark that if $h \in \mathrm{H}_\alpha^o[0,1]$, for every t in $[0,1]$

$$|h(t)| \le |h(0)| + |h(t) - h(0)| \le |h(0)| + \omega_\alpha(h,1)t^\alpha \le |h(0)| + \omega_\alpha(h,1),$$

whence $\|h\|_\infty \le \|h\|_\alpha$. Applying this to $h = x - y$ gives finally

$$\left| D^{1/2}(x) - D^{1/2}(y) \right| \le \left(\frac{1}{2\gamma} (\mathrm{e}^{2\gamma} - 1) \right)^{1/2} \|x - y\|_\alpha .$$

This implies that $D^{1/2}$ is continuous on $\mathrm{H}_\alpha^o[0,1]$, and so is D. Using the same arguments, we obtain the continuity of N on $\mathrm{H}_\alpha^o[0,1]$.

So the ratio N/D is continuous as ratio of two continuous functions except on the subset of $\mathrm{H}_\alpha^o[0,1]$, where $D(x) = 0$, that is at the null function on $[0,1]$.

As F is linear, it is enough to show its continuity at 0. Consider

$$\|F(x)\|_\alpha = |F(x)(0)| + \sup_{0 \le t' < t \le 1} \frac{|F(x)(t) - F(x)(t')|}{|t - t'|^\alpha}.$$

Noting $\|x\|_\infty \le \|x\|_\alpha$, we see that

$$\begin{aligned}|F(x)(t)-F(x)(t')| &= \left|\int_{t'}^{t}\left(x(r)+\gamma\int_0^r \mathrm{e}^{(r-s)\gamma}x(s)\,\mathrm{d}s\right)\,\mathrm{d}r\right|\\ &\le (1+\gamma\mathrm{e}^{\gamma})\,\|x\|_\alpha\,|t-t'|\,.\end{aligned}$$

Since $F(x)(0)=0$, we obtain

$$\|F(x)\|_\alpha \le (1+\gamma\mathrm{e}^{\gamma})\,\|x\|_\alpha \tag{4.43}$$

which gives the continuity of F

The continuity of $\widetilde{T}$ on H_T follows easily from the continuity of N, D and F. Finally, operator T is continuous on H_T as the difference of two continuous operators.

As the operator T is continuous on H_T and (4.42) holds, also the Hölderian invariance principle holds (see (3.10), page 23), we have

$$T(n^{-1/2}\widehat{W}_n^{\mathrm{pl}}) \xrightarrow[n\to\infty]{\mathrm{H}^{\mathrm{o}}_\alpha[0,1]} T(W) = W - A^{-1}BJ, \tag{4.44}$$

by continuous mapping theorem (for details see Billingsley [1986], Theorem 5.1)

Next we check that $\left\|n^{-1/2}\widehat{W}_n^{\mathrm{pl}} - T(n^{-1/2}W_n^{\mathrm{pl}})\right\|_\alpha$ goes to 0 in probability. Due to approximations of $n^{-1}\sum_{k=1}^n \varepsilon_k y_{k-1}$ and $n^{-2}\sum_{k=1}^n y_{k-1}^2$ by integrals (see Lemmas 4.2.6 and 4.2.7)

$$\begin{aligned}N\left(n^{-1/2}W_n^{\mathrm{pl}}\right) &:= \frac{1}{2}\left(n^{-1/2}W_n^{\mathrm{pl}}(1)+\gamma\int_0^1 \mathrm{e}^{(1-s)\gamma}n^{-1/2}W_n^{\mathrm{pl}}(s)\,\mathrm{d}s\right)^2\\ &\quad -\gamma\int_0^1\left(n^{-1/2}W_n^{\mathrm{pl}}(r)+\gamma\int_0^r \mathrm{e}^{(r-s)\gamma}n^{-1/2}W_n^{\mathrm{pl}}(s)\,\mathrm{d}s\right)^2\mathrm{d}r-\frac{1}{2},\\ D\left(n^{-1/2}W_n^{\mathrm{pl}}\right) &:= \int_0^1\left(n^{-1/2}W_n^{\mathrm{pl}}(r)+\gamma\int_0^r \mathrm{e}^{(r-s)\gamma}n^{-1/2}W_n^{\mathrm{pl}}(s)\,\mathrm{d}s\right)^2\mathrm{d}r\end{aligned}$$

we obtain

$$\begin{aligned}n^{-1}\sum_{k=1}^n \varepsilon_k y_{k-1} &:= N\left(n^{-1/2}W_n^{\mathrm{pl}}\right)+R_n\\ n^{-2}\sum_{k=1}^n y_{k-1}^2 &:= D\left(n^{-1/2}W_n^{\mathrm{pl}}\right)+\widetilde{R}_n,\end{aligned}$$

where $R_n = o_{\mathrm{P}}(n^{-\alpha})$ and $\widetilde{R}_n = o_{\mathrm{P}}(n^{-\alpha})$. We have also

$$n^{-3/2}S_n^{\mathrm{pl}}(t) = F(n^{-1/2}W_n^{\mathrm{pl}})(t)+\widetilde{\widetilde{R}}_n,\quad t\in[0,1]$$

where $\widetilde{\widetilde{R}}_n = o_{\mathrm{P}}(n^{-\alpha})$ (for details see the proof of theorem 4.1.3, page 33).

Further setting $W_n := n^{-1/2} W_n^{\mathrm{pl}}$ and writing formally

$$n^{-1/2}\widehat{W}_n^{\mathrm{pl}} = W_n - \frac{N(W_n) + R_n}{D(W_n) + \widetilde{R}_n} \cdot \left(F(W_n)(t) + \widetilde{\widetilde{R}}_n \right),$$

we have formally

$$\left\| n^{-1/2}\widehat{W}_n^{\mathrm{pl}} - T(W_n) \right\|_\alpha \leq \left| \frac{N(W_n) + R_n}{D(W_n) + \widetilde{R}_n} - \frac{N(W_n)}{D(W_n)} \right| \left\| F(W_n) + \widetilde{\widetilde{R}}_n \right\|_\alpha + \left| \frac{N(W_n)}{D(W_n)} \right| \left\| \widetilde{\widetilde{R}}_n \right\|_\alpha .$$

For the moment, such writing is just formal because here arises a problem : the denominators $D(W_n)$ and $D(W_n) + \widetilde{R}_n$ may vanish with a positive probability (unlike $D(W)$). This lead us to introduce the random variables Φ_n and $\widetilde{\Phi}_n$ defined by

$$\Phi_n := \begin{cases} \frac{N(W_n)}{D(W_n)} & \text{on } \{D(W_n) \neq 0\} \\ 0 & \text{on } \{D(W_n) = 0\} \end{cases} \qquad \widetilde{\Phi}_n := \begin{cases} \frac{N(W_n)+R_n}{D(W_n)+\widetilde{R}_n} & \text{on } \{D(W_n) + \widetilde{R}_n \neq 0\} \\ 0 & \text{on } \{D(W_n) + \widetilde{R}_n = 0\} \end{cases}$$

Consider the event $\{D(W_n) = 0\}$. It can occur if and only if the polygonal line is the null function on $[0,1]$, which is equivalent to $\varepsilon_i = 0$, $\forall i \in \{1, \dots, n\}$. Putting $p := P(\varepsilon_1 = 0)$ and discarding the degenerated case $p = 1$, we obtain by independence and identical distribution of the innovations that $P(D(W_n) = 0) = p^n$. So for every $p \in [0,1)$,

$$P(D(W_n) = 0) \xrightarrow[n\to\infty]{} 0. \tag{4.45}$$

Coming back to the decomposition of $n^{1/2}\widehat{W}_n^{\mathrm{pl}}$ and modifying the definition of $T(W_n)$ as $T(W_n) = W_n - \Phi_n F(W_n)$ (it suffices to define $T(0) := 0$ for that), we can recast the estimate of $\left\| n^{1/2}\widehat{W}_n^{\mathrm{pl}} - T(W_n) \right\|_\alpha$ as

$$\left\| n^{1/2}\widehat{W}_n^{\mathrm{pl}} - T(W_n) \right\|_\alpha \leq \left| \Phi_n - \widetilde{\Phi}_n \right| \left\| F(W_n) + \widetilde{\widetilde{R}}_n \right\|_\alpha + |\Phi_n| \left\| \widetilde{\widetilde{R}}_n \right\|_\alpha .$$

By continuous mapping, $(N(W_n), D(W_n))$ converges in distribution in $\mathbb{R}^2$ to $(N(W), D(W)) = (B, A)$. Accounting $P(D(W) = 0) = 0$ and (4.45), lemma 4.2.5 gives us the convergence in distribution of Φ_n to B/A and in particular Φ_n is stochastically bounded.

Since $\left\| \widetilde{\widetilde{R}}_n \right\|_\alpha$ converge to 0 in probability and $\|F(W_n)\|_\alpha$ is stochastically bounded, it remains only to check that $\left| \Phi_n - \widetilde{\Phi}_n \right|$ converges to 0 in probability.

Note that on the event $\{D(W_n) \neq 0\} \cap \{D(W_n) + \widetilde{R}_n \neq 0\}$,

$$\left|\Phi_n - \widetilde{\Phi}_n\right| \leq \frac{|R_n|}{\left|D(W_n) + \widetilde{R}_n\right|} + \left|\frac{N(W_n)}{D(W_n)}\right| \cdot \frac{\left|\widetilde{R}_n\right|}{\left|D(W_n) + \widetilde{R}_n\right|}$$

and that estimate remains true on the whole probability space if we redefine by 0 the fractions whose denominator vanishes in the above formula. So the problem reduces to proving that

$$\frac{|R_n|}{\left|D(W_n) + \widetilde{R}_n\right|} \xrightarrow[n\to\infty]{\mathrm{P}} 0 \quad \text{and} \quad \frac{\left|\widetilde{R}_n\right|}{\left|D(W_n) + \widetilde{R}_n\right|} \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

We detail only the first convergence. Let us fix an $\epsilon > 0$, we want to prove that $P(|R_n| / \left|D(W_n) + \widetilde{R}_n\right| \leq \epsilon)$ tends to 1. Let us fix an arbitrary $\delta \in (0, 1)$. Since the distribution function of $D(W)$ is null and continuous at 0, we can find $\eta > 0$ such that $P(D(W) \leq \eta) < \delta$ or equivalently $P(D(W) > \eta) > 1 - \delta$. There is no restriction in assuming that η is itself a point of continuity of the distribution function of $D(W)$. Hence by convergence in distribution of $D(W_n)$ to $D(W)$, there is an integer n_0 such that

$$\forall n \geq n_0, \quad P(D(W_n) > \eta) > 1 - 2\delta.$$

Next we can find $n_1 \geq n_0$ such that

$$\forall n \geq n_1, \quad P\left(\left|\widetilde{R}_n\right| > \frac{\eta}{2}\right) < \delta.$$

We can find a $n_2 \geq n_1$ such that

$$\forall n \geq n_2, \quad P\left(|R_n| > \eta\epsilon\right) < \delta.$$

From this we deduce that

$$P\left(\frac{|R_n|}{\left|D(W_n) + \widetilde{R}_n\right|} < 2\epsilon \text{ and } D(W_n) > 0\right) > 1 - 4\delta$$

and recalling that $P(D(W_n) = o)$ tends to 0, this establishes the expected convergence in probability.

And finally the convergence (4.35) is established.

Next step is to prove the necessity. From (4.35), the sequence $(n^{-1/2}\widehat{W}_n^{\mathrm{pl}})$ is tight on $\mathrm{H}^o_\alpha[0,1]$ and this implies that for every $\epsilon > 0$,

$$\lim_{\delta\to 0} \sup_{n\geq 1} P\left(\omega_\alpha(n^{-1/2}\widehat{W}_n^{\mathrm{pl}}, \delta) > \epsilon\right) = 0,$$

see e.g. Theorem 13 in Suquet [1999]. This clearly entails that

$$\omega_\alpha\left(n^{-1/2}\widehat{W}_n^{\mathrm{pl}}, \frac{1}{n}\right) \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

Observing that

$$\begin{aligned}\frac{n^{-1/2}\max_{1\leq k\leq n}|\widehat{\varepsilon}_k|}{\frac{1}{n^\alpha}} &= \frac{n^{-1/2}\max_{1\leq k\leq n}\left|\widehat{W}_n^{\mathrm{pl}}(k/n) - \widehat{W}_n^{\mathrm{pl}}((k-1)/n)\right|}{\frac{1}{n^\alpha}} \\ &\leq \omega_\alpha\left(n^{-1/2}\widehat{W}_n^{\mathrm{pl}}, \frac{1}{n}\right),\end{aligned}$$

we obtain $n^{-1/2}\max_{1\leq k\leq n}|\widehat{\varepsilon}_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0$.

Next decompose $\widehat{\varepsilon}_k = \varepsilon_k - (\widehat{\phi_n} - \phi_n)y_{k-1}$. Denote by $y_{[n\bullet]}$ the step process $(y_{[nt]},\ t \in [0,1])$. Recall that by Phillips [1987] Lemma 1, part (a), $n^{-1/2}y_{[n\bullet]}$ converges in distribution in $\mathrm{D}[0,1]$ to an Ornstein-Uhlenbeck process. As the supremum norm of such a step process is obviously reached at one of the points $t = k/n$, $0 \leq k \leq n$, this convergence implies the stochastic boundedness of $\max_{1\leq k\leq n}\left|n^{-1/2}y_{k-1}\right| = \left\|n^{-1/2}y_{[n\bullet]}\right\|_\infty$. Notice, that

$$n^{\alpha-1/2}\max_{1\leq k\leq n}\left|(\widehat{\phi_n} - \phi_n)y_{k-1}\right| \leq n^{\alpha-1}\left|n(\widehat{\phi_n} - \phi_n)\right|\max_{1\leq k\leq n}\left|n^{-1/2}y_{k-1}\right| \xrightarrow[n\to\infty]{\mathrm{P}} 0,$$

because from Phillips [1987] (Theorem 1, part (a)) $\left|n(\widehat{\phi_n} - \phi_n)\right|$ is also stochastically bounded. It follows then

$$n^{\alpha-1/2}\max_{1\leq k\leq n}|\varepsilon_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0,$$

which gives the condition (3.11), page 23, due to independence of (ε_k). □

4.2.2 Second type model

For the second type model we obtain the result of convergence $n^{-1/2}\widehat{W}_n^{\mathrm{pl}}$ to Wiener process in $\mathrm{H}^o_\beta[0,1]$ for $0 < \beta \leq \alpha$ assuming additionally some rate of divergence for γ_n.

Theorem 4.2.8. *Suppose $(y_{n,k})$ is generated by (4.1) and $\phi_n = 1 - \gamma_n/n$, where (γ_n) is a sequence of nonnegative numbers, $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$.*

Assume also that the innovations (ε_k) *are i.i.d. and satisfy condition* (3.11)*:*

$$\lim_{t\to\infty} t^p \mathbb{P}(|\varepsilon_0| > t) = 0$$

for some $p > 2$. *Put* $\alpha = \frac{1}{2} - \frac{1}{p}$. *Then for* $0 < \beta \leq \alpha$*,*

$$n^{-1/2}\widehat{W}_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}^o_\beta[0,1]} W, \tag{4.46}$$

if $y_{n,0} = o((1-\phi_n)^{-1/2})$ *and*

$$\liminf_{n\to\infty} \gamma_n n^{-\frac{2\beta}{1+2\alpha}} > 0. \tag{4.47}$$

Remark 4.2.9. If variance of innovations σ^2 is known, then under conditions of Theorem 4.2.8, we obtain

$$n^{-1/2}\sigma^{-1}\widehat{W}_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}^o_\beta[0,1]} W. \tag{4.48}$$

Remark 4.2.10. If variance is unknown by Slutsky's Theorem it can be replaced in (4.48) by its estimator defined by (4.7) (page 33) via Lemma 4.3.1.

Proof of Theorem 4.2.8. Condition (3.11), page 23, (see Račkauskas and Suquet [2004b]) gives the weak convergence of W_n^{pl}, defined by (3.8), page 23, to the standard Brownian motion in the space $\mathrm{H}^o_\alpha[0,1]$. By continuous embedding of Hölder spaces, the same convergence remains true in $H^0_\beta[0,1]$ for $0 < \beta \leq \alpha$. Therefore to obtain (4.46) it suffices to prove that

$$\Delta_{n,\beta} := \left\| n^{-1/2}\widehat{W}_n^{\mathrm{pl}} - n^{-1/2}W_n^{\mathrm{pl}} \right\|_\beta \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

We first establish the useful inequality:

$$\left\| S_n^{\mathrm{pl}} \right\|_\beta \leq \frac{n}{\gamma_n}\left[\left\| W_n^{\mathrm{pl}} \right\|_\beta + 2n^\beta \max_{1\leq k\leq n} |y_k| \right], \tag{4.49}$$

where S_n^{pl} is defined by (4.2), page 31. We have for $1 \leq j < k \leq n$,

$$S_n^{\mathrm{pl}}(k/n) - S_n^{\mathrm{pl}}(j/n) = (1-\phi_n)^{-1}\left(W_n^{\mathrm{pl}}(k/j) - W_n^{\mathrm{pl}}(j/n) - y_k + y_j \right).$$

Recalling that the Hölder norm of a polygonal line is reached at some pair of vertices (see Lemma 3.1.1, page 21) and that $S_n^{\mathrm{pl}}(0) = 0$, we have

$$\left\| S_n^{\mathrm{pl}} \right\|_\beta = \max_{1\leq j<k\leq n} \frac{\left| S_n^{\mathrm{pl}}(k/n) - S_n^{\mathrm{pl}}(j/n) \right|}{|k/n - j/n|^\beta}$$

$$
\begin{aligned}
&= \max_{1\le j<k\le n} \frac{\left|(1-\phi_n)^{-1}\left(W_n^{\mathrm{pl}}(k/j) - W_n^{\mathrm{pl}}(j/n) - y_k + y_j\right)\right|}{|k/n - j/n|^{\beta}} \\
&\le \frac{n}{\gamma_n}\left[\max_{1\le j<k\le n} \frac{\left|W_n^{\mathrm{pl}}(k/j) - W_n^{\mathrm{pl}}(j/n)\right|}{|k/n-j/n|^{\beta}} + \max_{1\le j<k\le n}\frac{|y_k-y_j|}{|k/n-j/n|^{\beta}}\right] \\
&= \frac{n}{\gamma_n}\left[\left\|W_n^{\mathrm{pl}}\right\|_{\beta} + \max_{1\le j<k\le n}\frac{|y_k-y_j|}{|k/n-j/n|^{\beta}}\right].
\end{aligned}
$$

This leads to (4.49) via the elementary estimate

$$
\max_{1\le j<k\le n}\frac{|y_k-y_j|}{|k/n-j/n|^{\beta}} \le 2n^{\beta}\max_{1\le k\le n}|y_k|. \tag{4.50}
$$

Note, that $\widehat{W}_n^{\mathrm{pl}} = W_n^{\mathrm{pl}} + (\phi_n - \widehat{\phi}_n)S_n^{\mathrm{pl}}$, see 4.39, page 44, thus we have

$$
\Delta_{n,\beta} = n^{-1/2}|\phi_n - \widehat{\phi}_n|\left\|S_n^{\mathrm{pl}}\right\|_{\beta}.
$$

By results in Giraitis and Phillips [2006], there is a positive random variable M not depending on n, such that $|\phi_n - \widehat{\phi}_n| \le Mn^{-1}\gamma_n^{1/2}$, so accounting (4.49), we can bound $\Delta_{n,\beta}$ by:

$$
\Delta_{n,\beta} \le Mn^{-1/2}\gamma_n^{-1/2}\left(\left\|W_n^{\mathrm{pl}}\right\|_{\beta} + 2n^{\beta}\max_{1\le k\le n}|y_k|\right).
$$

As $n^{-1/2}\left\|W_n^{\mathrm{pl}}\right\|_{\beta}$ is stochastically bounded, the proof of the Theorem is finally reduced in checking that

$$
n^{-1/2+\beta}\gamma_n^{-1/2}\max_{1\le k\le n}|y_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0.
$$

By Lemma 4.1.7, $\max_{1\le k\le n}|y_k| = o_P(n^{1/2}\gamma_n^{-\alpha})$, so the above convergence is satisfied provided that

$$
\limsup_{n\to\infty}\frac{n^{\beta}}{\gamma_n^{1/2+\alpha}} < \infty,
$$

which is equivalent to assumption (4.47). □

4.3 Supplementary results

In this section, we provide supplementary results. At first we give the proof, that estimate of the variance $\widehat{\sigma}^2$ is consistent for the second type model. Further we prove the maximal inequality (Lemma 4.1.7). At the end of this section we give the proofs of the technical lemmas used in the proof of Theorem 4.2.2.

4.3.1 Consistency of the estimate of variance

Here we show that for the second type model defined by (4.1), the estimate of variance is consistent.

Lemma 4.3.1. *Suppose (y_k) is generated by (4.1) and $\phi_n = 1 - \gamma_n/n$, where γ_n is sequence of non negative numbers, $\gamma_n/n \to 0$ and $\gamma_n \to \infty$ as $n \to \infty$. Assume also that the innovations (ε_k) are i.i.d. random variables with $\mathbb{E}\varepsilon_k = 0$, $\mathbb{E}\varepsilon_k^2 = \sigma^2$. Variance estimator $\widehat{\sigma}^2$ is defined by (4.7) (page 33). Then*

$$\widehat{\sigma}^2 \xrightarrow[n\to\infty]{\mathrm{P}} \sigma^2. \tag{4.51}$$

Proof. We can rearrange (4.7), page 33, using (4.32), page 41, in the following way

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{n}\widehat{\varepsilon_k}^2 = \frac{1}{n}\sum_{k=1}^{n}\varepsilon_k^2 - \frac{2}{n}(\widehat{\phi}_n - \phi_n)\sum_{k=1}^{n}\varepsilon_k y_{k-1} + \frac{1}{n}(\widehat{\phi_n} - \phi_n)^2\sum_{k=1}^{n}y_{k-1}^2.$$

By the weak law of large numbers we have

$$\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k^2 \xrightarrow[n\to\infty]{\mathrm{P}} \sigma^2. \tag{4.52}$$

Further we will use Giraitis and Phillips [2006] results:

$$\frac{n^{1/2}}{(1-\phi_n^2)^{1/2}}(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,1) \tag{4.53}$$

$$\frac{(1-\phi_n^2)^{1/2}}{n^{1/2}}\sum_{k=1}^{n}\varepsilon_k y_{k-1} \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,\sigma^4) \tag{4.54}$$

$$\frac{1-\phi_n^2}{n}\sum_{k=1}^{n}y_{k-1}^2 \xrightarrow[n\to\infty]{\mathrm{P}} \sigma^2 \tag{4.55}$$

So using (4.53) and (4.55) for $\frac{1}{n}(\widehat{\phi_n} - \phi_n)^2\sum_{k=1}^{n}y_{k-1}^2$ we obtain

$$\frac{1}{n}(\widehat{\phi_n} - \phi_n)^2\sum_{k=1}^{n}y_{k-1}^2 \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{4.56}$$

Also for $\frac{2}{n}(\widehat{\phi}_n - \phi_n)\sum_{k=1}^{n}\varepsilon_k y_{k-1}$ using (4.53) and (4.54) we find

$$\frac{2}{n}(\widehat{\phi}_n - \phi_n)\sum_{k=1}^{n}\varepsilon_k y_{k-1} \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{4.57}$$

Thus (4.52), (4.56) and (4.57) gives us (4.51). □

4.3.2 Maximal inequality

Here we give a detailed proof of Lemma 4.1.7, page 38. It is convenient to start with the following weaker result which already contains the estimate $\max_{1\leq k\leq n}|y_k| = O_P(n^{1/2}\gamma_n^{-\alpha})$ if $\mathbb{E}\,|\varepsilon_0|^p < \infty$.

Lemma 4.3.2. *Let $(\eta_j)_{j\geq 0}$ be a sequence of i.i.d. random variables, with $\mathbb{E}\eta_0 = 0$ and $\mathbb{E}\,|\eta_0|^q < \infty$ for some $q \geq 2$. Suppose $\phi_n = 1 - \frac{\gamma_n}{n}$, where $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$. Define*

$$z_k = \sum_{j=1}^{k} \phi_n^{k-j}\eta_j. \tag{4.58}$$

Then there exists an integer $n_0(q) \geq 1$ depending on q only, such that for every $n \geq n_0(q)$, $\gamma_n > \gamma_{n_0}(q)$, and every $\lambda > 0$,

$$P\left(\max_{1\leq k\leq n}|z_k| > \lambda\right) \leq \frac{4C_q \mathrm{e}^q \mathbb{E}\,|\eta_0|^q}{\lambda^q} n^{q/2}\gamma_n^{1-q/2}, \tag{4.59}$$

where C_q is the universal constant in the Rosenthal inequality of order q. Choosing $\lambda = n^{1/2}\gamma_n^{1/q-1/2}\tau$ for arbitrary $\tau > 0$ provides:

$$\max_{1\leq k\leq n}|z_k| = O_P\left(n^{1/2}\gamma_n^{1/q-1/2}\right).$$

The right hand side of (4.59) becomes smaller as q increases, subject to an optimal choice of λ. It seems difficult to say if the bound (4.59) is sharp. We can nevertheless remark that in the boundary case, where $\gamma_n = n$ and so the z_k's become i.i.d., our bound would lead to the estimate $\max_{1\leq k\leq n}|z_k| = O_{\mathrm{P}}(n^{1/q})$ which is optimal in this case.

Proof. The idea of the proof relies on the following observation. For $a < k \leq b$,

$$|z_k| = \phi_n^k \left|\sum_{j=1}^{k}\phi_n^{-j}\eta_j\right| \leq \phi_n^a \left|\sum_{j=1}^{k}\phi_n^{-j}\eta_j\right|.$$

Here $\{\sum_{j=1}^{k}\phi_n^{-j}\eta_j,\ a < k \leq b\}$ is a martingale adapted to its natural filtration and if we repeat this procedure with regularly spaced bounds a and b, we keep the structure of a geometric sum for the coefficients ϕ_n^a. To profit of these two features we are lead to the following splitting:

$$n = MK, \quad \max_{1\leq k\leq n}|z_k| = \max_{1\leq m\leq M}\ \max_{(m-1)K<k\leq mK}|z_k|,$$

where M and K (not necessarily integers) depend on n in a way which will be precised later. Applying this splitting we obtain first:

$$P\left(\max_{1\leq k\leq n}|z_k|>\lambda\right)\leq\sum_{1\leq m\leq M}P\left(\phi_n^{(m-1)K}\max_{1\leq k\leq mK}\left|\sum_{j=1}^{k}\phi_n^{-j}\eta_j\right|>\lambda\right).$$

Then using Markov's and Doob's inequalities at order q gives

$$P\left(\max_{1\leq k\leq n}|z_k|>\lambda\right)\leq\sum_{1\leq m\leq M}\frac{\phi_n^{q(m-1)K}T_m}{\lambda^q}\quad\text{where}\quad T_m:=\mathbb{E}\left|\sum_{1\leq j\leq mK}\phi_n^{-j}\eta_j\right|^q.\tag{4.60}$$

To bound T_m, we treat separately the special case $q=2$ with a simple variance computation and use Rosenthal inequality in the case $q>2$. In both cases, the following elementary estimate is useful.

$$\begin{aligned}\sum_{1\leq j\leq mK}\phi_n^{-jq}&=\phi_n^{-[qmK]}\sum_{j=1}^{[mK]}\phi_n^{[mK]q-jq}=\phi_n^{-[qmK]}\sum_{j=0}^{[mK]-1}\phi_n^{jq}\\&\leq\frac{\phi_n^{-[qmK]}}{1-\phi_n^q}\leq\frac{\phi_n^{-qmK}}{1-\phi_n}\end{aligned}$$

recalling that $0<\phi_n<1$, whence,

$$\sum_{1\leq j\leq mK}\phi_n^{-jq}\leq\frac{n}{\gamma_n}\phi_n^{-qmK}.\tag{4.61}$$

Now in the special case $q=2$, we have

$$T_m=\operatorname{Var}\left(\sum_{j=1}^{k}\phi_n^{-j}\eta_j\right)=\mathbb{E}\eta_0^2\sum_{1\leq j\leq mK}\phi_n^{-2j},$$

so by (4.61),

$$T_m\leq\frac{n}{\gamma_n}\phi_n^{-2mK}\mathbb{E}\eta_0^2.\tag{4.62}$$

When $q>2$, we apply Rosenthal inequality which gives here

$$T_m\leq C_q\left(\left(\mathbb{E}\eta_0^2\right)^{q/2}\left(\sum_{1\leq j\leq mK}\phi_n^{-2j}\right)^{q/2}+\mathbb{E}\left|\eta_0\right|^q\sum_{1\leq j\leq mK}\phi_n^{-jq}\right).$$

As $q>2$, $\left(\mathbb{E}\eta_0^2\right)^{q/2}\leq\mathbb{E}\left|\eta_0\right|^q$. Also we may assume without loss of generality that

$\frac{n}{\gamma_n} \geq 1$, so $\frac{n}{\gamma_n} \leq \left(\frac{n}{\gamma_n}\right)^{q/2}$. Then using (4.61), we obtain

$$T_m \leq 2C_q \mathbb{E} |\eta_0|^q n^{q/2} \gamma_n^{-q/2} \phi_n^{-qmK}. \tag{4.63}$$

Note that (4.62) obtained in the special case $q = 2$ can be included in this formula by defining $C_2 := 1/2$.

Going back to (4.60) with this estimate, we obtain

$$\begin{aligned} P\left(\max_{1\leq k\leq n} |z_k| > \lambda\right) &\leq 2C_q \mathbb{E} |\eta_0|^q n^{q/2} \gamma_n^{-q/2} \lambda^{-q} \sum_{1\leq m\leq M} \phi_n^{-Kq} \\ &\leq 2C_q \mathbb{E} |\eta_0|^q n^{q/2} \gamma_n^{-q/2} \lambda^{-q} M \phi_n^{-Kq}. \end{aligned}$$

Now, choosing $K = \frac{n}{\gamma_n}$, we see that ϕ_n^{-Kq} converges to e^q, so for $n \geq n_0(q)$, $\phi_n^{-Kq} \leq 2\mathrm{e}^q$. Then (4.59) follows by pluging this upper bound in the inequality above and noting that $M = \gamma_n$. □

Remark 4.3.3. Under assumptions of Lemma 4.3.2 there exists a constant c_q depending on q only, such that for every $n \geq 1$ and every $\lambda > 0$

$$P\left(\max_{1\leq k\leq n} |z_k| > \lambda\right) \leq \frac{c_q \mathbb{E} |\eta_0|^q}{\lambda^q} n^{q/2} \gamma_n^{1-q/2}.$$

Remark 4.3.4. The Lemma 4.3.2 can be proved by applying Hájek-Rényi type inequality (e.g. see Petrov [1975] section III.5, paragraph 6). In our opinion, the method applied in the proof of Lemma 4.3.2 seems more suitable for generalization, e.g. for dependent innovations.

Proof. of Lemma 4.1.7. It is convenient to rewrite the assumption (4.21), page 38, as

$$P(|\varepsilon_0| > t) = \frac{f(t)}{t^p}, \quad f(t) \xrightarrow[t\to\infty]{} 0.$$

Moreover

$$f^*(b) := \sup_{t\geq b} f(t) \xrightarrow[b\to\infty]{} 0.$$

In the special case where $p = 2$, (4.21) is replaced by $\mathbb{E}\varepsilon_0^2 < \infty$, but the above representation of $P(|\varepsilon_0| > t)$ remains valid since $f(t) = t^2 P(|\varepsilon_0| > t) \leq \mathbb{E}(\varepsilon_0^2 \mathbf{1}_{\{|\varepsilon_0|>t\}})$ by Markov inequality and this upper bound goes to zero by dominated convergence theorem.

Let us fix arbitrary positive numbers δ and ϵ, and introduce the truncated random variables

$$\varepsilon_j' = \varepsilon_j \mathbf{1}_{\{|\varepsilon_j| \le b_n\}} \qquad \widetilde{\varepsilon}_j' = \varepsilon_j' - \mathbb{E}\varepsilon_j'$$
$$\varepsilon_j'' = \varepsilon_j \mathbf{1}_{\{|\varepsilon_j| > b_n\}} \qquad \widetilde{\varepsilon}_j'' = \varepsilon_j'' - \mathbb{E}\varepsilon_j'',$$

where the truncation level b_n goes to infinity at a rate which will be precised later. Since $\mathbb{E}\varepsilon_j = 0$, $\varepsilon_j = \widetilde{\varepsilon}_j' + \widetilde{\varepsilon}_j''$. Now let us recall that

$$y_k = \sum_{j=1}^{k} \phi_n^{k-j} \varepsilon_j = \sum_{j=1}^{k} \phi_n^{k-j} (\widetilde{\varepsilon}_j' + \widetilde{\varepsilon}_j'') = \sum_{j=1}^{k} \phi_n^{k-j} \widetilde{\varepsilon}_j' + \sum_{j=1}^{k} \phi_n^{k-j} \widetilde{\varepsilon}_j'' = \widetilde{z}_k' + \widetilde{z}_k'',$$

where $\widetilde{z}_k'$ and $\widetilde{z}_k''$ are defined by substituting ε_j by $\widetilde{\varepsilon}_j'$ and $\widetilde{\varepsilon}_j''$ respectively in the definition of z_k, given by (4.58). Then for positive $\lambda = \lambda_n$, whose dependence on n will be precised later,

$$P\left(\max_{1\le k\le n} |y_k| > 2\lambda\right) \le P_n' + P_n'', \tag{4.64}$$

where

$$P_n' := P\left(\max_{1\le k\le n} |\widetilde{z}_k'| > \lambda\right), \quad P_n'' := P\left(\max_{1\le k\le n} |\widetilde{z}_k''| > \lambda\right).$$

To bound P_n', applying Lemma 4.3.2 to $\widetilde{z}_k'$ gives for any $q > p$

$$P_n' \le \frac{4\mathrm{e}^q C_q \mathbb{E}|\widetilde{\varepsilon_0'}|^q}{\lambda^q} n^{q/2} \gamma_n^{1-q/2} \le \frac{2^{q+2}\mathrm{e}^q C_q \mathbb{E}|\varepsilon_0'|^q}{\lambda^q} n^{q/2} \gamma_n^{1-q/2},$$

since by elementary convexity inequalities, $\mathbb{E}|\widetilde{\varepsilon}_0'|^q \le 2^q \mathbb{E}|\varepsilon_0'|^q$. Now

$$\begin{aligned}
\mathbb{E}|\varepsilon_0'|^q &= \int_0^\infty q t^{q-1} P\left(|\varepsilon_0| \,\mathbf{1}_{\{|\varepsilon_j|\le b_n\}} > t\right) \mathrm{d}t \\
&= \int_0^{b_n} q t^{q-1} P(t < |\varepsilon_0| \le b_n) \,\mathrm{d}t \le \int_0^{b_n} q t^{q-1} P(|\varepsilon_0| > t) \,\mathrm{d}t \\
&= \int_0^{b_n} q t^{q-1} \frac{f(t)}{t^p} \,\mathrm{d}t \le \frac{q\,\|f\|_\infty}{q-p} b_n^{q-p}.
\end{aligned}$$

Going back to P_n' we find that

$$P_n' \le \frac{2^{q+2}\mathrm{e}^q q C_q \|f\|_\infty}{q-p} \cdot \frac{n^{q/2}\gamma_n^{1-q/2} b_n^{q-p}}{\lambda^q}.$$

Now we choose $\lambda = n^{1/2}\gamma_n^{1/p-1/2}\delta$, $q = p+1$ and

$$b_n = \delta^{p+1}\epsilon\gamma_n^{1/p} \tag{4.65}$$

with arbitrary $\epsilon > 0$. Recalling that γ_n goes to infinity, the same holds for b_n. This choice gives

$$P'_n = P\left(n^{-1/2}\gamma_n^\alpha \max_{1\le k\le n} |\tilde{z}'_k| > \delta\right) \le C'_p\epsilon, \tag{4.66}$$

with $C'_p = 2^{p+3}\mathrm{e}^{p+1}(p+1)C_{p+1}\,\|f\|_\infty$.

To bound P''_n, we apply Lemma 4.3.2 with $z_k = \tilde{z}''_k$ and $q = 2$ (keeping the above choices of λ and b_n which do not depend on q):

$$P''_n \le \frac{8\mathrm{e}^2}{\delta^2}\gamma_n^{1-2/p}\mathbb{E}(\varepsilon''_0)^2.$$

In the special case where $p = 2$, this reduces to

$$P''_n \le \frac{8\mathrm{e}^2}{\delta^2}\mathbb{E}(\varepsilon_0^2\mathbf{1}_{\{|\varepsilon_0|>b_n\}})$$

and this bound goes to zero by Lebesgue's dominated convergence theorem, since b_n defined by (4.65) goes to infinity. When $p > 2$, we estimate $\mathbb{E}(\varepsilon''_0)^2$ as follows.

$$\begin{aligned}
\mathbb{E}(\varepsilon''_0)^2 &= \int_0^\infty 2tP\left(|\varepsilon_0|\mathbf{1}_{\{|\varepsilon_0|>b_n\}} > t\right)\,\mathrm{d}t \\
&= \int_0^{b_n} 2tP(|\varepsilon_0| > b_n)\,\mathrm{d}t + \int_{b_n}^\infty 2tP(|\varepsilon_0| > t)\,\mathrm{d}t \\
&= b_n^2P(|\varepsilon_0| > b_n) + \int_{b_n}^\infty 2t^{1-p}f(t)\,\mathrm{d}t \le f(b_n)b_n^{2-p} + \frac{2}{p-2}f^*(b_n)b_n^{2-p} \\
&\le \frac{p}{p-2}\delta^{(p+1)(2-p)}\epsilon^{2-p}\gamma_n^{2/p-1}f^*(b_n).
\end{aligned}$$

Finally, we see that there is a constant $C''_{\delta,\epsilon,p}$ such that for $p \ge 2$,

$$P''_n \le C''_{\delta,\epsilon,p}f^*(b_n). \tag{4.67}$$

Going back to (4.64) with (4.66) and (4.67), we obtain

$$Q_n := P\left(n^{-1/2}\gamma_n^\alpha \max_{1\le k\le n}|y_k| > \delta\right) \le C'_p\epsilon + C''_{\delta,\epsilon,p}f^*(b_n).$$

This gives $\limsup_{n\to\infty} Q_n \le C'_p\epsilon$ and as ϵ is arbitrary, so (4.22) (page 38) follows.

□

4.3.3 Lemmas for the proof of theorem 4.2.2

In this section we give proofs of Lemmas 4.2.5, 4.2.6 and 4.2.7.

Proof of Lemma 4.2.5. We note first that since $P(D = 0) = 0$, the limiting random variable N/D is well defined (up to an event of probability 0). We will check that for each real t such that $P(N/D = t)$ (i.e. for each point of continuity of the distribution function of the claimed limiting distribution), $P(\Phi_n \leq t)$ tends to $P(N/D \leq t)$.

For such a t we clearly have $P(N - tD = 0) = 0$. This combined with the convergence in distribution of (N_n, D_n) to (N, D) and continuous maping gives the convergence:

$$P(N_n - tD_n \leq 0) \underset{\to\infty}{\longrightarrow} P(N - tD \leq 0).$$

Now

$$\begin{aligned} P(\Phi_n \leq t) &= P(0 \leq t \text{ and } D_n = 0) + P\left(\frac{N_n}{D_n} \leq t \text{ and } D_n > 0\right) \\ &= o(1) + P(N_n - tD_n \leq 0 \text{ and } D_n > 0). \end{aligned}$$

Noting that

$$|P(N_n \leq tD_n) - P(N_n - tD_n \leq 0 \text{ and } D_n > 0)| \leq P(D_n = 0) = o(1)$$

we deduce that $P(\Phi_n \leq t)$ tends to $P(N/D \leq t)$. □

The next lemma is an auxilliary result used in the forthcoming proof of Lemma 4.2.6.

Lemma 4.3.5. *Suppose that the process* (y_k) *is defined by* (4.1) *with* $\phi_n = \mathrm{e}^{\gamma/n}$, $\gamma < 0$ *and* $y_0 = 0$. *Let* (ε_k) *be i.i.d. random variables with mean* 0 *and satisfies condition* (3.11) *(page 23). Define*

$$V_n(l) := W_n\left(\frac{l-1}{n}\right) + \gamma \int_0^{\frac{l}{n}} \mathrm{e}^{(\frac{l}{n}-s)\gamma} W_n(s)\,\mathrm{d}s \tag{4.68}$$

for $l \leq n$. *Then*

$$\left|n^{-1/2} y_{l-1} - V_n(l)\right| \leq \|W_n\|_\infty \frac{\gamma^2 \mathrm{e}^\gamma}{2n} + \frac{\mathrm{e}^\gamma}{n^\alpha} \omega_\alpha\left(W_n, \frac{1}{n}\right) + \frac{|2+\gamma|\,\mathrm{e}^\gamma}{n} \|W_n\|_\infty. \tag{4.69}$$

Proof. Denote

$$V_{l,1} := n^{-1/2} y_{l-1} = n^{-1/2} \sum_{j=1}^{l-1} \mathrm{e}^{(l-1-j)\gamma/n} \varepsilon_j.$$

Noting that $\varepsilon_l = W_n^{\mathrm{pl}}\left(\frac{l}{n}\right) - W_n^{\mathrm{pl}}\left(\frac{l-1}{n}\right)$ we can express

$$y_{l-1} = W_n^{\mathrm{pl}}\left(\frac{l-1}{n}\right) + \frac{\gamma}{n}\sum_{j=1}^{l-2} \mathrm{e}^{(l-1-j)\gamma/n} W_n^{\mathrm{pl}}\left(\frac{j}{n}\right) + \frac{\gamma^2 u_n}{2n^2}\sum_{j=1}^{l-2} \mathrm{e}^{(l-1-j)\gamma/n} W_n^{\mathrm{pl}}\left(\frac{j}{n}\right),$$

where u_n is defined by (4.13), page 35, and $u_n \to -1$, as $n \to \infty$. Then we define

$$V_{l,2} := W_n\left(\frac{l-1}{n}\right) + \frac{\gamma}{n}\sum_{j=1}^{l-2} \mathrm{e}^{(l-1-j)\gamma/n} W_n\left(\frac{j}{n}\right)$$

and for the approximation error we obtain the bound

$$|V_{l,2} - V_{l,1}| \le \|W_n\|_\infty \frac{\gamma^2 \mathrm{e}^\gamma}{2n}.$$

Further we approximate Riemann sum by integral (which is exactly (4.68))

$$V_n(l) := W_n\left(\frac{l-1}{n}\right) + \gamma\int_0^{\frac{l}{n}} \mathrm{e}^{(\frac{l}{n}-s)\gamma} W_n(s)\,\mathrm{d}s.$$

Now we estimate the error. For any $f \in \mathrm{C}[0,1]$, we have

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{l-2} f\Big(\frac{j+j_0}{n}\Big) - \int_0^{l/n} f(s)\,\mathrm{d}s \\ = \sum_{j=1}^{l-2}\int_{(j-1)/n}^{j/n}\Big(f\Big(\frac{j+j_0}{n}\Big) - f(s)\Big)\,\mathrm{d}s - \int_{(l-2)/n}^{l/n} f(s)\,\mathrm{d}s, \qquad (4.70)\end{aligned}$$

whence

$$\left|\frac{1}{n}\sum_{j=1}^{l-2} f\Big(\frac{j+j_0}{n}\Big) - \int_0^{l/n} f(s)\,\mathrm{d}s\right| \le \omega_0\Big(f, \frac{1+j_0}{n}\Big) + \|f\|_\infty \frac{2}{n}. \qquad (4.71)$$

Moreover,

$$\text{if } f \in \mathrm{H}_\alpha^o[0,1], \quad \omega_0(f,\delta) \le \omega_\alpha(f,\delta)\delta^\alpha. \qquad (4.72)$$

If $f(t) = g(t)h(t)$ with g of class C^1 and $h \in \mathrm{C}[0,1]$,

$$\omega_0(gh,\delta) \le \|g\|_\infty \omega_0(h,\delta) + \|g'\|_\infty \|h\|_\infty \delta. \qquad (4.73)$$

So from (4.71)-(4.73) we obtain the uniform bound

$$|V_n(l) - V_{l,2}| \le \frac{\mathrm{e}^\gamma}{n^\alpha}\omega_\alpha(W_n, \frac{1}{n}) + \frac{|2+\gamma|\,\mathrm{e}^\gamma}{n}\|W_n\|_\infty.$$

□

Proof of Lemma (4.2.6). Using lemma 4.3.5 we approximate $A_{n,0} := n^{-2}\sum_{k=1}^{n} y_{k-1}^2$ by

$$A_{n,1} := \frac{1}{n}\sum_{k=1}^{n}\left(W_n\left(\frac{k-1}{n}\right) + \gamma\int_0^{\frac{k}{n}} \mathrm{e}^{(\frac{k}{n}-s)\gamma} W_n(s)\,\mathrm{d}s\right)^2.$$

The approximation error is bounded by

$$|A_{n,1} - A_{n,0}| \le \max_{1\le k\le n}\left|n^{-1/2}y_{k-1} - V_n(k)\right|\left(\max_{1\le k\le n}\left|n^{-1/2}y_{k-1}\right| + \max_{1\le k\le n}|V_n(k)|\right). \tag{4.74}$$

From Lemma 4.3.5, $\max_{1\le k\le n}\left|n^{-1/2}y_{k-1} - V_n(k)\right| = o_{\mathrm{P}}(n^{-\alpha})$. As $V_n(l)$ is the image of W_n by a functional continuous on H_α^o, from continuous mapping theorem and Hölderian invariance principle, $\max_{1\le k\le n}|V_n(k)|$ is stochastically bounded. Also by Phillips [1987] $\max_{1\le k\le n}\left|n^{-1/2}y_{k-1}\right|$ is stochastically bounded.

Further $A_{n,1}$ might be approximated by A_n and the bound of approximation error is

$$|A_n - A_{n,1}| \le \omega\left(f, \frac{1}{n}\right)$$

where $f(r) := \left(W_n^{\mathrm{pl}}(r) + \gamma\int_0^r \mathrm{e}^{(r-s)\gamma}W_n^{\mathrm{pl}}(s)\,\mathrm{d}s\right)^2$. Denote $f(r) = g^2(r)$ and $g(r) := W_n^{\mathrm{pl}}(r) + \gamma\int_0^r \mathrm{e}^{(r-s)\gamma}W_n^{\mathrm{pl}}(s)\,\mathrm{d}s$. Then

$$\begin{aligned}\omega\left(f,\frac{1}{n}\right) &\le \frac{1}{n^\alpha}\omega_\alpha\left(f,\frac{1}{n}\right) \le \frac{2}{n^\alpha}\|g\|_\infty\,\omega_\alpha\left(g,\frac{1}{n}\right)\\ &\le \frac{2}{n^\alpha}\cdot\|W_n\|_\infty\,\mathrm{e}^\gamma\left(\omega_\alpha\left(W_n,\frac{1}{n}\right) + \frac{1}{n^{1-\alpha}}\mathrm{e}^\gamma\|W_n\|_\infty\right).\end{aligned} \tag{4.75}$$

So we obtain $|A_n - A_{n,0}| = o_{\mathrm{P}}(n^{-\alpha})$. □

Proof of Lemma 4.2.7. By squaring equation (4.1), page 31, subtracting y_{k-1}^2 from both sides and summing both sides over k we obtain:

$$y_n^2 = (\mathrm{e}^{2\gamma/n} - 1)\sum_{k=1}^{n} y_{k-1}^2 + 2\mathrm{e}^{\gamma/n}\sum_{k=1}^{n} y_{k-1}\varepsilon_k + \sum_{k=1}^{n}\varepsilon_k^2.$$

Then multiplying everything by n^{-1} we get:

$$B_{n,1} := 2n^{-1}\sum_{k=1}^{n} y_{k-1}\varepsilon_k = \frac{1}{\mathrm{e}^{\gamma/n}}\left(n^{-1}y_n^2 - \frac{2\gamma}{n^2}\sum_{k=1}^{n} y_{k-1}^2 - \frac{1}{n}\sum_{k=1}^{n}\varepsilon_k^2 - \frac{\gamma^2 u_n}{n^3}\sum_{k=1}^{n} y_{k-1}^2\right),$$

where $u_n \to -1$, $n \to \infty$. Further we can approximate $B_{n,1}$ by

$$B_{n,2} := \frac{1}{\mathrm{e}^{\gamma/n}} \left(n^{-1} y_n^2 - \frac{2\gamma}{n^2} \sum_{k=1}^{n} y_{k-1}^2 - \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k^2 \right)$$

and the bound of the approximation error is

$$|B_{n,2} - B_{n,1}| \le \frac{\gamma^2}{n} \left| \frac{1}{n^2} \sum_{k=1}^{n} y_{k-1}^2 \right| \xrightarrow[n\to\infty]{\mathrm{P}} 0,$$

because by Phillips [1987] $\left| \frac{1}{n^2} \sum_{k=1}^{n} y_{k-1}^2 \right|$ is stochastically bounded on $\mathbb{R}$ and $\frac{\gamma^2}{n} \to 0$, as $n \to \infty$. Further $B_{n,2}$ we can approximate by

$$B_{n,3} := \frac{1}{\mathrm{e}^{\gamma/n}} \left(n^{-1} y_n^2 - \frac{2\gamma}{n^2} \sum_{k=1}^{n} y_{k-1}^2 - 1 \right).$$

In this case for the approximation error we have

$$|B_{n,3} - B_{n,2}| \le \left| \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k^2 - 1 \right| \xrightarrow[n\to\infty]{\mathrm{P}} 0$$

by the weak law of large numbers since $\mathbb{E}\varepsilon_0^2 = 1$. Next we approximate $B_{n,3}$ by

$$B_{n,4} := n^{-1} y_n^2 - \frac{2\gamma}{n^2} \sum_{k=1}^{n} y_{k-1}^2 - 1.$$

As $\left| n^{-1} y_n^2 - \frac{2\gamma}{n^2} \sum_{k=1}^{n} y_{k-1}^2 - 1 \right|$ is stochastically bounded by Phillips [1987] Lemma 1, we obtain

$$|B_{n,4} - B_{n,3}| = \left| n^{-1} y_n^2 - \frac{2\gamma}{n^2} \sum_{k=1}^{n} y_{k-1}^2 - 1 \right| \cdot \left| 1 - \frac{1}{\mathrm{e}^{\gamma/n}} \right| \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

Finally, using Lemma 4.3.5 we obtain

$$\begin{aligned} B_n = &\frac{1}{2} \left(W_n(1) + \gamma \int_0^1 \mathrm{e}^{(1-s)\gamma} W_n(s) \, \mathrm{d}s \right)^2 \\ &- \gamma \int_0^1 \left(W_n(r) + \gamma \int_0^r \mathrm{e}^{(r-s)\gamma} W_n(s) \, \mathrm{d}s \right)^2 \mathrm{d}r - \frac{1}{2}. \end{aligned}$$

The bound

$$\begin{aligned} |B_n - B_{n,4}| \le &\frac{1}{2} \left| (n^{-1/2} y_n)^2 - \left(W_n(1) + \gamma \int_0^1 \mathrm{e}^{(1-s)\gamma} W_n(s) \, \mathrm{d}s \right)^2 \right| \\ &+ \gamma \left| \frac{1}{n^2} \sum_{k=1}^{n} y_{k-1}^2 - \int_0^1 \left(W_n(r) + \gamma \int_0^r \mathrm{e}^{(r-s)\gamma} W_n(s) \, \mathrm{d}s \right)^2 \mathrm{d}r \right| \end{aligned}$$

Further

$$\gamma \left| \frac{1}{n^2} \sum_{k=1}^{n} y_{k-1}^2 - \int_0^1 \left(W_n(r) + \gamma \int_0^r \mathrm{e}^{(r-s)\gamma} W_n(s) \,\mathrm{d}s \right)^2 \mathrm{d}r \right|$$

is bounded by (4.74) and (4.75) and

$$\frac{1}{2} \left| (n^{-1/2} y_n)^2 - \left(W_n(1) + \gamma \int_0^1 \mathrm{e}^{(1-s)\gamma} W_n(s) \,\mathrm{d}s \right)^2 \right|$$

is bounded by (4.74). □

5

Testing the epidemic change with statistics built on observations

In this chapter we investigate some epidemic change in the innovations of the first order nearly nonstationary autoregressive process. For $0 \le \alpha < 1/2$, we build the α-Hölderian uniform increments statistics based on the observations to detect a short epidemic change in the process under consideration. Under the assumptions for innovations we find the limit of the statistics under null hypothesis, some conditions of consistency and we perform a test power analysis. We also discuss the interplay between the various parameters to detect the shortest epidemics.

Assume we are given an n-sample $y_{n,1}, \dots, y_{n,n}$ generated by

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k + a_{n,k}, \quad k = 1, \dots, n, \quad n \ge 1, \quad y_{n,0} = 0, \tag{5.1}$$

where the parameter $\phi_n \in (0,1)$ satisfies $\phi_n \to 1$, as $n \to \infty$, $(\varepsilon_k, k \ge 1)$ are i.i.d. centered, at least square integrable random variables, $(a_{n,k})$ is a sequence that will be precised later. Throughout this chapter, the parameter ϕ_n is supposed to be known. The aim of this chapter is to propose tests for the null hypothesis

$$H_0: \quad a_{n,1} = \cdots = a_{n,n} = 0$$

against the epidemic or changed segment alternative:

$$H_A: \text{ there exist } \quad 1 \le k_n^*, \quad 1 \le m_n^* \le n \quad \text{such that}$$
$$a_{n,k} = a_n \mathbf{1}_{\mathbb{I}_n^*}(k), \quad a_n \ne 0, \quad 1 \le k \le n,$$

where $\mathbb{I}_n^*$ is the epidemics interval

$$\mathbb{I}_n^* = \{k_n^* + 1, \ldots, m_n^*\}$$

and $\mathbf{1}_{\mathbb{I}_n^*}$ denotes its indicator function. Under that type of alternative the values $a_{n,k}$ during the period $\mathbb{I}_n^*$ are interpreted as an epidemic deviation from the usual (zero) mean and $\ell_n^* = m_n^* - k_n^*$ is called the duration of the epidemic state.

To investigate such hypothesis, we build the test statistics

$$\widetilde{T}_{\alpha,n} = T_{\alpha,n}(y_{n,1}, \ldots, y_{n,n}), \tag{5.2}$$

where $T_{\alpha,n}(X_1, \ldots, X_n)$ is defined by (2.11), page 18:

$$T_{\alpha,n} = \max_{1\le \ell \le n} \ell^{-\alpha} \max_{1 \le k \le n-\ell} \left| \sum_{j=k+1}^{k+\ell} X_j - \frac{\ell}{n} \sum_{j=1}^{n} X_j \right|.$$

To motivate such choice, rewrite the model (5.1) in the following way

$$y_{n,k} - \tau_{n,k} = \phi_n(y_{n,k-1} - \tau_{n,k-1}) + \varepsilon_k,$$

where

$$\tau_{n,k} = \sum_{j=0}^{k-1} \phi_n^j a_{n,k-j} = \sum_{j=1}^{k} \phi_n^{k-j} a_{n,j}. \tag{5.3}$$

Define

$$z_{n,k} = y_{n,k} - \tau_{n,k}. \tag{5.4}$$

Note that $(z_{n,k})$ is a nearly nonstationary first order autoregressive process and satisfies the null hypothesis. So, due to (5.4), we have the epidemic change problem where a sequence of dependent random variables satisfying the null hypothesis is shifted by a deterministic sequence. This is the reason why statistics (5.2) seems natural in this situation.

We study limit behavior of $\widetilde{T}_{\alpha,n}$ for $\alpha = 0$ (Levin and Kline statistics) and $\alpha \in (0, 1/2 - 1/p)$, $p > 2$ (Račkauskas and Suquet statistics) trying to see how the use of Hölder weighting allows detection of shorter epidemics than the use of $\widetilde{T}_{0,n}$. Of course the range of detection will be smaller here than in the case of i.i.d. samples. If $\alpha = 0$, then the innovations are required to have finite second moment. For another case the innovations satisfy the stronger integrability condition (3.11):

$$\lim_{t\to\infty} t^p P(|\varepsilon_0| > t) = 0.$$

Here we also study two types of first order nearly nonstationary models with the coefficient ϕ_n close to 1 in the model (5.1). The first type model corresponds to the coefficient

$$\phi_n = \mathrm{e}^{\gamma/n}, \quad \gamma < 0. \tag{5.5}$$

The second type model corresponds to the coefficient

$$\phi_n = 1 - \frac{\gamma_n}{n} \quad \text{where} \quad \gamma_n \to \infty \text{ and } \frac{\gamma_n}{n} \to 0 \quad \text{as} \quad n \to \infty. \tag{5.6}$$

As we shall see the limit behavior of $\widetilde{T}_{\alpha,n}$ statistics differs for these two types of models.

5.1 Limit behavior of test statistics under null hypothesis

For any function $f \in C[0,1]$ or $f \in \mathrm{H}^o_\alpha[0,1]$ and $0 \le \alpha < 1/2$ set

$$T_{\alpha,\infty}(f) := \sup_{0<t<s<1} \frac{|f(t) - f(s) - (t-s)f(1)|}{|t-s|^\alpha}. \tag{5.7}$$

5.1.1 Levin and Kline statistics

We start the investigation from Levin and Kline statistics $\widetilde{T}_{0,n}$. First let us consider the model (5.1) under null hypothesis H_0 with the coefficient $\phi_n = \mathrm{e}^{\gamma/n}$, $\gamma < 0$. Under the assumption of square integrability of innovations, we obtain that the limit of such statistics is a functional depending on an integrated Ornstein-Uhlenbeck process.

Theorem 5.1.1. *Under H_0, for the first type model defined by* (5.1) *and* (5.5),

$$n^{-3/2}\sigma^{-1}\widetilde{T}_{0,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_{0,\infty}(J), \tag{5.8}$$

where $\sigma^2 = \mathbb{E}\varepsilon_1^2$ and J is an integrated Ornstein-Uhlenbeck process $J(t)=\int_0^t U_\gamma(r)\mathrm{d}r$, $t \in [0,1]$ with U_γ defined by (3.1) *(page 19).*

Proof. Consider the functionals g_n and g defined on the continuous function space $\mathrm{C}[0,1]$ by

$$g_n(x) := \max_{1\le i<j\le n} I_0(x, i/n, j/n), \quad g(x) := \sup_{0<s<t<1} I_0(x,s,t), \tag{5.9}$$

where

$$I_0(x,s,t) := |x(t) - x(s) - (t-s)x(1)|, \quad 0 < t-s < 1.$$

By the special case of Lemma 3.3.3, page 25, where $\alpha = 0$, the functionals g_n and g are Lipschitz on $G_0 = \{x \in \mathrm{C}[0,1] : x(0) = 0\}$. Note that

$$\widetilde{T}_{0,n} = g_n(S_n^{\mathrm{pl}}), \quad T_{0,\infty}(J) = g(J). \tag{5.10}$$

where $(S_n^{\mathrm{pl}}(t), t \in [0,1])$ is the polygonal line constructed from partial sums of observations $(y_{n,k-1})$ defined by (4.2), page 31:

$$S_n^{\mathrm{pl}}(t) := \sum_{k=1}^{[nt]} y_{n,k-1} + (nt - [nt])y_{n,[nt]}, \quad t \in [0,1].$$

It follows from Theorem 4.1.3 (see also remark 4.1.4, page 33), that

$$n^{-3/2}\sigma^{-1}S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{C}[0,1]} J. \tag{5.11}$$

Lemma 3.3.3 (page 25) now gives

$$g_n(n^{-3/2}\sigma^{-1}S_n^{\mathrm{pl}}) = g(n^{-3/2}\sigma^{-1}S_n^{\mathrm{pl}}) + o_{\mathrm{P}}(1) \tag{5.12}$$

and the convergence (5.8) follows from (5.10), (5.11) and (5.12) and continuous mapping theorem. □

Next we find the limit of test statistics $\widetilde{T}_{0,n}$ under null hypothesis for second type model.

Theorem 5.1.2. *Under H_0, for the second type model defined by* (5.1) *and* (5.6)*,*

$$n^{-1/2}(1-\phi_n)\sigma^{-1}\widetilde{T}_{0,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_{0,\infty}(W), \tag{5.13}$$

where $\sigma^2 = \mathbb{E}\varepsilon_1^2$.

Proof. The proof of this theorem is essentially the same as the proof of the Theorem 5.1.1 using the Theorem 4.1.8 instead of Theorem 4.1.3 and Lemma 3.3.3. □

5.1.2 $\widetilde{T}_{\alpha,n}$ statistics with $\alpha > 0$

Now we show that for the model (5.1) with $\phi_n = \mathrm{e}^{\gamma/n}$, $\gamma < 0$, the limit of $\widetilde{T}_{\alpha,n}$ ($\alpha > 0$) under null hypothesis H_0 is a functional of an integrated Ornstein-Uhlenbeck process, but we have to require a stronger integrability on innovations than just a second moment.

Theorem 5.1.3. *In the first type model defined by* (5.1) *and* (5.5)*, assume that* (ε_i) *satisfy condition* (3.11) *(page 23) for some $p > 2$. Then under H_0 for any $\alpha \in (0, 1/2 - 1/p)$*

$$n^{-3/2+\alpha}\sigma^{-1}\widetilde{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_{\alpha,\infty}(J), \tag{5.14}$$

where $\sigma^2 = \mathbb{E}\varepsilon_1^2$ and J is an integrated Ornstein-Uhlenbeck process $J(t)=\int_0^t U_\gamma(r)\mathrm{d}r$, $t \in [0,1]$ with U_γ defined by (3.1) *(page 19).*

Proof. Consider the functionals g_n, g, defined on $\mathrm{H}_\alpha^o[0,1]$ by (5.9) where

$$I_\alpha(x,s,t) := \frac{|x(t)-x(s)-(t-s)x(1)|}{|t-s|^\alpha}, \quad 0 < t-s < 1.$$

By Lemma 3.3.3 (page 25) g_n and g are Lipschitz on $G_\alpha = \{x \in \mathrm{H}_\alpha^o[0,1] : x(0) = 0\}$. Observe that

$$n^\alpha \widetilde{T}_{\alpha,n} = g_n(S_n^{\mathrm{pl}}), \quad T_{\alpha,\infty}(J) = g(J). \tag{5.15}$$

where $(S_n^{\mathrm{pl}}(t), t \in [0,1])$ is defined by (4.2), page 31. From Theorem 4.1.3 (page 33),

$$n^{-3/2}\sigma^{-1}S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\alpha^o[0,1]} J \tag{5.16}$$

holds. Now from Lemma 3.3.3 it follows that

$$g_n(n^{-3/2}\sigma^{-1}S_n^{\mathrm{pl}}) = g(n^{-3/2}\sigma^{-1}S_n^{\mathrm{pl}}) + o_{\mathrm{P}}(1) \tag{5.17}$$

and the convergence (5.14) follows from (5.15), (5.16) and (5.17) and continuous mapping theorem. □

Further we find the limit of test statistics $\widetilde{T}_{\alpha,n}$ under null hypothesis in the second type model, i.e., in model (5.1) the coefficient is defined by $\phi_n = 1 - \gamma_n/n$, $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$. The limit under null hypothesis H_0 of this statistics is a functional depending on Wiener process. Here the requirements involve not only integrability condition on innovations, but also the rate of divergence of γ_n.

Theorem 5.1.4. *In the second type model defined by* (5.1) *and* (5.6)*, assume that* (ε_i) *satisfy condition* (3.11) *(page 23), for some* $p > 2$*. Then for* $\alpha \in (0, 1/2 - 1/p)$ *under* H_0

$$n^{-1/2+\alpha}(1-\phi_n)\sigma^{-1}\widetilde{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_{\alpha,\infty}(W) \tag{5.18}$$

provided that

$$\liminf_{n\to\infty} \gamma_n n^{-\alpha/(1/2-1/p)} > 0.$$

Proof. The idea of the proof of this theorem is the same as the proof of the Theorem 5.1.3 using the Theorem 4.1.9 instead of Theorem 4.1.3 and Lemma 3.3.3. □

5.2 Consistency of test statistics

We investigate the consistency of the test statistics $\widetilde{T}_{n,\alpha}$. The practical results are given in corollaries 5.2.5 and 5.2.2. Proofs of these corollaries are based on the following generic result (Theorem 5.2.1) which has a broader scope. The consistency condition is expressed in terms of:

$$T_{\alpha,n}(\tau_{n,1},\ldots,\tau_{n,n}) = \max_{1\le\ell\le n} \ell^{-\alpha} \max_{1\le k\le n-\ell} \left| \sum_{j=k+1}^{k+\ell} \tau_{n,j} - \frac{\ell}{n}\sum_{j=1}^{n}\tau_{n,j} \right|, \tag{5.19}$$

where the $\tau_{n,k}$'s are defined by (5.3).

For notational simplicity we omit the index n in k_n^*, m_n^* and ℓ_n^*.

Theorem 5.2.1. *Suppose that in the first order nearly nonstationary process defined by* (5.1) *innovations are i.i.d. centred and satisfy condition* (3.11) *(page 23). Assume that for some normalizing sequence* $(b_n)_{n\geq 1}$ *the statistics* $b_n\widetilde{T}_{\alpha,n}$ *is stochastically bounded under* H_0*. Then under* H_A*,*

$$b_n\widetilde{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathrm{P}} \infty \tag{5.20}$$

if and only if

$$b_n T_{\alpha,n}(\tau_{n,1},\dots,\tau_{n,n}) \xrightarrow[n\to\infty]{} \infty. \tag{5.21}$$

A sufficient condition for (5.21) *is*

$$\frac{a_n b_n}{(1-\phi_n)^2\ell^{*\alpha}}\left(\ell^*(1-\phi_n)\left(1-\frac{\ell^*}{n}\right)\right.$$
$$\left. -(1-\phi_n^{\ell^*})\left(\phi_n-\frac{\ell^*}{n}\phi_n^{n-m^*+1}\right)\right) \xrightarrow[n\to\infty]{} \infty. \tag{5.22}$$

Proof. Recall that the process (z_n) is defined by $z_{n,k}=y_{n,k}-\tau_{n,k}$, $0\leq k\leq n$. The key point here is that when the process (y_n) satisfies H_A, the process (z_n) satisfies H_0 (when (y_n) satisfies H_0 both process are identical). Hence $b_nT_{\alpha,n}(z_{n,1},\dots,z_{n,n})$ is stochastically bounded. Now by triangle inequality for the sequential norm $T_{\alpha,n}$:

$$\begin{aligned}&|T_{\alpha,n}(y_{n,1},\dots,y_{n,n})-T_{\alpha,n}(\tau_{n,1},\dots,\tau_{n,n})|\\ &\leq T_{\alpha,n}(y_{n,1}-\tau_{n,1},\dots,y_{n,n}-\tau_{n,n})\\ &=T_{\alpha,n}(z_{n,1},\dots,z_{n,n}),\end{aligned}$$

so the stochastic boundedness of $b_nT_{\alpha,n}(z_{n,1},\dots,z_{n,n})$ gives the equivalence between (5.20) and (5.21).

Looking now for a practical sufficient condition for (5.21), we choose as a lower bound for $T_{\alpha,n}(\tau_{n,1},\dots,\tau_{n,n})$ the weighted increment corresponding to the epidemics interval $(k^*,m^*]$ with length $m^*-k^*=\ell^*$. With these notations,

$$\tau_{n,k}=\sum_{j=1}^{k}\phi_n^{k-j}a_n\mathbf{1}_{(k^*,m^*]}(j),\quad 1\leq k\leq n,\quad \tau_{n,0}:=0.$$

Since a_n will obviously be in factor in all computations of lower bounds below, it is enough to write the proof for the case where $a_n=1$.

Let us compute $\sum_{j=1}^{n} \tau_{n,k}$.

$$\sum_{k=1}^{n} \tau_{n,k} = \sum_{k \le k^*} \tau_{n,k} + \sum_{k^* < k \le m^*} \tau_{n,k} + \sum_{m^* < k \le n} \tau_{n,k}$$
$$= \underbrace{\sum_{k^* < k \le m^*} \sum_{k^* < j \le k} \phi_n^{k-j}}_{=:A} + \underbrace{\sum_{m^* < k \le n} \sum_{k^* < j \le m^*} \phi_n^{k-j}}_{=:B}.$$

We compute separately the double geometric sums A and B.

$$A = \sum_{k^* < k \le m^*} \sum_{i=0}^{k-k^*-1} \phi_n^i = \sum_{k^* < k \le m^*} \frac{1 - \phi_n^{k-k^*}}{1 - \phi_n} = \frac{1}{1 - \phi_n} \left(\ell^* - \sum_{i=1}^{\ell^*} \phi_n^i \right),$$

so

$$A = \frac{1}{(1 - \phi_n)^2} \left(\ell^*(1 - \phi_n) - \phi_n(1 - \phi_n^{\ell^*}) \right). \tag{5.23}$$

Similarly,

$$B = \sum_{m^* < k \le n} \frac{\phi_n^{k-m^*} - \phi_n^{k-k^*}}{1 - \phi_n} = \frac{\phi_n^{-m^*} - \phi_n^{-k^*}}{1 - \phi_n} \sum_{m^* < k \le n} \phi_n^k$$
$$= \frac{\phi_n^{-m^*} - \phi_n^{-k^*}}{1 - \phi_n} \times \frac{\phi_n^{m^*+1} - \phi_n^{n+1}}{1 - \phi_n}$$
$$= \frac{1}{(1 - \phi_n)^2} (\phi_n - \phi_n^{\ell^*+1} - \phi_n^{n-m^*+1} + \phi_n^{n-k^*+1}),$$

whence

$$B = \frac{1}{(1 - \phi_n)^2} \left(\phi_n(1 - \phi_n^{\ell^*}) - \phi_n^{n-m^*+1}(1 - \phi_n^{\ell^*}) \right). \tag{5.24}$$

Gathering (5.23) and (5.24), we obtain

$$\sum_{j=1}^{n} \tau_{n,j} = \frac{1}{(1 - \phi_n)^2} \left(\ell^*(1 - \phi_n) - \phi_n^{n-m^*+1}(1 - \phi_n^{\ell^*}) \right). \tag{5.25}$$

Finally

$$A - \frac{\ell^*}{n}(A + B)$$
$$= \frac{1}{(1 - \phi_n)^2} \left(\ell^*(1 - \phi_n) \left(1 - \frac{\ell^*}{n} \right) - (1 - \phi_n^{\ell^*}) \left(\phi_n - \frac{\ell^*}{n} \phi_n^{n-m^*+1} \right) \right), \tag{5.26}$$

which explains why (5.22) is a sufficient condition for (5.21). □

Corollary 5.2.2. *In the first type model defined by* (5.1) *and* (5.5)*, assume that for some* $p > 2$*,* (ε_i) *satisfy condition* (3.11)*. Let* $\alpha \in (0, 1/2 - 1/p)$*, then under*

H_A

$$n^{-3/2+\alpha}\widetilde{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathrm{P}} \infty \tag{5.27}$$

provided that $\ell^{*2-\alpha}n^{-3/2+\alpha}a_n \to \infty$*, as* $n \to \infty$ *and*

$$\liminf_{n\to\infty}\left|1+\frac{\gamma}{2}-\mathrm{e}^{\gamma(1-\frac{m^*}{n})}\right| > 0. \tag{5.28}$$

All this extends to the special case $\alpha = 0$*, assuming that* $\mathbb{E}\varepsilon_1^2 < \infty$*.*

Remark 5.2.3. From a statistical point of view, it is useful to find for which values of the parameter γ, condition (5.28) does not induce some extra restriction on the choice of the sequence $(m^*(n))_{n\geq 1}$. Writing $\theta_n := m^*(n)/n$, we see that (5.28) is not satisfied if and only if there exists some subsequence $(\theta_{n_j})_{j\geq 1}$ in $(0,1)$ such that $\mathrm{e}^{\gamma(1-\theta_{n_j})}$ tends to $1+\gamma/2$. Then any θ limit of some subsequence of $(\theta_{n_j})_{j\geq 1}$ (there is at least one such θ by compactness of $[0,1]$) must satisfy $1+\gamma/2 = \mathrm{e}^{\gamma(1-\theta)}$. Clearly this equation has no solution for $\gamma \leq -2$. For $-2 < \gamma < 0$, it has a unique solution

$$\theta = 1 - \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{2}\right).$$

It is easily seen that this solution belongs to $[0,1]$ only if $\gamma_0 \leq \gamma < 0$, where $\gamma_0 \simeq -1.5937$. From this we can conclude that if $\gamma < \gamma_0$, the condition (5.28) is satisfied without any extra restrictions on the choice of the sequence $(m^*(n))_{n\geq 1}$. For $\gamma_0 \leq \gamma < 0$, one can always find a sequence $(m^*(n))_{n\geq 1}$ for which (5.28) fails.

Remark 5.2.4. From the consistency condition $\ell^{*2-\alpha}n^{-3/2+\alpha}a_n \to \infty$, as $n \to \infty$, one can see that the bigger α the shorter change can be detected with the statistics. As expected, the detection is not so good as in the i.i.d. case, see Račkauskas and Suquet [2004b].

Proof. We keep the notations A and B already used in the previous proof. By Theorem 5.1.3, under H_0, $b_n\widetilde{T}_{\alpha,n}$ converges in distribution and hence is stochastically bounded for the normalization $b_n = n^{-3/2+\alpha}$. So it remains only to check condition (5.22). This require an estimate for the asymptotic order of magnitude of

$$\begin{aligned} &A - \frac{\ell^*}{n}(A+B) \\ &= \frac{1}{(1-\phi_n)^2}\left(\ell^*(1-\phi_n)\left(1-\frac{\ell^*}{n}\right) - \phi_n(1-\phi_n^{\ell^*})\left(1-\frac{\ell^*}{n}\phi_n^{n-m^*}\right)\right). \end{aligned}$$

Using the second order expansions

$$\begin{aligned} 1-\phi_n &= -\frac{\gamma}{n} - \frac{\gamma^2}{2n^2} + o(n^{-2}) \\ 1-\phi_n^{\ell^*} &= -\frac{\gamma\ell^*}{n} - \frac{\gamma^2\ell^{*2}}{2n^2} + o(\ell^{*2}/n^{-2}) \end{aligned}$$

we deduce

$$\begin{aligned} \left|A - \frac{\ell^*}{n}(A+B)\right| &\geq \frac{n^2}{\gamma^2}\Bigg| \left(\ell^* - \frac{\ell^{*2}}{n}\right)\left(-\frac{\gamma}{n} - \frac{\gamma^2}{2n^2} + o(n^{-2})\right) \\ &\quad - \left(1 + \frac{\gamma}{n} + o\left(\frac{1}{n}\right)\right)\left(-\frac{\gamma\ell^*}{n} - \frac{\gamma^2\ell^{*2}}{2n^2} + o\left(\frac{\ell^{*2}}{n^2}\right)\right) \\ &\quad \times \left(1 - \frac{\ell^*}{n}\mathrm{e}^{\gamma(1-\frac{m^*}{n})}\right)\Bigg| \\ &\geq \frac{n^2}{\gamma^2}\left|\frac{\ell^{*2}\gamma}{n^2} + \frac{\ell^{*2}\gamma^2}{2n^2} - \frac{\ell^{*2}\gamma}{n^2}\mathrm{e}^{\gamma(1-\frac{m^*}{n})} + o\left(\frac{\ell^{*2}}{n^2}\right)\right| \\ &\geq \ell^{*2}\left|\frac{1}{2} + \frac{1}{\gamma}\left(1 - \mathrm{e}^{\gamma(1-\frac{m^*}{n})}\right)\right|. \end{aligned}$$

So the divergence (5.22) follows from the condition $n^{-3/2+\alpha}\ell^{*2-\alpha}a_n \to \infty$ and (5.28).

□

Corollary 5.2.5. *In the second type model defined by* (5.1) *and* (5.6)*, assume that for some* $p > 2$*,* (ε_i) *satisfy condition* (3.11)*. Let* $\alpha \in (0, 1/2 - 1/p)$ *and assume that*

$$\liminf_{n\to\infty} \gamma_n n^{-\alpha/(1/2-1/p)} > 0.$$

Suppose that either of the following conditions is satisfied:

1. $\ell^*(1-\phi_n) \to \infty$*,* $\limsup_{n\to\infty} \ell^*/n < 1$ *and* $n^{-1/2+\alpha}\ell^{*1-\alpha}a_n \to \infty$*;*
2. $\ell^*(1-\phi_n) \to c > 0$ *and* $n^{-1/2+\alpha}\ell^{*1-\alpha}a_n \to \infty$*;*
3. $\ell^*(1-\phi_n) \to 0$ *and* $n^{-3/2+\alpha}\gamma_n\ell^{*2-\alpha}a_n \to \infty$*.*

Then under H_A*,*

$$n^{-1/2+\alpha}(1-\phi_n)\widetilde{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathrm{P}} \infty. \tag{5.29}$$

The conclusion extends to the special case $\alpha = 0$ *under the same assumptions provided that* (3.11) *is replaced by* $\mathbb{E}\varepsilon_1^2 < \infty$*.*

Proof. By Theorem 5.1.4, under H_0, $b_n\widetilde{T}_{\alpha,n}$ converges in distribution and hence is stochastically bounded for the normalization $b_n = n^{-1/2+\alpha}(1-\phi_n)$. So it remains only to check condition (5.22) in the three cases under consideration.

– If $\ell^*(1-\phi_n)$ tends to infinity, noting that

$$\left|(1-\phi_n^{\ell^*})(\phi_n - \ell^* n^{-1}\phi_n^{n-m^*+1})\right| \leq 1$$

and recalling that $\limsup \ell^*/n < 1$, we immediately see that for n large enough, there is some positive constant c such that:

$$\left|A - \frac{\ell^*}{n}(A+B)\right| \geq \frac{c\ell^*}{1-\phi_n}.$$

Then the divergence (5.22) follows clearly from the condition

$$n^{-1/2+\alpha}\ell^{*1-\alpha}a_n \to \infty.$$

– If $\ell^*(1-\phi_n)$ tends to some $c>0$, this implies in particular that ℓ^*/n tends to zero and

$$1-\phi_n^{\ell^*} \xrightarrow[n\to\infty]{} 1-\mathrm{e}^{-c}.$$

By strict convexity of the exponential function, $\mathrm{e}^{-c} \geq 1-c$ with equality only if $c=0$, hence $c-1+\mathrm{e}^{-c}>0$ since $c>0$ and

$$\left|A - \frac{\ell^*}{n}(A+B)\right| \sim \frac{c-1+\mathrm{e}^{-c}}{(1-\phi_n)^2} \sim \frac{(c-1+\mathrm{e}^{-c})\ell^*}{c(1-\phi_n)}.$$

Again the divergence (5.22) follows from the condition

$$n^{-1/2+\alpha}\ell^{*1-\alpha}a_n \to \infty.$$

– Assume finally that $\ell^*(1-\phi_n)$ tends to zero (this implies in particular that $\ell^* = o(n)$). Then in (5.26) the term $\ell^*(1-\phi_n)$ is compensated at the first order by $(1-\phi_n^{\ell^*})$. By second order expansion, we find that

$$1-\phi_n^{\ell^*} = \frac{\ell^*\gamma_n}{n} + \frac{\ell^{*2}\gamma_n^2}{2n^2}(1+o(1)).$$

This leads by elementary computation to

$$A - \frac{\ell^*}{n}(A+B) \sim -\frac{\ell^{*2}}{2},$$

so the the divergence (5.22) follows from the condition

$$n^{-3/2+\alpha}\gamma_n\ell^{*2-\alpha}a_n \to \infty.$$

□

Remark 5.2.6. The graphical interpretation presented in figure 5.1, 5.2 and 5.2 may provide a better understanding of the results in corollary 5.2.5. Assume for simplicity that $a_n = 1$, $\ell^* \asymp n^a$ (that is there are positive constants c_1 and c_2 such that for n large enough, $c_1 n^a \leq \ell^* \leq c_2 n^a$) and that $\phi_n \asymp n^b$ for some $0 < a, b < 1$. For a given value of p in condition (3.11), page 23, what are the pairs (a, b) for which corollary 5.2.5 allows detection of an epidemics of length $\ell^* \asymp n^a$, subject to an admissible choice of α ? The set of solutions is represented by the shadowed area of the unit square. The light grey part above the diagonal corresponds to the cases 1 and 2, that is $\lim_{n\to\infty} \ell^*(1 - \phi_n)$ belongs to $(0, \infty]$. Its West border is an arc of hyperbola with parametric representation $a = (1 - 2\alpha_p t)/(2 - 2\alpha_p t)$, $b = t$ where $t = \alpha/\alpha_p$ and $\alpha_p = 1/2 - 1/p$. The darker grey area corresponds to the case where $\ell^*(1 - \phi_n)$ tends to 0. It is the triangle delimited by the diagonal, the horizontal axis and the straight line D_{α_p}, where D_α has for Cartesian equation $(2 - \alpha)a + b - 3/2 + \alpha = 0$. All these lines have $F(1, -1/2)$ as a common point. Figure 5.1 is given with the $p = 8$. If p tends to 2, the detection region becomes smaller. This effect one may observe in figure 5.2, where $p = 3$. One can remark that when p tends to infinity the whole shadowed area converges to the trapezoid with upper basis the upper side of the unit square and lower basis the segment $[2/3, 1]$ on the horizontal axis (see figure 5.3).

5.3 Test power analysis

Here we perform the test power analysis. For this, we present the results of experiments in the tables 5.4 and 5.5. We computed empirical power on size-adjusted (not nominal size) basis, i.e., replaced the nominal value of significance level by the value of empirical distribution function for p-values under null hypothesis. For more details on size power curves see Davidson and MacKinnon [1994].

For different values of parameters γ, γ_n, α, k^*, ℓ^* and a_n we compute $N = 1000$ realizations of test statistics with the sample size n. Innovations have been generated as standard normally distributed random variables. For the limit distribution we compute $N = 5000$ realizations of test statistics with the sample size $n = 5000$.

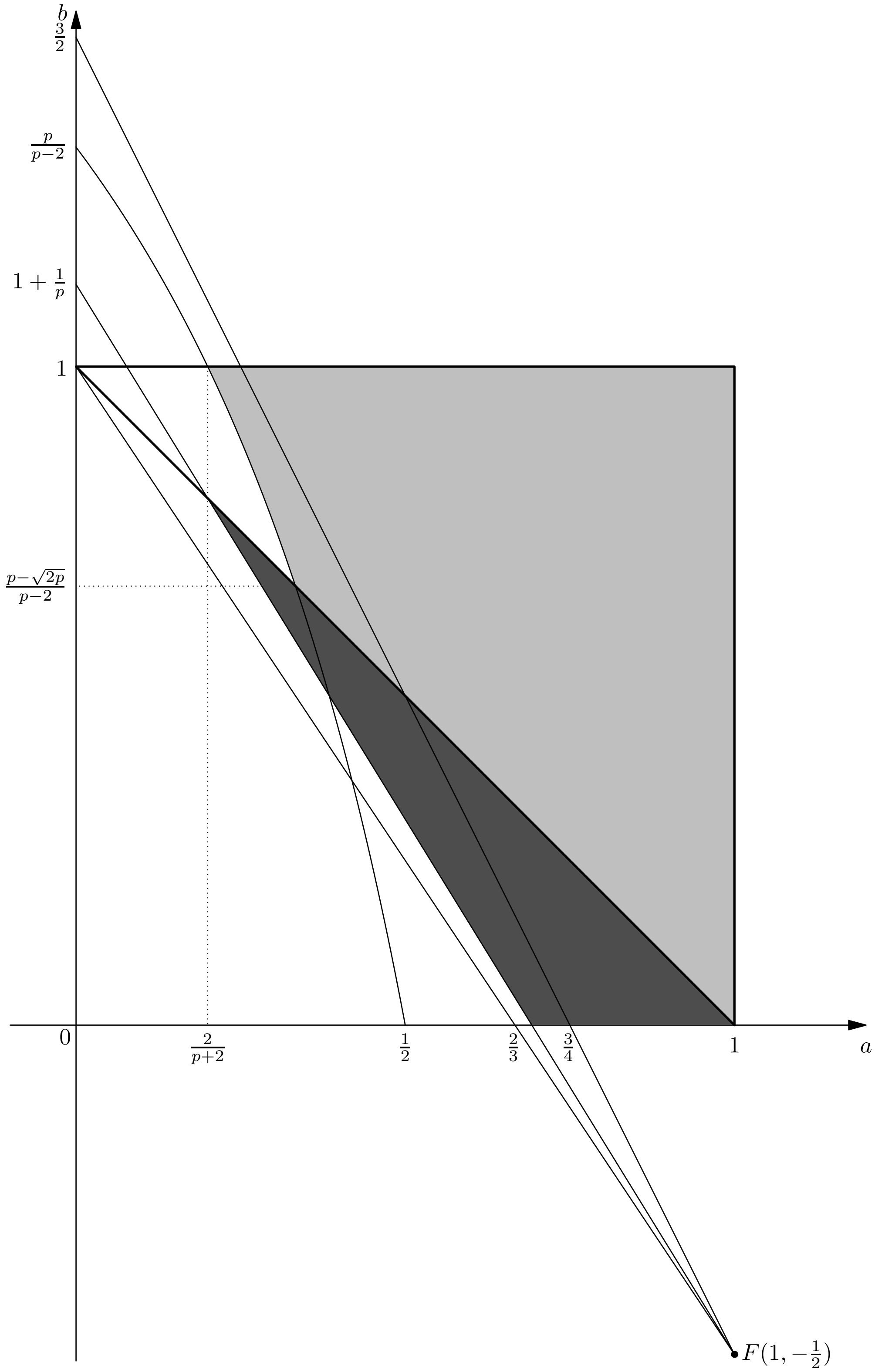


Figure 5.1: Detection area in the space of parameters ($\ell^* \asymp n^a, \gamma_n \asymp n^b$) for corollary 5.2.5 with $p = 8$.

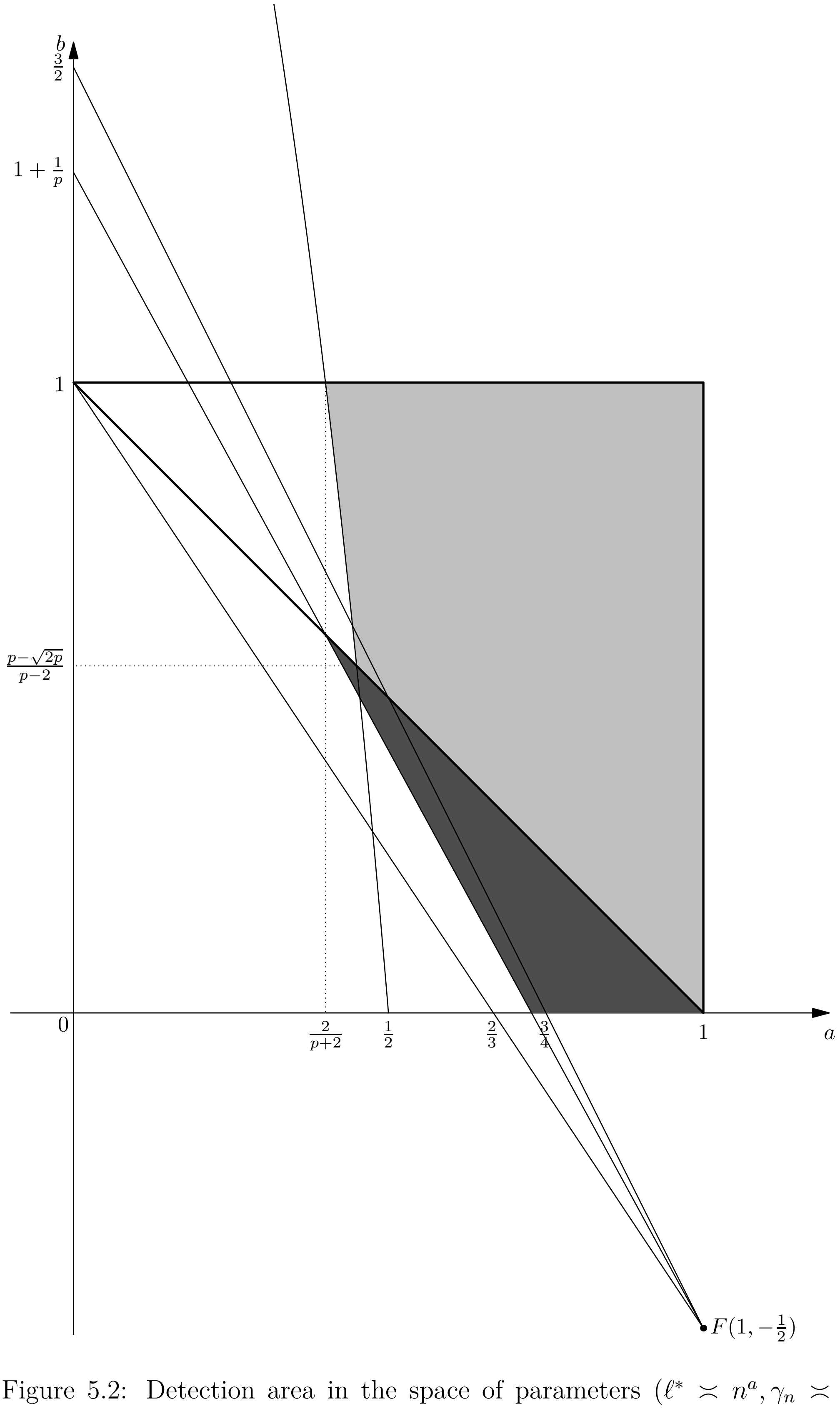


Figure 5.2: Detection area in the space of parameters ($\ell^* \asymp n^a, \gamma_n \asymp n^b$) for corollary 5.2.5 with $p = 3$.

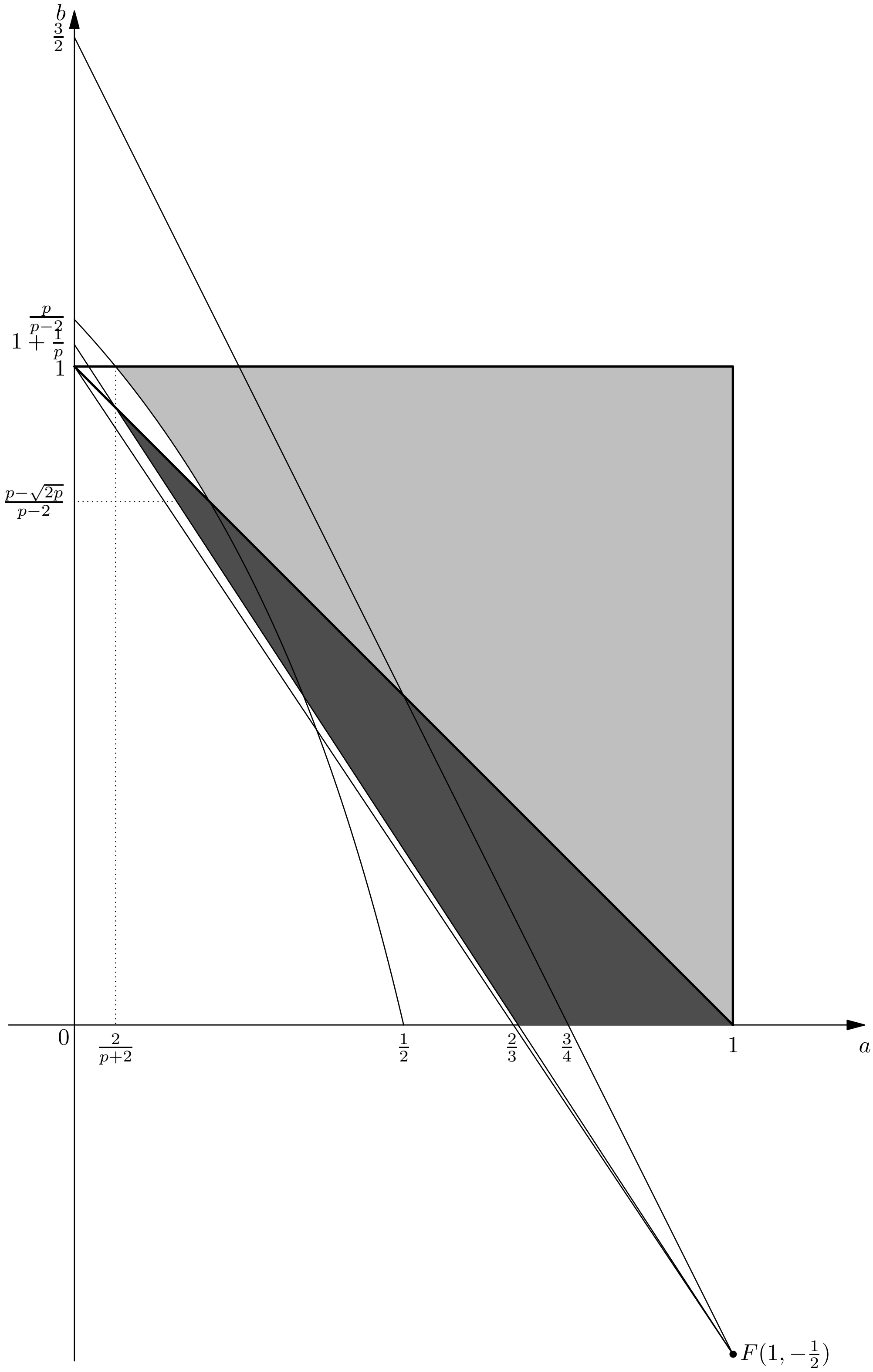


Figure 5.3: Detection area in the space of parameters ($\ell^* \asymp n^a, \gamma_n \asymp n^b$) for corollary 5.2.5 with $p = 30$.

We approximate the values of the standard Wiener process by

$$W\left(\frac{k}{5000}\right) = 5000^{-1/2}\sum_{j=1}^{k}\varepsilon(j), \quad k = 1, \ldots, 5000, \tag{5.30}$$

where $\varepsilon(j)$ are generated as standard normally distributed random variables. The Ornstein-Uhlenbeck process have been approximated by the the following discretization

$$S(j) = S(j-1)\mathrm{e}^{\gamma/n} + \sqrt{\frac{1-\mathrm{e}^{2\gamma/n}}{-2\gamma}} \cdot \varepsilon(j), \quad \varepsilon(j) \sim \mathfrak{N}(0,1). \tag{5.31}$$

For more details about (5.31), see van den Berg [2011]. Using values generated by (5.31), we approximate the integrated Ornstein-Uhlenbeck process by

$$J\left(\frac{k}{5000}\right) = 5000^{-1}\sum_{j=1}^{k} S(j), \quad k = 1, \ldots, 5000,$$

Next we define the basic parameter set for the first type model

$$\gamma = -2; \quad a_n = 1; \quad n = 1000; \quad \frac{\ell^*}{n} = 0.05; \quad \frac{k^*}{n} = 0.4, \quad y_{n,0} = 0.$$

Further modifying the separate parameters we compute the empirical size-power. We always keep all these parameters fixed except one (indicated in the first column in both tables) which we allow to vary. Note, that in order to compute the test power, we need to compute the empirical p-values. Usually, the estimate of empirical p-value is $\widehat{p} = s/N$, where s is the number of values (limit process) that are greater than or equal to the observed value (statistics), N is the number of values. Nevertheless, the previous formula is biased due to the finite sampling. Davison and Hinkley [1997] (see p.141) suggested to correct the bias with such formula $\widehat{p} = (s+1)/(N+1)$. One can observe, that these two formulas are essentially the same when the number of replications N is large, but we use unbiased estimate in this computations.

As one can see in the table 5.4 the test power is almost the same for all α. The test power increases with the length of epidemics, the location of epidemics makes the difference. The biggest power is for the epidemics in the middle of the observations. For this model, the test can detect the epidemic change best when $a_n = 1$ or bigger, for the smaller changes it has a lower power. Naturally, the test power increases with the number of observations. Further the bigger is γ, the bigger is test power. That is the test power increases when the coefficient is further removed from the unity.

Parameters	$\alpha = 0$	$\alpha = 2/32$	$\alpha = 6/32$	$\alpha = 12.5/32$
$\ell^*/n = 0.035$	0.442	0.440	0.446	0.421
$\ell^*/n = 0.050$	0.758	0.757	0.767	0.752
$\ell^*/n = 0.100$	1.000	1.000	1.000	1.000
$k^*/n = 0.2$	0.591	0.589	0.615	0.653
$k^*/n = 0.4$	0.758	0.757	0.767	0.752
$k^*/n = 0.8$	0.587	0.616	0.697	0.784
$a_n = 0.8$	0.554	0.549	0.556	0.534
$a_n = 1$	0.758	0.757	0.767	0.752
$a_n = 1.2$	0.907	0.908	0.920	0.914
$n = 500$	0.388	0.404	0.408	0.409
$n = 1000$	0.758	0.757	0.767	0.752
$n = 2000$	0.979	0.982	0.980	0.983
$\gamma = -2$	0.758	0.757	0.767	0.752
$\gamma = -12$	0.677	0.728	0.822	0.896
$\gamma = -100$	0.748	0.833	0.967	0.998

Table 5.4: Empirical power at the size-adjusted significance level 0.05 for the first type model

The basic parameter set for the second type model ($\phi_n = 1 - \gamma_n/n$) are

$$\gamma_n = n^{3/4}; \quad a_n = 1; \quad n = 1000; \quad \frac{\ell^*}{n} = 0.05; \quad \frac{k^*}{n} = 0.4, \quad y_{n,0} = 0.$$

For the second type model (table 5.5), the test power for all parameter values is the lowest, when $\alpha = 0$ and increases with α. For this model, detection of epidemic changes becomes better with the increasing length of epidemics, nevertheless the test detects short epidemic change very good for the bigger α ($\approx 1/3$). Note, that the test power does not depend on the place of epidemics. Also, it detects quite good even small changes as $a_n = 0.8$. The test power increases when the number of observations is increasing. The test power does not vary too much depending on the chosen γ_n.

Parameters	$\alpha = 0$	$\alpha = 2/32$	$\alpha = 6/32$	$\alpha = 10/32$
$\ell^*/n = 0.035$	0.373	0.441	0.675	0.909
$\ell^*/n = 0.050$	0.758	0.859	0.974	0.996
$\ell^*/n = 0.065$	0.980	0.990	0.999	1.000
$k^*/n = 0.2$	0.780	0.875	0.980	0.999
$k^*/n = 0.4$	0.758	0.859	0.974	0.996
$k^*/n = 0.8$	0.783	0.877	0.981	0.998
$a_n = 0.8$	0.478	0.565	0.780	0.929
$a_n = 1$	0.758	0.859	0.974	0.996
$a_n = 1.2$	0.949	0.985	0.999	1.000
$n = 500$	0.422	0.480	0.676	0.813
$n = 1000$	0.758	0.859	0.974	0.996
$n = 2000$	0.997	1.000	1.000	1.000
$\gamma_n = n/\ln(n)$	0.754	0.847	0.970	0.995
$\gamma_n = \ln^{2.5}(n)$	0.758	0.844	0.972	0.995
$\gamma_n = n^{3/4}$	0.758	0.859	0.974	0.996

Table 5.5: Empirical power at the size-adjusted significance level 0.05 for the second type model

6

Testing the epidemic change with statistics built on residuals

In the previous chapter we have studied test statistics built on the observations for the detection of a changed segment in the mean of innovations in a first order nearly nonstationary process. Another way to test such hypothesis is to build the test statistics on residuals, since innovations are not observed. Indeed, residuals are the estimated innovations and are supposed to have the same mean. In this chapter we find the limit behaviour of test statistics under null hypothesis, we investigate the conditions of consistency when the mean is shifted by a constant during the epidemics. Also, we discuss the interplay of various parameters to detect the shortest possible epidemics. Moreover, we perform test power analysis for our test statistics.

Here we investigate the same model as in the previous section. Suppose, that we observe an n-sample $y_{n,1}, \dots, y_{n,n}$ generated by

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k + a_{n,k}, \quad k \le n, \quad n \ge 1, \quad y_{n,0} = 0 \qquad (6.1)$$

where $\phi_n \to 1$, as $n \to \infty$, innovations $(\varepsilon_k, k \geq 1)$ are i.i.d. centered and at least square integrable random variables, $(a_{n,k})$ is a sequence that denotes the epidemic change in mean.

The goal of this chapter is to propose the test statistics that is devoted to test the null hypothesis

$$H_0: \quad a_{n,1} = \cdots = a_{n,n} = 0$$

against the changed segment alternative:

$$\begin{aligned} H_A: \ \text{there exist} \quad & 1 \leq k_n^*, \quad 1 \leq m_n^* \leq n \quad \text{such that} \\ & a_{n,k} = a_n \mathbf{1}_{\mathbb{I}_n^*}(k), \quad a_n \neq 0, \quad 1 \leq k \leq n, \end{aligned}$$

where $\mathbb{I}_n^*$ is the epidemics interval

$$\mathbb{I}_n^* = \{k_n^* + 1, \ldots, m_n^*\}$$

and $\mathbf{1}_{\mathbb{I}_n^*}$ denotes its indicator function.

To detect a short epidemic change in the mean of innovations of the first order nearly nonstationary autoregressive process, we build the α-Hölderian uniform increments statistics based on the residuals for $0 < \alpha \leq 1$:

$$\widehat{T}_{\alpha,n} = \max_{1 \leq \ell \leq n} \ell^{-\alpha} \max_{1 \leq k \leq n-\ell} \left| \sum_{j=k+1}^{k+\ell} \widehat{\varepsilon}_j - \frac{\ell}{n} \sum_{j=1}^{n} \widehat{\varepsilon}_j \right|. \tag{6.2}$$

Recall that residuals are defined by

$$\widehat{\varepsilon}_k = y_{n,k} - \widehat{y}_{n,k} = y_{n,k} - \widehat{\phi}_n y_{n,k-1}, \quad k \leq n, \quad n \geq 0,$$

where $\widehat{\phi}_n$ is the least squares estimate of the coefficient ϕ_n:

$$\widehat{\phi}_n = \frac{\sum_{k=1}^n y_{n,k} y_{n,k-1}}{\sum_{k=1}^n y_{n,k-1}^2}.$$

In this chapter we again investigate two type of models. First type model is defined by (6.1) with the coefficient

$$\phi_n = \mathrm{e}^{\gamma/n}, \quad \gamma < 0, \tag{6.3}$$

while second type model is defined by (6.1) with coefficient

$$\phi_n = 1 - \frac{\gamma_n}{n}, \quad \gamma_n \to \infty, \quad \gamma_n/n \to 0, \quad \text{as} \quad n \to \infty. \tag{6.4}$$

Here we assume that innovations are

i.i.d. centred and satisfies for some $p > 2$ the integrability condition

$$\lim_{t\to\infty} t^p \mathbb{P}\left(|\varepsilon_0| > t\right) = 0 \tag{6.5}$$

or

$$\text{i.i.d. centred and regularly varying random variables with index } p > 2. \tag{6.6}$$

Definition 6.0.1. *The random variable X is regularly varying with index $p > 0$ (denoted $X \in RV_p$) if there exists a slowly varying function L such that the distribution function $F(t) = P(X \le t)$ satisfies the tail balance condition*

$$F(-x) \sim bL(x)x^{-p} \quad \textit{and} \quad 1 - F(x) \sim aL(x)x^{-p}, \quad \textit{as} \quad x \to \infty,$$

where $a, b \in (0, 1)$ and $a + b = 1$.

We refer to Bingham et al. [1987] for an encyclopaedic treatment of regular variation. The assumption on regular variation with $p > 2$ allows us to investigate the test statistics in the whole range of $\alpha \in (0, 1]$ except one point $\alpha_p = 1/2 - 1/p$.

6.1 Limit under null hypothesis

For any function $f \in \mathrm{H}^o_\alpha[0, 1]$ and $0 < \alpha < 1/2$ we define

$$T_{\alpha,\infty}(f) := \sup_{0<t<s<1} \frac{|f(t) - f(s) - (t - s)f(1)|}{|t - s|^\alpha}.$$

6.1.1 Model with innovations satisfying condition (6.5)

Here we shall find the limit of the test statistics for two type of models.

Theorem 6.1.1. *In the first type model defined by* (6.1) *and* (6.3) *assume that innovations satisfy* (6.5) *for some $p > 2$. Then under H_0 for any $\alpha \in (0, \alpha_p)$*

$$n^{-1/2+\alpha}\sigma^{-1}\widehat{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_{\alpha,\infty}(Z), \tag{6.7}$$

where $\sigma^2 = E\varepsilon_1^2$*. Here*

$$Z(t) = W(t) - A^{-1}BJ(t), \tag{6.8}$$

where $A = \int_0^1 U_\gamma^2(t)\,\mathrm{d}t$*,* $B = \int_0^1 U_\gamma(t)\,\mathrm{d}W(t)$ *and* $J(t) = \int_0^t U_\gamma(r)\,\mathrm{d}r$*,* $t \in [0,1]$ *and* U_γ *is an Ornstein-Uhlenbeck process defined by* (3.1) *(page 19).*

Proof. Consider the functionals g_n, g, defined on $\mathrm{H}_\alpha^o[0,1]$ by

$$g_n(x) := \max_{1\le i<j\le n} I_\alpha(x, i/n, j/n), \quad g(x) := \sup_{0<s<t<1} I_\alpha(x,s,t), \tag{6.9}$$

where

$$I_\alpha(x,s,t) := \frac{|x(t)-x(s)-(t-s)x(1)|}{|t-s|^\alpha}, \quad 0 < t-s < 1.$$

By Lemma 3.3.3 (page 25), g_n and g are Lipschitz on $G_\alpha = \{x \in \mathrm{H}_\alpha^o[0,1] : x(0) = 0\}$. Observe that

$$n^\alpha \widehat{T}_{\alpha,n} = g_n(\widehat{W}_n^{\mathrm{pl}}), \quad T_{\alpha,\infty}(Z) = g(Z). \tag{6.10}$$

where $(\widehat{W}_n^{\mathrm{pl}}(t), t \in [0,1])$ is a polygonal line process built on residuals $(\widehat{\varepsilon}_k)$

$$\widehat{W}_n^{\mathrm{pl}}(t) := \sum_{k=1}^{[nt]} \widehat{\varepsilon}_k + (nt-[nt])\widehat{\varepsilon}_{[nt]+1}, \quad t \in [0,1].$$

From Theorem 4.2.2 (page 42) we have that

$$n^{-1/2}\sigma^{-1}\widehat{W}_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\alpha^o[0,1]} W - A^{-1}BJ. \tag{6.11}$$

Lemma 3.3.3 now gives

$$g_n(n^{-1/2}\sigma^{-1}\widehat{W}_n^{\mathrm{pl}}) = g(n^{-1/2}\sigma^{-1}\widehat{W}_n^{\mathrm{pl}}) + o_{\mathrm{P}}(1) \tag{6.12}$$

and the convergence (6.7) follows from (6.10), (6.11) and (6.12) and continuous mapping theorem. □

Theorem 6.1.2. *In the second type model defined by* (6.1) *and* (6.4) *assume that innovations satisfies* (6.5) *for some* $p > 2$*. Then under* H_0 *for any* $\alpha \in (0, \alpha_p)$

$$n^{-1/2+\alpha}\sigma^{-1}\widehat{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_{\alpha,\infty}(W), \tag{6.13}$$

where $\sigma^2 = E\varepsilon_1^2$, *provided that*

$$\liminf_{n\to\infty} \gamma_n n^{-2\alpha/(1+2\alpha_p)} > 0.$$

Proof. The proof of this theorem is essentially the same as the proof of Theorem 6.1.1 using the Theorem 4.2.8 (page 50) instead of Theorem 4.2.2 and Lemma 3.3.3. □

6.1.2 Model with regularly varying innovations

If $\varepsilon_1 \in RV_p$ we define

$$b_n = \inf\{x > 0 : P(|\varepsilon_1| \le x) \ge 1 - 1/n\}. \tag{6.14}$$

It easily follows from tail condition that there is a slowly varying function $v(n), n \in \mathbb{N}$ such that

$$b_n \sim \ n^{1/p} v(n) \quad \text{as} \quad n \to \infty. \tag{6.15}$$

Next theorem gives result for the first type model.

Theorem 6.1.3. *Let* $p > 2$. *If innovations* (ε_i) *satisfy* (6.6) *in the first type model defined by* (6.1) *and* (6.3), *then under* H_0

(a) for any $\alpha \in (\alpha_p, 1]$

$$b_n^{-1}\sigma^{-1}\widehat{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_p, \tag{6.16}$$

where T_p *is a random variable with Frechet distribution* $P(T_p \le x) = \mathrm{e}^{-x^{-p}}, x \in \mathbb{R}$.

(b) for any $\alpha \in (0, \alpha_p)$

$$n^{-1/2+\alpha}\sigma^{-1}\widehat{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_{\alpha,\infty}(Z), \tag{6.17}$$

where $Z(t)$ *is defined by* (6.8) *and* $A = \int_0^1 U_\gamma^2(t)\,\mathrm{d}t$, $B = \int_0^1 U_\gamma(t)\,\mathrm{d}W(t)$ *and* $J(t) = \int_0^t U_\gamma(r)\,\mathrm{d}r$, $t \in [0,1]$, U_γ *is an Ornstein-Uhlenbeck process.*

For the proof of Theorems 6.1.3 and 6.1.5 we use the following proposition whose proof is given in subsection 6.4.

Proposition 6.1.4. *Let* $p > 2$. *If* (ε_i) *are i.i.d. random variables,* $\varepsilon_1 \in RV_p$ *and* $\alpha \in (\alpha_p, 1]$ *and* $(y_{n,k})$ *is generated by* (4.1), *then*

1. *(a) for* $\phi_n = \mathrm{e}^{\gamma/n}$*,* $\gamma < 0$

$$T_{\alpha,n}(\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_n) = T_{\alpha,n}(\varepsilon_1, \ldots, \varepsilon_n) + o_P(b_n).$$

holds,

2. *(b) for* $\phi_n = 1 - \frac{\gamma_n}{n}$*, where* $\gamma_n \to \infty$ *and* $\gamma_n/n \to 0$ *as* $n \to \infty$

$$T_{\alpha,n}(\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_n) = T_{\alpha,n}(\varepsilon_1, \ldots, \varepsilon_n) + o_P(b_n).$$

holds, provided that

$$\gamma_n = O(n^{q(\alpha-\alpha_p)}), \quad 0 < q < 2. \tag{6.18}$$

Proof of Theorem 6.1.3. (a) Proposition 6.1.4 (page 86) indicates that the limit behaviors of both statistics $\widehat{T}_{\alpha,n}$ and $T_{\alpha,n}(\varepsilon_1, \ldots, \varepsilon_n)$ coincide. Hence the result follows by Theorem 1.1 in Mikosch and Račkauskas [2010] (see Theorem 3.3.8 and Corollary 3.3.9 on page 29).

(b) We notice that if $\varepsilon_1 \in RV_p$, then for any $p' < p$ we have $t^{p'}\mathbb{P}(|\varepsilon_1| > t) \to 0$, as $t \to \infty$. Hence for $\alpha < \alpha_p$ choosing $p' < p$ such that $\alpha \le \alpha_{p'}$ we deduce the result by Theorem 6.1.1.

□

Further for the second type model, we obtain the following result.

Theorem 6.1.5. *Let* $p > 2$*. If innovations* (ε_i) *satisfy* (6.6) *in the second type model defined by* (6.1) *and* (6.4)*, then under* H_0

(a) for any $\alpha \in (\alpha_p, 1]$

$$b_n^{-1}\sigma^{-1}\widehat{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_p, \tag{6.19}$$

provided that $\gamma_n = O(n^{q(\alpha-\alpha_p)})$ *for some* $0 < q < 2$.

(b) for any $\alpha \in (0, \alpha_p)$ *if*

$$\liminf_{n\to\infty} \gamma_n n^{\frac{-2\alpha}{1+2\alpha_p}} > 0,$$

then it holds

$$n^{-1/2+\alpha}\sigma^{-1}\widehat{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathbb{R}} T_{\alpha,\infty}(W). \tag{6.20}$$

Proof. **(a)** Proposition 6.1.4 (page 86) indicates that the limit behaviors of both statistics $\widehat{T}_{\alpha,n}$ and $T_{\alpha,n}(\varepsilon_1, \ldots, \varepsilon_n)$ coincide. Hence the result follows by Theorem 1.1 in Mikosch and Račkauskas [2010] (see Theorem 3.3.8 and Corollary 3.3.9 on page 29).

(b) To prove this part, we notice that if $\varepsilon_1 \in RV_p$, then $t^p \mathbb{P}(|\varepsilon_1| > t) \sim L(t)$ for some slowly varying function $L(t)$. So for every $0 < p' < p$ we have $t^{p'} \mathbb{P}(|\varepsilon_1| > t) \to 0$, as $t \to \infty$. Now for all $\alpha \in (0, \alpha_p)$ we choose $2 < p' < p$ such that $\alpha < \alpha_{p'} < \alpha_p$. It follows that $n^{-\alpha/\alpha_p} < n^{-\alpha/\alpha_{p'}}$ so that condition $\liminf_{n\to\infty} \gamma_n n^{-\alpha/\alpha_{p'}} > 0$ holds. Then we deduce the convergence (6.20) by Theorem 6.1.2.

□

6.2 Consistency analysis

In this section we find conditions for the consistency of test statistics for the second type model. We see further, that the methods we use to prove the consistency do not work for the first type model.

We again rewrite the model through the term $\tau_{n,k}$

$$y_{n,k} - \tau_{n,k} = \phi_n(y_{n,k-1} - \tau_{n,k-1}) + \varepsilon_k,$$

where $\tau_{n,k}$ is defined by (5.3) (page 65):

$$\tau_{n,k} = \sum_{j=0}^{k-1} \phi_n^j a_{n,k-j} = \sum_{j=1}^{k} \phi_n^{k-j} a_{n,j}.$$

Recall that

$$z_{n,k} = y_{n,k} - \tau_{n,k}, \quad k = 0, 1, \dots, n.$$

Note, that $z_{n,k}$ is a first order nearly nonstationary autoregressive process generated by (4.1), page 31.

The next theorem gives the result for consistency of test statistics $\widehat{T}_{\alpha,n}$ for the second type model with a_n constant and $\alpha \in (0, \alpha_p)$.

Theorem 6.2.1. *Under H_A, assume that $\ell^* \to \infty$, $\ell^*/n \to 0$ and for some $\alpha \in (0, \alpha_p)$,*

$$n^{-1/2+\alpha} \ell^{*(1-\alpha)} \xrightarrow[n\to\infty]{} \infty.$$

Then for the second type model defined by (6.1) *and* (6.4) *with innovations* (ε_i) *that satisfy* (6.5) *or* (6.6)

$$n^{-1/2+\alpha} \widehat{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathrm{P}} \infty \tag{6.21}$$

holds, provided that γ_n is increasing in n or regular varying sequence,

$$\liminf_{n\to\infty} \gamma_n n^{-\alpha/\alpha_p} > 0 \tag{6.22}$$

and

$$\frac{\widehat{\phi}_n - \phi_n}{1-\phi_n} = o_P(1). \tag{6.23}$$

Condition (6.23) holds provided that

- $\ell^* = o(\gamma_n)$ *if* $\ell^*(1-\phi_n) \to \infty$, *as* $n \to \infty$;
- $\ell^* = o(\gamma_n^3 n^{-1})$ *if* $\ell^*(1-\phi_n) \to 0$, *as* $n \to \infty$.

Further we give the proof of consistency of statistics $\widehat{T}_{\alpha,n}$ for the second type model with a_n constant and $\alpha \in (\alpha_p, 1]$.

Theorem 6.2.2. *Under H_A, assume that $\ell^* \to \infty$, $\ell^*/n \to 0$ and for some $\alpha \in (\alpha_p, 1]$,*

$$b_n^{-1} \ell^{*(1-\alpha)} \xrightarrow[n\to\infty]{} \infty.$$

Then for the second type model defined by (6.1) and (6.4) with innovations (ε_i) that satisfy (6.6)

$$b_n^{-1} \widehat{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathrm{P}} \infty \tag{6.24}$$

holds, provided that γ_n is increasing in n or regular varying sequence,

$$\gamma_n = O(n^{q(\alpha-\alpha_p)}), \quad 0 < q < 2 \tag{6.25}$$

and

$$\frac{\widehat{\phi}_n - \phi_n}{1-\phi_n} = o_P(1). \tag{6.26}$$

Condition (6.26) holds provided that

- $\ell^* = o(\gamma_n)$ *if* $\ell^*(1-\phi_n) \to \infty$, *as* $n \to \infty$;
- $\ell^* = o(\gamma_n^3 n^{-1})$ *if* $\ell^*(1-\phi_n) \to 0$, *as* $n \to \infty$.

The proofs of Theorems 6.2.1 and 6.2.2 are given at the end of this subsection on pages 98 and 100. Further supplementary results are given for both type of models. We start from the lower bound of the test statistics and with the estimates for some members of this lower bound.

Lemma 6.2.3. *In the first order nearly nonstationary autoregressive process defined by* (6.1) *and either* (6.3) *or* (6.4) *assume that innovation satisfies condition* (6.5) *or* (6.6). *Then under H_A for any $\alpha \in (0,1]$*

$$\widehat{T}_{\alpha,n} \geq T_{\alpha,n}(a_{n,1},\ldots,a_{n,n}) - \left|\widehat{\phi}_n - \phi_n\right| T_{\alpha,n}(\tau_{n,0},\ldots,\tau_{n,n-1}) \\ - T_{\alpha,n}(\varepsilon_1,\ldots,\varepsilon_n) - \left|\widehat{\phi}_n - \phi_n\right| T_{\alpha,n}(z_{n,0},\ldots,z_{n,n-1}) \tag{6.27}$$

and

$$T_{\alpha,n}(a_{n,1},\ldots,a_{n,n}) \geq |a_n| \, \ell^{*(1-\alpha)}. \tag{6.28}$$

Proof. We have under H_A

$$\begin{aligned}\widehat{T}_{\alpha,n} &= \max_{1\leq\ell\leq n} \ell^{-\alpha} \max_{1\leq k\leq n-\ell} \left| \sum_{j=k+1}^{k+\ell} \widehat{\varepsilon}_j - \frac{l}{n}\sum_{j=1}^{n}\widehat{\varepsilon}_j \right| \\ &= \max_{1\leq\ell\leq n} \ell^{-\alpha} \max_{1\leq k\leq n-\ell} \left| \sum_{j=k+1}^{k+\ell} \Big(\varepsilon_j + a_{n,j} - (\widehat{\phi}_n - \phi_n)\tau_{n,j-1} - (\widehat{\phi}_n - \phi_n) z_{n,j-1}\Big) \right. \\ &\quad \left. - \frac{l}{n}\sum_{j=1}^{n} \Big(\varepsilon_j + a_{n,j} - (\widehat{\phi}_n - \phi_n)\tau_{n,j-1} - (\widehat{\phi}_n - \phi_n) z_{n,j-1}\Big) \right| \\ &\geq T_{\alpha,n}(a_{n,1},\ldots,a_{n,n}) - \left|\widehat{\phi}_n - \phi_n\right| T_{\alpha,n}(\tau_{n,0},\ldots,\tau_{n,n-1}) \\ &\quad - T_{\alpha,n}(\varepsilon_1,\ldots,\varepsilon_n) - \left|\widehat{\phi}_n - \phi_n\right| T_{\alpha,n}(z_{n,0},\ldots,z_{n,n-1}).\end{aligned}$$

Further, assume that $\ell^* = o(n)$, then

$$\begin{aligned}T_{\alpha,n}(a_{n,1},\ldots,a_{n,n}) &= \max_{1\leq\ell\leq n} \ell^{-\alpha} \max_{1lek\leq n-\ell} \left| \sum_{j=k+1}^{k+\ell} a_n \mathbf{1}_{\mathbb{I}_n^*}(j) - \frac{l}{n}\sum_{j=1}^{n} a_n \mathbf{1}_{\mathbb{I}_n^*}(j) \right| \\ &\geq |a_n| \, \ell^{*(1-\alpha)} \left(1 - \frac{\ell^*}{n}\right) \geq |a_n| \, \ell^{*(1-\alpha)}.\end{aligned}$$

□

The next lemma gives us the estimate of $\left|\widehat{\phi}_n - \phi_n\right| / (1 - \phi_n)$.

Lemma 6.2.4. *Assume $k^* = [\lambda n]$ with some fixed $0 < \lambda < 1$. Suppose that first order nearly nonstationary process is defined by* (6.1) *and* (6.3) *or* (6.4) *with innovations satisfying* (6.5) *or* (6.6). *Then for the least squares estimator $\widehat{\phi}_n$*

under alternative H_A

$$\frac{\left|\widehat{\phi}_n - \phi_n\right|}{1-\phi_n} = \frac{\left|\widetilde{\phi}_n - \phi_n\right|}{1-\phi_n} O_P(1) + O_P\left(\frac{a_n^2 \ell^*}{n(1-\phi_n)}\right) + O_P\left(\frac{|a_n|}{n(1-\phi_n)^{3/2}}\right) + O_P\left(\max\left(\frac{|a_n|\sqrt{\ell^*}}{n(1-\phi_n)}, \frac{|a_n|\,\ell^*}{n(1-\phi_n)^{1/2}}\right)\right)$$

holds, assuming for the second type model that γ_n is increasing in n or regular varying. Here $\widetilde{\phi}_n$ denotes the least squares estimator under null hypothesis H_0.

To prove Lemma 6.2.4 we need the two following auxiliary lemmas whose proof is deferred to section 6.4 on pages 122 and 124.

Lemma 6.2.5. *Assume $k^* = [\lambda n]$ with some fixed $0 < \lambda < 1$. Then it holds*

$$\frac{\sum_{k=1}^{n} z_{n,k-1}^2}{\sum_{k=1}^{[n\lambda]} z_{n,k-1}^2} = O_P(1)$$

additionally assuming that γ_n is increasing in n or regular varying for the second type model.

Lemma 6.2.6. *Assume $k^* = [\lambda n]$ with some fixed $0 < \lambda < 1$. Then it holds*

$$\frac{1}{\sum_{k=1}^{[n\lambda]} z_{n,k-1}^2} \leq \frac{(1-\phi_n) O_P(1)}{n}.$$

Proof of Lemma 6.2.4. Since

$$\sum_{k=1}^{n} y_{n,k-1}^2 = \sum_{k=1}^{n} (z_{n,k-1} + \tau_{n,k-1})^2 \geq \sum_{k=1}^{[n\lambda]} (z_{n,k-1} + \tau_{n,k-1})^2 = \sum_{k=1}^{[n\lambda]} z_{n,k-1}^2$$

and

$$\widehat{\phi}_n - \phi_n = \frac{\sum_{k=1}^{n} z_{n,k-1}\varepsilon_k + \sum_{k=1}^{n} \tau_{n,k-1}\varepsilon_k + \sum_{k=1}^{n} y_{n,k-1} a_{n,k}}{\sum_{k=1}^{n} y_{n,k-1}^2}$$

we have by denoting $\widetilde{\phi}_n$ the least squares estimator of ϕ_n built on $(z_{n,k})$

$$\begin{aligned}\left|\widehat{\phi}_n - \phi_n\right| &\leq \left|\widetilde{\phi}_n - \phi_n\right| \frac{\sum_{k=1}^{n} z_{n,k-1}^2}{\sum_{k=1}^{[n\lambda]} z_{n,k-1}^2} + \frac{1}{\sum_{k=1}^{[n\lambda]} z_{n,k-1}^2} \left|\sum_{k=1}^{n} \tau_{n,k-1}\varepsilon_k + \sum_{k=1}^{n} y_{n,k-1} a_{n,k}\right| \\ &\leq \left|\widetilde{\phi}_n - \phi_n\right| O_P(1) + \frac{(1-\phi_n) O_P(1)}{n} \left|\sum_{k=1}^{n} \tau_{n,k-1}\varepsilon_k + \sum_{k=1}^{n} y_{n,k-1} a_{n,k}\right|.\end{aligned}$$

Hence by Lemmas 6.2.5 and 6.2.6 we obtain

$$\frac{\left|\widehat{\phi}_n - \phi_n\right|}{1-\phi_n} \le \underbrace{\frac{\left|\widetilde{\phi}_n - \phi_n\right|}{1-\phi_n} O_P(1)}_{:=A} + \underbrace{\frac{O_P(1)}{n}\left|\sum_{k=1}^{n} y_{n,k-1}a_{n,k}\right|}_{:=B} + \underbrace{\frac{O_P(1)}{n}\left|\sum_{k=1}^{n} \tau_{n,k-1}\varepsilon_k\right|}_{:=C}.$$

As $\widetilde{\phi}_n$ is a least squares estimator in the model under null hypothesis, we have

– by Phillips $n(\widetilde{\phi}_n - \phi_n) = O_P(1)$ and $\frac{n}{-\gamma} \sim \frac{1}{1-\phi_n}$, so $\frac{\left|\widetilde{\phi}_n-\phi_n\right|}{1-\phi_n} = O_P(1)$ for the first type model.
– by Giraitis and Phillips $\frac{n^{1/2}}{(1-\phi_n^2)^{1/2}}(\widetilde{\phi}_n - \phi_n) = O_P(1)$ and $\frac{(1+\phi_n)^{1/2}}{(n(1-\phi_n))^{1/2}} \to 0$, so $\frac{\left|\widetilde{\phi}_n-\phi_n\right|}{1-\phi_n} = o_P(1)$ for the second type model.

Thus

$$A = O_P(1) \quad \text{for the first type model} \tag{6.29}$$

$$A = o_P(1) \quad \text{for the second type model.} \tag{6.30}$$

Next we have for part B

$$\sum_{k=1}^{n} y_{n,k-1}a_{n,k} = a_n \sum_{k^*+1}^{m^*} y_{n,k-1} = \frac{a_n}{1-\phi_n}\left(\ell^* a_n + y_{n,k^*} - y_{n,m^*} + \sum_{k+1}^{m^*}\varepsilon_k\right).$$

Evidently

$$\operatorname{Var}\left(\sum_{k+1}^{m^*}\varepsilon_k\right) = \sigma^2\ell^* \quad \Rightarrow \quad \sum_{k+1}^{m^*}\varepsilon_k = O_P(\sqrt{\ell^*}).$$

As $y_{n,k^*} = \sum_{j=1}^{k^*}\phi_n^{k^*-j}(\varepsilon_j + a_{n,j}) = \sum_{j=1}^{k^*}\phi_n^{k^*-j}\varepsilon_j$ we obtain

$$\operatorname{Var}\left(\sum_{j=1}^{k^*}\phi_n^{k^*-j}\varepsilon_j\right) \le \frac{\sigma^2}{1-\phi_n} \quad \Rightarrow \quad y_{n,k^*} = O_P(1/(1-\phi_n)^{1/2}).$$

Since

$$\begin{aligned} y_{n,m^*} &= \sum_{j=1}^{m^*}\phi_n^{m^*-j}(\varepsilon_j + a_{n,j}) = \sum_{j=1}^{m^*}\phi_n^{m^*-j}\varepsilon_j + \sum_{j=1}^{m^*}\phi_n^{m^*-j}a_{n,j} \\ &= \sum_{j=1}^{m^*}\phi_n^{m^*-j}\varepsilon_j + a_n\sum_{j=k^*+1}^{m^*}\phi_n^{m^*-j} = \sum_{j=1}^{m^*}\phi_n^{m^*-j}\varepsilon_j + a_n\frac{1-\phi_n^{\ell*}}{1-\phi_n}, \end{aligned}$$

we have

$$|y_{n,m^*}| = O_P(1/(1-\phi_n)^{1/2}) + O_P(|a_n|\,\ell^*).$$

So the upper bound

$$\begin{aligned}
B &\le \frac{|a_n|\,O_P(1)}{n(1-\phi_n)}\left(\ell^*\,|a_n| + |y_{n,k^*}| + |y_{n,m^*}| + \left|\sum_{k+1}^{m^*}\varepsilon_k\right|\right) \\
&= \frac{a_n^2\ell^*}{n(1-\phi_n)}O_P(1) + O_P\left(\frac{|a_n|}{n(1-\phi_n)^{3/2}}\right) + O_P\left(\frac{a_n^2\ell^*}{n(1-\phi_n)}\right) + O_P\left(\frac{|a_n|\sqrt{\ell^*}}{n(1-\phi_n)}\right) \\
&= O_P\left(\frac{a_n^2\ell^*}{n(1-\phi_n)}\right) + O_P\left(\frac{|a_n|}{n(1-\phi_n)^{3/2}}\right) + O_P\left(\frac{|a_n|\sqrt{\ell^*}}{n(1-\phi_n)}\right).
\end{aligned}$$

Finally for C we have

$$\mathrm{Var}\left(\sum_{k=1}^{n}\tau_{n,k-1}\varepsilon_k\right) = \sigma^2\sum_{k=1}^{n}\tau_{n,k-1}^2 \quad\Rightarrow\quad \sum_{k=1}^{n}\tau_{n,k-1}\varepsilon_k = O_P\left(\left(\sum_{k=1}^{n}\tau_{n,k-1}^2\right)^{1/2}\right).$$

Seeing that

$$\begin{aligned}
\sum_{k=1}^{n}\tau_{n,k-1}^2 &= \sum_{k=1}^{n}\left(\sum_{j=1}^{k-1}\phi_n^{k-1-j}a_{n,j}\right)^2 = a_n^2\sum_{k=1}^{n}\left(\sum_{j=1}^{k-1}\phi_n^{k-1-j}\mathbf{1}_{\mathbb{I}_n^*}(j)\right)^2 \\
&= a_n^2\left[\sum_{k=k^*+1}^{m^*}\left(\sum_{j=k^*+1}^{k-1}\phi_n^{k-1-j}\right)^2 + \sum_{k=m^*+1}^{n}\left(\sum_{j=k^*+1}^{m^*}\phi_n^{k-1-j}\right)^2\right] \\
&\le a_n^2\left[\frac{l^*}{(1-\phi_n)^2} + \frac{l^{*2}}{1-\phi_n^2}\right],
\end{aligned}$$

we obtain

$$C \le \frac{O_P(1)}{n}\left|\sum_{k=1}^{n}\tau_{n,k-1}\varepsilon_k\right| = O_P\left(\max\left(\frac{|a_n|\sqrt{\ell^*}}{n(1-\phi_n)}, \frac{|a_n|\,\ell^*}{n(1-\phi_n)^{1/2}}\right)\right).$$

□

Remark 6.2.7. Clearly (6.29) shows that the condition (6.23) can not be satisfied for the first type model using this method. But this condition is required to have the consistency of statistics. Thus we can not obtain the result for the first type model.

In the next corollary we assume that a_n is constant and we investigate only second type model.

Corollary 6.2.8. *Assume* $k^* = [\lambda n]$ *with some fixed* $0 < \lambda < 1$ *and* γ_n *is increasing in* n *or regular varying in the second type model defined by* (6.1) *and* (6.4) *with innovations satisfying* (6.5) *or* (6.6). *Then it holds under alternative* H_A

$$\frac{\left|\widehat{\phi}_n - \phi_n\right|}{1-\phi_n} = o_P(1)$$

provided that

- $\ell^* = o(\gamma_n)$ *if* $\ell^*(1-\phi_n) \to \infty$, *as* $n \to \infty$*;*
- $\ell^* = o(\gamma_n^3 n^{-1})$ *if* $\ell^*(1-\phi_n) \to 0$, *as* $n \to \infty$.

Proof. Taking into account the estimate of $\left|\widehat{\phi}_n - \phi_n\right|/(1-\phi_n)$ in Lemma 6.2.4 we obtain

$$\frac{\left|\widehat{\phi}_n - \phi_n\right|}{1-\phi_n} = o_P(1) + \frac{\sqrt{\ell^*}}{n(1-\phi_n)^{1/2}}\left(O_P\left(\frac{\sqrt{\ell^*}}{(1-\phi_n)^{1/2}}\right) + O_P\left(\frac{1}{1-\phi_n}\right)\right.$$
$$\left. + O_P\left(\max\left(\frac{1}{(1-\phi_n)^{1/2}}, \sqrt{\ell^*}\right)\right)\right).$$

As $1/(1-\phi_n)^{1/2}$ and $\sqrt{\ell^*}$ are negligible compared with $\sqrt{\ell^*}/(1-\phi_n)^{1/2}$, thus we need to consider only two cases.

- The first case is

$$\frac{\frac{\sqrt{\ell^*}}{(1-\phi_n)^{1/2}}}{\frac{1}{1-\phi_n}} \to \infty \Leftrightarrow \ell^*(1-\phi_n) \to \infty,$$

  then

$$\frac{\left|\widehat{\phi}_n - \phi_n\right|}{1-\phi_n} = o_P(1) + \frac{\sqrt{\ell^*}}{n(1-\phi_n)^{1/2}}\left(O_P\left(\frac{\sqrt{\ell^*}}{(1-\phi_n)^{1/2}}\right)\right)$$
$$= o_P(1) + O_P\left(\frac{\ell^*}{n(1-\phi_n)}\right).$$

  The latter estimate gives us the condition $\ell^* = o(\gamma_n)$.
- The second case is

$$\frac{\frac{\sqrt{\ell^*}}{(1-\phi_n)^{1/2}}}{\frac{1}{1-\phi_n}} \to 0 \Leftrightarrow \ell^*(1-\phi_n) \to 0,$$

  so

$$\frac{\left|\widehat{\phi}_n - \phi_n\right|}{1-\phi_n} = o_P(1) + \frac{\sqrt{\ell^*}}{n(1-\phi_n)^{1/2}}\left(O_P\left(\frac{1}{(1-\phi_n)}\right)\right)$$
$$= o_P(1) + O_P\left(\frac{\sqrt{\ell^*}}{n(1-\phi_n)^{3/2}}\right).$$

Thus we obtain $\ell^* = o(\gamma_n^3 n^{-1})$.

□

Next lemma allows us to estimate $T_{\alpha,n}(\tau_{n,0},\dots,\tau_{n,n-1})$ for the both type models.

Lemma 6.2.9. *Let $\tau_{n,k}$ is defined by* (5.3)*, then with constant $c = 5$ it holds*

$$T_{\alpha,n}(\tau_{n,0},\dots,\tau_{n,n-1}) \le \frac{5\,|a_n|}{1-\phi_n}\ell^{*(1-\alpha)}. \tag{6.31}$$

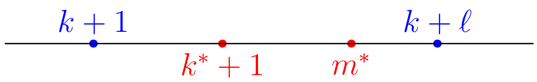
Proof. We shall use

$$\sum_{j=1}^{n} \tau_{n,k-1} = \frac{a_n}{1-\phi_n}\left(\ell^* - \phi_n^{n-m^*}\frac{1-\phi_n^{\ell^*}}{1-\phi_n}\right). \tag{6.32}$$

To prove (6.31) we have to consider all the possible configurations of the sets $\{k+1,\dots,k+\ell\}$ and $\{k^*+1,\dots,k^*+\ell^*\}$. There are six configurations $I_1,\dots,I_6$. Denote for $v = 1,\dots,6$

$$T_{\alpha,n}^{(v)} = \max_{k,\ell\in I_v} \ell^{-\alpha}\left|\sum_{j=k+1}^{k+\ell}\tau_{n,j-1} - \frac{\ell}{n}\sum_{j=1}^{n}\tau_{n,j-1}\right|.$$

First consider configuration $I_1 := \{k,\ell : [k^*+1, m^*] \subset [k+1, k+\ell]\}$



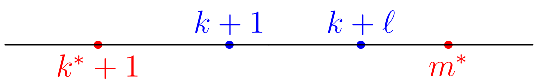
We easily obtain

$$\begin{aligned}
\sum_{j=k+1}^{k+\ell}\tau_{n,j-1} &= a_n\left[\sum_{j=k^*+1}^{m^*}\sum_{i=0}^{j-k^*-2}\phi_n^i + \phi_n^{-1}\sum_{j=m^*+1}^{k+\ell}\phi_n^j\sum_{i=k^*+1}^{m^*}\phi_n^{-i}\right]\\
&= a_n\left[\sum_{j=k^*+1}^{m^*}\frac{1-\phi_n^{j-k^*-1}}{1-\phi_n} + \phi_n^{-1}\frac{\phi_n^{m*+1}-\phi_n^{k+\ell+1}}{1-\phi_n}\,\frac{\phi_n^{-k*-1}-\phi_n^{-m*-1}}{1-\phi_n^{-1}}\right]\\
&= \frac{a_n}{1-\phi_n}\left[\ell^* - \frac{1-\phi_n^{\ell*}}{1-\phi_n} + \frac{1-\phi_n^{\ell*}-\phi_n^{k+\ell-m*}+\phi_n^{k+\ell-k*}}{1-\phi_n}\right]\\
&= \frac{a_n}{1-\phi_n}\left[\ell^* - \frac{1-\phi_n^{\ell*}}{1-\phi_n} + \frac{(1-\phi_n^{\ell*})(1-\phi_n^{k+\ell-m*})}{1-\phi_n}\right]\\
&= \frac{a_n}{1-\phi_n}\left[\ell^* - \phi_n^{k+\ell-m*}\frac{1-\phi_n^{\ell*}}{1-\phi_n}\right].
\end{aligned}$$

Together with (6.32) we find

$$T_{\alpha,n}^{(1)} = \frac{|a_n|}{1-\phi_n} \max_{k,\ell \in I_1} \ell^{-\alpha} \left| \ell^*(1-\ell/n) - \frac{1-\phi_n^{\ell*}}{1-\phi_n}(\phi_n^{k+\ell-m*} - (\ell/n)\phi_n^{n-m^*}) \right|$$
$$\leq \frac{3\,|a_n|}{1-\phi_n} \ell^{*(1-\alpha)}.$$

Now let us turn to second configuration $I_2 := \{k, \ell : [k+1, k+\ell] \subset [k^*+1, m^*]\}$



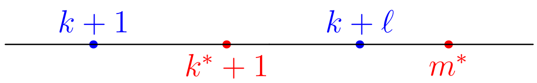
Obviously

$$\sum_{j=k+1}^{k+\ell} \tau_{n,j-1} = a_n \sum_{j=k+1}^{k+\ell} \sum_{i=1}^{j-1} \phi_n^{j-1-i} \mathbf{1}_{\mathbb{I}_n^*}(i) = a_n \sum_{j=k+1}^{k+\ell} \sum_{i=k^*+1}^{j-1} \phi_n^{j-1-i}$$
$$= a_n \sum_{j=k+1}^{k+\ell} \sum_{i=0}^{j-k^*-2} \phi_n^{i} = \frac{a_n}{1-\phi_n} \sum_{j=k+1}^{k+\ell} (1-\phi_n^{j-k^*-1})$$
$$= \frac{a_n}{1-\phi_n} \left(\ell - \frac{\phi_n^{k-k^*}(1-\phi_n^{\ell})}{1-\phi_n} \right),$$

so

$$T_{\alpha,n}^{(2)} = \frac{|a_n|}{1-\phi_n} \max_{k,\ell \in I_2} \ell^{-\alpha} \left| \ell - \frac{\phi_n^{k-k^*}(1-\phi_n^{\ell})}{1-\phi_n} - \frac{\ell}{n} \left(\ell^* - \phi_n^{n-m^*} \frac{1-\phi_n^{\ell^*}}{1-\phi_n} \right) \right|$$
$$\leq \frac{4\,|a_n|}{1-\phi_n} \ell^{*(1-\alpha)}.$$

If we consider the third configuration $I_3 := \{k, \ell : k+1 < k^*+1 \leq k+\ell < m^*\}$



we have

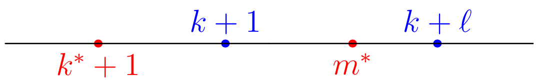
$$\sum_{j=k+1}^{k+\ell} \tau_{n,j-1} = a_n \sum_{j=k+1}^{k+\ell} \sum_{i=1}^{j-1} \phi_n^{j-1-i} \mathbf{1}_{\mathbb{I}_n^*}(i) = a_n \sum_{j=k^*+1}^{k+\ell} \sum_{i=k^*+1}^{j-1} \phi_n^{j-1-i}$$

$$= a_n \sum_{j=k^*+1}^{k+\ell} \sum_{i=0}^{j-k^*-2} \phi_n^i = \frac{a_n}{1-\phi_n} \sum_{j=k^*+1}^{k+\ell} (1-\phi_n^{j-k^*-1})$$
$$= \frac{a_n}{1-\phi_n} \left((k+\ell-k^*) - \frac{1-\phi_n^{k+\ell-k^*}}{1-\phi_n} \right).$$

Since $k+\ell-k^* \le \ell^*$, then it is easy to see, that

$$T_{\alpha,n}^{(3)} = \frac{|a_n|}{1-\phi_n} \max_{k,\ell \in I_3} \ell^{-\alpha} \left| (k+\ell-k^*) - \frac{1-\phi_n^{k+\ell-k^*}}{1-\phi_n} - \frac{\ell}{n} \left(\ell^* - \phi_n^{n-m^*} \frac{1-\phi_n^{\ell^*}}{1-\phi_n} \right) \right|$$
$$\le \frac{4\,|a_n|}{1-\phi_n} \ell^{*(1-\alpha)}.$$

Next, fourth configuration is $I_4 := \{k, \ell : k^*+1 < k+1 \le m^* < k+\ell\}$



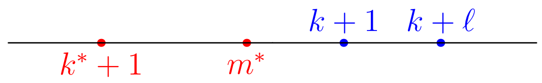
Now

$$\sum_{j=k+1}^{k+\ell} \tau_{n,j-1} = a_n \left[\sum_{j=k+1}^{m^*} \sum_{i=k^*+1}^{j-1} \phi_n^{j-1-i} + \sum_{j=m^*+1}^{k+\ell} \sum_{i=k^*+1}^{m^*} \phi_n^{j-1-i} \right]$$
$$= a_n \left[\sum_{j=k+1}^{m^*} \sum_{i=0}^{j-k^*-2} \phi_n^i + \phi_n^{-1} \sum_{j=m^*+1}^{k+\ell} \phi_n^j \sum_{i=k^*+1}^{m^*} \phi_n^{-i} \right]$$
$$= a_n \left[\sum_{j=k+1}^{m^*} \frac{1-\phi_n^{j-k^*-1}}{1-\phi_n} + \phi_n^{-1} \frac{\phi_n^{m^*+1} - \phi_n^{k+\ell+1}}{1-\phi_n} \frac{\phi_n^{-k^*-1} - \phi_n^{-m^*-1}}{1-\phi_n^{-1}} \right]$$
$$= \frac{a_n}{1-\phi_n} \left[(m^*-k) - \phi_n^{k-k^*} \frac{1-\phi_n^{m^*-k}}{1-\phi_n} + (1-\phi_n^{k+\ell-m^*}) \frac{1-\phi_n^{\ell^*}}{1-\phi_n} \right]$$

together with (6.32) and $m^*-k \le \ell^*$ gives the estimate

$$T_{\alpha,n}^{(4)} = \frac{|a_n|}{1-\phi_n} \max_{k,\ell \in I_4} \ell^{-\alpha} \Bigg| (m^*-k) - \phi_n^{k-k^*} \frac{1-\phi_n^{m^*-k}}{1-\phi_n} + (1-\phi_n^{k+\ell-m^*}) \frac{1-\phi_n^{\ell^*}}{1-\phi_n}$$
$$- \frac{\ell}{n} \left(\ell^* - \phi_n^{n-m^*} \frac{1-\phi_n^{\ell^*}}{1-\phi_n} \right) \Bigg| \le \frac{5\,|a_n|}{1-\phi_n} \ell^{*(1-\alpha)}.$$

From the fifth configuration $I_5 := \{k, \ell : m^* < k+1 < k+\ell\}$



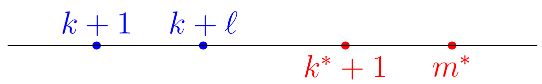
we get

$$\sum_{j=k+1}^{k+\ell}\tau_{n,j-1} = a_n\sum_{j=k+1}^{k+\ell}\sum_{i=k^*+1}^{m^*}\phi_n^{j-1-i} = a_n\phi_n^{-1}\sum_{j=k+1}^{k+\ell}\phi_n^j\sum_{i=k^*+1}^{m^*}\phi_n^{-i}$$
$$= a_n\phi_n^{-1}\cdot\frac{\phi_n^{k+1}-\phi_n^{k+\ell+1}}{1-\phi_n}\cdot\frac{\phi_n^{-k^*-1}-\phi_n^{-m^*-1}}{1-\phi_n^{-1}}$$
$$= \frac{a_n}{1-\phi_n}\cdot\phi_n^{k-m^*}\frac{(1-\phi_n^{\ell})(1-\phi_n^{\ell^*})}{1-\phi_n}$$

and together with (6.32) the estimate is

$$T_{\alpha,n}^{(5)} = \frac{|a_n|}{1-\phi_n}\max_{k,\ell\in I_5}\ell^{-\alpha}\left|\phi_n^{k-m^*}\frac{(1-\phi_n^{\ell})(1-\phi_n^{\ell^*})}{1-\phi_n} - \frac{\ell}{n}\left(\ell^* - \phi_n^{n-m^*}\frac{1-\phi_n^{\ell^*}}{1-\phi_n}\right)\right|$$
$$\leq \frac{3\,|a_n|}{1-\phi_n}\ell^{*(1-\alpha)}.$$

Finally sixth configuration $I_6 := \{k,\ell : k+1 < k+\ell \leq k^*\}$



gives us

$$\sum_{j=k+1}^{k+\ell}\tau_{n,j-1} = 0.$$

Thus

$$T_{\alpha,n}^{(6)} = \frac{|a_n|}{1-\phi_n}\left|\ell^* - \phi_n^{n-m^*}\frac{1-\phi_n^{\ell^*}}{1-\phi_n}\right|\max_{k,\ell\in I_6}\ell^{-\alpha}\frac{\ell}{n} \leq \frac{2\,|a_n|}{1-\phi_n}\ell^{*(1-\alpha)}.$$

So collecting all the estimates of $T_{\alpha,n}^{(v)}$, $v = 1,\ldots,6$ we obtain (6.31). □

Finally we give proofs of the Theorems 6.2.1 and 6.2.2.

Proof of Theorem 6.2.1. Since (ε_k)'s are i.i.d. centered random variables, that satisfies condition (6.5) and since consequently the partial sums polygonal line built on ε_k's satisfies the Hölderian invariance principle, then

$$n^{-1/2+\alpha} T_{\alpha,n}(\varepsilon_1, \dots, \varepsilon_n) = O_P(1). \tag{6.33}$$

For more details see Račkauskas and Suquet [2004b] (also see Theorem 3.3.6 and Corollary 3.3.7 on page 28). Besides we have

$$n^{-1/2+\alpha} T_{\alpha,n}(z_{n,0}, \dots, z_{n,n-1}) = O_P(1/(1-\phi_n)), \tag{6.34}$$

since by Theorem 5.1.4 (page 69) in previous section,

$$n^{-1/2+\alpha}(1-\phi_n) T_{\alpha,n}(z_{n,0}, \dots, z_{n,n-1}) = O_P(1)$$

when condition (6.22) holds. Taking into account (6.27), (6.33) and (6.34), we obtain the lower bound of test statistics

$$\begin{aligned} \widehat{T}_{\alpha,n} \geq\ & T_{\alpha,n}(a_{n,1}, \dots, a_{n,n}) - \left|\widehat{\phi}_n - \phi_n\right| T_{\alpha,n}(\tau_{n,0}, \dots, \tau_{n,n-1}) \\ & - O_p(n^{1/2-\alpha}) \left(1 + \frac{\left|\widehat{\phi}_n - \phi_n\right|}{1-\phi_n}\right). \end{aligned}$$

Further (6.28) gives

$$n^{-1/2+\alpha} \widehat{T}_{\alpha,n} \geq n^{-1/2+\alpha} \ell^{*(1-\alpha)} - \Delta_n,$$

where

$$\Delta_n = n^{-1/2+\alpha} \left|\widehat{\phi}_n - \phi_n\right| T_{\alpha,n}(\tau_{n,0}, \dots, \tau_{n,n-1}) + O_p(1) \left(1 + \frac{\left|\widehat{\phi}_n - \phi_n\right|}{1-\phi_n}\right).$$

Thus to get the condition of consistency we have to find the condition under which

$$\Delta_n = o_P(n^{-1/2+\alpha} \ell^{*(1-\alpha)}) \tag{6.35}$$

when

$$n^{-1/2+\alpha} \ell^{*(1-\alpha)} \xrightarrow[n\to\infty]{} \infty.$$

Next from the estimate (6.31) we obtain

$$n^{-1/2+\alpha}\ell^{*(1-\alpha)}\frac{\left|\widehat{\phi}_n-\phi_n\right|}{1-\phi_n}=o_P(n^{-1/2+\alpha}\ell^{*(1-\alpha)}),$$

thus Lemma 6.2.9 gives that condition (6.35) is satisfied when

$$\frac{\left|\widehat{\phi}_n-\phi_n\right|}{1-\phi_n}=o_p(1).$$

Finally, Corollary 6.2.8 says that the latter equality holds for the second type model provided that γ_n is increasing in n or regular varying sequence and

- $\ell^*=o(\gamma_n)$ if $\ell^*(1-\phi_n)\to\infty$, as $n\to\infty$;
- $\ell^*=o(\gamma_n^3 n^{-1})$ if $\ell^*(1-\phi_n)\to 0$, as $n\to\infty$.

□

Proof of Theorem 6.2.2. By Mikosch and Račkauskas [2010] (see Theorem 3.3.8 and Corollary 3.3.9 on page 29) we have that

$$b_n^{-1}T_{\alpha,n}(\varepsilon_1,\ldots,\varepsilon_n)=O_P(1). \tag{6.36}$$

Further from (6.40), page 117, we have that for the second type model

$$b_n^{-1}\gamma_n^{-1/2}T_{\alpha,n}(z_{n,0},\ldots,z_{n,n-1})=o_P(1)$$

holds if $\gamma_n=O(n^{q(\alpha-\alpha_p)})$ with some $0<q<2$. Taking into account (6.27), (6.36) and (6.40), we obtain the lower bound of test statistics $\widehat{T}_{\alpha,n}$

$$\begin{aligned}\widehat{T}_{\alpha,n}\geq T_{\alpha,n}(a_{n,1},\ldots,a_{n,n})&-\left|\widehat{\phi}_n-\phi_n\right|T_{\alpha,n}(\tau_{n,0},\ldots,\tau_{n,n-1})\\&-O_p(b_n)-\left|\widehat{\phi}_n-\phi_n\right|o_P(b_n\gamma_n^{1/2}).\end{aligned}$$

Further (6.28) gives

$$b_n^{-1}\widehat{T}_{\alpha,n}\geq b_n^{-1}\ell^{*(1-\alpha)}-\Delta_n,$$

where

$$\Delta_n=\left|\widehat{\phi}_n-\phi_n\right|T_{\alpha,n}(\tau_{n,0},\ldots,\tau_{n,n-1})-O_p(b_n)-\left|\widehat{\phi}_n-\phi_n\right|o_P(b_n\gamma_n^{1/2}).$$

Thus to get the condition of consistency we have to find the condition under which

$$\Delta_n=o_P(b_n^{-1}\ell^{*(1-\alpha)}) \tag{6.37}$$

when

$$b_n^{-1}\ell^{*(1-\alpha)} \xrightarrow[n\to\infty]{} \infty.$$

Next from the estimate (6.31) we obtain

$$b_n^{-1}\ell^{*(1-\alpha)}\frac{\left|\widehat{\phi}_n - \phi_n\right|}{1-\phi_n} = o_P(b_n^{-1}\ell^{*(1-\alpha)}),$$

thus Lemma 6.2.9 gives that condition (6.36) is satisfied when

$$\frac{\left|\widehat{\phi}_n - \phi_n\right|}{1-\phi_n} = o_p(1).$$

Finally, Corollary 6.2.8 says that the latter equality holds for the second type model provided that γ_n is increasing in n or regular varying sequence and

– $\ell^* = o(\gamma_n)$ if $\ell^*(1-\phi_n) \to \infty$, as $n \to \infty$;
– $\ell^* = o(\gamma_n^3 n^{-1})$ if $\ell^*(1-\phi_n) \to 0$, as $n \to \infty$.

□

Remark 6.2.10. We investigate the compatibility of the conditions obtained in Corollary 6.2.8 with the test statistics consistency condition obtained in Theorem 6.2.1. Put $\ell^* \asymp n^a$ and $\gamma_n \asymp n^b$. We draw the detection region in figures 6.1, 6.2 and 6.3. The two cases are considered:

– *Case* $\ell^*(1-\phi_n) \to \infty$. Then we obtain a set of parameters (a, b) by inequalities

$$\begin{cases} a + b > 1 \\ a < b \end{cases}$$

– *Case* $\ell^*(1-\phi_n) \to 0$. Evidently the set (a, b) that satisfies conditions is

$$\begin{cases} a + b < 1 \\ a < 3b - 1 \end{cases}$$

For a given value p in condition (3.11), page 23, in both cases the West border of the detection region is given as an arc of hyperbola with parametric representation $a = (1 - 2\alpha_p t)/(2 - 2\alpha_p t)$, $b = t$ where $t = \alpha/\alpha_p$ and $\alpha_p = 1/2 - 1/p$. The light grey area in figures 6.1, 6.2 and 6.3 corresponds to case $\ell^*(1-\phi_n) \to \infty$, while the dark grey area corresponds to case $\ell^*(1-\phi_n) \to 0$.

In the figure 6.1 one can see the detection region of the test statistics $\widehat{T}_{\alpha,n}$. To compare this detection area with the one in the figure 5.1, we see that it is smaller than for the statistics $\widetilde{T}_{\alpha,n}$. Figure 6.2 shows the detection region with $p = 3$. One can see, that this region is smaller than in figure 6.1 ($p = 8$), while in figure 6.3 the detection region is much bigger ($p = 30$). Thus, from this we may conclude, that as p tend to infinity (α_p tend to 1/2), we can detect shorter epidemics and we have more freedom in choosing the divergence rate of γ_n.

Remark 6.2.11. We also study the compatibility of the conditions obtained in Corollary 6.2.8 with the test statistics consistency condition obtained in Theorem 6.2.2. Put $\ell^* \asymp n^a$, $\gamma_n \asymp n^b$, and $b_n \asymp n^{1/p}$. We draw the detection region considering two cases:

– *Case* $\ell^*(1 - \phi_n) \to \infty$. The possible choice of the parameters (a, b) is given by inequalities:

$$\begin{cases} a + b > 1 \\ a < b \end{cases}$$

– *Case* $\ell^*(1 - \phi_n) \to 0$. For this case possible choice of parameters (a, b) is

$$\begin{cases} a + b < 1 \\ a < 3b - 1 \end{cases}$$

For a given value p in condition (3.11), page 23, in both cases the North border of the detection region is given as a parametric curve $a = (1)/(p(1/2 - t + 1/p))$, $b = qt$ where $t = \alpha - \alpha_p$ and $\alpha_p = 1/2 - 1/p$. The light grey area in figures 6.4, 6.5 and 6.6 corresponds to case $\ell^*(1-\phi_n) \to \infty$, while the dark grey area corresponds to case $\ell^*(1 - \phi_n) \to 0$.

The points marked in the figures are:

$$\kappa_a^{(1)} = 3 \cdot \frac{3pq - \sqrt{(3pq + 2p + 6q)^2 + 24p(-pq - 4q)} + 2p + 6q}{12p} - 1$$

$$\kappa_b^{(1)} = \frac{3pq - \sqrt{(3pq + 2p + 6q)^2 + 24p(-pq - 4q)} + 2p + 6q}{12p}$$

$$\kappa^{(2)} = \frac{\sqrt{q} \cdot \sqrt{(p^2q + 4pq - 16p + 4q)} + pq + 2q}{4p}$$

$$\kappa_a^{(3)} = 1 - \frac{-\sqrt{(pq + 2p + 2q)^2 - 8p^2q} + pq + 2p + 2q}{4p}$$

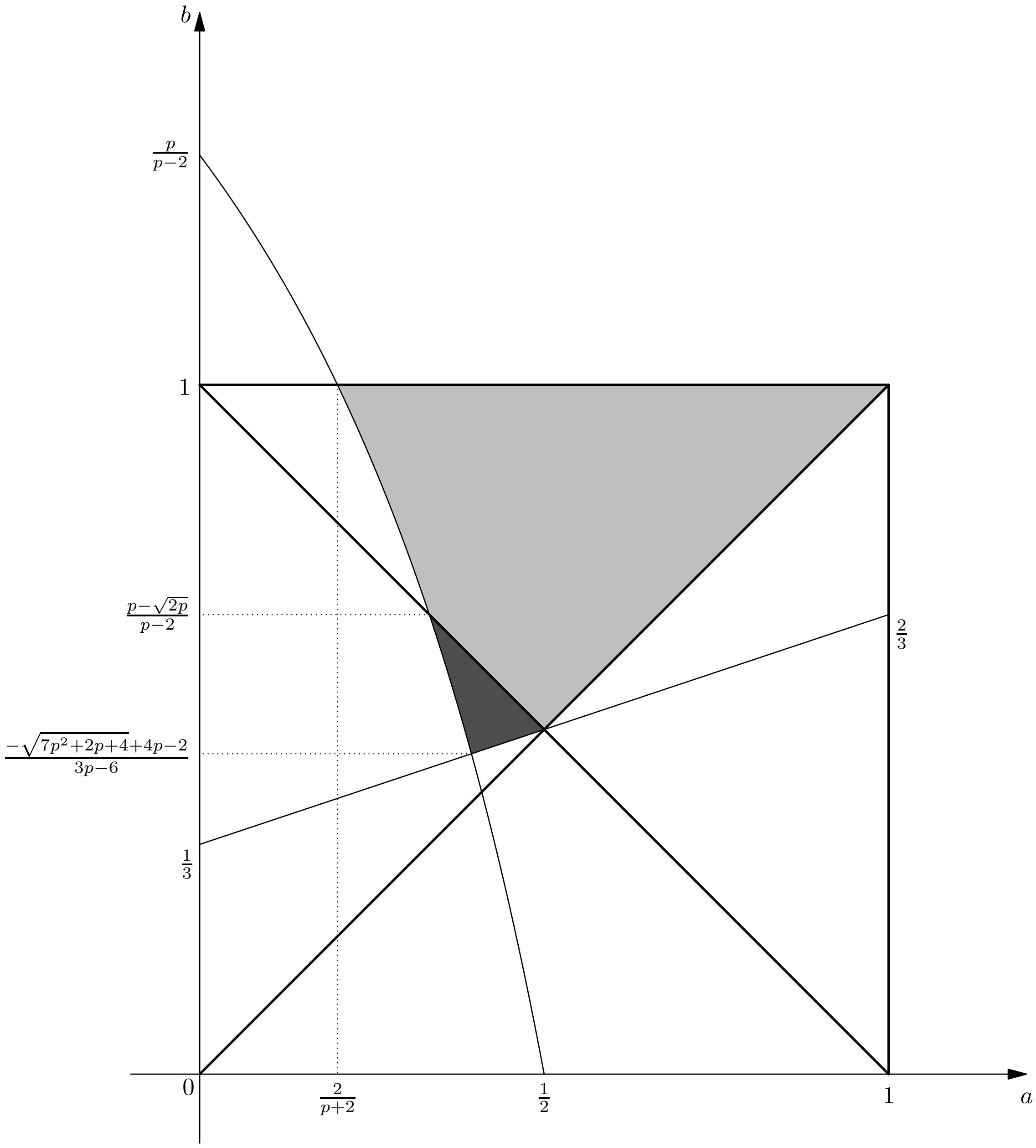


Figure 6.1: Detection areas in the space of parameters ($\ell^* \asymp n^a, \gamma_n \asymp n^b$) for Theorem 6.2.1 with $p = 8$ and $\alpha < \alpha_p$.
In light grey the case where $\ell^*(1 - \phi_n) \to \infty$.
In dark grey the case where $\ell^*(1 - \phi_n) \to 0$.

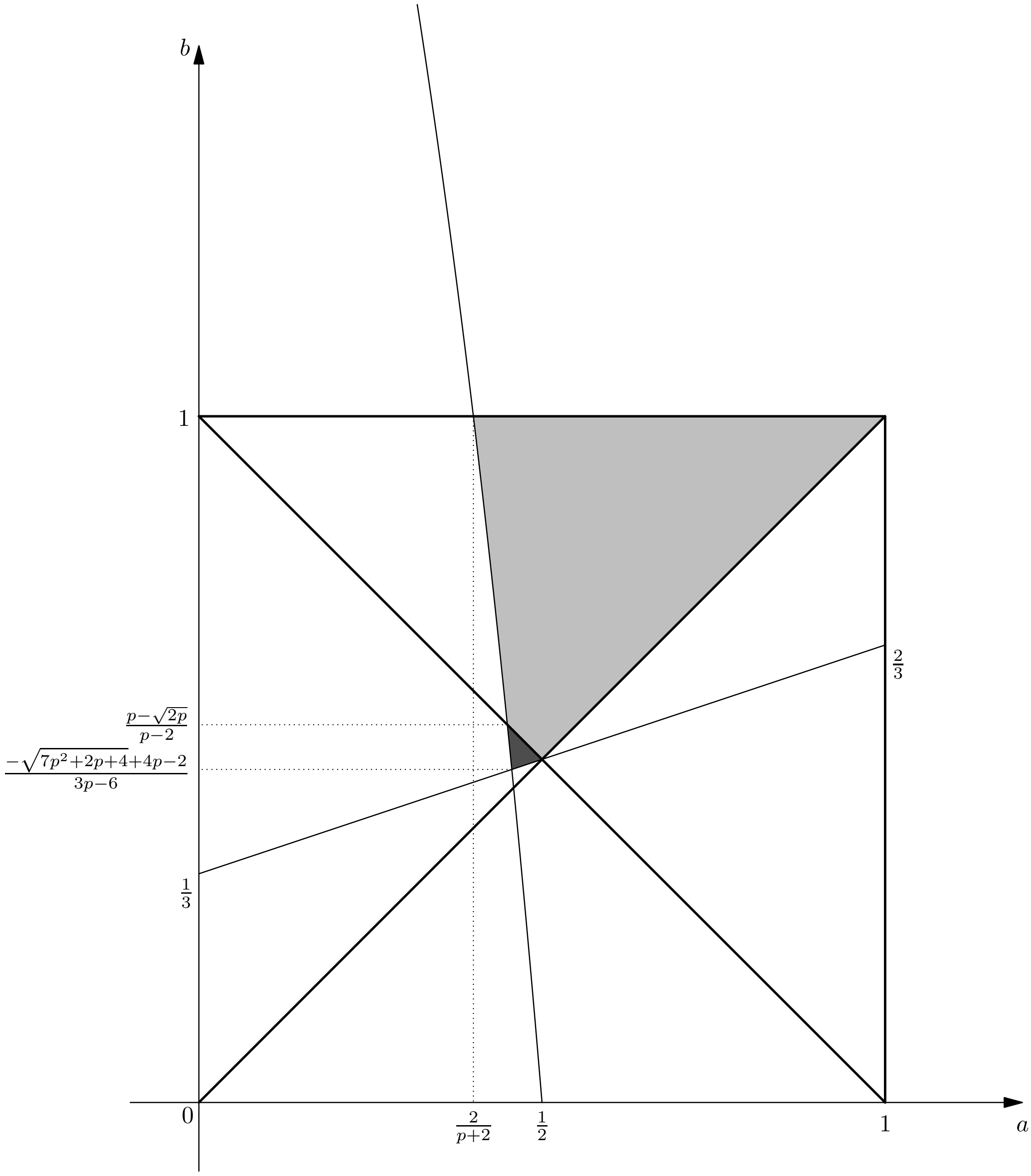


Figure 6.2: Detection areas in the space of parameters ($\ell^* \asymp n^a, \gamma_n \asymp n^b$) for Theorem 6.2.1 with $p = 3$ and $\alpha < \alpha_p$.
In light grey the case where $\ell^*(1-\phi_n) \to \infty$.
In dark grey the case where $\ell^*(1-\phi_n) \to 0$.

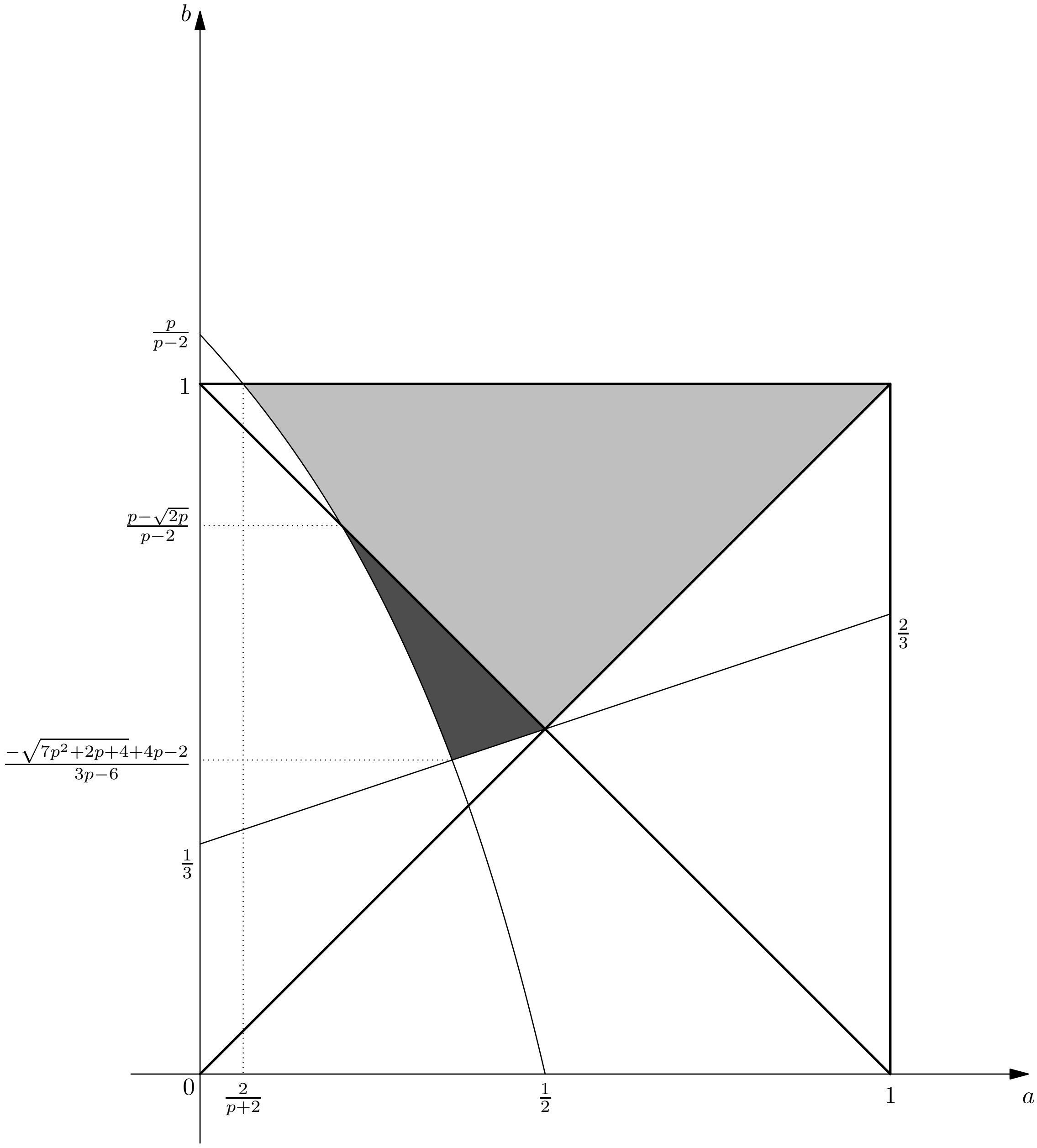


Figure 6.3: Detection areas in the space of parameters ($\ell^* \asymp n^a, \gamma_n \asymp n^b$) for Theorem 6.2.1 with $p = 30$ and $\alpha < \alpha_p$.
In light grey the case where $\ell^*(1 - \phi_n) \to \infty$.
In dark grey the case where $\ell^*(1 - \phi_n) \to 0$.

$$\kappa_b^{(3)} = \frac{-\sqrt{(pq + 2p + 2q)^2 - 8p^2 q} + pq + 2p + 2q}{4p}.$$

In the figure 6.4 one can see the detection region of the test statistics $\widehat{T}_{\alpha,n}$. To compare this detection area with the one in the figure 5.1 and 6.1, we see that it is smaller, but partially it covers different area. Figure 6.5 shows the detection region with $p = 12$. This region is bigger than in figure 6.4 ($p = 8$), while in figure 6.6 the detection region is even bigger ($p = 30$). Thus, from this we may conclude, that as p tend to infinity (α_p tends to 1/2), we can detect shorter epidemics.

6.3 Test power analysis

In this section we perform the test power analysis. Though the methodology we have used for consistency analysis have not worked for the first type model, but we perform power analysis for both type models and using numerical methods we see if this test statistics can detect epidemic change. The results are presented in the tables 6.7 and 6.8. As in the previous section we compute empirical power on size-adjusted (not nominal size) basis, i.e., replaced the nominal value of significance level by the value of empirical distribution function for p-values under null hypothesis.

Here we compute $N = 1000$ realizations of test statistics with the sample size n for different values of parameters γ, γ_n, α, k^*, ℓ^* and a_n. Innovations are generated as standard normally distributed random variables. For the limit distribution we compute $N = 5000$ realizations of test statistics with the sample size $n = 5000$. We approximate the values of the standard Wiener process by

$$W\left(\frac{k}{5000}\right) = 5000^{-1/2} \sum_{j=1}^{k} \varepsilon(j), \quad k = 1, \ldots, 5000,$$

where $\varepsilon(j)$ are generated as standard normally distributed random variables. The Ornstein-Uhlenbeck process have been approximated by the the following discretization

$$S(j) = S(j-1)\mathrm{e}^{\gamma/n} + \sqrt{\frac{1 - \mathrm{e}^{2\gamma/n}}{-2\gamma}} \cdot \varepsilon(j), \quad \varepsilon(j) \sim \mathfrak{N}(0,1). \tag{6.38}$$

Using values generated by (6.38), we approximate the integrated Ornstein-Uhlenbeck

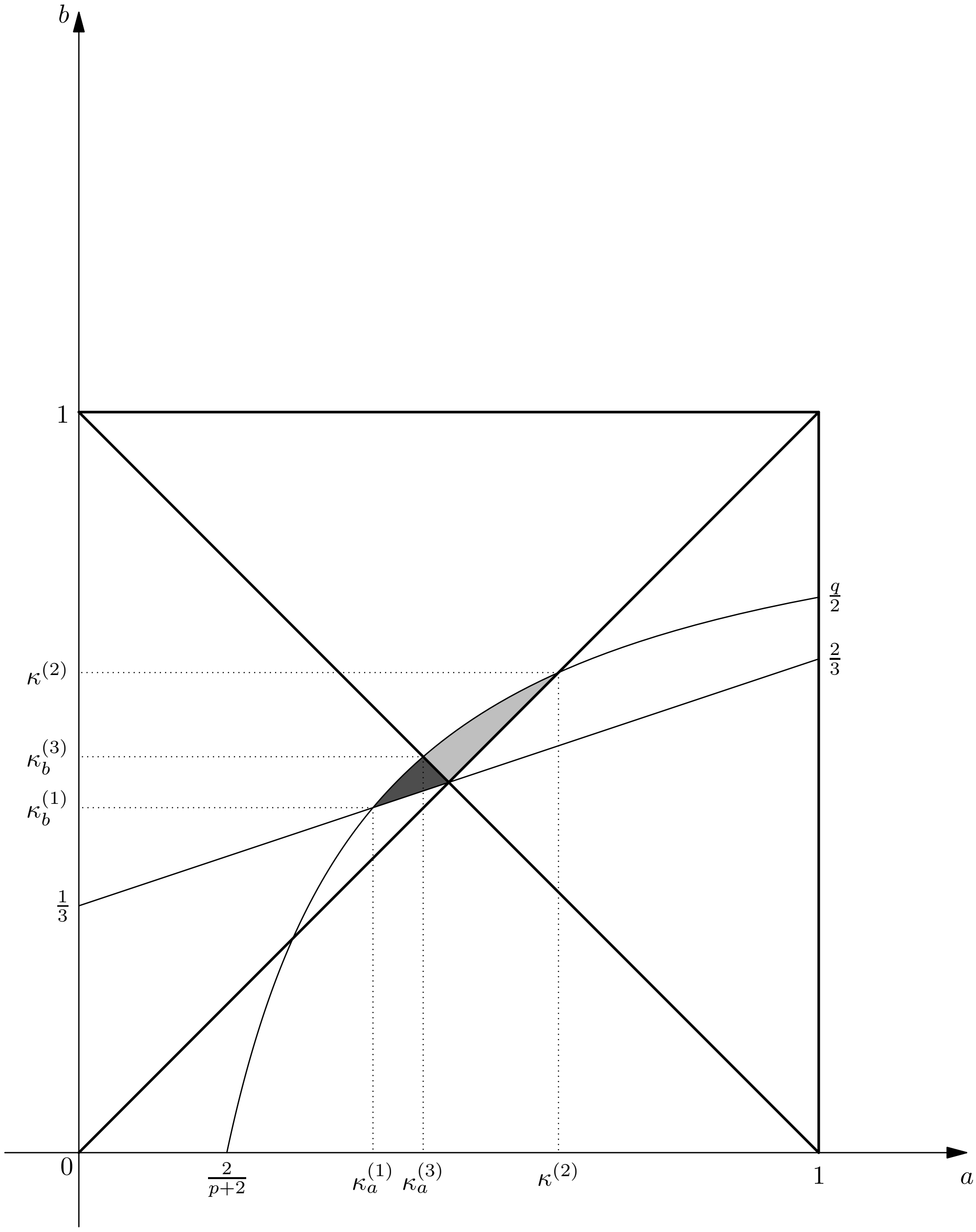


Figure 6.4: Detection areas in the space of parameters ($\ell^* \asymp n^a, \gamma_n \asymp n^b$) for Theorem 6.2.1 with $p = 8$, $q = 1.5$ and $\alpha > \alpha_p$.
In light grey the case where $\ell^*(1 - \phi_n) \to \infty$.
In dark grey the case where $\ell^*(1 - \phi_n) \to 0$.

Figure 6.5: Detection areas in the space of parameters ($\ell^* \asymp n^a, \gamma_n \asymp n^b$) for Theorem 6.2.1 with $p = 8$, $q = 1.5$ and $\alpha > \alpha_p$.
In light grey the case where $\ell^*(1 - \phi_n) \to \infty$.
In dark grey the case where $\ell^*(1 - \phi_n) \to 0$.

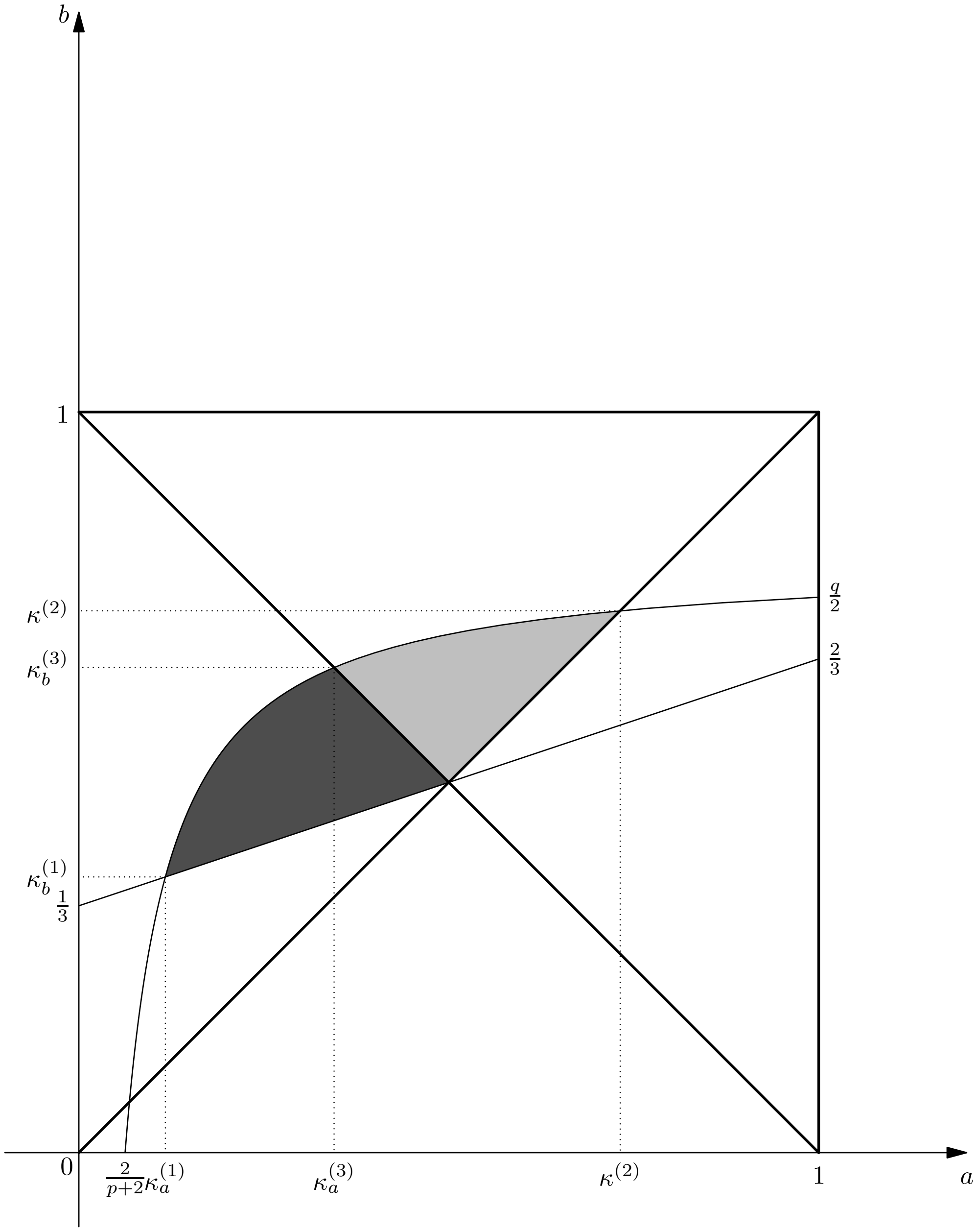


Figure 6.6: Detection areas in the space of parameters ($\ell^* \asymp n^a, \gamma_n \asymp n^b$) for Theorem 6.2.1 with $p = 8$, $q = 1.5$ and $\alpha > \alpha_p$.
In light grey the case where $\ell^*(1 - \phi_n) \to \infty$.
In dark grey the case where $\ell^*(1 - \phi_n) \to 0$.

process by

$$J\left(\frac{k}{5000}\right) = 5000^{-1}\sum_{j=1}^{k} S(j), \quad k = 1, \dots, 5000,$$

and values

$$A = 5000^{-1}\sum_{j=1}^{n} S^2(j), \quad B = \sum_{j=1}^{n} S(j)\left(W\left(\frac{j}{5000}\right) - W\left(\frac{j-1}{5000}\right)\right).$$

For the first type model ($\phi_n = \mathrm{e}^{\gamma/n}$) with innovations that satisfy integrability condition (3.11), page 23, the basic parameters are

$$\gamma = -2; \quad a_n = 1; \quad n = 1000; \quad \frac{\ell^*}{n} = 0.05; \quad \frac{k^*}{n} = 0.4, \quad y_{n,0} = 0.$$

We modify them separately and we compute the empirical size-power. We keep all the parameters fixed except one (indicated in the first column in both tables) which is allowed to vary. We use the same methodology for computing empirical p-values as in the previous chapter.

As one can see in the table 6.7 the test power increases with the α. Test statistics has a quite big power in detecting short epidemics with α closer to 1/2. Naturally, increasing n increases test power. In general, test has a quite big power for all chosen parameters.

For the second type model ($\phi_n = 1 - \gamma_n/n$) with innovations that satisfy integrability condition (3.11), the basic parameter set are

$$\gamma_n = n^{3/4}; \quad a_n = 1; \quad n = 1000; \quad \frac{\ell^*}{n} = 0.05; \quad \frac{k^*}{n} = 0.4, \quad y_{n,0} = 0.$$

For the second type model (table 6.8), the test power is very low for the small α. The test power increases with n, ℓ^* and the rate of divergence of γ_n.

Further we give the test power analysis for the model with regularly varying innovations. For this we generate innovations as symmetric Pareto random variables. Note, that symmetric Pareto probability density function for some constant $p > 0$ is

$$f_P(x) = \begin{cases} \frac{p}{2}\,|x|^{-(p+1)}, & \text{if} \quad |x| > 1 \\ 0, & \text{if} \quad |x| \le 1 \end{cases}$$

Parameters	$\alpha = 2/32$	$\alpha = 6/32$	$\alpha = 12.5/32$
$\ell^*/n = 0.035$	0.462	0.715	0.968
$\ell^*/n = 0.050$	0.879	0.981	0.998
$\ell^*/n = 0.065$	0.988	1.000	1.000
$k^*/n = 0.2$	0.903	0.981	1.000
$k^*/n = 0.4$	0.879	0.981	0.998
$k^*/n = 0.8$	0.784	0.967	0.997
$a_n = 0.8$	0.574	0.793	0.957
$a_n = 1$	0.879	0.981	0.998
$a_n = 1.2$	0.989	1.000	1.000
$n = 500$	0.498	0.700	0.884
$n = 1000$	0.879	0.981	0.998
$n = 2000$	1.000	1.000	1.000
$\gamma = -2$	0.879	0.981	0.998
$\gamma = -12$	0.831	0.976	0.998
$\gamma = -100$	0.010	0.267	0.975

Table 6.7: Empirical power at the size-adjusted significance level 0.05 for the first type model with Gaussian innovations.

and cumulative distribution function

$$F_P(x) = \begin{cases} \frac{1}{2}(-x)^{-p}, & \text{if} \quad x < -1 \\ \frac{1}{2}, & \text{if} \quad -1 \leq x \leq 1 \\ 1 - \frac{1}{2}x^{-p}, & \text{if} \quad x \geq 1. \end{cases}$$

Next, tables 6.9 and 6.10 shows the results of empirical size-adjusted test power for the first type model with regularly varying innovations. Thus we generate innovations as Pareto random variables with parameter p. The symmetric Pareto CDF gives that $b_n = n^{1/p}$. For the first type model, we use basic parameters:

$$\gamma = -2; \quad a_n = 1; \quad n = 1000; \quad \frac{\ell^*}{n} = 0.05; \quad \frac{k^*}{n} = 0.4, \quad y_{n,0} = 0.$$

Table 6.9 correspond to the Theorem 6.1.3 part (a), so we choose the values $\alpha = 17/32, 20/32, 26/32$ and $p = 8$. We see in this table that in general test power increases with the length of epidemics ℓ^*, epidemic change size a_n and number of observations n. Also, we see that test power increases when α and γ values decreases. Further, there is no difference for the test power if the epidemics

Parameters	$\alpha = 2/32$	$\alpha = 6/32$	$\alpha = 10/32$
$\ell^*/n = 0.035$	0.049	0.190	0.763
$\ell^*/n = 0.050$	0.093	0.573	0.965
$\ell^*/n = 0.065$	0.216	0.880	0.998
$k^*/n = 0.2$	0.077	0.589	0.974
$k^*/n = 0.4$	0.093	0.573	0.965
$k^*/n = 0.8$	0.105	0.615	0.974
$a_n = 0.8$	0.102	0.328	0.791
$a_n = 1$	0.093	0.573	0.965
$a_n = 1.2$	0.080	0.810	1.000
$n = 500$	0.062	0.171	0.552
$n = 1000$	0.093	0.573	0.965
$n = 2000$	0.660	0.997	1.000
$\gamma_n = n/\ln(n)$	0.035	0.416	0.950
$\gamma_n = \ln^{2.5}(n)$	0.020	0.353	0.935
$\gamma_n = n^{3/4}$	0.093	0.573	0.965

Table 6.8: Empirical power at the size-adjusted significance level 0.05 for the second type model with Gaussian innovations.

occur at the beginning, middle or end of the sample.

Table 6.10 correspond to the Theorem 6.1.3 part (b), so we choose the same α values as in Gaussian innovation case and $p = 20$ in order to compare the results. Thus basic parameters:

$$\gamma = -2; \quad a_n = 1; \quad n = 1000; \quad \frac{\ell^*}{n} = 0.05; \quad \frac{k^*}{n} = 0.4, \quad y_{n,0} = 0.$$

As we see in this table the test power increases with n, α, length of epidemics ℓ^*. Test can detect epidemics with bigger power at the beginning or middle of the sample. The bigger a_n, the bigger test power. To compare tables 6.10 and 6.7, we observe that in general test power is a little smaller for the model with regularly varying innovations (6.10).

For the second type model with regularly varying innovations, generated as Pareto random variables with parameters p and $b_n = n^{1/p}$, we use such basic parameters:

$$\gamma_n = n^{3/4}; \quad a_n = 1; \quad n = 1000; \quad \frac{\ell^*}{n} = 0.05; \quad \frac{k^*}{n} = 0.4, \quad y_{n,0} = 0.$$

Parameters	$\alpha = 17/32$	$\alpha = 20/32$	$\alpha = 26/32$
$\ell^*/n = 0.035$	0.934	0.802	0.064
$\ell^*/n = 0.050$	0.991	0.921	0.067
$\ell^*/n = 0.065$	0.998	0.975	0.069
$k^*/n = 0.2$	0.993	0.944	0.059
$k^*/n = 0.4$	0.991	0.921	0.067
$k^*/n = 0.8$	0.986	0.917	0.054
$a_n = 0.8$	0.893	0.662	0.058
$a_n = 1$	0.991	0.921	0.067
$a_n = 1.2$	1.000	0.994	0.092
$n = 500$	0.760	0.816	0.091
$n = 1000$	0.991	0.921	0.067
$n = 2000$	1.000	0.999	0.064
$\gamma = -2$	0.991	0.921	0.067
$\gamma = -12$	0.969	0.840	0.058
$\gamma = -20$	0.947	0.760	0.056

Table 6.9: Empirical power at the size-adjusted significance level 0.05 for the first type model with regular varying innovations, $\alpha > \alpha_p$.

The results size-adjusted empirical power we present in the tables 6.11 and 6.12. Table 6.11 correspond to the Theorem 6.1.5 part (a) and we choose $\alpha = 17/32$, $20/32$, $26/32$ and $p = 8$. We see, that test power increases when α decreases (i.e., α is close to $1/2$). Also test power increases with the number of observations n, length of epidemics ℓ^* and a_n.

Table 6.12 correspond to the Theorem 6.1.5 part (b), so we choose the same α values as in a case of Gaussian innovations, $p = 20$, normalization $n^{-1/2+\alpha}$. We see, that test has no power for the small α values, but it increases with α, n, ℓ^* and a_n. Comparing tables 6.12 and 6.8 we see, that generally test power is lower for the model with regularly varying innovations.

6.4 Supplementary results and notes

The Proposition 6.1.4 is the main tool in proving the Theorems 6.1.3 and 6.1.5 parts (a). The proof of Proposition 6.1.4 intensively exploits the following version of Hájek-Rényi inequality.

Parameters	$\alpha = 2/32$	$\alpha = 6/32$	$\alpha = 12.5/32$
$\ell^*/n = 0.035$	0.385	0.637	0.974
$\ell^*/n = 0.050$	0.790	0.965	0.998
$\ell^*/n = 0.065$	0.976	0.999	1.000
$k^*/n = 0.2$	0.768	0.973	0.999
$k^*/n = 0.4$	0.790	0.965	0.998
$k^*/n = 0.8$	0.679	0.942	0.995
$a_n = 0.8$	0.502	0.787	0.951
$a_n = 1$	0.790	0.965	0.998
$a_n = 1.2$	0.962	0.999	1.000
$n = 500$	0.476	0.621	0.876
$n = 1000$	0.790	0.965	0.998
$n = 2000$	1.000	1.000	1.000
$\gamma = -2$	0.790	0.965	0.998
$\gamma = -12$	0.793	0.972	0.995
$\gamma = -20$	0.562	0.930	0.990.

Table 6.10: Empirical power at the size-adjusted significance level 0.05 for the first type model with regular varying innovations and $\alpha < \alpha_p$.

Lemma 6.4.1. *For each $n \geq 1$ let $(X_{nk}, 1 \leq k \leq n)$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and $(a_{nk}, 1 \leq k \leq n)$ be a sequence of nonnegative real numbers and $r > 0$. If there exists $c > 0$ such that for any $n \geq 1$ and any $\epsilon > 0$*

$$\mathbb{P}\Big(\max_{k \leq n}\Big|\sum_{j=1}^{k} X_{nj}\Big| \geq \epsilon\Big) \leq c\epsilon^{-r}\sum_{k=1}^{n} a_{nk}$$

then there exists $c > 0$ such that for any $n \geq 1$ any sequence $(\beta_{nk}, 1 \leq k \leq n)$ such that $0 < \beta_{n1} \leq \cdots \leq \beta_{nn}$ and any $\epsilon > 0$ we have

$$\mathbb{P}\Big(\max_{k \leq n}\beta_{nk}^{-1}\Big|\sum_{j=1}^{k} X_{nj}\Big| \geq \epsilon\Big) \leq c\epsilon^{-r}\sum_{k=1}^{n} \beta_{nk}^{-r} a_{nk}.$$

Proof. The proof for the sequences and not triangular arrays is given in Tómács and Libor [2006]. We shall use the same idea of the proof. Fix $\epsilon > 0$ and $n \geq 0$. Without loss of generality assume that $\beta_{n1} = 1$. Let

$$A_i = \Big\{m : 1 \leq m \leq n \quad \text{and} \quad 2^i < \beta_{nm}^r < 2^{i+1}\Big\}, \quad i = 0, 1, 2, \ldots$$

Parameters	$\alpha = 17/32$	$\alpha = 20/32$	$\alpha = 26/32$
$\ell^*/n = 0.035$	0.831	0.590	0.055
$\ell^*/n = 0.050$	0.930	0.615	0.055
$\ell^*/n = 0.065$	0.966	0.548	0.051
$k^*/n = 0.2$	0.956	0.627	0.052
$k^*/n = 0.4$	0.930	0.615	0.055
$k^*/n = 0.8$	0.945	0.628	0.051
$a_n = 0.8$	0.726	0.354	0.051
$a_n = 1$	0.930	0.615	0.055
$a_n = 1.2$	0.991	0.827	0.059
$n = 500$	0.750	0.634	0.058
$n = 1000$	0.930	0.615	0.055
$n = 2000$	0.999	0.788	0.052
$\gamma_n = n/\ln(n)$	0.910	0.528	0.055
$\gamma_n = \ln^{2.5}(n)$	0.883	0.488	0.055
$\gamma_n = n^{3/4}$	0.930	0.615	0.055

Table 6.11: Empirical power at the size-adjusted significance level 0.05 for the second type model with regular varying innovations and $\alpha > \alpha_p$.

and

$$I = \max\{i : A_i \neq \emptyset\}.$$

Further denote

$$m_i = \begin{cases} \max A_i & \text{if } A_i \neq \emptyset \\ m_{i-1} & \text{if } A_i = \emptyset \end{cases} \qquad i = 0, 1, 2, \ldots, \qquad m_{-1} = 0.$$

Then we have

$$\begin{aligned} \mathbb{P}\left(\max_{k\le n} \beta_{nk}^{-1}\left|\sum_{j=1}^{k} X_{nj}\right| \ge \epsilon\right) &\le \sum_{i=0}^{I} \mathbb{P}\left(\max_{k\in A_i}\left|\sum_{j=1}^{k} X_{nj}\right| \ge \epsilon 2^{i/r}\right) \\ &\le \sum_{i=0}^{I} \mathbb{P}\left(\max_{k\le m_i}\left|\sum_{j=1}^{k} X_{nj}\right| \ge \epsilon 2^{i/r}\right) \le \sum_{i=0}^{I} c\epsilon^{-r}2^{-i}\sum_{k=1}^{m_i} a_{nk} \\ &\le 2c\epsilon^{-r}\sum_{k=0}^{I} 2^{-k}\sum_{j\in A_k} a_{nj} \le 2c\epsilon^{-r}\sum_{k=0}^{I}\sum_{j\in A_k} a_{nj}2\beta_{nk}^{-r} \\ &= 4c\epsilon^{-r}\sum_{k=1}^{n} a_{nk}\beta_{nk}^{-r}. \end{aligned}$$

Parameters	$\alpha = 2/32$	$\alpha = 6/32$	$\alpha = 10/32$
$\ell^*/n = 0.035$	0.094	0.226	0.792
$\ell^*/n = 0.050$	0.173	0.630	0.959
$\ell^*/n = 0.100$	0.368	0.912	0.994
$k^*/n = 0.2$	0.152	0.620	0.966
$k^*/n = 0.4$	0.173	0.630	0.959
$k^*/n = 0.8$	0.141	0.627	0.963
$a_n = 0.8$	0.154	0.389	0.805
$a_n = 1$	0.173	0.630	0.959
$a_n = 1.2$	0.172	0.854	0.997
$n = 500$	0.039	0.124	0.509
$n = 1000$	0.173	0.630	0.959
$n = 2000$	0.706	0.997	1.000
$\gamma_n = n/\ln(n)$	0.085	0.555	0.944
$\gamma_n = \ln^{2.5}(n)$	0.057	0.445	0.949
$\gamma_n = n^{3/4}$	0.173	0.630	0.95

Table 6.12: Empirical power at the size-adjusted significance level 0.05 for the second type model with regular varying innovations and $\alpha < \alpha_p$.

So the theorem is proved. □

Proof of Proposition 6.1.4. The proofs of both parts of this proposition are essentially the same, so we shall give a unified proof noting the differences in cases (a) and (b) where it is necessary. Since

$$\sum_{j=1}^{n} \widehat{\varepsilon}_j = \sum_{j=1}^{n} \varepsilon_j + (\phi_n - \widehat{\phi}_n) \sum_{j=1}^{n} y_{j-1}$$

and

$$\sum_{j=k+1}^{k+\ell} y_{j-1} = \frac{1}{1-\phi_n} \Big[\sum_{j=k+1}^{k+\ell} \varepsilon_j + y_k - y_{k+\ell} \Big],$$

we have

$$\Big| T_{\alpha,n}(\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_n) - T_{\alpha,n}(\varepsilon_1, \ldots, \varepsilon_n) \Big| \le \frac{|\widehat{\phi}_n - \phi_n|}{1-\phi_n} \Delta_n,$$

where

$$\Delta_n = \max_{1 \le \ell < n} \ell^{-\alpha} \max_{1 \le k \le n-\ell} \Big| \sum_{j=k+1}^{k+\ell} \varepsilon_j - (y_{k+\ell} - y_k) - \frac{\ell}{n} \sum_{j=1}^{n} \varepsilon_j + \frac{\ell}{n} y_n \Big|$$

Since $|\widehat{\phi}_n - \phi_n|/(1-\phi_n) = O_P(1)$ in case (a) (by Phillips Phillips [1987]) and $n\gamma_n^{-1/2}|\widehat{\phi}_n - \phi_n| = O_P(1)$ in case (b) (by Giraitis and Phillips Giraitis and Phillips [2006]) the proofs reduces to

$$b_n^{-1}\Delta_n = o_P(1) \quad \text{in case (a),} \tag{6.39}$$

$$b_n^{-1}\gamma_n^{-1/2}\Delta_n = o_P(1) \quad \text{in case (b).} \tag{6.40}$$

Writing

$$y_{k+\ell} - y_k = \sum_{j=k+1}^{k+\ell} \phi_n^{k+\ell-j}\varepsilon_j + \sum_{j=1}^{k}[\phi_n^{k+\ell-j} - \phi_n^{k-j}]\varepsilon_j$$

we have $\Delta_n \le \Delta_n' + \Delta_n'' + \Delta_n'''$, where

$$\Delta_n' = \max_{1\le \ell<n} \ell^{-\alpha} \max_{1\le k\le n-\ell}\Big|\sum_{j=k+1}^{k+\ell}[1-\phi_n^{k+\ell-j}]\varepsilon_j\Big|,$$

$$\Delta_n'' = \max_{1\le \ell<n} \ell^{-\alpha} \max_{1\le k\le n-\ell}\Big|\sum_{j=1}^{k}[\phi_n^{k-j} - \phi_n^{k+\ell-j}]\varepsilon_j\Big|,$$

$$\Delta_n''' = \max_{1\le \ell<n} \ell^{-\alpha}\frac{\ell}{n}\Big|\sum_{j=1}^{n}[1-\phi_n^{n-j}]\varepsilon_j\Big|.$$

1. Estimate for Δ_n'''. First we note that

$$\Delta_n''' = n^{-\alpha}\left|\sum_{j=1}^{n}(1-\phi_n^{n-j})\varepsilon_j\right|.$$

Since $\mathbb{E}(\sum_{j=1}^{n}(1-\phi_n^{n-j})\varepsilon_j)^2 = O(n)$, then $\Delta_n''' = O(n^{1/2-\alpha})$. As by assumption $1/2-\alpha < 1/p$ and as $b_n = n^{1/p}v(n)$ with v slowly varying function, this gives that $\Delta_n''' = o_P(b_n)$ in both cases.

2. Estimate for Δ_n''. Next consider Δ_n'' and note that

$$\Delta_n'' \le \max_{1\le \ell<n} \ell^{-\alpha}(1-\phi_n^{\ell}) \max_{1\le k\le n-\ell}\left|\sum_{j=1}^{k}\phi_n^{k-j}\varepsilon_j\right|.$$

(a) Using the convexity inequality $1-e^{-z}\le z$ for $z\ge 0$ gives

$$\Delta_n'' \le \max_{1\le \ell<n} \ell^{-\alpha}\frac{|\gamma|\,\ell}{n} \max_{1\le k\le n}\Big|\sum_{j=1}^{k}\phi_n^{k-j}\varepsilon_j\Big| \le |\gamma|\, n^{-\alpha} \max_{1\le k\le n}\Big|\sum_{j=1}^{k}\phi_n^{k-j}\varepsilon_j\Big|.$$

(b) Using the convexity inequality $1-(1-x)^y \le xy$ for $0 < x \le 1$ and $y \ge 1$, gives

$$\Delta_n'' \le \max_{1\le \ell<n} \ell^{-\alpha}\frac{\gamma_n \ell}{n} \max_{1\le k\le n}\left|\sum_{j=1}^{k}\phi_n^{k-j}\varepsilon_j\right| \le \gamma_n n^{-\alpha} \max_{1\le k\le n}\left|\sum_{j=1}^{k}\phi_n^{k-j}\varepsilon_j\right|.$$

Now we shall use Hájek-Rényi inequality (Lemma 6.4.1). Since

$$\mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{j=1}^{k}\phi_n^{-j}\varepsilon_j\right| > \varepsilon\right) \le \sum_{k=1}^{n}\mathbb{P}\left(\left|\phi_n^{-k}\varepsilon_k\right| > \varepsilon\right) \le \varepsilon^{-2}\sum_{k=1}^{n}\phi_n^{-2k}\sigma^2,$$

we have for any $\delta > 0$

(a)

$$\begin{aligned}\mathbb{P}(\Delta_n'' > \delta b_n) &\le \mathbb{P}(\max_{1\le k\le n}\phi_n^k\left|\sum_{j=1}^{k}\phi_n^{-j}\varepsilon_j\right| > \delta b_n n^{\alpha})\\ &\le \delta^{-2}n^{-2\alpha}b_n^{-2}\sigma^2\sum_{j=1}^{n}\phi_n^{2j}\phi_n^{-2j} = \delta^{-2}b_n^{-2}\sigma^2 n^{1-2\alpha} \to 0\end{aligned}$$

as $n \to \infty$ since $b_n = n^{1/p}v(n)$ with slowly varying function v and $\alpha > 1/2 - 1/p$.

(b)

$$\begin{aligned}\mathbb{P}(\Delta_n'' > \delta b_n\gamma_n^{1/2}) &\le \mathbb{P}(\max_{1\le k\le n}\phi_n^k\left|\sum_{j=1}^{k}\phi_n^{-j}\varepsilon_j\right| > \delta b_n n^{\alpha}\gamma_n^{-1/2})\\ &\le \delta^{-2}n^{-2\alpha}b_n^{-2}\gamma_n\sigma^2\sum_{j=1}^{n}\phi_n^{2j}\phi_n^{-2j} = \delta^{-2}\gamma_n b_n^{-2}\sigma^2 n^{1-2\alpha} \to 0,\end{aligned}$$

as $n \to \infty$ by the same argumentation as in case (a) provided that

$$\lim_{n\to\infty}\gamma_n^{1/2}n^{(\alpha_p-\alpha)}v(n)^{-1} = 0. \tag{6.41}$$

This part (6.18) $\Rightarrow$ (6.41) is correct, but I suggest to replace it by the following sentence. From (6.18), $\gamma_n^{1/2}n^{(\alpha_p-\alpha)} = O(n^{(q/2-1)(\alpha-\alpha_p)})$ and as $q/2-1 < 0$ and $v(n)$ is slowly varying, (6.41) is satisfied.

3. Estimate for Δ_n'. Finally it remains to prove

$$\Delta_n' = o_P(b_n) \quad \text{in case (a)}, \tag{6.42}$$

$$\Delta_n' = o_P(\gamma_n^{1/2}b_n) \quad \text{in case (b)}. \tag{6.43}$$

For simplicity now we shall write c_n noting that either $c_n = b_n$ or $c_n = b_n\gamma_n^{1/2}$.

First we decompose $\mathbb{P}(\Delta_n' > 2\delta c_n) \le P_{1n} + P_{2n}$, where

$$P_{1n} = \mathbb{P}\Big(\max_{1\le \ell < n} \ell^{-\alpha}(1-\phi_n^\ell) \max_{1\le k\le n-\ell} \Big|\sum_{j=k+1}^{k+\ell} \varepsilon_j\Big| > \delta c_n)$$

$$P_{2n} = \mathbb{P}\Big(\max_{1\le \ell < n} \ell^{-\alpha}\phi_n^\ell \max_{1\le k\le n-\ell} \Big|\sum_{j=1}^{\ell} (1-\phi_n^{-j})\varepsilon_{k+j}\Big| > \delta c_n).$$

We have for the first probability, using Doob inequality

(a)

$$\begin{aligned}P_{1n} &\le \mathbb{P}\Big(\max_\ell \max_k \Big|\sum_{j=k+1}^{k+\ell} \varepsilon_j\Big| > \delta b_n n^\alpha\Big) \le \mathbb{P}(2\max_{1\le k\le n} \Big|\sum_{j=1}^{k} \varepsilon_j\Big| > \delta b_n n^\alpha)\\ &\le 4\delta^{-2} b_n^{-2} n^{-2\alpha} \mathbb{E} \max_{1\le k\le n} \Big|\sum_{j=1}^{k} \varepsilon_j\Big|^2 \le 4\delta^{-2}\sigma^2 n^{1-2\alpha} b_n^{-2}\\ &\sim n^{1-2/p-2\alpha} v(n)^{-2} \to 0\end{aligned}$$

as $n\to\infty$, since $\alpha > 1/2 - 1/p$.

(b)

$$\begin{aligned}P_{1n} &\le \mathbb{P}\Big(\max_\ell \max_k \Big|\sum_{j=k+1}^{k+\ell} \varepsilon_j\Big| > \delta \gamma_n^{-1/2} b_n n^\alpha\Big)\\ &\le \mathbb{P}(2\max_{1\le k\le n} \Big|\sum_{j=1}^{k} \varepsilon_j\Big| > \delta\gamma_n^{-1/2} b_n n^\alpha)\\ &\le 4\delta^{-2} b_n^{-2} n^{-2\alpha}\gamma_n \mathbb{E} \max_{1\le k\le n} \Big|\sum_{j=1}^{k} \varepsilon_j\Big|^2\\ &\le 4\delta^{-2}\sigma^2 n^{1-2\alpha}\gamma_n b_n^{-2}\\ &\sim \gamma_n n^{1-2/p-2\alpha} v(n)^{-2} \to 0\end{aligned}$$

as $n\to\infty$, since $\alpha > 1/2-1/p$ and $\lim_{n\to\infty}\gamma_n^{1/2} n^{-\alpha+(1/2-1/p)} v(n)^{-1} = 0$.

To estimate P_{2n} we define truncated random variables:

$$\varepsilon_j' = \varepsilon_j \mathbf{1}\{|\varepsilon_j| \ge hb_n\}, \quad \varepsilon_j'' = \varepsilon_j\mathbf{1}\{|\varepsilon_j| \le hb_n\} - \mathbb{E}\varepsilon_j\mathbf{1}\{|\varepsilon_j| \le hb_n\},$$

for $j \ge 1$, where $h > 0$ is subject to a choice. Then $P_{2n}(\delta) \le P_{2n}' + P_{2n}''(\delta)$, where

$$P_{2n}' = \mathbb{P}\Big(\max_{1\le j\le n} |\varepsilon_j'| > hb_n\Big),$$

$$P_{2n}'' = \mathbb{P}\Big(\max_\ell \ell^{-\alpha}\phi_n^\ell \max_k \Big|\sum_{j=1}^{\ell}(1-\phi_n^{-j})\varepsilon_{k+j}\mathbf{1}\{|\varepsilon_{k+j}| \le hb_n\}\Big| > \delta c_n\Big).$$

Since

$$\mathbb{E}\varepsilon_{k+j}\mathbf{1}\{|\varepsilon_{k+j}| \le hb_n\} = \mathbb{E}\varepsilon_{k+j}\mathbf{1}\{|\varepsilon_{k+j}| \ge hb_n\},$$

we have

$$\max_\ell \ell^{-\alpha}\phi_n^\ell \max_k \Big|\sum_{j=1}^{\ell}(1-\phi_n^{-j})\mathbb{E}|\varepsilon_{k+j}\mathbf{1}\{|\varepsilon_{k+j}| \ge hb_n\}| \le cn^{1-\alpha}\mathbb{E}|\varepsilon_1|\mathbf{1}\{|\varepsilon_1| \ge hb_n\}.$$

By Karamata (see Bingham et al. [1987]) $\mathbb{E}|\varepsilon_1|\mathbf{1}\{|\varepsilon_1| \ge hb_n\} \sim cn^{-1}b_n h^{1-p}$. So we can center random variables in probability P''_{2n} and estimate for large n

$$P''_{2n} \le \mathbb{P}\Big(\max_{1\le\ell\le n} \ell^{-\alpha}\phi_n^\ell \max_k \Big|\sum_{j=1}^{\ell}(1-\phi_n^{-j})\varepsilon''_{k+j}\Big| > \frac{\delta}{2}c_n\Big).$$

By stationarity

$$P''_{2n} \le n\mathbb{P}\Big(\max_{1\le\ell\le n} \ell^{-\alpha}\phi_n^\ell\Big|\sum_{j=1}^{\ell}(1-\phi_n^{-j})\varepsilon''_j\Big| > \frac{\delta}{2}c_n\Big).$$

Fix $r > (\alpha-\alpha_p)^{-1}$ in case (a) and $r > ((1-q/2)(\alpha-\alpha_p))^{-1}$, $0<q<2$ in case (b). Using successively Markov's, Doob's and Rosenthal's inequalities, we obtain for each $a>0$ we have

$$\mathbb{P}\Big(\max_\ell\Big|\sum_{j=1}^{\ell}(1-\phi_n^{-j})\varepsilon''_j\Big| > a\Big) \le ca^{-r}\Big[\Big(\sum_{j=1}^{n}(1-\phi_n^{-j})^2\mathbb{E}(\varepsilon''_1)^2\Big)^{r/2} + \sum_{j=1}^{n}|1-\phi_n^{-j}|^r\mathbb{E}|\varepsilon''_1|^r\Big]$$

with a constant $c>0$ depending on r only. By Karamata $\mathbb{E}|\varepsilon''_1|^r \sim b_n^r n^{-1}h^{r-p}$. Hence, there is a constant $c>0$ such that

$$\begin{aligned}\mathbb{P}\Big(\max_\ell\Big|\sum_{j=1}^{\ell}(1-\phi_n^{-j})\varepsilon''_j\Big| > a\Big) &\le ca^{-r}\Big[\Big(\sum_{j=1}^{n}(1-\phi_n^{-j})^2\mathbb{E}(\varepsilon''_1)^2\Big)^{r/2} + \sum_{j=1}^{n}|1-\phi_n^{-j}|^r\mathbb{E}|\varepsilon''_1|^r\Big]\\ &\le ca^{-r}\sum_{j=1}^{n}\tau_{nj},\end{aligned}$$

where $\tau_{nj} = \sigma^r n^{r/2-1}(\phi_n^{-j}-1)^r + (\phi_n^{-j}-1)^r b_n^r n^{-1}h^{r-p}$. By Lemma 6.4.1, noting that the finite sequence $(\ell^\alpha\phi_n^{-\ell})_{1\le\ell\le n}$ is non decreasing, we obtain

$$\mathbb{P}\Big(\max_\ell \ell^{-\alpha}\phi_n^\ell\Big|\sum_{j=1}^{\ell}(1-\phi_n^{-j})\varepsilon''_j\Big| > a\Big) \le ca^{-r}\sum_{j=1}^{n}\phi_n^{rj}j^{-r\alpha}\tau_{nj}.$$

Finally we deduce

$$P''_{2n} \le c2^r\delta^{-r}nb_n^{-r}\left(\sigma^r n^{r/2-1} + n^{-1}b_n^r h^{r-p}\right)\sum_{j=1}^{n} j^{-r\alpha}(1-\phi_n^j)^r$$

(a) Using again convexity inequality $1 - \mathrm{e}^{-z} \leq z$, we note that

$$\sum_{j=1}^{n} j^{-r\alpha}(1-\phi_n^j)^r \leq \sum_{j=1}^{n} j^{-r\alpha}\left(\frac{-\gamma j}{n}\right)^r = \frac{|\gamma|^r}{n^r}\sum_{j=1}^{n} j^{r(1-\alpha)} = O_P(n^{1-r\alpha}).$$

This leads to

$$P''_{2n} \leq C\left(v(n)^{-r} n^{1-r\alpha+r/2-r/p} + n^{1-r\alpha}\right),$$

where $C = C(\delta, r, \alpha, \gamma, \sigma, h, p)$ is a positive constant. Now the choice done for r verifies $1 - r\alpha + r/2 - r/p < 0$, which implies also $1 - r\alpha < 0$, so finally

$$\lim_{n\to\infty} P''_{2n} = 0.$$

(b) Using the convexity inequality $1 - (1-x)^y \leq xy$, for $0 < x \leq 1$ and $y \geq 1$, we note that

$$\sum_{j=1}^{n} j^{-r\alpha}(1-\phi_n^j)^r \leq \sum_{j=1}^{n} j^{-r\alpha}\left(\frac{\gamma_n j}{n}\right)^r = \frac{\gamma_n^r}{n^r}\sum_{j=1}^{n} j^{r(1-\alpha)} = O_P(\gamma_n^r n^{1-r\alpha}).$$

This leads to

$$P''_{2n} \leq C\left(v(n)^{-r}\gamma_n^{r/2} n^{1-r\alpha+r/2-r/p} + n^{1-r\alpha}\right),$$

where $C = C(\delta, r, \alpha, \sigma, h, p)$ is a positive constant. Now we find that the condition to have

$$\lim_{n\to\infty} P''_{2n} = 0$$

is the existence of some $r > p$ such that

$$\lim_{n\to\infty} v(n)^{-1}\gamma_n^{1/2} n^{1/r+\alpha_p-\alpha} = 0. \tag{6.44}$$

This follows from our assumption (6.18), since we have for some constant K:

$$v(n)^{-1}\gamma_n^{1/2} n^{1/r+\alpha_p-\alpha} \leq K v(n)^{-1} n^{1/r+(q/2-1)(\alpha-\alpha_p)}.$$

then this upper bound tends to 0 for any $r > 0$ such that

$$\frac{1}{r} < \left(1 - \frac{q}{2}\right)(\alpha - \alpha_p).$$

Finally, since (Gnedenko [1943], see, for example, Embrechts et al. [1997], Theorem 3.3.7, for a more recent reference)

$$\lim_{n\to\infty} \mathbb{P}(\max_{1\le j\le n} |\varepsilon_j| \ge hb_n) = 1 - \mathrm{e}^{-h^{-p}}$$

the probability P'_{1n} can be made arbitrary small by choosing big h. So (6.42) and (6.43) as well as (6.39) and (6.40) are proved. □

Remark 6.4.2. There is no loss in the proof when we deduce (6.44) from (6.18) since the converse implication is true. Indeed assume that (6.44) holds true for some r. Then we can rewrite it as

$$(v(n)^{-1}n^{1/(2r)})\gamma_n^{1/2}n^{1/(2r)-(\alpha-\alpha_p)} \xrightarrow[n\to\infty]{} 0.$$

As $v(n)$ is slowly varying and r positive, $v(n)^{-1}n^{1/(2r)}$ tends to infinity, then necessarily $\gamma_n^{1/2}n^{1/(2r)-(\alpha-\alpha_p)}$ tends to zero and in particular is bounded. So for some positive constant K:

$$\gamma_n \le Kn^{-1/r+2(\alpha-\alpha_p)}.$$

Now we define q by

$$-\frac{1}{r} + 2(\alpha-\alpha_p) = q(\alpha-\alpha_p),$$

as γ_n tends to infinity, necessarily $-1/r + 2(\alpha-\alpha_p)$ is positive. Then $q \in (0,2)$ and we get (6.18).

Further we state proofs of two lemmas that are the main tools to prove the Lemma 6.2.4.

Proof of Lemma 6.2.5. For the first type model by Phillips it hold

$$n^{-1/2}\sigma^{-1}z_{[nt]} \xrightarrow[n\to\infty]{\mathrm{D}[0,1]} U_\gamma(t) \tag{6.45}$$

with the supremum norm $\|\cdot\|_\infty$. The map

$$\Psi : \quad (\mathrm{D}[0,1], \|\cdot\|_\infty) \mapsto \quad \mathbb{R}^2 : \quad f \mapsto \left(\int_0^1 f^2(t)\,\mathrm{d}t, \int_0^\lambda f^2(t)\,\mathrm{d}t\right)$$

is continuous. Obviously

$$\int_0^1 z_{n,[nt]}^2\,\mathrm{d}t = \frac{1}{n}\sum_{k=1}^n z_{n,k-1}^2. \tag{6.46}$$

Since

$$f \longmapsto \frac{\int_0^1 f^2(t)\,\mathrm{d}t}{\int_0^\lambda f^2(t)\,\mathrm{d}t}$$

is continuous on

$$\left\{ f \in \mathrm{D}[0,1]; \quad \int_0^\lambda f^2(t)\,\mathrm{d}t \neq 0 \right\}$$

and according to (6.45) the limiting process is Gaussian (Ornstein-Uhlenbeck), so

$$\mathbb{P}\left(\int_0^\lambda U_\gamma^2(t)\,\mathrm{d}t = 0 \right) = 0,$$

thus

$$\frac{\int_0^1 (n^{-1/2}\sigma^{-1} z_{n,[nt]})^2\,\mathrm{d}t}{\int_0^\lambda (n^{-1/2}\sigma^{-1} z_{n,[nt]})^2\,\mathrm{d}t} \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\int_0^1 U_\gamma^2(t)\,\mathrm{d}t}{\int_0^\lambda U_\gamma^2(t)\,\mathrm{d}t} = O_P(1).$$

For the second type model we have the following weak law of large numbers

$$\frac{1-\phi_n^2}{n} \sum_{k=1}^n z_{n,k-1}^2 \xrightarrow[n\to\infty]{\mathrm{P}} \sigma^2. \tag{6.47}$$

Seeing that

$$\frac{\sum_{k=1}^n z_{n,k-1}^2}{\sum_{k=1}^{[n\lambda]} z_{n,k-1}^2} = \frac{\frac{1-\phi_n^2}{n} \sum_{k=1}^n z_{n,k-1}^2}{\frac{1-\phi_{[n\lambda]}^2}{[n\lambda]} \sum_{k=1}^{[n\lambda]} z_{n,k-1}^2} \cdot \frac{1-\phi_{[n\lambda]}^2}{1-\phi_n^2} \cdot \frac{n}{[n\lambda]}$$

we obtain

$$\frac{\frac{1-\phi_n^2}{n} \sum_{k=1}^n z_{n,k-1}^2}{\frac{1-\phi_{[n\lambda]}^2}{[n\lambda]} \sum_{k=1}^{[n\lambda]} z_{n,k-1}^2} \xrightarrow[n\to\infty]{\mathrm{P}} 1$$

and

$$\frac{n}{[n\lambda]} \sim \frac{1}{\lambda}.$$

Further assuming that γ_n is increasing in n we have $\gamma_{[n\lambda]}/\gamma_n \leq 1$

$$\frac{1-\phi_{[n\lambda]}^2}{1-\phi_n^2} \sim c \frac{n}{[n\lambda]} \frac{\gamma_{[n\lambda]}}{\gamma_n} \leq \frac{c}{\lambda}.$$

If γ_n is regular varying then

$$\lim_{n\to\infty} \frac{\gamma_{[n\lambda]}}{\gamma_n} = c(\lambda)$$

so

$$\frac{1-\phi_{[n\lambda]}^2}{1-\phi_n^2}$$

remains bounded. □

Remark 6.4.3. We explain, why we need to put here an additional assumption on γ_n (increasing or regular varying). Let us define the sequence

$$\gamma_n = \begin{cases} n^{0.1} & \text{if } n \text{ is even,} \\ n^{0.9} & \text{if } n \text{ is odd.} \end{cases}$$

Then let us define the subsequence $n_k = (4k+2)$, $k = 0, 1, 2, \ldots$. As all n_k are even, we obtain $\gamma_{n_k} = n_k^{0.1} = (4k+2)^{0.1}$. Now we take $\lambda = 1/2$, then $[n_k\lambda]$ are odd

$$[n_k\lambda] = [n_k/2] = 2k+1$$

and so $\gamma_{[n_k\lambda]} = (n_k/2)^{0.9} = (2k+1)^{0.9}$. So we get

$$\frac{\gamma_{[n_k\lambda]}}{\gamma_{n_k}} = \frac{(2k+1)^{0.9}}{(4k+2)^{0.1}} = (1/2)^{0.9}\frac{(4k+2)^{0.9}}{(4k+2)^{0.1}} \to \infty \quad \text{as} \quad k \to \infty.$$

The latter result implies that

$$\frac{1-\phi_{[n\lambda]}^2}{1-\phi_n^2}$$

is not bounded in such case.

Proof of Lemma 6.2.6. Let us consider first type model. Then equation (6.46) gives us

$$\frac{1}{\sum_{k=1}^{[n\lambda]} z_{n,k-1}^2} = \frac{1}{n\int_0^\lambda (n^{-1/2}\sigma^{-1}z_{n,[nt]})^2\,\mathrm{d}t}(n^{-1/2}\sigma^{-1})^2 \le \frac{(1-\phi_n)O_P(1)}{n}$$

by the same argumentation as in Lemma 6.2.5 and equivalence $1-\phi_n \sim -\gamma/n$.

For the second type model applying the weak law of large numbers (6.47) we imediatelly obtain the inequality

$$\frac{1}{\sum_{k=1}^{[n\lambda]} z_{n,k-1}^2} \le \frac{(1-\phi_n)O_P(1)}{n}.$$

□

Conclusions

First order nearly nonstationary autoregressive processes $y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k$ are considered with coefficient ϕ_n defined in two ways:

– $\phi_n = \mathrm{e}^{\gamma/n}$, $\gamma < 0$;

– $\phi_n = 1 - \gamma_n/n$, $\gamma_n \to\to \infty$, $\gamma_n/n \to 0$, as $n \to \infty$.

Polygonal line processes S_n^{pl} build on observations $y_{n,k}$ and $\widehat{W}_n^{\mathrm{pl}}$ build on process residuals $\widehat{\varepsilon}_k$ are studied. The functional limit theorems for S_n^{pl} in the spaces $\mathrm{C}[0,1]$ and $\mathrm{H}_\alpha^o[0,1]$, $\alpha \in (0, 1/2)$ are proved. It is shown that the limiting distribution differs for the both type models. Properly normalized S_n^{pl} converge to integrated Ornstein-Uhlenbeck process in the first type model whereas to Wiener process in the second type model. Functional limit theorems for $\widehat{W}_n^{\mathrm{pl}}$ in $\mathrm{H}_\alpha^o[0,1]$ are proved. For the first type model it is shown that integrability condition $\lim_{t\to\infty} t^p \mathbb{P}(|\varepsilon_0| > t) = 0$ is necessary and sufficient for the convergence in distribution of $\widehat{W}_n^{\mathrm{pl}}$ in the $\mathrm{H}_\alpha^o[0,1]$ space. For the second type model, the convergence in distribution to Wiener process in $\mathrm{H}_\alpha^o[0,1]$ is derived.

Further epidemic change detection in mean of innovations is investigated. The model

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k + a_{n,k}, \quad n \geq 0, \quad k \leq n$$

is concerned. Uniform increments statistics is build on observations $y_{n,1}, \ldots, y_{n,n}$ and residuals $\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_n$. Under some assumptions on residuals we find the limit of both statistics. Consistency conditions for statistics $\widetilde{T}_{\alpha,n} = T_{\alpha,n}(y_{n,1}, \ldots, y_{n,n})$ and $\widehat{T}_{\alpha,n} = T_{\alpha,n}(\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_n)$ are found and test power analysis is performed. Both statistics are worth of further investigation. Table 6.13 shows size-adjusted test power for statistics $\widetilde{T}_{\alpha,n}$ and $\widehat{T}_{\alpha,n}$, where innovations satisfies integrability condition $\lim_{t\to\infty} t^p \mathbb{P}(|\varepsilon_0| > t) = 0$. The result shows that with different parameters for the second type model, different statistics give different results. In this exam-

$a_n = 1, \ell^* = 30, k^* = 400, n = 1000, \gamma = -2, \gamma_n = n^{0.45}$				
		$\alpha_1 = 0.0625$	$\alpha_2 = 0.1875$	$\alpha_3 = 0.39$ (I model); $\alpha_3 = 0.31$ (II model)
$\widetilde{T}_{\alpha,n}$	I model	0.318	0.327	0.306
	II model	0.276	0.330	0.429
$\widehat{T}_{\alpha,n}$	I model	0.335	0.526	0.914
	II model	0.061	0.452	0.836
$a_n = 1, \ell^* = 30, k^* = 400, n = 1000, \gamma = -20, \gamma_n = n^{0.8}$				
$\widetilde{T}_{\alpha,n}$	I model	0.280	0.322	0.467
	II model	0.314	0.505	0.796
$\widehat{T}_{\alpha,n}$	I model	0.088	0.502	0.913
	II model	0.073	0.213	0.682

Table 6.13: Comparing statistics $\widetilde{T}_{\alpha,n}$ and $\widehat{T}_{\alpha,n}$

ple, statistics $\widehat{T}_{\alpha,n}$ with $\gamma_n = n^{0.45}$ detects epidemics better, while with $\gamma_n = n^{0.8}$ statistics $\widetilde{T}_{\alpha,n}$ performs better. Further note, that with the chosen parameters statistics $\widehat{T}_{\alpha,n}$ for the first type model works better in both cases, but consistency of this case is still an open question.

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