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JOINT UNIVERSALITY FOR PERIODIC HURWITZ ZETA-FUNCTIONS

Master work

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Šiauliai, 2009

ŠIAULIŲ UNIVERSITETAS
MATEMATIKOS IR INFORMATIKOS FAKULTETAS
MATEMATIKOS KATEDRA

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PERIODINIŲ HURVICO DZETA FUNKCIJŲ JUNGTINIS UNIVERSALUMAS

Magistro darbas

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Šiauliai, 2009

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Introduction

In 1975, S. M. Voronin discovered [19] the universality of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$. Roughly speaking, this means that any analytic function can be approximated by shifts $\zeta(s + i\tau)$ with desired accuracy. A precise statement of the Voronin theorem is the following result.

Suppose that $0 < r < \frac{1}{4}$. *Let a function* $f(s)$ *be non-vanishing and continuous on the disc* $|s| \leq r$, *and analytic in the interior of this disc. Then, for every* $\varepsilon > 0$, *there exists a real number* $\tau = \tau(\varepsilon)$ *such that*

$$\max_{|s|\leq r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon.$$

Later, the Voronin theorem was improved. We recall a theorem obtained in [9]. Denote by meas $\{A\}$ the Lebesque measure of a measurable set $A \subset \mathbb{R}$.

Suppose that K *is a compact subset of the strip* $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ *with connected complement, and* $f(s)$ *is a non-vanishing continuous function on* K *which is analytic in the interior of* K. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

This theorem shows that the set of shifts $\zeta(s + i\tau)$ which approximate a given analytic function is sufficiently rich, it has a positive lower density.

The universality property was also obtained for other zeta and L-function, see a survey in [10].

Also, S. M. Voronin was the first who proved a point universality theorem for Dirichlet L-functions. We recall that a Dirichlet L-function $L(s, \chi)$ is defined, for $\sigma > 1$, by

$$L(s, \chi) = \sum_{m=0}^{\infty} \frac{\chi(m)}{m^s},$$

and by analytic continuation elsewhere. Here $\chi(m)$ is a Dirichlet character. In [18], S. M. Voronin proved the following theorem. We state its modern version. *Suppose that* $\chi_1, ..., \chi_n$ *are pairwise non-equivalent Dirichlet characters. Let* $K_1, ..., K_n$ *be compact subsets of the strip* D *with connected complements, and* $f_1(s), ..., f_n(s)$ *be non-vanishing continuous functions on* $K_1, ..., K_n$ *which are analytic in the interior of* $K_1, ..., K_n$, *respectively. Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{\substack{1\leq j\leq n \\ s\in K_j}} |L(s + i\tau, \alpha_j) - f(s)| < \varepsilon \right\} > 0.$$

The aim of our work is to obtain joint universality theorems for periodic Hurwitz zeta-functions.

The master work consist of two papers submisted for publication on joint universality of periodic Hurwitz zeta-functions. We recall the definition of the periodic Hurwitz zeta-function. Let $\mathfrak{A} = \{a_m : m \in \mathbb{N}_0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, be a periodic sequence with minimal period $k \in \mathbb{N}$ of complex numbers, and α, $0 < \alpha \leq 1$, be a fixed parameter. For $\sigma > 1$, the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{A})$ is defined by

$$\zeta(s, \alpha; \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$

The function $\zeta(s, \alpha; \mathfrak{A})$ is analytically continuable to the whole complex plane, except, maybe, for a simple pole at $s = 1$.

We prove two joint universality theorems for periodic Hurwitz zeta-functions $\zeta(s, \alpha_1; \mathfrak{A}_1), \dots, \zeta(s, \alpha_r; \mathfrak{A}_r)$. Define

$$L(\alpha_1, \dots, \alpha_r) = \{\log(m + \alpha_j) : j = 1, \dots, r, m \in \mathbb{N}_0\}.$$

Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over the field of national numbers. For every $j = 1, \dots, 2$, let K_j be a compact subset of the strip D with connected complement, and let $f(s)$ be a continuous function on K_j and analytic in the interior of K_j. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j; \mathfrak{A}_j) - f(s)| < \varepsilon \right\} > 0.$$

The later theorem is more stronger than that obtained in [8] because it does not use any hypothesis on the sequences $\mathfrak{A}_1, \dots, \mathfrak{A}_r$.

In the second joint universality theorem, we consider the use when parameter α_j corresponds general periodic sequences. Let l_j, $j = 1, \dots, r$, be positive integers. For every $l = 1, \dots, l_j$, let $\mathfrak{A}_{jl} = \{a_{mjl} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k_{jl} \in \mathbb{N}$, $0 < \alpha_j \leq 1$, and

$$\zeta(s, \alpha_j; \mathfrak{A}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}}{(m+\alpha_j)^s}.$$

Moreover, let k_j be the least common multiple of the periods $k_{j_1}, \dots, k_{jl_j}$, $j = 1, \dots, r$, of $k_{11}, \dots, k_{1l_1}, \dots, k_{r1}, \dots, k_{rl_r}$, and k be the least common multiple

$$B_j = \begin{pmatrix} a_{1j_1} & a_{1j_2} & \dots & a_{1jl_j} \\ a_{2j_1} & a_{2j_2} & \dots & a_{2jl_j} \\ \dots & \dots & \dots & \dots \\ a_{kj_1} & a_{kj_2} & \dots & a_{kjl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

The we have the following result.

Suppose that the system $L(\alpha_1, \dots, \alpha_r)$ *is linearly independent over the field of national numbers, and that* $\operatorname{rank}(B_j) = l_j$, $j = 1, \dots, r$. *For every* $j = 1, \dots, r$ *and* $l = 1, \dots, l_j$, *let* K_{jl} *be a compact subset of the strip* D *with connected complement, and let* $f_{jl}(s)$ *be a continuous on* K_{jl} *function which is analytic in the interior of* $K_j l$, *then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j; \mathfrak{A}_{jl}) - f_{jl}(s)| < \varepsilon \right\} > 0.$$

Note that the rank hypothesis in the later theorem is weaker then that in [13].

Chapter 1

A joint universality theorem for periodic Hurwitz zeta-functions

1.1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all positive integers, non-negative integers, integers, real and complex numbers, respectively. First, we recall the definition of the periodic Hurwitz zeta-function. Let $\{a_m : m \in \mathbb{N}_0\} \subset \mathbb{C}$ be a periodic sequence with minimal period $k \in \mathbb{N}$, and α, $0 < \alpha \leq 1$, be a fixed parameter. Then the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{A})$, $s = \sigma + it$, is defined, for $\sigma > 1$, by

$$\zeta(s, \alpha; \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s}.$$

The periodicity of the sequence $\{a_m\}$, for $\sigma > 1$, implies the equality

$$\zeta(s, \alpha; \mathfrak{A}) = \frac{1}{k^s} \sum_{l=0}^{k-1} a_l \zeta\left(s, \frac{\alpha + l}{k}\right), \tag{1.1}$$

where $\zeta(s, \alpha)$ is the classical Hurwitz zeta-function, for $\sigma > 1$, given by Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and can be continued analytically to the whole complex plane, except for a simple pole at $s = 1$ with residue 1. Therefore, equality (1.1) shows that the function $\zeta(s, \alpha; \mathfrak{A})$ also admits meromorphic continuation with a simple pole at $s = 1$ with residue

$$a \stackrel{def}{=} \frac{1}{k} \sum_{l=0}^{k-1} a_l$$

In the case $a = 0$, the function $\zeta(s, \alpha; \mathfrak{A})$ is entire.

Obviously, if $a_m \equiv 1$, then $\zeta(s, \alpha; \mathfrak{A}) = \zeta(s, \alpha)$.

In [6] and [7], the universality of the function $\zeta(s, \alpha; \mathfrak{A})$ with transcendental parameter α was investigated. Denote by meas $\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let, for $T > 0$,

$$\nu_T(\dots) = \frac{1}{T}\text{meas}\{\tau \in [0; T] : \dots\},$$

where in place of dots a condition satisfied by τ is to be written. Then in [7] the following statement was proved.

Theorem 1.1. *Suppose that α is transcendental. Let K be a compact subset of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{s\in K} \big|\zeta(s + i\tau, \alpha; \mathfrak{A}) - f(s)\big| < \varepsilon\Big) > 0.$$

Note that the transcendence of α is used only for the linear independence of the set

$$L(\alpha) \overset{def}{=} \{\log(m + \alpha) : m \in \mathbb{N}_0\}.$$

Since the fundamental Voronin work [18] on joint universality of Dirichlet L-functions, many authors considered the joint universality of other zeta- and L-functions. The first result in the field for periodic Hurwitz zeta-functions was obtained in [11]. Let $\mathfrak{A}_j = \{a_{mj} : m \in \mathbb{N}\} \subset \mathbb{C}$ be a periodic sequence with minimal period k_j, α_j, $0 < \alpha_j \leq 1$, be a fixed parameter, and, for $\sigma > 1$,

$$\zeta(s, \alpha_j; \mathfrak{A}_j) = \sum_{m=0}^{\infty} \frac{a_{mj}}{(m + \alpha_j)^s},$$

$j = 1, \dots, r, \quad r \in \mathbb{N}\backslash\{1\}$.

Definition. For every $j = 1, \dots, r$, let K_j be a compact subset of the strip D with connected complement, and let $f_j(s)$ be a function continuous on K_j and analytic in the interior of K_j. We say that the functions $\zeta(s, \alpha_1; \mathfrak{A}_1), \dots, \zeta(s, \alpha_r; \mathfrak{A}_r)$ are universal, if, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{1\leq j\leq r}\ \sup_{s\in K_j} \big|\zeta(s + i\tau, \alpha_j; \mathfrak{A}_j) - f_j(s)\big| < \varepsilon\Big) > 0.$$

Denote by k the least common multiple of the periods $k_1, \dots, k_r$, and define

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & a_{kr} \end{pmatrix}.$$

Then in [11] it was proved that if $k_j = k, \quad \alpha_j = \alpha, \quad j = 1, \dots, r, \quad \alpha$ is transcendental and $\text{rank}(A) = r$, then the functions $\zeta(s, \alpha; \mathfrak{A}_1), \dots, \zeta(s, \alpha; \mathfrak{A}_r)$ are universal. In [12], the requirement that $k_j = k, \quad j = 1, \dots, r$, was removed. Finally, in [8] the following joint universality theorem was proved.

Theorem 1.2. [8]. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$, and* $\text{rank}(A) = r$. *Then the functions $\zeta(s, \alpha_1; \mathfrak{A}_1), \dots, \zeta(s, \alpha_r; \mathfrak{A}_r)$ are universal.*

The aim of this paper is to prove a joint universality theorem for the functions $\zeta(s, \alpha_1; \mathfrak{A}_1), \dots, \zeta(s, \alpha_r; \mathfrak{A}_r)$ without using the hypothesis on the rank of the matrix A. Let

$$L(\alpha_1, \dots, \alpha_r) = \{\log(m + \alpha_j) : j = 1, \dots, r, m \in \mathbb{N}_0\}.$$

Theorem 1.3. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linear independent over $\mathbb{Q}$. Then the functions $\zeta(s, \alpha_1; \mathfrak{A}_1), \dots, \zeta(s, \alpha_r; \mathfrak{A}_r)$ are universal.*

We note that the linear independence of $L(\alpha_1, \dots, \alpha_r)$ holds whenever the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Therefore, the hypothesis on the numbers $\alpha_1, \dots, \alpha_r$ in Theorem 1.3 is weaker than that in Theorem 1.2.

1.2. A joint limit theorem

The proof of Theorem 1.3 is based on a joint limit theorem in the space of analytic functions for the functions $\zeta(s, \alpha_1; \mathfrak{A}_1), \dots, \zeta(s, \alpha_r; \mathfrak{A}_r)$. Its proof is independent on the matrix A, therefore it is similar to that of a limit theorem from [8], where the case of the algebraically independent numbers $\alpha_1, \dots, \alpha_r$ was considered.

Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta, and define

$$H^r(D) = \underbrace{H(D) \times \dots \times H(D)}_{r}.$$

Let, as usual, $\mathcal{B}(S)$ stand for the class of Borel sets of a space S. Moreover, let

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \{s \in \mathbb{C} : |s| = 1\} \stackrel{def}{=} \gamma$ for all $m \in \mathbb{N}_0$. Since γ is a compact, the Tikhonov theorem shows that the infinite-dimensional torus Ω with the product topology and point-wise multiplication is a compact topological Abelian group. Let

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \dots, r$. Then Ω^r is also a compact topological Abelian group. Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H^r can be defined, and we obtain a probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$. It is important to note that the Haar measure m_H^r is the product of the Haar measures m_{jH} on the coordinate spaces $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \dots, r$. Denote by $\omega_j(m)$ the projection of an element $\omega_j \in \Omega_j$ to the coordinate space γ_m, $m \in \mathbb{N}_0$. Let $\omega = (\omega_1, \dots, \omega_r) \in \Omega^r$, where $\omega_j \in \Omega_j$, $j = 1, \dots, r$, and let, for brevity,

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r), \quad \boldsymbol{\mathfrak{A}} = (\mathfrak{A}_1, \dots, \mathfrak{A}_r).$$

On the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$, define the $H^r(D)$-valued random element $\boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{A}})$ by

$$\boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{A}}) = (\zeta(s, \alpha_1, \omega_1; \mathfrak{A}_1), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{A}_r)),$$

where

$$\zeta(s, \alpha_j, \omega_j; \mathfrak{A}_j) = \sum_{m=0}^{\infty} \frac{a_{mj}\omega_j(m)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r,$$

and denote by $P_{\boldsymbol{\zeta}}$ the distribution of $\boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{A}})$, i.e.,

$$P_{\boldsymbol{\zeta}}(A) = m_H^r(\boldsymbol{\omega} \in \Omega^r : \boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{A}}) \in A), \quad A \in \mathcal{B}(H^r(D)).$$

Let, for $A \in \mathcal{B}(H^r(D))$,

$$P_T(A) = \nu_T(\boldsymbol{\zeta}(s + i\tau, \boldsymbol{\alpha}; \boldsymbol{\mathfrak{A}}) \in A),$$

where

$$\boldsymbol{\zeta}(s, \boldsymbol{\alpha}; \boldsymbol{\mathfrak{A}}) = (\zeta(s, \alpha_1; \mathfrak{A}_1), \dots, \zeta(s, \alpha_r; \mathfrak{A}_r)).$$

This section is devoted to the following probabilistic limit theorem.

Theorem 1.4. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linear independent over $\mathbb{Q}$. Then P_T converges weakly to the measure $P_{\boldsymbol{\zeta}}$ as $T \to \infty$.*

We will not give a proof of Theorem 4 in details, we will indicate only its principal points. The first important ingredient is a limit theorem on the torus Ω^r. Let, for $A \in \mathcal{B}(\Omega^r)$,

$$Q_T(A) = \nu_T((((m + \alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, ((m + \alpha_r)^{-i\tau} : m \in \mathbb{N}_0)) \in A).$$

Lemma 1.5. *Suppose that the set $L(\alpha_1,\dots,\alpha_r)$ is linear independent over $\mathbb{Q}$. Then Q_T converges weakly to the Haar measure m_H^r as $T\to\infty$.*

A proof of Lemma 5 is based on the Fonrier transforms method on compact topological groups, and is given in [13].

Now let $\sigma_1>\frac{1}{2}$ be fixed, and, for $m,n\in\mathbb{N}_0$,

$$\upsilon_n(m,\alpha_j)=\exp\left\{-\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\sigma_1}\right\},\quad j=1,\dots,r.$$

Define

$$\zeta_n(s,\alpha_j;\mathfrak{A}_j)=\sum_{m=0}^{\infty}\frac{a_{mj}\upsilon_n(m,\alpha_j)}{(m+\alpha_j)^s},\quad j=1,\dots,n,$$

and

$$\zeta_n(s,\alpha_j,\omega_j;\mathfrak{A}_j)=\sum_{m=0}^{\infty}\frac{a_{mj}\omega_j(m)\upsilon_n(m,\alpha_j)}{(m+\alpha_j)^s},\quad j=1,\dots,n,$$

the series being absolutely convergent for $\sigma>\frac{1}{2}$, see [6]. The next important step in the proof of Theorem 1.4 are limit theorems for

$$\boldsymbol{\zeta}_n(s,\boldsymbol{\alpha};\boldsymbol{\mathfrak{A}})=(\zeta_n(s,\alpha_1;\mathfrak{A}_1),\dots,\zeta_n(s,\alpha_r;\mathfrak{A}_r))$$

and

$$\boldsymbol{\zeta}_n(s,\boldsymbol{\alpha},\boldsymbol{\omega};\boldsymbol{\mathfrak{A}})=(\zeta_n(s,\alpha_1,\omega_1;\mathfrak{A}_1),\dots,\zeta_n(s,\alpha_r,\omega_r;\mathfrak{A}_r)).$$

Let, for $A\in\mathcal{B}(H^r(D))$,

$$P_{T,n}(A)=\nu_T(\boldsymbol{\zeta}_n(s+i\tau,\boldsymbol{\alpha};\boldsymbol{\mathfrak{A}})\in A),$$

and, for fixed $\boldsymbol{\omega_0}\in\Omega^r$,

$$Q_{T,n}(A)=\nu_T(\boldsymbol{\zeta}_n(s+i\tau,\boldsymbol{\alpha},\boldsymbol{\omega}_0;\boldsymbol{\mathfrak{A}})\in A),$$

Lemma 1.6. *Suppose that the set $L(\alpha_1,\dots,\alpha_r)$ is linearly independent over $\mathbb{Q}$. Then $P_{T,n}$ and $Q_{T,n}$ both converge weakly to the same probability measure P_n on $(H^r(D),\mathcal{B}(H^r(D)))$ as $T\to\infty$.*

Proof. Since the functions $h_n:\Omega^r\to H^r(D)$ and $h_{1,n}:\Omega^r\to H^r(D)$ given by $h_n(\omega)=\boldsymbol{\zeta}_n(s,\boldsymbol{\alpha},\boldsymbol{\omega};\boldsymbol{\mathfrak{A}})$ and

$$h_{1,n}(\boldsymbol{w})=\boldsymbol{\zeta}_n(s,\boldsymbol{\alpha},\boldsymbol{w}\boldsymbol{w}_0;\boldsymbol{\mathfrak{A}})$$

are continuous, and $P_{T,n}=Q_Th_n^{-1}$, $Q_{T,n}=Q_Th_{1,n}^{-1}$, we obtain from Lemma 1.5 and Theorem 5.1 of [3] that $P_{T,n}$ and $Q_{T,n}$ converge weakly to $m_H^rh_n^{-1}$ and $m_H^rh_{1,n}^{-1}$, respectively, as $T \to \infty$. Moreover, the invariance of the Haar measure m_H^r and the definitions of h_n and $h_{1,n}$ show that $m_H^r h_{1,n}^{-1} = m_H^r h_n^{-1}$.

In order to pass from $\boldsymbol{\zeta}_n(s, \boldsymbol{\alpha}; \mathfrak{A})$ to $\boldsymbol{\zeta}(s, \boldsymbol{\alpha}; \mathfrak{A})$, we need an approximation of $\boldsymbol{\zeta}(s, \boldsymbol{\alpha}; \mathfrak{A})$ and $\boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A})$ by $\boldsymbol{\zeta}_n(s, \boldsymbol{\alpha}; \mathfrak{A})$ and $\boldsymbol{\zeta}_n(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A})$, respectively. For this, we will use a metric on $H^r(D)$ which induces its topology. First we define a metric on $H(D)$. For $g_1, g_2 \in H(D)$, we set

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup\limits_{s \in K_l} \left| g_1(s) - g_2(s) \right|}{1 + \sup\limits_{s \in K_l} \left| g_1(s) - g_2(s) \right|},$$

where $\{K_l : l \in \mathbb{N}\}$ is a sequence of compact subsets of D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some l. It is not difficult to see that the metric ρ induces on $H(D)$ the topology of uniform convergence on compacta.

Now, for $\boldsymbol{g}_1 = (g_{11}, \dots, g_{1r}), \boldsymbol{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D)$, putting

$$\boldsymbol{\rho}(\boldsymbol{g}_1, \boldsymbol{g}_2) = \max_{1 \le j \le j} \rho(g_{1j}, g_{2j}),$$

we have the metric on $H^r(D)$.

Lemma 1.7. *The equality*

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \boldsymbol{\rho}(\boldsymbol{\zeta}(s + i\tau, \boldsymbol{\alpha}; \mathfrak{A}), \boldsymbol{\zeta}_n(s + i\tau, \boldsymbol{\alpha}; \mathfrak{A})) d\tau = 0$$

holds.

A proof does not depend on arithmetical nature of the numbers $\alpha_1, \dots, \alpha_r$ and is given in [8].

Lemma 1.8. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linear independent over $\mathbb{Q}$. Then, for almost all $\boldsymbol{\omega} \in \Omega^r$, the equality*

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \boldsymbol{\rho}(\boldsymbol{\zeta}(s + i\tau, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A}), \boldsymbol{\zeta}_n(s + i\tau, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A})) d\tau = 0$$

holds.

Proof. Let $a_\tau = \{(m + \alpha)^{-i\tau} : m \in \mathbb{N}_0\}, \quad \tau \in \mathbb{R}, \quad 0 < \alpha \le 1$, and define $\varphi_\tau : \Omega \to \Omega$ by $\varphi_\tau(\omega) = a_\tau \omega, \quad \omega \in \Omega$. Then $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on Ω. If $L(\alpha)$ is linear independent over $\mathbb{Q}$, there it follows, see, for example, [13], that the group $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is ergodic. Since the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, each set $L(\alpha_j)$, $j = 1, \dots, r$, is as well. Combining this with the classical Birkhoff-Khintchine ergodic theorem, we obtain in a standard way that, for every compact subset K of D, the equality

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} \left|\zeta(s+i\tau, \alpha_j, \omega_j; \mathfrak{A}_j) - \zeta_n(s+i\tau, \alpha_j, \omega_j; \mathfrak{A}_j)\right| d\tau = 0$$

holds for almost all $\omega_j \in \Omega$, $j = 1, \dots, r$. This, for almost all $\omega_j \in \Omega_j$, implies the equality

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho(\zeta(s+i\tau, \alpha_j, \omega_j; \mathfrak{A}_j), \zeta_n(s+i\tau, \alpha_j, \omega_j; \mathfrak{A}_j)) d\tau = 0$$

which together with the definition of $\boldsymbol{\rho}$ implies the assertion of the lemma.

For the proof of the theorem, we need one more lemma on a common limit measure. Let, for $A \in \mathcal{B}(H^r(D))$,

$$\widehat{P}_T(A) = \nu_T(\boldsymbol{\zeta}(s+i\tau, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{A}}) \in A.$$

Lemma 1.9. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then P_T and $\widehat{P}_T$ for almost all $\underline{\omega} \in \Omega^r$, both converge weakly to the same probability measure P on $(H^r(D), \mathcal{B}(H^r(D)))$ as $T \to \infty$.*

Proof. We take a random variable θ defined on a certain probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$ and uniformly distributed on $[0,1]$. On $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$, define the $H^r(D)$-valued random element $\boldsymbol{X}_{T,n} = \boldsymbol{X}_{T,n}(s, \boldsymbol{\alpha}; \boldsymbol{\mathfrak{A}})$ by the equality

$$\boldsymbol{X}_{T,n}(s, \boldsymbol{\alpha}; \boldsymbol{\mathfrak{A}}) = \boldsymbol{\zeta}_n(s + i\theta\tau, \boldsymbol{\alpha}; \boldsymbol{\mathfrak{A}}).$$

Then, denoting by $\xrightarrow{\mathcal{D}}$ the convergence in distribution, we have by Lemma 6 that

$$\boldsymbol{X}_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} \boldsymbol{X}_n, \tag{1.2}$$

where $\boldsymbol{X}_n$ is the $H^r(D)$-valued random element having the distribution P_n, and P_n is the limit measure in Lemma 6. By a standard method, see, for example, [14], it can be proved that the family of probability measures $\{P_n : n \in \mathbb{N}_0\}$ is tight, i.e., for every $\varepsilon > 0$ there exists a compact subset $H_\varepsilon \subset H^r(D)$ such that

$$P_n(H_\varepsilon) > 1 - \varepsilon$$

for all $n \in \mathbb{N}_0$. Then, by the Prokhorov theorem, see Theorem 6.1 of [3], we have that the family $\{P_n : n \in \mathbb{N}_0\}$ is relatively compact. Therefore, there exists a subsequence

$\{P_{n_k}\} \subset \{P_n\}$ such that P_{n_k} converges weakly to a certain probability measure P on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \to \infty$, so the relation

$$\boldsymbol{X}_n \xrightarrow[k\to\infty]{\mathcal{D}} P \tag{1.3}$$

holds.

On $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$, define one more $H^r(D)$-valued random element $\boldsymbol{X}_T = \boldsymbol{X}_T(s, \boldsymbol{\alpha}; \mathfrak{A})$ by the equality

$$\boldsymbol{X}_T(s, \boldsymbol{\alpha}; \mathfrak{A}) = \boldsymbol{\zeta}(s + i\theta\tau, \boldsymbol{\alpha}; \mathfrak{A}).$$

Then, for every $\varepsilon > 0$, Lemma 7 implies that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}(\boldsymbol{\rho}(\boldsymbol{X}_T(s, \boldsymbol{\alpha}; \mathfrak{A}), \boldsymbol{X}_{T,n}(s, \boldsymbol{\alpha}; \mathfrak{A})) \geq \varepsilon) =$$
$$\lim_{n\to\infty} \limsup_{T\to\infty} \nu_T(\boldsymbol{\rho}(\boldsymbol{\zeta}(s + i\tau, \boldsymbol{\alpha}; \mathfrak{A}), \boldsymbol{\zeta}_n(s + i\tau, \boldsymbol{\alpha}; \mathfrak{A})) \geq \varepsilon) \leq$$
$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T \boldsymbol{\rho}(\boldsymbol{\zeta}(s + i\tau, \boldsymbol{\alpha}; \mathfrak{A}), \boldsymbol{\zeta}_n(s + i\tau, \boldsymbol{\alpha}; \mathfrak{A})) d\tau = 0.$$

This, and relations (1.2) and (1.3) together with Theorem 4.2 of [3] lead to

$$\boldsymbol{X}_T \xrightarrow[T\to\infty]{\mathcal{D}} P, \tag{1.4}$$

and this is equivalent to the weak convergence of P_T to P as $T \to \infty$. Moreover, relation (1.4) shows that the probability measure P does not depend on the sequence $\{P_{n_k}\}$. Hence, taking into account the relative compactness of the family $\{P_n\}$, we obtain that

$$\boldsymbol{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{1.5}$$

It remains to shows, that $\widehat{P}_T$ also converges weakly to the same measure P as $T \to \infty$. For this, we define the random elements

$$\boldsymbol{X}_{T,n}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A}) = \boldsymbol{\zeta}_n(s + i\theta T, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A})$$

and

$$\boldsymbol{X}_T(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A}) = \boldsymbol{\zeta}(s + i\theta T, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A}).$$

Then, using (1.5) and repeating the above arguments for the elements $X_{T,n}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A})$ and $X_T(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A})$ with application of Lemma 8, we obtain the weak convergence of $\widehat{P}_T$ to P as $T \to \infty$.

Proof of Theorem 1.4. Let, for $\tau \in \mathbb{R}$,

$$\boldsymbol{a}_\tau = \{((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, ((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0)\},$$

and, on the torus Ω^r, define a family of transformations $\{\Phi_\tau : \tau \in \mathbb{R}\}$ by formula $\Phi_\tau(\boldsymbol{\omega}) = \boldsymbol{a}_\tau \boldsymbol{\omega}$, $\boldsymbol{\omega} \in \Omega^r$. Then $\{\Phi_\tau : \tau \in \mathbb{R}\}$ is a one - parameter group of measurable measure preserving transformations on Ω^r. Since the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, by Lemma 3 of [13], the group $\{\Phi_\tau : \tau \in \mathbb{R}\}$ is ergodic.

Now we fix a continuity set A of the measure P in Lemma 1.9. Then by Theorem 2.1 of [3] we find that

$$\lim_{T\to\infty} \nu_T(\boldsymbol{\zeta}(s+i\tau, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A}) \in A) = P(A). \tag{1.6}$$

Let ξ be a random variable on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$ given by

$$\xi = \xi(\boldsymbol{\omega}) = \begin{cases} 1 & \text{if } \boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A}) \in A, \\ 0 & \text{if } \boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A}) \notin A. \end{cases}$$

Then the ergodicity of $\{\Phi_\tau : \tau \in \mathbb{R}\}$ shows that the random process $\xi(\Phi_\tau(\boldsymbol{\omega}))$ is also ergodic. Therefore, the Birkhoff-Khinchine theorem mentioned already in the proof of Lemma 8, for almost all $\boldsymbol{\omega} \in \Omega^r$, implies the equality

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \xi(\Phi_\tau(\boldsymbol{\omega})) d\tau = \mathbb{E}\xi, \tag{1.7}$$

where $\mathbb{E}\xi$ denotes the expectation of ξ. On the other hand, the definition of ζ shows that

$$\mathbb{E}\xi = \int_{\Omega^r} \xi dm_H^r = m_H^r(\boldsymbol{\omega} \in \Omega^r : \boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A}) \in A),$$

that is

$$\mathbb{E}\xi = P_{\boldsymbol{\zeta}}(A). \tag{1.8}$$

Since, by the definitions of ξ and Φ_τ,

$$\frac{1}{T} \int_0^T \xi(\Phi_\tau(\boldsymbol{\omega})) d\tau = \nu_T(\boldsymbol{\zeta}(s+i\tau, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A}) \in A),$$

we see from relations (1.7) and (1.8) that, for almost all $\boldsymbol{\omega} \in \Omega^r$,

$$\lim_{T\to\infty} \nu_T(\boldsymbol{\zeta}(s+i\tau, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A}) \in A) = P_{\boldsymbol{\zeta}}(A).$$

This together with (1.6) shows that $P(A) = P_{\boldsymbol{\zeta}}(A)$ for continuity sets A of the measure. However, all continuity sets constitute a determining class. Thus, the measures P and $P_{\boldsymbol{\zeta}}$ coincide for all $A \in \mathcal{B}(H^r(D))$, and the theorem is proved.

1.3. Support of the measure $P_{\boldsymbol{\zeta}}$

Denote the support of the measure $P_{\boldsymbol{\zeta}}$ by $S_{P_{\boldsymbol{\zeta}}}$. Thus, $S_{P_{\boldsymbol{\zeta}}}$ is a minimal closed set of the space $H^r(D)$ such that $P_{\boldsymbol{\zeta}}(S_{P_{\boldsymbol{\zeta}}}) = 1$. The support $S_{P_{\boldsymbol{\zeta}}}$ consist of all points $\boldsymbol{g} \in H^r(D)$ such that $P_{\boldsymbol{\zeta}}(G) > 0$ for every neighbourhood G of $\boldsymbol{g}$.

Theorem 1.10. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then the support of the measures $P_{\boldsymbol{\zeta}}$ is the whole of $H^r(D)$.*

Proof. Let, for $A_j \in H(D), \quad j = 1, \dots, r,$

$$A = A_1 \times \cdots \times A_r. \tag{1.9}$$

Since the space $H^r(D)$ is separable, $\mathcal{B}(H^r(D))$ coincides with σ-field generated by sets (1.9). Moreover, the Haar measure m_H^r is the product of $m_{1H}, \dots, m_{rH}$. Therefore,

$$\begin{aligned} &P_{\boldsymbol{\zeta}}(A) = m_H^r(\boldsymbol{\omega} \in \Omega^r : \boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{A}}) \in A) = \\ &m_H^r(\boldsymbol{\omega} \in \Omega^r : \boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{A}}) \in A_1 \times \cdots \times A_r) = \\ &m_{1H}(\omega_1 \in \Omega_1 : \zeta(s, \alpha_1, \omega_1; \mathfrak{A}_1) \in A_1) \dots \\ &m_{rH}(\omega_r \in \Omega_r : \zeta(s, \alpha_r, \omega_r; \mathfrak{A}_r) \in A_r). \end{aligned} \tag{1.10}$$

Since the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, each set $L(\alpha_1), \dots, L(\alpha_r)$ is as well. Therefore, as in the case of transcendental α_j [7], we have that, for every $j = 1, \dots, r$,

$$\nu_T(\zeta(s, \alpha_j; \mathfrak{A}_j) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{j\zeta}$ as $T \to \infty$, where

$$P_{j\zeta}(A) = m_{jH}(\omega_j \in \Omega_j : \zeta(s, \alpha_j, \omega_j; \mathfrak{A}_j) \in A), \quad A \in \mathcal{B}(H(D)),$$

is the distribution of the $H(D)$-valued random element $\zeta(s, \alpha_j, \omega_j; \mathfrak{A}_j)$, and the support of $P_{j\zeta}$ is the whole of $H(D)$. This and equality (1.10) prove the theorem.

1.4. Proof of Theorem 1.3

In view of Theorem 1.4 and 1.10, Theorem 1.3 is proved by a standard way. The used arguments are based on the Mergelyan theorem on approximation of analytic functions by polynomials, see, for example, [20], and on the properties of weak convergence of probability measures as well as on those of a support.

Proof of Theorem 1.3. By the Mergelyan theorem, there exist polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1 \le j \le r} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{2}. \tag{1.11}$$

Let

$$G = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \le j \le r} \sup_{s \in K_j} |g_j(s) - p_j(s)| < \frac{\varepsilon}{2} \right\}.$$

Clearly, G is an open set. Moreover, in view of Theorem 10, $(p_1(s), \dots, p_r(s)) \in S_{P_{\boldsymbol{\zeta}}}$. Therefore, by properties of a support, the inequality $P_{\boldsymbol{\zeta}}(G) > 0$ holds. Since Theorem 1.4 and Theorem 2.1 of [3] imply

$$\liminf_{T \to \infty} P_T(G) \ge P_{\boldsymbol{\zeta}}(G),$$

hence we deduce that

$$\liminf_{T \to \infty} \nu_T \left(\sup_{1 \le j \le r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j; \mathfrak{A}_j) - p_j(s)| < \frac{\varepsilon}{2} \right) > 0. \tag{1.12}$$

However, inequality (1.11) shows that the inequality

$$\sup_{1 \le j \le r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j; \mathfrak{A}_j) - p_j(s)| < \frac{\varepsilon}{2}$$

implies

$$\sup_{1 \le j \le r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j; \mathfrak{A}_j) - f_j(s)| < \varepsilon.$$

Therefore,

$$\left\{ \tau \in [0, T] : \sup_{1 \le j \le r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j; \mathfrak{A}_j) - f_j(s)| < \varepsilon \right\} \supseteq$$

$$\left\{ \tau \in [0, T] : \sup_{1 \le j \le r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j; \mathfrak{A}_j) - p_j(s)| < \frac{\varepsilon}{2} \right\},$$

and this together with inequality (1.12) proves the theorem.

Chapter 2

Joint universality for periodic Hurwitz zeta-functions. II

2.1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of all positive integers, non-negative integers, integers, real and complex numbers, respectively, and let $\mathfrak{A} = \{a_m : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$, and α, $0 < \alpha \leq 1$, be a fixed parameter. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{A})$, $s = \sigma + it$, is defined, for $\sigma > 1$, by Dirichlet series

$$\zeta(s, \alpha, \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$

For $\mathfrak{A} = \{a_m = 1 : m \in \mathbb{N}_0\} \stackrel{def}{=} \mathbf{1}$, the function $\zeta(s, \alpha, \mathfrak{A})$ reduces to the classical Hurwitz zeta-function $\zeta(s, \alpha)$. In view of the periodicity of the sequence $\mathfrak{A}$, we have that, for $\sigma > 1$,

$$\zeta(s, \alpha; \mathfrak{A}) = \frac{1}{k^s} \sum_{l=0}^{k-1} a_l \zeta\left(s, \frac{l+\alpha}{k}\right). \tag{2.1}$$

Since the function $\zeta(s, \alpha)$ has meromorphic continuation to the whole complex plane and has a simple pole at $s = 1$ with $\operatorname*{Res}_{s=1} \zeta(s, \alpha) = 1$, equality (2.1) shows that the periodic Hurwitz zeta-function $\zeta(s, \alpha, \mathfrak{A})$ can be meromorphically continued to the whole complex plane with possible simple pole at $s = 1$. If

$$a \stackrel{def}{=} \frac{1}{k} \sum_{l=0}^{k-1} a_l = 0,$$

then $\zeta(s, \alpha, \mathfrak{A})$ is an entire function, while if $a \neq 0$, then $\operatorname*{Res}_{s=1} \zeta(s, \alpha; \mathfrak{A}) = a$.

In [6] and [7], the universality of the function $\zeta(s, \alpha, \mathfrak{A})$ in the Voronin sense was considered. For brevity, we will use the following notation. Let meas $\{A\}$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and, for $T > 0$, we put

$$\nu_T(\dots) = \frac{1}{T} \text{meas}\{\tau \in [0; T] : \dots\},$$

where in place of dots a condition satisfied by τ is to be written. We also denote $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$.

Theorem 2.1 [7]. *Suppose that the number α is transcendental. Let K be a compact subset of the strop D with connected complement, and let $f(s)$ be a continuous on K function which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T \left(\sup_{s\in K} |\zeta(s+i\tau, \alpha; \mathfrak{A}) - f(s)| < \varepsilon \right) > 0.$$

We recall that the universality of the Riemann zeta-function $\zeta(s) = \zeta(s, 1; \mathbf{1})$ was discovered by Voronin in [19]. The improved version of his theorem can be found in [9],[10], [15] and [16].

Voronin also obtained [18] the first joint universality theorem for Dirichlet L-functions. A similar result was also given by Gonek [5] and Bagchi [1], [2]. A modern version of that theorem is contained in [17].

In [11], the first author began to study the joint universality of periodic Hurwitz zeta-functions. For $j = 1, \dots, r$, $r > 1$, let $\mathfrak{A}_j = \{a_{mj} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k_j = \mathbb{N}$, $\alpha_j \in \mathbb{R}$, $0 < \alpha_j \leq 1$, and, for $\sigma > 1$,

$$\zeta(s, \alpha_j; \mathfrak{A}_j) = \sum_{m=0}^{\infty} \frac{a_{mj}}{(m+\alpha_j)^s}.$$

In [11], the simplest case $k_j = k$ and $\alpha_j = \alpha$, $j = 1, \dots, r$, was considered. The requirement that $k_j = k$, $j = 1, \dots, r$, was removed in [12]. Let k be the least common multiple of the periods $k_1, \dots, k_r$, and

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & a_{kr} \end{pmatrix}.$$

Then in [8], the following statement was obtained.

Theorem 2.2 [8]. *Suppose that the number $\alpha_1, \dots, \alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$, and that $rank(A) = r$. For every $j = 1, \dots, r$, let K_j be a compact subset of the strip D with connected complement, and let $f_j(s)$ be a continuous on K_j function which is analytic in the interior of K_j. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T \left(\sup_{1\leq j\leq r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j; \mathfrak{A}_j) - f_j(s)| < \varepsilon \right) > 0.$$

The rank condition of Theorem 2.2 was removed in [15].

A more general case was considered in [13]. Let l_j, $j = 1, \dots, r$, be positive integers. For every $l = 1, \dots, l_j$, let $\mathfrak{A}_{jl} = \{a_{mjl} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k_{jl} \in \mathbb{N}$. Suppose that $\alpha_j \in \mathbb{R}$, $0 < \alpha_j \le 1$, and, for $\sigma > 1$,

$$\zeta(s, \alpha_j; \mathfrak{A}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}}{(m+\alpha_j)^s}.$$

Denote by k the least common multiple of $k_{11}, \dots, k_{1l_1}, \dots, k_{r1}, \dots, k_{rl_r}$, and define

$$B = \begin{pmatrix} a_{111} & a_{112} & \dots & a_{11l_1} & a_{122} & \dots & a_{12l_2} & \dots & a_{1r1} & a_{1r2} & \dots & a_{1rl_r} \\ a_{211} & a_{212} & \dots & a_{21l_1} & a_{222} & \dots & a_{22l_2} & \dots & a_{2r1} & a_{2r2} & \dots & a_{2rl_r} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{k11} & a_{k12} & \dots & a_{k1l_1} & a_{k22} & \dots & a_{k2l_2} & \dots & a_{kr1} & a_{kr2} & \dots & a_{krl_r} \end{pmatrix}.$$

Moreover, let

$$\kappa = \sum_{j=1}^{r} l_j,$$

and

$$L(\alpha_1, \dots, \alpha_r) = \{\log(m+\alpha_j) : m \in \mathbb{N}_0, j = 1, \dots, r\}.$$

Then we have the following result.

Theorem 2.3 [13]. *Suppose that the system $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, and that $\operatorname{rank}(B) = \kappa$. For every $j = 1, \dots, r$ and $l = 1, \dots, l_j$, let K_{jl} be a compact subset of the strip D with connected complement, and let $f_{jl}(s)$ be a continuous on K_{jl} function which is analytic in the interior of K_{jl}. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T \left(\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau, \alpha_j; \mathfrak{A}_{jl}) - f_{jl}(s)| < \varepsilon \right) > 0.$$

The aim of this note is to make weaker the rank condition in Theorem 3. Let k_j be the least common multiple of the periods $k_{j1}, k_{j2}, \dots, k_{jl_j}$, $j = 1, \dots, r$, and define

$$B_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \dots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \dots & a_{2jl_j} \\ \dots & \dots & \dots & \dots \\ a_{kj1} & a_{kj2} & \dots & a_{kjl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

Theorem 2.4. *Suppose that the system $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, and that $rank(B_j) = l_j$, $j = 1, \dots, r$. Let K_{jl} and f_{jl} be the same as in Theorem 3. Then the assertion of Theorem 2.3 is true.*

We note that if the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, then the system $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. In the case $r = 1$, the linear independence of $L(\alpha_1)$ follows from the transcendence of α_1.

2.2. A limit theorem

The proof of Theorem 2.4 is based on the method of limit theorems in the space of analytic functions proposed by Bagchi in [1], and later developed by Kohji Matsumoto, Mishou, Nagoshi, Nakamura, Steuding, the first author, and others.

Denote by $H(D)$ the space of analytic on D functions endowed with the topology of uniform convergence on compacta, and let

$$H^\kappa(D) = \underbrace{H(D) \times \cdots \times H(D)}_{\kappa},$$

where κ is defined in Introduction. Moreover, we define

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \{s \in \mathbb{C} : |s| = 1\}$ for all $m \in \mathbb{N}_0$. By the Tikhonow theorem, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, denoting by $\mathcal{B}(S)$ the class of Borel sets of a space S, we have that on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H exists, and this leads to the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Putting

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$, by the Tikhonov theorem again we have that Ω^r is also a compact topological Abelian group. Similarly as above, we obtain the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$, where m_H^r is the probability Haar measure on $(\Omega^r, \mathcal{B}(\Omega^r))$.

Let, for brevity $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\boldsymbol{\mathfrak{A}} = (\mathfrak{A}_{11}, \dots, \mathfrak{A}_{1l_1}, \dots, \mathfrak{A}_{r1}, \dots, \mathfrak{A}_{rl_r})$, denote the elements of Ω^r by $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$, and on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$ define the $H^\kappa(D)$-valued random element $\boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{A}})$ by

$$\boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{A}}) = (\zeta(s, \alpha_1, \omega_1; \mathfrak{A}_1), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{A}_{1l_1}), \dots,$$

$$\zeta(s,\alpha_r,\omega_r;\mathfrak{A}_{r1}),\dots,\zeta(s,\alpha_r,\omega_r;\mathfrak{A}_{rl_r})),$$

where

$$\zeta(s,\alpha_j,\omega_j;\mathfrak{A}_{jl})=\sum_{m=0}^{\infty}\frac{a_{mjl}\omega_j(m)}{(m+\alpha_j)^s},\qquad j=1,\dots,r,\qquad l=1,\dots,l_j,$$

and $\omega_j(m)$ is the projection of $\omega_j\in\Omega_j$, $j=1,\dots,r$, to the coordinate space γ_m, $m\in\mathbb{N}_0$. Denote by $P_{\boldsymbol{\zeta}}$ the distribution of the random element $\boldsymbol{\zeta}(s,\boldsymbol{\alpha},\boldsymbol{\omega};\boldsymbol{\mathfrak{A}})$.

Let, for $A\in\mathcal{B}(H^\kappa(D))$,

$$P_T(A)=\nu_T(\boldsymbol{\zeta}(s+i\tau,\boldsymbol{\alpha};\boldsymbol{\mathfrak{A}})\in A).$$

Theorem 2.5. *Suppose that the system $L(\alpha_1,\dots,\alpha_r)$ is linearly independent over $\mathbb{Q}$. Then the probability measure P_T converges weakly to $P_{\boldsymbol{\zeta}}$ as $T\to\infty$.*

The proof of Theorem does not depend of the hypothesis on the rank of the matrices B_j, therefore, it remains the same as in [13], and we will recall only the principal steps of that proof.

First it is proved that the probability measure

$$\begin{aligned}Q_T(A)\ &\stackrel{def}{=}\ \nu_T((((m+\alpha_1)^{-it}:m\in\mathbb{N}_0),\dots,((m+\alpha_r)^{-it}:m\in\mathbb{N}_0))\in A),\\ &A\in\mathcal{B}(\Omega^r),\end{aligned}$$

converges weakly to the Haar measure m_H^r as $T\to\infty$. For this, the method of Fourier transforms is applied in which the hypothesis on the linear independence of the system $L(\alpha_1,\dots,\alpha_r)$ is essentially used. A limit theorem for Q_T together with the well-known properties of weak convergence of probability measures leads to limit theorems for vectors whose components are absolutely convergent Dirichlet series.

Let, for a fixed $\sigma_1>\frac{1}{2}$, $m,n\in\mathbb{N}_0$, and $0<\alpha\le 1$,

$$v_n(m,\alpha)=\exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\sigma_1}\right\}.$$

Then it is not difficult to see that the series

$$\zeta_n(s,\alpha_j;\mathfrak{A}_{jl})=\sum_{m=0}^{\infty}\frac{a_{mjl}v_n(m,\alpha_j)}{(m+\alpha_j)^s}$$

and

$$\zeta_n(s,\alpha_j,\omega_j;\mathfrak{A}_{jl})=\sum_{m=0}^{\infty}\frac{a_{mjl}\omega_j(m)v_n(m,\alpha_j)}{(m+\alpha_j)^s}$$

are absolutely convergent for $\sigma > \frac{1}{2}$, $j = 1, \dots, r$, $l = 1, \dots, l_j$. Define

$$\boldsymbol{\zeta}_n(s, \boldsymbol{\alpha}; \mathfrak{A}) = (\zeta_n(s, \alpha_1; \mathfrak{A}_{11}), \dots, \zeta_n(s, \alpha_1; \mathfrak{A}_{1l_1}), \dots,$$

$$\zeta_n(s, \alpha_r; \mathfrak{A}_{r1}), \dots, \zeta_n(s, \alpha_r; \mathfrak{A}_{rl_r})),$$

and

$$\boldsymbol{\zeta}_n(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A}) = (\zeta_n(s, \alpha_1, \omega_1; \mathfrak{A}_{11}), \dots, \zeta_n(s, \alpha_1, \omega_1; \mathfrak{A}_{1l_1}), \dots,$$

$$\zeta_n(s, \alpha_r, \omega_r; \mathfrak{A}_{rl_1}), \dots, \zeta_n(s, \alpha_r, \omega_r; \mathfrak{A}_{rl_r})).$$

Then the next step of the proof consists of the observation that the probability measures

$$\nu_T(\boldsymbol{\zeta}_n(s, \boldsymbol{\alpha}; \mathfrak{A}) \in A), \quad A \in \mathcal{B}(H^\kappa(D)), \tag{2.2}$$

and

$$\nu_T(\boldsymbol{\zeta}_n(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A}) \in A), \quad A \in \mathcal{B}(H^\kappa(D)), \tag{2.3}$$

both converge weakly simultaneously to the same probability measure on $(H^\kappa(D), \mathcal{B}(H^\kappa(D)))$ as $T \to \infty$.

Now it remains to pass from the vector $\boldsymbol{\zeta}_n(s, \boldsymbol{\alpha}; \mathfrak{A})$ to $\boldsymbol{\zeta}(s, \boldsymbol{\alpha}; \mathfrak{A})$. For this, an approximation in the mean of $\boldsymbol{\zeta}(s, \boldsymbol{\alpha}; \mathfrak{A})$ by $\boldsymbol{\zeta}_n(s, \boldsymbol{\alpha}; \mathfrak{A})$ as well as of $\boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A})$ by $\boldsymbol{\zeta}_n(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \mathfrak{A})$ is needed. First we define a metric on the space $H^\kappa(D)$ which induces the topology of uniform convergence on compacta. Let $\{K_m : m \in \mathbb{N}\}$ be a sequence of compacta subsets of D such that

$$\bigcup_{m=1}^{\infty} K_m = D,$$

$K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$, and, for every compact $K \subset D$, $K \subset K_m$ for some m. For the existence of the sequence $\{K_m\}$, see, for example, [4]. Let, for $f, g \in H(D)$,

$$\rho(f, g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup_{s \in K_m} |f(s) - g(s)|}{1 + \sup_{s \in K_m} |f(s) - g(s)|}.$$

Then ρ is a metric on $H(D)$ which induces its topology of uniform convergence on compacta. Now if $\boldsymbol{f} = (f_{11}, \dots, f_{1l_1}, f_{r1}, \dots, f_{rl_r})$, $\boldsymbol{g} = (g_{11}, \dots, g_{1l_1}, g_{r1}, \dots, g_{rl_r}) \in H^\kappa(D)$ and

$$\rho_\kappa(\boldsymbol{f}, \boldsymbol{g}) = \max_{1 \le j \le r} \max_{1 \le j \le l_j} \rho(f_{jl}, g_{jl}),$$

then ρ_κ is a desired metric on $H^\kappa(D)$.

Furthermore, it is proved that

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho_\kappa(\boldsymbol{\zeta}(s + i\tau, \boldsymbol{\alpha}; \mathfrak{A}) - \boldsymbol{\zeta}_n(s + i\tau, \boldsymbol{\alpha}; \mathfrak{A})) d\tau = 0. \tag{2.4}$$

For this, the mean square estimate [6]

$$\int_0^T |\zeta(\sigma+it,\alpha;\mathfrak{A})|^2\,dt \ll T, \quad \frac{1}{2}<\sigma<1, \tag{2.5}$$

is applied.

A more complicated situation is in the case of $\boldsymbol{\zeta}(s,\boldsymbol{\alpha},\boldsymbol{\omega};\boldsymbol{\mathfrak{A}})$, because we have to obtain an analogue of estimate (2.5) for $\zeta(\sigma+it,\alpha,\omega;\mathfrak{A})$. To get such an analogue, elements of ergodic theory are applied. Let $\boldsymbol{a}_{\tau_k}=\{((m+\alpha_1)^{-i\tau}:m\in\mathbb{N}_0),\ldots,((m+\alpha_r)^{-i\tau}:m\in\mathbb{N}_0)\}$, $\tau\in\mathbb{R}$, and define a family of transformations $\{\Phi_\tau:\tau\in\mathbb{R}\}$ on Ω^r by $\Phi_\tau(\boldsymbol{\omega})=\boldsymbol{a}_\tau\boldsymbol{\omega}$, $\boldsymbol{\omega}\in\Omega^r$. Then $\{\Phi_\tau:\tau\in\mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on Ω^r. We say that a set $A\in\mathcal{B}(\Omega^r)$ is invariant with respect to the group $\{\Phi_\tau:\tau\in\mathbb{R}\}$ if, for every $\tau\in\mathbb{R}$, the sets A and $A_\tau=\Phi_\tau(A)$ coincide up to a set of m_H^r-measure zero. The invariant sets form a σ-field which is a σ-subfield of $\mathcal{B}(\Omega^r)$.

A one - parameter group $\{\Phi_\tau:\tau\in\mathbb{R}\}$ is ergodic if its σ-field of invariant sets consists only of sets of m_H^r-measure zero or one.

Using the linear independence of the system $L(\alpha_1,\ldots,\alpha_r)$, it is not difficult to prove that the one-parameter group $\{\Phi_\tau:\tau\in\mathbb{R}\}$ is ergodic. This together with the classical Birkhoff-Khintchine theorem, for almost all $\omega\in\Omega$, implies a bound

$$\int_0^T |\zeta(\sigma+it,\alpha,\omega;\mathfrak{A})|^2\,dt \ll T, \quad \sigma>\frac{1}{2}.$$

Now, similarly to equality (2.4), we obtain that, for almost all $\boldsymbol{\omega}\in\Omega^r$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho_\kappa(\boldsymbol{\zeta}(s+i\tau,\boldsymbol{\alpha},\boldsymbol{\omega};\boldsymbol{\mathfrak{A}}),\boldsymbol{\zeta}_n(s+i\tau,\boldsymbol{\alpha},\boldsymbol{\omega};\boldsymbol{\mathfrak{A}}))d\tau=0. \tag{2.6}$$

The weak convergence of the probability measures (2.2) and (2.3) to the same probability measure as $T\to\infty$, relations (2.4) and (2.6), and Theorem 4.2 of [3] allow to prove that the probability measures P_T and

$$\nu_T(\boldsymbol{\zeta}(s+i\tau,\boldsymbol{\alpha},\boldsymbol{\omega};\boldsymbol{\mathfrak{A}})\in A), \quad A\in\mathcal{B}(H^\kappa(D)),$$

both converge weakly to the same probability measure P on $(H^\kappa(D))$, $\mathcal{B}(H^\kappa(D))$ as $T\to\infty$. This, the ergodicity of the group $\{\Phi_\tau:\tau\in\mathbb{R}\}$, and the Birkhoff-Khintchine theorem show that the measure P coincides with $P_{\boldsymbol{\zeta}}$. This completes the proof Theorem 2.4.

2.3. Support of $P_{\boldsymbol{\zeta}}$

For the proof of Theorem 2.3, we need to know the support of the limit measure $P_{\boldsymbol{\zeta}}$ in Theorem 2.4. Let S be a separable metric space, and P be a probability measure on $(S, \mathcal{B}(S))$. We recall that the support of the measure P is a minimal closed set $S_P \in \mathcal{B}(S)$ such that $P(S_P) = 1$. Moreover, the support S_P consists of all $x \in S$ such that $P(G) > 0$ for every neighbourhood G of x.

Theorem 2.6. *Suppose that the system $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then the support of the measure $P_{\boldsymbol{\zeta}}$ is the whole of $H^{\kappa}(D)$.*

Proof. Let, for $A_j \in \mathcal{B}(H^{l_j}(D))$, $j = 1, \dots, r$,

$$A = A_1 \times \dots \times A_r. \tag{2.7}$$

Since the space $H^{\kappa}(D)$ is separable, we have [3] that the σ-field $\mathcal{B}(H^{\kappa}(D))$ coincides with

$$\mathcal{B}(H^1(D)) \times \dots \times \mathcal{B}(H^{l_r}(D)),$$

that is, it coincides with a σ-field generated by the sets (2.7). We also note that the measure m_H^r is the product of the measures m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \dots, r$. Therefore, we have that

$$\begin{aligned} &P_{\boldsymbol{\zeta}}(A) = m_H^r(\boldsymbol{\omega} \in \Omega^r : \boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{A}}) \in A) = \\ &m_{1H}(\omega_1 \in \Omega_1 : (\zeta(s, \alpha_1, \omega_1; \mathfrak{A}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{A}_{1l_1})) \in A_1) \times \dots \\ &\times m_{rH}(\omega_r \in \Omega_r : \zeta(s, \alpha_r, \omega_r; \mathfrak{A}_{r1}) \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{A}_{rl_r}) \in A_r). \end{aligned} \tag{2.8}$$

Since $rank(B_j) = l_j$, $j = 1, \dots, r$, in view of Lemma 11 from [12] we have that the support of the measure

$$P_{j\boldsymbol{\zeta}}(A_j) = m_{jH}(\omega_j \in \Omega_j : (\zeta(s, \alpha_j, \omega_j; \mathfrak{A}_{j1}), \dots, \zeta(s, \alpha_j, \omega_j; \mathfrak{A}_{jl_j})) \in A_j)$$

is the whole of $H(D)$, $j = 1, \dots, r$. Therefore, the theorem is a consequence of equality (2.8).

Remark. In [12], it was assumed that α is transcendental, however, the proofs of all statements remains the same if the system $L(\alpha)$ is linearly independent over $\mathbb{Q}$, because the transcendence of α is used only for the linear independence of $L(\alpha)$.

2.4. Proof of Theorem 2.4

A proof of Theorem 2.4 is short and standard. By the Mergelyan theorem, see, for example, [20], there exist polynomials $p_{jl}(s)$ such that

$$\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|f_{jl}(s)-p_{jl}(s)|<\frac{\varepsilon}{2}. \tag{2.9}$$

Let

$$G=\Bigg\{(g_{11},\dots,g_{1l_1},\dots,g_{r1},\dots,g_{rl_r})\in H^r(D):$$
$$\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|p_{jl}(s)-g_{jl}(s)|<\frac{\varepsilon}{2}\Bigg\}.$$

The set G is open in $H^\kappa(D)$, and, by Theorem 2.6, $(p_{11},\dots,p_{1l_1},\dots,p_{r1},\dots,p_{rl_r})\in S_{P_{\boldsymbol{\zeta}}}$. Therefore, Theorem 2.5 and Theorem 2.1 of [3] show that

$$\liminf_{T\to\infty}\nu_T\left(\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|\zeta(s+i\tau,\alpha_j;\mathfrak{A}_{jl})-p_{jl}(s)|<\frac{\varepsilon}{2}\right)\ge P_{\boldsymbol{\zeta}}(G)>0.$$

This and inequality (2.9) complete the proof of Theorem 2.4.

Santrauka

Periodinių Hurvico dzeta funkcijų jungtinis universalumas

Tarkime, kad $s = \sigma + it$ yra kompleksinis kintamasis, α, $0 < \alpha \leq 1$, yra fiksuotas parametras, o $\mathfrak{A} = \{a_m : m \in \mathbb{N}_0\}$ yra periodinų kompleksinių skaičių seka. Periodinė Hurvico dzeta funkcija $\zeta(s, \alpha; \mathfrak{A})$ pusplokštumėje $\sigma > 1$ yra apibrėžiama Dirichlė eilute ir yra analiziškai pratęsiama į visą kompleksinę plokštumą, išskyrus, gal būt, tašką $s = 1$.

Magistro darbe yra nagrinėjamas Hurvico dzeta funkcijų rinkinio jungtinis universalumas. Tarkime, kad K_j yra juostos $\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ kompaktinė aibė, turinti jungųjį papildinį, o $f_j(s)$ yra tolydi aibėje K_j funkcija ir analizinė aibės K_j viduje, $j = 1, \ldots, r$. Sakome, kad funkcijos $\zeta(s, \alpha_1; \mathfrak{A}_1), \ldots, \zeta(s, \alpha_r; \mathfrak{A}_r)$ yra universalios, jei su kiekvienu $\varepsilon > 0$

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j; \mathfrak{A}_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Čia meas$A$ yra mačios aibės $A \subset \mathbb{R}$ Lebego matas.

Darbe yra įrodytos dvi jungtinės universalumo teoremos. Pirmoji teorema tvirtina, kad jei aibė $L(\alpha_1, \ldots, \alpha_r) = \{log(m+\alpha_j) : j = 1, \ldots, r, \quad m \in \mathbb{N}_0\}$ yra tiesiškai nepsiklausoma virš racionaliųjų skaičių kūno, tai funkcijos $\zeta(s+i\tau, \alpha_j; \mathfrak{A}_j), \ldots, \zeta(s+i\tau, \alpha_r; \mathfrak{A}_r)$ yra universalios. Ši teorema žymiai susilpnina sąlygas, kurioms esant, buvo gautas analogiškas rezultatas A. Javtoko ir A. Laurinčiko 2008 m. darbe.

Antroje teoremoje yra nagrinėjamas atvejis, kai kiekvieną skaičių α_j atitinka periodinių sekų rinkinys. Kai sistema $L(\alpha_1, \ldots, \alpha_r)$ yra tiesiškai nepriklausoma virš racionaliųjų skaičių kūno ir galioja vieno rango tipo sąlyga, silpnesnė negu A. Laurinčiko darbe (2008), tai funkcijų rinkinys $\zeta(s, \alpha_j; \mathfrak{A}_{jl})$, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$ yra taip pat universalus.

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