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A discrete limit theorem for the periodic Hurwitz zeta-function. II

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Abstract. In the paper, we prove a limit theorem of discrete type on the weak convergence of probability measures on the complex plane for the periodic Hurwitz zeta-function.

Keywords: Hurwitz zeta-function,limit theorem, probability measure, weak convergence.

The periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$, $s=\sigma+it$, where α, $0<\alpha\leqslant 1$, is a fixed parameter, and $\mathfrak{a}=\{a_m : m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\}$ is a periodic sequence of complex numbers, is defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s,\alpha;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m}{(m+\alpha)^s}.$$

Moreover, the function $\zeta(s,\alpha;\mathfrak{a})$ is meromorphically continued to the whole complex plane with unique possible simple pole at the point $s=1$. In a series of works [5, 6, 1, 4], limit theorems for the function $\zeta(s,\alpha;\mathfrak{a})$ were obtained on the weak convergence, for $\sigma>\frac{1}{2}$, of

$$\frac{1}{T}\mathrm{meas}\bigl\{\tau\in[0,T] : \zeta(\sigma+i\tau,\alpha;\mathfrak{a})\in A\bigr\},\quad A\in\mathcal{B}(\mathbb{C}), \tag{1}$$

as $T\to\infty$, where $\mathcal{B}(X)$ denotes the Borel σ-field of the space X, and $\mathrm{meas}A$ is the Lebesgue measure of a measurable set $A\subset\mathbb{R}$.

The limit measures depend on the arithmetical nature of the parameter α. The cases of rational and transcendental α are completely investigated, while in the case of algebraic irrational α only certain conditional results are obtained. Limit theorems on the weak convergence of the measure (1) are of continuous character because the shift τ can take arbitrary real values. In [7], a discrete limit theorem for the function $\zeta(s,\alpha;\mathfrak{a})$, for $\sigma>\frac{1}{2}$, on the weak convergence of

$$\frac{1}{N+1}\#\bigl\{0\leqslant k\leqslant N : \zeta(\sigma+ikh,\alpha;\mathfrak{a})\in A\bigr\},\quad A\in\mathcal{B}(\mathbb{C}), \tag{2}$$

as $N\to\infty$, has been obtained with $h>0$ provided the set

$$\left\{\bigl(\log(m+\alpha) : m\in\mathbb{N}_0\bigr),\frac{\pi}{h}\right\}$$

is linearly independent over the field of rational numbers $\mathbb{Q}$. Here $\#A$ denotes the number of elements of the set A. For example, $\alpha = \frac{1}{\pi}$ and rational h can be taken.

The aim of this paper is to replace the set $\{kh : k \in \mathbb{N}_0\}$ by a more complicated set $\{k^\beta h : k \in \mathbb{N}_0\}$ with a fixed β, $0 < \beta < 1$. Let

$$L(\alpha) = \big\{ \log(m+\alpha) : m \in \mathbb{N}_0 \big\},$$

and, for $A \in \mathcal{B}(\mathbb{C})$,

$$P_N(A) = \frac{1}{N+1} \#\big\{ 0 \leqslant k \leqslant N : \zeta\big(\sigma + ik^\beta h, \alpha; \mathfrak{a}\big) \in A \big\}.$$

Moreover, let

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \{s \in \mathbb{C} : |s| = 1\}$ for all $m \in \mathbb{N}_0$. The torus Ω is a compact topological Abelian group, therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of an element $\omega \in \Omega$ to the coordinate space γ_m, $m \in \mathbb{N}$, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define, for $\sigma > \frac{1}{2}$, the complex-valued random variable $\zeta(\sigma, \alpha, \omega; \mathfrak{a})$ by the formula

$$\zeta(\sigma, \alpha, \omega; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m+\alpha)^\sigma}.$$

We note that the latter series, for almost all $\omega \in \Omega$, converges for any fixed $\sigma > \frac{1}{2}$. Let P_ζ be the distribution of the random variable $\zeta(\sigma, \alpha, \omega; \mathfrak{a})$, i.e.,

$$P_{\zeta,\sigma}(A) = m_H\big(\omega \in \Omega : \zeta(\sigma, \alpha, \omega; \mathfrak{a}) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

Then the following statement is valid.

Theorem 1 *Suppose that $\sigma > \frac{1}{2}$ and β, $0 < \beta < 1$, are fixed numbers, and the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then the measure P_N converges weakly to $P_{\zeta,\sigma}$ as $N \to \infty$.*

A proof of Teorem 1 is based on the following lemma. Let, for $A \in \mathcal{B}(\Omega)$,

$$Q_N(A) = \frac{1}{N+1} \#\big\{ 0 \leqslant k \leqslant N : \big((m+\alpha)^{-ik^\beta h} : m \in \mathbb{N}_0\big) \in A \big\}.$$

Lemma 2 *Suppose that β, $0 < \beta < 1$, is a fixed number, and that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. Let $g_N(\underline{k})$, $\underline{k} = (k_0, k_1, \ldots)$, be the Fourier transform of the measure Q_N, i.e.,

$$g_N(\underline{k}) = \int_\Omega \omega^{k_m}(m)\, dQ_N = \frac{1}{N+1} \sum_{k=0}^{N} \prod_{m=0}^{\infty} (m+\alpha)^{-ik_m k^\beta h},$$

where only a finite number of integers k_m are distinct from zero. Thus, we have that

$$g_N(\underline{k}) = \frac{1}{N+1}\sum_{k=0}^{N}\exp\left\{ -ik^\beta h\sum_{m=0}^{\infty}k_m\log(m+\alpha)\right\}. \tag{3}$$

We remind that a sequence $\{x_m : m \in \mathbb{N}\}$ of real numbers is said to be uniformly distributed modulo 1, if, for each interval $I = [a, b) \subset [0, 1)$ of length $|I|$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\chi_I\bigl(\{x_m\}\bigr) = |I|,$$

where u is the fractional part of $u \in \mathbb{R}$, and χ_I denotes the indicator function of the interval I. It is known [2] that the sequence $\{k^\beta a : k \in \mathbb{N}\}$ with a fixed β, $0 < \beta < 1$, and $a \neq 0$, is uniformly distributed modulo 1. Since the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, we have that

$$\sum_{m=0}^{\infty}k_m\log(m+\alpha) = 0$$

if and only if $\underline{k} = \underline{0}$. Clearly, in view of (3),

$$g_N(\underline{0}) = 1. \tag{4}$$

Moreover, the uniform distribution modulo 1 of the sequence $\{k^\beta a\}$ and the Weil criterion [2] show that

$$\lim_{N\to\infty} g_N(\underline{k}) = 0$$

for $\underline{k} \neq \underline{0}$. Therefore, by (4),

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Since the right-hand side of this equality is the Fourier transform of the Haar measure m_H, the lemma follows from a continuity theorem for probability measures on compact groups. □

Proof of Theorem 1. The proof uses Lemma 2 and is quite standard. First, for a fixed $\sigma_0 > \frac{1}{2}$, consider the function

$$\zeta_n(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty}\frac{a_m v_n(m, \alpha)}{(m+\alpha)^s},$$

where $v_n(m, \alpha) = \exp\{-(\frac{m+\alpha}{n+\alpha})^{\sigma_0}\}$ for all $n \in \mathbb{N}$. Let the function $u_n : \Omega \to \mathbb{C}$ be given by the formula

$$u_n(m) = \zeta_n(\sigma, \alpha, \omega; \mathfrak{a}),$$

where

$$\zeta_n(s, \alpha, \omega; \mathfrak{a}) = \sum_{m=0}^{\infty}\frac{a_m\omega(m)v_n(m, \alpha)}{(m+\alpha)^s}.$$

We note that both the series for $\zeta_n(s,\alpha;\mathfrak{a})$ and $\zeta_n(s,\alpha,\omega;\mathfrak{a})$ are absolutely convergent for $\sigma > \frac{1}{2}$. Therefore, the function u_n is continuous one. This, Lemma 2 and properties of the weak convergence show

$$\frac{1}{N+1}\#\bigl\{0 \leqslant k \leqslant N : \zeta_n\bigl(\sigma + ik^\beta h, \alpha;\mathfrak{a}\bigr) \in A\bigr\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the measure $m_H u_n^{-1}$ as $N \to \infty$. Here the measure $m_H u_n^{-1}$ is defined by the formula

$$m_H u_n^{-1}(A) = m_H\bigl(u_n^{-1}A\bigr), \quad A \in \mathcal{B}(\mathbb{C}).$$

Let, for brevity, $\widehat{P}_n = m_H u_n^{-1}$. Then it is proved by a standard way that the measure P_N, as $N \to \infty$, converges weakly to the measure P, where P is the limit measure of $\widehat{P}_n$ as $n \to \infty$.

On the other hand, in [3], it is obtained that if the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$ then, for $\sigma_0 > \frac{1}{2}$, the measure

$$\frac{1}{T}\mathrm{meas}\bigl\{\tau \in [0,T] : \zeta(\sigma + i\tau, \alpha;\mathfrak{a}) \in A\bigr\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

as $T \to \infty$ also converges weakly to the limit measure P of $\widehat{P}_n$, and that P coincides with $P_{\zeta,\sigma}$. Therefore, P_N also converges weakly to $P_{\zeta,\sigma}$ as $N \to \infty$. The theorem is proved. □

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REZIUMĖ

Diskreti ribinė teorema periodinei Hurvico dzeta funkcijai. II
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Straipsnyje gauta diskreti ribinė teorema periodinei Hurvico dzeta funkcijai apie tikimybinių matų kompleksinėje plokštumoje silpnąjį konvergavimą. Yra nurodytas ribinio mato pavidalas. Įrodymas remiasi sekų tolygaus pasiskirstymo moduliu 1 savybėmis.

Raktiniai žodžiai: periodinė Hurvico dzeta funkcija, ribinė teorema, silpnasis konvergavimas, tikimybinis matas.