VILNIUS UNIVERSITY

DONATA PUPLINSKAITĖ

AGGREGATION OF AUTOREGRESSIVE PROCESSES AND RANDOM FIELDS WITH FINITE OR INFINITE VARIANCE

Doctoral dissertation

Physical sciences, Mathematics (01 P)

Vilnius, 2013

The scientific work was carried out during 2009-2013 at Vilnius University and Université de Nantes

Scientific supervisor:

prof. habil. dr. Donatas Surgailis (Vilnius University, Physical Sciences, Mathematics – 01 P)

Scientific adviser:

prof. habil. dr. Anne Philippe (Université de Nantes, Physical Sciences, Mathematics – 01 P)

VILNIAUS UNIVERSITETAS

DONATA PUPLINSKAITĖ

AUTOREGRESINIŲ PROCESŲ IR ATSITIKTINIŲ LAUKŲ SU BAIGTINE ARBA BEGALINE DISPERSIJA AGREGAVIMAS

Daktaro disertacija

Fiziniai mokslai, matematika (01 P)

Vilnius, 2013

Disertacija rengta 2009-2013 metais Vilniaus universitete bei Nanto universitete, Prancūzijoje.

Mokslinis vadovas:

prof. habil. dr. Donatas Surgailis (Vilniaus universitetas, fiziniai mokslai, matematika – 01 P)

Mokslinis konsultantas:

prof. habil. dr. Anne Philippe (Nanto universitetas, fiziniai mokslai, matematika – 01 P)

Notations and Abbreviations

$\mathbb{N}$ - the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$

$\mathbb{N}^*$ - the set of positive natural numbers, $\mathbb{N} = \{1, 2, \ldots\}$

$\mathbb{Z}$ - the set of integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{R}$ - the set of real numbers

$\mathbb{Z}^d_+ := \{(t_1, \ldots, t_d) \in \mathbb{Z}^d : t_i \geq 0, i = 1, \ldots, d\}$

$\mathbb{R}^d_+ := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \geq 0, i = 1, \ldots, d\}$

$\mathbb{Z}_+ := \mathbb{Z}^1_+$

$\mathbb{R}_+ := \mathbb{R}^1_+$

$\mathbb{R}^2_0 := \mathbb{R}^2 \setminus \{(0, 0)\}$

$a_+ := \max(0, a)$, for $a \in \mathbb{R}$.

$a_- := (-a)_+ = \max(0, -a)$, for $a \in \mathbb{R}$.

$\|x\| := \sqrt{x_1^2 + x_2^2}$, for $x = (x_1, x_2) \in \mathbb{R}^2$.

$E = \mathrm{diag}(\gamma_1, \ldots, \gamma_d)$ denotes the diagonal $d \times d$ matrix with entries $\gamma_1, \ldots, \gamma_d$ on the diagonal.

C, $C(K)$, denote generic constants, possibly depending on the variables in brackets, which may be different at different locations.

$\mathrm{E}X$ denotes the mean of random variable X.

$\mathrm{Var}(X)$ denotes the variance of random variable X.

$D(\alpha)$ is the domain of attraction of an $\alpha-$stable law.

L is the lag operator, $X(t-1) = LX(t)$.

$\to_{\mathrm{d}}$ denotes convergence in distribution.

$\to_{\mathrm{p}}$ denotes convergence in probability.

$\to_{\mathrm{L_p}}$ denotes convergence of random variables in L_p space. We write $\xi_n \to_{\mathrm{L_p}} \xi$, if $\mathrm{E}|\xi_n - \xi|^p \to 0$.

$\to_{\mathrm{L_p}(\mathcal{A})}$ denotes conditional convergence of random variables in L_p space. We write $\xi_n \to_{\mathrm{L_p}(\mathcal{A})} \xi$, if $\mathrm{E}\Big[|\xi_n - \xi|^p \Big| \mathcal{A}\Big] \to 0$ almost surely.

$\to_{\mathrm{fdd}}$ denotes weak convergence of finite dimensional distributions.

fdd-lim denotes weak convergence of finite dimensional distributions.

$\to_{\mathrm{D}[0,1]}$ denotes convergence in Skorohod space with the J_1 Skorohod topology.

$x \uparrow a$ means that x approaches a from the left.

$\stackrel{\mathrm{fdd}}{=}$ denotes equality of finite dimensional distributions.

$\stackrel{\mathrm{d}}{=}$ denotes equality of distributions.

$\mathbf{1}(\cdot)$ denotes the indicator function.

$\mathrm{sign}(\cdot)$ is the sign function.

$[x]$ denotes integer part of real number x.

$x \wedge y$ denotes $\min(x, y)$ for real numbers x and y.

$x \vee y$ denotes $\max(x, y)$ for real numbers x and y.

$t \stackrel{\mathrm{mod}\,2}{=} s$ means that $t + s$ is even, for $t \in \mathbb{Z}$, $s \in \mathbb{Z}$.

$t \stackrel{\mathrm{mod}\,2}{\neq} s$ means that $t + s$ is odd, for $t \in \mathbb{Z}$, $s \in \mathbb{Z}$.

i.i.d. independent identicaly distributed

i.d. identicaly distributed

r.v. random variable

a.s. almost surely

a.e. almost every

r.h.s. right hand side

l.h.s. left hand side

1

Introduction

Aggregation as an object of research. The aggregation problem is concerned with the relationship between individual (micro) behaviour and aggregate (macro) statistics. There are different types of aggregation: small-scale, large-scale, temporal aggregation, aggregation in time and space (see Chapter 2, page 15, also [19], [42]). We concentrate on the large-scale contemporaneous aggregation. The scheme of contemporaneous aggregation was firstly proposed by P. Robinson (1978, [90]) and C.W.J. Granger (1980, [41]) in order to obtain the long memory phenomena in aggregated time series. Suppose we have a group of N heterogeneous individuals, each of which is described by some model $X_i(t)$, $i = 1, \ldots, N$. Then *the aggregated process* is defined as a normalised sum over all individuals at fixed time point t:

$$\bar{X}_N(t) := \frac{1}{A_N} \sum_{i=1}^{N} X_i(t), \qquad t \in \mathbb{Z}, \tag{1.1}$$

where A_N is some normalizing sequence. The fundamental statistical problem of large-scale contemporaneous aggregation is to determine the limit distribution of the aggregated process $\{\bar{X}_N(t),\, t \in \mathbb{Z}\}$ in (1.1), as the number of individuals N grows to infinity, and to explore main properties of *the limit aggregated process* $\mathfrak{X}(t) := \lim_{N\to\infty} \bar{X}_N(t)$, $t \in \mathbb{Z}$. The limit aggregated process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$, may have a completely different structure than the individual processes have. The most important properties, which the limit aggregated process may admit, are ergodicity and long memory. Ergodicity is a quality of the stochastic process that allows estimation of characteristics of the process using only one sufficiently long realization of the process, and we do not need to observe separate independent realizations of this process. Whilemean, the long memory property shows the dependence of a series at long lags, dependence between observations occurring now and after an amount of time. In the scientific literature appear various definitions for long momery (see Section 2.3, page 33). In general, as is written in [93], the memory is something that lasts.

Another important problem is so called disaggregation problem: having a sample $\mathfrak{X}(1)$, $\mathfrak{X}(2)$, ..., $\mathfrak{X}(n)$, $n \in \mathbb{N}^*$, of the limit aggregated process at hand, to recover the properties of the individual processes $\{X_i(t),\, t \in \mathbb{Z}\}$, $i = 1, \ldots, N$. For example, suppose we have a sample of the limit aggregated process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$, which is accumulated from independent AR(1) random processes:

$$X_i(t) = a_i X_i(t-1) + \varepsilon_i(t), \qquad t \in \mathbb{Z}, \quad i = 1, \ldots, N, \tag{1.2}$$

where $\{\varepsilon_i(t),\, t \in \mathbb{Z}\}$ is white noise and a_i, $i = 1, \ldots, N$, are random coefficients with generic distribution a. The aim of the disaggregation problem in this case is to find a "good" estimate of the density function of random variable a, using observed data $\mathfrak{X}(1)$, $\mathfrak{X}(2)$, ..., $\mathfrak{X}(n)$.

The (dis)aggregation problem was discussed in [13], [14], [19], [21], [39], [40], [41], [42], [52], [53], [60], [62], [64], [65], [69], [77], [90], [102], [103], et.al. A short review of literature is given in Chapter 2, page 15. Almost all of the above-mentioned papers investigate aggregation schemes when (micro) level data have finite variance. It is well known the aggregation scheme of independent processes with finite variance, which leads to the Gaussian case, i.e. the limit aggregated process is the Gaussian process. The aim of our research was to extend these results to infinite variance case or finite variance but not necessarily Gaussian case.

Actuality. Aggregated data is most often found, collected and used in many areas such as economics, applied statistics, sociology, geography, etc. Whilemean, disaggregate (panel) data are difficult to obtain and not always available. This motivates an importance of studying the aggregation and disaggregation problem.

One of the most important reasons why the contemporaneous aggregation become an object of research is the possibility of obtaining the long memory phenomena in processes. The aggregation provides an explanation of the long-memory effect in time series and a similation method of such series as well. Accumulation of short-memory non-ergodic random processes can lead to the long memory ergodic process, that can be used for the forecasts of the macro and micro variables.

Aims and problems. One of the main goals of the PhD thesis is to explore the aggregation scheme of random processes and fields with infinite variance. Another aim of our study is to get a non-Gaussian limit aggregated process by the aggregation of independent processes with finite variance (in the scientific literature is given only the aggregation scheme of independent processes, which leads to the Gaussian case). The disaggregation problem is also the problem of our interest. More precisely, our aim is to solve the following problems:

- *Aggregation of AR(1) models with infinite variance (Chapters 3 and 4).*

The main goal of this research is to extend results of P. Zaffaroni paper [102] from finite variance case to infinite variance case. Following the idea of this paper, we discuss the aggregation of autoregressive random-coefficient AR(1) processes with innovations belonging to the domain of attraction of an α-stable law. We investigate separately the aggregation of AR(1) processes with common innovations and idiosyncratic innovations. We obtain conditions under which the limit aggregated process exists and exhibits long memory in a certain sense. Since in our case the variance of the aggregated process is infinite and second order properties as spectral density or covariance function are not defined, we use alternative definitions of long memory which do not require finite variance: distributional long memory, LRD(SAV) and codifference (see Section 2.3, page 33). Results of this research are given in Chapters 3, 4 and published in papers [85], [86].

- *Aggregation of a triangular array of AR(1) processes (Chapter 5)*. The aim of this research is to investigate the aggregation scheme, which generalize previous results and leads to the case of the finite variance but not necessary Gaussian or infinite variance but not necessary stable limit aggregated process $\mathfrak{X}(t) := \lim_{n\to\infty} \bar{X}_N(t)$, $t \in \mathbb{Z}$. For this reason we discuss an aggregation of independent random-coefficient AR(1) models with innovations belonging to the domain of attraction of an infinitely divisible law W. We obtain conditions under which the limit aggregated process exists and is represented as a mixed infinitely divisible moving average $\mathfrak{X}(t)$ in (5.4), page 84. Using Cox's definition of distributional long memory (Definition 2.3.6, page 36) and assuming that the limit aggregated process admits finite variance, we investigate its long memory properties. In short, we study partial sums of the limit aggregated process and show that these partial sums may exhibit four different limit behaviors depending on the distribution of random coeffitient of AR(1) model and the Lévy triplet of infinitely divisible law W [1]. Results of this research are given in Chapter 5 and in submitted paper [82]. But, it should be noted here that this generalisation problem is not fully finished. The questions for the future: What is the limit of

1. Lévy triplet (μ, σ, π) completely determines the characteristic function of the infinitely divisible law W, see (5.6), page 84.

partial sums of the limit aggregated process (5.4) in infinite variance case? What is the limit aggregated process and what properties it have if we include common innovations belonging to the domain of attraction of an infinitely divisible law? What happens if the random coefficient of AR(1) models depends on time?

- *Aggregation of random fields (Chapter 6).* The goal of this research is to extend the aggregation scheme from one-dimensional processes to two-dimensional random fields. The (dis)aggregation problem for finite-variance random fields was investigated in [60], [61], [65], while we focus on the aggregation of independent random fields with infinite variance (innovations belong to the domain of attraction of an α-stable law). First, we explore the aggregation scheme of nearest-neighbor autoregressive random fields and specify what is the limit aggregated field. Another question of our interest is the dependence structure of the limit aggregated field. The dependence structure of random field is more complicated than in a univariate process case, because dependence for random fields extends in all directions and can have different intensity in different directions. Since properties of the limit aggregated random field are highly dependent on the assumptions put on micro level (individual) fields, we investigate the long memory property of the limit aggregated field in two special cases of individual models (see (6.14)-(6.15), page 115). In order to describe the dependence structure of the aggregated random field we introduce the notion of anisotropic/isotropic distributional long memory (see Definition 6.2.2, page 119, and Definition 6.2.3, page 119). Results of this research are given in Chapter 6 and in submitted paper [84]. The new interesting question for the future: the aggregation scheme of autoregressive random fields with common innovations.

- *Disaggregation problem (Section 5.4).* The main idea of the disaggregation problem is: having data from the limit aggregated process at hand to recover the distribution of individual processes. Suppose we have sample of the limit aggregated process, which is obtained via aggregation of independent random-coefficient AR(1) processes. Let $\phi(a)$ be an unknown density function of random coefficient of AR(1) model. The disaggregation problem in this case is to find a "good" estimator of the density function $\phi(a)$. The authors of papers [22], [62] proposed consistent estimator of this density function via Gegenbauer polynomials, under assumption that the limit aggregated process is Gaussian. Our aim was to show that this density estimator, proposed in [22], [62] is consistent not only in Gaussian case. We showed that for the consistency of the density estimator via Gegenbauer polynomials (or Jacobi polynomials (5.53), page 105) it is enough to have finite fourth moment of the limit aggregated process. This result is small extension of the disaggregation problem. It remains many interesting questions for the future. The main of them is how to solve disaggregation problem in infinite variance case.

• *Asymptotics of the ruin probability (Chapter 7).* The goal of this research is to find asymptotics of the ruin probability in a discrete time risk insurance model with stationary claims modeled by the aggregated heavy-tailed process (4.4) in page 62. Using the asymptotics of the ruin probability, we can describe the long memory properties of heavy-tailed claims. Results of this research are given in Chapter 7 and in paper [81].

The novelty of the results presented in this PhD thesis is:

- the scheme of the aggregation of independent autoregressive processes, which leads to the finite variance but not necessarily Gaussian aggregated process;
- the scheme of the aggregation of autoregressive random processes with infinite variance.
- the scheme of the aggregation of nearest-neighbor autoregressive random fields with infinite variance.
- The notion of anisotropic/isotropic long memory for random fields on $\mathbb{Z}^2$.

These problems have not been investigated before in the scientific literature.

Methods. Methods of probability theory, mathematical statistics, functional analysis and time series analysis are applied. Used tools: Cramér-Wold device (to prove finite dimensional convergence), Dominated convergence theorem (to prove convergence of integrals), Kolmogorov tightness criterion (to prove tightness), Law of large numbers (to show convergence of the sample average), Moivre-Laplace theorem (normal approximation to the binomial distribution), Hunt's interpolation theorem (a result bounding the norms of operators acting in L_p spaces), well-known inequalities (Minkowski's, Hölder's, Jensen's, Hoeffding's), and etc.

Approbation of results. The main dissertation results were presented in the following conferences:

- 50th Conference of the Lithuanian Mathematical Society, Vilnius, Lithuania, June 18 - 19, 2009.
- 10th international Vilnius conference on probability theory and mathematical statistics, Vilnius, Lithuania, June 28 - July 2, 2010.
- 1st Conference by Lithuanian Academy of Sciences "Interdisciplinary research in physical and technological sciences", Vilnius, Lithuania, February 8, 2011.

- 2nd Conference by Lithuanian Academy of Sciences "Interdisciplinary research in physical and technological sciences", Vilnius, Lithuania, February 14, 2012.
- Journée des doctorants, Nantes, France, April 26, 2012.
- 53rd Conference of the Lithuanian Mathematical Society, Klaipėda, Lithuania, June 11 - 12, 2012.
- Conference "Non-stationarity in Statistics and Risk Management", Luminy, Marseille, France, January 21 - 25, 2013.
- The First German-Polish Joint Conference on Probability Theory and Mathematical Statistics, Torun, Poland, June 6-9, 2013.

Publications. The main results are published in the following articles:

1. D. Puplinskaitė, D. Surgailis, *Aggregation of random-coefficient AR(1) process with infinite variance and common innovations.* Lithuanian Math. J., **49** (4), 446-463, 2009.
2. D. Puplinskaitė, D. Surgailis, *Aggregation of a random-coefficient AR(1) process with infinite variance and idiosyncratic innovations.* Adv. Appl. Probab., **42** (2), 509-527, 2010.
3. K. Perilioglu, D. Puplinskaitė, *Asymptotics of the ruin probability with claims modeled by α-stable aggregated AR(1) process.* Turkish J. Math., **37** (1), 129-138, 2013.
4. A. Philippe, D. Puplinskaitė, D. Surgailis, *Contemporaneous aggregation of triangular array of random-coefficient AR(1) processes.* 2013, to appear in J. Time Ser. Anal.

Structure of the thesis. Dissertation consists of eight chapters and bibliography. An introduction and the review of aims and problems is given in Chapter 1. Chapter 2 contains a short review of the scientific literature on this topic. Chapter 3 provides the aggregation scheme of autoregressive random-coefficient AR(1) processes with infinite variance and common innovations. Chapter 4 provides the aggregation scheme of autoregressive random-coefficient AR(1) processes with infinite variance and idiosyncratic innovations. Chapter 5 is dedicated to the contemporaneous aggregation of tringular array of random-coefficient AR(1) processes. Chapter 6 presents the aggregation scheme of random fields and the notion of the anisotropic long memory. In Chapter 7 we discuss asymptotics of the ruin probability with claims modeled by α-stable aggregated AR(1) process. Finally, the main results of the thesis are summarized in the Chapter 8.

Acknowledgments. I would like to thank my supervisors, prof. Anne Philippe and prof. Donatas Surgailis, for the patient guidance, encouragement and advice throughout the years of my PhD thesis project.

2

Review of the State of the Art

In this section, firstly we give a brief review of main types of the aggregation, then we focus on the main results obtained by other authors, which are dealing with the problem of aggregation and disaggregation of linear models. Finally, in the last section of this chapter we review different definitions of long memory.

2.1 Aggregation

The aggregation problem is concerned with the relationship between individual (micro) behaviour and aggregate (macro) statistics. One of the important properties of aggregation is the possibility to get long memory phenomenon in the aggregated series. There are several types of aggregation that occur in the time series analysis: small-scale aggregation, large-scale aggregation, temporal aggregation, aggregation in time and space.

A small-scale aggregation involves sums of finite number individual processes. For example, suppose $\{X_1(t),\, t \in \mathbb{Z}\}$ is ARMA(p_1,q_1) process and $\{X_2(t),\, t \in \mathbb{Z}\}$ is ARMA (p_2,q_2) process:

$$\begin{aligned} X_1(t) + \sum_{k=1}^{p_1} a_k X_1(t-k) &= \varepsilon_1(t) + \sum_{k=1}^{q_1} \theta_k \varepsilon_1(t-k) \\ X_2(t) + \sum_{k=1}^{p_2} b_k X_2(t-k) &= \varepsilon_2(t) + \sum_{k=1}^{q_2} c_k \varepsilon_2(t-k), \end{aligned}$$

where $(\varepsilon_1(t), \varepsilon_2(t))_{t\in\mathbb{Z}}$ is bivariate white noise, then the aggregated process $\mathfrak{X}(t) := X_1(t) + X_2(t)$, $t \in \mathbb{Z}$, is autoregressive ARMA(m,n) process with $m \le p_1 + p_2$

and $n \leq \max(p_1 + q_2, p_2 + q_1)$, see [42]. The small-scale aggregation helps us to develop new time series models. Note, that if the number of individual processes increases, we get more complicated dynamics. And this is the result of large-scale aggregation. In the context of large-scale aggregation, the aggregated process is the sum of large number of individual processes.

Another type of aggregation is temporal aggregation. The temporal aggregation is the relationship between high and low frequency. The problem of temporal aggregation arises when the data are observed at a lower frequency than the frequency of the data generating model. Suppose that the unit is the basic time interval for which a time series is generated. If observations are fixed every k, $k > 1$, units, then it is said that the series is "systematically sampled". Systematic sampling is a type of temporal aggregation for "stock" variables (see [42]). The temporal aggregation for "flow" variable is a summation of observations over k unit before the systematic sampling. Suppose we have time series $\{X(i),\, i \in \mathbb{Z}\}$, then the temporal aggregation is summation over k units:

$$\mathfrak{X}(t) := \sum_{i=k(t-1)+1}^{kt} X(i), \qquad t \in \mathbb{Z}.$$

Here arises the question, what model can be used to describe temporal aggregated series, what properties it has. Such questions of temporal aggregation have been studied in [23], [24] and in other articles.

The combining both spacial and temporal aggregation creates so called time-space models (see [37], [83] and references therein), which take into account dependence lagged in time and in space.

The main attention in the thesis is devoted to the crosssectional large-scale contemporaneous aggregation of linear models, but the aggregation of non-linear and heteroskedastic models is also an interesting and popular object of research. Contemporaneous aggregation of heterogeneous heteroscedastic models was discussed in [29], [39], [53], [64], [103], [104]. It is proved that the contemporaneous large-scale aggregation of ARCH/GARCH models do not lead to the long memory processes in the sense of a non-summable autocovariance function of the squared aggregate. For the GARCH(1,1) process $\{X_i(t),\, t \in \mathbb{Z}\}$ the limit of $N^{-1}\sum_{i=1}^{N} X_i^2(t)$ exhibits a summable hyperbolically decaying autocovariance function under condition for covariance stationarity (see [53], [103]). However, stochastic volatility models as a nonlinear moving average model (see [103]) and linear ARCH/GARCH models (see [39]) were found to reproduced the long memory via contemporaneous aggregation (in the sense of summing and averaging across observation).

More detailed review of the types of the aggregation can be found in a doctoral thesis of D. Celov [19], and in [42]. Now let's take a look at the main results of the aggregation of linear models.

2.1.1 Aggregation of ARMA(p, q) processes

First of all, we review here the aggregation of AR(1) processes. Then we describe the aggregation of AR(p) models and at the end the aggregation of ARMA(p, q) processes.

Aggregation of AR(1) processes. Observed macroeconomic time series often represent the result of aggregating over a huge number of heterogeneous units. An individual (micro) behavior can be described usually by autoregressive model. This motivates the importance of investigating the asymptotic behaviour of the aggregated process of heterogeneous autoregressive models. The initial interest for aggregation was prompted by the possibility of obtaining long memory. This idea was first introduced by Robinson (1978, [90]) and developed by Granger (1980, [41]). C.W.J. Granger investigated the contemporaneous aggregation of autoregressive AR(1) models:

$$X_i(t) = a_i X_i(t-1) + \rho_i u(t) + \varepsilon_i(t), \qquad i = 1, 2, \dots, N, \quad t \in \mathbb{Z}, \tag{2.1}$$

where $\{X_i(t),\, t \in \mathbb{Z}\}$ describes an evolution of ith micro-unit, N is the number of units, $\{\varepsilon_i(t),\, t \in \mathbb{Z}\}$ is a white noise specific to each agent (idiosyncratic innovations) and $\{u(t),\, t \in \mathbb{Z}\}$ is a white noise, which is common to all agents (common innovations); the coefficients $\theta_i := (a_i,\, \rho_i)$, $i = 1, \dots, N$, are i.i.d. drawings from $\Theta := [0,\, 1) \times \mathbb{R}$; a_i and ρ_i are independent and $\mathrm{E}|\rho_i| \neq 0$, $\mathrm{E}\rho_i^2 < \infty$. Additionally assume that parameters a_i, $i = 1, \dots, N$, are Beta distributed with the density function

$$\phi(a) = \frac{2}{B(p,q)} a^{2p-1}(1-a^2)^{q-1}, \quad a \in [0,1), p > 0, q > 0. \tag{2.2}$$

C.W.J. Granger showed that in the case of aggregation of independent series

$$X_i(t) = a_i X_i(t-1) + \varepsilon_i(t), \qquad i = 1, 2, \dots, N, \quad t \in \mathbb{Z},$$

the aggregated process $\bar{X}_N(t) := N^{-1/2} \sum_{i=1}^{N} X_i(t)$ can have long memory property, in the sense of non-summable autocovariance function. He showed that the covariance function of the aggregated process $\{\bar{X}_N(t),\, t \in \mathbb{Z}\}$ is equal to

$$\mathrm{Cov}(\bar{X}_N(t), \bar{X}_N(t+h)) = \sigma_\varepsilon^2 \mathrm{E}\Big[\frac{a^{|h|}}{1-a^2}\Big] =: r(h),$$

and the conditional covariance

$$\mathrm{Cov}(\bar{X}_N(t), \bar{X}_N(t+h)|\mathcal{A}) = \sigma_\varepsilon^2 \frac{1}{N} \sum_{i=1}^{N} \frac{a_i^{|h|}}{1-a_i^2} \to r(h), \qquad \text{a.s., as } N \to \infty,$$

here $\sigma_\varepsilon^2 := \mathrm{Var}(\varepsilon_i(t))$, and $\mathcal{A} = \sigma\{a_1, a_2, \dots\}$ denote the $\sigma-$algebra generated by r.v.'s $a_1, a_2, \dots$. Assuming that coefficients a_i have a density function as in (2.2), the covariance of the aggregated process decays hyperbolically,

$$r(h) \sim Ch^{1-q}, \qquad \text{as } h \to \infty. \tag{2.3}$$

From the last relation (2.3), it follows that if $1 < q < 2$, $\sum_{h \in \mathbb{Z}} r(h) = \infty$ and the process with such covariance function exhibits long memory. [1] Note, that the decay rate of the covariance function (2.3) does not depend on parameter p. The long memory property depends on the behavior of a_i's density near unity.

If individual processes have dependent innovations

$$X_i(t) = a_i X_i(t-1) + \rho_i u(t), \qquad i = 1, 2, \dots, N, \quad t \in \mathbb{Z},$$

and assumption (2.2) is satisfied, then the conditional covariance of the aggregated process $\bar{X}_N(t) := N^{-1} \sum_{i=1}^{N} X_i(t)$ converges a.s., as $N \to \infty$,

$$\begin{aligned} \mathrm{Cov}(\bar{X}_N(t), \bar{X}_N(t+h)|\mathcal{A}) &= \sigma_u^2 \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1}^{N} \rho_i \rho_j \frac{a_j^{|h|}}{1 - a_i a_j} \\ &\to \sigma_u^2 (\mathrm{E}\rho)^2 \sum_{k=0}^{\infty} \mathrm{E}a^k \mathrm{E}a^{|h|+k} =: r(h), \end{aligned}$$

where $\sigma_u^2 := \mathrm{Var}(u)$, and $\mathrm{E}a^k \sim k^{-q}$, as $k \to \infty$. It is not difficult to see, that in this case $r(h) \sim Ch^{1-2q}$, as $h \to \infty$, and the process with such covariance function exhibits long memory, if $0 < q < 1$.

As we see, with contemporaneous aggregation scheme (summing and averaging across observations), based on the AR(1) model near the nonstationarity regime, Granger provided an explanation of the long-memory effect. He also showed that the common and idiosyncratic components exhibit a different degree of long memory.

Zaffaroni (2004, [102]) generalized results obtained in [41]. Rather than limiting the attention to the limit behavior of the autocovariance function, P. Zaffaroni studies the limit of the aggregated process $A_N^{-1} \sum_{i=1}^{N} X_i(t)$. The author assumes that units are generated by AR(1) equations of the form (2.1). He does not put

1. If $0 < q \le 1$, $r(h)$ is not defined because in this case $\mathrm{E}[(1-a^2)^{-1}] = \infty$.

an assumption that a_i are Beta distributed, but assumes only that

$$\phi(a) \sim C(1-a)^{\beta}, \quad \text{as } a \uparrow 1, \text{ with } 0 < C < \infty,\ \beta \in (-1, \infty). \tag{2.4}$$

Define the aggregated process as

$$\bar{X}_N(t) := \frac{1}{N} \sum_{i=1}^{N} X_i(t) = U_{N,t} + E_{N,t}, \tag{2.5}$$

where

$$U_{N,t} = \frac{1}{N} \sum_{i=1}^{N} \rho_i \frac{1}{1-a_i L} u(t), \qquad E_{N,t} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{1-a_i L} \varepsilon_i(t), \tag{2.6}$$

are common and idiosyncratic components, respectively. The conditional variances [2] of the idiosyncratic $E_{N,t}$ and common $U_{N,t}$ components are equal to

$$V_N^E := \frac{\sigma_\varepsilon^2}{N^2} \sum_{j=1}^{N} \frac{1}{1-a_j^2}, \qquad V_N^U := \frac{\sigma_u^2}{N^2} \sum_{h,j=1}^{N} \frac{\rho_h \rho_j}{1-a_h a_j}.$$

P. Zaffaroni studied the behavior of the common component $U_{N,t}$ and the idiosyncratic component $E_{N,t}$ separately. The following theorems show what is the limit of common and idiosyncratic components of the aggregated process in (2.5).

Theorem 2.1.1. *([102], Th.3 (stationary case), p. 84) Assume that $\varepsilon_i(t)$, $t \in \mathbb{Z}$, $i \in \mathbb{N}$, are i.i.d. innovations with zero mean and finite variance. Assume, the density function of random coefficient a satisfies* (2.4)*. If $\beta > 0$, then for a.e. $\{\theta_i = (a_i, \rho_i),\ i = 1, 2, \ldots\}$,*

$$\frac{E_{N,t}}{\sqrt{V_N^E}} \to_{\mathrm{d}} E_t, \qquad \text{as } N \to \infty, \tag{2.7}$$

where $\{E_t,\ t \in \mathbb{Z}\}$ is a stationary zero-mean Gaussian process with long memory parameter [3] $d^E = (1-\beta)/2$ and covariance function:

$$\mathrm{Cov}(E_t,\, E_{t+h}) = \Big(\mathrm{E}\Big[\frac{1}{1-a^2}\Big]\Big)^{-1} \mathrm{E}\Big[\frac{a^{|h|}}{1-a^2}\Big], \quad h \in \mathbb{Z}. \tag{2.8}$$

To prove the limit in (2.7), P. Zaffaroni use the Lindeberg-Lévy central limit theorem (CLT) and calculates the limit of the conditional covariance function of the idiosyncratic component $E_{N,t}$. Note, that the Theorem 2.1.1 is proved under

2. With respect to σ-algebra generated by $\{(a_i,\ \rho_i),\ i = 1, 2, \ldots\}$.

3. We say, that the stationary stochastic process $\{Y_t,\ t \in \mathbb{Z}\}$ has memory parameter d ($d < 1/2$), if $\mathrm{Cov}(Y_t, Y_{t+u}) \sim c u^{2d-1}$, as $u \to \infty$. It is not difficult to see, that Y_t have long memory (in the sense of non-summable autocovariance function), if $d > 0$.

assumption, that $\beta > 0$. If $\beta \leq 0$, the covariance function in (2.8) is not well defined, because $\mathrm{E}[(1-a^2)^{-1}] = \infty$. In such case, P. Zaffaroni investigates the truncation of $E_{N,t}$:

$$\widetilde{E}_{N,t} := \frac{1}{N}\sum_{k=0}^{t-1}\sum_{i=1}^{N} a_i^k \varepsilon_i(t-k),$$

which is a non-stationary process. Zaffaroni [102] showed, that the limit of $\widetilde{E}_{N,t}/\sqrt{\mathrm{Var}_N(\widetilde{E}_{N,t})}$ is the non-stationary Gaussian process. Here, $\sqrt{\mathrm{Var}_N(\widetilde{E}_{N,t})}$ denotes the conditional variance of $\widetilde{E}_{N,t}$.

Now let's take a look at what is the limit of common component $U_{N,t}$.

Theorem 2.1.2. *([102], Th.5 (stationary case), p. 86). If $\beta > -\frac{1}{2}$, then for a.e. $\{\theta_i = (a_i, \rho_i),\ i = 1, 2, \ldots\}$,*

$$U_{N,t} \to_{L_2(\theta)} U_t := \mathrm{E}\rho \sum_{k=0}^{\infty} \mathrm{E}a^k u(t-k), \qquad \text{as } N \to \infty, \tag{2.9}$$

here $\to_{L_2(\theta)}$ means conditional convergence in L_2. The process $\{U_t,\, t \in \mathbb{Z}\}$ has the long memory parameter $d^U = -\beta$ and is not Gaussian unless the $\{u(t),\, t \in \mathbb{Z}\}$ is a Gaussian white-noise.

It is not difficult to see, that $\sum_{k=0}^{\infty}(\mathrm{E}a^k)^2 < \infty$ and the moving average U_t in (2.9) is well defined in L_2, if $\beta > -1/2$. While for $\beta \leq -1/2$, this moving avegare is not well defined. Therefore in this case, P. Zaffaroni investigates the truncation of $U_{N,t}$:

$$\widetilde{U}_{N,t} = \sum_{k=0}^{t-1}\Big(\frac{1}{N}\sum_{i=1}^{N}\rho_i a_i^k\Big)u(t-k).$$

Theorem 2.1.3. *([102], Th.5 (non-stationary case), p. 86). Assume that $\{u(t),\, t \in \mathbb{Z}\}$ are i.i.d. and $\mathrm{E}|u(t)|^q < \infty$ for real $q > max(2,\ -2/(2\beta+1))$. Set $d^U := -\beta$. If $\beta < -1/2$, then for a.e. $\{\theta_i = (a_i, \rho_i),\ i = 1, 2, \ldots\}$,*

$$\widetilde{U}_{N,t} \to_{\mathrm{d}} \widetilde{U}_t := \mathrm{E}\rho \sum_{k=0}^{t-1} \mathrm{E}a^k u(t-k), \qquad \text{as } N \to \infty,$$

and, for any real $0 \leq r \leq 1$,

$$t^{\beta+1/2}\widetilde{U}_{[rt]} \to_{\mathrm{D}[0,1]} (2d^U - 1)U(d^U; r), \qquad \text{as } t \to \infty.$$

The process $\{\widetilde{U}_t,\, t \in \mathbb{Z}\}$ is not Gaussian unless $\{u(t),\, t \in \mathbb{Z}\}$ is Gaussian white-noise; $\{U(d; r),\ r \in \mathbb{R}_+\}$, $1/2 < d < 1$, is type II fractional Brownian motion

$$U(d; r) = \int_0^r (r-s)^{d-1}\,\mathrm{d}B(s), \qquad r > 0,$$

Here $B(s)$ denotes standard Brownian motion. The process $\{U(d;r),\ r \in \mathbb{R}_+\}$ is self-similar with Hurst index $H = d - 1/2$.

Limits of the idiosyncratic and common components have $d^E = (1-\beta)/2$ and $d^U = -\beta$ long memory parameters respectively. Therefore it follows, that the the distribution of the random coefficient a is more concentrated near the unit, the stronger long memory of the limit aggregated process is. If $|a| \le \alpha < 1$ a.s. for some constant α, then the limit aggregated process has short memory. Note that for $\beta > -1/2$ the limit of the aggregated process in (2.5) is stationary process and depends only on the common componet. The idiosyncratic component disappears in the limit, because its variance V_N^E converges a.s. to zero, as $N \to \infty$, for $\beta > -1/2$ (see [102], Th. 1). The spectral density of the limit aggregated process has the same properties as the spectral density of the U_t process in (2.9):

$$s^U(\lambda) = \frac{\sigma_u^2(\mathrm{E}\rho)^2}{2\pi}\left|\sum_{k=0}^{\infty}\mathrm{E}a^k\mathrm{e}^{-\mathrm{i}\lambda k}\right|^2 \sim \begin{cases} C\lambda^{2\beta}, & \beta < 0, \\ C\log\left(\frac{1}{\lambda}\right), & \beta = 0, \\ C, & \beta > 0. \end{cases} \quad \text{as } \lambda \to 0.$$

Therefore, in the presence of common innovations, the limit aggregated process is stationary and exhibits long memory property when $-1/2 < \beta < 0$. If we aggregate independent processes only with idiosyncratic innovations, then the limit aggregated process E_t in (2.7) is stationary and has long memory for $0 < \beta < 1$.

Following the frame-work of [102], we worked out the aggregation problem of autoregressive AR(1) processes with innovations belonging to the domain of attraction of an $\alpha-$stable law, $0 < \alpha \le 2$ (see Chapters 3, 4).

Aggregation of AR(p) processes. The aggregation of AR(p) processes was investigated by G. Oppenheim and M.C. Viano [77]. Assume that the behavior of unit is described by the stationary autoregressive model of order p:

$$X(t) - \sum_{k=1}^{p} a_k X(t-k) = \varepsilon(t), \qquad t \in \mathbb{Z}, \tag{2.10}$$

where $\{\varepsilon(t), t \in \mathbb{Z}\}$ is zero-mean second-order strong white noise with variance σ_ε^2. Let α_j, $j = 1, \ldots, p$, denote the inverse of the roots of the polynomial $1-\sum_{k=1}^{p} a_k z^k$ and D is an open unit disc. Assume that the random vector $\alpha = (\alpha_1, \ldots, \alpha_p)$ is in D^p almost surely and that α is independent of the innovations $\{\varepsilon(t), t \in \mathbb{Z}\}$. Given α, let $A_\alpha(z)$ be the characteristic polynomial of the autoregressive process $X(t)$:

$$A_\alpha(z) = \prod_{k=1}^{p}(1-\alpha_k z), \qquad A_\alpha(z)^{-1} = 1 + \sum_{k=1}^{\infty} b_k z^k. \tag{2.11}$$

The moving avarage representation of $X(t)$ is

$$X(t) = \varepsilon(t) + \sum_{k=1}^{\infty} b_k \varepsilon(t-k), \qquad t \in \mathbb{Z}. \tag{2.12}$$

This series converges almost surely[4]. $\{X(t), t \in \mathbb{Z}\}$ is stationary but not ergodic process with a covariance function

$$\mathrm{Cov}(X(t), X(t+h)) = \sigma_\varepsilon^2 \mathrm{E}\Big[\sum_{k=1}^{\infty} b_k b_{k+h}\Big] = \sigma_\varepsilon^2 \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}h\lambda} \mathrm{E}\left|A_\alpha(\mathrm{e}^{\mathrm{i}\lambda})\right|^{-2} \mathrm{d}\lambda,$$

and a spectral density

$$f(\lambda) = \frac{\sigma_\varepsilon^2}{2\pi} \mathrm{E}\left|A_\alpha(\mathrm{e}^{\mathrm{i}\lambda})\right|^{-2}. \tag{2.13}$$

The process $X(t)$ is in L_2, i.e. $\mathrm{E}(X(t))^2 < \infty$, if and only if

$$\mathrm{E}\int_{-\pi}^{\pi} \left|A_\alpha(\mathrm{e}^{\mathrm{i}\lambda})\right|^{-2} \mathrm{d}\lambda < \infty.$$

Now assume, that all units are independent and the behavior of them is described by N independent copies of (2.10). Define the aggregated process as cross-sectional average with normalisation $\sqrt{N}$:

$$\bar{X}_N(t) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i(t), \qquad t \in \mathbb{Z}. \tag{2.14}$$

$\{\bar{X}_N(t), t \in \mathbb{Z}\}$ has the same second order characteristics as $\{X(t), t \in \mathbb{Z}\}$ process (the same covariance function and the same spectral density). In [77] it is proved, that $\{\bar{X}_N(t), t \in \mathbb{Z}\}$ converges to zero-mean Gaussian process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$,

$$\bar{X}_N(t) \to_{\mathrm{fdd}} \mathfrak{X}(t). \tag{2.15}$$

The limit process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ is ergodic, has a spectral density as in (2.13) and can be seasonally long-range-dependent, i.e.

$$\mathrm{Cov}(\mathfrak{X}(t), \mathfrak{X}(t+h)) \sim |h|^{-2d-1} \beta(h), \qquad \text{as} \quad h \to \infty,$$

for some $d \in (-1/2, 0)$, where $\beta(h)$ is an oscillating function. To show that the limit aggregated process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ can obtain seasonal long memory,

4. From the independence hypotheses and because $\mathrm{P}(|\alpha_j < 1|) = 1$, the series (2.12) convergece conditionaly a.s. for almost all α, and consequently it convergeces unconditionaly a.s.

G. Oppenheim and M.C. Viano [77] assumed that

$$A_{\alpha_i}(z) = (1-\alpha_{i,1}z)(1-\alpha_{i,2}z)\prod_{j=3}^{p+1}(1-\rho_{i,j}\mathrm{e}^{\mathrm{i}\theta_j}z)(1-\rho_{i,j}\mathrm{e}^{-\mathrm{i}\theta_j}z), \qquad i=1,\dots,N,$$

are the characteristic polynomials of independent AR($2p$) processes $\{X_i(t), t\in\mathbb{Z}\}$, $i=1,\dots,N$. Here θ_j, $j=3,\dots,p+1$, are fixed arguments in $(-\pi,\pi)\setminus\{0\}$; $\alpha_i := \{\alpha_{i,1}, -\alpha_{i,2}, \rho_{i,3},\dots,\rho_{i,p+1}\}$, $i=1,\dots,N$, are independent copies of random vector $\alpha := \{\alpha_1, -\alpha_2, \rho_3,\dots,\rho_{p+1}\}$, which components are independent and have the following density functions:

$$g_j(s) = (1-s)^{d_j}\psi_j(s), \qquad j=1,\dots,p+1,$$

where $\psi_j(a)$ is a continuous function at the point $s=1$, $\psi_j(1)>0$, $0<d_j<1$. In this case, the limit aggregated process $\{\mathfrak{X}(t), t\in\mathbb{Z}\}$ in (2.15) is zero-mean Gaussian process with the covariance function

$$\mathrm{Cov}(\mathfrak{X}(t),\mathfrak{X}(t+h)) = h^{-d}\Big(\sum_{\{k:\ d_k=d\}}\gamma_k\cos(h\theta_k)+o(1)\Big), \qquad \text{as } h\to\infty,$$

where $d=\min(d_j,\ 1\le j\le p+1)$ and γ_k, $k=1,\dots,p+1$, are some constants.

The obtained result shows that if the characteristic polynomial $A_\alpha(z)$ has complex conjugate roots, the covariance function of $\{\mathfrak{X}(t), t\in\mathbb{Z}\}$ has an oscillating component, the spectral density has singular points other than zero, and the limit aggregated process $\{\mathfrak{X}(t), t\in\mathbb{Z}\}$ obtains seasonal long memory.

Aggregation of ARMA(p, q) processes. P. Zaffaroni [102] noticed that the results of aggregation of AR(1) processes generalize to the case of aggregation of ARMA(p, q) processes. The ARMA(p, q) model contains autoregressive AR(p) and moving average MA(q) models:

$$A(L)X(t) = \Pi(L)Z(t), \qquad t\in\mathbb{Z}, \tag{2.16}$$

where

$$\begin{aligned} A(L) &= (1-a_1L-a_2L^2-\dots-a_pL^p),\\ \Pi(L) &= (1+\pi_1L+\pi_2L^2+\dots+\pi_pL^q). \end{aligned}$$

Assume, that $q<p$, $A(z)$ has distinct roots, the polynomials $A(z)$ and $\Pi(x)$ have no common zeroes and $A(z)\neq 0$, $\Pi(z)\neq 0$ for all $z\in\mathbb{C}$ such that $|z|\le 1$. Under these assumptions the process $\{X(t), t\in\mathbb{Z}\}$ is causal, invertible and the model (2.16) can be rewritten as

$$X(t) = \Big(\frac{\beta_1}{1-\alpha_1 L} + \cdots + \frac{\beta_p}{1-\alpha_p L}\Big) Z(t), \qquad t \in \mathbb{Z}, \tag{2.17}$$

where α_j, $j = 1, \ldots, p$, denotes the inverse of the roots of $A(z)$ and β_j, $j = 1, \ldots, p$, are constants depending on α_j, $j = 1, \ldots, p$, and π_j, $j = 1, \ldots, q$.

Suppose $\{X_i(t), t \in \mathbb{Z}\}$, $i = 1, \ldots, N$, are independent copies of (2.17) with $Z_i(t) = \rho_i u(t) + \varepsilon_i(t)$, where $u(t)$ is a common noise for all units and $\varepsilon_i(t)$ is a noise specific to each unit. The aggregated process can be splitted in two parts:

$$\bar{X}_N(t) = \frac{1}{N}\sum_{i=1}^{N} X_i(t) = U_{N,t} + E_{N,t},$$

where

$$\begin{aligned} U_{N,t} &= \frac{1}{N}\sum_{i=1}^{N} \rho_i \Big(\frac{\beta_{i,1}}{1-\alpha_{i,1} L} + \cdots + \frac{\beta_{i,p}}{1-\alpha_{i,p} L}\Big) u(t), \\ E_{N,t} &= \frac{1}{N}\sum_{i=1}^{N} \Big(\frac{\beta_{i,1}}{1-\alpha_{i,1} L} + \cdots + \frac{\beta_{i,p}}{1-\alpha_{i,p} L}\Big) \varepsilon_i(t). \end{aligned}$$

We can see, that $U_{N,t}$ and $E_{N,t}$ are very similar to components in (2.6). The results of aggregation of AR(1) processes generalize to the case of aggregation of ARMA(p, q) processes. The properties of the aggregated process depend on the distribution of the autoregressive root with the more dense near 1.

If $q = 0$, then the ARMA(p, q) process is the AR(p) process described in (2.10). In this case, the aggregated process (2.14) is equal to $\bar{X}_N(t) = \sqrt{N} E_{N,t}$, $t \in \mathbb{Z}$, and the limit aggregated process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ obtains seasonal long memory if the polynomial $A(z)$ has complex conjugate roots. The autocovariance function of the limit aggregated process has an oscillating component and the spectral density has singular points other than zero.

It should be noticed here that the moving average component has no effect on the memory of the limit aggregated process. In [41], [102] it is shown that if $p = 0$ and the behavior of units is described by the moving average MA(q) model

$$X_i(t) = \Pi_i(L)(\rho_i u(t) + \varepsilon_i(t)), \qquad t \in \mathbb{Z}, \quad i = 1, \ldots, N,$$

then the idiosyncratic component $E_{N,t}$ converges to 0 conditionally in L_2 and the limit of the common component $U_{N,t}$ is equal to

$$U_t = \mathrm{E}(\rho)(u(t) + \mathrm{E}(\pi_1) u(t-1) + \cdots + \mathrm{E}(\pi_q) u(t-q)).$$

2.1.2 Aggregation of random fields

Models of random fields were introduced by P. Whittle in 1954, [101]. Basic results about random fields can be found in [43], [51]. Long memory properties of random fields was investigated in [57], [58], [59], [61]. And the aggregation procedure of autoregressive random fields with finite variance was discussed in [57], [60], [61].

Consider the autoregressive random field

$$\sum_{k,l\in D} a_{k,l} X(t-k, s-l) = \varepsilon(t,s), \qquad (t,\ s)\in\mathbb{Z}^2, \tag{2.18}$$

where D is a finite subset of $\mathbb{Z}^2$, $(a_{k,l})_{(k,l)\in D}$ are real random coefficients and $\{\varepsilon(t,s),\ (t,s)\in\mathbb{Z}^2\}$ is a white noise in L_2 space. Let L_1 and L_2 be lag operators, i.e. $L_1X(t,s) = X(t-1,s)$, $L_2X(t,s) = X(t,s-1)$, and denote

$$P(z_1,z_2) := \sum_{k,l\in D} a_{k,l} z_1^k z_2^l$$

Then (2.18) can be rewritten in more compact form

$$P(L_1,L_2)X(t,s) = \varepsilon(t,s), \qquad (t,\ s)\in\mathbb{Z}^2. \tag{2.19}$$

If for every $(a_{k,l})$, $P(\mathrm{e}^{\mathrm{i}\lambda_1}, \mathrm{e}^{\mathrm{i}\lambda_2}) \neq 0$ for all $(\lambda_1,\lambda_2)\in[-\pi,\pi]^2$, (2.19) admits unique stationary solution (see [43], [57]), which is given by the series

$$X(t,s) = \sum_{k,l\in\mathbb{Z}^2} b_{k,l}\varepsilon(t-k,s-l), \qquad (t,\ s)\in\mathbb{Z}^2, \tag{2.20}$$

where $(b_{k,l})_{(k,l)\in\mathbb{Z}^2}$ are random coefficients of the Laurent expantion $P(z_1,z_2)^{-1} = \sum_{k,l\in\mathbb{Z}^2} b_{k,l} z_1^k z_2^l$. The series (2.20) converges in L_2 if and only if

$$\sum_{k,l\in\mathbb{Z}^2} \mathrm{E}(|b_{k,l}|^2) < \infty.$$

The spectral density of the random field (2.20) is

$$f(\lambda_1,\lambda_2) = \frac{\sigma_\varepsilon^2}{(2\pi)^2}\mathrm{E}\left|P(\mathrm{e}^{\mathrm{i}\lambda_1}, \mathrm{e}^{\mathrm{i}\lambda_2})\right|^{-2}, \qquad (\lambda_1,\lambda_2)\in[-\pi,\pi]^2, \tag{2.21}$$

where σ_ε^2 is the variance of the white noise.

Now suppose we have N independent copies $X_j(t,s)$, $j=1,\dots,N$, of (2.19). Define the aggregated random field

$$\bar{X}_N(t,s) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} X_j(t,s), \qquad (t,\ s) \in \mathbb{Z}^2. \tag{2.22}$$

From the central limit theorem it follows, that the limit of the aggregated process $\bar{X}_N(t,s)$, as $N \to \infty$, is a Gaussian random field $\mathfrak{X}(t,s)$, which has the same spectral density (2.21) as the aggregated field $\bar{X}_N(t,s)$ and individual fields $X_j(t,s)$, $j = 1, \ldots, N$ (see [60]).

Long memory properties and the dependence structure of the limit aggregated random field $\{\mathfrak{X}(t,s),\ (t,s) \in \mathbb{Z}^2\}$ strongly depends on what model of fields one uses to describe the behavior of individual fields. Lavancier[60] investigates the long memory properties of the limit aggregated random field $\{\mathfrak{X}(t,s),\ (t,s) \in \mathbb{Z}^2\}$ under assumption that the individual fields are described by the nearest-neighbor autoregressive random fields with finite variance [5]. Suppose, for example, we have N independent copies of the nearest-neighbor random field

$$X_j(t,s) = \frac{A}{4}(X_j(t-1,s)+X_j(t+1,s)+X_j(t,s-1)+X_j(t,s+1))+\varepsilon_j(t,s), \tag{2.23}$$

where $(t,\ s) \in \mathbb{Z}^2$, $j = 1, \ldots, N$, A is random coefficient and $\varepsilon_j(t,s)$ is white noise with variance $\sigma_\varepsilon^2 > 0$. If $|A| < 1$ almost surely, (2.23) admits stationary solution [6]. Define the aggregated random field as in (2.22). Then the limit of the aggregated random field (in the sense of finite dimensional distributions) is Gaussian random field:

$$\mathfrak{X}(t,s) = \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_j(t,s), \qquad (t,s) \in \mathbb{Z}^2.$$

Now the main question is: does $\{\mathfrak{X}(t,s),\ (t,s) \in \mathbb{Z}^2\}$ have the long memory property and in which sense? It is well known, that in finite variance case, the long memory property of the stationary random field can be described using its spectral density or covariance function. When the spectral density of the random field is unbounded or autocovariance function is non-summable, then the random field is said to exhibit long memory.

Definition 2.1.4. *([57], Def. 1)A stationary random field exhibits isotropic long memory if it admits a spectral density which is continuous everywhere except at 0, i.e. for $\lambda = (\lambda_1, \lambda_2) \in [-\pi, \pi]^2$,*

$$f(\lambda) \sim ||\lambda||^\alpha L\Big(\frac{1}{||\lambda||}\Big) b\Big(\frac{\lambda_1}{||\lambda||}, \frac{\lambda_2}{||\lambda||}\Big), \qquad as\ ||\lambda|| := \sqrt{\lambda_1^2 + \lambda_2^2} \to 0, \tag{2.24}$$

5. In Section 6, we investigate the aggregation of such fields in the case of infinite variance, i.e. we assume that innovations belong to the domain of attraction of an $\alpha-$stable law.

6. Such stationary solution converges conditionally in L_2. Under additional assumptions, it converges unconditionally in L_2.

where $-2 < \alpha < 0$*,* $L(\cdot)$ *- slowly varying function at infinity and* $b(\cdot)$ *is continuous function on the unit sphere in* $\mathbb{R}^2$*.*

Lavancier [60] proved, that the limit aggregated random field $\{\mathfrak{X}(t,s), (t,s) \in \mathbb{Z}^2\}$, accumulated from independent nearest-neighbor random fields (2.23), can admit isotropic long memory in the sense of Definition 2.1.4. Indeed, assume, that the density function of the coefficient A has the form

$$\phi(a) \sim \psi(a)(1-a)^{\beta}, \qquad \text{as } a \uparrow 1, \tag{2.25}$$

where $\psi(a)$ is bounded function, continuous at 1 with $\psi(1) > 0$, $\beta > -1$. Then the spectral density of the limit aggregated field is equal to

$$f(\lambda_1, \lambda_2) = \frac{\sigma_\varepsilon^2}{(2\pi)^2} \int_0^1 \frac{\psi(a)(1-a)^{\beta}}{(1 - 2a(\cos(\lambda_1) + \cos(\lambda_2)))^2} \mathrm{d}a. \tag{2.26}$$

In [60], it is proved, that this spectral density satisfies the condition (2.24),

$$f(\lambda_1, \lambda_2) \sim \begin{cases} C(\lambda_1^2 + \lambda_2^2)^{\beta - 1}, & \text{if } -1 < \beta < 1, \\ C \ln(\lambda_1^2 + \lambda_2^2), & \text{if } \beta = 1, \end{cases} \qquad \text{as } \sqrt{\lambda_1^2 + \lambda_2^2} \to 0,$$

and the limit aggregated random field $\mathfrak{X}(t,s)$ exibits *isotropic long memory*, if $-1 < \beta < 1$. Note, that when $\beta = 1$, the asymptotic of the spectral density does not exactly suit the latter definition, but it is unbounded function of $||\lambda||$. Therefore, in this case, we could also say, that random field exhibits isotropic long memory. For $\beta > 1$, the spectral density is continuous everywhere and $\mathfrak{X}(t,s)$ is *short-range dependent*.

To describe the dependence structure of a random field is more complicated than in a univariate process case, since dependence for a random field extends in all direction, while a univariate time series has only one direction. Actually, in the scientific literature there are many definitions of long memory property (see Subsection 2.3 for details). The usual definition of long memory is based on the spectral density function or the covariance function. However, in infinite variance case these definitions are not applicable. The best way to describe the dependence structure of random fields and processes is probably the investigation of partial sums and its limits under the suitable normalization. In this PhD thesis, the main definition of long memory is so called *distributional long memory* (Cox [28]). We say, that the random process has distributional long memory, if its normalized partial sums tend to the random process with dependent increments. In Section 6, we discuss the aggregation of nearest-neighbor autoregressive random fields with infinite variance and introduce the notion of anisotropic/isotropic distributional long memory for random fields on $\mathbb{Z}^2$.

2.2 Disaggregation

Studies of the aggregation problem showed that accumulation of short-memory processes can lead to long memory phenomena and that the aggregated process may exhibit long memory property. But the weak point of the aggregation is that by the accumulation of data we lose some information about the attributes of individual processes and the aggregated data are not so informative as the micro level data are. It is clear that if we have the samples of the individual processes, we can easy aggregate them and get an aggregated process. But what we can say about the behavior of individual processes if we have only a sample of the limiting aggregated process and samples of the individual processes remain unobserved? This is an interesting problem, which is so-called disaggregation problem. The disaggregation problem has been studied in [22], [25], [62], [65], [69] and by other authors under assumption that the individual processes have known structure, for instance AR(1), GARCH(1,1) and etc. The recovering the attributes of the individual behavior from panel data is also called as the disaggregation problem. Such approach of the disaggregation problem was discussed in [14], [90]. Let's now review methods of disaggregation in autoregressive aggregation scheme.

Disaggregation in AR(1) aggregation scheme. Suppose, the behavior of micro-units is described by AR(1) processes:

$$X_i(t) = a_i X_i(t-1) + \varepsilon_i(t), \qquad i = 1, 2, \dots, N, \quad t \in \mathbb{Z}, \tag{2.27}$$

where $X_i(t)$ describes an evolution of ith micro-unit; N is the number of units; $\varepsilon_i(t)$, $i = 1, \dots, N$, $t \in \mathbb{Z}$, are independent identically distributed random variables with $\mathrm{E}\varepsilon_i(t) = 0$ and $\sigma_\varepsilon^2 = \mathrm{E}\varepsilon_i(t)^2 < \infty$; a, a_i, $i = 1, \dots, N$, are i.i.d. random variables independent of innivations $\varepsilon_i(t)$, supported by $[-1, 1]$ and satisfying

$$\mathrm{E}\Big[\frac{1}{1-a^2}\Big] < \infty. \tag{2.28}$$

Under these conditions the equation (2.27) admits a stationary solution and the aggregated process

$$\bar{X}_N(t) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i(t).$$

converges to zero mean Gaussian process $\mathfrak{X}(t)$. Note, that the limit aggregated process $\mathfrak{X}(t)$ and the individual processes $X_i(t)$ have the same covariance function

$$r(t) := \mathrm{Cov}(X(0), X(t)) = \mathrm{Cov}(\mathfrak{X}(0), \mathfrak{X}(t)) = \sigma_\varepsilon^2 \, \mathrm{E}\Big[\frac{a^{|t|}}{1-a^2}\Big]. \tag{2.29}$$

Our goal is to construct an algorithm to estimate the density function $\phi(a)$ of random coefficient a (we call it a mixing density). The way of solution of this disaggregation problem depends on the assumptions put on the mixing density function. If we assume, that distribution of random coefficient belongs to some parametric family of distributions, for example is Beta distributed, then the main task is to find the estimate of unknown parameters. Robinson [90], Beran et al [14] gives the solution of this problem under assumption, that the samples of individual processes are known. Consider a panel of N independent AR(1) processes, each of length n. Assume also that a_i, $i = 1, \ldots, N$, are i.i.d. with a density function

$$\phi_{p,q}(a) = \frac{2}{B(p,q)} a^{2p-1}(1-a^2)^{q-1}, \quad a \in [0,1), p > 1, q > 1, \tag{2.30}$$

where the parameters p and q are unknown. To construct an estimator of these parameters, first of all define estimates of random coefficients a_i of autoregressive processes $X_i(t)$, $i = 1, \ldots, N$, as truncated version of lag-one correlation coefficient

$$\widehat{a}_{i,n,h} = \min\{\max\{\widehat{a}_{i,n},\ h\},\ 1-h\}, \qquad h = h(N,n) > 0,\ h \to 0,\ \text{as } N, n \to \infty,$$

where

$$\widehat{a}_{i,n} = \frac{\sum_{t=1}^n X_i(t) X_i(t-1)}{\sum_{t=1}^n X_i^2(t)}, \quad n \geq 1.$$

In this way we obtain N "pseudo" observations $\widehat{a}_{1,n,h}$, $\widehat{a}_{2,n,h}$, ..., $\widehat{a}_{N,n,h}$ of r.v. a based on observations $X_i(t)$, $i = 1, \ldots, N$, $t = 0, \ldots, n$. The unobserved AR(1) coefficients are replaced by their estimates. In the second step, the parameters p and q of the mixing distribution in (2.30) are estimated by maximizing the likelihood, viz. $(\widehat{p}, \widehat{q}) = \arg\max_{p,q} \prod_{i=1}^N \phi_{p,q}(\widehat{a}_{i,n,h})$. Beran et al. [14] proved the consistency in probability of the above maximum likelihood estimator and its asymptotic normality with the convergence rate $\sqrt{N}$ under the following conditions on the sample sizes and the truncation parameter h: $n \to \infty$, $N \to \infty$, $h \to 0$, $(\log(h))^2/\sqrt{N} \to 0$, $\sqrt{N} h^{\min(p,q)} \to 0$ and $\sqrt{N} h^{-2} n^{-1} \to 0$.

Now let us discuss the disaggregation problem under assumption that only the aggregated data are at hand and samples of the individual processes remain unobserved. Such disaggregation approach has been studied in [22], [25], [62], [69].

Leipus et al [62] assumed the following semiparametric form of the mixing density:

$$\phi(a) = (1-a)^{d_1}(1+a)^{d_2}\psi(a), \qquad d_1 > 0, \quad d_2 > 0, \tag{2.31}$$

where $\psi(a)$ is continuous on $[-1,1]$ and does not vanishes at $+1$, -1, and proposed an estimator of $\phi(a)$, which is based on the expantion of the density function on the basis of orthogonal Gegenbauer polynomials:

$$\widehat{\phi}_n(a) := (1-a^2)^{\alpha}\frac{1}{\sigma_\varepsilon^2}\sum_{k=0}^{K_n}\widehat{\zeta}_{n,k}G_k^{(\alpha)}(a), \tag{2.32}$$

where

- The coefficients $\widehat{\zeta}_{n,k}$ are defined as follows

$$\widehat{\zeta}_{n,k} := \sum_{j=0}^{k} g_{k,j}^{(\alpha)}(\widehat{r}_n(j) - \widehat{r}_n(j+2)), \tag{2.33}$$

  where $\widehat{r}_n(j) = \frac{1}{n}\sum_{i=1}^{n-j}\mathfrak{X}(i)\mathfrak{X}(i+j)$ is the sample covariance of the zero mean aggregated process $\{\mathfrak{X}(t),\, t\in\mathbb{Z}\}$ and n is the number of observations, $\mathfrak{X}(1)$, $\mathfrak{X}(2)$, ..., $\mathfrak{X}(n)$.
- $G_k^{(\alpha)}(x) = \sum_{j=0}^{k} g_{k,j}^{(\alpha)}x^j$, $k = 0, 1, \ldots$, $\alpha > -1$, are orthogonal Gegenbauer polynomials,

$$\int_{-1}^{1} G_j^{(\alpha)}(x)G_k^{(\alpha)}(x)(1-x^2)^{\alpha}\,\mathrm{d}x = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k; \end{cases}$$

- $\sigma_\varepsilon^2 = \mathrm{Var}(\varepsilon) = \mathrm{E}\varepsilon^2$ is known variance of zero mean innovations;
- (K_n) is a nondecreasing sequence which tends to infinity at rate $[\gamma\log(n)]$, $0 < \gamma < (2\log(1+\sqrt{2}))^{-1}$. This assumption on K_n convergence rate is needed to get convergence to zero of the mean integrated square error of $\widehat{\phi}_n(x)$, i.e.

$$\lim_{n\to\infty}\int_{-1}^{1}\frac{\mathrm{E}(\widehat{\phi}_n(x) - \phi(x))^2}{(1-x^2)^{\alpha}}\,\mathrm{d}x = 0; \tag{2.34}$$

It should be noted here, that the estimator (2.32) is correct under assumption that the individual process and the aggregated process have the same autocovariance function in (2.29). If micro-units depend on common innovations, these covariance functions are not the same. Therefore common innovations in this case are not allowed here.

Leipus et al [62] showed the consistensy of the estimator (2.32) under assumption that the variance of the noise, $\sigma_\varepsilon^2 = r(0) - r(2)$, is known. But usually in practice σ_ε^2 is unknown and we need to estimate it. Celov et al [22] used the following eastimator of σ_ε^2,

$$\widehat{\sigma}_\varepsilon^2 = \widehat{r}_n(0) - \widehat{r}_n(2),$$

where $\widehat{r}_n(h)$ is a sample covariance function of the aggregated process, and, under mild conditions on the (semiparametric) form of the mixing density (2.31), proved the asymptotic normality of the estimator (2.32):

$$\frac{\widehat{\phi}_n(x) - \mathrm{E}\widehat{\phi}_n(x)}{\sqrt{\mathrm{Var}(\widehat{\phi}_n(x))}} \to_{\mathrm{d}} N(0,1),$$

for every fixed $x \in (-1;1)$, such that $\phi(x) \neq 0$.

Results in [62] and [22] were obtained for a Gaussian aggregated processes. In Section 5.4, we extend these results to the case when the aggregated process is a mixed ID moving-average (5.4), page 84. Under the finiteness of 4th moment, we obtained the weak consistensy of the mixing density estimator in a suitable L_2-space (Theorem 5.4.4, page 109).

As was noticed above, the estimator (2.32) of the mixing density is not correct in the precence of common innovation, because the covariance functions of the aggregated process and the underlying process do not coincide. Chong [25] proposed another estimator of the mixing density $\phi(x)$, assuming, that it belongs to the class of polynomial densities, i.e.

$$\phi(x) = \sum_{k=0}^{m} c_k x^k \mathbf{1}_{x\in[0,1)}, \quad m \in \mathbb{N}, \quad \phi(x) \geq 0, \quad \int_0^1 \phi(x)\,\mathrm{d}x = \sum_{k=0}^{m} \frac{c_k}{k+1}. \tag{2.35}$$

It is not difficult to see, that in this case,

$$\mathrm{E}a^r = \sum_{k=0}^{m} \frac{c_k}{k+r+1}, \quad r = 1, \dots, m. \tag{2.36}$$

In order to have an estimator of mixing density in (2.35), we need to estimate unknown coefficients c_k, $k = 0, \dots, m$, and the polynomial order m. Consider the case of AR(1) aggregation with common innovations,

$$X_i(t) = a_i X_i(t-1) + u(t) + \varepsilon_i(t), \qquad i = 1, 2, \dots, N, \quad t \in \mathbb{Z}.$$

The limit of the aggregated process $\bar{X}_N(t) := \frac{1}{N}\sum_{i=1}^N X_i(t)$ is

$$\mathfrak{X}(t) := \sum_{r=0}^{\infty} \mathrm{E}a^r u(t-r) = \Phi(L)u(t),$$

where $\Phi(L) := \sum_{r=0}^{\infty} \mathrm{E}a^r L^r$. If $\mathfrak{X}(t)$ is invertible, we can rewrite

$$\mathfrak{X}(t) = \sum_{j=1}^{\infty} A_j \mathfrak{X}(t-j) + u(t).$$

Since it is impossible to estimate an autoregression of infinite order, we have to make a truncation at a fixed order H,

$$\mathfrak{X}(t) = \sum_{j=1}^{H} A_j \mathfrak{X}(t-j) + u(t).$$

Given the data of the aggregated process $\mathfrak{X}(t)$, coefficients A_j can be estimated, for example, by solving the Yule-Walker equations. Then the estimates of $\mu_s := \mathrm{E}a^s$ can be found from recursive equations

$$\widehat{\mu}_s = \sum_{r=0}^{s-1} \widehat{\mu}_r \widehat{A}_{s-r}, \qquad \widehat{\mu}_0 = 1.$$

(The last equality follows from the relation between coefficients of AR and MA representations.) Having estimators of moments $\mu_s := \mathrm{E}a^s$ and using the relation (2.36), it is not difficult to calculate estimates of coefficients c_k in (2.35). The estimate of an unknown polynomial order m could be defined as a value, which minimize the distance between empirical and theoretical autocorrelation functions (for more details the reader is referred to [25]).

The Chong's estimator of the mixing density function $\phi(x)$ is justified only for the class of polynomial densities. But the advantage of this estimator is that it remains correct in the presence of common innovations, whilemean the estimator in (2.32) is not valid in this case. The comparison of these estimation methods is given in [20]. Examining results of Monte-Carlo simulations it is shown (in [20]) that none of the methods was found to outperform another.

Disaggregation of autoregressive fields. The disaggregation problem of autoregressive random fields was discussed in [65]. N. Leonenko and E. Taufer [65] extended results of Leipus et al [62] from one-dimensional to spatial autoregressive processes. The authors assumed that the aggregated Gaussian random field

$$\mathfrak{X}(t,s) = \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i(t,s)$$

is obtained by accumulation of i.i.d. random fields:

$$X_i(t,s) = \theta_{1,i} X_i(t-1,s) + \theta_{2,i} X_i(t,s-1) - \theta_{1,i}\theta_{2,i} X_i(t-1,s-1) + \varepsilon(t,s),$$

where $i = 1, 2, \ldots, N$, $(t,s) \in \mathbb{Z}^2$, $\{\varepsilon(t,s),\ (t,s) \in \mathbb{Z}^2\}$ is a white noise with zero mean and finite variance σ^2; coefficients $(\theta_{1,i}, \theta_{2,i})$, $i = 1, 2, \ldots, N$ are independent copies of a random vector (θ_1, θ_2) supported on $[-1,1]^2$ with density function $\phi(\theta_1, \theta_2)$. It is proved under some assumptions in [65], that the mean integrated square error of the estimator $\widehat{\phi}_n(\theta_1, \theta_2)$ (which is based on the expantion of the density funtion on the basis of two-dimensional orthogonal Gegenbauer polynomials) converge to zero, as in one-dimensional case (2.34). For more details we refer the reader to [65].

2.3 Long memory

The phenomenon of long memory is a widely studied subject and has long history. There are many publications addressed to detection of long memory in the data, limit theorems under long memory, statistical estimation of memory parameters, simulation of long memory processes, and many others. But the first main question is what is the long memory. There are many definition of long memory, they vary from author to author and are not always equivalent. As it was noted in [93], the history of long memory as a concrete phenomenon begins in the 1960s with a series of papers of B. Mandelbrot and his co-authors, when the Hurst phenomenon was explaned. British hydrologist H. Hurst studied the flow of water in Nile river and wanted to model them so that architects could construct a reservoir system. In 1951, H. Hurst [49] showed that the aggregated water flows in year depends not only on the flows in recent year but also on flows in year before the present year. He introduced the rescaled range statistic R/S:

$$\frac{R}{S}(X_1, X_2, \dots, X_n) = \frac{\max_{0\le i\le n}(\sum_{i=1}^n X_i - i\bar{X}) - \min_{0\le i\le n}(\sum_{i=1}^n X_i - i\bar{X})}{\sqrt{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2}},$$

where X_1, X_2, ..., X_n are observations, $\bar{X} = n^{-1}\sum_{i=1}^n X_i$ is sample mean of the data. H. Hurst got that the empirical rate of growth of R/S statistic on the Nile river data is close to $n^{0.74}$. This phenomenon, called Hurst phenomenon, was explain and advanced by Mandelbrot and co-workers [71], [72], [73]. It is known that if X_1, X_2, ..., X_n are finite-variance independent and identically distributed random variables, then the rate of growth of R/S statistic is $n^{0.5}$. The idea to explain the Hurst phenomenon was to take a stationary process $\{X_t,\ t \in \mathbb{Z}\}$ with slowly decaying covariance function (see [73]). And this idea was successful. It was proved that for the fractional Gaussian noise (the unit difference of fractional Brownian motion B_H)

$$X_j := B_H(j) - B_H(j-1),$$

with the covariance function

$$\mathrm{Cov}(X_{j+n}, X_j) = \frac{\sigma}{2}[(n+1)^{2H} + |n-1|^{2H} - 2n^{2H}],$$

the R/S statistic grows at the rate n^H. In this way the term of "long memory" came into beying.

Most of the definitions of long memory are based on the second-order properties (covariance, spectral density) of a stochastic process $\{X(t),\, t \in \mathbb{Z}\}$. Such properties are relatively simple and it is not difficult to estimate them from the given data. However, when the process does not have finite variance, the usual definitions of long memory in terms of covariance/spectrum are not applicable. Among the alternative notions of long memory, which do not require finite variance, we mention the (decay rate of) codifference (see Samorodnitsky and Taqqu [94]), distributional long memory (see Cox, [28]), and long-range dependence (sample Allen variance) (LRD(SAV)) (see Heyde and Yang[45]), also characteristics of dependence, like covariation or $\alpha-$covariance, for stable processes expressed in terms of the spectral measure (Samorodnitsky and Taqqu [94], Paulauskas [79]).

Before introducing detailed definitions of long memory, let us take a look to some properties of functions.

Definition 2.3.1.

- *A positive measurable function $L(h)$ defined on some neighborhood $[a, \infty)$ of infinity is said to be slowly varying if for any c, $c > 0$,*

$$\frac{L(cx)}{L(x)} \to 1, \qquad \text{as } x \to \infty.$$

- *Let $B \subseteq (0, \infty)$ be a compact set, the total variation of the real-valued function f on B is*

$$v(f, B) = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

here the supremum is over all finite sequences $x_0 \le x_1 \le \cdots \le x_n$ in B.

- *A function f is said to be of locally bounded variation on $(0, \infty)$, if $v(f, B) < \infty$ for each compact set $B \subseteq (0, \infty)$.*
- *A positive function f of locally bounded variation on $(0, \infty)$ is said to be quasi-monotone, if for some $\delta > 0$,*

$$\int_0^x t^\delta \, |\mathrm{d}f(t)| = O(x^\delta f(x)), \qquad \text{as } x \to \infty.$$

Now we can discuss definitions of long memory.

Definition 2.3.2. *A stationary process* $\{X(t),\, t \in \mathbb{Z}\}$ *has a long memory property, if the autocovariance function* $r(h) = \mathrm{Cov}(X(t), X(t+h))$ *is not absolutely summable*

$$\sum_{h \in \mathbb{Z}} |r(h)| = \infty. \tag{2.37}$$

Definition 2.3.3. *A stationary process* $\{X(t),\, t \in \mathbb{Z}\}$ *has a long memory property, if the autocovariance functions decays hyperbolically, as* $h \to \infty$*,*

$$r(h) \sim h^{2d-1} L(h), \qquad 0 < d < 1/2, \tag{2.38}$$

where d *is long-memory parameter,* $L(\cdot)$ *is a slowly varying function at infinity.*

The covariance function of a stationary process can be written in such form:

$$r(h) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}h\lambda}\, \mathrm{d}F(\lambda),$$

where the function F is non-decreasing, right-continuous, bounded over $[-\pi, \pi]$, and $F(-\pi) = 0$. Such function F is called *the spectral distribution*, and if

$$F(\lambda) = \int_{-\pi}^{\lambda} f(\omega)\, \mathrm{d}\omega,$$

the function $f(\cdot)$ is called *the spectral density* of $r(\cdot)$. The spectral density function can also be used to describe the dependence in time series.

Definition 2.3.4. *A stationary process* $\{X(t),\, t \in \mathbb{Z}\}$ *has a long memory property, if its spectral density function satisfies*

$$f(\lambda) \sim |\lambda|^{-2d} L(1/|\lambda|), \qquad 0 < d < 1/2, \qquad \textit{as } |\lambda| \to 0, \tag{2.39}$$

and $L(\cdot)$ *is a slowly varying function at infinity.*

Another definition of long memory is based on the $X(t)$'s Wold decomposition $X(t) = \sum_{j=0}^{\infty} \psi_j \varepsilon(t-j)$.

Definition 2.3.5. *A stationary time series* $\{X(t),\, t \in \mathbb{Z}\}$ *is a long memory time series, if the coefficient* ψ_j *in purely non-deterministic part of the* $X(t)$*'s Wold decomposition satisfies*

$$\psi_j \sim j^{d-1} L(j), \qquad 0 < d < 1/2, \tag{2.40}$$

where $L(h)$ *is a slowly varying function at infinity.*

Palma [78] described all above mentioned Definitions 2.3.2 - 2.3.5 of long memory and compared them. These four definitions are not necessarily equivalent.

Palma (see [78], Thm 3.1) proved the following relations between these definitions:

– If the process $\{X(t),\, t \in \mathbb{Z}\}$ satisfies (2.38), it also satisfies (2.37).

– If the process $\{X(t),\, t \in \mathbb{Z}\}$ satisfies (2.40), it also satisfies (2.38).

– If the function $L(\cdot)$ in (2.38) is quasi-monotone slowly varying, then (2.38) implies (2.39).

Let us discuss now two definitions of long memory, which are based on limits of partial sums of the process.

Definition 2.3.6. *(See [28]). A strictly stationary time series $\{X(t),\, t \in \mathbb{Z}\}$, is said to have distributional long memory (respectively, distributional short memory) if there exist some constants $A_n \to \infty$, $n \to \infty$, and B_n and a stochastic process $\{J(t), t \geq 0\} \not\equiv 0$ with dependent increments (respectively, with independent increments), such that*

$$A_n^{-1} \sum_{s=1}^{[nt]} (X(s) - B_n) \ \to_{\mathrm{fdd}} \ J(t), \tag{2.41}$$

Lamperti [56] showed that under mild additional assumptions the normalizing constant A_n in (2.41) grows as n^H (with some $H > 0$), more precisely, $A_n = L(n)n^H$, where $L(n)$ is a slowly varying function at infinity, and the limit process $\{J(t), t \geq 0\}$ is self-similar with index H.

Definition 2.3.7. *(See [45]). A strictly stationary time series $\{X(t),\, t \in \mathbb{Z}\}$, is called LRD(SAV) if*

$$\frac{\left(\sum_{t=1}^{n} X(t)\right)^2}{\sum_{t=1}^{n} X^2(t)} \ \to_{\mathrm{p}} \ \infty; \tag{2.42}$$

otherwise $\{X(t),\, t \in \mathbb{Z}\}$ is called SRD(SAV).

Now, for a strictly stationary process $\{X(t),\, t \in \mathbb{Z}\}$, define a quantity

$$\mathrm{Cod}(X(0), X(t)) := \log \mathrm{E}e^{\,\mathrm{i}(X(t)-X(0))} - \log \mathrm{E}e^{\,\mathrm{i}X(t)} - \log \mathrm{E}e^{-\,\mathrm{i}X(0)}, \tag{2.43}$$

which is called the *codifference* of random variables's $X(0)$ and $X(t)$. Long memory of $\{X(t),\, t \in \mathbb{Z}\}$ can be characterized by the decay rate of $\mathrm{Cod}(X(0), X(t))$ (see [94]).

Definition 2.3.8. *A strictly stationary time series $\{X(t),\, t \in \mathbb{Z}\}$ has long memory property, if its codifference satisfies*

$$\sum_{h \in \mathbb{Z}} |\mathrm{Cod}(X(0), X(h))| = \infty. \tag{2.44}$$

Note that the existence of $\mathrm{Cod}(X(0), X(t))$ does not require any moments. For stationary stable or heavy tailed moving averages and some other processes with long memory, the asymptotics of $\mathrm{Cod}(X(0), X(t))$ was investigated in [7], [9], [55]. In particularly, if $\{X(t),\, t \in \mathbb{Z}\}$, is a stationary Gaussian process, with zero mean, unit variance, then $\mathrm{Cod}(X(0), X(t)) = (1/2)\mathrm{Cov}(X(0), X(t))$.

The dependence structure of random fields is more complicated than in a univariate processes, because the intensity of long memory can be different for different directions. In the case of finite variance, the long memory of the stationary random fields can be described using its second order properties (covariance function or spectral density). We say that a stationary random field $X(t_1, t_2)$ has long memory if its covariance function $r(h) := \mathrm{Cov}(X(t_1, t_2), X(t_1 + h_1, t_2 + h_2))$, $h = (h_1, h_2) \in \mathbb{Z}^2$, is not absolutely summable,

$$\sum_{h \in \mathbb{Z}^2} |r(h)| = \infty. \tag{2.45}$$

or behaves at infinity as

$$r(h) \sim \|h\|^{\alpha-1} L\Big(\frac{1}{\|h\|}\Big) b\Big(\frac{h}{\|h\|}\Big), \qquad \text{as } \|h\| \to \infty, \tag{2.46}$$

where $0 < \alpha < 2$, $\|.\|$ denotes the Euclidean norm, $L(\cdot)$ is a slowly varying function at infinity and $b(\cdot)$ is continuous function on the unit sphere in $\mathbb{R}^2$. An alternative definition of long memory involves properties of the spectral density function. A random field is said to exhibit isotropic long memory if its spectral density is unbounded and

$$f(\lambda) \sim \|\lambda\|^{-\alpha} L\Big(\frac{1}{\|\lambda\|}\Big) b\Big(\frac{\lambda}{\|\lambda\|}\Big), \qquad \text{as } \|\lambda\| \to 0, \tag{2.47}$$

where $\lambda := (\lambda_1, \lambda_2)$, $0 < \alpha < 2$, $\|.\|$ denotes the Euclidean norm, $L(\cdot)$ is a slowly varying function at infinity and $b(\cdot)$ is continuous function on the unit sphere in $\mathbb{R}^2$. Note that conditions (2.47) and (2.46) are not equivalent. The random field exhibits isotropic long memory and its spectral density satisfies condition (2.47) if the covariance of random field satisfies the condition (2.46) and the spectral density is continuous outside 0. If spectral density is unbounded and not continuous outside 0, then the long memory is non-isotropic, for example, if we investigate random field

$$X(t, s) = aX(t + 1, s - 1) + \varepsilon(t, s),$$

where a is random coefficient with the density function $\phi(x) \sim c(1 - x)^{\beta}$, as $x \uparrow 1$,

$\beta > -1$, then the spectral density of the random field $X(t,s)$ satisfies

$$f(\lambda_1, \lambda_2) \sim c\, |\lambda_2 - \lambda_1|^{\beta-1}, \qquad \text{as } |\lambda_2 - \lambda_1| \to 0.$$

Therefore, the long memory is non-isotropic in this case (see [57]).

In the Chapter 6 we introduce the new notion of anisotropic/isotropic long memory for random fields on $\mathbb{Z}^2$, which is based on the behavior of partial sums and does not require finite variance of random field.

The notion of long memory is polysemous, especially for infinite-variance processes, and is not limited to the characterization properties mentioned above. There are many definitions of long memory, and they are not always equivalent.

3

Aggregation of AR(1) process with infinite variance and common innovations

Abstract. Aggregation of random-coefficient AR(1) processes

$$X_i(t) = a_i X_i(t-1) + \varepsilon(t), \quad t \in \mathbb{Z}, \quad i = 1, \dots, N,$$

with i.i.d. coefficients $a_i \in (-1, 1)$ and common i.i.d. innovations $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ belonging to the domain of attraction of an $\alpha-$stable law, $0 < \alpha \leq 2$, is discussed. Particular attention is given to the case of slope coefficient having probability density growing regularly to infinity at points $a = 1$ and $a = -1$. Conditions are obtained under which the limit aggregated process $\mathfrak{X}(t) = \lim_{N\to\infty} N^{-1} \sum_{i=1}^{N} X_i(t)$ exists and exhibits long memory, in certain sense. In particularly, we show that suitably normalized partial sums of the $\mathfrak{X}(t)$'s tend to fractional $\alpha-$stable motion, and that $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ satisfies the LRD(SAV) property of Heyde and Yang [45], and can have distributional long memory of Cox [28].

3.1 Introduction

The present chapter extends the results of Zaffaroni [102] on aggregation of random-coefficient AR(1) processes from finite variance case to infinite variance case. Here, we discuss only the case of *common* innovations of the aggregated series. The case of *idiosyncratic* innovations belonging to the domain of attraction of a stable distribution will be discussed in Chapter 4 (see also [86]).

Let us describe the main results of this Chapter. Suppose, the behavior of micro units is described by random-coefficient AR(1) processes

$$X_i(t) = a_i X_i(t-1) + \varepsilon(t), \qquad i = 1, 2, \ldots, \quad t \in \mathbb{Z},$$

where $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ are common i.i.d. innovations with generic distribution ε, satisfying $\mathrm{E}|\varepsilon|^p < \infty$, for some $0 < p \le 2$, and $\mathrm{E}\varepsilon = 0$, $1 \le p \le 2$; $\{a_i, i = 1, \ldots, N\}$ are i.i.d. r.v.'s independent of $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ and having a common distribution a, $a \in (-1, 1)$ almost surely. Theorem 3.2.4 obtains sufficient conditions for convergence in probability of the aggregated process $\bar{X}_N(t) := N^{-1} \sum_{i=1}^N X_i(t)$ to a stationary moving average

$$\mathfrak{X}(t) = \sum_{j=0}^{\infty} \bar{a}_j \varepsilon(t-j), \qquad \bar{a}_j = \mathrm{E}a^j. \tag{3.1}$$

In the case $1 \le p \le 2$, the sufficient condition for such convergence is

$$\mathrm{E}\Big[\frac{1}{(1-|a|^p)^{1/p}}\Big] \;<\; \infty. \tag{3.2}$$

The last condition also implies $\sum_{j=0}^{\infty} (\mathrm{E}|a^j|)^p < \infty$ so that the process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ is well-defined.

In Sections 3.3 - 3.5, we study the case when the innovations $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ belong to the domain of attraction of an $\alpha-$stable law, $0 < \alpha \le 2$, and the probability density ϕ of r.v. $a \in (-1, 1)$ takes the form

$$\phi(x) \;=\; (1-x)^{-d_1}(1+x)^{-d_2}\psi(x), \qquad -1 < x < 1 \tag{3.3}$$

where parameters d_1, d_2 satisfy $0 < d_1, d_2 < 1$ and where $\psi \ge 0$ is an integrable function on the interval $(-1, 1)$ having finite limits $\psi_1 = \lim_{x \to 1} \psi(x)$, $\psi_2 = \lim_{x \to -1} \psi(x)$. A particular case of (3.3) is Beta distributed $a \in (0, 1)$ with the density function

$$\phi(x) = B(d_1, 1-d_1)^{-1} x^{d_1 - 1}(1-x)^{-d_1}, \; 0 < x < 1.$$

In the latter case,

$$\bar{a}_j \;=\; \frac{1}{B(d_1, 1-d_1)} \int_0^1 x^{d_1+j-1}(1-x)^{-d_1}\,\mathrm{d}x \;=\; \frac{\Gamma(j+d_1)}{\Gamma(j+1)\Gamma(d_1)}, \quad j = 0, 1, .. \tag{3.4}$$

are FARIMA$(0, d_1, 0)$ coefficients. More generally, if (3.3) holds with $0 < d_2 < d_1 < 1$, $\psi_1 > 0$, then the coefficients $\bar{a}_j$ decay as j^{d_1-1} similarly as in the case of FARIMA$(0, d_1, 0)$ process (see Proposition 3.3.1, page 49). Section 3.3 introduces a time domain generalization of I(d) filter (Definition 3.3.3, page 52). We show that, under some regularity conditions of the function ψ in (3.3) at the ends of the interval $(-1, 1)$, the 'mixed' coefficients $\bar{a}_j = \mathrm{E}a^j$ form an I(d_1) filter in the sense of this definition.

The most interesting case which can lead to long memory of the limit aggregated process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ in (3.1) is $1 < \alpha \le 2$. In this case, condition (3.2) for mixing density in (3.3) with $\psi_i > 0, i = 1, 2$ is satisfied if and only if

$$d_i < 1 - (1/\alpha), \quad i = 1, 2. \tag{3.5}$$

Section 3.4 studies long memory properties of the corresponding limit aggregated process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ in (3.1). Since we are dealing with infinite variance processes, the usual definitions of long memory in terms of covariance/spectrum are not applicable. According to Corollary 3.4.2, page 57, if (3.5) holds (and $1 < \alpha \le 2$, $\psi_1 > 0$), then $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ enjoys the so-called long-range dependence (sample Allen variance) property of Heyde and Yang [45], and the distributional long memory of Cox [28]; in particularly, its normalized partial sums process converges to a fractional stable motion with self-similarity parameter $H = d_1 + 1/\alpha \in (1/\alpha, 1)$. See Section 3.4 for definitions and precise formulations.

Section 3.5 considers the case of $1 - (1/\alpha) < d_1 < 1$, or nonstationary limit aggregate. In this case, the stationary infinite order moving average process in (3.1) is not defined. Following Zaffaroni [102], we consider aggregation of random coefficient AR(1) processes $\{Y_i(t),\, t = 1, 2, \ldots\}$, $i = 1, \ldots, N$, with zero initial condition $Y_i(0) = 0$. According to Proposition 3.5.1, page 58, in such case the limit aggregated process $\bar{Y}(t) = \lim_{N\to\infty} N^{-1} \sum_{i=1}^{N} Y_i(t)$ is nonstationary and the normalized process $\frac{1}{n^{d_1+1/\alpha-1}} \bar{Y}([n\tau]), \tau \in [0, \infty)$ converges, in the sense of weak convergence of finite dimensional distributions, to an $\alpha-$stable self-similar process given by a stochastic integral with respect to stable motion.

3.2 The limit of the aggregated process

Consider a random-coefficient AR(1) process

$$X(t) = aX(t-1) + \varepsilon(t), \qquad t \in \mathbb{Z}, \tag{3.6}$$

where $\{\varepsilon, \varepsilon(t), t \in \mathbb{Z}\}$ are i.i.d. r.v.'s and where a is a r.v., independent of innovations $\{\varepsilon(t), t \in \mathbb{Z}\}$ and satisfying $|a| < 1$ a.s.

Definition 3.2.1. *Write $\varepsilon \in D(\alpha)$, $0 < \alpha \le 2$, if*

(i) $\alpha = 2$ *and* $\mathrm{E}\varepsilon = 0$, $\sigma^2 := \mathrm{E}\varepsilon^2 < \infty$, *or*

(ii) $0 < \alpha < 2$ *and there exist some constants* $c_1, c_2 \ge 0, c_1 + c_2 \ne 0$ *such that*

$$\lim_{x\to\infty} x^\alpha \mathrm{P}(\varepsilon > x) \;=\; c_1 \quad \textit{and} \quad \lim_{x\to-\infty} |x|^\alpha \mathrm{P}(\varepsilon \le x) \;=\; c_2.$$

moreover, $\mathrm{E}\varepsilon = 0$ *whenever* $1 < \alpha < 2$, *while, for* $\alpha = 1$, *we assume that the distribution of* ε *is symmetric.*

Remark 3.2.2. (i) Condition $\varepsilon \in D(\alpha)$ means that r.v. ε belongs to the domain of normal attraction of an $\alpha-$stable law; in other words,

$$n^{-1/\alpha} \sum_{i=1}^{n} \varepsilon_i \;\to_{\mathrm{d}}\; Z, \tag{3.7}$$

where Z is an $\alpha-$stable r.v., see [36]. The characteristic function of r.v. Z is given by

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta Z} \;=\; \mathrm{e}^{-|\theta|^\alpha \omega(\theta)}, \quad \theta \in \mathbb{R}, \tag{3.8}$$

where

$$\omega(\theta) := \begin{cases} \frac{\Gamma(2-\alpha)}{1-\alpha}\Big((c_1+c_2)\cos(\pi\alpha/2) -\, \mathrm{i}(c_1-c_2)\mathrm{sign}(\theta)\sin(\pi\alpha/2)\Big), & \alpha \ne 1, 2, \\ (c_1+c_2)(\pi/2), & \alpha = 1, \\ \sigma^2/2, & \alpha = 2. \end{cases} \tag{3.9}$$

(ii) Condition $\varepsilon \in D(\alpha)$ implies $E|\varepsilon|^p < \infty$ for any $0 < p < \alpha$.

Proposition 3.2.3. *(i) Assume* $\mathrm{E}|\varepsilon|^p < \infty$, *for some* $0 < p \le 2$ *and* $\mathrm{E}\varepsilon = 0$, $p \ge 1$. *Then there exists a unique strict stationary solution to equation* (3.6) *given by the series*

$$X(t) \;=\; \sum_{k=0}^{\infty} a^k \varepsilon(t-k). \tag{3.10}$$

The series in (3.10) *converge conditionally a.s. and in* L_p, *for a.e.* $a \in (-1, 1)$. *Moreover, if*

$$\mathrm{E}\Big[\frac{1}{1-|a|^p}\Big] \;<\; \infty, \tag{3.11}$$

then the series in (3.10) *converge unconditionally in* L_p.

(ii) Assume that $\varepsilon \in D(\alpha)$, *for some* $\alpha \in (0, 2]$, *and condition* (3.11), *for some* $0 < p < \alpha$. *Moreover, if* $\alpha = 1$, *assume additionally that* $\mathrm{E}(1-|a|^p)^{-1-2(1-p)/p} < \infty$ *for some* $0 < p < 1$. *Then* $X(t) \in D(\alpha)$.

Proof. (i) Let us prove first that equation (3.6) admits a unique stationary solution. Let $\{X(t)\}$, $\{X'(t)\}$ be two such solutions. By iteration we have that for any $n > 0$

$$X(0) = \varepsilon(0) + a\varepsilon(-1) + \cdots + a^{n-1}\varepsilon(-n+1) + a^n X(-n)$$

and a similar equation holds for $X'(0)$. Hence

$$X(0) - X'(0) = a^n(X(-n) - X'(-n)),$$

or

$$|X(0) - X'(0)| \le |a|^n(|X(-n)| + |X'(-n)|).$$

For any $\epsilon > 0,\ 0 < \delta < 1,\ K > 0$ we can write

$$\begin{aligned} \mathrm{P}(|X(0) - X'(0)| > \epsilon) \ &\le\ \mathrm{P}(|a| > 1-\delta) + \mathrm{P}(|X(-n)| > K) + \mathrm{P}(|X'(-n)| > K) \\ &\quad + \mathrm{P}(2(1-\delta)^n K > \epsilon). \end{aligned}$$

Since $|a| < 1$ a.s., so $\mathrm{P}(|a| > 1-\delta)$ can be made arbitrarily small by a suitable choice of δ. Next,

$$\mathrm{P}(|X(-n)| > K) = \mathrm{P}(|X(0)| > K)$$

and

$$\mathrm{P}(|X'(-n)| > K) = \mathrm{P}(|X'(0)| > K)$$

do not depend on n by stationarity and can be made arbitrarily small by choosing K large enough. Clearly, $\mathrm{P}(2(1-\delta)^n K > \epsilon) = 0$ for n large enough. This proves $\mathrm{P}(|X(0) - X'(0)| > 0) = 0$.

We shall use the following inequality. Let $0 < p \le 2$, and let $\xi_1, \xi_2, \ldots$ be random variables with $\mathrm{E}|\xi_i|^p < \infty$. Moreover, in the case $1 < p \le 2$ we assume that the r.v.'s ξ_i form a martingale difference sequence:

$$\mathrm{E}[\xi_{i+1}|\xi_i, \ldots, \xi_1] = 0, \quad i = 1, 2, \ldots.$$

Then there exists a constant $C_p < \infty$, which depends only on p, such that

$$\mathrm{E}\Big|\sum_i \xi_i\Big|^p \ \le\ C_p \sum_i \mathrm{E}|\xi_i|^p. \tag{3.12}$$

In fact, inequality (3.12) holds with $C_p = 1$ for $0 < p \le 1$ and with $C_p = 2$ for $1 < p \le 2$ (see [11]).

From (3.12), for any $a \in (-1,1)$ we obtain

$$\mathrm{E}\Big[\Big|\sum_{k=0}^{\infty} a^k \varepsilon(t-k)\Big|^p \Big| a\Big] \;\le\; C_p \mathrm{E}|\varepsilon|^p \sum_{k=0}^{\infty} |a|^{kp} \;=\; \frac{C_p \mathrm{E}|\varepsilon|^p}{1-|a|^p} \;<\; \infty. \tag{3.13}$$

This proves the conditional convergence in L_p of the series in (3.10). The a.s. convergence of (3.10) follows from (3.13). Clearly, (3.13) and (3.11) imply that (3.10) converges unconditionally in L_p. This proves part (i).

(ii) We need to prove that $X(t) \in D(\alpha)$, $0 < \alpha \le 2$. For this it suffices to prove, that

$$\mathrm{E}X^2(t) < \infty, \qquad \text{for } \alpha = 2, \tag{3.14}$$

and for $0 < \alpha < 2$,

$$\lim_{x\to\infty} x^\alpha \mathrm{P}(X(t) > x) = \sum_{j=1}^{\infty} \mathrm{E}\Big[\big|a^j\big|^\alpha \{c_1 \mathbf{1}(a^j > 0) + c_2 \mathbf{1}(a^j < 0)\}\Big] = C < \infty, \tag{3.15}$$

$$\lim_{x\to-\infty} |x|^\alpha \mathrm{P}(X(t) \le x) = \sum_{j=1}^{\infty} \mathrm{E}\Big[\big|a^j\big|^\alpha \{c_1 \mathbf{1}(a^j < 0) + c_2 \mathbf{1}(a^j > 0)\}\Big] = C < \infty.$$

Here, (3.14) immediately follows from the condition (3.11). To prove (3.15), we use Theorem 3.1 of [48]. Accordingly, it suffices to check that there exists $\epsilon > 0$ such that

$$\sum_{j=1}^{\infty} \mathrm{E}\big|a^j\big|^{\alpha-\epsilon} < \infty \qquad \text{and} \qquad \sum_{j=1}^{\infty} \mathrm{E}\big|a^j\big|^{\alpha+\epsilon} < \infty, \qquad \text{for } \alpha \in (0,2) \setminus \{1\}, \tag{3.16}$$

$$\mathrm{E}\Big(\sum_{j=1}^{\infty} \big|a^j\big|^{\alpha-\epsilon}\Big)^{\frac{\alpha+\epsilon}{\alpha-\epsilon}} < \infty, \qquad \text{for } \alpha = 1. \tag{3.17}$$

The condition (3.16) is satisfied because of (3.11). And (3.17) follows from

$$\mathrm{E}\Big(\sum_{j=1}^{\infty} \big|a^j\big|^{1-\epsilon}\Big)^{\frac{1+\epsilon}{1-\epsilon}} = \mathrm{E}(1-|a|^{1-\epsilon})^{-1-\frac{2(1-(1-\epsilon))}{1-\epsilon}} < \infty,$$

and from the condition of this proposition in part (ii) with $p = 1-\epsilon$. Proposition 3.2.3 is proved. □

Assume, that the behavior of individuals is described by random-coefficient AR(1) equations

$$X_i(t) = a_i X_i(t-1) + \varepsilon(t), \qquad i = 1, 2, \dots, \tag{3.18}$$

where $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ are i.i.d. r.v.'s satisfying the same conditions as in Proposition 3.2.3, and where $\{a_i\}$ are i.i.d. r.v.'s independent of $\{\varepsilon(t),\, t \in \mathbb{Z}\}$ and having a common distribution a. Define the aggregated process by

$$\bar{X}_N(t) \ := \ N^{-1}\sum_{i=1}^{N} X_i(t), \qquad t \in \mathbb{Z}. \tag{3.19}$$

Let $\mathcal{A} = \sigma\{a_1, a_2, \ldots\}$ denote the $\sigma-$algebra generated by r.v.'s $a_1, a_2, \ldots$. For r.v.'s $\xi, \xi_1, \xi_2, \ldots$, we write $\xi_n \to_{L_p(\mathcal{A})} \xi$ (respectively, $\xi_n \to_{L_p} \xi$) if $\mathrm{E}\Big[|\xi_n - \xi|^p \Big| \mathcal{A}\Big] \to 0$ a.s. as $n \to \infty$ (respectively, $\mathrm{E}|\xi_n - \xi|^p \to 0$). Note the convergence $\xi_n \to_{L_p(\mathcal{A})} \xi$ implies $\xi_n \to \xi$ in probability. (In general, none of the convergences $\to_{L_p(\mathcal{A})}$ or $\to_{L_p}$ implies the other.) For real a, denote $a_+ := \max(0, a)$, $a_- := (-a)_+ = \max(0, -a)$.

Theorem 3.2.4. *Assume that* $\mathrm{E}|\varepsilon|^p < \infty$, *for some* $0 < p \le 2$, *and* $\mathrm{E}\varepsilon = 0$, $p \ge 1$, *as in Proposition 3.2.3 (page 42).*

(i) Let $1 \le p \le 2$ *and*

$$\mathrm{E}\Big[\frac{1}{(1-|a|^p)^{1/p}}\Big] \ < \ \infty. \tag{3.20}$$

Then for any $t \in \mathbb{Z}$, *as* $N \to \infty$,

$$\bar{X}_N(t) \ \to_{L_p(\mathcal{A})} \ \mathfrak{X}(t), \tag{3.21}$$

where the limit process is given by

$$\mathfrak{X}(t) \ := \ \sum_{j=0}^{\infty} \bar{a}_j \varepsilon(t-j), \qquad \bar{a}_j := \mathrm{E}[a^j]. \tag{3.22}$$

(ii) Let $0 < p < 1$ *and*

$$\sum_{j=0}^{\infty} (\mathrm{E}|a^j|)^p \ < \ \infty. \tag{3.23}$$

Then for any $t \in \mathbb{Z}$, *as* $N \to \infty$,

$$\bar{X}_N(t) \ \to_{L_p} \ \mathfrak{X}(t), \tag{3.24}$$

where the limit process is given by (3.22).

In both cases (i) and (ii), the limit process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ *is strict stationary, ergodic, and the series in* (3.22) *converges a.s. and in* L_p.

Remark 3.2.5. Note that for $1 \le p \le 2$, condition (3.20) implies convergence of the series in (3.23), while for $0 < p < 1$, condition (3.23) implies finiteness of the expectation in (3.11). To show the first implication, we use Minkowski's inequality: let $f_j \in L_p(\mathcal{X}, \mu)$, $j = 0, 1, \ldots$, where $(\mathcal{X}, \mu)$ is a measurable space,

$p \geq 1$. Then

$$\sum_{j=0}^{\infty} \Big| \int_{\mathcal{X}} f_j(x)\mu(\mathrm{d}x) \Big|^p \leq \Big(\int_{\mathcal{X}} \Big(\sum_{j=0}^{\infty} |f_j(x)|^p \Big)^{1/p} \mu(\mathrm{d}x) \Big)^p. \tag{3.25}$$

Applying (3.25) with $(\mathcal{X}, \mu) = (\Omega, \mathrm{P})$, $f_j = a^j$ we obtain

$$\sum_{j=0}^{\infty} (\mathrm{E}|a^j|)^p \;\leq\; \Big(\mathrm{E}\Big(\sum_{j=0}^{\infty} |a|^{jp} \Big)^{1/p} \Big)^p \;=\; \Big(\mathrm{E} \frac{1}{(1-|a|^p)^{1/p}} \Big)^p \;<\; \infty.$$

The second implication follows by Jensen's inequality: since for $0 < p < 1$,

$$(\mathrm{E}|a|^j)^p \geq \mathrm{E}|a|^{jp},$$

we have

$$\mathrm{E}\Big[\frac{1}{1-|a|^p} \Big] \;=\; \sum_{j=0}^{\infty} \mathrm{E}|a|^{jp} \;\leq\; \sum_{j=0}^{\infty} (\mathrm{E}|a|^j)^p \;<\; \infty.$$

Remark 3.2.6. Assume $\varepsilon \in D(\alpha)$, for some $\alpha \in (0, 2]$, and condition (3.23), for some $0 < p < \alpha$. Then from Theorem 3.1 of [48] (similarly as in the proof of Proposition 3.2.3(ii), page 42), follows that $\mathfrak{X}(t) \in D(\alpha)$ and

$$\begin{aligned} \lim_{x \to \infty} x^\alpha \mathrm{P}(\mathfrak{X}(t) > x) \;&=\; \sum_{j=0}^{\infty} \Big(c_1 (\mathrm{E}a^j)_+^\alpha + c_2 (\mathrm{E}a^j)_-^\alpha \Big), \\ \lim_{x \to -\infty} |x|^\alpha \mathrm{P}(\mathfrak{X}(t) \leq x) \;&=\; \sum_{j=0}^{\infty} \Big(c_1 (\mathrm{E}a^j)_-^\alpha + c_2 (\mathrm{E}a^j)_+^\alpha \Big). \end{aligned}$$

Proof of Theorem 3.2.4. Note that the series in (3.22) converges in L_p, due to (3.23) and Remark 3.2.5, and defines a stationary and ergodic process.

(i) Let us prove (3.21). Write

$$\bar{X}_N(t) - \mathfrak{X}(t) \;=\; \sum_{j=0}^{\infty} \varepsilon(t-j) \sum_{i=1}^{N} N^{-1}(a_i^j - \mathrm{E}a_i^j) \;=\; \sum_{j=1}^{4} Y_{Nj}, \tag{3.26}$$

where

$$\begin{aligned} Y_{N1} \;&:=\; N^{-1} \sum_{j=0}^{s} \varepsilon(t-j) \sum_{i=1}^{N} (a_i^j - \mathrm{E}a_i^j), \\ Y_{N2} \;&:=\; N^{-1} \sum_{j=s+1}^{\infty} \varepsilon(t-j) \sum_{i=1}^{N} a_i^j \mathbf{1}(0 < a_i < 1), \\ Y_{N3} \;&:=\; N^{-1} \sum_{j=s+1}^{\infty} \varepsilon(t-j) \sum_{i=1}^{N} a_i^j \mathbf{1}(-1 < a_i < 0), \end{aligned}$$

$$Y_{N4} \;:=\; -N^{-1}\sum_{j=s+1}^{\infty}\varepsilon(t-j)\sum_{i=1}^{N}\mathrm{E}a_i^j \;=\; -\sum_{j=s+1}^{\infty}\varepsilon(t-j)\mathrm{E}a^j$$

and where $s \ge 1$ will be chosen later. Here, Y_{N4} does not depend on N and

$$\mathrm{E}\Big[|Y_{N4}|^p\Big|\mathcal{A}\Big] \;\le\; 2\mathrm{E}|\varepsilon|^p\sum_{j=s+1}^{\infty}\Big|\mathrm{E}a^j\Big|^p \;<\; \epsilon \tag{3.27}$$

can be made arbitrary small in view of (3.23) and Remark 3.2.5, by choosing s large enough. Next,

$$\mathrm{E}\Big[|Y_{N2}|^p\Big|\mathcal{A}\Big] \;\le\; 2N^{-p}\mathrm{E}|\varepsilon|^p\sum_{j=s+1}^{\infty}\Big|\sum_{i=1}^{N}a_i^j\mathbf{1}(0<a_i<1)\Big|^p.$$

Applying Minkowski's inequality in (3.25), with $\mathcal{X} = \{1,\ldots,N\}$ and the counting measure μ on $\mathcal{X}$, we obtain

$$\begin{aligned}\sum_{j=s+1}^{\infty}\Big|\sum_{i=1}^{N}a_i^j\mathbf{1}(0<a_i<1)\Big|^p \;&\le\; \Big(\sum_{i=1}^{N}\Big(\sum_{j=s+1}^{\infty}a_i^{jp}\mathbf{1}(0<a_i<1)\Big)^{1/p}\Big)^p\\ &=\; \Big(\sum_{i=1}^{N}\frac{a_i^{s+1}}{(1-a_i^p)^{1/p}}\mathbf{1}(0<a_i<1)\Big)^p\end{aligned}$$

and therefore

$$\mathrm{E}\Big[|Y_{N2}|^p\Big|\mathcal{A}\Big] \;\le\; 2\mathrm{E}|\varepsilon|^p\Big(N^{-1}\sum_{i=1}^{N}\frac{a_i^{s+1}}{(1-a_i^p)^{1/p}}\mathbf{1}(0<a_i<1)\Big)^p.$$

Note that

$$\xi_i(s) := a_i^{s+1}(1-a_i^p)^{-1/p}\mathbf{1}(0<a_i<1), \quad i=1,2,\ldots$$

are i.i.d. r.v.'s, for any $s \ge 1$ fixed, and

$$\mathrm{E}\xi_1(s) = \mathrm{E}a^{s+1}(1-a^p)^{-1/p}\mathbf{1}(0<a<1) \le \mathrm{E}(1-|a|^p)^{-1/p} < \infty$$

according to condition (3.20). Moreover, $\xi_i(s) \le \xi_i(s')$ a.s. for any $s' \le s$ and therefore $\lim_{s\to\infty}\mathrm{E}\xi_i(s) = 0$ by the dominated convergence theorem. From these facts and the strong law of large numbers we infer, that, for any $\epsilon > 0$, there exist integers $s_0 \ge 1$ and $N_0(\omega) \ge 1$ such that

$$N^{-1}\sum_{i=1}^{N}\frac{a_i^{s+1}}{(1-a_i^p)^{1/p}}\mathbf{1}(0<a_i<1) \;<\; \epsilon, \quad \text{for any } \; N > N_0(\omega) \; \text{ and any } \; s > s_0.$$

The above argument applies also to $\mathrm{E}\Big[|Y_{N3}|^p\Big|\mathcal{A}\Big]$ by symmetry. Consequently, we obtain that for any $1 \le p \le 2$ and any $\epsilon > 0$ there exist integers $s_0 \ge 1$ and

$N_0(\omega) \ge 1$ such that

$$\mathrm{E}\Big[|Y_{Ni}|^p \Big| \mathcal{A}\Big] \ < \ \epsilon, \quad i = 2,3, \qquad \text{holds for any} \ \ N > N_0(\omega) \ \ \text{and any} \ \ s > s_0. \tag{3.28}$$

Finally, according to (3.12) and the strong law of large numbers,

$$\mathrm{E}\Big[|Y_{N1}|^p \Big| \mathcal{A}\Big] \ \le \ 2\mathrm{E}|\varepsilon|^p \sum_{j=0}^{s} \Big| N^{-1} \sum_{i=1}^{N} (a_i^j - \mathrm{E} a_i^j) \Big|^p \ \to \ 0 \qquad \text{a.s.} \tag{3.29}$$

for any $s < \infty$. It is clear that (3.27), (3.28), and (3.29) imply

$$\mathrm{E}\Big[\Big|\bar{X}_N(t) - \mathfrak{X}(t)\Big|^p \Big| \mathcal{A}\Big] \to 0 \ \text{ a.s., as } N \to \infty,$$

and relation (3.21). This proves part (i).

(ii) Let us prove (3.24). Consider the decomposition as in (3.26). It suffices to show that for any $s < \infty$, $\mathrm{E}|Y_{N1}|^p \to 0$, as $N \to \infty$, and that $\mathrm{E}|Y_{Ni}|^p, \ i = 2,3,4$, can be made arbitrary small by an appropriate choice of s uniformly in N. The first fact follows similarly as in the case (i) above, with the difference that the strong law of large numbers in (3.29) above must be replaced by the convergence in L_p. The proof of the second fact for Y_{N2} follows by Jensen's inequality:

$$\begin{aligned}
\mathrm{E}|Y_{N2}|^p \ &\le \ \mathrm{E}|\varepsilon|^p \sum_{j=s+1}^{\infty} \mathrm{E}\Big| N^{-1} \sum_{i=1}^{N} a_i^j \mathbf{1}(0 < a_i < 1)\Big|^p \\
&\le \ \mathrm{E}|\varepsilon|^p \sum_{j=s+1}^{\infty} \Big| \mathrm{E} N^{-1} \sum_{i=1}^{N} a_i^j \mathbf{1}(0 < a_i < 1)\Big|^p \\
&= \ \mathrm{E}|\varepsilon|^p \sum_{j=s+1}^{\infty} \Big(\mathrm{E} a^j \mathbf{1}(0 < a < 1)\Big)^p \le \mathrm{E}|\varepsilon|^p \sum_{j=s+1}^{\infty} \Big(\mathrm{E}|a|^j\Big)^p
\end{aligned}$$

and by the convergence of the series in (3.23). Since $\mathrm{E}|Y_{Ni}|^p, \ i = 3,4$ can be similarly estimated, this proves part (ii) and Theorem 3.2.4, too. □

Remark 3.2.7. (i) If condition (3.20) in Theorem 3.2.4 (i) is replaced by condition (3.11), then similarly as above, the conditional convergence in (3.21) can be replaced by unconditional convergence as in (3.24). However, condition (3.11) excludes the case of aggregated process with long memory which is discussed below.

(ii) For $p \ge 1$, the limit process $\mathfrak{X}(t)$ in Theorem 3.2.4, (3.22) can be defined as conditional expectation:

$$\mathfrak{X}(t) \ = \ \mathrm{E}[X(t)|\varepsilon(t), t \in \mathbb{Z}], \qquad t \in \mathbb{Z},$$

where $\{X(t),\, t \in \mathbb{Z}\}$ is the random-coefficient AR(1) process in (3.10).

3.3 Asymptotics of the aggregated moving average coefficients

The most interesting case of aggregation occurs when the mixing density is singular at points $+1$ and/or -1. From now on, in this chapter, we shall assume that the distribution of r.v. a has a density ϕ of the form

$$\phi(x) \;=\; (1-x)^{-d_1}(1+x)^{-d_2}\psi(x), \qquad -1 < x < 1, \tag{3.30}$$

where parameters d_1, d_2 satisfy $0 < d_1, d_2 < 1$ and where $\psi \geq 0$ is an integrable function on the interval $(-1, 1)$ such that the limits

$$\lim_{x\to 1} \psi(x) =: \psi_1 \geq 0 \qquad \text{and} \qquad \lim_{x\to -1} \psi(x) =: \psi_2 \geq 0 \tag{3.31}$$

exist.

Proposition 3.3.1, below, describes the asymptotics as $j \to \infty$ of the moving average coefficients $\bar{a}_j = \mathrm{E}a^j$ of the limit aggregated process in (3.22) under the assumption (3.30) on the mixing density. Clearly,

$$\begin{aligned} \mathrm{E}a^j \;&=\; \mathrm{E}a^j \mathbf{1}(0 < a < 1) + (-1)^j \mathrm{E}(-a)^j \mathbf{1}(-1 < a < 0) \\ &=\; \mathrm{E}a_+^j + (-1)^j \mathrm{E}a_-^j, \end{aligned}$$

so that it suffices to consider the asymptotics of $\mathrm{E}a_+^j$ and $\mathrm{E}a_-^j$.

Proposition 3.3.1. *Let the probability density ϕ of r.v. a satisfy the assumptions in* (3.30)-(3.31)*. Moreover, assume that there exist $\beta_i \in (0, 1], i = 1, 2$ such that*

$$\psi(x) - \psi_1 \;=\; O\Big(|1-x|^{\beta_1}\Big), \qquad \psi(x) - \psi_2 \;=\; O\Big(|1+x|^{\beta_2}\Big). \tag{3.32}$$

Then, as $j \to \infty$,

$$\mathrm{E}a_+^j \;=\; \frac{c(d_1, d_2)}{j^{1-d_1}}\Big(\psi_1 + O\Big(j^{-\beta_1}\Big)\Big), \tag{3.33}$$

$$\mathrm{E}a_-^j \;=\; \frac{c(d_2, d_1)}{j^{1-d_2}}\Big(\psi_2 + O\Big(j^{-\beta_2}\Big)\Big), \tag{3.34}$$

where $c(d_1, d_2) := 2^{-d_2}\Gamma(1-d_1)$. If conditions in (3.32) *are replaced by conditions in* (3.31)*, then relations in* (3.33), (3.34) *hold with $O(j^{-\beta_i})$ replaced by $o(1)$, $i = 1, 2$.*

Proof. We shall discuss the asymptotics of $\mathrm{E}a_+^j$ only, since $\mathrm{E}a_-^j$ is analogous. Write $\mathrm{E}a_+^j = \sum_{i=1}^2 \ell_i(j)$, where $\ell_1(j) := \int_{1-\epsilon}^1 x^j\phi(x)\,\mathrm{d}x$, $\ell_2(j) := \int_0^{1-\epsilon} x^j\phi(x)\,\mathrm{d}x$, and where $0<\epsilon<1$ is a small number. Since $|\ell_2(j)| \le (1-\epsilon)^j = o(j^{d-1})$ for any $d<1$, it suffices to show the limit

$$\lim_{j\to\infty} j^{1-d_1}\ell_1(j) = c(d_1,d_2)\psi_1. \tag{3.35}$$

Rewrite

$$\begin{aligned} j^{1-d_1}\ell_1(j) &= \int_0^{\epsilon j}\Big(1-\frac{z}{j}\Big)^j\psi\Big(1-\frac{z}{j}\Big)\Big(2-\frac{z}{j}\Big)^{-d_2} z^{-d_1}\,\mathrm{d}z \\ &\to \psi_1 2^{-d_2}\int_0^\infty \mathrm{e}^{-z}z^{-d_1}\,\mathrm{d}z = \psi_1 c(d_1,d_2) \end{aligned}$$

by the dominated convergence theorem, proving the limit in (3.35). Next, write

$$j^{1-d_1}\ell_1(j) - \psi_1 c(d_1,d_2) = \sum_{i=1}^4 \nu_i(j),$$

where

$$\begin{aligned} \nu_1(j) &:= \psi_1 2^{-d_2}\int_0^{\epsilon j}\Big[\Big(1-\frac{z}{j}\Big)^j - \mathrm{e}^{-z}\Big]z^{-d_1}\,\mathrm{d}z, \\ \nu_2(j) &:= -\psi_1 2^{-d_2}\int_{\epsilon j}^\infty \mathrm{e}^{-z}z^{-d_1}\,\mathrm{d}z, \\ \nu_3(j) &:= 2^{-d_2}\int_0^{\epsilon j}\Big(1-\frac{z}{j}\Big)^j\Big(\psi\Big(1-\frac{z}{j}\Big)-\psi_1\Big)z^{-d_1}\,\mathrm{d}z, \\ \nu_4(j) &:= \int_0^{\epsilon j}\psi\Big(1-\frac{z}{j}\Big)\Big(1-\frac{z}{j}\Big)^j\Big(\Big(2-\frac{z}{j}\Big)^{-d_2} - 2^{-d_2}\Big)z^{-d_1}\,\mathrm{d}z. \end{aligned}$$

It suffices to show that

$$\nu_1 = O(j^{-1}), \quad \nu_2 = o(j^{-1}), \quad \nu_3 = O(j^{-\beta_1}), \quad \nu_4 = O(j^{-1}). \tag{3.36}$$

Split $\nu_1 = \nu_{11}+\nu_{12}$, where

$$\begin{aligned} \nu_{11} &:= \psi_1 2^{-d_2}\int_0^{\sqrt{\epsilon j}}\Big[\Big(1-\frac{z}{j}\Big)^j - \mathrm{e}^{-z}\Big]z^{-d_1}\,\mathrm{d}z, \\ \nu_{12} &:= \psi_1 2^{-d_2}\int_{\sqrt{\epsilon j}}^{\epsilon j}\Big[\Big(1-\frac{z}{j}\Big)^j - \mathrm{e}^{-z}\Big]z^{-d_1}\,\mathrm{d}z. \end{aligned}$$

Since $\Big|\Big(1-\frac{z}{j}\Big)^j - \mathrm{e}^{-z}\Big| = \mathrm{e}^{-z}|\mathrm{e}^{z+j\log(1-z/j)}-1| = \mathrm{e}^{-z}\Big|\mathrm{e}^{O(z^2/j)}-1\Big| = \mathrm{e}^{-z}O(z^2/j)$ for $z\in(0,\sqrt{\epsilon j})$, so

$$\nu_{11} = j^{-1}O\Big(\int_0^\infty z^{-d_1}\,\mathrm{d}z\Big) = O(j^{-1}).$$

Next, since $(1-z/j)^j \le \mathrm{e}^{-z}$ for $z \in (0, j)$, so

$$\nu_{12} = O\Big(\int_{\sqrt{\epsilon j}}^{\infty} \mathrm{e}^{-z} z^{-d_1}\,\mathrm{d}z\Big) = O(\mathrm{e}^{-\sqrt{\epsilon j}}) = o(j^{-1})$$

for any $\epsilon > 0$ fixed. Similarly, $\nu_2(j) = o(j^{-1})$ and

$$\nu_3(j) = j^{-\beta_1} O\Big(\int_0^{\infty} \mathrm{e}^{-z} z^{1-d_1}\,\mathrm{d}z\Big) = O(j^{-\beta_1}).$$

Finally,

$$\nu_4(j) = j^{-1} O\Big(\int_0^{\infty} \mathrm{e}^{-z} z^{1-d_1}\,\mathrm{d}z\Big) = O(j^{-1})$$

by Taylor expansion. This proves (3.36) and Proposition 3.3.1, too. □

Remark 3.3.2. Note, for $1 \le p \le 2$ and mixing density ϕ as in (3.30),

$$\int_{-1}^{1} \frac{\phi(x)\,\mathrm{d}x}{(1-|x|^p)^{1/p}} \quad\le\quad 2\Big[\int_0^1 \frac{\psi(x)\,\mathrm{d}x}{(1-x)^{d_1+1/p}} + \int_{-1}^{0} \frac{\psi(x)\,\mathrm{d}x}{(1+x)^{d_2+1/p}}\Big].$$

Therefore, for $1 \le p \le 2$, condition (3.20) is satisfied if

$$d_i < 1 - \frac{1}{p}, \qquad i = 1, 2. \tag{3.37}$$

Moreover, if $\psi_i > 0$ then condition (3.37) is also necessary for (3.20). Also note that, for $0 < p < 1$, conditions (3.20) and (3.23) are not satisfied unless $d_i < 0$ or $\psi_i = 0$ hold, $i = 1, 2$.

Any sequence $\{a_j\} = \{a_j, j = 0, 1, \ldots\}$ of real numbers will be called a *filter*. Given two filters $\{a_j\}$ and $\{b_j\}$, their convolution $\{(a \star b)_j\}$ is the filter defined by $(a \star b)_j = \sum_{i=0}^{j} a_i b_{j-i}$. For $d \in (-1, 1)$, the FARIMA$(0, d, 0)$ filter $\{b_j(d)\}$ is defined by

$$b_j(d) := \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}, \quad j = 0, 1, \ldots, \tag{3.38}$$

or by the generating series:

$$\sum_{j=0}^{\infty} z^j b_j(d) \;=\; (1-z)^{-d}, \qquad |z| < 1.$$

Clearly, $b_j(0) = \delta_{0j} := 1$ $(j = 0)$, $:= 0$ $(j \ge 1)$ is the trivial filter and $\{(b(d) \star b(-d))_j\} = \{b_j(0)\}$ for any $-1 < d < 1$. Since $\{b_j(d)\}$ for $0 < d < 1$ is a particular case of $\{\bar{a}_j\}$, see (3.4), Proposition 3.3.1 implies

$$b_j(d) = \frac{1}{\Gamma(d)} j^{d-1}\Big(1 + O\big(j^{-1}\big)\Big), \qquad 0 < d < 1. \tag{3.39}$$

Let us note that relation (3.39) holds for any $d \in (-1,1), d \neq 0$, which fact easily follows from (3.38) and the Stirling formula (see also [54]).

The following definition was inspired by Granger [41].

Definition 3.3.3. *A filter $\{a_j\}$ is said an I(0) filter if $\sum_{j=0}^{\infty} |a_j| < \infty$ and $\sum_{j=0}^{\infty} a_j \neq 0$ hold. A filter $\{a_j\}$ will be said an I(d) filter (where $-1 < d < 1, d \neq 0$) if the convolution $\{a \star b(-d)\}$ is an I(0) filter.*

Proposition 3.3.4. *Let the mixing density ϕ have the form as in (3.30), where $0 < d_i < 1,\ i = 1,2,\ \psi_1 > 0, \psi_2 = 0$ and ψ satisfies conditions in (3.32) with $1 \geq \beta_i > d_i,\ i = 1,2$. Then $\{\bar{a}_j\}$ is an I(d_1) filter.*

Proof. Write

$$\bar{a}_j = \bar{a}_{j1} + \bar{a}_{j2}, \qquad \bar{a}_{j1} := \mathrm{E}a_+^j, \quad \bar{a}_{j2} := \mathrm{E}a^j \mathbf{1}(-1 < a < 0) = (-1)^j \mathrm{E}a_-^j.$$

From (3.33), (3.34), and (3.39) we obtain

$$\begin{aligned} \bar{a}_{j1} &= \kappa_1 b_j(d_1)\Big(1 + O\big(j^{-\beta_1}\big)\Big) = \kappa_1 b_j(d_1) + w_{j1}, \\ \kappa_1 &:= \psi_1 c(d_1,d_2)\Gamma(d_1), \qquad w_{j1} = O\Big(\frac{1}{j^{1+\beta_1-d_1}}\Big). \end{aligned} \tag{3.40}$$

Consider the convolution

$$\begin{aligned} (\bar{a}_1 \star b(-d_1))_k &= \sum_{j=0}^{k} \bar{a}_{j1} b_{k-j}(-d_1) \\ &= \kappa_1 \sum_{j=0}^{k} b_j(d_1) b_{k-j}(-d_1) + \sum_{j=0}^{k} w_{j1} b_{k-j}(-d_1) \\ &= \kappa_1 \delta_k + \sum_{j=0}^{k} w_{j1} b_{k-j}(-d_1). \end{aligned}$$

From (3.39) and (3.40) we obtain

$$\begin{aligned} \Big|\sum_{j=0}^{k} w_{j1} b_{k-j}(-d_1)\Big| &\leq C \sum_{j=0}^{k} \frac{1}{(j+1)^{1+\beta_1-d_1}} \frac{1}{(k+1-j)^{1+d_1}} \\ &\leq C k^{-1-\min(d_1,\beta_1-d_1)}. \end{aligned}$$

Since $\min(d_1, \beta_1 - d_1) > 0$, this proves the convergence $\sum_{k=0}^{\infty} \Big|(\bar{a}_1 \star b(-d_1))_k\Big| < \infty$. The convergence $\sum_{k=0}^{\infty} \Big|(\bar{a}_2 \star b(-d_1))_k\Big| < \infty$ follows similarly using the fact that $\psi_2 = 0$.

It remains to show that

$$A := \sum_{k=0}^{\infty} (\bar{a} \star b(-d_1))_k \neq 0. \tag{3.41}$$

Consider the power series $A(z) := \sum_{k=0}^{\infty}(\bar{a} \star b(-d_1))_k z^k,\ |z| \le 1$. Since the series in (3.41) absolutely converges, so

$$A = \lim_{z\uparrow 1} A(z).$$

We have

$$A(z) \;=\; (1-z)^{d_1}\sum_{j=0}^{\infty}\bar{a}_j z^j \;=\; \int_{-1}^{1}\frac{(1-z)^{d_1}\psi(x)\,\mathrm{d}x}{(1-xz)(1-x)^{d_1}(1+x)^{d_2}}.$$

Decompose $A(z) = \int_0^1 \cdots + \int_{-1}^0 \cdots =: A_1(z) + A_2(z)$. Clearly, $\lim_{z\uparrow 1} A_2(z) = 0$. Let $\delta = 1 - z \downarrow 0$. Then

$$\begin{aligned} A_1(z) \;&\sim\; 2^{-d_2}\psi_1\int_0^1 \frac{\delta^{d_1}\,\mathrm{d}y}{(1-(1-y)(1-\delta))y^{d_1}} \\ &=\; 2^{-d_2}\psi_1\int_0^{1/\delta}\frac{\mathrm{d}u}{(1+u-u\delta)u^{d_1}} \\ &\sim\; 2^{-d_2}\psi_1 B(d_1, 1-d_1) \;\neq\; 0. \end{aligned}$$

Proposition 3.3.4 is proved. □

3.4 Long memory properties of the limit aggregated process

In this chapter, we discuss two notions of long memory which do not require finite variance. The first notion - *distributional long memory* - was introduced in Cox [28] (see Definition 2.3.6, page 36). The second notion - *long-range dependence (sample Allen variance) (LRD(SAV))* and its antonym *short-range dependence (sample Allen variance) (SRD(SAV))* - was introduced in Heyde and Yang [45] (see Definition 2.3.7, page 36).

For $0 < \alpha \le 2, -1/\alpha < d < 1 - 1/\alpha, d \neq 0$, introduce fractional Lévy motion, $L_{\alpha,d}$, written as stochastic integral

$$L_{\alpha,d}(t) \;:=\; \int_{-\infty}^{t}\Big((t-x)^d - (-x)_+^d\Big)\,\mathrm{d}Z_\alpha(x), \quad t \ge 0, \tag{3.42}$$

where $\{Z_\alpha(x), x \in \mathbb{R}\}$ is Lévy $\alpha-$stable process, with characteristic function

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta Z_\alpha(x)} \;=\; \mathrm{e}^{-|\theta|^\alpha \omega(\theta;\alpha,c_1,c_2)|x|}, \qquad \theta, x \in \mathbb{R}, \tag{3.43}$$

where $\omega(\theta;\alpha,c_1,c_2)$ is defined in (3.8). Recall that $L_{\alpha,d}$ has stationary increments, $\alpha-$stable finite dimensional distributions and is $H-$self-similar with self-similarity parameter $H = d + 1/\alpha$. Moreover, for $1 < \alpha \le 2$ and $0 < d < 1 - 1/\alpha$, the process $L_{\alpha,d}$ has a.s. continuous trajectories, while for $-1/\alpha < d < 0$, trajectories of $L_{\alpha,d}$ are a.s. unbounded on any finite interval. See [94] for these and other properties of fractional Lévy motion.

Proposition 3.4.1. *Let* $\{\mathfrak{X}(t),\ t \in \mathbb{Z}\}$ *be the limit aggregated process in* (3.22), *with i.i.d. innovations* $\varepsilon(t) \in D(\alpha)$, $0 < \alpha \le 2$.

(i) Let $1 < \alpha \le 2$ *and the distribution of r.v.* a *have a probability density as in* (3.30), *such that* $d_1 > 0, \psi_1 > 0$, *and*

$$d_i < 1 - \frac{1}{\alpha}, \qquad i = 1, 2. \tag{3.44}$$

Then

$$\frac{1}{n^{d_1+1/\alpha}} \sum_{k=1}^{[n\tau]} \mathfrak{X}(k) \ \to_{D[0,1]}\ \kappa_1 L_{\alpha,d_1}(\tau), \tag{3.45}$$

where $\kappa_1 := \psi_1 c(d_1, d_2)/d_1$.

(ii) Let $0 < \alpha < 2$ *and* $\sum_{j=1}^{\infty} (\mathrm{E}|a|^j)^p < \infty$ *for some* $p < \alpha$. *Then*

$$\frac{1}{n^{2/\alpha}} \sum_{k=1}^{[n\tau]} \mathfrak{X}^2(k) \ \to_{\mathrm{fdd}}\ Z^+_{\alpha/2}(\tau), \tag{3.46}$$

where $\{Z^+_{\alpha/2}(t), t \ge 0\}$ *is a homogeneous* $\alpha/2-$*stable Lévy process with positive jumps and characteristic function*

$$\mathrm{E}e^{\mathrm{i}\theta Z^+_{\alpha/2}(1)} \ = \ \exp\Big\{ - |\theta|^{\alpha/2} A^{\alpha/2} \omega(\theta; \alpha/2, c_1 + c_2, 0)\Big\}, \quad \theta \in \mathbb{R}, \quad A := \sum_{k=0}^{\infty} (\mathrm{E}a^k)^2.$$

Proof. (i) Denote

$$\begin{aligned} \bar a_{j1} \ &:= \ \mathrm{E}a^j \mathbf{1}(0 < a < 1), \qquad \bar a_{j2} \ := \ \mathrm{E}a^j \mathbf{1}(-1 < a < 0), \\ \mathfrak{X}_i(t) \ &:= \ \sum_{j=0}^{\infty} \bar a_{ji} \varepsilon(t - j), \qquad i = 1, 2. \end{aligned} \tag{3.47}$$

Since $\mathfrak{X}(t) = \mathfrak{X}_1(t) + \mathfrak{X}_2(t)$, for convergence of finite-dimensional distributions in (3.45), it suffices to check that

$$\frac{1}{n^{d_1+1/\alpha}} \sum_{k=1}^{[n\tau]} \mathfrak{X}_1(k) \ \to_{\mathrm{fdd}}\ \kappa_1 L_{\alpha,d_1}(\tau), \tag{3.48}$$

$$\sum_{k=1}^{n} \mathfrak{X}_2(k) \ = \ O_p(n^{1/p}), \qquad p < \alpha. \tag{3.49}$$

Relation (3.48) immediately follows from Theorem 1 (ii) of Astrauskas [7] and the asymptotics of $\bar{a}_{j1}$ in Proposition 3.3.1 (page 49).

Theorem 1 of Astrauskas [7]: *Let* $\{X_k, k \in \mathbb{N}\}$ *have the form*

$$X_k = \sum_j a(k-j)\varepsilon_j, \qquad k \in \mathbb{N},$$

where $\varepsilon_j \in D(\alpha)$, $0 < \alpha \le 2$.

(i) *Assume, that the series* $\sum_j a(j)$ *converges absolutely and* $A \equiv \left|\sum_j a(j)\right| > 0$. *Then*

$$\frac{1}{A_n}\sum_{k=1}^{[nt]} X_k \to_{\mathrm{fdd}} Z_\alpha(t)$$

where $Z_\alpha(t)$ *is* $\alpha-$*stable process with independent increments,* $A_n = C^{1/\alpha}An^{1/\alpha}H_\alpha^{1/\alpha}(n)$, $C = (c_1+c_2)\Gamma(|1-\alpha|)\cos(\alpha\pi/2)$, H_α *is a slowly varying function.*

(ii)*Let* $\alpha > 1$, $1/\alpha < \beta < 1$ *and* $a(k) = 0$, *for* $k = 0, -1, -2, \ldots$. *Assume,* $a(k) = k^{-\beta}L(k)$, *for* $k > 0$. *Here* L *is a slowly varying function. Then*

$$\frac{1}{A_n}\sum_{k=1}^{[nt]} X_k \to_{\mathrm{fdd}} L_{\alpha,1-\beta}(t)$$

where $A_n = |1-\beta|^{-1}C^{1/\alpha}n^{1/\alpha+1-\beta}L(n)H_\alpha^{1/\alpha}(n)$, $C = (c_1+c_2)\Gamma(|1-\alpha|)\cos(\alpha\pi/2)$, H_α *is a slowly varying function, and* $L_{\alpha,1-\beta}(t)$ *is the same as in* (3.42).

Next, continuing the proof of Proposition 3.4.1 (i), note that it suffices to show (3.49) for $1 < p < \alpha$ and p sufficiently close to α (to have $1/p < d_1 + 1/\alpha$). According to inequality (3.12),

$$\begin{aligned}
\mathrm{E}|\sum_{k=1}^{n}\mathfrak{X}_2(k)|^p &\le 2\mathrm{E}|\varepsilon|^p\sum_{s\le n}\Big|\sum_{t=\max(1,s)}^{n}\mathrm{E}a^{t-s}\mathbf{1}(-1<a<0)\Big|^p \\
&= 2\mathrm{E}|\varepsilon|^p\Big(\sum_{s=0}^{\infty}|\sum_{t=1}^{n}\mathrm{E}a^{t+s}\mathbf{1}(-1<a<0)|^p \\
&\quad + \sum_{s=1}^{n}|\sum_{i=0}^{n-s}\mathrm{E}a^{i}\mathbf{1}(-1<a<0)|^p\Big),
\end{aligned}$$

where

$$\begin{aligned}
\sum_{s=0}^{\infty}\Big|\mathrm{E}\sum_{t=1}^{n}a^{t+s}\mathbf{1}(-1<a<0)\Big|^p &= \sum_{s=0}^{\infty}\Big|\mathrm{E}\frac{a^{1+s}(1-a^{n-1})}{1-a}\mathbf{1}(-1<a<0)\Big|^p \\
&\le 2\sum_{s=0}^{\infty}(\mathrm{E}|a|^s)^p < \infty,
\end{aligned}$$

and the last series converges in view of Proposition 3.3.1 (page 49), provided p is chosen so that $d_i < 1 - 1/p$, $i = 1, 2$. In a similar way,

$$\begin{aligned}\sum_{s=1}^{n} \Big| \sum_{i=0}^{n-s} \mathrm{E} a^i \mathbf{1}(-1 < a < 0) \Big|^p &= \sum_{t=0}^{n-1} \Big| \mathrm{E} \sum_{i=0}^{t} a^i \mathbf{1}(-1 < a < 0) \Big|^p \\ &= \sum_{t=0}^{n-1} \Big| \mathrm{E} \frac{1 - a^{t+1}}{1 - a} \mathbf{1}(-1 < a < 0) \Big|^p \\ &\le n,\end{aligned}$$

proving (3.49) and the convergence of finite-dimensional distributions in (3.45), too. The tightness in $D[0,1]$ follows by the well-known Kolmogorov's criterion. Namely, it suffices to show that there exist $C, \Gamma > 0$ and $p < \alpha$ such that for any $n \ge 1$ and any $0 \le t < t + h \le 1$

$$\mathrm{E} \Big| \sum_{k=[nt]+1}^{[n(t+h)]} \mathfrak{X}(k) \Big|^p \le C h^{1+\Gamma} n^{(d_1 + 1/\alpha)p}. \tag{3.50}$$

By stationarity of $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$, it suffices to show (3.50) for $t = 0$ and $h = 1$. Furthermore, it suffices to check (3.50) separately for $\{\mathfrak{X}_1(t),\, t \in \mathbb{Z}\}$ and $\{\mathfrak{X}_2(t),\, t \in \mathbb{Z}\}$ as defined in (3.47). Again, for $\{\mathfrak{X}_1(t),\, t \in \mathbb{Z}\}$, (3.50) follows from Astrauskas[7] [1], while for $\{\mathfrak{X}_2(t),\, t \in \mathbb{Z}\}$, we have

$$\mathrm{E} \Big| \sum_{k=1}^{n} \mathfrak{X}_2(k) \Big|^p \le Cn, \text{ for any } p < \alpha,$$

implying (3.50) by the fact that $1 < (d_1 + 1/\alpha)p$ for suitably chosen p. This proves part (i).

(ii) Rewrite $\sum_{k=1}^{[n\tau]} \mathfrak{X}^2(k) = I_1(\tau) + 2I_2(\tau)$, where

$$\begin{aligned}I_1(\tau) &:= \sum_{k=1}^{[n\tau]} \sum_{j=-\infty}^{k} (\mathrm{E} a^{k-j})^2 \varepsilon^2(j), \\ I_2(\tau) &:= \sum_{k=1}^{[n\tau]} \sum_{-\infty < j < i \le k} \mathrm{E} a^{k-j} \mathrm{E} a^{k-i} \varepsilon(j) \varepsilon(i).\end{aligned}$$

Note $\varepsilon^2 \in D(\alpha/2)$ and $\sum_{j=-\infty}^{k} (\mathrm{E} a^{k-j})^2 = A < \infty$. The convergence

$$n^{-2/\alpha} I_1(\tau) \to_{\mathrm{fdd}} Z^+_{\alpha/2}(\tau)$$

1. In the proof of Theorem 2 of [7], A. Astrauskas proves the tightness in $C[0,1]$ for processes such as $\{\mathfrak{X}_1(t),\, t \in \mathbb{Z}\}$. He uses the well-known Kolmogorov criterion.

follows from (Astrauskas [7], Theorem 1 (i)). Thus, part (ii) follows from

$$\mathrm{E}|I_2(1)|^p = o(n^{2p/\alpha}). \tag{3.51}$$

Using (3.12) and Minkowski's (3.25) inequalities, for any $1 \le p < \alpha$ we obtain

$$\begin{aligned}
\mathrm{E}|I_2(1)|^p &\le 2\mathrm{E}|\varepsilon|^p \sum_{i=-\infty}^{n} \mathrm{E}\Big| \sum_{k=1\vee i}^{n} \sum_{j=-\infty}^{i-1} \mathrm{E}a^{k-j}\mathrm{E}a^{k-i}\varepsilon(j)\Big|^p \\
&\le (2\mathrm{E}|\varepsilon|^p)^2 \sum_{i=-\infty}^{n} \sum_{j=-\infty}^{i-1} \Big| \sum_{k=1\vee i}^{n} \mathrm{E}a^{k-j}\mathrm{E}a^{k-i}\Big|^p \\
&\le (2\mathrm{E}|\varepsilon|^p)^2 \Big(\sum_{k=1}^{n} \Big(\sum_{i=-\infty}^{n\wedge k} \sum_{j=-\infty}^{i-1} \Big|\mathrm{E}a^{k-j}\mathrm{E}a^{k-i}\Big|^p \Big)^{1/p} \Big)^p \\
&\le (2\mathrm{E}|\varepsilon|^p)^2 A_p^2 n^p \ = \ O(n^p),
\end{aligned}$$

where $A_p := \sum_{i=0}^{\infty} |\mathrm{E}a^i|^p < \infty$. Whence, (3.51) follows for $1 < \alpha < 2$. For $0 < \alpha \le 1$, relation (3.51) follows similarly. Proposition 3.4.1 is proved. □

Corollary 3.4.2. *Let* $\{\mathfrak{X}(t),\ t \in \mathbb{Z}\}$ *be the limit aggregated process* (3.22) *satisfying the conditions as in Proposition 3.4.1 (i). Then*

(i) $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ *has distributional long memory.*

(ii) $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ *is LRD(SAV).*

Proof. Part (i) follows from (3.45) and the fact that the limit process, L_{α,d_1}, has dependent increments. Part (ii) follows from (3.45), (3.46) and the fact that $2/\alpha < 2(d_1 + 1/\alpha)$. □

Remark 3.4.3. A natural question, which remains open, is whether the finite-dimensional convergence in (3.46) can be replaced by functional convergence in $D[0,1]$. Let us note that the usual J_1-topology in $D[0,1]$ apparently is not suitable here. See [10].

3.5 Nonstationary limit aggregate

Following Zaffaroni [102], consider aggregation of nonstationary AR(1) processes:

$$Y_i(t) \ := \ \sum_{j=0}^{t-1} a_i^j \varepsilon(t-j), \qquad t \ = \ 1, 2, \ldots, \quad i \ = \ 1, \ldots, N,$$

where $\{a_i\}$ and $\{\varepsilon(t), t \in \mathbb{Z}\}$ satisfy the same conditions as in (3.18). Similarly to (3.19), define

$$\bar{Y}_N(t) := N^{-1} \sum_{i=1}^{N} Y_i(t), \quad t = 1, 2, \ldots.$$

Proposition 3.5.1. *(i) Assume the same conditions as in Proposition 3.2.3. Then for any $t = 1, 2, \ldots,$*

$$\bar{Y}_N(t) \to_{L_p(\mathcal{A})} \bar{Y}(t) \qquad \textit{and} \qquad \bar{Y}_N(t) \to_{L_p} \bar{Y}(t),$$

where

$$\bar{Y}(t) \;:=\; \sum_{j=0}^{t-1} \bar{a}_j \varepsilon(t-j), \qquad \bar{a}_j := \mathrm{E}a^j.$$

(ii) Let $1 < \alpha \le 2$ and let the mixing density have the form as in (3.30) such that $\psi_1 > 0$ and

$$1 - \frac{1}{\alpha} < d_1 < 1 \quad \textit{and} \quad d_2 < d_1. \tag{3.52}$$

Then

$$\frac{1}{n^{d_1+1/\alpha-1}} \bar{Y}([n\tau]) \;\to_{\mathrm{fdd}}\; \psi_1 c(d_1, d_2) U_{d_1,\alpha}(\tau), \tag{3.53}$$

where

$$U_{d,\alpha}(\tau) \;:=\; \int_0^\tau (\tau - s)^{d-1}\, \mathrm{d}Z_\alpha(s), \qquad \tau \ge 0 \tag{3.54}$$

and where Z_α is the same Lévy process as in (3.42).

Proof. (i) The proof is analog to the proof of Theorem 3.2.4 (page 45), so we omit the details.
(ii) Similarly to (3.47), decompose $\bar{Y}(t) = \bar{Y}_1(t) + \bar{Y}_2(t)$, where

$$\bar{Y}_i(t) := \sum_{j=0}^{t-1} \bar{a}_{ji} \varepsilon(t-j) \quad i = 1, 2,$$

and where $\bar{a}_{ji}$ are defined as in (3.47). Relation (3.53) follows from

$$\tfrac{1}{n^{d_1+1/\alpha-1}} \bar{Y}_1([n\tau]) \;\to_{\mathrm{fdd}}\; \psi_1 c(d_1, d_2) U_{\alpha,d_1}(\tau), \tag{3.55}$$

$$\bar{Y}_2(n) \;=\; o_p(n^{d_1+1/\alpha-1}). \tag{3.56}$$

The proof of (3.55) follows the argument in [8] and [18]. As in these papers, it suffices to show the convergence of one-dimensional distributions in (3.55). To this end, we write the left-hand side of (3.55) as a 'discrete stochastic integral'

$$\begin{aligned}
\frac{1}{n^{d_1+1/\alpha-1}} \bar{Y}_1([n\tau]) &= \frac{1}{n^{d_1+1/\alpha-1}} \sum_{j=1}^{[n\tau]} \mathrm{E}a_+^{[n\tau]-j} \varepsilon(j) \\
&= \frac{1}{n^{d_1+1/\alpha-1}} \int_1^{[n\tau]+1} \mathrm{E}a_+^{[n\tau]-[s]} \varepsilon([s])\, \mathrm{d}s \\
&= \int_0^\infty \frac{1}{n^{d_1-1}} \mathrm{E}a_+^{[n\tau]-[ns]} \mathbf{1}(s \in (1/n, [n\tau]/n)) \frac{\varepsilon([ns])}{n^{1/\alpha}}\, \mathrm{d}ns \\
&=: \int_0^\infty f_n(\tau, s) Z_n(\,\mathrm{d}s)
\end{aligned}$$

where $Z_n(s', s''] = n^{-1/\alpha} \sum_{s'n<t\le s''n} \varepsilon_t$ is a discrete random measure defined on finite intervals $(s', s''] \subset (0, \infty)$, and where the integrand $f_n(\tau, \cdot)$ is a piecewise constant function:

$$f_n(\tau, s) := \frac{1}{n^{d_1-1}} \mathrm{E} a_+^{[n\tau]-[ns]} \mathbf{1}(s \in (1/n, [n\tau]/n]).$$

From Proposition 3.3.1 (page 49), it is clear that for any $\tau, s > 0, \tau \ne s$

$$f_n(\tau, s) \ \to \ \psi_1 c(d_1, d_2)(\tau - s)^{d_1-1}\mathbf{1}(s \in (0, \tau]) \ =: \ f(\tau, s), \qquad \text{as } n \to \infty.$$

Moreover, the last convergence extends to the convergence in $L_{\alpha\pm\epsilon}(\mathbb{R})$, for any sufficiently small $\epsilon > 0$, i.e.

$$\int_{-\infty}^{+\infty} |f_n(\tau, s) - f(\tau, s)|^{\alpha\pm\epsilon} \,\mathrm{d}s \to 0, \text{ as } n \to \infty.$$

This guarantees the convergence in finite dimensional distributions of the discrete stochastic integral $\int_0^\infty f_n(\tau, s) Z_n(\,\mathrm{d}s)$ towards the limiting $\alpha-$stable integral $\int_0^\infty f(\tau, s)\,\mathrm{d}Z_\alpha(s) = \psi_1 c(d_1, d_2) U_{\alpha,d_1}(\tau)$ (see [8] for details). Next, (3.56) can be proved analogously as (3.55), using expression of 'discrete stochastic integral' and the fact that $d_2 < d_1$. Proposition 3.5.1 is proved. □

Remark 3.5.2. The process $U_{\alpha,d}$ in (3.54) is well-defined for any $1 < \alpha \le 2$, $1 - 1/\alpha < d < 1$, as a stochastic integral with respect to Lévy process Z_α. It has $\alpha-$stable finite-dimensional distributions and is self-similar with index $H = d + 1/\alpha - 1 \in (0, 1/\alpha)$. These facts are easy consequences from the definition of stochastic integral with respect to $\alpha-$stable random measure and its properties; see e.g. [94].

Let us also note that, for $\alpha = 2$, the process $U_{\alpha,d}$ is a.s. continuous while for $\alpha < 2$, it is a.s. discontinuous and nowhere bounded (a.s. unbounded on every finite interval). The last fact follows from a general result in [91]. In particularly, the convergence in (3.53) cannot be replaced by a functional convergence in $D[0, 1]$.

Remark 3.5.3. If inequality $d_2 < d_1$ in (3.52) is reversed, then $\bar{Y}_2([n\tau]) = O_p(n^{d_2+1/\alpha-1})$ dominates $\bar{Y}_1([n\tau])$, and one can ask if the convergence in (3.53) holds with d_1 replaced by d_2. Somewhat surprisingly, in turns out that the process $n^{-d_2-1/\alpha+1}\bar{Y}_2([n\tau])$ does not converge in the sense of finite dimensional distributions.

The last fact can be observed for $\alpha = 2$ and Gaussian innovations $\varepsilon(t) \sim$

$\mathcal{N}(0,1)$, by considering the covariance function

$$\begin{aligned}
\mathrm{Cov}\Big(n^{1/2-d_2}\bar{Y}_2(n), n^{1/2-d_2}\bar{Y}_2(2n)\Big) &= n^{1-2d_2}\sum_{s=1}^{n}(-1)^{n-s}(-1)^{2n-s}\mathrm{E}a_-^{n-s}\mathrm{E}a_-^{2n-s} \\
&= (-1)^n n^{1-2d_2}\sum_{s=1}^{n}\mathrm{E}a_-^{n-s}\mathrm{E}a_-^{2n-s} \\
&\sim C(-1)^n n^{1-2d_2}\sum_{s=1}^{n}(n-s)^{d_2-1}(2n-s)^{d_2-1} \\
&\sim C(-1)^n\int_0^1(1-x)^{d_2-1}(2-x)^{d_2-1}\,\mathrm{d}x,
\end{aligned}$$

which oscillates with n and has no limit as $n\to\infty$.

4

Aggregation of AR(1) process with infinite variance and idiosyncratic innovations

Abstract. Contemporaneous aggregation of N independent copies of random-coefficient AR(1) process with random coefficient $a \in (-1, 1)$ and independent identically distributed innovations belonging to the domain of attraction of an $\alpha-$stable law, $0 < \alpha < 2$, is discussed. We show that, under normalization $N^{1/\alpha}$, the limit aggregated process exists, in the sense of weak convergence of finite-dimensional distributions, and is a mixed stable moving average as studied in [100]. We focus on the case where the slope coefficient a has probability density vanishing regularly at $a = 1$ with exponent $\beta \in (0, \alpha - 1)$, for $\alpha \in (1, 2)$. We show that in this case, the limit aggregated process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ exhibits long memory. In particular, for $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$, we investigate the decay of codifference, the limit of partial sums, and the long-range dependence (sample Allen variance) property of Heyde and Yang [45].

4.1 Introduction

In Chapter 3, we discussed contemporaneous aggregation of heterogenous random-coefficient AR(1) models with *common* innovations in the domain of attraction of $\alpha-$stable law, $0 < \alpha < 2$, and long-memory properties of the limit aggregated process. We showed that in such case, the limit aggregated process is a moving average with independent identilly distributed innovations, whose coefficients decay hyperbolically j^{d-1}, for $0 < d < 1 - 1/\alpha$, $1 < \alpha < 2$. Let us note that the above aggregation scheme with a particular choice of beta-distributed slope coefficient leads to FARIMA$(0, d, 0)$ process with $\alpha-$stable innovations (see Chapter 3).

In the present chapter (also in [86]) we discuss contemporaneous aggregation of infinite-variance heterogeneous AR(1) processes with *idiosyncratic* innovations (in other words, aggregation of *independent* copies of random-coefficient AR(1) processes). We show that, under some natural assumptions on the AR(1) noise and distribution of the slope coefficient, the limit aggregated process exists and is a so-called *mixed stable moving average* given in (4.4) below. The class of mixed stable moving average processes, introduced in [100] extends (usual) $\alpha-$stable moving average processes, and plays an important role in the general theory of stationary $\alpha-$stable processes (see [92]).

Let us describe the main results of this chapter. Let $\{X(t), t \in \mathbb{Z}\}$ be a stationary solution of the AR(1) equation

$$X(t) = aX(t-1) + \varepsilon(t), \tag{4.1}$$

where $\{\varepsilon(t), t \in \mathbb{Z}\}$ are i.i.d. random variables in the domain of the (normal) attraction of an $\alpha-$stable law, $0 < \alpha < 2$, and where a is an r.v., independent of $\{\varepsilon(t), t \in \mathbb{Z}\}$ and satisfying $|a| < 1$ almost surely. Let the

$$X_i(t) = a_i X_i(t-1) + \varepsilon_i(t), \ i = 1, 2, \dots, N,$$

be independent copies of (4.1). If the distribution of a satisfies the condition that, for some $p < \alpha$,

$$\mathrm{E}\Big[\frac{1}{1-|a|^p}\Big] \ < \ \infty \tag{4.2}$$

then

$$N^{-1/\alpha} \sum_{i=1}^{N} X_i(t) \ \to_{\mathrm{fdd}} \ \mathfrak{X}(t), \tag{4.3}$$

in the sense of weak convergence of finite-dimensional distributions, where the limit process is written as stochastic integral

$$\mathfrak{X}(t) \ = \ \sum_{s \le t} \int_{(-1,1)} a^{t-s} M_s(\,\mathrm{d}a), \tag{4.4}$$

where $\{M_s, s \in \mathbb{Z}\}$ are i.i.d. copies of an $\alpha-$stable random measure M on $(-1, 1)$ with control measure proportional to the distribution Φ of r.v. a (Theorem 4.2.1, page 64). Below, we call Φ the *mixing distribution* of $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$. The class of processes in (4.4) is quite numerous since different mixing distributions Φ yield different processes $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ (Proposition 4.2.4, page 66).

The main incentive of the research was answering the question of whether aggregation of the infinite-variance AR(1) series can lead to long memory. To this end, similarly to Zaffaroni [102], we assume that the mixing distribution is concentrated in the interval $(0, 1)$ and has a density ϕ such that

$$\phi(x) \ \sim \psi(1)\ (1-x)^{\beta}, \qquad \text{as } x \to 1, \tag{4.5}$$

for some $\psi(1) > 0$, $\beta > -1$. In Section 4.3 we study the long-memory properties of the mixed $\alpha-$stable moving average in (4.4).

Clearly, the usual definitions of long memory in terms of covariance/spectrum do not apply in infinite-variance case. Therefore, we use alternative notions of long memory: the decay rate of codifference (see Samorodnitsky and Taqqu [94], pp. 103-106), distributional long memory (see Cox, [28]), and the long-range dependence (sample Allen variance) property of Heyde and Yang [45]. These three properties are established for the aggregated process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ in (4.4) under assumption (4.5) in the parameter range

$$0 < \beta < \alpha - 1, \quad 1 < \alpha < 2;$$

see Theorems 4.3.1, 4.3.2 and 4.3.3 (68, 69 and 69 pages respectively). In particular, normalized partial sums of $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ in (4.4) tend to an $\alpha-$stable stationary increment process $\{\Lambda_{\alpha,\beta}(\tau), \tau > 0\}$, which is self-similar with index $H = 1 - \beta/\alpha \in (1/\alpha, 1)$ and is written as a stochastic integral

$$\begin{aligned} \Lambda_{\alpha,\beta}(\tau) \ &:= \ \int_{\mathbb{R}_+\times\mathbb{R}} \Big(f(x, \tau - s) - f(x, -s)\Big) N(\,\mathrm{d}x,\,\mathrm{d}s), \qquad (4.6) \\ f(x,t) \ &:= \ \begin{cases} 1 - \mathrm{e}^{-xt}, & \text{if } x > 0 \text{ and } t > 0, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

with respect to an independently scattered $\alpha-$stable random measure N on $(0, \infty) \times \mathbb{R}$ with control measure $\psi(1)x^{\beta-\alpha}\,\mathrm{d}x\,\mathrm{d}s$; see Theorem 4.3.1 (page 68) for precise formulations. The value $\beta = \alpha - 1$ seems to separate long memory and short memory in the above aggregation scheme; indeed, in the case $\beta > \alpha - 1$ the aggregated process has the short-range dependence (sample Allen variance) property and its partial sums tend to an $\alpha-$stable Lévy process with independent increments (see Section 4.3). Let us note that $\alpha-$stable self-similar processes of the type in (4.6) were discussed in [26], [87], [99]. Also, note that (4.6) is different from the (more usual) $\alpha-$stable fractional Lévy motion. Since the latter process arises in a similar context by aggregating AR(1) processes with common infinite-variance innovations (see Chapter 3), we can conclude that, in the infinite-variance case, the distinctions between dependent and independent aggregation schemes are deeper than in the case of finite variance; see also Remark 4.2.6, page 67. On the other hand, there are certain similarities between the two aggregation schemes and long-memory properties of the limit aggregated processes, including the relation in (4.21), below, between exponents of the mixing density near $a = 1$. See Remarks 4.3.4 and 4.3.5 (page 70).

The notion of long memory is polysemous, especially for infinite-variance processes, and is not limited to the three characterization properties mentioned above. Another interesting characterization of long memory by the behavior of ruin probabilities in risk insurance models with $\alpha-$stable claims is given in Mikosch and Samorodnitsky [75]. See Remark 4.3.6, page 71, also Chapter 7.

4.2 Existence of the limit aggregated process

Let $\{X_i(t),\, t \in \mathbb{Z}\}, i = 1, 2, \ldots,$ be independent copies of AR(1) process $X(t)$ in (4.1). From the Proposition 3.2.3, page 42, it follows that the solution of the equation (4.1) is the series

$$X(t) \;=\; \sum_{k=0}^{\infty} a^k \varepsilon(t-k), \tag{4.7}$$

which converges conditionally a.s. and in L_p for any $p < \alpha$ and almost every $a \in (-1, 1)$. Moreover, if the condition (4.2) is satisfied, the series in (4.7) converges unconditionally in L_p.

We are interested in the existence and properties of the limit aggregated process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ defined by (4.3).

Introduce independently scattered $\alpha-$stable random measure $M = \{M_s(\,\mathrm{d}a), s \in \mathbb{Z}, a \in (-1, 1)\}$ on $\mathbb{Z} \times (-1, 1)$ with the characteristic functional

$$\mathrm{E}\exp\Big\{\,\mathrm{i}\sum_{s\in\mathbb{Z}} \theta_s M_s(A_s)\Big\} \;=\; \exp\Big\{-\sum_{s\in\mathbb{Z}} |\theta_s|^{\alpha}\omega(\theta_s)\Phi(A_s)\Big\}, \tag{4.8}$$

where $\theta_s \in \mathbb{R}$ and $A_s \subset (-1, 1)$ are arbitrary Borel sets.

We write $\varepsilon \in D(\alpha)$, $0 < \alpha \le 2$, when ε belongs to the domain of normal attraction of an α-stable law (see Definition 3.2.1, page 42).

Theorem 4.2.1. *Let $\varepsilon \in D(\alpha)$ for some $0 < \alpha \le 2$, and let condition* (4.2) *be satisfied. Then the limit aggregated process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ in* (4.3) *exists. It is* *stationary, ergodic, has $\alpha-$stable finite-dimensional distributions, and a stochastic integral representation as in* (4.4)*, where M is an $\alpha-$stable random measure as defined in* (4.8)*.*

The proof of theorem is given in Section 4.4, page 71.

Remark 4.2.2. If the distribution Φ is concentrated at a finite number of points $a_1, \ldots, a_k \in (-1,1)$ and $\phi_i := \mathrm{P}(a = a_i) > 0$, the process in (4.4) can be written as a sum of independent $\alpha-$stable AR(1) processes:

$$\mathfrak{X}(t) = \sum_{i=1}^{k} Y_i(t), \qquad Y_i(t) := \sum_{s\le t} a_i^{t-s}\zeta_i(s), \tag{4.9}$$

where $\{\zeta_i(s) := M_s(\{a_i\}), s \in \mathbb{Z}\}$ is an i.i.d. sequence of $\alpha-$stable r.v.'s with $\mathrm{E}\mathrm{e}^{\mathrm{i}\zeta_i(s)\theta} = \mathrm{e}^{-|\theta|^{\alpha}\omega(\theta)\phi_i}$. For a general mixing distribution Φ, the process in (4.4) can be approximated by finite sums of AR(1) processes as in (4.9). The process in (4.4) is well defined (see [100]) if and only if

$$\sum_{s\in\mathbb{Z}} \mathrm{E}|a^{t-s}|^{\alpha}\mathbf{1}(s\le t) \;=\; \sum_{k=0}^{\infty}\mathrm{E}|a|^{\alpha k} \;=\; \mathrm{E}\Big[\frac{1}{1-|a|^{\alpha}}\Big] \;<\; \infty,$$

which agrees with (4.2). The characteristic function of (4.4) is given by

$$\mathrm{E}\exp\Big\{\,\mathrm{i}\sum_{t=1}^{m}\theta_t\mathfrak{X}(t)\Big\} = \exp\Big\{-\sum_{s\in\mathbb{Z}}\mathrm{E}\Big[\Big|\sum_{t=1}^{m}\theta_t a^{t-s}\mathbf{1}(s\le t)\Big|^{\alpha}\omega\Big(\sum_{t=1}^{m}\theta_t a^{t-s}\mathbf{1}(s\le t)\Big)\Big]\Big\}. \tag{4.10}$$

Remark 4.2.3. For $\alpha = 2$ the limit process in (4.4) is Gaussian and its covariance function is given by

$$\mathrm{cov}(\mathfrak{X}(0),\mathfrak{X}(t)) = \sigma^2\sum_{s\le 0}\int_{(-1,1)} a^{t-s}a^{-s}\Phi(\,\mathrm{d}a) \;=\; \sigma^2\mathrm{E}\Big[\frac{a^t}{1-a^2}\Big] \;=\; \mathrm{cov}(X(0),X(t)) \tag{4.11}$$

and coincides with the covariance of the original series in (4.7). For $\alpha = 2$, the statement of Theorem 4.2.1 is well known; see [77] and [102].

It is clear from (4.10) that the distribution (i.e. finite-dimensional distributions) of $\{\mathfrak{X}(t),\, t\in\mathbb{Z}\}$ is uniquely determined by the distributions of r.v.'s a and Z in (3.7), page 42. It is also clear that the distribution of $\{\mathfrak{X}(t),\, t\in\mathbb{Z}\}$ (particularly, the marginal $\alpha-$stable distribution of $\mathfrak{X}(0)$) uniquely determines the parameter α. Part (i) of Proposition 4.2.4, below, shows that the class of mixed stable moving averages in (4.4) is nonparametric and very large, since different mixing distributions lead to different processes. Part (ii) says that this class is different from (usual) $\alpha-$stable moving averages, except for a trivial mixing distribution Φ.

Proposition 4.2.4. *Let* $0 < \alpha < 2$.

(i) The distribution of $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ *in* (4.4) *uniquely determines the distribution* Φ.

(ii) Let $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\} \stackrel{\mathrm{fdd}}{=} \{Y(t),\, t \in \mathbb{Z}\},\ \ Y(t) := \sum_{j=0}^{\infty} c_j \zeta(t-j)$, *where* $\{\zeta(t),\, t \in \mathbb{Z}\}$ *is an i.i.d. sequence having the same distribution as the* $\alpha-$*stable r.v. in* (3.7), *page 42, and* $c_j, j \geq 0$, *are real coefficients with* $\sum_{j=0}^{\infty} |c_j|^{\alpha} < \infty$. *Then there exist* $a_0 \in (-1, 1)$ *and* $\epsilon \in \{-1, 1\}$ *such that* $c_j = \epsilon a_0^j$ *and* $\Phi = \delta_{a_0}$.

The proof of the Proposition 4.2.4 is given in Section 4.4, page 73.

Let us note that condition (4.2) is crucial for the existence of nontrivial limit of aggregated AR(1) processes. Note also that condition (4.2) does not depend on $p > 0$ since

$$\sup_{0 \leq a < 1} \frac{1 - a^q}{1 - a^p} < \infty,$$

for any $p, q > 0$. Below we show that if condition (4.2) is violated and the mixing density has a power-law behavior at $a = 1$ with negative exponent $\beta \in (-1, 0)$, the limit aggregated process is a random $\alpha(1+\beta)-$stable constant whose stability index $\alpha(1 + \beta) < \alpha$. For notational simplicity, we assume that the noise belongs to the domain of attraction of a symmetric $\alpha-$stable law.

Proposition 4.2.5. *Assume that* $\varepsilon \in D(\alpha),\ 0 < \alpha \leq 2$, *and that* $\omega(\theta) \equiv 1$ *in* (4.23), *page 71. Moreover, assume that the mixing density has the form*

$$\phi(a) \;=\; \psi(a)(1-a)^{\beta}, \quad a \in (0, 1), \tag{4.12}$$

where $\beta \in (-1, 0)$ *and* ψ *is an integrable function on* $(0, 1)$ *having a limit*

$$\psi(1) := \lim_{a \to 1} \psi(a) > 0.$$

Then

$$N^{-1/\alpha(1+\beta)} \sum_{i=1}^{N} X_i(t) \;\to_{\mathrm{fdd}}\; \tilde{Z},$$

where the limit process $\tilde{Z}$ *does not depend on* t *and is an* $\alpha(1+\beta)-$*stable r.v. with characteristic function* $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta\tilde{Z}} = \mathrm{e}^{-K|\theta|^{\alpha(1+\beta)}}$, *where* K *is given in* (4.31), *page 74.*

The proof of the Proposition 4.2.5 is given in Section 4.4, page 74.

Note that, for the mixing density in (4.12) with $\beta > 0$, Theorem 4.2.1, page 64, applies and, therefore, $\beta = 0$ is a critical point resulting in completely different limits of the aggregated process in the cases $\beta > 0$ and $\beta < 0$. The fact that the limit is degenerate in the latter case can be explained as follows. It is clear that, with β decreasing, the dependence increases in the random-coefficient AR(1) process $\{X(t),\, t \in \mathbb{Z}\}$, as well as in the limit aggregated process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$. In Section 4.3 we show that the dependence in $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ decays hyperbolically with the lag, with an exponent which depends on β and α and which tends to 0 as $\beta \downarrow 0$. Therefore, for negative $\beta < 0$, the dependence in the aggregated process becomes extremely strong so that the limit process is degenerate and completely dependent.

Remark 4.2.6. Let M be the $\alpha-$stable random measure in (4.8), and $\{\zeta(s) := M_s(-1,1),\ s \in \mathbb{Z}\}$ be the corresponding i.i.d. sequence of $\alpha-$stable r.v.'s. Let $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ be the aggregated mixed $\alpha-$stable moving average in (4.4), and let $1 < \alpha \le 2$. Then

$$\mathrm{E}[\mathfrak{X}(t)|\zeta(s), s \in \mathbb{Z}] \;=\; \sum_{s \le t} \mathrm{E}[a^{t-s}]\zeta(s), \quad t \in \mathbb{Z}. \tag{4.13}$$

Relation (4.13) follows from a general 'interpolation formula' for independently scattered random measures (see [96], Proposition 1.3). For the reader's convenience, we present this formula for the $\alpha-$stable measure M in Proposition 4.2.7, below. Recall from Chapter 3 that the right-hand side of (4.13) represents the limit aggregated process in the AR(1) aggregation scheme with *common* $\alpha-$stable innovations $\varepsilon(s) = \zeta(s), s \in \mathbb{Z}$. Thus, (4.13) establishes a link between the aggregated processes in the two aggregation schemes. It also suggests that the latter aggregation scheme leads to a simpler aggregated process when compared to the process (4.4) in the present chapter. In particular, the moving average on the right-hand side of (4.13) may be invertible (which occurs, e.g. in the case of FARIMA(0,d,0) coefficients $\mathrm{E}[a^{t-s}]$ mentioned in the introduction), while, for the mixed moving average in (4.4) the usual definition of invertibility does not apply and the possibility of 'recovering' $M_t(A)$ from $\mathfrak{X}(s), s \le t$, seems unlikely. On the other hand, in the finite-variance case, $\alpha = 2$, the limit aggregated process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ is Gaussian with covariance given in (4.11); hence, it is also invertible under known conditions on the spectral density. (A particular form of the mixing density ϕ leading to the FARIMA(0, d, 0) Gaussian process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ was found in [21].) The above discussion complies with the remark in the introduction that the distinctions between dependent and independent aggregation schemes in the infinite-variance case are deeper than in the finite-variance case.

Let $L^\alpha(\mathbb{Z} \times (-1,1))$ denote the class of all measurable functions $h : \mathbb{Z} \times (-1,1) \to \mathbb{R}$ with

$$\sum_{s \in \mathbb{Z}} \mathrm{E}|h(s,a)|^\alpha < \infty,\ 1 < \alpha \le 2.$$

The stochastic integral

$$M(h) := \sum_{s\in\mathbb{Z}} \int_{(-1,1)} h(s,a) M_s(\,\mathrm{d}a)$$

is well defined for any $h \in L^\alpha(\mathbb{Z}\times(-1,1))$; see ([94], Ch. 3, pp. 111-167).

Proposition 4.2.7. *Let M and $\{\zeta(s), s\in\mathbb{Z}\}$ be the same as in Remark 4.2.6, and let $1<\alpha\le 2$. Then, for any $h\in L^\alpha(\mathbb{Z}\times(-1,1))$,*

$$\mathrm{E}\Big[M(h)|\zeta(s), s\in\mathbb{Z}\Big] \;=\; \sum_{s\in\mathbb{Z}} \bar{h}(s)\zeta(s); \qquad \bar{h}(s) := \mathrm{E}h(s,a). \tag{4.14}$$

The proof of the Proposition 4.2.7 is given in Section 4.4, page 75.

4.3 Long memory properties of the limit aggregated process

Recall the definition of the process $\{\Lambda_{\alpha,\beta}(\tau), \tau\in\mathbb{R}\}$ in (4.6). This process is well defined for any $0<\beta<\alpha-1$ and $\alpha\in(1,2)$ and its characteristic functional is given by

$$\begin{aligned}\mathrm{E}\exp\Big\{\,\mathrm{i}\sum_{i=1}^m \theta_i\Lambda_{\alpha,\beta}(\tau_i)\Big\} \;&=\; \exp\Big\{-\psi(1)\int_{\mathbb{R}}\int_{\mathbb{R}_+}\Big|\sum_{i=1}^m \theta_i(f(x,\tau_i-s)-f(x,-s))\Big|^\alpha \\ &\quad\times\,\omega\Big(\sum_{i=1}^m \theta_i(f(x,\tau_i-s)-f(x,-s))\Big)x^{\beta-\alpha}\,\mathrm{d}s\,\mathrm{d}x\Big\},\end{aligned} \tag{4.15}$$

where $\tau_i,\theta_i\in\mathbb{R},\ i=1,\dots,m,\ m=1,2,\dots$. The process $\{\Lambda_{\alpha,\beta}(\tau),\,\tau\ge 0\}$ is self-similar with index

$$H = 1-\frac{\beta}{\alpha}\;\in\;\Big(\frac{1}{\alpha},1\Big), \tag{4.16}$$

which follows from (4.15) by the change of variables $s\to\lambda s$, $x\to x/\lambda$, $\lambda>0$, and has $\alpha-$stable finite-dimensional distributions and stationary increments. From these facts and Kolmogorov's moment criterion, it follows that $\{\Lambda_{\alpha,\beta}(\tau),\,\tau\ge 0\}$ has a sample continuous version. See also ([99], Corollary 4).

Theorem 4.3.1. *Let $\{\mathfrak{X}(t),\,t\in\mathbb{Z}\}$ be the aggregated process in (4.4) with mixing density as in (4.12), where $\beta>0$ and ψ is integrable on $(0,1)$ and has a limit $\lim_{a\to 1-}\psi(a) =: \psi(1)>0$.*

(i) Let $1<\alpha<2$ and $0<\beta<\alpha-1$. Let $H=1-\beta/\alpha$, as in (4.16). Then

$$\frac{1}{n^H}\sum_{t=1}^{[n\tau]}\mathfrak{X}(t) \;\to_{\mathrm{fdd}}\; \Lambda_{\alpha,\beta}(\tau), \tag{4.17}$$

where the limit process is given in (4.6)*.*
(ii) Let $0 < \alpha < 2$ *and* $\beta > \max(\alpha - 1, 0)$*. Then*

$$\frac{1}{n^{1/\alpha}} \sum_{t=1}^{[n\tau]} \mathfrak{X}(t) \ \to_{\rm fdd} \ L(\tau), \tag{4.18}$$

where $\{L(\tau), \tau \geq 0\}$ *is an* $\alpha-$*stable homogeneous Lévy process with characteristic function*

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta L(\tau)} = \mathrm{e}^{-K|\theta|^{\alpha}\omega(\theta)\tau}, \quad K := \int_0^1 (1-x)^{-\alpha}\phi(x)\,\mathrm{d}x.$$

The proof of Theorem 4.3.1 is given in Section 4.4, page 76.

Since the process $\{\Lambda_{\alpha,\beta}(\tau), \tau \geq 0\}$ in (4.17) has dependent increments while the Lévy process $\{L(\tau), \tau \geq 0\}$ in (4.18) has independent increments, from Theorem 4.3.1 we conclude that the limit aggregated process $\{\mathfrak{X}(t),\, t \in Z\}$ with mixing density as in (4.12) has distributional long memory (see Definition 2.3.6, page 36) for $0 < \beta < \alpha - 1,\ 1 < \alpha < 2$, and distributional short memory for $\beta > \max(\alpha - 1, 0)$.

Next, we turn to the study of the LRD(SAV) property defined in Heyde and Yang [45] (see Definition 2.3.7, page 36).

Theorem 4.3.2. *Let* $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ *satisfy the conditions of Theorem 4.3.1.*
(i) Let $1 < \alpha < 2$ *and* $0 < \beta < \alpha - 1$*. Then* $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ *is LRD(SAV).*
(ii) Let $1 < \alpha < 2$ *and* $\beta > \alpha - 1$*. Then* $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ *is SRD(SAV).*

The proof of Theorem 4.3.2 is given in Section 4.4, page 78.

The *codifference* of a strictly stationary process $\{Y(t),\, t \in \mathbb{Z}\}$,

$$\mathrm{Cod}(Y(0), Y(t)) := \log \mathrm{E}\mathrm{e}^{\mathrm{i}(Y(t)-Y(0))} - \log \mathrm{E}\mathrm{e}^{\mathrm{i}Y(t)} - \log \mathrm{E}\mathrm{e}^{-\mathrm{i}Y(0)},$$

can also be used to characterize the long memory of $\{Y(t),\, t \in \mathbb{Z}\}$ (see [94], pp. 384-387). Theorem 4.3.3, below, gives the decay rate of the codifference of the mixed stable moving average in (4.4) and the mixing density in (4.19), below.

Theorem 4.3.3. *Let* $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ *be the aggregated process in* (4.4)*, with characteristic functional as in* (4.10)*,* $0 < \alpha < 2$*, and mixing density*

$$\phi(a) \ = \ \psi(a) \begin{cases} (1-a)^{\beta_1}, & 0 < a < 1, \\ (1+a)^{\beta_2}, & -1 < a \leq 0, \end{cases} \qquad a \in (-1,1), \tag{4.19}$$

where $1 > \beta_1 > 0$*,* $1 > \beta_2 > 0$*, are parameters and* ψ *is continuous at* ± 1 *with*

$\lim_{a\to\pm 1}\psi(a) =: \psi(\pm 1) \ge 0$. *Then, as* $t \to \infty$,

$$\mathrm{Cod}(\mathfrak{X}(0), \mathfrak{X}(t)) \;=\; \Big(C_1 + o(1)\Big)t^{-\beta_1} + \Big(C_2(t) + o(1)\Big)t^{-\beta_2}, \qquad (4.20)$$

where

$$\begin{aligned} C_1 &:= \psi(1)\alpha^{-1}\int_0^\infty [\omega(1)\mathrm{e}^{-y\alpha} + \overline{\omega(1)}(1-(1-\mathrm{e}^{-y})^\alpha)]y^{\beta_1-1}\,\mathrm{d}y, \\ C_2(t) &:= \psi(-1)\alpha^{-1}\mathrm{Re}(\omega(1))\int_0^\infty [\mathrm{e}^{-y\alpha} + 1 - (1-(-1)^t\mathrm{e}^{-y})^\alpha]y^{\beta_2-1}\,\mathrm{d}y. \end{aligned}$$

The proof of Theorem 4.3.3 is given in Section 4.4, page 80.

Remark 4.3.4. For $1 < \alpha \le 2$ and $0 < \beta < \alpha - 1$, introduce the parameter

$$d \;:=\; \frac{\alpha - 1 - \beta}{\alpha}, \qquad (4.21)$$

or $\beta = \alpha - 1 - \alpha d$. Note $\beta = 0$ if and only if $d = 1 - 1/\alpha$, and $\beta = \alpha - 1$ if and only if $d = 0$. Recall from ([94], Theorem 7.13.4) that, for the FARIMA$(0, d, 0)$ process $\{Y(t),\, t \in \mathbb{Z}\}$ with $\alpha-$stable innovations, $0 < d < 1 - 1/\alpha$, and $1 < \alpha \le 2$,

$$\mathrm{Cod}(Y(0), Y(t)) \;\sim\; C\, t^{1+\alpha d-\alpha} \quad \text{as} \quad t \to \infty. \qquad (4.22)$$

Therefore, Theorem 4.3.3 implies that the codifference of the aggregated process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ in (4.4) with the mixing density in (4.12) and $0 < \beta < \alpha - 1$ decays similarly as the codifference of an $\alpha-$stable FARIMA$(0, d, 0)$ process with parameter d given in (4.21). From Theorem 4.3.1 we see that the above similarity between $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ and FARIMA$(0, d, 0)$ with parameter d in (4.21) also extends to the normalization exponent H of partial sums of both processes: for the former process, $H = 1 - \beta/\alpha$ and, for the latter process, $H = d + 1/\alpha$. Clearly, $1 - \beta/\alpha = d + 1/\alpha$ is equivalent to (4.21). In other words, if β and d are related as in (4.21), then partial sums of $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ and partial sums of the FARIMA$(0, d, 0)$ process converge under the same normalization and the limits are self-similar processes with the same parameter H.

Remark 4.3.5. Recall that a second-order stationary process is said to have covariance long memory if the sum of the absolute values of covariances diverges. In the case of an infinite-variance process, the divergence of the absolute values of codifferences also indicates the presence of long memory. From Theorems 4.3.1-4.3.3 we see that the codifference of $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ is nonsummable for any $0 < \beta < 1$, irrespective of the value of α, while at the same time this process may have the SRD(SAV) property and distributional short memory, provided $\alpha - 1 < \beta < 1$ and $1 < \alpha < 2$. These results might look strange and a peculiarity of the process in (4.4) at first glance; however, similar facts also hold for moving averages $Y(t) = \sum_{j=0}^{\infty} c_j \varepsilon(t-j)$ in i.i.d. innovations $\varepsilon(t) \in D(\alpha)$ with regularly decaying coefficients $c_j \sim j^{d-1}$. Indeed, for such $\{Y(t),\, t \in \mathbb{Z}\}$, the codifference decays as in (4.22), for any $0 < \alpha < 2$ and $d < 1 - 1/\alpha$, so that $\sum_{j=0}^{\infty} |\mathrm{Cod}(Y(0), Y(j))| = \infty$ and $\sum_{j=0}^{\infty} |c_j| < \infty$ hold for $1 - 2/\alpha < d < 0$. Since $\{Y(t),\, t \in \mathbb{Z}\}$ has distributional short memory for $d < 0$ and $\sum_{j=0}^{\infty} c_j \neq 0$ (see, e.g. [7]), we have exactly the same situation as in the case of $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$, with parameters d and β related as in (4.21).

Remark 4.3.6. Mikosch and Samorodnitsky [75] discussed the asymptotic behavior of the *ruin probability*

$$\psi(u) := \mathrm{P}\Big(\sup_{n\ge 0}(X(1) + \cdots + X(n) - n\mu) > u\Big)$$

as $u \to \infty$, where 'claims' $\{X(t),\, t \in \mathbb{Z}\}$ form a stationary $\alpha-$stable process, $1 < \alpha < 2$, and $\mu > \mathrm{E}X(1)$ is a given constant. They associated the 'classical' decay rate $\psi(u) = O(u^{-(\alpha-1)})$ with short-range dependence and the decay rate $\psi(u) = O(u^{-\nu})$ with exponent $\nu < \alpha - 1$ with long-range dependence of the claim sequence $\{X(t),\, t \in \mathbb{Z}\}$. In the case when the $X(t)$'s are stationary increments of a linear $\alpha-$stable fractional motion with self-similarity parameter $H \in (1/\alpha, 1)$, Mikosch and Samorodnitsky ([75], Proposition 4.4) obtained a decay rate $\psi(u) \sim (\text{constant})\, u^{-\alpha(1-H)}$ of the ruin probability. Let us note that increments of an $\alpha-$stable fractional motion satisfy the distributional long-memory property and also exhibit the decay of codifference as in (4.22), with d and H related as in Remark 4.3.4 (see ([94], pp. 380-387)). Therefore, the above characterization of long memory via ruin probabilities seems to agree with other characterizations of long memory discussed in this paper, at least for $\alpha-$stable moving averages. In Chapter 7 (see also [81]), we find the asymptotics of the ruin probability, when 'claims' are modeled by the limit aggregated process $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ in (4.4).

4.4 Proofs

Proof of Theorem 4.2.1, page 64. The characteristic function of the r.v. $\varepsilon \in D(\alpha)$ has the following representation in a neighborhood of the origin (see, e.g. ([50], Theorem 2.6.5)): there exists an $\epsilon > 0$ such that

$$\mathrm{E}e^{\mathrm{i}\theta\varepsilon} = \mathrm{e}^{-|\theta|^{\alpha}\omega(\theta)h(\theta)}, \qquad |\theta| < \epsilon, \tag{4.23}$$

where h is a positive function tending to 1 as $\theta \to 0$. Denote

$$\vartheta(s,a) := \sum_{t=1}^{m} \theta_t a^{t-s} \mathbf{1}(s \le t). \tag{4.24}$$

Then $N^{-1/\alpha}\sum_{t=1}^m \theta_t X(t) = N^{-1/\alpha}\sum_{s\in\mathbb{Z}}\vartheta(s,a)\varepsilon(s)$. Since m and $\theta_t, t = 1,\dots,m$ are fixed and a is bounded, it is clear that $|\vartheta(s,a)| \le C$ for a constant C independent of a and s, and, therefore, $|N^{-1/\alpha}\vartheta(s,a)| < \epsilon$ for all $N > N_0$ large enough. Therefore, using (4.23), we can write

$$\begin{aligned}
&\mathrm{E}\exp\Big\{\,\mathrm{i}N^{-1/\alpha}\sum_{i=1}^{N}\sum_{t=1}^{m}\theta_t X_i(t)\Big\} \\
&\qquad = \Big(\mathrm{E}\exp\Big\{\,\mathrm{i}N^{-1/\alpha}\sum_{t=1}^{m}\theta_t X(t)\Big\}\Big)^N \\
&\qquad = \Big(\mathrm{E}\exp\Big\{-N^{-1}\sum_{s\in\mathbb{Z}}\Big|\vartheta(s,a)\Big|^\alpha h\Big(N^{-1/\alpha}\vartheta(s,a)\Big)\omega\Big(\vartheta(s,a)\Big)\Big\}\Big)^N.
\end{aligned}$$

Clearly, for any $a \in (-1,1)$,

$$\sum_{s\in\mathbb{Z}}\Big|\vartheta(s,a)\Big|^\alpha h\Big(N^{-1/\alpha}\vartheta(s,a)\Big)\omega\Big(\vartheta(s,a)\Big) \;\to\; \sum_{s\in\mathbb{Z}}\Big|\vartheta(s,a)\Big|^\alpha \omega\Big(\vartheta(s,a)\Big) \tag{4.25}$$

as $N \to \infty$, and

$$\Big|\sum_{s\in\mathbb{Z}}\Big|\vartheta(s,a)\Big|^\alpha h\Big(N^{-1/\alpha}\vartheta(s,a)\Big)\omega\Big(\vartheta(s,a)\Big)\Big| \;\le\; \frac{C}{1-|a|^\alpha} \tag{4.26}$$

for a constant $C < \infty$ independent of a. Define

$$\Theta_N := N\mathrm{E}\Big[\exp\Big\{-N^{-1}\sum_{s\in\mathbb{Z}}\Big|\vartheta(s,a)\Big|^\alpha h\Big(N^{-1/\alpha}\vartheta(s,a)\Big)\omega\Big(\vartheta(s,a)\Big)\Big\} - 1\Big].$$

Using (4.25), (4.26), condition (4.2), the fact that $0 \le h(\theta) \le C$, the inequality $|\mathrm{e}^z - 1| \le |z|$ $z \in \mathbb{C}$, $\mathrm{Re}z \le 0$, and the dominated convergence theorem, we obtain

$$\lim_{N\to\infty}\Theta_N \;=\; -\sum_{s\in\mathbb{Z}}\mathrm{E}[|\vartheta(s,a)|^\alpha\omega(\vartheta(s,a))].$$

Therefore,

$$\begin{aligned}
\lim_{N\to\infty}\mathrm{E}\exp\Big\{\,\mathrm{i}N^{-1/\alpha}\sum_{i=1}^{N}\sum_{t=1}^{m}\theta_t X_i(t)\Big\} \;&=\; \lim_{N\to\infty}\Big(1+\frac{\Theta_N}{N}\Big)^N \\
&=\; \exp\Big\{-\sum_{s\in\mathbb{Z}}\mathrm{E}[|\vartheta(s,a)|^\alpha\omega(\vartheta(s,a))]\Big\},
\end{aligned}$$

which coincides with (4.10). The properties of $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ mentioned in the statement of the theorem follow from [100]. This completes the proof. □

Proof of Proposition 4.2.4, page 66. (i) By separately considering the real and imaginary parts of the logarithm of the characteristic function in (4.10), we see that it suffices to prove the proposition for the symmetric case $\omega \equiv 1$ only.

Let $L^\alpha(\mathbb{Z})$ be the space of all real sequences $g = (g_t, t \in \mathbb{Z})$ with

$$\|g\|_\alpha^\alpha := \sum_{t\in\mathbb{Z}} |g_t|^\alpha < \infty.$$

Let $\mathcal{B}(L^\alpha(\mathbb{Z}))$ be the $\sigma-$algebra of Borel sets of $L^\alpha(\mathbb{Z})$. A Borel set $A \subset L^\alpha(\mathbb{Z})$ is said to be *symmetric* if $-A = A$ and *shift invariant* if $U_t A = A$ for every $t \in \mathbb{Z}$, where $U_s, s \in \mathbb{Z}$, is the group of shift operators on $L^\alpha(\mathbb{Z})$, $(U_s g)_t := g_{t-s}$. Let $\mathcal{B}_{\mathrm{inv}}(L^\alpha(\mathbb{Z}))$ denote the class of all open symmetric and shift-invariant sets $A \subset L^\alpha(\mathbb{Z})$.

According to ([100], Theorem 2 and Lemma 1), the characteristic function in (4.10) uniquely determines the measure

$$\mu(A) := \int_{L^\alpha(\mathbb{Z})} \mathbf{1}\Big(\frac{g}{\|g\|_\alpha} \in A\Big) \|g\|_\alpha^\alpha \, \lambda(\mathrm{d}g), \tag{4.27}$$

on open symmetric and shift-invariant sets $A \in \mathcal{B}_{\mathrm{inv}}(L^\alpha(\mathbb{Z}))$ and vice versa; here

$$\lambda(A) := \mathrm{P}\Big((a^{-t}\mathbf{1}(t \le 0), t \in \mathbb{Z}) \in A\Big), \qquad A \in \mathcal{B}(L^\alpha(\mathbb{Z})) \tag{4.28}$$

is a probability measure concentrated on the set

$$\{g = (g_t, t \in \mathbb{Z}) \in L^\alpha(\mathbb{Z}) : g_t = a^{-t}\mathbf{1}(t \le 0), \ \ \text{there exists } a \in (-1,1)\}$$

of geometric progressions.

Let $V \subset (-1,1)$ be an open set, and let

$$\begin{aligned} A(V) &:= \bigcup_{s\in\mathbb{Z}} \bigcup_{\delta=\pm 1} A_{s,\delta}(V), \qquad (4.29)\\ A_{s,\delta}(V) &:= \Big\{f = (f_t, t \in \mathbb{Z}) \in L^\alpha(\mathbb{Z}) : f_t = \delta(1-|v|^\alpha)^{1/\alpha} v^{s-t}\mathbf{1}(t \le s),\ \exists v \in V\Big\}. \end{aligned}$$

Note that, $A_{s,\delta}(V)$ are disjoint sets for distinct pairs (s,δ), the set $A(V)$ is open, symmetric and shift invariant and $\mu(A_{s,\delta}(V)) = 0$ unless $(s,\delta) = (0,1)$. Moreover,

$$\mu(A(V)) \ = \ \mu(A_{(0,1)}(V)) \ = \ \mathrm{E}\Big[\frac{\mathbf{1}(a \in V)}{1-|a|^\alpha}\Big] \ = \ \int_V \frac{\Phi(\mathrm{d}a)}{1-|a|^\alpha} \ =: \ G(V)$$

according to the definitions in (4.27)-(4.28). Therefore, the characteristic function in (4.10) uniquely determines the measure G on the interval $(-1,1)$. Since $\Phi(V) = \int_V (1-|a|^\alpha) G(\mathrm{d}a)$, part (i) of the proposition follows.

(ii) As in (i), it suffices to discuss the case $\omega \equiv 1$. Let $\mu = \mu_{\mathfrak{X}}$ be defined in (4.27), and let

$$\mu_Y(A) := \|c\|_\alpha^\alpha \mathbf{1}\Big(\frac{c}{\|c\|_\alpha} \in A\Big), \quad c := (c_{-t}\mathbf{1}(t \le 0), t \in \mathbb{Z}) \in L^\alpha(\mathbb{Z}),$$

be the measure on the unit sphere of $L^\alpha(\mathbb{Z})$, corresponding to the moving average $\{Y(t),\, t \in \mathbb{Z}\}$. By definition, μ_Y is concentrated on a single element $c/\|c\|_\alpha \in L^\alpha(\mathbb{Z})$.

As mentioned above in the proof of (i), $\{\mathfrak{X}(t)\} \overset{\mathrm{fdd}}{=} \{Y(t)\}$ implies that

$$\mu_Y(A) = \mu_{\mathfrak{X}}(A), \qquad A \in \mathcal{B}_{\mathrm{inv}}(L^\alpha(\mathbb{Z})). \tag{4.30}$$

Consider the set $A = A(-1,1)$, as defined in (4.29), consisting of all signed translations of normalized geometric progressions. Clearly, $c/\|c\|_\alpha \in A(-1,1)$ if and only if $c_j = \epsilon a_0^j$, $j \ge 0$ for some $a_0 \in (-1,1)$ and $\epsilon \in \{-1,1\}$. It also easily follows from (4.30) that $\Phi = \delta_{a_0}$. This completes the proof. □

Proof of Proposition 4.2.5, page 66. Let

$$\Theta_N := N\mathrm{E}\Big[\exp\Big\{-N^{-1/(1+\beta)}\sum_{s\in\mathbb{Z}}|\vartheta(s,a)|^\alpha h\Big(N^{-1/\alpha(1+\beta)}\vartheta(s,a)\Big)\Big\} - 1\Big],$$

where $\vartheta(s,a)$ is defined as in (4.24), i.e.

$$\vartheta(s,a) := \sum_{t=1}^m \theta_t a^{t-s}\mathbf{1}(s \le t).$$

Then,

$$\mathrm{E}\exp\Big\{\mathrm{i}N^{-1/\alpha(1+\beta)}\sum_{i=1}^N\sum_{t=1}^m \theta_t X_i(t)\Big\} = \Big(1 + \frac{\Theta_N}{N}\Big)^N.$$

Similarly as in the proof of Theorem 4.2.1, it suffices to show that

$$\lim_{N\to\infty}\Theta_N = -K\Big|\sum_{t=1}^m \theta_m\Big|^{\alpha(1+\beta)}, \quad K := \alpha^{-(\beta+1)}\psi(1)\int_0^\infty (1-\mathrm{e}^{-z})z^{-(\beta+2)}\,\mathrm{d}z. \tag{4.31}$$

To prove (4.31), split

$$\sum_{s\in\mathbb{Z}}|\vartheta(s,a)|^\alpha h(N^{-1/\alpha(1+\beta)}\vartheta(s,a)) = \sum_{s\le 0}\cdots + \sum_{s=1}^m \cdots =: \Sigma_1 + \Sigma_2.$$

Note that Σ_2 is uniformly bounded in $a \in [0,1)$ and $N \ge 1$ and $N^{-1/(1+\beta)} =$

$o(N^{-1})$ for $\beta < 0$. Therefore, it suffices to prove (4.31) for Θ_N replaced by

$$\Theta_{N1} := N\mathrm{E}[\mathrm{e}^{-N^{-1/(1+\beta)}\Sigma_1} - 1].$$

We have

$$\begin{aligned}\Theta_{N1} &= N\int_{1-\epsilon}^{1}\Big(\exp\Big\{-N^{-1/(1+\beta)}\frac{1}{\alpha(1-a)}\Big|\sum_{t=1}^{m}\theta_t\Big|^{\alpha}\Big\} - 1\Big)(1-a)^{\beta}\psi(a)\,\mathrm{d}a + o(1)\\ &= N\int_{0}^{\epsilon}\Big(\exp\Big\{-\frac{1}{\alpha x N^{1/(1+\beta)}}\Big|\sum_{t=1}^{m}\theta_t\Big|^{\alpha}\Big\} - 1\Big)\psi(1-x)x^{\beta}\,\mathrm{d}x + o(1)\\ &= -K_N(\theta)\Big|\sum_{t=1}^{m}\theta_t\Big|^{\alpha(1+\beta)} + o(1),\end{aligned}$$

where

$$K_N(\theta) := \alpha^{-(\beta+1)}\psi(1)\int_0^{\infty}\mathbf{1}(z > \delta_N(\theta))(1-\mathrm{e}^{-z})z^{-(\beta+2)}\,\mathrm{d}z$$

and

$$\delta_N(\theta) := (\alpha\epsilon)^{-1}N^{-1/(1+\beta)}|\sum_{t=1}^{m}\theta_t|^{\alpha} \to 0, \quad \text{as } N\to\infty.$$

Since $\lim_{N\to\infty}K_N(\theta) = K$ by the dominated convergence theorem, this proves (4.31) and the proposition. □

Proof of Proposition 4.2.7, page 68. It suffices to prove the proposition for simple functions $h \in L^{\alpha}(\mathbb{Z}\times(-1,1))$ of the form $h(t,a) = \sum_{i=1}^{n} h_{it}\mathbf{1}(|t| \le n, a \in A_i)$, where $A_i \subset (-1,1)$, $i = 1,\dots,n$, are disjoint Borel sets. For such h,

$$M(h) = \sum_{|t|\le n}\sum_{i=1}^{n} h_{it}M_t(A_i)$$

is a finite sum of $\alpha-$stable r.v.'s. By linearity of both sides of (4.14) in h and independence of $M_t(A_i)$ and $M_s(A_j), s \ne t$, it suffices to check (4.14) for $h(t,a) = \mathbf{1}(t = s, a \in A)$, or

$$\mathrm{E}[M_s(A)|M_s(-1,1)] \;=\; \Phi(A)M_s(-1,1) \qquad (4.32)$$

for any Borel set $A \subset (-1,1)$. By standard arguments, (4.32) is equivalent to

$$\mathrm{E}[M_s(A)\mathrm{e}^{\mathrm{i}\theta M_s(-1,1)}] \;=\; \Phi(A)\mathrm{E}[M_s(-1,1)\mathrm{e}^{\mathrm{i}\theta M_s(-1,1)}], \quad \theta \in \mathbb{R}. \qquad (4.33)$$

Let $\kappa_A(\theta) := \mathrm{E}\Big[\mathrm{e}^{\mathrm{i}\theta M_s(-1,1)}\Big]$, $\kappa(\theta) := \kappa_{(-1,1)}(\theta)$ and $A^c := (-1,1)\backslash A$. Then (4.33) can be rewritten as

$$\kappa_A'(\theta)\kappa_{A^c}(\theta) = \Phi(A)\kappa'(\theta).$$

The above equality is immediate from $\kappa_A(\theta)\kappa_{A^c}(\theta) = \kappa(\theta)$ and $\kappa_A(\theta) = (\kappa(\theta))^{\Phi(A)}$

(the last relation follows from the form of the characteristic functional in (4.8) and the fact that $\omega(\theta)$ in (3.9), page 42, depends only on the sign of θ). □

Proof of Theorem 4.3.1, page 68. (i) We will prove the one-dimensional convergence in (4.17) at $\tau = 1$ only, since the general case in (4.17) follows analogously. In view of (4.10) and (4.15), it suffices to prove that, for any $\theta \in \mathbb{R}$,

$$n^{-H\alpha} \sum_{s \in \mathbb{Z}} \mathrm{E} \Big| \sum_{t=1}^{n} a^{t-s} \mathbf{1}(s \le t) \Big|^{\alpha} \omega\Big(\theta \sum_{t=1}^{n} a^{t-s} \mathbf{1}(s \le t)\Big) \tag{4.34}$$
$$\to c \int_{\mathbb{R}} \int_{\mathbb{R}_+} |f(x, 1-s) - f(x, -s)|^{\alpha} \omega(\theta(f(x, 1-s) - f(x, -s))) x^{\beta-\alpha} \, \mathrm{d}s \, \mathrm{d}x.$$

Note that the expressions inside ω on both sides of (4.34) are positive or negative depending on the sign of θ and $\omega(\theta) = \omega(\mathrm{sign}(\theta))$. Therefore, it suffices to show (4.34) for $\theta = 1$ alone. To this end, let us denote the left- and right-hand sides of (4.34) (with $\theta = 1$) by J_n and J, respectively. Split $J = J_1 + J_2$, where

$$\begin{aligned} J_1 &:= \psi(1)\omega(1) \int_{-\infty}^{0} \mathrm{d}s \int_0^{\infty} |f(x, 1-s) - f(x, -s)|^{\alpha} x^{\beta-\alpha} \, \mathrm{d}x \\ &= \psi(1)\omega(1)\alpha^{-1} \int_0^{\infty} (1 - \mathrm{e}^{-y})^{\alpha} y^{\beta-\alpha-1} \, \mathrm{d}y, \\ J_2 &:= \psi(1)\omega(1) \int_0^1 \mathrm{d}s \int_0^{\infty} |f(x, 1-s)|^{\alpha} x^{\beta-\alpha} \, \mathrm{d}x \\ &= \psi(1)\omega(1) \int_0^1 \mathrm{d}u \int_0^{\infty} (1 - \mathrm{e}^{-x(1-u)})^{\alpha} x^{\beta-\alpha} \, \mathrm{d}x, \end{aligned}$$

according to the definition of f in (4.6). Next, write $J_n = J_{n1} + J_{n2}$, where

$$\begin{aligned} J_{n1} &:= n^{-H\alpha}\omega(1) \sum_{s=-\infty}^{0} \int_0^1 \Big| \sum_{t=1}^{n} a^{t-s} \Big|^{\alpha} (1-a)^{\beta} \psi(a) \, \mathrm{d}a \\ &= n^{-H\alpha}\omega(1) \int_0^1 \frac{1}{1-a^{\alpha}} \Big| \frac{a(1-a^n)}{1-a} \Big|^{\alpha} (1-a)^{\beta} \psi(a) \, \mathrm{d}a \\ &= \omega(1) \int_0^{\infty} \frac{(1-y/n)^{\alpha}}{n(1-(1-y/n)^{\alpha})} \Big(1 - \Big(1 - \frac{y}{n}\Big)^{n}\Big)^{\alpha} y^{\beta-\alpha} \\ &\quad \times \psi\Big(1 - \frac{y}{n}\Big) \mathbf{1}(0 < y < \epsilon n) \, \mathrm{d}y + o(1) \\ &\to \frac{\psi(1)\omega(1)}{\alpha} \int_0^{\infty} (1 - \mathrm{e}^{-y})^{\alpha} y^{\beta-\alpha-1} \, \mathrm{d}y = J_1 \end{aligned}$$

by the dominated convergence theorem as $n \to \infty$. In a similar way,

$$\begin{aligned} J_{n2} &:= n^{-H\alpha}\omega(1)\sum_{s=1}^{n}\int_0^1 \Big|\sum_{t=1}^{n} a^{t-s}\mathbf{1}(s \le t)\Big|^{\alpha}(1-a)^{\beta}\psi(a)\,\mathrm{d}a \\ &= n^{-H\alpha}\omega(1)\int_0^1 \sum_{s=1}^{n}\Big|\frac{1-a^{n-s+1}}{1-a}\Big|^{\alpha}(1-a)^{\beta}\psi(a)\,\mathrm{d}a \\ &= \omega(1)\int_0^{\infty}\frac{1}{n}\sum_{s=1}^{n}\Big(1-\Big(1-\frac{y}{n}\Big)^{n-s+1}\Big)^{\alpha}y^{\beta-\alpha}\psi\Big(1-\frac{y}{n}\Big)\mathbf{1}(0<y<\epsilon n)\,\mathrm{d}y + o(1) \\ &\to \psi(1)\omega(1)\int_0^{\infty}\int_0^1\Big(1-\mathrm{e}^{-y(1-u)}\Big)^{\alpha}y^{\beta-\alpha}\,\mathrm{d}y\,\mathrm{d}u = J_2. \end{aligned}$$

This proves part (i).

(ii) Denote by $\{L_n(\tau),\tau\}$ the process on the left-hand side of (4.18). It suffices to prove that, for any $m \ge 1$ and any $0 =: \tau_0 < \tau_1 < \cdots < \tau_m,\ \theta_1 \in \mathbb{R},\ldots,\theta_m \in \mathbb{R}$,

$$\sum_{k=1}^{m}\theta_k(L_n(\tau_k)-L_n(\tau_{k-1})) \ \to_{\mathrm{d}}\ \sum_{k=1}^{m}\theta_k(L(\tau_k)-L(\tau_{k-1})).$$

Rewrite $L_n(\tau_k)-L_n(\tau_{k-1}) = \Delta L'_n(\tau_k)+\Delta L''_n(\tau_k)$, where

$$\begin{aligned} \Delta L'_n(\tau_k) &:= n^{-1/\alpha}\sum_{[n\tau_{k-1}]<s\le[n\tau_k]}\ \sum_{s\le t\le[n\tau_k]}\int_0^1 a^{t-s}M_s(\,\mathrm{d}a), \\ \Delta L''_n(\tau_k) &:= n^{-1/\alpha}\sum_{s\le[n\tau_{k-1}]}\ \sum_{[n\tau_{k-1}]<t\le[n\tau_k]}\int_0^1 a^{t-s}M_s(\,\mathrm{d}a). \end{aligned}$$

Since $\Delta L'_n(\tau_k),\ k=1,\ldots,m$, are independent, it suffices to prove that, for any $k=1,\ldots,m$,

$$\Delta L'_n(\tau_k) \to_{\mathrm{d}} L(\tau_k)-L(\tau_{k-1}), \qquad \Delta L''_n(\tau_k) \ = \ o_p(1).$$

Moreover, it suffices to prove the last relations for $k=1$ and $\tau_k=1$ only; in other words, to prove that, for any $\theta \in \mathbb{R}$,

$$\begin{aligned} n^{-1}\sum_{s=1}^{n}\mathrm{E}\Big(\sum_{t=s}^{n}a^{t-s}\Big)^{\alpha}\omega\Big(\theta\sum_{t=s}^{n}a^{t-s}\Big) &\to K\omega(\theta), \\ n^{-1}\sum_{s\le 0}\mathrm{E}\Big(\sum_{t=1}^{n}a^{t-s}\Big)^{\alpha}\omega\Big(\theta\sum_{t=1}^{n}a^{t-s}\Big) &\to 0. \end{aligned}$$

Similarly as in the proof of (4.17), it suffices to prove the above relations for $\omega(\theta) \equiv 1$, viz.

$$J_{n1} := n^{-1}\sum_{s\le 0}\mathrm{E}\Big(\sum_{t=1}^{n}a^{t-s}\Big)^{\alpha} \to 0, \quad J_{n2} := n^{-1}\sum_{s=1}^{n}\mathrm{E}\Big(\sum_{t=s}^{n}a^{t-s}\Big)^{\alpha} \to K. \tag{4.35}$$

Consider

$$J_{n1} \;=\; n^{-1}\int_0^1 \frac{(1-x)^\alpha}{1-(1-x)^\alpha}\,(1-(1-x)^n)^\alpha x^{\beta-\alpha}\psi(1-x)\,\mathrm{d}x.$$

If $\beta > \alpha$ then, clearly,

$$J_{n1} \le Cn^{-1}\int_0^1 x^{\beta-\alpha-1}\psi(1-x)\,\mathrm{d}x \;=\; O(n^{-1})$$

since the last integral converges. Let $0 < \beta < \alpha$. Then, for any $\epsilon > 0$, similarly as above

$$J_{n1} \;=\; \frac{1}{n^{\beta-\alpha+1}}\int_0^{\epsilon n} \frac{(1-y/n)^\alpha}{n(1-(1-y/n)^\alpha)}\Big(1-\Big(1-\frac{y}{n}\Big)^n\Big)^\alpha y^{\beta-\alpha}\psi\Big(1-\frac{y}{n}\Big)\,\mathrm{d}y + O\Big(\frac{1}{n}\Big),$$

where the last integral tends to

$$\psi(1)\alpha^{-1}\int_0^\infty (1-\mathrm{e}^{-y})^\alpha y^{\beta-\alpha-1}\,\mathrm{d}y < \infty$$

implying that $J_{n1} = O(1/n^{\beta-\alpha+1}) = o(1)$. For $\beta = \alpha$, a similar argument yields $J_{n1} = O(n^{-1}\log n) = o(1)$. This proves the first convergence in (4.35).

Next, by the dominated convergence theorem,

$$J_{n2} \;=\; n^{-1}\sum_{k=0}^{n-1}\int_0^1 x^{\beta-\alpha}(1-(1-x)^k)^\alpha\psi(1-x)\,\mathrm{d}x \;\to\; \int_0^1 x^{\beta-\alpha}\psi(1-x)\,\mathrm{d}x \;=\; K,$$

proving the second relation in (4.35) and the theorem. □

Proof of Theorem 4.3.2, page 69. (i) In view of Theorem 4.3.1 (i), it suffices to show that $n^{-2H}\sum_{t=1}^n \mathfrak{X}^2(t) = o_p(1)$, with H as in (4.16). The last relation follows from $H > 1/\alpha$ and ([70], p. 387). See also ([45], proof of Theorem 1). This proves part (i).

(ii) According to Theorem 4.3.1(ii), it suffices to show that D_n^{-1} is bounded in probability, where

$$D_n := n^{-2/\alpha}\sum_{t=1}^n \mathfrak{X}^2(t).$$

Decompose

$$D_n = \sum_{i=1}^3 D_{ni},$$

where D_{ni} are defined in (4.36), below. Then $D_n^{-1} = O_p(1)$ follows from the following three facts:

$$\begin{aligned}&\text{(d1)}\quad D_{n1} = o_p(1),\\ &\text{(d2)}\quad D_{n2} \ge 0, \qquad \text{a.s.},\\ &\text{(d3)}\quad D_{n3} \to_{\mathrm{d}} Z, \quad \text{where } Z > 0 \text{ a.s.}\end{aligned}$$

To this end, let $\mathfrak{X}(t) = \sum_{s\le t} U_{t,s},\ U_{t,s} := \int_0^1 a^{t-s} M_s(\,\mathrm{d}a)\mathbf{1}(s \le t)$, and

$$\begin{aligned} D_{n1} &:= n^{-2/\alpha}\sum_{t=1}^n \sum_{s_1\neq s_2} U_{t,s_1}U_{t,s_2}, \qquad (4.36)\\ D_{n2} &:= n^{-2/\alpha}\sum_{t=1}^n\sum_{s\neq t} U_{t,s}^2, \qquad D_{n3} := n^{-2/\alpha}\sum_{t=1}^n U_{t,t}^2. \end{aligned}$$

Fact (d2) is obvious. Fact (d3) holds since $U_{t,t}, t = 1,\dots,n$ are i.i.d. $\alpha-$stable r.v.'s, so that $U_{t,t}^2 \in D(\alpha/2)$ and $D_{n3} \to_{\mathrm{d}} Z$, where Z is a strictly positive $\alpha/2-$stable r.v.

Let us prove (d1). Write $D_{n1} = \sum_{s\le n}\Gamma_{n,s}$, where

$$\Gamma_{n,s} := 2n^{-2/\alpha}\sum_{t=1}^n\sum_{v<s} U_{t,s}U_{t,v}.$$

Let $\mathcal{F}_s$ be the $\sigma-$algebra generated by r.v.'s $M_v(A), v \le s, A \subset (-1,1)$. Then $\{\Gamma_{n,s}, \mathcal{F}_s, s \in \mathbb{Z}\}$ is a martingale difference sequence. Hence, for any $1 < r < \alpha$, we have

$$\mathrm{E}|D_{n1}|^r \le 2\sum_{s\le n}\mathrm{E}|\Gamma_{n,s}|^r.$$

By a similar backward martingale property,

$$\mathrm{E}|\Gamma_{n,s}|^r \le 2\sum_{v<s} n^{-2r/\alpha}\mathrm{E}\Big|\sum_{t=1}^n U_{t,s}U_{t,v}\Big|^r.$$

Hence, using independence of $U_{t,s}$ and $U_{t,v},\ v < s$, and Hölder's inequality, for any $1 < r < \alpha$, we obtain

$$\begin{aligned} \mathrm{E}|D_{n1}|^r &\le 4n^{-2r/\alpha}\sum_{v<s\le n}\mathrm{E}\Big(\sum_{t=1}^n U_{t,s}U_{t,v}\Big)^r\\ &\le 4n^{-2r/\alpha}n^{r-1}\sum_{v<s\le n}\sum_{t=1}^n \mathrm{E}|U_{t,s}|^r\mathrm{E}|U_{t,v}|^r\\ &\le 4n^{-2r/\alpha}n^rQ_r, \end{aligned}$$

Where

$$Q_r := (\sum_{s\ge 0} \mathrm{E}|U_{s,0}|^r)^2.$$

Since $r - 2r/\alpha < 0$, for (d1), it suffices to show that $Q_r < \infty$. From ([94], Property 1.2.17) we have

$$\mathrm{E}|U_{s,0}|^r \le C(\mathrm{E}|a^s|^\alpha)^{r/\alpha},$$

where

$$\mathrm{E}|a^s|^\alpha \le C\int_0^1 x^\beta(1-x)^{s\alpha}\,\mathrm{d}x \le Cs^{-1-\beta}$$

and, therefore, $Q_r < \infty$ for $\alpha/(1+\beta) < r < \alpha$. This completes the proof. □

Proof of Theorem 4.3.3, page 69. From (4.10) and the definition of the codifference for $t \ge 1$, we obtain

$$\mathrm{Cod}(\mathfrak{X}(0), \mathfrak{X}(t)) \;=\; \mathrm{Re}(\omega(1))\Lambda_1(t) - \mathrm{i}\,\mathrm{Im}(\omega(1))\Lambda_2(t), \qquad (4.37)$$

where $\Lambda_i(t) := \mathrm{E}R_i,\ i = 1, 2$, and

$$\begin{aligned} R_1 &:= \frac{1 - |1-a^t|^\alpha + |a^t|^\alpha}{1-|a|^\alpha}, \\ R_2 &:= \sum_{s\le 0} |a^{t-s} - a^{-s}|^\alpha \mathrm{sign}(a^{t-s} - a^{-s}) + \sum_{s=1}^{t} |a^{t-s}|^\alpha \mathrm{sign}(a^{t-s}). \end{aligned}$$

Next, decompose $\Lambda_i(t) = \sum_{j=1}^4 \Lambda_{ij}(t)$, where

$$\begin{aligned} &\Lambda_{i1}(t) := \mathrm{E}R_i\mathbf{1}(1-\epsilon < a < 1), &&\Lambda_{i3}(t) := \mathrm{E}R_i\mathbf{1}(0 < a < 1-\epsilon), \\ &\Lambda_{i2}(t) := \mathrm{E}R_i\mathbf{1}(-1 < a < -1+\epsilon), &&\Lambda_{i4}(t) := \mathrm{E}R_i\mathbf{1}(-1+\epsilon < a < 0), \end{aligned}$$

and $\epsilon > 0$ is a small number. It is easy to check that, for any $\epsilon > 0$,

$$\Lambda_{ij}(t) = O(\mathrm{e}^{-\tilde c t}) = o(t^{-\beta_1\vee\beta_2}), \quad i = 1,2,\ j = 3,4, \quad \text{there exists } \tilde c > 0, \quad (4.38)$$

decay exponentially and, hence, are negligible in (4.20). Consider the terms $\Lambda_{ij}(t), i, j = 1, 2$. We have

$$
\begin{aligned}
\Lambda_{11}(t) &= \int_{1-\epsilon}^{1} \frac{1-|1-a^t|^\alpha + |a^t|^\alpha}{1-|a|^\alpha}(1-a)^{\beta_1}\psi(a)\,\mathrm{d}a \\
&= \int_0^{\epsilon} \frac{1-(1-(1-x)^t)^\alpha + (1-x)^{t\alpha}}{1-(1-x)^\alpha} x^{\beta_1}\psi(1-x)\,\mathrm{d}x \\
&= C_{11}(t)t^{-\beta_1}, \qquad (4.39)
\end{aligned}
$$

where

$$
\begin{aligned}
C_{11}(t) &:= \psi(1)\alpha^{-1}\int_0^\infty f(t,y)(1-(1-\mathrm{e}^{-y})^\alpha + \mathrm{e}^{-y\alpha})y^{\beta_1-1}\,\mathrm{d}y, \\
f(t,y) &:= \frac{1-(1-(1-y/t)^t)^\alpha + (1-y/t)^{t\alpha}}{1-(1-\mathrm{e}^{-y})^\alpha + \mathrm{e}^{-y\alpha}} \\
&\quad \times \frac{\alpha(y/t)}{1-(1-y/t)^\alpha}\cdot\frac{\psi(1-y/t)}{\psi(1)}\cdot \mathbf{1}(0<y<\epsilon t).
\end{aligned}
$$

Observe that $f(t,y) \to 1$, $t\to\infty$, for any $y \in (0,\infty)$, and, moreover, $|f|$ is bounded in $y \in (0,\infty)$ uniformly in $t \to \infty$. Hence, by the dominated convergence theorem,

$$
C_{11}(t) = \psi(1)\alpha^{-1}\int_0^\infty (1-(1-\mathrm{e}^{-y})^\alpha + \mathrm{e}^{-y\alpha})y^{\beta_1-1}\,\mathrm{d}y \; + \; o(1). \qquad (4.40)
$$

In a similar way,

$$
\begin{aligned}
\Lambda_{12}(t) &= \int_0^{\epsilon} \frac{1-(1-(-1)^t(1-x)^t)^\alpha + (1-x)^{t\alpha}}{1-(1-x)^\alpha} x^{\beta_2}\psi(x-1)\,\mathrm{d}x \\
&= C_{12}(t)t^{-\beta_2}, \qquad (4.41)
\end{aligned}
$$

where

$$
C_{12}(t) = \psi(-1)\alpha^{-1}\int_0^\infty [\mathrm{e}^{-y\alpha} + 1 - (1-(-1)^t\mathrm{e}^{-y})^\alpha]y^{\beta_2-1}\,\mathrm{d}y \; + \; o(1). \qquad (4.42)
$$

Next, using $\mathrm{sign}(a^{t-s}) = \mathrm{sign}(a^t)\mathrm{sign}(a^{-s})$ and

$$
\begin{aligned}
&\mathrm{sign}(a^{t-s}-a^{-s}) = -1, \quad \mathrm{sign}(a^{t-s}) = +1, \quad \text{for} \quad a>0, \\
&\mathrm{sign}(a^{t-s}-a^{-s}) = -((-1)^{-s}), \quad \mathrm{sign}(a^{t-s}) = (-1)^t((-1)^{-s}), \quad \text{for} \quad a<0,
\end{aligned}
$$

we can rewrite

$$R_2 = \frac{1-(1-a^t)^\alpha - a^{t\alpha}}{1-a^\alpha}\mathbf{1}(a>0) + \frac{1-(1-a^t)^\alpha - (-1)^t|a^t|^\alpha}{1+|a|^\alpha}\mathbf{1}(a<0).$$

Whence, similarly as above,

$$\begin{aligned}\Lambda_{21}(t) &= \int_0^\epsilon \frac{1-(1-(1-x)^t)^\alpha - (1-x)^{t\alpha}}{1-(1-x)^\alpha} x^{\beta_1}\psi(1-x)\,\mathrm{d}x \\ &= C_{21}(t)t^{-\beta_1}, \end{aligned} \qquad (4.43)$$

where

$$C_{21}(t) = \psi(1)\alpha^{-1}\int_0^\infty (1-(1-\mathrm{e}^{-y})^\alpha - \mathrm{e}^{-y\alpha})y^{\beta_1-1}\,\mathrm{d}y + o(1). \qquad (4.44)$$

Finally,

$$\begin{aligned}\Lambda_{22}(t) &= \int_0^\epsilon \frac{1-(1-(-1)^t(1-x)^t)^\alpha - (-1)^t(1-x)^{t\alpha}}{1+(1-x)^\alpha} x^{\beta_2}\psi(x-1)\,\mathrm{d}x \\ &= C_{22}(t)t^{-\beta_2-1} = o(t^{-\beta_2}), \end{aligned} \qquad (4.45)$$

where $C_{22}(t) = \psi(-1)2^{-1}\int_0^\infty(1-(1-(-1)^t\mathrm{e}^{-y})^\alpha - \mathrm{e}^{-y\alpha})y^{\beta_2}\,\mathrm{d}y + o(1)$.

The asymptotics in (4.20) follows from (4.37) and (4.38) - (4.45). □

5

Aggregation of a triangular array of AR(1) processes

Abstract. We discuss contemporaneous aggregation of independent copies of a triangular array of random-coefficient AR(1) processes with i.i.d. innovations belonging to the domain of attraction of an infinitely divisible law W. The limit aggregated process is shown to exist, under general assumptions on W and the mixing distribution, and is represented as a mixed infinitely divisible moving-average $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ in (5.4). Partial sums process of $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ is discussed under the assumption $\mathrm{E}W^2 < \infty$ and a mixing density regularly varying at the "unit root" $x = 1$ with exponent $\beta > 0$. We show that the above partial sums process may exhibit four different limit behaviors depending on β and the Lévy triplet of W. Finally, we study the disaggregation problem for $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ in spirit of Leipus et al. (2006, [62]) and obtain the weak consistency of the corresponding estimator of $\phi(x)$ in a suitable L_2-space.

5.1 Introduction

The present chapter discusses contemporaneous aggregation of N independent copies

$$X_i^{(N)}(t) = a_i X_i^{(N)}(t-1) + \varepsilon_i^{(N)}(t), \qquad t \in \mathbb{Z}, \quad i = 1, 2, \ldots, N \tag{5.1}$$

of random-coefficient AR(1) process $X^{(N)}(t) = aX^{(N)}(t-1)+\varepsilon^{(N)}(t),\ t \in \mathbb{Z}$, where $\{\varepsilon^{(N)}(t),\ t \in \mathbb{Z}\}$, $N = 1, 2, \dots$ is a triangular array of i.i.d. random variables in the domain of attraction of an infinitely divisible law W:

$$\sum_{t=1}^{N} \varepsilon^{(N)}(t) \quad \to_{\mathrm{d}} \quad W \tag{5.2}$$

and where a is a r.v., independent of $\{\varepsilon^{(N)}(t), t \in \mathbb{Z}\}$ and satisfying $0 \le a < 1$ almost surely. The limit aggregated process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ is defined as the limit in distribution:

$$\sum_{i=1}^{N} X_i^{(N)}(t) \ \to_{\mathrm{fdd}} \ \mathfrak{X}(t). \tag{5.3}$$

A particular case of (5.1)-(5.3) corresponding to $\varepsilon^{(N)}(t) = N^{-1/2}\zeta(t)$, where $\{\zeta(t), t \in \mathbb{Z}\}$ are i.i.d. r.v.'s with zero mean and finite variance, leads to the classical aggregation scheme of Robinson (1978, [90]), Granger (1980, [41]) and a Gaussian limit process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$. Chapters 3 and 4 (see also [85], [86]) discussed aggregation of random-coefficient AR(1) processes with infinite variance and innovations $\varepsilon^{(N)}(t) = N^{-1/\alpha}\zeta(t)$, where $\{\zeta(t), t \in \mathbb{Z}\}$ are i.i.d. r.v.'s in the domain of attraction of an $\alpha-$stable law W, $0 < \alpha \le 2$.

The present chapter discusses the existence and properties of the limit process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ in the general triangular aggregation scheme (5.1)-(5.3). Let us describe our main results. Theorem 5.2.7 (Section 5.2) says that under condition (5.2) and some mild additional conditions, the limit process in (5.3) exists and is written as a stochastic integral

$$\mathfrak{X}(t) \ := \ \sum_{s \le t} \int_{[0,1)} x^{t-s} M_s(\mathrm{d}x), \qquad t \in \mathbb{Z}, \tag{5.4}$$

where $\{M_s, s \in \mathbb{Z}\}$ are i.i.d. copies of an infinitely divisible (ID) random measure M on $[0, 1)$ with control measure $\Phi(\mathrm{d}x) := \mathrm{P}(a \in \mathrm{d}x)$ and Lévy characteristics (μ, σ, π) the same as of r.v. W in (5.2) (denote $M \sim W$), i.e., for any Borel set $A \subset [0, 1)$

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta M(A)} \ = \ \mathrm{e}^{\Phi(A)V(\theta)}, \qquad \theta \in \mathbb{R}. \tag{5.5}$$

Here and in the sequel, $V(\theta)$ denotes the log-characteristic function of r.v. W:

$$V(\theta) := \log \mathrm{E}\mathrm{e}^{\mathrm{i}\theta \mathrm{W}} \ = \ \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta y \mathbf{1}(|y| \le 1))\pi(\mathrm{d}y) - \frac{1}{2}\theta^2\sigma^2 + \mathrm{i}\theta\mu, \tag{5.6}$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and π is a Lévy measure (see Section 5.2 for details). In the particular case when W is $\alpha-$stable, $0 < \alpha \leq 2$, Theorem 5.2.7 agrees with the Theorem 4.2.1 from Chapter 4. We note that the process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ in (5.4) is stationary, ergodic and has ID finite-dimensional distributions. According to the terminology in [88], (5.4) is called a *mixed ID moving-average*.

Section 5.3 discusses partial sums limits and long memory properties of the aggregated process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ in (5.4) under the assumption that the mixing distribution Φ has a probability density ϕ varying regularly at $x = 1$ with exponent $\beta > 0$:

$$\phi(x) \;\sim\; C(1-x)^{\beta}, \qquad x \to 1 \tag{5.7}$$

for some $C > 0$. In the finite variance case $\sigma_W^2 := \mathrm{Var}(W) < \infty$ the aggregated process in (5.4) is covariance stationary provided $\mathrm{E}(1-a^2)^{-1} < \infty$, with covariance

$$r(t) \;:=\; \mathrm{Cov}(\mathfrak{X}(t), \mathfrak{X}(0)) \;=\; \sigma_W^2\, \mathrm{E}\Big[\sum_{s \leq 0} a^{t-s} a^{-s}\Big] \;=\; \sigma_W^2\, \mathrm{E}\Big[\frac{a^t}{1-a^2}\Big] \tag{5.8}$$

depending on σ_W^2 and the mixing distribution only. It is well-known that for $0 < \beta < 1$ (5.7) implies that $r(t) \sim Ct^{-\beta}$, $t \to \infty$, with some $C > 0$, in other words, the aggregated process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ has nonsummable covariances $\sum_{t \in \mathbb{Z}} |r(t)| = \infty$, or *covariance long memory*.

The main result of Section 5.3 is Theorem 5.3.1 which shows that under conditions (5.7) and $\mathrm{E}W^2 < \infty$, partial sums of the limit aggregated process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ in (5.4) may exhibit four different limit behaviors, depending on parameters β, σ and the behavior of the Lévy measure π at the origin. Write

$$W \sim ID_2(\sigma, \pi), \quad \text{if}$$

$$\mathrm{E}W = 0, \qquad \text{and} \qquad \mathrm{E}W^2 = \sigma^2 + \int_{\mathbb{R}} x^2 \pi(\,\mathrm{d}x) < \infty,$$

in which case $V(\theta)$ of (5.6) can be written as

$$V(\theta) \;=\; \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta y)\pi(\,\mathrm{d}y) - \frac{1}{2}\theta^2\sigma^2. \tag{5.9}$$

The Lévy measure π is completely determined by two nonincreasing functions

$$\Pi^+(x) := \pi(\{u > x\}), \quad \Pi^-(x) := \pi(\{u \leq -x\}), \quad x > 0.$$

Assume that there exist $\alpha > 0$ and $c^{\pm} \geq 0, c^+ + c^- > 0$ such that

$$\lim_{x\to 0} x^{\alpha}\Pi^{+}(x) = c^{+}, \qquad \lim_{x\to 0} x^{\alpha}\Pi^{-}(x) = c^{-}. \tag{5.10}$$

Under these assumptions, the four limit behaviors of $S_n(\tau) := \sum_{t=1}^{[n\tau]} \mathfrak{X}(t)$ correspond to the following parameter regions:

(i) $0 < \beta < 1,\ \sigma > 0$,

(ii) $0 < \beta < 1,\ \sigma = 0,\ 1 + \beta < \alpha < 2$,

(iii) $0 < \beta < 1,\ \sigma = 0,\ 0 < \alpha < 1 + \beta$,

(iv) $\beta > 1$.

According to Theorem 5.3.1, the limit process of $\{S_n(\tau),\ \tau \geq 0\}$, in respective cases (i) - (iv), is a

(i) fractional Brownian motion with parameter $H = 1 - \beta/2$,

(ii) $\alpha-$stable self-similar process $\Lambda_{\alpha,\beta}$ with dependent increments and self-similarity parameter $H = 1 - \beta/\alpha$, defined in (5.28) below,

(iii) $(1 + \beta)-$stable Lévy process with independent increments,

(iv) Brownian motion.

See Theorem 5.3.1 for precise formulations. Accordingly, the process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ in (5.4) has distributional long memory in cases (i) and (ii) and distributional short memory (see Definition 2.3.6, page 36) in case (iii). At the same time, $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ has covariance long memory in all three cases (i)-(iii). Case (iv) corresponds to distributional and covariance short memory. As α increases from 0 to 2, the Lévy measure in (5.10) increases its "mass" near the origin, the limiting case $\alpha = 2$ corresponding to $\sigma > 0$ or a positive "mass" at 0. We see from (i)-(ii) that distributional long memory is related to α being large enough, or small jumps of the random measure M having sufficient high intensity. Note that the critical exponent $\alpha = 1 + \beta$ separating the long and short memory "regimes" in (ii) and (iii) decreases with β, which is quite natural since smaller β means the mixing distribution putting more weight near the unit root $a = 1$.

Since aggregation leads to a natural loss of information about aggregated "micro" series, an important statistical problem arises to recover the lost information from the observed sample of the aggregated process. In the context of the AR(1) aggregation scheme (5.1)-(5.3) this leads to the so-called the disaggregation problem, or reconstruction of the mixing density $\phi(x)$ from observed sample $\mathfrak{X}(1), \ldots, \mathfrak{X}(n)$ of the aggregated process in (5.4). For Gaussian process (5.4), the disaggregation problem was investigated in [22] and [62], who constructed an estimator of the mixing density based on its expansion in an orthogonal polynomial basis. In Section 5.4, we extend the results in [62] to the case when the aggregated process is a mixed ID moving-average of (5.4) with finite 4th moment and obtain the weak consistency of the mixing density estimator in a suitable L_2-space (Theorem 5.4.4).

These results could be developed in several directions. We expect that Theorem 5.3.1 can be extended to the aggregation scheme with common innovations and to infinite variance ID moving-averages of (5.4), generalizing the results in Chapters 3 and 4 (and in [85], [86]). An interesting open problem is generalizing Theorem 5.3.1 to the random field set-up of [60] and [84].

5.2 Existence of the limit aggregated process

Consider random-coefficient AR(1) equation

$$X(t) = aX(t-1) + \varepsilon(t), \qquad t \in \mathbb{Z}, \tag{5.11}$$

where $\{\varepsilon(t), t \in \mathbb{Z}\}$ are i.i.d. r.v.'s with generic distribution ε, and $a \in [0,1)$ is a random coefficient independent of $\{\varepsilon(t), t \in \mathbb{Z}\}$. Assume that $\mathrm{E}|\varepsilon|^p < \infty$ for some $0 < p \leq 2$ and $\mathrm{E}\varepsilon = 0$, $p \geq 1$. Then, acording to the Proposition 3.2.3, page 42, there exists a unique strictly stationary solution to the AR(1) equation (5.11) given by the series

$$X(t) \ = \ \sum_{k=0}^{\infty} a^k \varepsilon(t-k), \tag{5.12}$$

which converge conditionally a.s. and in L_p for a.e. $a \in [0,1)$. Moreover, if

$$\mathrm{E}\Big[\frac{1}{1-a}\Big] \ < \ \infty \tag{5.13}$$

then the series in (5.12) converges unconditionally in L_p.

We will write

$$W \sim ID(\mu, \sigma, \pi),$$

if r.v. W is infinitely divisible having the log-characteristic function in (5.6), where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and π is a measure on $\mathbb{R}$ satisfying $\pi(\{0\}) = 0$ and $\int_{\mathbb{R}} (x^2 \wedge 1) \pi(\,\mathrm{d}x) < \infty$, called the Lévy measure of W. It is well-known that the distribution of W is completely determined by the (characteristic) triplet (μ, σ, π) and vice versa. See, e.g., [95].

Definition 5.2.1. *Let $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\}$ be a sequence of r.v.'s tending to 0 in probability, and $W \sim ID(\mu, \sigma, \pi)$ be an ID r.v. We say that the sequence $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\}$ belongs to the domain of attraction of W, denoted $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\} \in D(W)$, if*

$$(\mathcal{C}_N(\theta))^N \ \to \ \mathrm{E}\mathrm{e}^{\mathrm{i}\theta \mathrm{W}}, \qquad \forall\, \theta \in \mathbb{R}, \tag{5.14}$$

where $\mathcal{C}_N(\theta) := \mathrm{E}\exp\{\,\mathrm{i}\theta\varepsilon^{(\mathrm{N})}\}$, $\theta \in \mathbb{R}$, *is the characteristic function of* $\varepsilon^{(N)}$.

Remark 5.2.2. Sufficient and necessary conditions for $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\} \in D(W)$ in terms of the distribution functions of $\varepsilon^{(N)}$ are well-known. See, e.g., [95], Ch. 17 of [35]. In particular, these conditions include the convergences

$$N\mathrm{P}(\varepsilon^{(N)} > x) \ \to \ \Pi^+(x), \qquad N\mathrm{P}(\varepsilon^{(N)} < -x) \ \to \ \Pi^-(x) \tag{5.15}$$

at each continuity point $x > 0$ of Π^+, Π^-, respectively, where $\Pi^{\pm}$ are defined as in (5.10).

Remark 5.2.3. By taking logarithms of both sides, condition (5.14) can be rewritten as

$$N \log \mathcal{C}_N(\theta) \ \to \ \log \mathrm{E}e^{\mathrm{i}\theta \mathrm{W}} \ = \ \mathrm{V}(\theta), \quad \forall\, \theta \in \mathbb{R}, \tag{5.16}$$

with the convention that the l.h.s. of (5.16) is defined for $N > N_0(\theta)$ sufficiently large only, since for a fixed N, the characteristic function $\mathcal{C}_N(\theta)$ may vanish at some points θ. In the general case, (5.16) can be precised as follows: For any $\epsilon > 0$ and any $K > 0$ there exists $N_0(K, \epsilon) \in \mathbb{N}^*$ such that

$$\sup_{|\theta|<K} \Big| N \log \mathcal{C}_N(\theta) - V(\theta) \Big| \ < \ \epsilon, \qquad \forall\, N > N_0(K, \epsilon). \tag{5.17}$$

The following definitions introduce some technical conditions, in addition to $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\} \in D(W)$, needed to prove the convergence towards the aggregated process in (5.3).

Definition 5.2.4. *Let* $0 < \alpha \le 2$ *and* $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\}$ *be a sequence of r.v.'s. Write* $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\} \in T(\alpha)$ *if there exists a constant* C *independent of* N *and* x *and such that one of the two following conditions hold: either*

(i) $\alpha = 2$ *and* $\mathrm{E}\varepsilon^{(N)} = 0$, $N\mathrm{E}(\varepsilon^{(N)})^2 \le C$, *or*

(ii) $0 < \alpha < 2$ *and* $N\mathrm{P}(|\varepsilon^{(N)}| > x) \le Cx^{-\alpha}$, $x > 0$; *moreover,* $\mathrm{E}\varepsilon^{(N)} = 0$ *whenever* $1 < \alpha < 2$, *while, for* $\alpha = 1$ *we assume that the distribution of* $\varepsilon^{(N)}$ *is symmetric.*

Definition 5.2.5. *Let* $0 < \alpha \le 2$ *and* $W \sim ID(\mu, \sigma, \pi)$. *Write* $W \in \mathcal{T}(\alpha)$ *if there exists a constant* C *independent of* x *and such that one of the two following conditions hold: either*

(i) $\alpha = 2$ *and* $\mathrm{E}W = 0$, $\mathrm{E}W^2 < \infty$, *or*

(ii) $0 < \alpha < 2$ *and* $\Pi^+(x) + \Pi^-(x) \le Cx^{-\alpha}$, $\forall\, x > 0$; *moreover,* $\mathrm{E}W = 0$ *whenever* $1 < \alpha < 2$, *while, for* $\alpha = 1$ *we assume that the distribution of* W *is symmetric.*

Corollary 5.2.6. *Let* $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\} \in D(W)$, $W \sim ID(\mu, \sigma, \pi)$. *Assume that* $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\} \in T(\alpha)$ *for some* $0 < \alpha \le 2$. *Then* $W \in \mathcal{T}(\alpha)$.

Proof. Let $\alpha = 2$ and R_N denote the l.h.s. of (5.2). Then $R_N^2 \to_{\mathrm{d}} W^2$ and

$$\mathrm{E}W^2 \le \liminf_{N\to\infty} \mathrm{E}R_N^2 = \liminf_{N\to\infty} N\mathrm{E}(\varepsilon^{(N)})^2 < \infty$$

follows by Fatou's lemma. Then, relation

$$\mathrm{E}W = \lim_{N\to\infty} \mathrm{E}R_N = 0$$

follows by the dominated convergence theorem. For $0 < \alpha < 2$, relation $\Pi^{\pm}(x) \le Cx^{-\alpha}$ at each continuity point x of $\Pi^{\pm}$ follows from $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\} \in T(\alpha)$ and (5.15) and then extends to all $x > 0$ by monotonicity. Verification of the remaining properties of W in the cases $1 < \alpha < 2$ and $\alpha = 1$ is easy and is omitted. □

The main result of this section is the following theorem. Recall that $\{X_i(t) \equiv X_i^{(N)}(t)\}$, $i = 1, 2, \ldots, N$ are independent copies of AR(1) process in (5.11) with i.i.d. innovations $\{\varepsilon(t) \equiv \varepsilon^{(N)}(t)\}$ and random coefficient $a \in [0, 1)$. Write $M \sim W$ if M is an ID random measure on $[0, 1)$ with characteristic function as in (5.5)-(5.6).

Theorem 5.2.7. *Let condition* (5.13) *holds. In addition, assume that the generic sequence* $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\}$ *belongs to the domain of attraction of ID r.v.* $W \sim ID(\mu, \sigma, \pi)$ *and there exists an* $0 < \alpha \le 2$ *such that* $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\} \in T(\alpha)$. *Then the limit aggregated process* $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ *in* (5.3) *exists. It is stationary, ergodic, has infinitely divisible finite-dimensional distributions, and a stochastic integral representation as in* (5.4), *where* $M \sim W$.

Proof. We follow the proof of Theorem 4.2.1, page 64. Fix $m \ge 1$ and $\theta(1), \ldots, \theta(m) \in \mathbb{R}$. Denote

$$\vartheta(s, a) := \sum_{t=1}^{m} \theta(t) a^{t-s} \mathbf{1}(s \le t).$$

Then $\sum_{t=1}^{m} \theta(t) X_i^{(N)}(t) = \sum_{s\in\mathbb{Z}} \vartheta(s, a_i) \varepsilon_i^{(N)}(s)$, $i = 1, \ldots, N$, and

$$\mathrm{E}\exp\Big\{\mathrm{i}\sum_{i=1}^{N}\sum_{t=1}^{m} \theta(t) X_i^{(N)}(t)\Big\} = \Big(\mathrm{E}\exp\Big\{\mathrm{i}\sum_{t=1}^{m} \theta(t) X^{(N)}(t)\Big\}\Big)^N = \Big(1 + \frac{\Theta(N)}{N}\Big)^N, \tag{5.18}$$

where

$$\Theta(N) := N\Big(\mathrm{E}\Big[\prod_{s\in\mathbb{Z}} \mathcal{C}_N(\vartheta(s, a))\Big] - 1\Big).$$

From definitions (5.4), (5.6) it follows that

$$\mathrm{E}\exp\Big\{\mathrm{i}\sum_{t=1}^{m}\theta(t)\mathfrak{X}(t)\Big\} \ = \ \mathrm{e}^{\Theta}, \quad \text{where} \quad \Theta \ := \ \mathrm{E}\sum_{\mathrm{s}\in\mathbb{Z}}\mathrm{V}(\vartheta(\mathrm{s},\mathrm{a})). \tag{5.19}$$

The convergence in (5.3) to the aggregated process of (5.4) follows from (5.18), (5.19) and the limit

$$\lim_{N\to\infty}\Theta(N) = \Theta, \tag{5.20}$$

which will be proved below.

Note first that

$$\sup_{a\in[0,1),s\in\mathbb{Z}}|\vartheta(s,a)| \le \sum_{t=1}^{m}|\theta(t)| =: K$$

is bounded and therefore the logarithm $\log\mathcal{C}_N(\vartheta(s,a))$ is well-defined for $N > N_0(K)$ large enough, see (5.17), and $\Theta(N)$ can be rewritten as

$$\Theta(N) \ = \ \mathrm{E}N\Big(\exp\Big\{N^{-1}\sum_{s\in\mathbb{Z}}N\log\mathcal{C}_N(\vartheta(s,a))\Big\}-1\Big).$$

Then (5.20) follows if we show that for each $a\in[0,1)$,

$$\lim_{N\to\infty}\sum_{s\in\mathbb{Z}}N\log\mathcal{C}_N(\vartheta(s,a)) \ = \ \sum_{s\in\mathbb{Z}}V(\vartheta(s,a)), \qquad \forall\, a\in[0,1), \tag{5.21}$$

and

$$\sum_{s\in\mathbb{Z}}\Big|N\log\mathcal{C}_N(\vartheta(s,a))\Big| \ \le \ \frac{C}{1-a^{\alpha}}, \qquad \forall\, a\in[0,1), \tag{5.22}$$

where C does not depend on N, a.

Let us prove (5.22). It suffices to check the bound

$$N|1-\mathcal{C}_N(\theta)| \ \le \ C|\theta|^{\alpha}. \tag{5.23}$$

Indeed, since $|\mathcal{C}_N(\vartheta(s,a))-1|<\epsilon$ for N large enough (see above), so

$$\Big|N\log\mathcal{C}_N(\vartheta(s,a))\Big| \ \le \ CN\Big|1-\mathcal{C}_N(\vartheta(s,a))\Big|$$

and (5.23) implies

$$\sum_{s\in\mathbb{Z}}\Big|N\log\mathcal{C}_N(\vartheta(s,a))\Big| \ \le \ C\sum_{s\in\mathbb{Z}}|\vartheta(s,a)|^{\alpha} \ \le \ \frac{C}{1-a^{\alpha}}, \tag{5.24}$$

see ((4.26), page 72), proving (5.22).

Consider (5.23) for $1 < \alpha < 2$. Since $\mathrm{E}\varepsilon^{(N)} = 0$ so

$$\mathcal{C}_N(\theta) - 1 = \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x)\,\mathrm{d}F_N(x)$$

and

$$\begin{aligned} N|1 - \mathcal{C}_N(\theta)| &\le N\Big|\int_{-\infty}^{0} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x)\,\mathrm{d}F_N(x)\Big| \\ &\quad + N\Big|\int_{0}^{\infty} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x)\,\mathrm{d}(1 - F_N(x))\Big| \\ &= |\theta|\Big(\Big|\int_{-\infty}^{0} NF_N(x)(\mathrm{e}^{\mathrm{i}\theta x} - 1)\,\mathrm{d}x\Big| \\ &\quad + \Big|\int_{0}^{\infty} N(1 - F_N(x))(\mathrm{e}^{\mathrm{i}\theta x} - 1)\,\mathrm{d}x\Big|\Big) \\ &\le C|\theta|\int_{0}^{\infty} x^{-\alpha}((|\theta|x) \wedge 1)\,\mathrm{d}x \ \le\ C|\theta|^{\alpha}, \end{aligned} \tag{5.25}$$

since

$$NF_N(x)\mathbf{1}(x < 0) + N(1 - F_N(x))\mathbf{1}(x > 0) \le C|x|^{-\alpha}$$

and the integral

$$\begin{aligned} \int_{0}^{\infty} x^{-\alpha}((|\theta|x) \wedge 1)\,\mathrm{d}x &= |\theta|\int_{0}^{1/|\theta|} x^{1-\alpha}\,\mathrm{d}x + \int_{1/|\theta|}^{\infty} x^{-\alpha}\,\mathrm{d}x \\ &= |\theta|^{\alpha-1}\Big(\frac{1}{2-\alpha} + \frac{1}{\alpha-1}\Big) \end{aligned}$$

converges. In the case $\alpha = 2$, we have

$$N|\mathcal{C}_N(\theta) - 1| \le \frac{1}{2}\theta^2 N\mathrm{E}(\varepsilon^{(N)})^2 \le C\theta^2$$

and (5.23) follows.

Next, let $0 < \alpha < 1$. Then

$$\begin{aligned} N|1 - \mathcal{C}_N(\theta)| &\le N\int_{-\infty}^{0} |\mathrm{e}^{\mathrm{i}\theta x} - 1|\,\mathrm{d}F_N(x) + N\int_{0}^{\infty} |\mathrm{e}^{\mathrm{i}\theta x} - 1|\,|\,\mathrm{d}(1 - F_N(x))| \\ &=: I_1 + I_2. \end{aligned}$$

Here,

$$\begin{aligned} I_1 &\le 2N\int_{-\infty}^{0} ((|\theta|\,|x|) \wedge 1)\,\mathrm{d}F_N(x) \\ &= 2N\int_{-\infty}^{-1/|\theta|} \mathrm{d}F_N(x) + 2N|\theta|\int_{-1/|\theta|}^{0} |x|\,\mathrm{d}F_N(x) =: 2(I_{11} + I_{12}). \end{aligned}$$

We have $I_{11} = NF_N(-1/|\theta|) \le C|\theta|^\alpha$ and

$$\begin{aligned} I_{12} &= -|\theta| N \int_{-1/|\theta|}^{0} x \,\mathrm{d}F_N(x) = -|\theta| N \Big(x F_N(x) \Big|_{x=-1/|\theta|}^{x=0} - \int_{-1/|\theta|}^{0} F_N(x)\,\mathrm{d}x \Big) \\ &= |\theta| N \Big(-\frac{F_N(-1/|\theta|)}{|\theta|} + \int_{-1/|\theta|}^{0} F_N(x)\,\mathrm{d}x \Big) \\ &\le C|\theta|^\alpha + C|\theta| \int_{-1/|\theta|}^{0} |x|^{-\alpha}\,\mathrm{d}x \ \le \ C|\theta|^\alpha. \end{aligned}$$

Since I_2 can be evaluated analogously, this proves (5.23) for $0 < \alpha < 1$.

It remains to prove (5.23) for $\alpha = 1$. Since, by symmetry of $\varepsilon^{(N)}$,

$$\int_{\{|x| \le 1/|\theta|\}} x \,\mathrm{d}F_N(x) = 0,$$

so $\mathcal{C}_N(\theta) - 1 = J_1 + J_2 + J_3 + J_4$, where

$$\begin{aligned} J_1 &:= \int_{-\infty}^{-1/|\theta|} (\mathrm{e}^{\mathrm{i}\theta x} - 1)\,\mathrm{d}F_N(x), \\ J_2 &:= \int_{-1/|\theta|}^{0} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x)\,\mathrm{d}F_N(x), \\ J_3 &:= \int_{0}^{1/|\theta|} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x)\,\mathrm{d}F_N(x), \\ J_4 &:= \int_{1/|\theta|}^{\infty} (\mathrm{e}^{\mathrm{i}\theta x} - 1)\,\mathrm{d}F_N(x). \end{aligned}$$

We have

$$N|J_1| \le 2NF_N(-1/|\theta|) \le C|\theta|$$

and a similar bound follows for $J_i, i = 2, 3, 4$. This proves (5.23). Then (5.21) and the remaining proof of (5.20) and Theorem 5.2.7 follow as the proof of Thm. 4.2.1 in page 64. □

Remark 5.2.8. Theorem 5.2.7 applies in the case of innovations belonging to the domain of attraction of an $\alpha-$stable law. Let $\varepsilon^{(N)} = N^{-1/\alpha}\zeta$, where $\zeta \in D(\alpha)$, $0 < \alpha \le 2$ (see Definition 3.2.1, page 42). Then $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\} \in T(\alpha)$ and $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\} \in D(W)$, where W is an $\alpha-$stable r.v. with the characteristic function

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta W} \ = \ \mathrm{e}^{-|\theta|^\alpha \omega(\theta;\alpha,c_1,c_2)}, \quad \theta \in \mathbb{R}, \tag{5.26}$$

here $\omega(\theta; \alpha, c_1, c_2) \equiv \omega(\theta)$ is defined in (3.9), page 42. In this case, the statement of Theorem 5.2.7 coincides with Theorem 4.2.1, page 64.

5.3 Long memory properties of the limit aggregated process

In this section we study partial sums limits and distributional long memory property of the aggregated mixed ID moving-average in (5.4) under condition (5.7) on the mixing density ϕ. More precisely, we shall assume that ϕ has the form

$$\phi(x) \ = \psi(x)(1-x)^{\beta}, \qquad x \in (0,1), \tag{5.27}$$

where $\beta > 0$ and $\psi(x)$ is an bounded function having a finite limit $\psi(1) := \lim_{x\to 1} \psi(x) > 0$.

Consider an independently scattered $\alpha-$stable random measure $N(\,\mathrm{d}x,\,\mathrm{d}s)$ on $(0,\infty)\times\mathbb{R}$ with control measure $\nu(\,\mathrm{d}x,\,\mathrm{d}s) := \psi(1)x^{\beta-\alpha}\,\mathrm{d}x\,\mathrm{d}s$ and characteristic function

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta N(A)} = \mathrm{e}^{-|\theta|^{\alpha}\omega(\theta;\alpha,\mathrm{c}^{+},\mathrm{c}^{-})\nu(\mathrm{A})},\ \theta\in\mathbb{R},$$

where $A \subset (0,\infty)\times\mathbb{R}$ is a Borel set with $\nu(A) < \infty$ and ω is defined at (3.9), page 42. For $1 < \alpha \le 2$, $0 < \beta < \alpha - 1$, introduce the process

$$\begin{aligned} \Lambda_{\alpha,\beta}(\tau) \ &:= \ \int_{\mathbb{R}_+\times\mathbb{R}} \Big(f(x,\tau-s) - f(x,-s)\Big)N(\,\mathrm{d}x,\,\mathrm{d}s), \quad \tau\ge 0, \quad \text{where} \\ f(x,t) \ &:= \ \begin{cases} 1-\mathrm{e}^{-xt}, & \text{if } x>0 \text{ and } t>0, \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{5.28}$$

defined as a stochastic integral with respect to the above random measure N. The process $\Lambda_{\alpha,\beta}$ was also introduced in Chaper 4 (see (4.6), page 63). It has stationary increments, $\alpha-$stable finite-dimensional distributions, a.s. continuous sample paths and is self-similar with parameter $H = 1-\beta/\alpha \in (1/\alpha, 1)$. Note that for $\alpha = 2$, $\Lambda_{2,\beta}$ is a fractional Brownian motion.

Theorem 5.3.1. *Let* $\{\mathfrak{X}(t),\, t\in\mathbb{Z}\}$ *be the limit aggregated process in* (5.4)*, where* $M \sim W \sim ID_2(\sigma,\pi)$ *and the mixing distribution satisfies* (5.27).

(i) Let $0<\beta<1$ *and* $\sigma>0$*. Then*

$$\frac{1}{n^{1-\frac{\beta}{2}}}\sum_{t=1}^{[n\tau]}\mathfrak{X}(t) \ \to_{D[0,1]} \ B_H(\tau), \tag{5.29}$$

where B_H *is a fractional Brownian motion with parameter* $H := 1-\beta/2$ *and variance* $\mathrm{E}B_H^2(\tau) = \sigma^2\psi(1)\Gamma(\beta-2)\tau^{2H}$.

(ii) Let $0<\beta<1$, $\sigma=0$ *and there exist* $1+\beta<\alpha<2$ *and* $c^{\pm}\ge 0$, $c^{+}+c^{-}>0$

such that (5.10) *hold. Then*

$$\frac{1}{n^{1-\frac{\beta}{\alpha}}} \sum_{t=1}^{[n\tau]} \mathfrak{X}(t) \ \to_{D[0,1]} \ \Lambda_{\alpha,\beta}(\tau), \tag{5.30}$$

where $\Lambda_{\alpha,\beta}$ *is defined in* (5.28).

(iii) Let $0 < \beta < 1, \sigma = 0$, $\pi \neq 0$ *and there exists* $0 < \alpha < 1 + \beta$ *such that*

$$\int_{\mathbb{R}} |x|^{\alpha} \pi(\,\mathrm{d}x) < \infty. \tag{5.31}$$

Then

$$\frac{1}{n^{\frac{1}{1+\beta}}} \sum_{t=1}^{[n\tau]} \mathfrak{X}(t) \ \to_{\mathrm{fdd}} \ L_{1+\beta}(\tau), \tag{5.32}$$

where $\{L_{1+\beta}(\tau), \tau \geq 0\}$ *is an* $(1+\beta)-$*stable Lévy process with log-characteristic function given in* (5.49) *below.*

(iv) Let $\beta > 1$. *Then*

$$\frac{1}{n^{1/2}} \sum_{t=1}^{[n\tau]} \mathfrak{X}(t) \ \to_{\mathrm{fdd}} \ \sigma_{\Phi} B(\tau), \tag{5.33}$$

where B *is a standard Brownian motion with* $\mathrm{E}B^2(1) = 1$ *and* σ_{Φ} *is defined in* (5.50) *below. Moreover, if* $\beta > 2$ *and* π *satisfies* (5.31) *with* $\alpha = 4$*, the convergence* $\to_{\mathrm{fdd}}$ *in* (5.33) *can be replaced by* $\to_{D[0,1]}$.

Remark 5.3.2. Note that the normalization exponents in Theorem 5.3.1 decrease from (i) to (iv):

$$1 - \frac{\beta}{2} \ > \ 1 - \frac{\beta}{\alpha} \ > \ \frac{1}{1+\beta} \ > \ \frac{1}{2}.$$

Hence, we may conclude that the dependence in the aggregated process decreases from (i) to (iv). Also note that while $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ has finite variance in all cases (i) - (iv), the limit of its partial sums may have infinite variance as it happens in (ii) and (iii). Apparently, the finite-dimensional convergence in (5.32) cannot be replaced by the convergence in $D[0,1]$ with the J_1-topology. See ([74], p.40), ([63], Remark 4.1) for related discussion.

Proof. (i) The statement is true if $\pi = 0$, or $W \sim \mathcal{N}(0, \sigma^2)$. In the case $\pi \neq 0$, split

$$\mathfrak{X}(t) = \mathfrak{X}_1(t) + \mathfrak{X}_2(t),$$

where $\mathfrak{X}_1(t), \mathfrak{X}_2(t)$ are defined following the decomposition of the measure $M = M_1 + M_2$ into independent random measures $M_1 \sim W_1 \sim ID_2(\sigma, 0)$ and $M_2 \sim$

$W_2 \sim ID_2(0,\pi)$. Let us prove that

$$S_{n2} \ := \ \sum_{t=1}^{n} \mathfrak{X}_2(t) \ = \ o_p(n^{1-\frac{\beta}{2}}). \tag{5.34}$$

Let

$$V_2(\theta) := \log \mathrm{E} \mathrm{e}^{\mathrm{i}\theta W_2} = \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x)\pi(\,\mathrm{d}x).$$

Then

$$|V_2(\theta)| \ \le \ C\theta^2, \quad \forall\, \theta \in \mathbb{R}, \qquad \text{and} \qquad |V_2(\theta)| \ = \ o(\theta^2), \quad |\theta| \to \infty. \tag{5.35}$$

Indeed, for any $\epsilon > 0$,

$$|V_2(\theta)| \le \ \theta^2 I_1(\epsilon) + 2|\theta| I_2(\epsilon),$$

where

$$I_1(\epsilon) := \theta^{-2} \int_{|x|\le\epsilon} |\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x|\pi(\,\mathrm{d}x) \le \int_{|x|\le\epsilon} x^2 \pi(\,\mathrm{d}x) \to 0, \ \epsilon \to 0,$$

and

$$I_2(\epsilon) := (2|\theta|)^{-1} \int_{|x|>\epsilon} |\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x|\pi(\,\mathrm{d}x) \le \int_{|x|>\epsilon} |x|\pi(\,\mathrm{d}x) < \infty, \ \forall\, \epsilon > 0.$$

Hence, (5.35) follows.

Relation (5.34) follows from $J_n := \log \mathrm{E} \exp\Big\{ \mathrm{i}\theta n^{-1+\frac{\beta}{2}} S_{n2} \Big\} = o(1)$. We have

$$J_n \ = \ \sum_{s\in\mathbb{Z}} \int_0^1 V_2\Big(\theta n^{-1+\beta/2} \sum_{t=1}^{n} (1-z)^{t-s} \mathbf{1}(t \ge s)\Big) z^\beta \psi(1-z)\,\mathrm{d}z \ = \ J_{n1} + J_{n2},$$

where

$$J_{n1} := \sum_{s\le 0} \int_0^1 V_2(\cdots) z^\beta \psi(1-z)\,\mathrm{d}z, \qquad J_{n2} := \sum_{s=1}^{n} \int_0^1 V_2(\cdots) z^\beta \psi(1-z)\,\mathrm{d}z.$$

By change of variables: $nz = w, n-s+1 = nu$, J_{n2} can be rewritten as

$$\begin{aligned}
J_{n2} \ &= \ \sum_{s=1}^{n} \int_0^1 V_2\Big(\frac{\theta(1-(1-z)^{n-s+1})}{n^{1-\beta/2} z}\Big) z^\beta \psi(1-z)\,\mathrm{d}z \\
&= \ \frac{1}{n^\beta} \int_{1/n}^1 \mathrm{d}u \int_0^n V_2\Big(\frac{\theta n^{\beta/2}(1-(1-\frac{w}{n})^{[un]})}{w}\Big) w^\beta \psi\Big(1-\frac{w}{n}\Big)\,\mathrm{d}w \\
&= \ \theta^2 \int_0^1 \mathrm{d}u \int_0^\infty G_n(u,w) w^{\beta-2} \psi\Big(1-\frac{w}{n}\Big)\,\mathrm{d}w,
\end{aligned}$$

where

$$G_n(u,w) := \Big(1-(1-\frac{w}{n})^{[un]}\Big)^2 \kappa\Big(\frac{\theta n^{\beta/2}(1-(1-\frac{w}{n})^{[un]})}{w}\Big)\mathbf{1}(1/n < u < 1, 0 < w < n)$$

and where $\kappa(\theta) := V_2(\theta)/\theta^2$ is a bounded function vanishing as $|\theta| \to \infty$; see (5.35). Therefore $G_n(u,w) \to 0$, $n \to \infty$, for any $u \in (0,1]$, $w > 0$ fixed. We also have

$$|G_n(u,w)| \le C\Big(1-(1-\frac{w}{n})^{[un]}\Big)^2 \le C(1-\mathrm{e}^{-wu})^2 =: \bar{G}(u,w),$$

where

$$\int_0^1 \mathrm{d}u \int_0^\infty \bar{G}(u,w) w^{\beta-2}\,\mathrm{d}w < \infty.$$

Thus, $J_{n2} = o(1)$ follows by the dominated convergence theorem. The proof $J_{n1} = o(1)$ using (5.35) follows by a similar argument. This proves $J_n = o(1)$, or (5.34). The tightness of the partial sums process in $D[0,1]$ follows from $\beta < 1$ and Kolmogorov's criterion since

$$\mathrm{E}\Big(\sum_{t=1}^n \mathfrak{X}(t)\Big)^2 = O(n^{2-\beta}),$$

the last relation is an easy consequence of $r(t) = O(t^{-\beta})$, see (5.8) and the discussion below it.

(ii) Let $S_n(\tau) := \sum_{t=1}^{[n\tau]} \mathfrak{X}(t)$. Let us prove that for any $0 < \tau_1 < \cdots < \tau_m \le 1$, $\theta_1 \in \mathbb{R}, \ldots, \theta_m \in \mathbb{R}$,

$$J_n := \log \mathrm{E}\exp\Big\{\mathrm{i}\frac{1}{n^{1-\frac{\beta}{\alpha}}}\sum_{j=1}^m \theta_j S_n(\tau_j)\Big\} \to J, \qquad \text{where} \tag{5.36}$$

$$J := -\psi(1)\int_{\mathbb{R}_+\times\mathbb{R}} \Big|\sum_{j=1}^m \theta_j(f(w,\tau_j-u)-f(w,-u))\Big|^\alpha \tag{5.37}$$

$$\times\, \omega\Big(\sum_{j=1}^m \theta_j(f(w,\tau_j-u)-f(w,-u)); \alpha, c^+, c^-\Big)\frac{\mathrm{d}w\,\mathrm{d}u}{w^{\alpha-\beta}}.$$

We have

$$J = \log \mathrm{E}\mathrm{e}^{\mathrm{i}\sum_{j=1}^m \theta_j \Lambda_{\alpha,\beta}(\tau_j)}$$

by definition (5.28) of $\Lambda_{\alpha,\beta}$. We shall restrict the proof of (5.36) to $m = \tau_1 = 1$, since the general case follows analogously. Let $V(\theta)$ be defined as in (5.9), where

$\sigma = 0$. Then,

$$\begin{aligned} J_n &= \sum_{s\in\mathbb{Z}} \int_0^1 V\Big(\theta \frac{1}{n^{1-\frac{\beta}{\alpha}}} \sum_{t=1}^n (1-z)^{t-s}\mathbf{1}(t \ge s)\Big) z^\beta \psi(1-z)\,\mathrm{d}z \\ &= \sum_{s\le 0} \int_0^\epsilon V(...) z^\beta \psi(1-z)\,\mathrm{d}z + \sum_{s=1}^n \int_0^\epsilon V(...) z^\beta \psi(1-z)\,\mathrm{d}z \\ &\quad + \sum_{s\in\mathbb{Z}} \int_\epsilon^1 V(...) z^\beta \psi(1-z)\,\mathrm{d}z \ =: \ J_{n1} + J_{n2} + J_{n3}, \end{aligned}$$

Similarly, split $J = J_1 + J_2$, where

$$\begin{aligned} J_1 &:= -|\theta|^\alpha \psi(1) \omega(\theta; \alpha, c^+, c^-) \int_{-\infty}^0 \mathrm{d}u \int_0^\infty (f(w, 1-u) - f(w, -u))^\alpha w^{\beta-\alpha}\,\mathrm{d}w, \\ J_2 &:= -|\theta|^\alpha \psi(1) \omega(\theta; \alpha, c^+, c^-) \int_0^1 \mathrm{d}u \int_0^\infty (f(w, u))^\alpha w^{\beta-\alpha}\,\mathrm{d}w. \end{aligned}$$

To prove (5.36) we need to show $J_{n1} \to J_1$, $J_{n2} \to J_2$, $J_{n3} \to 0$. We shall use the following facts:

$$\lim_{\lambda\to+0} \lambda V\Big(\lambda^{-1/\alpha}\theta\Big) \ = \ -|\theta|^\alpha \omega(\theta; \alpha, c^+, c^-), \qquad \forall\, \theta \in \mathbb{R} \tag{5.38}$$

and

$$|V(\theta)| \ \le \ C|\theta|^\alpha, \qquad \forall\, \theta \in \mathbb{R} \qquad (\exists\, C < \infty). \tag{5.39}$$

Here, (5.39) follows from (5.10), $\int_\mathbb{R} x^2 \pi(\mathrm{d}x) < \infty$ and integration by parts. To show (5.38), let $\chi(x), x \in \mathbb{R}$ be a bounded continuously differentiable function with compact support and such that $\chi(x) \equiv 1, |x| \le 1$. Then the l.h.s. of (5.38) can be rewritten as

$$\lambda V\Big(\lambda^{-1/\alpha}\theta\Big) \ = \ \int_\mathbb{R} (\mathrm{e}^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta y \chi(y)) \pi_\lambda(\mathrm{d}y) + \mathrm{i}\theta \mu_{\chi,\lambda},$$

where

$$\pi_\lambda(\mathrm{d}y) := \lambda \pi(\mathrm{d}\lambda^{1/\alpha} y), \qquad \mu_{\chi,\lambda} := \int_\mathbb{R} y(\chi(y) - 1) \pi_\lambda(\mathrm{d}y).$$

The r.h.s. of (5.38) can be rewritten as

$$-|\theta|^\alpha \omega(\theta; \alpha, c^+, c^-) \ = \ V_0(\theta) \ := \ \int_\mathbb{R} (\mathrm{e}^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta y \chi(y)) \pi_0(\mathrm{d}y) + \mathrm{i}\theta \mu_{\chi,0},$$

where

$$\begin{aligned} \pi_0(\mathrm{d}y) &:= -c^+\,\mathrm{d}y^{-\alpha} \mathbf{1}(y > 0) + c^-\,\mathrm{d}(-y)^{-\alpha} \mathbf{1}(y < 0), \\ \mu_{\chi,0} &:= \int_\mathbb{R} y(\chi(y) - 1) \pi_0(\mathrm{d}y). \end{aligned}$$

Let $C_\natural$ be the class of all bounded continuous functions on $\mathbb{R}$ vanishing in a neighborhood of 0. According to ([95], Thm. 8.7), relation (5.38) follows from

$$\lim_{\lambda\to 0}\int_{\mathbb{R}} f(y)\pi_\lambda(\mathrm{d}y) \ = \ \int_{\mathbb{R}} f(y)\pi_0(\mathrm{d}y), \quad \forall f\in C_\natural, \tag{5.40}$$

$$\lim_{\lambda\to 0}\mu_{\chi,\lambda} = \mu_{\chi,0}, \qquad \lim_{\epsilon\downarrow 0}\lim_{\lambda\to 0}\int_{|y|\le\epsilon} y^2\pi_\lambda(\mathrm{d}y) \ = \ 0. \tag{5.41}$$

Relations (5.40) is immediate from (5.10) while (5.41) follows from (5.10) by integration by parts.

Coming back to the proof of (5.36), consider the convergence $J_{n2}\to J_2$. By change of variables: $nz = w, n-s+1 = nu$, J_{n2} can be rewritten as

$$\begin{aligned} J_{n2} \ &= \ \int_{1/n}^1 \mathrm{d}u\int_0^{\epsilon n} n^{-\beta}V\Big(\theta n^{\frac{\beta}{\alpha}}\frac{1-(1-\frac{w}{n})^{[un]}}{w}\Big)w^\beta\psi\Big(1-\frac{w}{n}\Big)\,\mathrm{d}w \\ &= \ -|\theta|^\alpha\omega(\theta;\alpha,c^+,c^-)\int_0^1\mathrm{d}u\int_0^\infty\Big(\frac{1-\mathrm{e}^{-wu}}{w}\Big)^\alpha\kappa_{n2}(\theta;u,w)w^\beta\psi\Big(1-\frac{w}{n}\Big)\,\mathrm{d}w, \end{aligned}$$

where $\kappa_{n2}(u,w)$ is written as

$$\begin{aligned} \kappa_{n2}(\theta;u,w) \ &:= \ -\Big(\frac{1-\mathrm{e}^{-wu}}{w}\Big)^{-\alpha}n^{-\beta}\frac{V\Big(\theta n^{\frac{\beta}{\alpha}}w^{-1}(1-(1-\frac{w}{n})^{[un]})\Big)}{|\theta|^\alpha\omega(\theta;\alpha,c^+,c^-)} \\ &\quad\times\mathbf{1}(n^{-1}<u\le 1, 0<w<\epsilon n) \\ &= \ \frac{\lambda V(\lambda^{-1/\alpha}\theta)}{-|\theta|^\alpha\omega(\theta;\alpha,c^+,c^-)}\Big(\frac{1-(1-\frac{w}{n})^{[un]}}{1-\mathrm{e}^{-wu}}\Big)^\alpha \\ &\quad\times\mathbf{1}(n^{-1}<u\le 1, 0<w<\epsilon n) \end{aligned} \tag{5.42}$$

with

$$\lambda \equiv \lambda_n(u,w) \ := \ n^{-\beta}\Big(\frac{w}{1-(1-\frac{w}{n})^{[un]}}\Big)^\alpha \ \to \ 0$$

for each $u\in(0,1], w>0$ fixed. Hence and with (5.38) in mind, it follows that $\kappa_{n2}(\theta;u,w)\to 1$ for each $\theta\in\mathbb{R}, u\in(0,1], w>0$ and therefore the convergence $J_{n2}\to J_2$ by the dominated convergence theorem provided we establish a dominating bound

$$|\kappa_{n2}(\theta;u,w)| \ \le \ C \tag{5.43}$$

with C independent of $n, u\in(0,1], w\in(0,\epsilon n)$. From (5.39) it follows that the first ratio on the r.h.s. of (5.42) is bounded by an absolute constant. Next, for any $0\le x\le 1/2$, $s>0$ we have

$$1-x\ge\mathrm{e}^{-2x} \implies (1-x)^s\ge\mathrm{e}^{-2xs} \implies 1-(1-x)^s\le 2(1-\mathrm{e}^{-xs})$$

and hence

$$\frac{1-(1-\frac{w}{n})^{[un]}}{1-\mathrm{e}^{-wu}} \le \frac{1-(1-\frac{w}{n})^{un}}{1-\mathrm{e}^{-wu}} \le 2, \text{ for any } 0 \le w \le n/2,\ u > 0$$

so that the second ratio on the r.h.s. of (5.42) is also bounded by 2, provided $\epsilon \le 1/2$. This proves (5.43) and concludes the proof of $J_{n2} \to J_2$. The proof of the convergence $J_{n1} \to J_1$ is similar and is omitted. Using inequality (5.39) it is not difficult to prove that $|J_{n3}| < Cn^{\beta-(\alpha-1)}$. Since $\beta - (\alpha - 1) < 0$, $J_{n3} \to 0$. This concludes the proof of (5.36), and finite-dimensional convergence in (5.30).

To prove the tightness part of (5.30), it suffices to verify the well-known criterion in ([17], Thm.12.3): there exists $C > 0$ such that, for any $n \ge 1$ and $0 \le \tau < \tau + h \le 1$

$$\sup_{u>0} u^{\alpha}\mathrm{P}\Big(n^{\frac{\beta}{\alpha}-1}|S_n(\tau+h) - S_n(\tau)| > u\Big) \ < \ Ch^{\alpha-\beta}, \tag{5.44}$$

where $\alpha - \beta > 1$. By stationarity of increments of $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ it suffices to prove (5.44) for $\tau = 0, h = 1$, in which case it becomes

$$\sup_{u>0} u^{\alpha}\mathrm{P}\Big(|S_n| > u\Big) \ < \ Cn^{\alpha-\beta}, \qquad S_n := S_n(1). \tag{5.45}$$

The proof of (5.45), below, requires inequality in (5.46) for tail probabilities of stochastic integrals with respect to ID random measure. Let $L^{\alpha}(\mathbb{Z}\times(0,1))$ be the class of measurable functions $g : \mathbb{Z}\times(0,1) \to \mathbb{R}$ with

$$\|g\|_{\alpha}^{\alpha} := \sum_{s\in\mathbb{Z}} \mathrm{E}|g(s,a)|^{\alpha} < \infty.$$

Also, introduce the weak space $L^{\alpha}_{w}(\mathbb{Z}\times(0,1))$ of measurable functions $g : \mathbb{Z}\times(0,1) \to \mathbb{R}$ with

$$\|g\|_{\alpha,w}^{\alpha} := \sup_{t>0} t^{\alpha} \sum_{s\in\mathbb{Z}} \mathrm{P}(|g(s,a)| > t) < \infty.$$

Note $L^{\alpha}(\mathbb{Z}\times(0,1)) \subset L^{\alpha}_{w}(\mathbb{Z}\times(0,1))$ and $\|g\|_{\alpha,w}^{\alpha} \le \|g\|_{\alpha}^{\alpha}$. Let $\{M_s, s \in \mathbb{Z}\}$ be the random measure in (5.4), $M \sim W \sim ID_2(0,\pi)$ with zero mean and the Lévy measure π satisfying the assumptions in (ii). It is well-known (see, e.g., [97]) that the stochastic integral

$$M(g) := \sum_{s\in\mathbb{Z}} \int_{(0,1)} g(s,a) M_s(\,\mathrm{d}a)$$

is well-defined for any $g \in L^{p}(\mathbb{Z}\times(0,1))$, $p = 1, 2$ and satisfies

$$\mathrm{E}M^2(g) = C_2\|g\|_2^2, \qquad \text{and} \qquad \mathrm{E}|M(g)| \le C_1\|g\|_1,$$

for some constants $C_1, C_2 > 0$. The above facts together with Hunt's interpolation theorem, see ([89], Theorem IX.19) imply that $M(g)$ extends to all $g \in L^\alpha_w(\mathbb{Z} \times (0,1))$, $1 < \alpha < 2$ and satisfies the bound

$$\sup_{u>0} u^\alpha \mathrm{P}(|M(g)| > u) \ \le \ C\|g\|^\alpha_{\alpha,w} \ \le \ C\|g\|^\alpha_\alpha, \tag{5.46}$$

with some constant $C > 0$ depending on α, C_1, C_2 only. Using (5.46) and the representation $S_n = M(g)$ with

$$g(s,a) = \sum_{t=1}^n a^{t-s}\mathbf{1}(t \ge s)$$

we obtain

$$\sup_{u>0} u^\alpha \mathrm{P}\Big(|S_n| > u\Big) \ \le \ C\sum_{s\le n} \mathrm{E}\Big(\sum_{t=1}^n a^{t-s}\Big)^\alpha \ = \ O(n^{\alpha-\beta}),$$

where the last relation is proved in Chapter 4 (proof of Theorem 4.3.1, page 68). This proves (5.45) and part (ii).

(iii) It suffices to prove that for any $0 < \tau_1 < \cdots < \tau_m \le 1$, $\theta_1 \in \mathbb{R}, \ldots, \theta_m \in \mathbb{R}$,

$$J_n \ := \ \log \mathrm{E}\exp\Big\{ \mathrm{i}\frac{1}{n^{1/(1+\beta)}}\sum_{j=1}^m \theta_j S_n(\tau_j)\Big\} \to J \ := \ \log \mathrm{E}\exp\{ \mathrm{i}\sum_{j=1}^m \theta_j L_{1+\beta}(\tau_j)\}. \tag{5.47}$$

Similarly as in (i)-(ii), we shall restrict the proof of (5.47) to the case $m = 1$ since the general case follows analogously. Then

$$J_n \ = \ \sum_{s\in\mathbb{Z}} \int_0^1 V\Big(n^{-1/(1+\beta)}\theta\sum_{t=1}^{[n\tau]}(1-z)^{t-s}\mathbf{1}(t \ge s)\Big) z^\beta \psi(1-z)\,\mathrm{d}z \ = \ J_{n1} + J_{n2},$$

where

$$J_{n1} \ := \ \sum_{s\le 0}\int_0^1 V(\cdots) z^\beta\psi(1-z)\,\mathrm{d}z,$$
$$J_{n2} \ := \ \sum_{s=1}^{[n\tau]}\int_0^1 V(\cdots) z^\beta\psi(1-z)\,\mathrm{d}z.$$

Let $\theta > 0$. By the change of variables: $n^{1/(1+\beta)}z = \theta/y$, $[n\tau] - s + 1 = nu$, J_{n2} can be rewritten as

$$\begin{aligned} J_{n2} &= \sum_{s=1}^{[n\tau]} \int_0^1 V\Big(\frac{\theta(1-(1-z)^{[n\tau]-s+1})}{n^{1/(1+\beta)}z}\Big) z^\beta \psi(1-z)\,\mathrm{d}z \\ &= \theta^{1+\beta} \int_0^\tau \mathrm{d}u \int_0^\infty \frac{\mathrm{d}y}{y^{\beta+2}} V\Big(y(1-(1-\frac{\theta}{n^{1/(1+\beta)}y})^{[un]})\Big) \qquad (5.48) \\ &\quad \times \psi\Big(1-\frac{\theta}{n^{1/(1+\beta)}y}\Big)\mathbf{1}(1/n < u < [n\tau]/n, y > \theta n^{-1/(1+\beta)}), \end{aligned}$$

where

$$\mathbf{1}_n(\theta; y, u) := \mathbf{1}(1/n < u < [n\tau]/n, y > \theta n^{-1/(1+\beta)}) \to \mathbf{1}(0 < u < \tau, y > 0).$$

As $(1-\frac{\theta}{n^{1/(1+\beta)}y})^{un} \to 0$ for any $u, y > 0$ due to $n/n^{1/(1+\beta)} \to \infty$, we see that the integrand in (5.48) tends to $y^{-\beta-2}V(y)\psi(1)$. We will soon prove that this passage to the limit under the sign of the integral in (5.48) is legitimate. Therefore,

$$J_{n2} \to J := \tau|\theta|^{1+\beta}\psi(1)\int_0^\infty V(y)y^{-\beta-2}\,\mathrm{d}y = -\tau|\theta|^{1+\beta}\psi(1)\omega(\theta; 1+\beta, \pi_\beta^-, \pi_\beta^+), \qquad (5.49)$$

$$\pi_\beta^+ := \frac{1}{1+\beta}\int_0^\infty x^{1+\beta}\pi(\,\mathrm{d}x), \qquad \pi_\beta^- := \frac{1}{1+\beta}\int_{-\infty}^0 |x|^{1+\beta}\pi(\,\mathrm{d}x),$$

and the last equality in (5.49) follows from the definition of $V(y)$ and ([50], Thm. 2.2.2).

For justification of the above passage to the limit, note that the function

$$V(y) = \int_{\mathbb{R}} (\mathrm{e}^{\,\mathrm{i}yx} - 1 - \,\mathrm{i}yx)\pi(\,\mathrm{d}x)$$

satisfies $|V(y)| \le V_1(y) + V_2(y)$, where

$$V_1(y) := y^2 \int_{|x|\le 1/|y|} x^2\pi(\,\mathrm{d}x), \qquad V_2(y) := 2|y|\int_{|x|>1/|y|} |x|\pi(\,\mathrm{d}x).$$

We have

$$\begin{aligned} \int_0^\infty (V_1(y)+V_2(y))y^{-\beta-2}\,\mathrm{d}y &\le \int_{\mathbb{R}} x^2\pi(\,\mathrm{d}x)\int_0^{1/|x|} y^{-\beta}\,\mathrm{d}y \\ &\quad + 2\int_{\mathbb{R}} |x|\pi(\,\mathrm{d}x)\int_{1/|x|}^\infty y^{-1-\beta}\,\mathrm{d}y \\ &\le C\int_{\mathbb{R}} |x|^{1+\beta}\pi(\,\mathrm{d}x) < \infty. \end{aligned}$$

Next,

$$\sup_{1/2\le c\le 1} V_1(cy) \le y^2 \int_{|x|\le 2/|y|} x^2 \pi(\mathrm{d}x) =: \bar V_1(y), \qquad \sup_{1/2\le c\le 1} V_2(cy) \le V_2(y)$$

and $\int_0^\infty \bar V_1(y) y^{-\beta-2}\,\mathrm{d}y < \infty$. Denote

$$\zeta_n(\theta; y, u) := (1-\theta n^{-1/(1+\beta)} y^{-1})^{[un]}.$$

Then $\zeta_n(\theta; y, u) \ge 0$ and we split the integral in (5.48) into two parts corresponding to $\zeta_n(\theta; y, u) \le 1/2$ and $\zeta_n(\theta; y, u) > 1/2$, viz., $J_{n2} = J^+_{n2} + J^-_{n2}$, where

$$\begin{aligned} J^+_{n2} &:= \theta^{1+\beta} \int_0^\tau \mathrm{d}u \int_0^\infty y^{-\beta-2}\,\mathrm{d}y V\Big(y(1-\zeta_n(\theta; y, u))\Big) \\ &\quad \times \psi\Big(1-\frac{\theta}{n^{1/(1+\beta)}y}\Big) \mathbf{1}(\zeta_n(\theta; y, u) \le 1/2) \mathbf{1}_n(\theta, y, u), \\ J^-_{n2} &:= \theta^{1+\beta} \int_0^\tau \mathrm{d}u \int_0^\infty y^{-\beta-2}\,\mathrm{d}y V\Big(y(1-\zeta_n(\theta; y, u))\Big) \\ &\quad \times \psi\Big(1-\frac{\theta}{n^{1/(1+\beta)}y}\Big) \mathbf{1}(\zeta_n(\theta; y, u) > 1/2) \mathbf{1}_n(\theta; y, u). \end{aligned}$$

Since

$$\Big|V\Big(y(1-\zeta_n(\theta; y, u))\Big)\mathbf{1}(\zeta_n(\theta; y, u) \le 1/2)\Big| \le \bar V_1(y) + V_2(y)$$

is bounded by integrable function (see above), so $J^+_{n2} \to J$ by the dominated convergence theorem. It remains to prove $J^-_{n2} \to 0$. From inequalities $1-x \le \mathrm{e}^{-x}$, $x > 0$, and $[un] \ge un/2$, $u > 1/n$, it follows that

$$\zeta_n(\theta; y, u) \le \mathrm{e}^{-\theta un/2n^{1/(1+\beta)}y}$$

and hence

$$\mathbf{1}(\zeta_n(\theta; y, u) > 1/2) \le \mathbf{1}(\mathrm{e}^{-\theta un/2n^{1/(1+\beta)}y} > 1/2) = \mathbf{1}((u/y) < c_1 n^{-\gamma}),$$

where $\gamma := \beta/(1+\beta) > 0$, $c_1 := (2\log 2)/\theta$. Without loss of generality, we can assume that $1 < \alpha < 1+\beta$ in (5.31). Condition (5.31) implies

$$|V(y)| \ \le\ \int_{|xy|\le 1} |yx|^\alpha \pi(\mathrm{d}x) + 2\int_{|yx|>1} |yx|^\alpha \pi(\mathrm{d}x) \ \le\ C|y|^\alpha, \qquad \forall\, y \in \mathbb{R}.$$

Hence

$$|J^-_{n2}| \ \le\ C \int_0^\tau \mathrm{d}u \int_0^\infty \mathbf{1}\Big(\frac{u}{y} < c_1 n^{-\gamma}\Big) \frac{\mathrm{d}y}{y^{2+\beta-\alpha}} \ \le\ K n^{-\gamma(1+\beta-\alpha)} \ \to\ 0$$

where

$$K := C \int_0^\tau u^{\alpha-1-\beta} \, \mathrm{d}u < \infty.$$

This proves $J_{n2} \to J$, or (5.49). The proof of $J_{n1} \to 0$ follows similarly and hence is omitted.

(iv) The proof of finite-dimensional convergence is similar the proof of Theorem 4.3.1 (ii), page 68. Below, we present the proof of the one-dimensional convergence of $n^{-1/2} S_n = n^{-1/2} \sum_{t=1}^n \mathfrak{X}(t)$ towards $N(0, \sigma_\Phi^2)$ with $\sigma_\Phi^2 > 0$ given in (5.50). Similarly as above, consider

$$J_n := \log \mathrm{E} \exp\{ \mathrm{i}\theta n^{-1/2} S_n \} = J_{n1} + J_{n2},$$

where

$$J_{n1} := \sum_{s \le 0} \mathrm{E} V\Big(\theta n^{-1/2} \sum_{t=1}^n a^{t-s}\Big), \qquad J_{n2} := \sum_{s=1}^n \mathrm{E} V\Big(\theta n^{-1/2} \sum_{t=s}^n a^{t-s}\Big).$$

We have

$$\begin{aligned} J_{n2} &= \sum_{k=1}^n \int_0^1 V\Big(\theta \frac{1-(1-z)^k}{z n^{1/2}}\Big) \phi(1-z) \, \mathrm{d}z \\ &= -\theta^2 \sigma_W^2 \, n^{-1} \sum_{k=1}^n \int_0^1 (1-(1-z)^k)^2 z^{-2} \kappa_n(\theta; k, z) \phi(1-z) \, \mathrm{d}z, \end{aligned}$$

where

$$\kappa_n(\theta; k, z) := \kappa\Big(\theta \frac{1-(1-z)^k}{z n^{1/2}}\Big), \qquad \kappa(y) := -V(y) \sigma_W^{-2} y^{-2},$$

and the function $\kappa(y)$ satisfies

$$\lim_{y \to 0} \kappa(y) = 1, \qquad \sup_{y \in \mathbb{R}} |\kappa(y)| < \infty.$$

These facts together with $\beta > 1$ imply

$$n^{-1} \sum_{k=1}^n \int_0^1 (1-(1-z)^k)^2 z^{-2} \kappa_n(\theta; k, z) \phi(1-z) \, \mathrm{d}z \to \int_0^1 z^{-2} \phi(1-z) \, \mathrm{d}z$$

and hence $J_{n2} \to -(1/2)\theta^2 \sigma_\Phi^2$, with

$$\sigma_\Phi^2 := 2\sigma_W^2 \int_0^1 z^{-2} \phi(1-z) \, \mathrm{d}z \;=\; 2\sigma_W^2 \mathrm{E}(1-a)^{-2}. \tag{5.50}$$

The proof of $J_{n1} \to 0$ follows similarly (see Chapter 4 for details). This proves (5.33).

Let us prove the tightness part in (iv). It suffices to show the bound

$$\mathrm{E}S_n^4 \ \le\ Cn^2. \tag{5.51}$$

We have $S_n = M(g)$, where M is the stochastic integral discussed in the proof of (ii) above and

$$g \equiv g(s,a) = \sum_{t=1}^{n} a^{t-s}\mathbf{1}(t \ge s) \in L^2(\mathbb{Z}\times(0,1)).$$

Then

$$\mathrm{E}M^4(g) = \mathrm{cum}_4(M(g)) + 3(\mathrm{E}M^2(g))^2,$$

where $\mathrm{E}M^2(g) = \mathrm{E}S_n^2$ satisfies $\mathrm{E}S_n^2 \le Cn$ (the last fact follows by a similar argument as above). Hence, $(\mathrm{E}M^2(g))^2 \le Cn^2$ in agreement with (5.51). It remains to evaluate the 4th cumulant

$$\mathrm{cum}_4(S_n) = \mathrm{cum}_4(M(g)) = \pi_4 \sum_{s\in\mathbb{Z}} \mathrm{E}g^4(s,a),$$

where $\pi_4 := \int_{\mathbb{R}} x^4 \pi(\,\mathrm{d}x)$. Then $\mathrm{cum}_4(S_n) = \pi_4(L_{n1} + L_{n2})$, where

$$L_{n1} \ := \ \sum_{s\le 0} \mathrm{E}\Big(\sum_{t=1}^{n} a^{t-s}\Big)^4, \qquad L_{n2} \ := \ \sum_{s=1}^{n} \mathrm{E}\Big(\sum_{t=s}^{n} a^{t-s}\Big)^4.$$

We have

$$L_{n2} \ \le \ n\sum_{k=1}^{n} \mathrm{E}\Big(\sum_{t=0}^{k} a^t\Big)^3 \ \le \ n\sum_{k=1}^{n}\int_0^1 z^{\beta-3}\psi(1-z)\,\mathrm{d}z \ \le \ Cn^2,$$

since $\beta > 2$. Similarly,

$$L_{n1} \ \le \ n^2 \sum_{s\le 0} \mathrm{E}\Big(\sum_{t=1}^{n} a^{t-s}\Big)^2 \ \le \ n^2 \int_0^1 \frac{z^{\beta-2}\psi(1-z)\,\mathrm{d}z}{1-(1-z)^2} \ \le \ Cn^2.$$

This proves (5.51) and part (iv). Theorem 5.3.1 is proved. □

5.4 Disaggregation

Following [62], let us define an estimator of ϕ, the density of the mixing distribution Φ. Its starting point is the equality (5.8), implying

$$\sigma_W^{-2}(r(k) - r(k+2)) \ = \ \int_0^1 x^k \phi(x)\,\mathrm{d}x, \qquad k = 0,1,\ldots, \tag{5.52}$$

where $r(k) = \mathrm{Cov}(\mathfrak{X}(k), \mathfrak{X}(0))$ and $\sigma_W^2 = \mathrm{Var}(W) = r(0) - r(2)$. The l.h.s. of (5.52), hence the integrals on the r.h.s. of (5.52), or moments of Φ, can be estimated from the observed sample, leading to the problem of recovering the density from its moments, as explained below.

For a given $q > 0$, consider a finite measure on $(0,1)$ having density $w^{(q)}(x) := (1-x)^{q-1}$. Let $L_2(w^{(q)})$ be the space of functions $h : (0,1) \to \mathbb{R}$ which are square integrable with respect to this measure. Denote by $\left\{J_n^{(q)},\ n = 0,1,\dots\right\}$ the orthonormal basis in $L_2(w^{(q)})$ consisting of normalized Jacobi polynomials:

$$J_n^{(q)}(x) \ := \ \sum_{j=0}^{n} g_{n,j}^{(q)} x^j, \tag{5.53}$$

with coefficients

$$g_{n,j}^{(q)} := (-1)^{n-j} \frac{\sqrt{2n+q}}{\Gamma(n+q)} \frac{\Gamma(n+1)}{\Gamma(n-j+1)} \frac{\Gamma(q+n+j)}{\Gamma(j+1)^2}, \tag{5.54}$$

$0 \le j \le n$. See ([1], p.774, formula 22.2.2). Thus,

$$\int_0^1 J_j^{(q)}(x) J_k^{(q)}(x) w^{(q)}(x)\,\mathrm{d}x \ = \ \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \tag{5.55}$$

Any function $h \in L_2(w^{(q)})$ can be expanded in Jacobi's polynomials:

$$h(x) = \sum_{k=0}^{\infty} h_k J_k^{(q)}(x), \tag{5.56}$$

with

$$h_k = \int_0^1 h(x) J_k^{(q)}(x) w^{(q)}(x)\,\mathrm{d}x = \sum_{j=0}^{k} g_{k,j}^{(q)} \int_0^1 h(x) x^j w^{(q)}(x)\,\mathrm{d}x.$$

Below, we call (5.56) the q-Jacobi expansion of h.

Consider the function

$$\zeta(x) := \frac{\phi(x)}{(1-x)^{q-1}}, \qquad \text{with} \quad \int_0^1 \zeta(x)(1-x)^{q-1}\,\mathrm{d}x = \int_0^1 \phi(x)\,\mathrm{d}x = 1. \tag{5.57}$$

Under the condition

$$\int_0^1 \frac{\phi(x)^2}{(1-x)^{q-1}}\,\mathrm{d}x < \infty, \tag{5.58}$$

the function ζ in (5.57) belongs to $L_2(w^{(q)})$, and has a $q-$Jacobi expansion with coefficients

$$\zeta_k = \sum_{j=0}^{k} g_{k,j}^{(q)} \int_0^1 \phi(x) x^j \, \mathrm{d}x = \frac{1}{\sigma_W^2} \sum_{j=0}^{k} g_{k,j}^{(q)} \left(r(j) - r(j+2) \right), \qquad k = 0, 1, \ldots; \tag{5.59}$$

see (5.52). Equations (5.56), (5.59) lead to the following estimates of the function $\zeta(x)$:

$$\widehat{\zeta}_n(x) \; := \; \sum_{k=0}^{K_n} \widehat{\zeta}_{n,k} J_k^{(q)}(x), \qquad \widetilde{\zeta}_n(x) \; := \; \sum_{k=0}^{K_n} \widetilde{\zeta}_{n,k} J_k^{(q)}(x), \tag{5.60}$$

where $K_n, n \in \mathbb{N}^*$ is a nondecreasing sequence tending to infinity at a rate which is discussed below, and

$$\widehat{\zeta}_{n,k} := \frac{1}{\widehat{\sigma}_W^2} \sum_{j=0}^{k} g_{k,j}^{(q)} (\widehat{r}_n(j) - \widehat{r}_n(j+2)), \qquad \widetilde{\zeta}_{n,k} \; := \; \frac{1}{\sigma_W^2} \sum_{j=0}^{k} g_{k,j}^{(q)} (\widehat{r}_n(j) - \widehat{r}_n(j+2)) \tag{5.61}$$

are natural estimates of the ζ_k's in (5.59) in the case when σ_W^2 is unknown or known, respectively. Here and below,

$$\overline{\mathfrak{X}} := \frac{1}{n} \sum_{k=1}^{n} \mathfrak{X}(k), \qquad \widehat{r}_n(j) := \frac{1}{n} \sum_{i=1}^{n-j} \left(\mathfrak{X}(i) - \overline{\mathfrak{X}} \right) \left(\mathfrak{X}(i+j) - \overline{\mathfrak{X}} \right), \quad j = 0, 1, \ldots, n \tag{5.62}$$

are the sample mean and the sample covariance, respectively, and the estimate of $\sigma_W^2 = r(0) - r(2)$ is defined as

$$\widehat{\sigma}_W^2 := \widehat{r}_n(0) - \widehat{r}_n(2).$$

The corresponding estimators of $\phi(x)$ is constructed following relation (5.57):

$$\widehat{\phi}_n(x) \; := \; \widehat{\zeta}_n(x)(1-x)^{q-1}, \qquad \widetilde{\phi}_n(x) \; := \; \widetilde{\zeta}_n(x)(1-x)^{q-1}. \tag{5.63}$$

The above estimators were essentially constructed in [62] and [22]. The modifications in (5.63) differ from the original ones in the above mentioned papers by the choice of a more natural estimate (5.62) of the covariance function $r(j)$, which allows for non-centered observations and makes both estimators in (5.63) location and scale invariant. Note also that the first estimator in (5.63) satisfies $\int_0^1 \widehat{\phi}_n(x) \, \mathrm{d}x = 1$, while the second one does not have this property and can be used only if σ_W^2 is known.

Proposition 5.4.1. *Let $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ be an aggregated process in* (5.4) *with finite 4th moment $\mathrm{E}\mathfrak{X}(0)^4 < \infty$ and $M \sim W \sim ID(\mu, \sigma, \pi)$. Assume that the mixing density $\phi(x)$ satisfies conditions* (5.13) *and* (5.58)*, with some $q > 0$. Let $\widetilde{\zeta}_n(x)$ be* *the estimator of* $\zeta(x)$ *as defined in* (5.60), *where* K_n *satisfy*

$$K_n = [\gamma \log n] \quad \text{with} \quad 0 < \gamma < (4\log(1+\sqrt{2}))^{-1}. \tag{5.64}$$

Then

$$\int_0^1 \mathrm{E}(\widetilde{\zeta}_n(x) - \zeta(x))^2 (1-x)^{q-1}\,\mathrm{d}x \ \to\ 0. \tag{5.65}$$

Proof. Denote v_n the l.h.s. of (5.65). From the orthonormality property (5.55), similarly as in ([62], (3.3)),

$$v_n \ = \ \sum_{k=0}^{K_n} \mathrm{E}(\widetilde{\zeta}_{n,k} - \zeta_k)^2 + \sum_{k=K_n+1}^{\infty} \zeta_k^2, \tag{5.66}$$

where the second sum on the r.h.s. tends to 0. By the location invariance mentioned above, w.l.g. we can assume below that $\mathrm{E}\mathfrak{X}(t) = 0$. Let $\widehat{r}_n^\circ(j) := \frac{1}{n}\sum_{i=1}^{n-j} \mathfrak{X}(i)\mathfrak{X}(i+j),\ 0 \le j < n$, then $\mathrm{E}\widehat{r}_n^\circ(j) - r(j) = (j/n)r(j)$ and

$$\begin{aligned}
\mathrm{E}\Big\{\widetilde{\zeta}_{n,k} - \zeta_k\Big\}^2 \ &= \ \sigma_W^{-4}\mathrm{E}\Big\{\sum_{j=0}^{k} g_{k,j}^{(q)}\Big(\widehat{r}_n(j) - \widehat{r}_n(j+2) - r(j) + r(j+2)\Big)\Big\}^2 \\
&= \ \sigma_W^{-4}\mathrm{E}\Big\{\sum_{j=0}^{k} g_{k,j}^{(q)}\Big(\widehat{r}_n^\circ(j) - \widehat{r}_n^\circ(j+2) - r(j) + r(j+2) + 2n^{-1}\overline{\mathfrak{X}}^2 \\
&\qquad - n^{-1}\overline{\mathfrak{X}}\Big[\mathfrak{X}(n-j-1) + \mathfrak{X}(n-j) + \mathfrak{X}(j+1) + \mathfrak{X}(j+2)\Big]\Big)\Big\}^2 \\
&\le \ Ck\Big(\max_{0\le j\le k}|g_{k,j}^{(q)}|\Big)^2 \sum_{j=0}^{k}\Big(\frac{j^2}{n^2} + \mathrm{Var}(\widehat{r}_n^\circ(j) - \widehat{r}_n^\circ(j+2)) + \frac{C}{n^2}\Big),
\end{aligned} \tag{5.67}$$

where we used the trivial bound $\mathrm{E}\overline{\mathfrak{X}}^4 < C$. The rest of the proof of Proposition 5.4.1 follows from (5.66), (5.67) and Lemmas 5.4.2 and 5.4.3 below. See ([62], pp.2552-2553) for details. □

Lemma 5.4.2 generalizes ([62], Lemma 4) for a non-Gaussian aggregated process with finite 4th moment.

Lemma 5.4.2. *Let* $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ *be an aggregated process in* (5.4) *with* $\mathrm{E}\mathfrak{X}(0)^4 < \infty$, $\mathrm{E}\mathfrak{X}(0) = 0$. *There exists a constant* $C > 0$ *independent of* n, k *and such that*

$$\mathrm{Var}(\widehat{r}_n^\circ(k) - \widehat{r}_n^\circ(k+2)) \le \frac{C}{n}. \tag{5.68}$$

Proof. Let $D(k) := \mathfrak{X}(k) - \mathfrak{X}(k+2)$. Similarly as in ([62], p.2560),

$$\mathrm{Var}(\widehat{r}_n^\circ(k) - \widehat{r}_n^\circ(k+2)) \ \le \ Cn^{-2}\Big(\mathrm{Var}\Big(\sum_{j=1}^{n-k-2}\mathfrak{X}(j)D(j+k)\Big) + 1\Big).$$

Here,

$$\mathrm{Var}\Big(\sum_{j=1}^{n-k-2}\mathfrak{X}(j)D(j+k)\Big)=\sum_{j,l=1}^{n-k-2}\mathrm{Cov}\Big(\mathfrak{X}(j)D(j+k),\mathfrak{X}(l)D(l+k)\Big),$$

where

$$\begin{aligned}\mathrm{Cov}(\mathfrak{X}(j)D(j+k),\mathfrak{X}(l)D(l+k)) &= \mathrm{Cum}(\mathfrak{X}(j),D(j+k),\mathfrak{X}(l),D(l+k))\\ &\quad+\mathrm{E}[\mathfrak{X}(j)\mathfrak{X}(l)]\mathrm{E}[D(j+k)D(l+k)]\\ &\quad+\mathrm{E}[\mathfrak{X}(j)D(k+l)]\mathrm{E}[\mathfrak{X}(l)D(j+k)].\end{aligned}$$

The two last terms in the above representation of the covariance are estimated in [62]. Hence the lemma follows from

$$\sum_{j,l=1}^{n-k-2}\mathrm{Cum}(\mathfrak{X}(j),D(j+k),\mathfrak{X}(l),D(l+k)) \ \le\ Cn. \tag{5.69}$$

We have for $k_1,k_2\ge 0, l\ge j$,

$$\begin{aligned}\mathrm{Cum}(\mathfrak{X}(j),\mathfrak{X}(j+k_1),\mathfrak{X}(l),\mathfrak{X}(l+k_2)) &= \pi_4\mathrm{E}\Big[\sum_{s\le j}a^{j-s}a^{j-s+k_1}a^{l-s}a^{l-s+k_2}\Big]\\ &= \pi_4\mathrm{E}\Big[\frac{a^{k_1+k_2+2(l-j)}}{1-a^4}\Big]\end{aligned}$$

and hence

$$c_{j,l,k} \ :=\ \mathrm{Cum}(\mathfrak{X}(j),D(j+k),\mathfrak{X}(l),D(l+k)) \ =\ \pi_4\mathrm{E}\Big[\frac{a^{2k+2(l-j)}(1-a^2)}{1+a^2}\Big],$$

where

$$\pi_4:=\int_{\mathbb{R}}x^4\pi(\,\mathrm{d}x).$$

Then

$$\sum_{j,l=1}^{n-k-2}|c_{j,l,k}|\le C\sum_{1\le j\le l\le n}\mathrm{E}\Big[\frac{(1-a^2)}{1+a^2}a^{2(l-j)}\Big]\le C\sum_{1\le j\le n}\mathrm{E}\Big[\frac{1}{1+a^2}\Big] \ \le\ Cn,$$

proving (5.69) and the lemma, too. □

Lemma 5.4.3. *Consider the coefficients $g_{n,j}^{(q)}$ (5.54) of the normalized Jacobi polynomial $J_n^{(q)}$ in (5.53). There exists a constant $C_q>0$ such that for all sufficiently large n,*

$$G_n^{(q)}:=\max_{0\le j\le n}|g_{n,j}^{(q)}|\le C_q n^{13/2}\mathrm{e}^{n\kappa}\quad \textit{with}\quad \kappa:=2\log(1+\sqrt{2}).$$

Proof is similar to ([62], proof of Lemma 5). We have

$$\left|\frac{g^{(q)}_{n,n-(m+1)}}{g^{(q)}_{n,n-m}}\right| = R(m), \qquad \text{where} \qquad R(z) := \frac{(n-z)^2}{(z+1)(q+2n-z-1)}. \tag{5.70}$$

The roots z_-, z_+ of $|R(z)| = 1$, or $(n-z)^2 - (z+1)(q+2n-z-1) = 0$, are equal

$$z_\pm \;=\; n + \frac{q-2}{4} \pm n\frac{\sqrt{2}}{2}\sqrt{1+\frac{p}{n}+\frac{q^2+4q-4}{8n^2}}.$$

A straightforward verification shows that for any $q > 0$ and all sufficiently large n the following bounds are true:

$$n\Big(1-\frac{\sqrt{2}}{2}\Big) - \frac{(\sqrt{2}-1)p}{4} - 1 \;\le\; z_- \le n\Big(1-\frac{\sqrt{2}}{2}\Big) - \frac{(\sqrt{2}-1)p}{4} =: z^*. \tag{5.71}$$

Since z_- is the only root satisfying $0 \le z_- \le n$ and

$$|R(z)| \ge 1 \quad \text{for} \quad z \le z_-; \quad |R(z)| \le 1 \quad \text{for} \quad z_- \le z \le n, \tag{5.72}$$

(5.71)–(5.72) imply that

$$G^{(q)}_n = \max_{0\le m\le n} |g^{(q)}_{n,n-m}| = \max(|g^{(q)}_{n,n-m^*}|, |g^{(q)}_{n,n-(m^*+1)}|),$$

where m^* is the integer satisfying $m^* \le z_- \le m^*+1$. Hence the statement of the lemma follows from Stirling's formula similarly to [62]. Lemma 5.4.3 is proved. □

The main result of this Section is the following theorem.

Theorem 5.4.4. *Let* $\{\mathfrak{X}(t),\, t\in\mathbb{Z}\}$, $\phi(x)$ *and* K_n *satisfy the conditions of Proposition 5.4.1, and* $\widehat{\phi}_n(x), \widetilde{\phi}_n(x)$ *be the estimators of* $\phi(x)$ *as defined in* (5.63). *Then*

$$\int_0^1 \frac{(\widehat{\phi}_n(x)-\phi(x))^2}{(1-x)^{q-1}}\,\mathrm{d}x \;\to_{\mathrm{p}}\; 0 \qquad \textit{and} \qquad \int_0^1 \frac{\mathrm{E}(\widetilde{\phi}_n(x)-\phi(x))^2}{(1-x)^{q-1}}\,\mathrm{d}x \;\to\; 0. \tag{5.73}$$

Proof. The second relation in (5.73) is immediate from (5.63) and (5.65). Next,

$$\widehat{\phi}_n(x) - \phi(x) = \frac{\sigma^2_W}{\widehat{\sigma}^2_W}\Big(\widetilde{\phi}_n(x)-\phi(x)\Big) + \phi(x)\Big(\frac{\sigma^2_W}{\widehat{\sigma}^2_W} - 1\Big),$$

where

$$\widehat{\sigma}^2_W = \widehat{r}_n(0) - \widehat{r}_n(2) = (g^{(q)}_{0,0})^{-1}\sigma^2_W\widetilde{\zeta}_{n,0} = \sigma^2_W\int_0^1 \widetilde{\zeta}_n(x)(1-x)^{q-1}\,\mathrm{d}x,$$

see (5.55), (5.56), (5.60), (5.61). Hence the first relation in (5.73) follows from the second one and the fact that $\widehat{\sigma}^2_W - \sigma^2_W \to_{\mathrm{p}} 0$. We have

$$
\begin{aligned}
\mathrm{E}(\widehat{\sigma}^2_W - \sigma^2_W)^2 &= \sigma^4_W \mathrm{E}\Big(\int_0^1 (\widetilde{\zeta}_n(x) - \zeta(x))(1-x)^{q-1}\,\mathrm{d}x \Big)^2 \\
&\le \sigma^4_W \mathrm{E}\Big(\int_0^1 (\widetilde{\zeta}_n(x) - \zeta(x))^2 (1-x)^{q-1}\,\mathrm{d}x \int_0^1 (1-x)^{q-1}\,\mathrm{d}x \Big) \\
&= \frac{\sigma^4_W}{q} \int_0^1 \mathrm{E}(\widetilde{\zeta}_n(x) - \zeta(x))^2 (1-x)^{q-1}\,\mathrm{d}x \to 0, \quad \text{as } n \to \infty,
\end{aligned}
$$

see (5.65). Theorem 5.4.4 is proved. □

Remark 5.4.5. The optimal choice of q (minimizing the integrated MISE in (5.73)) is not clear. If ϕ satisfies (5.27) then (5.58) is satisfied with any $0 < q < 2 + 2\beta$. Simulations in [62] and [22] show the "optimal" choice of q might be close to β which is generally unknown.

Remark 5.4.6. An interesting open question is asymptotic normality of the mixing density estimators in (5.63) for non-Gaussian process $\{\mathfrak{X}(t)\}$ (5.4), extending Theorem 2.1 in [22]. The proof of the last result relies on a central limit theorem for quadratic forms of moving-average processes due to [15]. Generalizing this theorem to mixed ID moving averages is an open problem at this moment.

6

Aggregation of autoregressive random-fields and anisotropic long memory

Abstract. We introduce the notion of anisotropic long memory for random fields on $\mathbb{Z}^2$ whose partial sums on incommensurate rectangles with sides growing at different rates $O(n)$ and $O(n^{H_1/H_2}), H_1 \neq H_2$ tend to an operator scaling random field on $\mathbb{R}^2$ with two scaling indices H_1, H_2. The random fields with such behavior are obtained by aggregating independent copies of a random-coefficient nearest-neighbor autoregressive random fields on $\mathbb{Z}^2$ with i.i.d. innovations belonging to the domain of attraction of an $\alpha-$stable law, $0 < \alpha \le 2$, with a scalar random coefficient A (the spectral radius of the corresponding autoregressive operator) having a regularly varying probability density near the 'unit root' $A = 1$. The proofs are based on a study of scaling limits of the corresponding lattice Green functions.

6.1 Introduction

Following Biermé et al. [16], a scalar random field $\{V(x), x \in \mathbb{R}^d\}$ is called *operator scaling random field* (OSRF) if there exist a $H > 0$ and a $d \times d$ real matrix E whose all eigenvalues have positive real parts, such that for any $\lambda > 0$

$$\{V(\lambda^E x)\} \stackrel{\mathrm{fdd}}{=} \{\lambda^H V(x)\}. \tag{6.1}$$

(See the end of this section for all unexplained notation.) In the case when $E = I$ is the unit matrix, (6.1) agrees with the definition of $H-$self-similar random field (SSRF), the latter referred to as self-similar process when $d = 1$. OSRFs may exhibit strong anisotropicity and play an important role in various physical theories, see [16] and the references therein. Several classes of OSRFs were constructed and discussed in [16], [27].

It is well-known that the class of self-similar processes is very large, SSRFs and OSFRs being even more numerous. According to a popular view, the 'value' of a concrete self-similar process depends on its 'domain of attraction'. In the case $d = 1$, the domain of attraction of a self-similar stationary increment (sssi) process $\{V(\tau), \tau \geq 0\}$ is usually defined as the class of all stationary processes $\{Y(t), t \in \mathbb{Z}_+\}$ whose normalized partial sums tend to $\{V(\tau), \tau \geq 0\}$, viz.,

$$B_n^{-1} \sum_{t=1}^{[n\tau]} Y(t) \ \to_{\mathrm{fdd}} \ V(\tau), \qquad \tau \in \mathbb{R}_+. \tag{6.2}$$

The classical Lamperti's theorem [56] says that in the case of (6.2), the normalizing constants B_n necessarily grow as n^H (modulus a slowly varying factor) and the limit random process in (6.2) is $H-$sssi. The limit process $\{V(\tau), \tau \geq 0\}$ in (6.2) characterizes large-scale and dependence properties of $\{Y(t), t \in \mathbb{Z}\}$, leading to the important concept of *distributional short/long memory* (Cox [28]). There exists a large probabilistic literature devoted to studying the partial sums limits of various classes of strongly and weakly dependent processes and random fields. See, e.g., the monographs [12], [33], [38] and the references therein. In particular, several works ([30], [31], [66], [98], [32]) discussed the partial sums limits of (stationary) random fields indexed by $t \in \mathbb{Z}^d$:

$$B_n^{-1} \sum_{t \in K_{[nx]}} Y(t) \ \to_{\mathrm{fdd}} \ V(x), \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d, \tag{6.3}$$

where $K_{[nx]} := \{t = (t_1, \ldots, t_d) \in \mathbb{Z}^d : 1 \leq t_i \leq nx_i\}$ is a sequence of rectangles whose all sides increase as $O(n)$. Related results for Gaussian or linear (shot-noise) and their subordinated random fields, with a particular emphasis on large-time behavior of statistical solutions of partial differential equations, were obtained in [2], [3], [4], [66], [68]. Most of the above mentioned studies deal with 'nearly isotropic' models of random fields characterized by a single memory parameter H

and a limiting SSRF $\{V(x)\}$ in (6.3).

In this Chapter we study *anisotropic distributional long memory*, by exhibiting a natural class of models whose partial sums tend to OSRFs. Related notion of *anisotropic long memory in spectral domain* and its implications is discussed in [61]. The present study is limited to the case $d = 2$ and random fields with the horizontal anisotropicity axis and a diagonal matrix E. Note that for $d = 2$ and $E = \mathrm{diag}(1, \gamma)$, $0 < \gamma \neq 1$, relation (6.1) writes as $\{V(\lambda x, \lambda^\gamma y)\} \overset{\mathrm{fdd}}{=} \{\lambda^H V(x, y)\}$, $(x, y) \in \mathbb{R}^2$, or

$$\{\lambda V(x, y)\} \overset{\mathrm{fdd}}{=} \{V(\lambda^{1/H_1} x, \lambda^{1/H_2} y)\}, \qquad \forall\, \lambda > 0, \tag{6.4}$$

where $H_1 := H$, $H_2 := H/\gamma \neq H_1$. The OSRFs (6.4) discussed in this Chapter are obtained by taking the partial sums limits

$$B_n^{-1} \sum_{(t,s)\in K_{[nx, n^{H_1/H_2} y]}} Y(t, s) \ \to_{\mathrm{fdd}}\ V(x, y), \qquad (x, y) \in \mathbb{R}^2_+ \tag{6.5}$$

on *incommensurate* rectangles $K_{[nx, n^{H_1/H_2} y]} := \{(t, s) \in \mathbb{Z}^2 : 1 \le t \le nx, 1 \le s \le n^{H_1/H_2} y\}$ with sides growing at different rates $O(n)$ and $O(n^{H_1/H_2})$. The convergence in (6.5) is established for a natural class of aggregated random-coefficient autoregressive random fields, see (6.6)-(6.9) below, with finite and infinite variance.

Consider a nearest-neighbor autoregressive random field $\{X(t, s), (t, s) \in \mathbb{Z}^2\}$ satisfying the difference equation

$$X(t, s) \ = \sum_{|u|+|v|=1} a(u, v) X(t + u, s + v) + \varepsilon(t, s), \qquad (t, s) \in \mathbb{Z}^2, \tag{6.6}$$

where $\{\varepsilon(t, s), (t, s) \in \mathbb{Z}^2\}$ are i.i.d. r.v.'s whose generic distribution ε belongs to the domain of (normal) attraction of an $\alpha-$stable law, $0 < \alpha \le 2$, and $a(t, s) \ge 0$, $|t| + |s| = 1$, are *random* coefficients, independent of $\{\varepsilon(t, s), (t, s) \in \mathbb{Z}^2\}$ and satisfying the following condition for the existence of a stationary solution of (6.6):

$$A \ := \sum_{|t|+|s|=1} a(t, s) \ < \ 1, \qquad \text{a.s.} \tag{6.7}$$

(Note, that this condition is sufficient but not necessary, see [80].) The stationary solution of (6.6) is given by the convergent series

$$X(t, s) \ = \sum_{(u,v)\in\mathbb{Z}^2} g(t - u, s - v, a)\varepsilon(u, v), \qquad (t, s) \in \mathbb{Z}^2, \tag{6.8}$$

where $a = (a(t,s), |t| + |s| = 1)$, and $g(t,s,a)$, $(t,s) \in \mathbb{Z}^2$, is the (random) lattice Green function solving the equation

$$g(t,s,a) - \sum_{|u|+|v|=1} a(u,v) g(t+u, s+v, a) = \delta(t,s),$$

where $\delta(t,s)$ is the delta function (see Section 6.2 for precise statement). Let $\{X_i(t,s), (t,s) \in \mathbb{Z}^2\}$, $i = 1, 2, \ldots$, be independent copies of (6.8). The aggregated field $\{\mathfrak{X}(t,s), (t,s) \in \mathbb{Z}^2\}$ is defined as the limit:

$$N^{-1/\alpha} \sum_{i=1}^{N} X_i(t,s) \quad \to_{\mathrm{fdd}} \quad \mathfrak{X}(t,s), \qquad (t,s) \in \mathbb{Z}^2. \tag{6.9}$$

Let Φ denote the distribution of the random vector $a = (a(t,s), |t|+|s| = 1)$ taking values in $\boldsymbol{A} := \{a(t,s) \in [0,1), \sum_{|t|+|s|=1} a(t,s) < 1\} \subset \mathbb{R}^4$ and called below the *mixing distribution*. Under mild additional conditions, the limit in (6.9) exists and is written as

$$\mathfrak{X}(t,s) \quad = \quad \sum_{(u,v) \in \mathbb{Z}^2} \int_{\boldsymbol{A}} g(t-u, s-v, a) M_{u,v}(\,\mathrm{d}a), \qquad (t,s) \in \mathbb{Z}^2. \tag{6.10}$$

In (6.10), $\{M_{u,v}(\,\mathrm{d}a),\ (u,v) \in \mathbb{Z}^2\}$ are i.i.d. copies of an $\alpha-$stable random measure M on $\boldsymbol{A}$ with control measure Φ, see (6.37). The random field $\{\mathfrak{X}(t,s),\ (t,s) \in \mathbb{Z}^2\}$ in (6.10) is $\alpha-$stable and a particular case of mixed stable moving-average fields introduced in [100]. In the case $\alpha = 2$, or a Gaussian limit in (6.10), the covariance function and the spectral density of this random field are given by

$$r(t,s) = \sigma^2 \sum\nolimits_{(u,v) \in \mathbb{Z}^2} \mathrm{E}\Big[g(u,v,a) g(t+u, s+v, a)\Big], \qquad (t,s) \in \mathbb{Z}^2, \tag{6.11}$$

and

$$f(x,y) = \frac{\sigma^2}{4\pi^2} \mathrm{E}|\hat{g}(x,y,a)|^2, \qquad (x,y) \in [-\pi,\pi]^2, \tag{6.12}$$

respectively, where $\hat{g}(x,y,a) = \Big(1 - \sum_{|t|+|s|=1} a(t,s) \mathrm{e}^{\mathrm{i}(xt+ys)}\Big)^{-1}$ is the Fourier transform of $g(t,s,a)$ and $\sigma^2 := \mathrm{E}\varepsilon^2$.

It is not surprising that large-scale and long memory properties of the aggregated field $\{\mathfrak{X}(t,s), (t,s) \in \mathbb{Z}^2\}$ strongly depend on the behavior of Φ near the 'unit root' $A = 1$. We assume in Sections 6.4 and 6.5 that $A \in [0,1)$ is random and has a regularly varying probability density ϕ at $a = 1$:

$$\phi(a) \sim \phi_1 (1-a)^{\beta}, \qquad a \uparrow 1, \quad \exists\, \phi_1 > 0,\ 0 < \beta < \alpha - 1,\ 1 < \alpha \le 2. \tag{6.13}$$

The case $0 < \alpha < 1$ apparently cannot lead to long-range dependence (see Chapters 3, 4 and papers [85], [86]). The long memory properties of the limit aggregated random field $\{\mathfrak{X}(t,s),\ (t,s)\in\mathbb{Z}^2\}$ strongly depend also on the model, which describes the behavior of individual fields. We investigate long memory properties of the limit aggregated field in two special cases of individual fields:

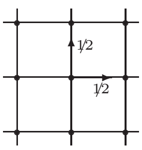
$$X(t,s) = \frac{A}{3}\Big(X(t-1,s)+X(t,s+1)+X(t,s-1)\Big)+\varepsilon(t,s), \tag{6.14}$$

$$X(t,s) = \frac{A}{4}\Big(X(t-1,s)+X(t+1,s)+X(t,s+1)+X(t,s-1)\Big)+\varepsilon(t,s). \tag{6.15}$$

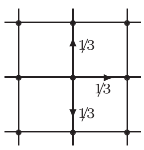
In the sequel, we refer to (6.14) and (6.15) as 3N and 4N models, N standing for 'Neighbor'. Stationary solution of the above equations in these two cases is given by (6.8), the Green function being written as

$$g(t,s,a) \;=\; \sum_{k=0}^{\infty} A^k p_k(t,s), \qquad (t,s)\in\mathbb{Z}^2, \qquad a\in\boldsymbol{A}, \tag{6.16}$$

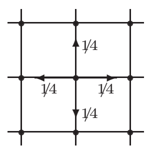
where $p_k(t,s) = \mathrm{P}(W_k = (t,s)|W_0 = (0,0))$ is the $k-$step probability of the nearest-neighbor random walk $\{W_k, k=0,1,\ldots\}$ on the lattice $\mathbb{Z}^2$ with one-step transition probabilities shown in Figure 6.1 (b), (c).



(a) 2N



(b) 3N



(c) 4N

Figure 6.1: One-step transition probabilities

Relation (6.12) implies (see also Remark 6.3.4 below) that for these two models (3N and 4N), $\alpha = 2$, and a mixing density as in (6.13), the aggregated spectral density $f(x,y)$ in (6.12) is unbounded for all $0<\beta<1$, meaning that the corresponding Gaussian random field in (6.10) has long memory. [61] obtained the asymptotics of $f(x,y)$, as $(x,y)\to(0,0)$, in an arbitrary way and showed that the 3N model satisfies spectral anisotropic long memory property (a spectral analog of the anisotropic distributional long memory property of Definition 6.2.2, page 119), in contrast to the 4N model having isotropic long memory spectrum ([60], [61]). The above mentioned works use the spectral approach which is applicable in the case $\alpha = 2$ only. Asymptotics of spectral density and covariance functions for some long-range dependent random fields was also studied in [67].

In this Chapter, we study also the asymptotics of the lattice Green function in (6.16) for models 3N and 4N, using classical probabilistic tools (the Moivre-Laplace theorem and the Hoeffding inequality for tails of binomial distribution, see [35], [36], [46]). In particular, Lemmas 6.4.2 and 6.5.1 obtain the following point-wise convergences: as $\lambda \to \infty$,

– for $t > 0$, $s \in \mathbb{R}$, $z > 0$,

$$\sqrt{\lambda} g_3\Big([\lambda t], [\sqrt{\lambda} s], 1 - \frac{z}{\lambda}\Big) \to h_3(t, s, z), \tag{6.17}$$

– for $(t, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $z > 0$,

$$g_4\Big([\lambda t], [\lambda s], 1 - \frac{z}{\lambda^2}\Big) \to h_4(t, s, z), \tag{6.18}$$

respectively, together with dominating bounds of the left-hand sides of (6.17), (6.18) (see (6.49), page 127, and (6.75), page 142). Here, g_3 and g_4 denote the Green functions of the 3N and 4N models, respectively, and the limit functions h_3 and h_4 in (6.17)-(6.18) are given by

$$h_3(t, s, z) \ := \ \frac{3}{2\sqrt{\pi t}} \mathrm{e}^{-3zt - \frac{s^2}{4t}}, \qquad h_4(t, s, z) := \frac{2}{\pi} K_0\Big(2\sqrt{z(t^2 + s^2)}\Big), \tag{6.19}$$

where K_0 is the modified Bessel function of second kind. Note that h_3 in (6.19) is the Green function of one-dimensional heat equation (modulus constant coefficients), while h_4 is the Green function of the Helmholtz equation in $\mathbb{R}^2$. Kernels h_3 and h_4 appear in the stochastic integral representation of scaling limits of models (6.14)-(6.15).

Let us summarize the remaining contents of the Chapter. Section 6.2 introduces the notions of anisotropic/isotropic distributional long memory, in terms of scaling behavior of partial sums limits (6.3), (6.5). An important feature of Definitions 6.2.2 and 6.2.3 is the requirement of dependence of increments of the limit random field in arbitrary direction. This requirement is analogous to the dependence of increments requirement in the definition of distributional long memory for processes indexed by $t \in \mathbb{Z}$, and helps to separate between isotropic and anisotropic scaling behaviors. See also Proposition 6.4.6.

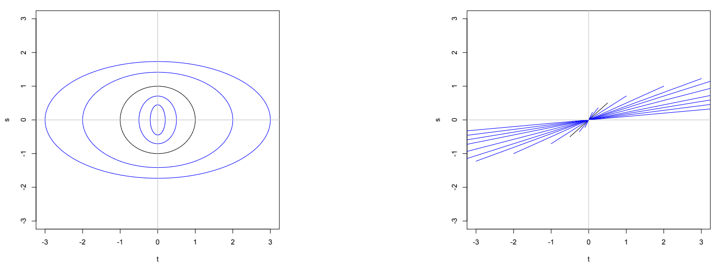
Section 6.3 discusses the existence of stationary solution in L_p, $0 < p \le 2$, of the nearest-neighbor random-coefficient equation (6.6), and the limit aggregated field in (6.9) as a mixed $\alpha-$stable moving average field of (6.10). Sections 6.4 and 6.5 are devoted to the study of scaling limits of the aggregated 3N and 4N models, respectively. The convergence in (6.5) with $B_n = n^{H_1}$, $H_1 := \frac{\frac{1}{2}+\alpha-\beta}{\alpha}$, $H_2 := 2H_1$ and the anisotropic long memory property are established in Theorem 6.4.3 for the aggregated 3N model $\{\mathfrak{X}(t,s) \equiv \mathfrak{X}_3(t,s)\}$ of (6.10). The limit random field $\{V_3(x,y),\ (x,y) \in \mathbb{R}^2_+\}$ is an $\alpha-$stable OSRF and satisfies (6.1). It is represented in (6.46) as a stochastic integral with respect to an $\alpha-$stable random measure with integrand involving the kernel h_3 in (6.19). For the same random field $\{\mathfrak{X}_3(t,s),\ (t,s) \in \mathbb{Z}^2\}$, Theorem 6.4.4 obtains a 'commensurate' scaling limit of (6.3) towards a different random field $\{V_{3\star}(x,y),\ (x,y) \in \mathbb{R}^2_+\}$ in (6.60), which is self-similar with $H_* := \frac{1+\alpha-\beta}{\alpha}$ and has independent increments in the vertical direction (see Definition 6.2.1). In the finite variance case $\alpha = 2$, Proposition 6.4.7 obtains the asymptotic decay of the covariance

$$r_3(t,s) = \mathrm{E}[\mathfrak{X}_3(0,0)\mathfrak{X}_3(t,s)]$$

as $t \to \infty$ and $s = O(\sqrt{t})$ increase 'parabolically', complementing the result in [61] on anisotropic asymptotics of the spectral density.

Section 6.5 discusses the lattice isotropic aggregated 4N model $\{\mathfrak{X}_4(t,s),\ (t,s) \in \mathbb{Z}^2\}$. We show that this field satisfies the isotropic distributional long memory property of Definition 6.2.3 and its scaling limit $\{V_4(x,y),\ (x,y) \in \mathbb{R}^2_+\}$ is an $\alpha-$stable SSRF with exponent $H = \frac{2(\alpha-\beta)}{\alpha}$, see Theorem 6.5.2 and Proposition 6.5.3. The isotropic covariance long memory property for $\{\mathfrak{X}_4(t,s),\ (t,s) \in \mathbb{Z}^2\}$ and $\alpha = 2$ is proved in Proposition 6.5.4. In the Gaussian case $\alpha = 2$, Theorem 6.5.2 and Proposition 6.5.4 agree with [60]. Section 6.6 (Appendix) contains the proofs of the technical Lemmas 6.4.2 and 6.5.1.

Notation. For $\lambda > 0$ and a $d \times d$ matrix E, $\lambda^E := \mathrm{e}^{E \log \lambda}$, where $\mathrm{e}^A = \sum_{k=0}^{\infty} A^k/k!$ is the matrix exponential. $E = \mathrm{diag}(\gamma_1, \ldots, \gamma_d)$ denotes the diagonal $d \times d$ matrix with entries $\gamma_1, \ldots, \gamma_d$ on the diagonal. Figure 6.2 shows a simple



(a) (b)

Figure 6.2: Linear scaling $x \mapsto \lambda^E x$, where $E = diag(1,\ 1/2)$

scaling example when $E = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$. Blue lines show the transformation of black one for different values of λ.

For integers t, s, $t \overset{\mathrm{mod}\,2}{=} s$ and $t \overset{\mathrm{mod}\,2}{\neq} s$ means that $t+s$ is even and odd, respectively. All equalities and inequalities between random variables are assumed to hold almost surely.

6.2 Isotropic and anisotropic long memory of random fields in $\mathbb{Z}^2$

Let $\ell = \{(x,y) \in \mathbb{R}^2 : ax + by = c\}$ be a line in $\mathbb{R}^2$. A line $\ell' = \{(x,y) \in \mathbb{R}^2 : a'x + b'y = c'\}$ is said *perpendicular to* ℓ (denoted $\ell' \perp \ell$) if $aa' + bb' = 0$. A *rectangle* is a set $K_{(u,v);(x,y)} := \{(s,t) \in \mathbb{R}^2_+ : u < s \le x,\ v < t \le y\}$; $K_{x,y} := K_{(0,0);(x,y)}$. We say that two rectangles $K = K_{(u,v);(x,y)}$ and $K' = K_{(u',v');(x',y')}$ are *separated by line* ℓ' if they lie on different sides of ℓ', in which case K and K' are necessarily disjoint: $K \cap K' = \emptyset$ (see Fig. 6.3 below).

Let $\{V(x,y)\} = \{V(x,y), (x,y) \in \mathbb{R}^2_+\}$ be a random field and $K = K_{(u,v);(x,y)} \subset \mathbb{R}^2_+$ be a rectangle. By *increment of* $\{V(x,y)\}$ *on rectangle* K we mean the difference

$$V(K) := V(x,y) - V(u,y) - V(x,v) + V(u,v).$$

Definition 6.2.1. *Let* $\{V(x,y), (x,y) \in \mathbb{R}^2_+\}$ *be a random field with* $V(x,0) = V(0,y) \equiv 0$, $x, y \ge 0$, *and* $\ell \subset \mathbb{R}^2$, *be a given line passing through the origin. We say that* $\{V(x,y)\}$ has independent increments in direction ℓ *if for any orthogonal line* $\ell' \perp \ell$ *and any two rectangles* $K, K' \subset \mathbb{R}^2_+$ *separated by* ℓ', *increments* $V(K)$ *and* $V(K')$ *are independent. Else, we say that* $\{V(x,y)\}$ *has dependent increments in direction* ℓ.

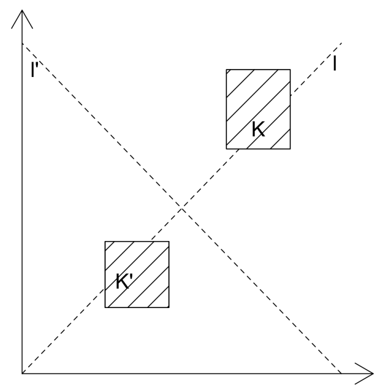


Figure 6.3: Independent increments

Definition 6.2.2. *We say that a stationary random field* $\{Y(t,s),(t,s)\in\mathbb{Z}^2\}$ *has* anisotropic distributional long memory with parameters $H_1,H_2>0, H_1\neq H_2$ *if*

$$n^{-H_1}\sum_{t=1}^{[nx]}\sum_{s=1}^{[n^{H_1/H_2}y]} Y(t,s) \quad \to_{\mathrm{fdd}} \quad V(x,y), \qquad (x,y)\in\mathbb{R}^2_+, \tag{6.20}$$

where $\{V(x,y)\}$ *is a random field having dependent increments in arbitrary direction.*

Definition 6.2.3. *We say that a stationary random field* $\{Y(t,s),(t,s)\in\mathbb{Z}^2\}$ *has* isotropic distributional long memory with parameter $H>0$ *if*

$$n^{-H}\sum_{t=1}^{[nx]}\sum_{s=1}^{[ny]} Y(t,s) \quad \to_{\mathrm{fdd}} \quad V(x,y), \qquad (x,y)\in\mathbb{R}^2_+, \tag{6.21}$$

where $\{V(x,y)\}$ *is a random field having dependent increments in arbitrary direction.*

Proposition 6.2.4. *(i) Let* $\{Y(t,s),\,(t,s)\in\mathbb{Z}^2\}$ *has anisotropic distributional long memory with parameters* $H_1\neq H_2$*. Then the limit random field* $\{V(x,y)\}$ *in* (6.20) *satisfies the self-similarity property* (6.4)*. In particular,* $\{V(x,y)\}$ *is OSRF corresponding to* $H:=H_1, E:=\mathrm{diag}(1,H_1/H_2)$*.*

(ii) Let $\{Y(t,s),\,(t,s)\in\mathbb{Z}^2\}$ *has isotropic distributional long memory with parameter* H*. Then the limit random field* $\{V(x,y)\}$ *in* (6.21) *satisfies the self-similarity property* (6.4) *with* $H_1=H_2:=H$*, i.e.,* $\{V(x,y)\}$ *is SSRF with parameter* H*.*

Proof. Fix $\lambda>0$ and let $m:=[n\lambda^{1/H_1}]$. We have

$$\begin{aligned} V(\lambda^{1/H_1}x,\lambda^{1/H_2}y) &= \text{fdd-lim}\,\frac{1}{n^{H_1}}\sum_{0<t\le x\lambda^{1/H_1}n,\,0<s\le y\lambda^{1/H_2}n^{H_1/H_2}} Y(t,s) \\ &= \text{fdd-lim}\,\frac{\lambda}{m^{H_1}}\sum_{0<t\le xm,\,0<s\le ym^{H_1/H_2}} Y(t,s) \\ &\overset{\mathrm{fdd}}{=} \lambda V(x,y). \end{aligned}$$

Proposition 6.2.4 is proved. □

6.3 The existence of the limit aggregated random field

We first discuss the solvability of the nearest-neighbor random-coefficient autoregressive equation (6.6) and the convergence of the series (6.8). The Green function of (6.6) is written as

$$g(t,s,a) \;=\; \sum_{k=0}^{\infty} a^{\star k}(t,s), \tag{6.22}$$

where $a^{\star k}(t,s)$ is the $k-$fold convolution of the function $a(t,s)$, $(t,s)\in\mathbb{Z}^2$, $a(t,s) := 0$, $|t|+|s|\neq 1$, defined recursively by

$$\begin{aligned} a^{\star 0}(t,s) &= \delta(t,s) := \begin{cases} 1, & (t,s)=(0,0), \\ 0, & (t,s)\neq(0,0), \end{cases} \\ a^{\star k}(t,s) &= \sum_{(u,v)\in\mathbb{Z}^2} a^{\star(k-1)}(u,v)a(t-u,s-v), \qquad k\geq 1. \end{aligned}$$

Note that (6.22) can be rewritten as (6.16), where

$$p_k(t,s) = \mathrm{P}(W_k=(t,s)|W_0=(0,0))$$

is the $k-$step probability of the nearest-neighbor random walk $\{W_k, k=0,1,\ldots\}$ on $\mathbb{Z}^2$ with one-step transition probabilities

$$p(t,s)\equiv p(t,s,a) \;=\; p_1(t,s) \;:=\; \begin{cases} \frac{a(t,s)}{A}, & (t,s)\in\mathbb{Z}^2,\ |t|+|s|=1 \\ 0, & (t,s)\in\mathbb{Z}^2,\ |t|+|s|\neq 1. \end{cases} \tag{6.23}$$

Generally, the $p_k(t,s)$'s depend also on $a=(a(t,s),|t|+|s|=1)\in\boldsymbol{A}$ but this dependence is suppressed for brevity. Write ε for generic $\varepsilon(t,s),(t,s)\in\mathbb{Z}^2$. Let

$$q_1 := p(0,1)+p(0,-1) = 1-p(1,0)-p(-1,0) =: 1-q_2, \qquad q := \min(q_1,q_2). \tag{6.24}$$

Note $q_i\in[0,1]$ and $q_1=0$ (respectively, $q_2=0$) means that the random walk $\{W_k\}$ is concentrated on the horizontal (respectively, vertical) axis of the lattice $\mathbb{Z}^2$.

Proposition 6.3.1. *(i) Assume there exists* $0<p\leq 2$ *such that*

$$\mathrm{E}|\varepsilon|^p<\infty \qquad \textit{and} \quad \mathrm{E}\varepsilon=0 \quad \textit{for} \quad 1\leq p\leq 2, \tag{6.25}$$

and condition (6.7)*. Then there exists a stationary solution of random-coefficient equation* (6.6) *given by* (6.8)*, where the series converges conditionally a.s. and in* L_p *for every* $a\in\boldsymbol{A}$*.*

(ii) In addition to (6.25)*, assume that* $q > 0$ *a.s. and*

$$\begin{cases} \mathrm{E}\big[\frac{1}{q^{2(p-1)}(1-A)}\big] \ < \ \infty, & \textit{if} \quad 1 < p \le 2, \\ \mathrm{E}\big[\frac{1}{(1-A)^{3-2p}}\big] \ < \ \infty, & \textit{if} \quad 0 < p \le 1. \end{cases} \tag{6.26}$$

Then the series in (6.8) *converges unconditionally in* L_p*.*

Proof. (i) Let us prove the convergence of (6.8). We shall use the following inequality. Let $0 < p \le 2$, and let $\xi_1, \xi_2, \dots$ be random variables with $\mathrm{E}|\xi_i|^p < \infty$. For $1 \le p \le 2$, assume in addition that the ξ_i's are independent and have zero mean $\mathrm{E}\xi_i = 0$. Then

$$\mathrm{E}\Big|\sum_i \xi_i\Big|^p \ \le \ 2\sum_i \mathrm{E}|\xi_i|^p. \tag{6.27}$$

Accordingly,

$$\mathrm{E}\Big[|X(t,s)|^p\Big|a\Big] \ \le \ 2\mathrm{E}|\varepsilon|^p \sum_{(u,v)\in\mathbb{Z}^2} |g(u,v,a)|^p. \tag{6.28}$$

By (6.16),

$$0 \ \le \ g(t,s,a) \ \le \ \sum_{k=|t|+|s|}^{\infty} A^k p_k(t,s) \ \le \ \frac{A^{(|t|+|s|)}}{1-A} \tag{6.29}$$

From above we obtain

$$\mathrm{E}\Big[|X(t,s)|^p\Big|a\Big] \ \le \ C \sum_{(u,v)\in\mathbb{Z}^2} A^{p(|u|+|v|)} \ \le \ C\sum_{k=0}^{\infty} A^{pk}(4k+1) \ < \ \infty, \tag{6.30}$$

proving the conditional convergence in L_p of the series in (6.8).

Let prove part (ii). According to the bound in (6.28), it suffices to prove that

$$\mathrm{E}\sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,a)|^p \ < \ \infty. \tag{6.31}$$

Let

$$\hat{a}(x,y) := \sum_{|t|+|s|=1} \mathrm{e}^{-\mathrm{i}(tx+sy)} a(t,s), \quad (x,y) \in \Pi^2, \quad \Pi := [-\pi, \pi].$$

Then

$$a(t,s) = \frac{1}{4\pi^2}\int_{\Pi^2} \mathrm{e}^{\mathrm{i}(tx+sy)} \hat{a}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$g(t,s,a) \ = \ \frac{1}{(2\pi)^2}\int_{\Pi^2} \mathrm{e}^{\mathrm{i}(tx+sy)} \frac{\mathrm{d}x\,\mathrm{d}y}{1-\hat{a}(x,y)} \ = \ \frac{1}{(2\pi)^2}\int_{\Pi^2} \mathrm{e}^{\mathrm{i}(tx+sy)} \frac{\mathrm{d}x\,\mathrm{d}y}{1-A\hat{p}(x,y)},$$

where

$$\hat{p}(x,y) := \frac{\hat{a}(x,y)}{A} = \sum_{|t|+|s|=1} \mathrm{e}^{-\,\mathrm{i}(tx+sy)} p(t,s)$$

satisfies $|\hat{p}(x,y)| \le \sum_{|t|+|s|=1} p(t,s) = 1$. From Parseval's identity,

$$\sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,a)|^2 \;=\; C\int_{\Pi^2} \frac{\mathrm{d}x\,\mathrm{d}y}{|1-A\hat{p}(x,y)|^2}. \tag{6.32}$$

We shall need the inequality

$$|1-A\hat{p}(x,y)| \;\ge\; \frac{q}{24}\Big[(1-A)+x^2+y^2\Big], \qquad (x,y)\in\Pi^2, \tag{6.33}$$

which is proved below. We have

$$\begin{aligned} 1-A\hat{p}(x,y) &= (1-A) + A\sum_{|t|+|s|=1} p(t,s)(1-\mathrm{e}^{\,\mathrm{i}(tx+sy)}) \\ &= (1-A) + A\Big[q_2(1-\cos(x)) + q_1(1-\cos(y))\Big] \\ &\quad -\; \mathrm{i}A\sum_{|t|+|s|=1} p(t,s)\sin(tx+sy) \end{aligned}$$

and therefore

$$|1-A\hat{p}(x,y)| \;\ge\; (1-A) + Aq\Big[(1-\cos(x)) + (1-\cos(y))\Big],$$

proving (6.33) (we used the inequalities $1-\cos(x) \ge x^2/8$ and $x^2 \le 10,\ \ |x|\le\pi$). From (6.32) and (6.33) we obtain

$$\begin{aligned} \sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,a)|^2 &\le \frac{C}{q^2}\int_{\Pi^2} \frac{\mathrm{d}x\,\mathrm{d}y}{\Big((1-A)+x^2+y^2\Big)^2} \\ &\le \frac{C}{q^2}\int_0^\infty \frac{r\,\mathrm{d}r}{\Big((1-A)+r^2\Big)^2} = \frac{C}{q^2(1-A)}. \end{aligned} \tag{6.34}$$

On the other hand, (6.16) immediately gives

$$\sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,a)| = \sum_{k=0}^\infty A^k \sum_{(t,s)\in\mathbb{Z}^2} p_k(t,s) = \sum_{k=0}^\infty A^k \;=\; \frac{1}{1-A}.$$

Therefore for any $1<p<2$, by Hölder's inequality,

$$\begin{aligned} \sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,a)|^p \;\le\; & \sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,a)|^{2(p-1)}|g(t,s,a)|^{2-p}\mathbf{1}(|g(t,s,a)|>1) \\ &+ \sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,a)|\mathbf{1}(|g(t,s,a)|\le 1) \end{aligned}$$

$$
\begin{aligned}
&\le \Big(\sum_{(t,s)\in\mathbb{Z}^2}|g(t,s,a)|^2\Big)^{p-1}\Big(\sum_{(t,s)\in\mathbb{Z}^2}|g(t,s,a)|\mathbf{1}(|g(t,s,a)|>1)\Big)^{2-p}\\
&\quad+\sum_{(t,s)\in\mathbb{Z}^2}|g(t,s,a)|,
\end{aligned}
$$

Therefore, using (6.34),

$$
\begin{aligned}
\sum_{(t,s)\in\mathbb{Z}^2}|g(t,s,a)|^p &\le C\Big(\frac{1}{q^2(1-A)}\Big)^{p-1}\Big(\sum_{(t,s)\in\mathbb{Z}^2}|g(t,s,a)|\Big)^{2-p}+\sum_{(t,s)\in\mathbb{Z}^2}|g(t,s,a)|\\
&\le \frac{C}{q^{2(p-1)}(1-A)}+\frac{C}{1-A}\le\frac{C}{q^{2(p-1)}(1-A)},
\end{aligned}
$$

proving (6.31) and the unconditional convergence of (6.8) under the first condition in (6.26).

Next, consider the case $0<p\le 1$. Using (6.16) and Hölder's inequality, we obtain

$$
\begin{aligned}
\sum_{(t,s)\in\mathbb{Z}^2}|g(t,s,a)|^p &\le \sum_{k=0}^{\infty}A^{kp}\sum_{|t|+|s|\le k}p_k^p(t,s)\\
&\le \sum_{k=0}^{\infty}A^{kp}\Big\{\sum_{|t|+|s|\le k}p_k(t,s)\Big\}^p\Big\{\sum_{|t|+|s|\le k}1\Big\}^{1-p}\\
&\le C\sum_{k=0}^{\infty}A^{kp}k^{2(1-p)}\le\frac{C}{(1-A^p)^{3-2p}}\le\frac{C}{(1-A)^{3-2p}}.
\end{aligned}
$$

This completes the proof of part (ii) and the proposition. □

In this chapter, we also use the notation $\varepsilon\in D(\alpha)$, $0<\alpha\le 2$ (see Definition 3.2.1, page 42), which means that innovations belong to the domain of normal attraction of an α-stable law.

Remark 6.3.2. Condition $\varepsilon\in D(\alpha)$ implies that the r.v. ε belongs to the domain of normal attraction of an $\alpha-$stable law; in other words,

$$
N^{-1/\alpha}\sum_{i=1}^{N}\varepsilon_i \ \to_{\mathrm{d}}\ Z, \tag{6.35}
$$

where Z is an $\alpha-$stable r.v., see ([35], pp.574-581). The characteristic function of the r.v. Z in (6.35) is given by

$$
\mathrm{E}e^{\mathrm{i}\theta Z} = \mathrm{e}^{-|\theta|^\alpha\omega(\theta)},\quad \theta\in\mathbb{R},
$$

where

$$\omega(\theta) := \begin{cases} \frac{\Gamma(2-\alpha)}{1-\alpha}\Big((c_1+c_2)\cos\left(\frac{\pi\alpha}{2}\right) - \mathrm{i}(c_1-c_2)\mathrm{sign}(\theta)\sin\left(\frac{\pi\alpha}{2}\right)\Big), & \alpha \neq 1, 2, \\ (c_1+c_2)\frac{\pi}{2}, & \alpha = 1, \\ \frac{\sigma^2}{2}, & \alpha = 2. \end{cases} \tag{6.36}$$

Introduce independently scattered $\alpha-$stable random measure M on $\mathbb{Z}^2 \times \boldsymbol{A}$ with characteristic functional

$$\mathrm{E}\exp\Big\{\mathrm{i}\sum_{(t,s)\in\mathbb{Z}^2}\theta_{t,s}M_{t,s}(A_s)\Big\} = \exp\Big\{-\sum_{(t,s)\in\mathbb{Z}^2}|\theta_{t,s}|^\alpha\omega(\theta_{t,s})\Phi(A_{t,s})\Big\}, \tag{6.37}$$

where $\theta_{t,s} \in \mathbb{R}$ and $A_{t,s} \subset \boldsymbol{A}$ are arbitrary Borel sets.

Proposition 6.3.3. *Let $\varepsilon \in D(\alpha)$, $0 < \alpha \le 2$. Assume that the mixing distribution satisfies condition* (6.26) *of Proposition 6.3.1 (ii) with some $0 < p \le 2$ and such that*

$$\begin{cases} p > \alpha, & \text{if} \quad 1 < \alpha < 2, \\ p < \alpha, & \text{if} \quad 0 < \alpha < 1, \\ p = 2, & \text{if} \quad \alpha = 2. \end{cases} \tag{6.38}$$

In the case $\alpha = 1$ we assume that

$$\mathrm{E}\frac{1}{(1-A)^p} < \infty \qquad \textit{for some} \quad p > 1. \tag{6.39}$$

Then the limit aggregated random field in (6.9) *exists and has the stochastic integral representation of* (6.10).

Proof. Let $T \subset \mathbb{Z}^2$ be a finite set, $\theta_{t,s} \in \mathbb{R}$, $(t,s) \in T$. It suffices to prove that $S_N \to_{\mathrm{d}} S$, where $S := \sum_{(t,s)\in T}\theta_{t,s}\mathfrak{X}(t,s)$ is a $\alpha-$stable r.v. with characteristic function

$$\mathrm{E}e^{\mathrm{i}wS} = \exp\Big\{-|w|^\alpha\sum_{(u,v)\in\mathbb{Z}^2}\mathrm{E}\Big[\Big|G(u,v,a)\Big|^\alpha\omega\Big(wG(u,v,a)\Big)\Big]\Big\},$$

$$G(u,v,a) := \sum_{(t,s)\in T}\theta_{t,s}g(t-u,s-v,a),$$

and $S_N = N^{-1/\alpha}\sum_{i=1}^N U_i$ is a sum of i.i.d. r.v.'s with common distribution

$$U := \sum_{(t,s)\in T}\theta_{t,s}X(t,s) = \sum_{(u,v)\in\mathbb{Z}^2}G(u,v,a)\varepsilon(u,v).$$

It suffices to prove that r.v. U belongs to the domain of attraction of r.v. S (in the sense of (6.35)); in other words, that

$$\mathrm{E}U^2 = \mathrm{E}S^2 < \infty \qquad \text{for } \alpha = 2, \tag{6.40}$$

and, for $0 < \alpha < 2$,

$$\begin{aligned} \lim_{x\to\infty} x^\alpha \mathrm{P}(U > x) &= \sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}\Big[\Big|G(u,v,a)\Big|^\alpha \Big\{c_1\mathbf{1}(G>0) + c_2\mathbf{1}(G<0)\Big\}\Big], \\ \lim_{x\to-\infty} |x|^\alpha \mathrm{P}(U \le x) &= \sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}\Big[\Big|G(u,v,a)\Big|^\alpha \Big\{c_1\mathbf{1}(G<0) + c_2\mathbf{1}(G>0)\Big\}\Big], \end{aligned} \tag{6.41}$$

where, $\mathbf{1}(G > 0) \equiv \mathbf{1}(G(u,v,a) > 0)$ and $\mathbf{1}(G < 0) \equiv \mathbf{1}(G(u,v,a) < 0)$. Here, (6.40) follows from definitions of U and S. To prove (6.41), we use ([48], Theorem 3.1). Accordingly, it suffices to check that there exists $\epsilon > 0$ such that

$$\begin{aligned} &\sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}\Big|G(u,v,a)\Big|^{\alpha+\epsilon} < \infty \text{ and } \sum_{s\in\mathbb{Z}^2} \mathrm{E}\Big|G(u,v,a)\Big|^{\alpha-\epsilon} < \infty, \text{ for } \alpha \in (0,2)\setminus\{1\} \\ &\mathrm{E}\Big(\sum_{(u,v)\in\mathbb{Z}^2} \Big|G(u,v,a)\Big|^{\alpha-\epsilon}\Big)^{\frac{\alpha+\epsilon}{\alpha-\epsilon}} < \infty, \text{ for } \quad \alpha = 1. \end{aligned} \tag{6.42}$$

Since $T \subset \mathbb{Z}^2$ is a finite set, it suffices to show (6.42) with $G(u,v,a)$ replaced by $g(u,v,a)$. Let $1 < \alpha < 2$ and $p = \alpha + \epsilon > \alpha$ in (6.38). Then

$$\sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}\Big|g(u,v,a)\Big|^{\alpha+\epsilon} \le C\mathrm{E}[q^{-2(\alpha+\epsilon-1)}(1-A)^{-1}] < \infty$$

follows from (6.35) and (6.38). Similarly, if $1 < \alpha < 2$ and $1 < p = \alpha - \epsilon \in (1, \alpha)$, then

$$\sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}\Big|g(u,v,a)\Big|^{\alpha-\epsilon} \le C\mathrm{E}[q^{-2(\alpha-\epsilon-1)}(1-A)^{-1}] \le C\mathrm{E}[q^{-2(\alpha+\epsilon-1)}(1-A)^{-1}] < \infty,$$

thus proving (6.42) for $1 < \alpha < 2$. In the case $0 < \alpha < 1$, relations (6.42) immediately follow from (6.35) and (6.38) with $p = \alpha \pm \epsilon \in (0, 1)$. Finally, for $\alpha = 1$, (6.42) follows from (6.35) and (6.39). □

Remark 6.3.4. For the 3N and 4N models in (6.14) and (6.15), we have $q = 1/3$ and $q = 1/2$, respectively. Hence, for $1 < \alpha \le 2$, condition (6.38) of Proposition 6.3.3 for the existence of the aggregated random field $\{\mathfrak{X}(t,s),\ (t,s) \in \mathbb{Z}^2\}$ in (6.10) reduces to

$$\mathrm{E}(1-A)^{-1} \;=\; \int_{[0,1)} (1-a)^{-1}\Phi(\,\mathrm{d}a) \;<\; \infty. \tag{6.43}$$

For regularly varying mixing density as in (6.13), condition (6.43) is equivalent to $\beta > 0$. In the Gaussian case $\alpha = 2$ the spectral density f of (6.9) is given in (6.12). For the 3N and 4N models we have that

$$f(x,y) = \frac{\sigma^2}{(2\pi)^2} \int_{[0,1)} \frac{1}{|1 - a\hat{p}(x,y)|^2} \Phi(\,\mathrm{d}a)$$

and hence $f(x,y)$ is bounded at the origin if and only if

$$f(0,0) = (\sigma/2\pi)^2 \mathrm{E}(1-A)^{-2} \; < \; \infty. \tag{6.44}$$

In particular, for Φ as in (6.13) and any $0 < \beta \le 1$, the spectral density f of the aggregated random field is unbounded.

6.4 Aggregation of the 3N model

In this section we prove the anisotropic long memory properties, in the sense of Definition 6.2.2 (page 119), of the aggregated 3N model given by

$$\mathfrak{X}_3(t,s) \;\; = \sum_{(u,v)\in\mathbb{Z}^2} \int_0^1 g_3(t-u, s-v, a) M_{u,v}(\,\mathrm{d}a), \qquad (t,s) \in \mathbb{Z}^2, \tag{6.45}$$

where $\{M_{u,v}(\,\mathrm{d}a),\, (u,v) \in \mathbb{Z}^2\}$ are i.i.d. copies of $\alpha-$stable random measure M on $[0,1)$ with control measure $\Phi(\,\mathrm{d}a) = \mathrm{P}(A \in \,\mathrm{d}a)$ and the characteristic function $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta M(B)} = \mathrm{e}^{-|\theta|^\alpha \omega(\theta)\Phi(B)}$, $B \subset [0,1)$, see (6.36), (6.37); and where $g_3(t,s,a)$ is the Green function of the random walk $\{W_k\}$ on $\mathbb{Z}^2$ with one-step transition probabilities shown in Figure 6.1 (b). For $1 < \alpha \le 2$, (6.45) is well-defined, provided the mixing distribution satisfies (6.43).

Introduce a random field $\{V_3(x,y), (x,y) \in \mathbb{R}^2_+\}$ as a stochastic integral

$$V_3(x,y) := \int_{\mathbb{R}^2\times\mathbb{R}_+} \mathcal{M}(\,\mathrm{d}u,\,\mathrm{d}v,\,\mathrm{d}z) \int_0^x \int_0^y h_3(t-u, s-v, z)\,\mathrm{d}t\,\mathrm{d}s, \tag{6.46}$$

where $\mathcal{M}$ is an $\alpha-$stable random measure on $\mathbb{R}^2 \times \mathbb{R}_+$ with the control measure $\mathrm{d}\mu(u,v,z) := \phi_1\, z^\beta\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}z$ and characteristic function $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta\mathcal{M}(B)} = \mathrm{e}^{-|\theta|^\alpha\omega(\theta)\mu(B)}$, where $B \subset \mathbb{R}^2 \times \mathbb{R}_+$ is a measurable set with $\mu(B) < \infty$. As shown in the proof of Theorem 6.4.3 below, the stochastic integral in (6.46) is well-defined. The random field in (6.46) has $\alpha-$stable finite-dimensional distributions and stationary increments in the sense that for any $(u,v) \in \mathbb{R}^2_+$

$$\{V_3(x,y)\} \overset{\mathrm{fdd}}{=} \{V_3(u+x, v+y) - V_3(u, v+y) - V_3(u+x, v) + V_3(u,v)\}. \tag{6.47}$$

Moreover, (6.46) is OSRF and satisfies (6.4), viz.,

$$\{V_3(\lambda x, \sqrt{\lambda} y)\} \stackrel{\mathrm{fdd}}{=} \{\lambda^H V_3(x,y)\}, \tag{6.48}$$

with H given in (6.51). Property (6.48) is immediate from the scaling properties

$$h_3(\lambda u, \sqrt{\lambda} v, \lambda^{-1} z) = \lambda^{-1/2} h_3(u,v,z)$$

and

$$\{\mathcal{M}(\,\mathrm{d}\lambda u,\,\mathrm{d}\sqrt{\lambda} v,\,\mathrm{d}\lambda^{-1} z)\} \stackrel{\mathrm{fdd}}{=} \{\lambda^{\frac{\frac{1}{2}-\beta}{\alpha}} \mathcal{M}(\,\mathrm{d}u,\,\mathrm{d}v,\,\mathrm{d}z)\},$$

the last property being a consequence of the scaling property

$$\mu(\,\mathrm{d}\lambda u,\,\mathrm{d}\sqrt{\lambda} v,\,\mathrm{d}\lambda^{-1} z) = \lambda^{\frac{1}{2}-\beta} \mu(\,\mathrm{d}u,\,\mathrm{d}v,\,\mathrm{d}z)$$

of the control measure μ.

Remark 6.4.1. The random field (6.46) is different from the class of $\alpha-$stable OSRFs defined in ([16], (3.1)) because the latter fields satisfy a different stationary increment property, see ([16], (3.5)). Moreover, (6.46) have a mixed moving average representation in contrast to the moving average representation in ([16], (3.1)).

The main result of this Section is Theorem 6.4.3. Its proof is based on the asymptotics of the Green function g_3 in Lemma 6.4.2, below. The proof of Lemma 6.4.2 is given in Section 6.6, page 154.

Lemma 6.4.2. *For any* $(t,s,z) \in (0,\infty) \times \mathbb{R} \times (0,\infty)$ *the point-wise convergence in* (6.17) *holds. This convergence is uniform on any relatively compact set*

$$\{\epsilon < t < 1/\epsilon,\, \epsilon < |s| < 1/\epsilon,\, \epsilon < z < 1/\epsilon\} \subset (0,\infty) \times \mathbb{R} \times (0,\infty), \quad \epsilon > 0.$$

Moreover, there exist constants $C, c > 0$ *such that for all sufficiently large* λ *and any* (t,s,z), $t > 0$, $s \in \mathbb{R}$, $0 < z < \lambda$ *the following inequality holds:*

$$\sqrt{\lambda} g_3\Big([\lambda t], [\sqrt{\lambda} s], 1 - \frac{z}{\lambda}\Big) \;<\; C\Big(\bar{h}_3(t,s,z) + \sqrt{\lambda} \mathrm{e}^{-zt - c(\lambda t)^{1/3} - c(\sqrt{\lambda}|s|)^{1/2}}\Big), \tag{6.49}$$

where $\bar{h}_3(t,s,z) := \frac{1}{\sqrt{t}} \mathrm{e}^{-zt - \frac{s^2}{16t}}$, $(t,s,z) \in (0,\infty) \times \mathbb{R} \times (0,\infty)$.

Theorem 6.4.3. *Assume that the mixing density* ϕ *is bounded on* $[0,1)$ *and satisfies* (6.13), *where*

$$0 < \beta < \alpha - 1, \qquad 1 < \alpha \le 2. \tag{6.50}$$

Let $\{\mathfrak{X}_3(t,s),\,(t,s)\in\mathbb{Z}^2\}$ *be the aggregated random field in* (6.45)*. Then*

$$n^{-H}\sum_{t=1}^{[nx]}\sum_{s=1}^{[\sqrt{n}y]}\mathfrak{X}_3(t,s)\to_{\mathrm{fdd}} V_3(x,y),\qquad x,y>0,\qquad H:=\frac{\frac{1}{2}+\alpha-\beta}{\alpha}. \tag{6.51}$$

Proof. Write $S_n(x,y)$ for the l.h.s. of (6.51). We prove the convergence of one-dimensional distributions in (6.51) at $x=y=1$ only, since the general case of (6.51) is completely analogous. We have

$$\begin{aligned}
\mathrm{E}e^{\mathrm{i}\theta V_3(1,1)} &= \exp\Big\{-|\theta|^\alpha\int_{\mathbb{R}^2\times\mathbb{R}_+}(G(u,v,z))^\alpha\omega\Big(\theta G(u,v,z)\Big)\,\mathrm{d}\mu(u,v,z)\Big\},\\
\mathrm{E}e^{\mathrm{i}\theta S_n(1,1)} &= \exp\Big\{-|\theta|^\alpha n^{-H\alpha}\sum_{(u,v)\in\mathbb{Z}^2}\mathrm{E}\Big[\mathcal{G}_n^\alpha(u,v,A)\omega\Big(\theta\mathcal{G}_n(u,v,A)\Big)\Big]\Big\},\qquad \theta\in\mathbb{R},
\end{aligned}$$

where

$$\begin{aligned}
G(u,v,z) &:= \int_0^1\int_0^1 h_3(t-u,s-v,z)\,\mathrm{d}t\,\mathrm{d}s, \qquad (6.52)\\
\mathcal{G}_n(u,v,a) &:= \sum_{1\le t\le n,\,1\le s\le[\sqrt{n}]} g_3(t-u,s-v,a).
\end{aligned}$$

Since $\omega(\theta)$ in (6.36) depends on the sign of θ only and $G\ge 0$, $\mathcal{G}_n\ge 0$, in the rest of the proof we can assume $\omega(\cdot)\equiv 1$ without loss of generality, c.f. (Chapter 4, proof of Theorem 4.3.1, page 68). Hence, it suffices to show

$$J_n := n^{-H\alpha}\sum_{(u,v)\in\mathbb{Z}^2}\mathrm{E}(\mathcal{G}_n(u,v,A))^\alpha \;\to\; \int_{\mathbb{R}^2\times\mathbb{R}_+}(G(u,v,z))^\alpha\,\mathrm{d}\mu \;=:\; J. \tag{6.53}$$

Let us first check that $J<\infty$, i.e., that $V_3(1,1)$ is well-defined as a stochastic integral with respect to $\mathcal{M}$. We have

$$\begin{aligned}
J &= C\int_{\mathbb{R}^2\times\mathbb{R}_+}\Big(\int_0^1\int_0^1\frac{1}{\sqrt{(t-u)}}\,e^{-(s-v)^2/4(t-u)}e^{-3z(t-u)}\,\mathbf{1}(u<t)\,\mathrm{d}t\,\mathrm{d}s\Big)^\alpha z^\beta\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}z\\
&= C(J_1+J_2),
\end{aligned}$$

where, by Minkowski's inequality,

$$
\begin{aligned}
J_1 &:= \int_0^\infty \mathrm{d}u \int_{\mathbb{R}} \mathrm{d}v \int_0^\infty z^\beta \,\mathrm{d}z \Big(\int_0^1 \int_0^1 \frac{1}{\sqrt{(t+u)}} \,\mathrm{e}^{-(s-v)^2/4(t+u)} \mathrm{e}^{-3z(t+u)} \,\mathrm{d}t \,\mathrm{d}s \Big)^\alpha \\
&\le \Big\{ \int_0^1 \int_0^1 \mathrm{d}t \,\mathrm{d}s \Big(\int_0^\infty \mathrm{d}u \int_{\mathbb{R}} \mathrm{d}v \int_0^\infty z^\beta \,\mathrm{d}z \frac{1}{(t+u)^{\alpha/2}} \,\mathrm{e}^{-\alpha(s-v)^2/4(t+u)} \mathrm{e}^{-3\alpha z(t+u)} \Big)^{1/\alpha} \Big\}^\alpha \\
&= C \Big\{ \int_0^1 \mathrm{d}t \Big(\int_0^\infty \mathrm{d}u \int_0^\infty z^\beta \,\mathrm{d}z \frac{1}{(t+u)^{\frac{\alpha-1}{2}}} \,\mathrm{e}^{-3\alpha z(t+u)} \Big)^{1/\alpha} \Big\}^\alpha \\
&= C \Big\{ \int_0^1 \mathrm{d}t \Big(\int_0^\infty \frac{\mathrm{d}u}{(t+u)^{\frac{\alpha-1}{2}+1+\beta}} \Big)^{1/\alpha} \Big\}^\alpha \\
&= C \Big\{ \int_0^1 \mathrm{d}t \Big(\frac{1}{t^{\frac{\alpha-1}{2}+\beta}} \Big)^{1/\alpha} \Big\}^\alpha \ < \ \infty
\end{aligned}
$$

since $\frac{\frac{\alpha-1}{2}+\beta}{\alpha} < 1$ holds because of (6.50) and $\alpha < 3$. Next,

$$
\begin{aligned}
J_2 &:= \int_0^1 \mathrm{d}y \int_{\mathbb{R}} \mathrm{d}v \int_0^\infty z^\beta \,\mathrm{d}z \Big\{ \int_0^1 \mathrm{d}s \int_0^y \frac{1}{\sqrt{x}} \,\mathrm{e}^{-(s-v)^2/4x} \mathrm{e}^{-3zx} \,\mathrm{d}x \Big\}^\alpha \\
&= \int_0^1 \mathrm{d}y \int_{|v|\le 2} \mathrm{d}v \int_0^\infty z^\beta \,\mathrm{d}z \Big\{ \cdots \Big\}^\alpha + \int_0^1 \mathrm{d}y \int_{|v|>2} \mathrm{d}v \int_0^\infty z^\beta \,\mathrm{d}z \Big\{ \cdots \Big\}^\alpha \\
&=: J_{21} + J_{22}.
\end{aligned}
$$

Here,

$$
J_{21} \ \le \ C \int_0^\infty z^\beta \,\mathrm{d}z \Big\{ \int_0^1 \mathrm{e}^{-3zx} \,\mathrm{d}x \Big\}^\alpha \ = \ C \int_0^\infty z^{\beta-\alpha} \Big(1 - \mathrm{e}^{-z}\Big)^\alpha \mathrm{d}z \ < \ \infty
$$

since $\alpha > 1 + \beta$. Finally, since $(s-v)^2 \ge v^2/4$ for $|s| < 1$, $|v| > 2$, so

$$
\int_0^1 \mathrm{e}^{-(s-v)^2/4x} \,\mathrm{d}s \ \le \ \mathrm{e}^{-v^2/16x} \ \le \ C \frac{x}{v^2}, \quad |v| > 2, \ 0 < x < 1,
$$

and

$$
\begin{aligned}
J_{22} &\le C \int_{|v|>2} |v|^{-2\alpha} \,\mathrm{d}v \int_0^\infty z^\beta \,\mathrm{d}z \Big\{ \int_0^1 x^{1/2} \,\mathrm{e}^{-3zx} \,\mathrm{d}x \Big\}^\alpha \\
&\le C \Big\{ \int_0^1 x^{1/2} \,\mathrm{d}x \Big(\int_0^\infty \mathrm{e}^{-3\alpha zx} z^\beta \,\mathrm{d}z \Big)^{1/\alpha} \Big\}^\alpha \ = \ C \Big\{ \int_0^1 \frac{x^{1/2} \,\mathrm{d}x}{x^{\frac{1+\beta}{\alpha}}} \Big\}^\alpha \ < \ \infty,
\end{aligned}
$$

since $-\frac{1}{2} + \frac{1+\beta}{\alpha} < 1$. This proves $J < \infty$, or $G \in L^\alpha(\mu)$.

Let us prove the convergence in (6.53). For notational simplicity we can assume $\phi(a) = (1-a)^\beta$, c.f. (Chapter 4, proof of Theorem 4.3.1, page 68). Then

$$
J_n \ = \ \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_n(u,v,z))^\alpha \,\mathrm{d}\mu(u,v,z),
$$

where

$$G_n(u,v,z) \ := \ \int_{(0,1]^2} \sqrt{n} g_3\Big([nt]-[nu],[\sqrt{n}s]-[\sqrt{n}v],1-\frac{z}{n}\Big)\mathbf{1}(0<z<n)\,\mathrm{d}t\,\mathrm{d}s.$$

Let

$$W_\epsilon := \{(u,v,z)\in\mathbb{R}^2\times\mathbb{R}_+ : |u|,|v|<1/\epsilon, \epsilon<z<1/\epsilon\}.$$

We claim that

$$\lim_{n\to\infty}\sup_{(u,v,z)\in W_\epsilon}|G_n(u,v,z)-G(u,v,z)| \ = \ 0, \qquad \forall\,\epsilon>0. \tag{6.54}$$

To show (6.54), for given $\epsilon_1>0$ split

$$G_n(u,v,z)-G(u,v,z) \ = \ \sum_{j=1}^{3}\Gamma_{nj}(u,v,z),$$

where, for $0<z<n$,

$$\begin{aligned}
\Gamma_{n1}(u,v,z) \ &:= \ \int_{(0,1]^2\cap D(\epsilon_1)}\Big\{\sqrt{n}g_3\Big([nt]-[nu],[\sqrt{n}s]-[\sqrt{n}v],1-\frac{z}{n}\Big)- \\
&\qquad - h_3(t-u,s-v,z)\Big\}\,\mathrm{d}t\,\mathrm{d}s,\\
\Gamma_{n2}(u,v,z) \ &:= \ \int_{(0,1]^2\cap D(\epsilon_1)^c}\sqrt{n}g_3\Big([nt]-[nu],[\sqrt{n}s]-[\sqrt{n}v],1-\frac{z}{n}\Big)\,\mathrm{d}t\,\mathrm{d}s,\\
\Gamma_{n3}(u,v,z) \ &:= \ -\int_{(0,1]^2\cap D(\epsilon_1)^c} h_3(t-u,s-v,z)\,\mathrm{d}t\,\mathrm{d}s,
\end{aligned}$$

and where the sets $D(\epsilon), D(\epsilon)^c$ (depending on u,v) are defined by

$$\begin{aligned}
D(\epsilon) \ &:= \ \{(t,s)\in(0,1]^2 : t-u>\epsilon,\ |s-v|>\epsilon\},\\
D(\epsilon)^c \ &:= \ (0,1]^2\setminus D(\epsilon).
\end{aligned}$$

To show (6.54), it suffices to verify that for any $\epsilon>0$, $\delta>0$ there exists $\epsilon_1>0$, $n_1\ge1$ such that

$$\lim_{n\to\infty}\sup_{(u,v,z)\in W_\epsilon}\Gamma_{n1}(u,v,z) \ = \ 0, \tag{6.55}$$

$$\sup_{(u,v,z)\in W_\epsilon}|\Gamma_{ni}(u,v,z)| \ < \ \delta, \qquad i=2,3, \quad \forall n\ge n_1. \tag{6.56}$$

Relation (6.55) follows from Lemma 6.4.2. Next,

$$|\Gamma_{n3}(u,v,z)|\le C\int_0^{\epsilon_1}t^{-1/2}\,\mathrm{d}t + C\int_{\epsilon_1}^1 t^{-1/2}\,\mathrm{d}t\int_{|s|<\epsilon_1}\mathrm{d}s = O(\sqrt{\epsilon_1}),$$

implying (6.56) for $i = 3$ with $\epsilon_1 = C\delta^2$. Finally, using (6.49) we obtain

$$|\Gamma_{n2}(u, v, z)| \le C\sqrt{\epsilon_1} + C\sqrt{n}\int_0^1 \mathrm{e}^{-c(nt)^{1/3}}\,\mathrm{d}t \le C\sqrt{\epsilon_1} + C/\sqrt{n} < \delta$$

provided $\sqrt{\epsilon_1} < \delta/(2C)$, $n > n_1 = (2C/\delta)^2$ hold. This proves (6.56) for $i = 2$ and hence (6.54), too.

Let

$$G'_n(u, v, z) \;:=\; \sqrt{n}\,\mathbf{1}(0 < z < n)\int_{(0,1]^2} \mathrm{e}^{-z(t-u)-c(n(t-u))^{1/3}-c(\sqrt{n}|s-v|)^{1/2}}\mathbf{1}(t > u)\,\mathrm{d}t\,\mathrm{d}s,$$

where $c > 0$ is the same as in (6.49). Let us show that

$$J'_n \;:=\; \int_{\mathbb{R}^2\times\mathbb{R}_+} (G'_n(u, v, z))^\alpha\,\mathrm{d}\mu \;=\; o(1). \tag{6.57}$$

Split $J'_n = \sum_{i=1}^3 I_{ni}$, where

$$I_{n1} := \int_{(-\infty,0]\times\mathbb{R}_+\times\mathbb{R}_+} (G'_n)^\alpha\,\mathrm{d}\mu, \quad I_{n2} := \int_{(0,1]\times[-2,2]\times\mathbb{R}_+} (G'_n)^\alpha\,\mathrm{d}\mu,$$
$$I_{n3} := \int_{(0,1]\times[-2,2]^c\times\mathbb{R}_+} (G'_n)^\alpha\,\mathrm{d}\mu,$$

$[-2, 2]^c := \mathbb{R}\backslash[-2, 2]$. Using the fact that $\int_{\mathbb{R}} \mathrm{e}^{-cn^{1/4}|s-v|^{1/2}}\,\mathrm{d}v = C/\sqrt{n}$ and Minkowski's inequality,

$$\begin{aligned} I_{n1} &\le Cn^{\alpha/2}\Big\{\int_{(0,1]^2}\mathrm{d}t\,\mathrm{d}s\Big(\int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+}\mathrm{e}^{-\alpha z(t+u)-c\alpha(n(t+u))^{1/3}-c\alpha(\sqrt{n}|s-v|)^{1/2}}z^\beta\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}z\Big)^{1/\alpha}\Big\}^\alpha \\ &\le Cn^{\frac{\alpha-1}{2}}\Big\{\int_0^1\mathrm{d}t\Big(\int_0^\infty \mathrm{e}^{-c\alpha(n(t+u))^{1/3}}\frac{\mathrm{d}u}{(t+u)^{1+\beta}}\Big)^{1/\alpha}\Big\}^\alpha \\ &\le Cn^{-(\frac{\alpha+1}{2}-\beta)}I, \end{aligned}$$

where $\frac{\alpha+1}{2} - \beta > 0$ and

$$I := \Big\{\int_0^\infty \mathrm{d}t\Big(\int_0^\infty \mathrm{e}^{-c\alpha(t+u)^{1/3}}(t+u)^{-1-\beta}\,\mathrm{d}u\Big)^{1/\alpha}\Big\}^\alpha < \infty.$$

Next,

$$\begin{aligned} I_{n2} \;&\le\; Cn^{\alpha/2}\int_0^\infty z^\beta\,\mathrm{d}z\Big\{\int_{(0,4]^2}\mathrm{e}^{-zt-c(nt)^{1/3}-c(\sqrt{n}|s|)^{1/2}}\,\mathrm{d}t\,\mathrm{d}s\Big\}^\alpha \\ &\le\; C\Big\{\int_0^4 \mathrm{e}^{-c(nt)^{1/3}}\,\mathrm{d}t\Big(\int_0^\infty \mathrm{e}^{-\alpha zt}z^\beta\,\mathrm{d}z\Big)^{1/\alpha}\Big\}^\alpha \\ &\le\; C\Big\{\int_0^\infty \mathrm{e}^{-c(nt)^{1/3}}t^{-\frac{1+\beta}{\alpha}}\,\mathrm{d}t\Big\}^\alpha \;\le\; Cn^{-(\alpha-1-\beta)} \;=\; o(1). \end{aligned}$$

Finally, using $\mathrm{e}^{-c(\sqrt{n}|s-v|)^{1/2}} \le \mathrm{e}^{-(c/2)(\sqrt{n}|v|)^{1/2}}$ for $|v| \ge 2$, $|s| \le 1$, it easily follows $I_{n3} = O(\mathrm{e}^{-c'n^{1/4}}) = o(1)$, $\exists\, c' > 0$, thus completing the proof of (6.57).

With (6.54) and (6.57) in mind, write

$$\begin{aligned} |J_n - J| &\le \int_{W_\epsilon} |G_n^\alpha - G^\alpha| \,\mathrm{d}\mu + \int_{W_\epsilon^c} |G_n|^\alpha \,\mathrm{d}\mu + \int_{W_\epsilon^c} |G|^\alpha \,\mathrm{d}\mu \qquad (6.58)\\ &\le \int_{W_\epsilon} |G_n^\alpha - G^\alpha| \,\mathrm{d}\mu + C\int_{\mathbb{R}^2\times\mathbb{R}_+} |G_n'|^\alpha \,\mathrm{d}\mu + C\int_{W_\epsilon^c} |\bar G|^\alpha \,\mathrm{d}\mu + \int_{W_\epsilon^c} |G|^\alpha \,\mathrm{d}\mu, \end{aligned}$$

where $\bar G(u,v,z) := \int_0^1\int_0^1 \bar h_3(t-u, s-v, z)\,\mathrm{d}t\,\mathrm{d}s$, $W_\epsilon^c := \mathbb{R}^2 \times \mathbb{R}_+ \setminus W_\epsilon$. Since $G, \bar G \in L^\alpha(\mu)$, the third and fourth terms on the r.h.s. of (6.58) can be made arbitrary small by choosing $\epsilon > 0$ small enough. Next, for a given $\epsilon > 0$, the first term on the r.h.s. of (6.58) vanishes in view of (6.54), and the second term tends to zero, see (6.57). This proves (6.53), thus concluding the proof Theorem 6.4.3. □

The next Theorem 6.4.4 shows that when partial sums of $\{\mathfrak{X}_3(t,s),\ (t,s) \in \mathbb{Z}^2\}$ in (6.45) are taken on 'commensurate' rectangles (the number of summands in the horizontal and the vertical directions grow at the same rate $O(n)$) the limit field is different.

Theorem 6.4.4. *Assume the conditions and notation of Theorem 6.4.3. Then*

$$n^{-H_*} \sum_{t=1}^{[nx]} \sum_{s=1}^{[ny]} \mathfrak{X}_3(t,s) \to_{\mathrm{fdd}} V_{3\star}(x,y), \qquad x,y > 0, \qquad H_* := \frac{1+\alpha-\beta}{\alpha} \qquad (6.59)$$

where

$$\begin{aligned} V_{3\star}(x,y) &:= \int_{\mathbb{R}^2\times\mathbb{R}_+} \mathcal{M}(\,\mathrm{d}u, \mathrm{d}v, \mathrm{d}z)\mathbf{1}(0 < v \le y)\int_0^x h_{3\star}(t-u, z)\,\mathrm{d}t, \qquad (6.60)\\ h_{3\star}(u,z) &:= \int_{\mathbb{R}} h_3(u,v,z)\,\mathrm{d}v \;=\; 12\mathrm{e}^{-3uz}\mathbf{1}(u > 0), \qquad (6.61) \end{aligned}$$

where $\mathcal{M}$ *is the same as in Theorem 6.4.3.*

Proof. Similarly as in the case of Theorem 6.4.3, we prove one-dimensional convergence in (6.59) at $x = y = 1$ only, and assume $\Phi(\,\mathrm{d}a) = (1-a)^\beta\,\mathrm{d}a$. Correspondingly, it suffices to show the limit $\lim J_{n\star} = J_\star$, where

$$J_{n\star} := \int_{\mathbb{R}^2\times\mathbb{R}_+} (G_{n\star}(u,v,z))^\alpha \,\mathrm{d}\mu(u,v,z), \quad J_\star := \int_{\mathbb{R}^2\times\mathbb{R}_+} (G_\star(u,v,z))^\alpha \,\mathrm{d}\mu(u,v,z),$$

where

$$G_\star(u,v,z) := \mathbf{1}(0 < v < 1)\int_0^1 \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}s\; h_3(t-u, s, z),$$

$$
\begin{aligned}
G_{n\star}(u,v,z) &:= \int_0^1 \mathrm{d}t \sum_{s=1}^{n} g_3\Big([nt]-[nu], s-[nv], 1-\frac{z}{n}\Big)\mathbf{1}(0<z<n), \\
&= \int_0^1 \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}s\ \sqrt{n}\ g_3\Big([nt]-[nu], [\sqrt{n}s], 1-\frac{z}{n}\Big) \\
&\quad \times \mathbf{1}(0<z<n, 1-[nv] \le [\sqrt{n}s] \le n-[nv]), \\
&=: \int_0^1 \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}s\ f_n(t,s,u,v,z).
\end{aligned}
$$

Define

$$
J'_{n\star} := \int_{\mathbb{R}^2\times\mathbb{R}_+} (G_{n\star}(u,v,z))^{\alpha}\mathbf{1}(|v|\le 3)\,\mathrm{d}\mu,
$$

$$
J''_{n\star} := \int_{\mathbb{R}^2\times\mathbb{R}_+} (G_{n\star}(u,v,z))^{\alpha}\mathbf{1}(|v|> 3)\,\mathrm{d}\mu,
$$

$$
J'_{n\star} + J''_{n\star} = J_{n\star}.
$$

Then $\lim J_{n\star} = J_\star$ follows from $\lim J'_{n\star} = J_\star$ and $\lim J''_{n\star} = 0$.

Note that for any $u \in \mathbb{R}$, $u < t$, $v \in \mathbb{R}\backslash\{0,1\}$, s, $z > 0$, we have pointwise convergence

$$
\begin{aligned}
\mathbf{1}(1-[nv] \le [\sqrt{n}s] \le n-[nv]) &\to \mathbf{1}(0<v<1),\ \text{as } n\to\infty, \\
\sqrt{n}\ g_3\Big([nt]-[nu], [\sqrt{n}s], 1-\frac{z}{n}\Big)\mathbf{1}(0<z<n) &\to h_3(t-u,s,z),\ \text{as } n\to\infty,
\end{aligned}
$$

and therefore

$$
f_n(t,s,u,v,z) \to h_3(t-u,s,z)\mathbf{1}(0<v<1),\ \text{as } n\to\infty. \tag{6.62}
$$

We claim that for any $u\in\mathbb{R}$, $v\in\mathbb{R}\backslash\{0,1\}$, $z>0$,

$$
G_{n\star}(u,v,z) \to G_\star(u,v,z), \qquad \text{as } n\to\infty. \tag{6.63}
$$

To show (6.63), for given $\epsilon_1 > 0$ split

$$
G_{n\star}(u,v,z) - G_\star(u,v,z) = \sum_{j=1}^{3} \Gamma^\star_{nj}(u,v,z),
$$

where, for $0<z<n$,

$$
\begin{aligned}
\Gamma^\star_{n1}(u,v,z) &:= \int_0^1 \int_{|s|>\epsilon_1} \Big(f_n(t,s,u,v,z) - h_3(t-u,s,z)\mathbf{1}(0<v<1)\Big)\,\mathrm{d}t\,\mathrm{d}s, \\
\Gamma^\star_{n2}(u,v,z) &:= \int_0^1 \int_{|s|\le\epsilon_1} f_n(t,s,u,v,z)\,\mathrm{d}t\,\mathrm{d}s,
\end{aligned}
$$

$$\Gamma^\star_{n3}(u,v,z) \;:=\; -\int_0^1\int_{|s|\le\epsilon_1} h_3(t-u,s,z)\mathbf{1}(0<v<1)\,\mathrm{d}t\,\mathrm{d}s,$$

To show (6.63), it suffices to verify that for any $\epsilon>0$, $\delta>0$ there exists $\epsilon_1>0$, $n_1\ge 1$ such that

$$\lim_{n\to\infty}\Gamma^\star_{n1}(u,v,z) \;=\; 0, \tag{6.64}$$

$$|\Gamma^\star_{ni}(u,v,z)| \;<\; \delta, \qquad i=2,3,\quad \forall n\ge n_1. \tag{6.65}$$

Relation (6.65) follows from Lemma 6.4.2,

$$\begin{aligned}|\Gamma^\star_{n2}(u,v,z)| &\le C_u\epsilon_1 + C_u\epsilon_1\sqrt{n}\int_0^1 \mathrm{e}^{-c(nt)^{1/3}}\,\mathrm{d}t\\ &\le C_u\epsilon_1 + C_u\epsilon_1/\sqrt{n} < \delta\end{aligned}$$

provided $\epsilon_1<\delta/(2C_u)$. $|\Gamma^\star_{n3}(u,v,z)|\le C_u\epsilon_1$, implying (6.65) for $i=3$ with $\epsilon_1=\delta/C_u$. Relation (6.64) follows from (6.62) and the dominated convergence theorem. For this we need to find the dominated integrable function for $f_n(t,s,u,v,z)$. Using inequality from Lemma 6.4.2 and inequalities

$$\mathrm{e}^{-x}\le x^{-3/2},\quad \text{for } x>0,\quad \text{and}\quad \sqrt{x}\mathrm{e}^{-x}\le \mathrm{e}^{-x/2},\quad \text{for } x>0,$$

we have for fixed u, $t-u>0$, v, z:

$$\begin{aligned}|f_n(t,s,u,v,z)| &\le C\frac{1}{\sqrt{\frac{[nt]-[nu]}{n}}}\,\mathrm{e}^{-\frac{s^2}{16\frac{[nt]-[nu]}{n}}} + C\sqrt{n}\mathrm{e}^{-cn^{1/3}(t-u)^{1/3}-c|s|^{1/2}}\\ &\le C\frac{1}{|s|}\,\mathrm{e}^{-\frac{s^2}{24\frac{[nt]-[nu]}{n}}} + C\sqrt{n}(n^{1/3}(t-u)^{1/3})^{-3/2}\mathrm{e}^{-c|s|^{1/2}}\\ &\le \frac{1}{|s|}\,\mathrm{e}^{-\frac{s^2}{24(1+|u|)}} + C(t-u)^{-1/2}\mathrm{e}^{-c|s|^{1/2}} =: \bar f(t,s).\end{aligned}$$

It is not difficult to see, that

$$\int_0^1\int_{|s|>\epsilon_1}\bar f(t,s)\,\mathrm{d}t\,\mathrm{d}s<\infty.$$

Therefore pointwise convergence in (6.63) is proved. Using (6.49), we also get

$$\begin{aligned}G_{n\star}(u,v,z) &= \int_0^1\mathrm{d}t\int_{\mathbb{R}}\mathrm{d}s\; f_n(t,s,u,v,z)\\ &\le \int_0^1\mathrm{d}t\int_{\mathbb{R}}\mathrm{d}s\Big(\bar h_3\Big(\frac{[nt]-[nu]}{n},s,z\Big)+\sqrt{n}\mathrm{e}^{-z\frac{[nt]-[nu]}{n}-c([nt]-[nu])^{1/3}-c(\sqrt{n}|s|)^{1/2}}\Big)\\ &\le C\int_0^1\mathrm{d}t\,\mathrm{e}^{-z(t-u)}\mathbf{1}(u<t)\end{aligned}$$

The integral of the function on the right side of last inequality is finite. Indeed,

$$\int_{\mathbb{R}^2\times\mathbb{R}_+}\Big(\int_0^1 \mathrm{d}t\,\mathrm{e}^{-z(t-u)}\Big)^\alpha \mathbf{1}(u<t,\ |v|\le 3)z^\beta\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}z \le$$
$$\le C\int_{\mathbb{R}}\mathrm{d}u\int_0^\infty \mathrm{d}z\ z^\beta\Big(\int_0^1 \mathrm{d}t\,\mathrm{e}^{-z(t-u)}\Big)^\alpha \mathbf{1}(u<t) =: I_1+I_2,$$

where

$$\begin{aligned} I_1 &\le C\int_0^1 \mathrm{d}u\int_0^\infty \mathrm{d}z\ z^\beta\Big(\int_u^1 \mathrm{d}t\,\mathrm{e}^{-z(t-u)}\Big)^\alpha \\ &\le C\int_0^1 \mathrm{d}u\int_0^\infty \mathrm{d}z\ z^{\beta-\alpha}\Big(1-\mathrm{e}^{-z(1-u)}\Big)^\alpha \\ &\le C\int_0^\infty z^{\beta-\alpha}\Big(1-\mathrm{e}^{-z}\Big)^\alpha\ \mathrm{d}z \le C, \end{aligned}$$

$$\begin{aligned} I_2 &\le C\int_0^{+\infty} \mathrm{d}u\int_0^\infty \mathrm{d}z\ z^\beta\Big(\int_0^1 \mathrm{d}t\,\mathrm{e}^{-z(t+u)}\Big)^\alpha \\ &\le C\Big\{\int_0^1 \mathrm{d}t\Big(\int_0^{+\infty}\mathrm{d}u\int_0^\infty \mathrm{d}z\ z^\beta\,\mathrm{e}^{-\alpha z(t+u)}\Big)^{1/\alpha}\Big\}^\alpha \\ &\le C\Big\{\int_0^1 \mathrm{d}t\Big(\int_0^{+\infty}(u+t)^{-1-\beta}\ \mathrm{d}u\Big)^{1/\alpha}\Big\}^\alpha \\ &\le C\Big\{\int_0^1 t^{-\beta/\alpha}\,\mathrm{d}t\Big\}^\alpha \le C, \text{ since } 1-\beta/\alpha>0. \end{aligned}$$

From the last fact, the limit in (6.63) and the dominated convergence theorem follows $\lim J'_{n\star}=J_\star$. Now we will show $\lim J''_{n\star}=0$. Again using inequality in (6.49), we have $J''_{n\star}\le I_{1,n}+I_{2,n}$, where

$$\begin{aligned} I_{1,n} &:= \int_{\mathbb{R}}\mathrm{d}u\int_{|v|>3}\mathrm{d}v\int_0^\infty \mathrm{d}z\ z^\beta \\ &\quad\times\Big(\int_0^1 \mathrm{d}t\int_{\mathbb{R}}\mathrm{d}s\sqrt{n}\mathrm{e}^{-z\frac{[nt]-[nu]}{n}-c([nt]-[nu])^{1/3}-c(\sqrt{n}|s|)^{1/2}}\Big)^\alpha \mathbf{1}_n(t,u,z,s,v), \\ I_{2,n} &:= \int_{\mathbb{R}}\mathrm{d}u\int_{|v|>3}\mathrm{d}v\int_0^\infty \mathrm{d}z\ z^\beta\Big(\int_0^1 \mathrm{d}t\int_{\mathbb{R}}\mathrm{d}s\bar{h}_3\Big(\frac{[nt]-[nu]}{n},s,z\Big)\Big)^\alpha \mathbf{1}_n(t,u,z,s,v), \end{aligned}$$

here

$$\mathbf{1}_n(t,u,z,s,v) := \mathbf{1}([nt]-[nu]>0, 0<z<n, 1-[nv]\le[\sqrt{n}s]\le n-[nv]).$$

Note that

$$\int_{\mathbb{R}}\mathrm{d}s\ \mathrm{e}^{-c(\sqrt{n}|s|)^{1/2}}\mathbf{1}(1-[nv]\le[\sqrt{n}s]\le n-[nv],\ |v|>3)\le$$
$$\le C\sqrt{n}\mathrm{e}^{-c\sqrt{n}(\min(|v|,\ |v-2|))^{1/2}}\mathbf{1}(|v|>3).$$

Therefore,

$$
\begin{aligned}
I_{1,n} &\le Cn^{\alpha}\int_{|v|>3} \mathrm{e}^{-c\alpha\sqrt{n}(\min(|v|,\ |v-2|)^{1/2}}\,\mathrm{d}v \\
&\quad\times \int_{\mathbb{R}} \mathrm{d}u \int_0^\infty \mathrm{d}z z^{\beta}\Big(\int_0^1 \mathrm{d}t\ \mathrm{e}^{-z(t-u)-c(t-u)^{1/3}}\Big)^{\alpha}\mathbf{1}(t-u>0, 0<z<n) \\
&\le Cn^{\alpha+\beta+1}\int_{|v|>1} \mathrm{e}^{-c\alpha\sqrt{n}|v|^{1/2}}\,\mathrm{d}v \int_{\mathbb{R}} \mathrm{d}u\Big(\int_0^1 \mathrm{d}t\ \mathrm{e}^{-c(t-u)^{1/3}}\Big)^{\alpha}\mathbf{1}(t-u>0) \\
&\le Cn^{\alpha+\beta}\mathrm{e}^{-c\sqrt{n}} \to 0, \text{ as } n\to\infty,
\end{aligned}
$$

$$
\begin{aligned}
I_{2,n} &\le Cn^{\frac{\alpha}{2}}\int_{\mathbb{R}} \mathrm{d}u \int_{|v|>1} \mathrm{d}v \int_0^\infty \mathrm{d}z z^{\beta}\Big(\int_0^1 \mathrm{d}t \frac{1}{\sqrt{t-u}}\,\mathrm{e}^{-z(t-u)-c\frac{nv^2}{t-u}}\Big)^{\alpha} \\
&\quad\times \mathbf{1}(t-u>0, 0<z<n) \\
&\le Cn^{\frac{\alpha}{2}}\Big(\int_0^1 \mathrm{d}t\Big(\int_{\mathbb{R}} \mathrm{d}u \int_{|v|>1} \mathrm{d}v \int_0^\infty \mathrm{d}z\, z^{\beta}(t-u)^{-\frac{\alpha}{2}}\,\mathrm{e}^{-z\alpha(t-u)-c\alpha\frac{nv^2}{t-u}}\Big)^{\frac{1}{\alpha}}\Big)^{\alpha} \\
&\quad\times \mathbf{1}(t-u>0, 0<z<n) \\
&\le Cn^{\frac{\alpha}{2}}\Big(\int_0^1 \mathrm{d}t\Big(\int_{\mathbb{R}} \mathrm{d}u \int_{|v|>1} \mathrm{d}v\,(t-u)^{-\frac{\alpha}{2}-\beta-1}\,\mathrm{e}^{-c\alpha\frac{nv^2}{t-u}}\Big)^{\frac{1}{\alpha}}\Big)^{\alpha}\mathbf{1}(t-u>0) \\
&\le Cn^{-\beta}\int_{v>1} \mathrm{d}v\ v^{-2(\frac{\alpha}{2}+\beta)}\int_0^\infty \mathrm{d}y\, y^{-\frac{\alpha}{2}-\beta-1}\,\mathrm{e}^{-\frac{c\alpha}{y}} = Cn^{-\beta} \to 0, \text{ as } n\to\infty,
\end{aligned}
$$

since $1-2(\frac{\alpha}{2}+\beta)<0$ and $\int_0^\infty y^{-\frac{\alpha}{2}-\beta-1}\,\mathrm{e}^{-c\alpha\frac{1}{y}}\,\mathrm{d}y<\infty$. This proves $\lim J''_{n\star}=0$ and Theorem 6.4.4 too. ☐

Remark 6.4.5. It is not difficult to show that the random fields $\{V_3(x,y)\}$ and $\{V_{3\star}(x,y)\}$ in Theorems 6.4.3 and 6.4.4 are related by

$$
\lambda^{-1/\alpha}V_3(x,\lambda y) \to_{\mathrm{fdd}} V_{3\star}(x,y),\ x,y>0,\ \lambda\to\infty.
$$

Proposition 6.4.6. *Let the conditions of Theorem 6.4.3 be satisfied. Then:*

(i) The random field $\{\mathfrak{X}_3(t,s), (t,s)\in\mathbb{Z}^2\}$ *in* (6.45) *has anisotropic distributional long memory with parameters* $H_1=H=\frac{\frac{1}{2}+\alpha-\beta}{\alpha}, H_2=2H_1$.

(ii) The random field $\{\mathfrak{X}_3(t,s), (t,s)\in\mathbb{Z}^2\}$ *in* (6.45) *does not have isotropic distributional long memory.*

Proof. (i) With Theorem 6.4.3 in mind, it suffices to check that the random field $\{V_3(x,y)\}$ in (6.46) has dependent increments in arbitrary direction. To this end, consider arbitrary rectangles $K_i=K_{(\xi_i,\eta_i);(x_i,y_i)}\subset\mathbb{R}^2_+, i=1,2$. Then

$$
V_3(K_i)=\int_{\mathbb{R}^2\times\mathbb{R}_+} G_{K_i}(u,v,z)\,\mathrm{d}\mathcal{M},
$$

where

$$G_{K_i}(u,v,z) := \int_{K_i} h_3(t-u, s-v, z)\,\mathrm{d}t\,\mathrm{d}s.$$

Note $G_{K_i} \ge 0$ and $G_{K_i}(u,v,z) > 0$ for any $u < x_i$ implying

$$\mathrm{supp}(G_{K_1}) \cap \mathrm{supp}(G_{K_2}) \ne \emptyset.$$

Hence and from ([94], Th 3.5.3, p. 128) it follows that the increments $V_3(K_i), i = 1,2$ on arbitrary nonempty rectangles K_1, K_2 are dependent, thus concluding the proof of (i).

(ii) With Theorem 6.4.4 in mind, it suffices to check that the random field $\{V_{3\star}(x,y)\}$ in (6.60) has independent increments in the vertical directions. Similarly as in the proof of (i), for any rectangle $K = K_{(\xi,\eta);(x,y)} \subset \mathbb{R}^2_+$,

$$V_{3\star}(K) = \int_{\mathbb{R}^2\times\mathbb{R}_+} G^\star_K(u,v,z)\,\mathrm{d}\mathcal{M},$$

where

$$G^\star_K(u,v,z) := \mathbf{1}(\eta < v \le y)\int_\xi^\eta h_{3\star}(t-u,z)\,\mathrm{d}t.$$

Clearly, if $K_i, i = 1,2$ are two rectangle separated by a horizontal line, then

$$\mathrm{supp}(G_{K_1}) \cap \mathrm{supp}(G_{K_2}) = \emptyset,$$

implying the independence of $V_{3\star}(K_1)$ and $V_{3\star}(K_2)$. Proposition 6.4.6 is proved. □

Let $\alpha = 2$ and $r_3(t,s) = \mathrm{E}\mathfrak{X}_3(t,s)\mathfrak{X}_3(0,0)$ be the covariance function of the aggregated Gaussian random field in (6.45). Using the representation of $r_3(t,s)$ in (6.11) and Lemma 6.4.2, the following proposition obtains the asymptotics of $r_3(t,s)$ as $|t|+|s| \to \infty$.

Proposition 6.4.7. *Assume $\alpha = 2$ and the conditions of Theorem 6.4.3. Then for any $(t,s) \in \mathbb{R}^2_0$*

$$\lim_{\lambda\to\infty} \lambda^{\beta+1/2} r_3([\lambda t],[\sqrt{\lambda}s]) = \rho(t,s) := \begin{cases} C_3|s|^{-2\beta-1}\gamma(\beta+1/2, s^2/4|t|), & t \ne 0, s \ne 0,\\ C_3|s|^{-2\beta-1}\Gamma(\beta+1/2), & t = 0,\\ C_4|t|^{-\beta-1/2}, & s = 0 \end{cases} \tag{6.66}$$

and

$$\lim_{\lambda\to\infty} \lambda^{\beta+1/2} r_3([\lambda t],[\lambda s]) = \rho_*(t,s) := \begin{cases} 0, & s \ne 0,\\ C_4|t|^{-\beta-1/2}, & s = 0, t \ne 0, \end{cases} \tag{6.67}$$

where $\gamma(\alpha, x) := \int_0^x y^{\alpha-1}\mathrm{e}^{-y}\,\mathrm{d}y$ *is incomplete gamma function and*

$$C_3 = \pi^{-\frac{1}{2}}2^{2\beta-1}3^{1-\beta}\sigma^2\phi_1\Gamma(\beta+1), \qquad C_4 = 4^{-\frac{1}{2}-\beta}(\beta+1/2)^{-1}C_3.$$

Notice that under the 'parabolic scaling' in (6.66) we have a non-degenerated limit $\rho(t,s)$ which is a generalized homogeneous function (see, e.g., [44] for a general account) satisfying

$$\lambda^{2(1+\frac{H_1}{H_2}-H_1)}\rho(\lambda t, \lambda^{H_1/H_2}s) \;=\; \rho(t,s), \qquad \forall \lambda > 0,$$

with H_1, H_2 as in Proposition 6.4.6 (i) ($\alpha = 2$). On the other hand, the 'isotonic scaling' in (6.67) leads to a degenerated limit concentrated on the anisotropicity axis $s = 0$ of the 3N model and vanishing elsewhere. It is clear that the corresponding integrated Gaussian random field must have independent increments in the vertical direction, in accordance with Proposition 6.4.6 (ii).

Proof of Proposition 6.4.7. We have

$$r_3(t,s) = \sigma^2 \sum_{(u,v)\in\mathbb{Z}^2}\int_{[0,1)} g_3(t+u, s+v, a)g_3(u,v,a)\Phi(\,\mathrm{d}a),\ (t,s)\in\mathbb{Z}^2, \quad (6.68)$$

where $\sigma^2 = \mathrm{E}\varepsilon^2$. For ease of notation, assume $\phi(a) = (1-a)^\beta$, $a \in [0,1)$, in the rest of the proof. Then

$$\begin{aligned} r_3([\lambda t],[\sqrt{\lambda}s]) \;&=\; \sigma^2\int_0^\infty \mathrm{d}u\int_{\mathbb{R}}\mathrm{d}v\int_0^1 (1-a)^\beta\,\mathrm{d}a\, g_3([u],[v],a) \\ &\quad\times\, g_3([\lambda t]+[u],[\sqrt{\lambda}s]+[v],a) \\ &=\; \lambda^{1/2-\beta}\sigma^2\int_0^\infty \mathrm{d}x\int_{\mathbb{R}}\mathrm{d}y\int_0^\lambda z^\beta\,\mathrm{d}z\, g_3([\lambda x],[\sqrt{\lambda}y],1-\frac{z}{\lambda}) \\ &\quad\times\, g_3([\lambda t]+[\lambda x],[\sqrt{\lambda}s]+[\sqrt{\lambda}y],1-\frac{z}{\lambda}). \end{aligned}$$

Hence,

$$\lambda^{\beta+1/2}r_3([\lambda t],[\sqrt{\lambda}s]) \;=\; \int_0^\infty\int_{\mathbb{R}}\int_0^\infty \mathcal{K}_\lambda(x,y,z)\,\mathrm{d}\mu,$$

where $\mathrm{d}\mu(x,y,z) = z^\beta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and

$$\mathcal{K}_\lambda(x,y,z) := \lambda\sigma^2 g_3([\lambda x],[\sqrt{\lambda}y],1-\frac{z}{\lambda})g_3([\lambda t]+[\lambda x],[\sqrt{\lambda}s]+[\sqrt{\lambda}y],1-\frac{z}{\lambda})\mathbf{1}(0<z<\lambda).$$

By Lemma 6.4.2, for any $(x,y,z) \in (0,\infty)\times\mathbb{R}\times(0,\infty)$ fixed,

$$\mathcal{K}_\lambda(x,y,z) \to \mathcal{K}(x,y,z) := \sigma^2 h_3(x,y,z)h_3(x+t,y+s,z),$$

where the integral $I_{\mathcal{K}} := \int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+} \mathcal{K}(x,y,z)\,\mathrm{d}\mu$ is equal to

$$
\begin{aligned}
I_{\mathcal{K}} &= \sigma^2 \int_0^\infty \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}y \int_0^\infty z^\beta\,\mathrm{d}z\,\frac{3}{2\sqrt{\pi x}}\,\mathrm{e}^{-3zx-\frac{y^2}{4x}}\,\frac{3}{2\sqrt{\pi(t+x)}}\,\mathrm{e}^{-3z(t+x)-\frac{(s+y)^2}{4(t+x)}} \\
&= \frac{9\sigma^2}{4\pi}\int_0^\infty \mathrm{d}x\Big\{\int_0^\infty z^\beta \mathrm{e}^{-3z(2x+t)}\,\mathrm{d}z\Big\}\Big\{\int_{\mathbb{R}}\frac{1}{\sqrt{x(t+x)}}\,\mathrm{e}^{-\frac{y^2}{4x}}\,\mathrm{e}^{-\frac{(s-y)^2}{4(t+x)}}\,\mathrm{d}y\Big\} \\
&= \frac{9\sigma^2}{4\pi}\int_0^\infty \mathrm{d}x\Big\{\frac{\Gamma(\beta+1)}{(3(2x+t))^{1+\beta}}\Big\}\Big\{\frac{2\sqrt{\pi}}{\sqrt{2x+t}}\,\mathrm{e}^{-\frac{s^2}{4(2x+t)}}\Big\}
\end{aligned}
$$

and, continuing equality,

$$
\begin{aligned}
I_{\mathcal{K}} &= \frac{9\sigma^2\Gamma(\beta+1)}{2\sqrt{\pi}3^{1+\beta}}\int_0^\infty \frac{1}{(2x+t)^{3/2+\beta}}\mathrm{e}^{-\frac{s^2}{4(2x+t)}}\,\mathrm{d}x \\
&= \frac{3^{1-\beta}\sigma^2\Gamma(\beta+1)}{4\sqrt{\pi}}\int_t^\infty \frac{1}{x^{3/2+\beta}}\mathrm{e}^{-\frac{s^2}{4x}}\,\mathrm{d}x \\
&= \begin{cases} \frac{3^{1-\beta}\sigma^2\Gamma(\beta+1)}{4^{1/2-\beta}\sqrt{\pi}}|s|^{-2\beta-1}\gamma(\beta+1/2, s^2/4t), & s\neq 0, \\ \frac{3^{1-\beta}\sigma^2\Gamma(\beta+1)}{4\sqrt{\pi}(\frac{1}{2}+\beta)}t^{-\beta-1/2}, & s=0. \end{cases}
\end{aligned}
$$

The legitimacy of the passage to the limit $\lambda\to\infty$ under the sign of the integral follows from Lemma 6.4.2. Indeed, the bound (6.49) implies

$$|\mathcal{K}_\lambda(x,y,z)| \le C(\mathcal{K}'(x,y,z)+\mathcal{K}''_\lambda(x,y,z)),$$

where

$$0\le \mathcal{K}'(x,y,z) := \bar{h}_3(x,y,z)\bar{h}_3(x+t,y+s,z)$$

does not depend on λ and satisfies $\int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+}\mathcal{K}'(x,y,z)\,\mathrm{d}\mu<\infty$, see above, while

$$0\le \mathcal{K}''_\lambda(x,y,z) := \lambda\,\mathrm{e}^{-zx-c(\lambda x)^{1/3}-c(\sqrt{\lambda}|y|)^{1/2}}\mathrm{e}^{-z(x+t)-c(\lambda(x+t))^{1/3}-c(\sqrt{\lambda}|s+y|)^{1/2}}$$

satisfies $\lim_{\lambda\to\infty}\int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+}\mathcal{K}''_\lambda(x,y,z)\,\mathrm{d}\mu = 0$ for any $(t,s)\in\mathbb{R}^2_0$ fixed. The last fact can be easily verified by separately considering the two cases $t>0$ and $t=0, s\neq 0$. E.g., in the first case, we have

$$\mathcal{K}''_\lambda(x,y,z)\le\lambda\,\mathrm{e}^{-c(\lambda t)^{1/3}}\mathrm{e}^{-zx-c(\lambda x)^{1/3}-c(\sqrt{\lambda}|y|)^{1/2}}$$

and

$$\int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+}\mathcal{K}''_\lambda(x,y,z)\,\mathrm{d}\mu\le C\mathrm{e}^{-c'(\lambda t)^{1/3}}, \qquad 0<c'<c$$

easily follows. The convergence in (6.67) can be proved in a similar way. Proposition 6.4.7 is proved.

Remark 6.4.8. Suppose, the individual behavior is described by two-neighbor (2N) random field:

$$X(t,s)=\frac{A}{2}\Big(X(t-1,s)+X(t,s-1)\Big)+\varepsilon(t,s),\quad (t,s)\in\mathbb{Z}^2,$$

where $\varepsilon\in D(\alpha)$, $1<\alpha\le 2$, and A is random coefficient with the mixing density ϕ satisfying (6.13), where $0<\beta<\alpha-1$. The stationary solution of this equation is given by (6.8), with the Green function:

$$g_2(t,s,a)=\sum_{k=0}^{\infty}A^k p_k(t,s)=\begin{cases}a^{t+s}b\Big(t,t+s,\frac{1}{2}\Big), & t+s\ge 0,\ |t-s|\le t+s,\\ 0, & \text{otherwise},\end{cases}$$

where $p_k(t,s)=\mathrm{P}(W_k=(t,s)|W_0=(0,0))$ is the $k-$step probability of the nearest-neighbor random walk $\{W_k,k=0,1,\ldots\}$ on the lattice $\mathbb{Z}^2$ with one-step transition probabilities shown in Figure 6.1 (a), page 115, and

$$b(t;k,p):=\frac{k!}{t!(k-t)!}p^t(1-p)^{k-t},\qquad k=0,1,\ldots,\quad t=0,1,\ldots,k.$$

is the binomial probability.

Using the Moivre-Laplace theorem (see [36], vol.I, ch.7, §2, Thm.1), similarly as in the proof of Lemma 6.4.2 we can show, that for $t>0$, $s\in\mathbb{R}$, $z>0$,

$$\sqrt{\lambda}g_2\Big(\frac{[\lambda t]+[\sqrt{\lambda}s]}{2},\frac{[\lambda t]-[\sqrt{\lambda}s]}{2},1-\frac{z}{\lambda}\Big)\mathbf{1}([\lambda t]\overset{\mathrm{mod}\,2}{=}[\sqrt{\lambda}s])\to h_2(t,s,z),$$

as $\lambda\to\infty$, where

$$h_2(t,s,z):=\sqrt{\frac{2}{\pi t}}\,\mathrm{e}^{-zt-\frac{s^2}{2t}}. \tag{6.69}$$

The obvious similarity between kernels h_2 in (6.69) and h_3 in (6.19) suggest that large-scale properties of the 2N and 3N models should be similar, modulus a rotation of the plane by angle $\pi/4$. We can show, that in 2N case the partial sums of the limit aggregated process

$$\mathfrak{X}_2(t,s)\ =\ \sum_{(u,v)\in\mathbb{Z}^2}\int_0^1 g_2(t-u,s-v,a)M_{u,v}(\mathrm{d}a),\qquad (t,s)\in\mathbb{Z}^2, \tag{6.70}$$

(the general form of the limit aggregated field is given in (6.10)) have the following limits:

-
$$n^{-H} \sum_{\substack{1 \le t+s \le [nx],\\ 1 \le t-s \le [\sqrt{n}y]}} \mathfrak{X}_2(t,s) \to_{\mathrm{fdd}} L_2(x,y), \quad x,y>0, \quad H := \frac{\frac{1}{2}+\alpha-\beta}{\alpha},$$

where

$$L_2(x,y) := \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+} \Big\{ \int_0^x \int_0^y h_2(t-u, s-v, z)\,\mathrm{d}t\,\mathrm{d}s \Big\} \mathcal{M}(\,\mathrm{d}u,\,\mathrm{d}v,\,\mathrm{d}z);$$

-
$$n^{-H_\star} \sum_{\substack{1 \le t+s \le [nx],\\ 1 \le t-s \le [ny]}} \mathfrak{X}_2(t,s) \to_{\mathrm{fdd}} L_{2\star}(x,y), \quad x,y>0, \quad H_\star := \frac{1+\alpha-\beta}{\alpha},$$

where

$$L_{2\star}(x,y) := \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+} \mathcal{M}(\,\mathrm{d}u,\,\mathrm{d}v,\,\mathrm{d}z)\mathbf{1}(0<v<y)\int_0^x 2\mathrm{e}^{-(t-u)z}\mathbf{1}(t-u>0)\,\mathrm{d}t,$$

here $\mathcal{M}$ is an $\alpha-$stable random measure on $\mathbb{R}^2 \times \mathbb{R}_+$.

The random field $L_2(x,y)$ has dependent increments in arbitrary direction, while the random field $L_{2\star}(x,y)$ has independent increments in vertical direction. Therefore, we can conclude that the limit aggregated field $\{\mathfrak{X}_2(t,s),\,(t,s)\in\mathbb{Z}^2\}$ in (6.70) has anisotropic distributional long memory with parameters $H_1 = (1/2+\alpha-\beta)/\alpha$, $H_2 = 2H_1$.

We do not give proofs of these results here, because after the change of coordinates

$$u = t+s, \qquad v = t-s,$$

the proof of these results is quite similar to the proofs of Theorem 6.4.3 and Theorem 6.4.4.

6.5 Aggregation of the 4N model

The stationary solution of (6.15) is given by

$$X_4(t,s) \;=\; \sum_{(u,v)\in\mathbb{Z}^2} g_4(t-u,s-v,A)\varepsilon(u,v), \qquad (t,s)\in\mathbb{Z}^2, \tag{6.71}$$

where

$$g_4(t,s,a) \;=\; \sum_{k=0}^{\infty} a^k p_k(t,s), \quad p_k(t,s) \;=\; \mathrm{P}(W_k=(t,s)|W_0=(0,0)) \quad (6.72)$$

and $\{W_k\}$ is a random walk on $\mathbb{Z}^2$ with one-step transition probabilities in Fig. 6.1 (c). Under the assumptions of Proposition 6.3.3, page 124, the aggregated random field of (6.71) exists and is written as

$$\mathfrak{X}_4(t,s) \;=\; \sum_{(u,v)\in\mathbb{Z}^2} \int_0^1 g_4(t-u,s-v,a) M_{u,v}(\,\mathrm{d}a), \qquad (t,s)\in\mathbb{Z}^2, \quad (6.73)$$

where $\{M_{u,v}(\,\mathrm{d}a),\,(u,v)\in\mathbb{Z}^2\}$ is the same $\alpha-$stable random measure as in Section 6.4. For $1<\alpha\le 2$ and a regularly varying mixing density as in (6.13), the random field in (6.73) is well-defined under the same condition $0<\beta<\alpha-1$ as in Theorem 6.4.3, page 127. Recall $\mathbb{R}_0^2=\mathbb{R}^2\setminus\{(0,0)\}$.

Lemma 6.5.1. *For any* $(t,s,z)\in\mathbb{R}_0^2\times(0,\infty)$*,*

$$\lim_{\lambda\to\infty} g_4\Big([\lambda t],[\lambda s],1-\frac{z}{\lambda^2}\Big) \;=\; h_4(t,s,z)=\frac{2}{\pi}K_0\Big(2\sqrt{z(t^2+s^2)}\Big). \quad (6.74)$$

The convergence in (6.74) *is uniform on any relatively compact set*

$$\{\epsilon<|t|+|s|<1/\epsilon\}\times\{\epsilon<z<1/\epsilon\}\subset\mathbb{R}_0^2\times\mathbb{R}_+,\ \epsilon>0.$$

Moreover, there exists constants $C,\,c>0$ *such that for all sufficiently large* λ *and any* $(t,s,z)\in\mathbb{R}_0^2\times(0,\lambda^2)$ *the following inequality holds:*

$$g_4\Big([\lambda t],[\lambda s],1-\frac{z}{\lambda^2}\Big) \;<\; C\Big\{h_4(t,s,z)+\mathrm{e}^{-c\sqrt{\lambda}(|t|^{1/2}+|s|^{1/2})}\Big\}. \quad (6.75)$$

The proof of this lemma is given in Section 6.6, page 154. The main result of this Section is Theorem 6.5.2 below.

Theorem 6.5.2. *Let* $\{\varepsilon(t,s),\,(t,s)\in\mathbb{Z}^2\}$ *and* Φ *satisfy the same conditions as in Theorem 6.4.3 (page 127), and* $\{\mathfrak{X}_4(t,s),\,(t,s)\in\mathbb{Z}^2\}$ *be the aggregated 4N random field in* (6.73)*. Then*

$$n^{-H}\sum_{t=1}^{[nx]}\sum_{s=1}^{[ny]}\mathfrak{X}_4(t,s) \;\to_{\mathrm{fdd}}\; V_4(x,y), \qquad x,y>0, \quad (6.76)$$

where $H:=\frac{2(\alpha-\beta)}{\alpha}$ *and*

$$V_4(x,y) \;:=\; \int_{\mathbb{R}^2\times\mathbb{R}_+}\mathcal{M}(\,\mathrm{d}u,\,\mathrm{d}v,\,\mathrm{d}z)\int_0^x\int_0^y h_4(t-u,s-v,z)\,\mathrm{d}t\,\mathrm{d}s \quad (6.77)$$

and where $\mathcal{M}$ is the same $\alpha-$stable random measure on $\mathbb{R}^2 \times \mathbb{R}_+$ as in Theorem 6.4.3 and $h_4(t,s,z)$ is given in (6.74).

Proof. As in all previous theorems, we prove the convergence of one-dimensional distributions in (6.76) at $x = y = 1$. Accordingly, it suffices to show the limit $\lim J_n = J$, where

$$J_n := \frac{1}{n^{H\alpha}} \sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}\Big(\sum_{t,s=1}^{n} g_4(t-u, s-v, A)\Big)^{\alpha},$$

$$J := \int_{\mathbb{R}^2\times\mathbb{R}_+} \Big(\int_{(0,1]^2} h_4(t-u, s-v, z)\,\mathrm{d}t\,\mathrm{d}s\Big)^{\alpha} \mathrm{d}\mu.$$

Let us first check that

$$J = C\int_{\mathbb{R}^2\times\mathbb{R}_+} \Big(\int_{(0,1]^2} K_0(2\sqrt{z}\|v-w\|)\,\mathrm{d}v\Big)^{\alpha} z^{\beta}\,\mathrm{d}w\,\mathrm{d}z < \infty,$$

here, $\|x\|^2 := x_1^2 + x_2^2$, for $x = (x_1, x_2) \in \mathbb{R}^2$. To this end, split $J = J_1 + J_2$, where

$$\begin{aligned} J_1 &:= \int_{\{\|w\|\le\sqrt{2}\}\times\mathbb{R}_+} \Big(\int_{(0,1]^2} K_0(2\sqrt{z}\|v-w\|)\,\mathrm{d}v\Big)^{\alpha} z^{\beta}\,\mathrm{d}w\,\mathrm{d}z, \\ J_2 &:= \int_{\{\|w\|>\sqrt{2}\}\times\mathbb{R}_+} \Big(\int_{(0,1]^2} K_0(2\sqrt{z}\|v-w\|)\,\mathrm{d}v\Big)^{\alpha} z^{\beta}\,\mathrm{d}w\,\mathrm{d}z. \end{aligned}$$

By Minkowski inequality,

$$\begin{aligned} J_2 &\le C\Big\{\int_{\{\|v\|\le\sqrt{2}\}} \mathrm{d}v\Big[\int_{\{\|w\|>\sqrt{2}\}\times\mathbb{R}_+} K_0^{\alpha}(2\sqrt{z}\|v-w\|) z^{\beta}\,\mathrm{d}z\,\mathrm{d}w\Big]^{1/\alpha}\Big\}^{\alpha} \\ &\le C\Big\{\int_{\{\|v\|\le\sqrt{2}\}} \mathrm{d}v\Big[\int_{\{\|w\|>\sqrt{2}\}} \|v-w\|^{-2-2\beta}\,\mathrm{d}w\Big]^{1/\alpha}\Big\}^{\alpha} \\ &\le C\Big\{\int_{\{\|v\|\le\sqrt{2}\}} (\sqrt{2}-\|v\|)^{-2\beta/\alpha}\,\mathrm{d}v\Big\}^{\alpha} < \infty, \end{aligned}$$

where we used the facts that

$$\int_0^{\infty} K_0^{\alpha}(2\sqrt{z}) z^{\beta}\,\mathrm{d}z < \infty \qquad \text{and} \qquad 0 < \beta < \alpha - 1 \le 2.$$

Next,

$$\begin{aligned} J_1 &\le C\int_{\{\|w\|\le\sqrt{2}\}} \mathrm{d}w \int_0^{\infty} z^{\beta}\,\mathrm{d}z\Big(\int_{\{\|v\|\le\sqrt{2}\}} K_0(2\sqrt{z}\|v\|)\,\mathrm{d}v\Big)^{\alpha} \\ &\le C\int_0^{\infty} z^{\beta}\,\mathrm{d}z\Big(\int_0^{\sqrt{2}} K_0(2\sqrt{z}r) r\,\mathrm{d}r\Big)^{\alpha} \\ &\le C\int_0^{\infty} z^{\beta}\Big(z^{-\alpha/2}\mathbf{1}(0<z<1) + z^{-\alpha}\mathbf{1}(z\ge 1)\Big)\,\mathrm{d}z < \infty, \end{aligned}$$

where we used $0 < \beta < \alpha - 1$ and the inequality

$$\int_0^{\sqrt{2}} K_0(2\sqrt{z}r) r \,\mathrm{d}r \le C \begin{cases} z^{-1/2}, & 0 < z \le 1, \\ z^{-1}, & z > 1, \end{cases}$$

which is a consequence of the fact that the function $r \mapsto rK_0(r)$ is bounded and integrable on $(0, \infty)$. This proves $J < \infty$.

Next, we prove the convergence $J_n \to J$. The proof uses Lemma 6.5.1. Assume for simplicity $\phi(a) = (1-a)^\beta$. Then

$$J_n = \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G_n(u, v, z))^\alpha \,\mathrm{d}\mu(u, v, z), \qquad J = \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G(u, v, z))^\alpha \,\mathrm{d}\mu(u, v, z),$$

where

$$G(u, v, z) := \int_{(0,1]^2} h_4(t - u, s - v, z) \,\mathrm{d}t \,\mathrm{d}s,$$

$$G_n(u, v, z) := \int_{(0,1]^2} g_4\Big([nt] - [nu], [ns] - [nv], 1 - \frac{z}{n^2}\Big) \,\mathrm{d}t \,\mathrm{d}s.$$

Let $G'_n(u, v, z) := \mathbf{1}(0 < z < n^2) \int_{(0,1]^2} \mathrm{e}^{-c(\sqrt{n|t-u|} + \sqrt{n|s-v|})} \,\mathrm{d}t \,\mathrm{d}s$, where $c > 0$ is the same as in (6.75). Then

$$J'_n := \int_{\mathbb{R}^2 \times \mathbb{R}_+} (G'_n(u, v, z))^\alpha \,\mathrm{d}\mu(u, v, z) = O(n^{2(\beta - \alpha + 1)}) = o(1). \tag{6.78}$$

Indeed, $J'_n \le Cn^{2\beta+2}\Big\{ \int_{\mathbb{R}} \Big(\int_0^1 \mathrm{e}^{-c\sqrt{n|t-u|}} \,\mathrm{d}t \Big)^\alpha \,\mathrm{d}u \Big\}^2$, where

$$\int_{\mathbb{R}} \Big(\int_0^1 \mathrm{e}^{-c\sqrt{n|t-u|}} \,\mathrm{d}t \Big)^\alpha \,\mathrm{d}u \le \int_{\{|u|<2\}} \Big(\int_0^1 \mathrm{e}^{-c\sqrt{n|t-u|}} \,\mathrm{d}t \Big)^\alpha \,\mathrm{d}u + \int_{\{|u|\ge 2\}} \Big(\int_0^1 \mathrm{e}^{-c\sqrt{n|t-u|}} \,\mathrm{d}t \Big)^\alpha \,\mathrm{d}u =: i'_n + i''_n.$$

Here, $i'_n \le C\Big(\int_0^3 \mathrm{e}^{-c\sqrt{nv}} \,\mathrm{d}v \Big)^\alpha \le C/n^\alpha$ and $i''_n \le C \int_2^\infty \mathrm{e}^{-c\alpha\sqrt{n(u-1)}} \,\mathrm{d}u = O(\mathrm{e}^{-c'\sqrt{n}})$, $c' > 0$. This proves (6.78). The rest of the proof is similar as in the case of Theorem 6.4.3. Theorem 6.5.2 is proved. □

Proposition 6.5.3. *Let the conditions of Theorem 6.5.2 be satisfied. Then the random field $\{\mathfrak{X}_4(t, s)\}$ in (6.73) has isotropic distributional long memory.*

Proof. Similar to the proof of Proposition 6.4.6 (page 136) we need to show that the random field $\{V_4(x, y)\}$ in (6.77) has dependent increments in arbitrary direction. Consider arbitrary rectangles $K_i = K_{(\xi_i, \eta_i);(x_i, y_i)} \subset \mathbb{R}^2_+, i = 1, 2$. Then

$V_4(K_i) = \int_{\mathbb{R}^2 \times \mathbb{R}_+} G_{K_i}(u, v, z)\, \mathrm{d}\mathcal{M}$, $i = 1, 2$, where

$$\begin{aligned} G_{K_i}(u, v, z) &:= \int_{K_i} h_4(t-u, s-v, z)\, \mathrm{d}t\, \mathrm{d}s \\ &= \int_{K_i} \Big(\frac{1}{\pi} \int_0^\infty \frac{1}{x} \exp\Big\{ -zx - \frac{(t-u)^2 + (s-v)^2}{x} \Big\}\, \mathrm{d}x \Big)\, \mathrm{d}t\, \mathrm{d}s > 0. \end{aligned}$$

Therefore $\mathrm{supp}(G_{K_1}) \cap \mathrm{supp}(G_{K_2}) \neq \emptyset$. Hence it follows that the increments $V_4(K_i), i = 1, 2$, on arbitrary nonempty rectangles K_1, K_2 are dependent and random field in (6.77) has isotropic long memory. □

The following proposition obtains an asymptotic behavior of the covariance function of the Gaussian aggregated random field in (6.77) ($\alpha = 2$). The proof of Proposition 6.5.4 uses Lemma 6.5.1 and is omitted.

Proposition 6.5.4. *Assume $\alpha = 2$ and the conditions of Theorem 6.5.2. Then for any $(t, s) \in \mathbb{R}_0^2$,*

$$\lim_{\lambda \to \infty} \lambda^{2\beta} r_4([\lambda t], [\lambda s]) \;=\; \frac{\sigma^2 \phi_1 \Gamma(\beta+1)\Gamma(\beta)}{\pi} (t^2 + s^2)^{-\beta}. \tag{6.79}$$

6.6 Appendix. Proofs of Lemmas.

Let us note that the asymptotics of some lattice Green functions as $|t| + |s| \to \infty$ and $a \uparrow 1$ simultaneously was derived in Montroll and Weiss [76] using Laplace's method, see, e.g., ([76], (II.16)), ([47], (3.185)), however in the literature we did not find dominating bounds needed for our purposes. As noted in Section 6.1, our proofs use probabilistic tools and are completely independent.

Proof of Lemma 6.4.2. Let us first explain the idea behind the derivation of (6.17). Write $W_k = (W_{1k}, W_{2k}) \in \mathbb{Z}^2$. Note W_{1k} has the binomial distribution with success probability 1/3 and, conditioned on $W_{1k} = t$, W_{2k} is a sum of $k - t$ Bernoulli r.v.'s taking values ± 1 with probability 1/2. Hence for $k \geq t, k - t \geq |s|$ and $k - t + s$ even,

$$\begin{aligned} p_k(t, s) &= \mathrm{P}(W_{1k} = t, W_{2k} = s) \;=\; \mathrm{P}(W_{k1} = t)\mathrm{P}(W_{k2} = s | W_{k1} = t) \\ &= b\Big(t; k, \frac{1}{3}\Big) p(k - t, s). \end{aligned} \tag{6.80}$$

Here and below, $b(t; k, p)$ denote the binomial distribution with success probability $p \in (0, 1)$:

$$b(t; k, p) := \frac{k!}{t!(k-t)!} p^t (1-p)^{k-t}, \qquad k = 0, 1, \dots, \quad t = 0, 1, \dots, k, \tag{6.81}$$

and

$$p(u,v) \;:=\; b\Big(\frac{u+v}{2}; u; \frac{1}{2}\Big) = \begin{cases} \frac{1}{2^u}\frac{u!}{\big(\frac{u+v}{2}\big)!\big(\frac{u-v}{2}\big)!}, & \text{if} \quad u \ge 0,\ |v| \le u,\ u+v \text{ is even}, \\ 0, & \text{otherwise.} \end{cases} \tag{6.82}$$

We shall need the following version of the Moivre-Laplace theorem (see [36], vol.I, ch.7, §2, Thm.1): *There exists a constant C such that when $k \to \infty$ and $t \to \infty$ vary in such a way that*

$$\frac{(t-pk)^3}{k^2} \;\to\; 0, \tag{6.83}$$

then

$$\left|\frac{b(t;k,p)}{\frac{1}{\sqrt{2\pi kp(1-p)}}\exp\{-\frac{(t-kp)^2}{2kp(1-p)}\}} - 1\right| \;<\; \frac{C}{k} + \frac{C|t-pk|^3}{k^2}. \tag{6.84}$$

For $p(u,v)$ in (6.82), (6.83)-(6.84) imply that there exist $K_0 > 0$ and $C > 0$ such that

$$\sup_{u>0,\, v\in\mathbb{Z}} \left|\frac{p(u,v)}{\sqrt{\frac{2}{\pi u}}\mathrm{e}^{-v^2/2u}} - 1\right| \mathbf{1}\Big(u^2 > K|v|^3,\ u > K,\ u \overset{\mathrm{mod}\,2}{=} v\Big) \;<\; \frac{C}{K}, \qquad \forall\ K > K_0. \tag{6.85}$$

Using (6.80) and the Moivre-Laplace approximation in (6.84), we can write

$$\begin{aligned}
&\sqrt{\lambda} g_3([\lambda t], [\sqrt{\lambda}s], 1-\lambda^{-1}z) \\
&\qquad = \sqrt{\lambda}\sum_{k=[\lambda t]}^{\infty}\Big(1-\frac{z}{\lambda}\Big)^k p_k([\lambda t], [\sqrt{\lambda}s]) \\
&\qquad \sim \frac{3}{2\lambda}\sum_{k=[\lambda t]}^{\infty} \mathrm{e}^{-z(k/\lambda)}\sqrt{\frac{\lambda}{4\pi(\frac{k}{\lambda})}}\,\mathrm{e}^{-\frac{(3\lambda t-k)^2}{12\lambda t\frac{k}{3\lambda t}}}\,\frac{1}{\sqrt{(\pi/2)(\frac{k}{\lambda}-t)}}\,\mathrm{e}^{-\frac{(\frac{s}{2})^2}{(1/2)(\frac{k}{\lambda}-t)}} \\
&\qquad \sim \frac{3}{2}\int_t^{\infty} \mathrm{e}^{-zx}\sqrt{\frac{\lambda}{4\pi x}}\,\mathrm{e}^{-\frac{\lambda(3t-x)^2}{4x}}\,\frac{1}{\sqrt{(\pi/2)(x-t)}}\,\mathrm{e}^{-\frac{(\frac{s}{2})^2}{(1/2)(x-t)}}\,\mathrm{d}x \\
&\qquad \sim \frac{3}{2}\int_t^{\infty} \mathrm{e}^{-zx}\sqrt{\frac{\lambda}{12\pi t}}\,\mathrm{e}^{-\frac{\lambda(3t-x)^2}{12t}}\,\frac{1}{\sqrt{(\pi/2)(x-t)}}\,\mathrm{e}^{-\frac{(\frac{s}{2})^2}{(1/2)(x-t)}}\,\mathrm{d}x \\
&\qquad \sim \frac{3}{2\sqrt{\pi t}}\,\mathrm{e}^{-3zt-\frac{s^2}{4t}} \;=\; h_3(t,s,z).
\end{aligned} \tag{6.86}$$

Here, factor $\frac{1}{2}$ in front of the second sum comes from the fact that $p_k(t,s) = 0$ for $k - t \overset{\mathrm{mod}\,2}{\neq} s$, while factor

$$\sqrt{\frac{\lambda}{4\pi(\frac{k}{\lambda})}}\exp\{-\frac{(3\lambda t-k)^2}{(12\lambda t)(\frac{k}{3\lambda t})}\} \sim \sqrt{\frac{\lambda}{12\pi t}}\exp\{-\frac{\lambda(3t-x)^2}{12t}\}$$

behaves as a delta-function in a neighborhood of $k = 3\lambda t$ or $x = 3t$, resulting in the asymptotic formula (6.86).

Let us turn to the rigorous proof of (6.86) and Lemma 6.4.2. Split

$$h_\lambda(t,s,z) := \sqrt{\lambda} g_3([\lambda t],[\sqrt{\lambda}s], 1-\lambda^{-1}z) = \sum_{i=0}^{5} h_{\lambda i}(t,s,z), \tag{6.87}$$

where

$$\begin{aligned}
h_{\lambda 0}(t,s,z) &:= \sqrt{\lambda}\sum_{k=[\lambda t]}^{\infty}\Big(1-\frac{z}{\lambda}\Big)^k p_k([\lambda t],[\sqrt{\lambda}s])\mathbf{1}(\lambda t \le K),\\
h_{\lambda 1}(t,s,z) &:= \sqrt{\lambda}\sum_{k=[\lambda t]}^{\infty}\Big(1-\frac{z}{\lambda}\Big)^k p_k([\lambda t],[\sqrt{\lambda}s])\mathbf{1}(K|3\lambda t-k|^3 \ge k^2,\ \lambda t > K),\\
h_{\lambda 2}(t,s,z) &:= \sqrt{\lambda}\sum_{k=[\lambda t]}^{\infty}\Big(1-\frac{z}{\lambda}\Big)^k b([\lambda t];k,\frac{1}{3})\Big\{p(k-[\lambda t],[\sqrt{\lambda}s]) - \bar p([2\lambda t],[\sqrt{\lambda}s])\Big\}\\
&\qquad\times\ \mathbf{1}(K|3\lambda t-k|^3 < k^2,\ \lambda t > K),\\
h_{\lambda 3}(t,s,z) &:= \sqrt{\lambda}\,\bar p([2\lambda t],[\sqrt{\lambda}s])\sum_{k=[\lambda t]}^{\infty}\Big\{\Big(1-\frac{z}{\lambda}\Big)^k - \Big(1-\frac{z}{\lambda}\Big)^{3\lambda t}\Big\} b([\lambda t];k,\frac{1}{3})\\
&\qquad\times\ \mathbf{1}(K|3\lambda t-k|^3 < k^2,\ \lambda t > K),\\
h_{\lambda 4}(t,s,z) &:= \sqrt{\lambda}\,\bar p([2\lambda t],[\sqrt{\lambda}s])\Big(1-\frac{z}{\lambda}\Big)^{3\lambda t}(V_\lambda(t)-3),\\
h_{\lambda 5}(t,s,z) &:= 3\sqrt{\lambda}\,\bar p([2\lambda t],[\sqrt{\lambda}s])\Big(1-\frac{z}{\lambda}\Big)^{3\lambda t},
\end{aligned}$$

and where $\bar p(t,s) := (p(t,s)+p(t,s+1))/2$, $t\in\mathbb{N}$, $s\in\mathbb{Z}$ and

$$V_\lambda(t) \ := \sum_{k=[\lambda t]}^{\infty} b([\lambda t];k,\frac{1}{3})\mathbf{1}(K|3\lambda t-k|^3 < k^2,\ \lambda t > K).$$

Here, $h_{\lambda 5}$ is the main term and $h_{\lambda i}, i = 0,1,\dots,4$ are remainder terms. In particular, we shall prove that

$$\lim_{K\to\infty}\limsup_{\lambda\to\infty}\sup_{\epsilon<t,|s|,z<1/\epsilon}|h_{\lambda i}(t,s,z)| = 0, \qquad \forall i = 0,1,2,3,4,\ \forall\epsilon>0. \tag{6.88}$$

Relations (6.88) are used to prove (6.17). The proof of (6.49) also uses the decomposition (6.87), with $K > 0$ a fixed large number.

<u>*Step 1* (estimation of $h_{\lambda 5}$).</u> For any $\epsilon > 0$,

$$\lim_{\lambda\to\infty}\sup_{\epsilon<t,|s|,z<1/\epsilon}|h_{\lambda 5}(t,s,z) - h_3(t,s,z)| = 0. \tag{6.89}$$

Moreover, there exist constants $C, c > 0$ such that for all sufficiently large λ and any $(t,s,z)\in\mathbb{R}^3$, $t>0$, $s\in\mathbb{R}$, $0<z<\lambda$ the following inequality holds

$$|h_{\lambda5}(t,s,z)| \ < \ C\Big(\bar h_3(t,s,z)+\sqrt{\lambda}\mathrm{e}^{-zt-c(\lambda t)^{1/3}-c(\sqrt{\lambda}|s|)^{1/2}}\Big). \tag{6.90}$$

Relations (6.85) and $\lim_{\lambda\to\infty}\sup_{\epsilon<t,z<1/\epsilon}|(1-\frac{z}{\lambda})^{3\lambda t}-\mathrm{e}^{-3zt}|=0$ easily imply (6.89).

Consider (6.90). Split

$$h_{\lambda5}(t,s,z)\le\sum_{i=1}^{3}h^i_{\lambda5}(t,s,z),$$

where

$$\begin{aligned}
h^1_{\lambda5}(t,s,z) \ &:= \ h_{\lambda5}(t,s,z)\mathbf{1}\Big(\sqrt{\lambda}t^2>K|s|^3,\ \lambda t>K\Big),\\
h^2_{\lambda5}(t,s,z) \ &:= \ h_{\lambda5}(t,s,z)\mathbf{1}\Big(\sqrt{\lambda}t^2\le K|s|^3,\ \lambda t>K\Big),\\
h^3_{\lambda5}(t,s,z) \ &:= \ h_{\lambda5}(t,s,z)\mathbf{1}\Big(\lambda t\le K\Big).
\end{aligned}$$

Then, (6.85) together with $0\le 1-\frac{z}{\lambda}\le\mathrm{e}^{-z/\lambda}$, $0<z<\lambda$ imply that

$$h^1_{\lambda5}(t,s,z)<\frac{C}{\sqrt{t}}\,\mathrm{e}^{-3zt-\frac{[\sqrt{\lambda}s]^2}{4\lambda t}}\Big(1+\frac{1}{K}\Big),\qquad \forall\,K>K_0,\quad \forall\,t>0,\ s\in\mathbb{R},\ 0<z<\lambda. \tag{6.91}$$

Note that $\sqrt{\lambda}|s|\ge 2$ implies $[\sqrt{\lambda}s]^2\ge(1/4)\lambda s^2$, while $\sqrt{\lambda}|s|<2$ and $\lambda t>K\ge 1$ imply $\mathrm{e}^{-s^2/16t}>\mathrm{e}^{-1/4}$. Hence and from (6.91) we obtain

$$h^1_{\lambda5}(t,s,z) \ < \ C\bar h_3(t,s,z),\qquad \forall\,t>0,\ s\in\mathbb{R},\ 0<z<\lambda. \tag{6.92}$$

To estimate $h^2_{\lambda5}$, we use the well-known Hoeffding's inequality [46]. Let $b(t;k,p)$ be the binomial distribution in (6.81). Then for any $\tau>0$

$$\sum_{|t-kp|>\tau\sqrt{k}} b(t;k,p) \ \le \ 2\mathrm{e}^{-2\tau^2}. \tag{6.93}$$

In terms of $p(u,v)$ of (6.82), inequality (6.93) writes as

$$\sum_{|v|>2\tau\sqrt{u}} p(u,v) \ \le \ 2\mathrm{e}^{-2\tau^2},\qquad \forall\,\tau>0. \tag{6.94}$$

We shall also use the following bound

$$p(u,v) \ \le \ 2\mathrm{e}^{-v^2/2u},\qquad \forall\,u,v\in\mathbb{Z},\ u\ge 0,\ |v|\le u, \tag{6.95}$$

which easily follows from (6.94). Using (6.95), for any $t>0$, $s\in\mathbb{R}$, $0<z<\lambda$,

$\lambda > 0,\ K > 0$ we obtain

$$\begin{aligned} h^2_{\lambda 5}(t,s,z) &< 2\sqrt{\lambda}\mathrm{e}^{-3zt-\frac{[\sqrt{\lambda}|s|]^2}{2[2\lambda t]}}\mathbf{1}\Big(\sqrt{\lambda}t^2 \le K|s|^3,\ \lambda t > K\Big) \\ &\le C(K)\sqrt{\lambda}\exp\Big\{-3zt-(1/16)\max\Big(\frac{(\lambda t)^{1/3}}{K^{2/3}},\frac{(\sqrt{\lambda}|s|)^{1/2}}{K^{1/2}}\Big)\Big\} \\ &\le C(K)\sqrt{\lambda}\exp\Big\{-3zt-\frac{(\lambda t)^{1/3}}{32K^{2/3}}-\frac{(\sqrt{\lambda}|s|)^{1/2}}{32K^{1/2}}\Big\}. \end{aligned} \tag{6.96}$$

Indeed, $[\sqrt{\lambda}|s|] \ge \sqrt{\lambda}|s| - 1 \ge \frac{\sqrt{\lambda}|s|}{2}$ for $|s| > 2/\sqrt{\lambda}$ and hence

$$\frac{[\sqrt{\lambda}|s|]^2}{2[2\lambda t]} \ge \frac{s^2}{16t} \ge \frac{1}{16}\max\Big(\frac{(\lambda t)^{1/3}}{K^{2/3}},\frac{(\sqrt{\lambda}|s|)^{1/2}}{K^{1/2}}\Big) \ge \frac{(\lambda t)^{1/3}}{32K^{2/3}}+\frac{(\sqrt{\lambda}|s|)^{1/2}}{32K^{1/2}} \tag{6.97}$$

holds for $\sqrt{\lambda}t^2 \le K|s|^3,\ |s| > 2/\sqrt{\lambda}$. On the other hand,

$$\Big(\frac{\sqrt{\lambda}t^2}{K}\Big)^{1/3} < |s| < 2/\sqrt{\lambda}$$

implies $\lambda t < 2^{3/2}K^{1/2}$ in which case the r.h.s. of (6.97) does not exceed

$$\frac{\sqrt{2}}{32}\Big(1+\frac{1}{\sqrt{K}}\Big) =: c(K)$$

and (6.96) holds with $C(K) = 2\mathrm{e}^{c(K)}$. A similar bound as in (6.96) follows for $h^3_{\lambda 5}(t,s,z)$, using

$$\begin{aligned} h^2_{\lambda 5}(t,s,z) &\le C\sqrt{\lambda}\mathrm{e}^{-3zt}(p([\lambda t],[\sqrt{\lambda}s]) + p([\lambda t],[\sqrt{\lambda}s]+1))\mathbf{1}(\lambda t \le K) \\ &\le C\sqrt{\lambda}\mathrm{e}^{-3zt}\mathbf{1}(\lambda t \le K,\ |[\sqrt{\lambda}s]| \le K). \end{aligned}$$

The desired inequality in (6.90) now follows by combining (6.92) and (6.96) and taking $K > K_0$ a fixed and sufficiently large number.

Step 2 (estimation of $h_{\lambda 4}$). Let us show (6.88) for $i = 4$ and that there exist constants $C, c > 0$ such that for all sufficiently large λ and any $(t,s,z) \in \mathbb{R}^3$, $t > 0,\ s \in \mathbb{R},\ 0 < z < \lambda$ the following inequality holds:

$$|h_{\lambda 4}(t,s,z)| < C\Big(\bar{h}_3(t,s,z) + \sqrt{\lambda}\mathrm{e}^{-zt-c(\lambda t)^{1/3}-c(\sqrt{\lambda}|s|)^{1/2}}\Big). \tag{6.98}$$

Indeed,

$$|h_{\lambda 4}(t,s,z)| \le Ch_{\lambda 5}(t,s,z)|V_\lambda(t) - 3|.$$

Therefore the above facts ((6.88) for $i = 4$ and (6.98)) follow from Step 1 and the following bound: There exist $C, c > 0$ and $K_0 > 0$ such that

$$|V_\lambda(t) - 3| \ < C\Big(K^{-1/3} + \mathrm{e}^{-c(\sqrt{\lambda t}/K)^{2/3}}\Big), \qquad \forall\ \lambda > 0,\ t > 0,\ \lambda t > K,\ K > K_0. \tag{6.99}$$

To show (6.99) we use the Moivre-Laplace approximation in (6.84). Accordingly, $V_\lambda(t) = V_{\lambda 1}(t) + V_{\lambda 2}(t)$, where

$$V_{\lambda 1}(t) := \frac{3}{2\sqrt{\pi}} \sum_{k=[\lambda t]}^{\infty} \frac{1}{\sqrt{k}}\, \mathrm{e}^{-(3[\lambda t]-k)^2/4k} \mathbf{1}(K|3\lambda t - k|^3 < k^2,\ \lambda t > K)$$

and where $V_{\lambda 2}(t)$ satisfies

$$|V_{\lambda 2}(t)| \ < \ \frac{C}{K} V_{\lambda 1}(t)$$

for all $\lambda > 0$, $t > 0$, $\lambda t > K$, $K > K_0$ and some $C > 0$ and $K_0 > 0$ independent of λ, t, and K. Hence, it suffices to prove (6.99) for $V_{\lambda 1}(t)$ instead of $V_\lambda(t)$.

Let

$$\mathcal{D}_K(\tau) := \{k \in \mathbb{N} : K|3\tau - k|^3 < k^2\}, \qquad \tau > 0.$$

There exist $C > 0$ and $\tau_0 > 0$ such that $k \in \mathcal{D}_K(\tau)$ implies

$$|k - 3\tau| < C\tau^{2/3}/K^{1/3} \quad \text{and} \quad 2\tau < k < 4\tau, \qquad \forall\, \tau > \tau_0. \tag{6.100}$$

Indeed, let $k \le 3\tau$. Then

$$|k - 3\tau| < k^{2/3}/K^{1/3} \le 3^{2/3}\tau^{2/3}/K^{1/3}$$

and the first inequality in (6.100) holds with $C = 3^{2/3}$. Next, let $k > 3\tau$. Then $k^{2/3}/K^{1/3} < k/4$ for $\tau > \tau_0$ and some $\tau_0 > 0$ and hence $k - 3\tau < k/4$ implying $k < 4\tau$. In turn this implies $|k - 3\tau| < (4\tau)^{2/3}/K^{1/3}$ and (6.100) holds with $C = 4^{2/3}$.

Consider

$$\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{3\lambda t}} = \frac{1}{\sqrt{3\lambda t}}\Big(\frac{1}{\sqrt{1 + \frac{k - 3\lambda t}{3\lambda t}}} - 1\Big).$$

Using $|1 - \frac{1}{\sqrt{1+x}}| \le |x|$ and $|1 - \frac{1}{1+x}| \le 2|x|$ for $|x| \le 1/2$ we obtain

$$\Big|\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{3\lambda t}}\Big| \le \frac{C}{\sqrt{\lambda t}}\, \frac{1}{K^{1/3}(\lambda t)^{1/3}} \ < \ \frac{C}{\sqrt{\lambda t} K^{2/3}}, \quad \Big|\frac{1}{k} - \frac{1}{3\lambda t}\Big| \ < \ \frac{C}{(\lambda t)^{4/3} K^{1/3}}, \tag{6.101}$$

for some constant $C < \infty$ and all $|k - 3\lambda t| < C(\lambda t)^{2/3}/K^{1/3}$, $\lambda t > K > K_0$ and $K_0 > 0$ large enough. From (6.101) and (6.100), for the above values of k, λ, t, K

we obtain

$$\Big|\frac{1}{\sqrt{k}}\,\mathrm{e}^{-(3[\lambda t]-k)^2/4k} - \frac{1}{\sqrt{3\lambda t}}\,\mathrm{e}^{-(3[\lambda t]-k)^2/12\lambda t}\Big| \ < \ \frac{C}{K^{1/3}}\frac{1}{\sqrt{\lambda t}}\mathrm{e}^{-(3[\lambda t]-k)^2/12\lambda t}.$$

Hence, $|V_{\lambda 1}(t) - 3| \le |U_{\lambda 1}(t) - 3| + U_{\lambda 2}(t) + \big(\frac{C}{K^{1/3}}\big)U_{\lambda 3}(t)$, where

$$\begin{aligned}
U_{\lambda 1}(t) \ &:= \ \frac{3}{2\sqrt{3\pi\lambda t}}\sum_{k=[\lambda t]}^{\infty}\mathrm{e}^{-(3[\lambda t]-k)^2/12\lambda t}\mathbf{1}(\lambda t > K),\\
U_{\lambda 2}(t) \ &:= \ \frac{3}{2\sqrt{3\pi\lambda t}}\sum_{k=[\lambda t]}^{\infty}\mathrm{e}^{-(3[\lambda t]-k)^2/12\lambda t}\mathbf{1}(K|3\lambda t-k|^3 \ge k^2,\, \lambda t > K),\\
U_{\lambda 3}(t) \ &:= \ \frac{1}{\sqrt{\lambda t}}\sum_{k\in\mathbb{Z}}\mathrm{e}^{-k^2/12\lambda t}\mathbf{1}(\lambda t > K).
\end{aligned}$$

It is easy to show that $U_{\lambda 3}(t) < C$ and

$$|U_{\lambda 1}(t) - 3| = \Big|U_{\lambda 1}(t) - \frac{3}{2\sqrt{3\pi}}\int_{\mathbb{R}}\mathrm{e}^{-x^2/12}\,\mathrm{d}x\Big| < C/\sqrt{\lambda t} < C/K^{1/2}$$

uniformly in $\lambda > 0, t > 0, K > K_0$. Next, with $j = k - 3[\lambda t]$ and using the fact that $k^2 = (j + 3[\lambda t])^2 \ge [\lambda t]^2 \ge (\lambda t)^2/2$

$$\begin{aligned}
|U_{\lambda 2}(t)| \ &\le \ \frac{C}{\sqrt{\lambda t}}\sum_{j\ge -2[\lambda t]}\mathrm{e}^{-(\frac{j}{\sqrt{\lambda t}})^2/12}\mathbf{1}\Big(K|\frac{j}{\sqrt{\lambda t}}|^3 \ge \frac{(\lambda t)^2}{2(\lambda t)^{3/2}}\Big)\\
&\le \ C\int\mathbf{1}(K|x|^3 > \sqrt{\lambda t}/2)\mathrm{e}^{-x^2}\,\mathrm{d}x \ \le \ C\mathrm{e}^{-c(\sqrt{\lambda t}/K)^{2/3}}.
\end{aligned}$$

This proves (6.99) and hence (6.98), too.

Step 3 (estimation of $h_{\lambda 3}$). First we estimate the difference inside the curly brackets. There exist $C, K_0, \tau_0 > 0$ such that $k \in \mathcal{D}_K(\tau)$, $K > K_0$, $\tau > \tau_0$ imply

$$|a^k - a^{3\tau}| \le Ca^{2\tau}\frac{\tau^{2/3}}{K^{1/3}}|1-a|, \quad \forall\, a \in [0,1]. \tag{6.102}$$

Indeed, let $k \le 3\tau$. Using (6.100) and

$$1 - a^\tau \le (1+\tau)(1-a), \qquad \forall\,\tau \ge 0, \quad \forall\, a \in [0,1],$$

for sufficiently large $\tau > K$ we obtain

$$\begin{aligned}
|a^k - a^{3\tau}| \ &\le \ a^k|a^{3\tau-k} - 1| \le a^{2\tau}|3\tau - k + 1||1-a|\\
&\le \ Ca^{2\tau}\frac{\tau^{2/3}+1}{K^{1/3}}|1-a| < \frac{C}{K^{1/3}}a^{2\tau}\tau^{2/3}|1-a|.
\end{aligned}$$

The case $k > 3\tau$ in (6.102) follows analogously. Using (6.102) and (6.99), together with the inequality $z\mathrm{e}^{-2z} < C\mathrm{e}^{-z}$, $z > 0$, we obtain

$$\begin{aligned}|h_{\lambda 3}(t,s,z)| &< \sqrt{\lambda}\,\bar p([2\lambda t],[\sqrt{\lambda}s])\Big(1-\frac{z}{\lambda}\Big)^{2\lambda t}\frac{(\lambda t)^{2/3}(z/\lambda)}{K^{1/3}}V_\lambda(t)\\ &< C\sqrt{\lambda}\,\bar p([2\lambda t],[\sqrt{\lambda}s])\mathrm{e}^{-2zt}\frac{(\lambda t)^{2/3}(zt)}{\lambda t}\\ &< \frac{C}{(\lambda t)^{1/3}}\sqrt{\lambda}\,\bar p([2\lambda t],[\sqrt{\lambda}s])\mathrm{e}^{-zt}.\end{aligned}$$

Therefore as in Step 2 we obtain the convergence in (6.88) for $i = 3$ together with the bound

$$|h_{\lambda 3}(t,s,z)| < C\Big(\bar h_3(t,s,z) + \sqrt{\lambda}\mathrm{e}^{-zt-c(\lambda t)^{1/3}-c(\sqrt{\lambda}|s|)^{1/2}}\Big). \tag{6.103}$$

Step 4 (estimation of $h_{\lambda 2}$). First we estimate the difference inside the curly brackets. There exist $C > 0$ and $K_0 > 0$ such that for any λ, t, s, k, K satisfying

$$\lambda > 0,\ t > 0,\ s \in \mathbb{R},\ k \in \mathbb{N},\ K > K_0,\ \lambda t > K,\ K|k-3\lambda t|^3 \le k^2,\ \lambda^{1/2}t^2 > K|s|^3, \tag{6.104}$$

the following inequality holds

$$\Big|\bar p(k-[\lambda t],[\sqrt{\lambda}s]) - \bar p([2\lambda t],[\sqrt{\lambda}s])\Big| \le \frac{C}{(\lambda t)^{1/2}K^{2/3}}\mathrm{e}^{-s^2/10t}. \tag{6.105}$$

In the proof of (6.105), below, assume that $k-[\lambda t] \overset{\mathrm{mod}\,2}{=} [\sqrt{\lambda}s]$, $[2\lambda t] \overset{\mathrm{mod}\,2}{=} [\sqrt{\lambda}s]$; the remaining cases can be discussed analogously. Using the Moivre-Laplace formula (6.85) we have that

$$\begin{aligned}p(k-[\lambda t],[\sqrt{\lambda}s]) - p([2\lambda t],[\sqrt{\lambda}s]) &= \frac{2}{\sqrt{2\pi(k-[\lambda t])}}\mathrm{e}^{-\frac{[\sqrt{\lambda}s]^2}{2(k-[\lambda t])}}\Big(1+O\Big(\frac{1}{K}\Big)\Big)\\ &\quad - \frac{2}{\sqrt{2\pi[2\lambda t]}}\mathrm{e}^{-\frac{[\sqrt{\lambda}s]^2}{2[2\lambda t]}}\Big(1+O\Big(\frac{1}{K}\Big)\Big).\end{aligned}$$

As in (6.101),

$$\Big|\frac{1}{\sqrt{k-[\lambda t]}} - \frac{1}{\sqrt{[2\lambda t]}}\Big| \le \frac{C}{(\lambda t)^{1/2}K^{2/3}}, \qquad \Big|\frac{1}{k-[\lambda t]} - \frac{1}{[2\lambda t]}\Big| \le \frac{C}{(\lambda t)^{4/3}K^{1/3}}.$$

Hence it easily follows

$$\mathrm{e}^{-\frac{[\sqrt{\lambda}s]^2}{2(k-[\lambda t])}} < C\mathrm{e}^{-\frac{s^2}{10t}},$$

where the arguments satisfy (6.104). The above facts imply (6.105). Using (6.105) for s satisfying (6.104) we can write

$$\begin{aligned}|h_{\lambda 2}(t,s,z)| &< \frac{C}{t^{1/2}K^{2/3}}\mathrm{e}^{-s^2/10t}\sum_{k=[\lambda t]}^{\infty}\Big(1-\frac{z}{\lambda}\Big)^k b\Big([\lambda t];k,\frac{1}{3}\Big)\\ &\qquad\times \mathbf{1}(K|3\lambda t-k|^3<k^2,\ \lambda t>K)\\ &< \frac{C}{K^{2/3}}\bar{h}_3(t,s,z)V_\lambda(t) \ < \ \frac{C}{K^{2/3}}\bar{h}_3(t,s,z), \qquad (6.106)\end{aligned}$$

see (6.99).

Next we will evaluate $h_{\lambda 2}(t,s,z)$ for $\lambda^{1/2}t^2 < K|s|^3$ (and λ, t, k, K satisfying (6.104)). Using the inequality in (6.95), the bound $k < 4\lambda t$, see (6.100), and arguing as in (6.97) we have that

$$\frac{[\sqrt{\lambda}s]^2}{2(k-[\lambda t])} > \frac{s^2}{6t} > \max\left\{\frac{(\sqrt{\lambda}s)^{1/2}}{6K^{1/2}}, \frac{(\lambda t)^{1/3}}{6K^{2/3}}\right\}$$

and hence

$$p([\lambda t],[\sqrt{\lambda}s]) \le C\mathrm{e}^{-c(K)(\lambda t)^{1/3}-c(K)(\sqrt{\lambda}|s|)^{1/2}},$$

where $c(K)>0$ depends only on K. Therefore,

$$\begin{aligned}|h_{\lambda 2}(t,s,z)| &< C\sqrt{\lambda}\mathrm{e}^{-2zt-c(K)(\lambda t)^{1/3}-c(K)(\sqrt{\lambda}|s|)^{1/2}}V_\lambda(t)\\ &< C\sqrt{\lambda}\mathrm{e}^{-2zt-c(K)(\lambda t)^{1/3}-c(K)(\sqrt{\lambda}|s|)^{1/2}}. \qquad (6.107)\end{aligned}$$

The resulting bound

$$|h_{\lambda 2}(t,s,z)| \ < \ C\Big(\bar{h}_3(t,s,z)+\sqrt{\lambda}\mathrm{e}^{-zt-c(\lambda t)^{1/3}-c(\sqrt{\lambda}|s|)^{1/2}}\Big) \qquad (6.108)$$

follows from (6.106) and (6.107) by taking $K > K_0$ sufficiently large but fixed.

Step 5 (estimation of $h_{\lambda 1}$). From (6.93), we have

$$b([\lambda t];k,1/3) \le 2\mathrm{e}^{-(2/9)|3[\lambda t]-k|^2/k}.$$

Using this and a similar inequality (6.95) for $p(k-[\lambda t],[\sqrt{\lambda}s])$ we see that

$$\begin{aligned}|h_{\lambda 1}(t,s,z)| &< C\sqrt{\lambda}\mathrm{e}^{-zt}\sum_{k=[\lambda t]}^{\infty}\mathrm{e}^{-(2/9)\frac{|3[\lambda t]-k|^2}{k}}\mathrm{e}^{-(1/2)\frac{[\sqrt{\lambda}s]^2}{k-[\lambda t]}}\mathbf{1}(K|3\lambda t-k|^3\ge k^2, \lambda t>K)\\ &< C\sqrt{\lambda}\mathrm{e}^{-zt}\sum_{k\ge\lambda t}\mathrm{e}^{-ck^{1/3}-c\frac{(\sqrt{\lambda}s)^2}{k-\lambda t}}\end{aligned}$$

for some positive constant $c > 0$ depending on K. Split the sum

$$\sum_{k\ge\lambda t} \mathrm{e}^{-ck^{1/3}-c\frac{(\sqrt{\lambda}s)^2}{k-\lambda t}} = \sum_{\lambda t<k\le\lambda t+(\sqrt{\lambda}|s|)^{3/2}} \mathrm{e}^{-ck^{1/3}-c\frac{(\sqrt{\lambda}s)^2}{k-\lambda t}} + \sum_{k>\lambda t+(\sqrt{\lambda}|s|)^{3/2}} \mathrm{e}^{-ck^{1/3}-c\frac{(\sqrt{\lambda}s)^2}{k-\lambda t}}$$
$$=: \Sigma_1 + \Sigma_2.$$

Then

$$\Sigma_1 < C\mathrm{e}^{-c(\sqrt{\lambda}|s|)^{1/2}} \sum_{k\ge\lambda t} \mathrm{e}^{-ck^{1/3}} < C\mathrm{e}^{-c(\lambda t)^{1/3}-c(\sqrt{\lambda}|s|)^{1/2}}$$

and

$$\Sigma_2 < C \sum_{k\ge\lambda t+(\sqrt{\lambda}|s|)^{3/2}} \mathrm{e}^{-ck^{1/3}} < C\mathrm{e}^{-c(\lambda t+(\sqrt{\lambda}|s|)^{3/2})^{1/3}} < C\mathrm{e}^{-c(\lambda t)^{1/3}-c(\sqrt{\lambda}|s|)^{1/2}}.$$

By taking $K > K_0$ sufficiently large but fixed the above calculations lead to the bound

$$|h_{\lambda 1}(t,s,z)| \ < \ C\sqrt{\lambda}\mathrm{e}^{-zt-c(\lambda t)^{1/3}-c(\sqrt{\lambda}|s|)^{1/2}}. \tag{6.109}$$

Step 6 (estimation of $h_{\lambda 0}$). Similarly as in Step 5 we obtain

$$|h_{\lambda 0}(t,s,z)| \ < \ C\sqrt{\lambda}\mathrm{e}^{-zt-c(\lambda t)^{1/3}-c(\sqrt{\lambda}|s|)^{1/2}}. \tag{6.110}$$

The proof of Lemma 6.4.2 follows from Steps 1 - 6. □

Proof of Lemma 6.5.1. Let $W_k = (W_{1k}, W_{2k}) \in \mathbb{Z}^2$ and

$$\widetilde{W}_{1k} := W_{1k} + W_{2k}, \qquad \widetilde{W}_{1k} := W_{1k} - W_{2k}.$$

Then $\widetilde{W}_k = (\widetilde{W}_{1k}, \widetilde{W}_{2k})$, $k = 0, 1, \ldots$, is a random walk on the even lattice

$$\widetilde{\mathbb{Z}}^2 := \{(u,v) \in \mathbb{Z}^2 : u+v \text{ is even}\} \ = \ \{(u,v) \in \mathbb{Z}^2 : u \overset{\text{mod } 2}{=} v\} \tag{6.111}$$

with one-step transition probabilities

$$\mathrm{P}(\widetilde{W}_1 = (i,j)|\widetilde{W}_0 = (0,0)) \ = \ 1/4, \qquad i,j = \pm 1.$$

Note that $\{\tilde{W}_{1k}\}$ and $\{\tilde{W}_{2k}\}$ are independent symmetric random walks on $\mathbb{Z}$ and therefore

$$\tilde{p}_k(u,v) := \mathrm{P}(\widetilde{W}_k = (u,v)|\widetilde{W}_0 = (0,0)) = p(k,u)p(k,v), \quad (u,v) \in \widetilde{\mathbb{Z}}^2, \quad k = 0,1,\ldots,$$

where $p(u,v)$ is the $u-$th step transition probability for the symmetric random walk on $\mathbb{Z}$ as given in (6.82). The above facts imply the following factorization property:

$$p_k(t,s) \ = \ \tilde p_k(t+s,t-s) \ = \ p(k,t+s)p(k,t-s), \quad t,s\in\mathbb{Z},\ k=0,1,\dots. \quad (6.112)$$

In particular, $p_k(t,s)=0$ if $k \overset{\mathrm{mod}\,2}{\neq} t+s$. Split

$$g_4([\lambda t],[\lambda s],1-\frac{z}{\lambda^2}) = \sum_{i=1}^3 \gamma_{\lambda i}(t,s,z),$$

where

$$\gamma_{\lambda i}(t,s,z) := \lambda^2\int_0^\infty \Big(1-\frac{z}{\lambda^2}\Big)^{[\lambda^2 x]} p_{[\lambda^2 x]}([\lambda t],[\lambda s])\mathbf{1}(x\in I_{\lambda i}(t,s))\,\mathrm{d}x, \quad i=1,2,3,$$

and where

$$\begin{aligned} I_{\lambda 1}(t,s) &:= \Big\{x>0:\ \lambda x^2 > K(|t|^3+|s|^3),\ \lambda^2 x > K,\ [\lambda^2 x] \overset{\mathrm{mod}\,2}{=} [\lambda t]+[\lambda s]\Big\}, \\ I_{\lambda 2}(t,s) &:= \Big\{x>0:\ \lambda x^2 \le K(|t|^3+|s|^3),\ \lambda^2 x > K,\ [\lambda^2 x] \overset{\mathrm{mod}\,2}{=} [\lambda t]+[\lambda s]\Big\}, \\ I_{\lambda 3}(t,s) &:= \Big\{x>0:\ \lambda^2 x \le K,\ [\lambda^2 x] \overset{\mathrm{mod}\,2}{=} [\lambda t]+[\lambda s]\Big\} \end{aligned}$$

satisfy $\bigcup_{i=1}^3 I_{\lambda i}(t,s) = I_{\lambda 0}(t,s) := \Big\{x>0:\ [\lambda^2 x] \overset{\mathrm{mod}\,2}{=} [\lambda t]+[\lambda s]\Big\}$. Also split

$$h_4(t,s,z) = \pi^{-1}\int_0^\infty x^{-1}\mathrm{e}^{-zx-\frac{t^2+s^2}{x}}\ \mathrm{d}x = \sum_{i=0}^3 h_{\lambda i}(t,s,z),$$

where

$$\begin{aligned} h_{\lambda 0}(t,s,z) &:= \pi^{-1}\int_0^\infty x^{-1}\mathrm{e}^{-zx-\frac{t^2+s^2}{x}}(1-2\mathbf{1}(x\in I_{\lambda 0}(t,s)))\,\mathrm{d}x, \\ h_{\lambda i}(t,s,z) &:= 2\pi^{-1}\int_0^\infty x^{-1}\mathrm{e}^{-zx-\frac{t^2+s^2}{x}}\ \mathbf{1}(x\in I_{\lambda i}(t,s))\,\mathrm{d}x, \qquad i=1,2,3. \end{aligned}$$

We shall prove below

$$\lim_{\lambda,K\to\infty}\ \sup_{\epsilon<|t|+|s|<1/\epsilon,\ \epsilon<z<1/\epsilon} (|\gamma_{\lambda 1}(t,s,z)-h_{\lambda 1}(t,s,z)|+|h_{\lambda 0}(t,s,z)|) = 0, \qquad \forall\,\epsilon>0, \quad (6.113)$$

and that for any sufficiently large $K>K_0$ there exist $c(K),C(K)<\infty$ independent of t,s,z,λ and such that for any $(t,s)\in\mathbb{R}^2_0$, $0<z<\lambda^2$ the following inequalities hold:

$$\gamma_{\lambda 1}(t,s,z) + |h_{\lambda 0}(t,s,z)| \le C(K)h_4(t,s,z), \quad (6.114)$$

$$\gamma_{\lambda i}(t,s,z)+h_{\lambda i}(t,s,z)\le C(K)\mathrm{e}^{-c(K)(|\lambda t|^{1/2}+|\lambda s|^{1/2})},\qquad i=2,3. \tag{6.115}$$

Relations (6.114)-(6.115) imply statement (6.75) of the lemma. Statement (6.74) follows from

$$\begin{aligned}|g_4([\lambda t],[\lambda s],1-\frac{z}{\lambda^2})-h_4(t,s,z)| &\le |h_{\lambda 0}(t,s,z)|+|\gamma_{\lambda 1}(t,s,z)-h_{\lambda 1}(t,s,z)|\\ &+ \sum_{i=2}^{3}(\gamma_{\lambda i}(t,s,z)+h_{\lambda i}(t,s,z))\end{aligned}$$

and using (6.113) and the bounds in (6.114)-(6.115).

Let us prove (6.114). Clearly, $|h_{\lambda 0}(t,s,z)|\le 2h_4(t,s,z)$ by the definition of $h_{\lambda 0}$ so that we need to estimate $\gamma_{\lambda 1}$ only. Note (6.85) and (6.112) imply

$$\sup_{x,t,s}\Big|\frac{p_{[\lambda^2x]}([\lambda t],[\lambda s])}{\frac{2}{\pi[\lambda^2 x]}\mathrm{e}^{-\frac{[\lambda t]^2+[\lambda s]^2}{[\lambda^2 x]}}}-1\Big|\mathbf{1}(x\in I_{\lambda 1}(t,s))<\frac{C}{K},\qquad \forall\, K>K_0. \tag{6.116}$$

We also need the bound

$$\sup_{x,t,s}\Big|\frac{\frac{2}{\pi[\lambda^2 x]}\mathrm{e}^{-\frac{[\lambda t]^2+[\lambda s]^2}{[\lambda^2 x]}}}{\frac{2}{\pi\lambda^2 x}\mathrm{e}^{-\frac{t^2+s^2}{x}}}-1\Big|\mathbf{1}(x\in I_{\lambda 1}(t,s))<\frac{C}{K^{2/3}},\qquad \forall\, K>K_0. \tag{6.117}$$

which follows from

$$\Big|\frac{\lambda^2 x}{[\lambda^2 x]}-1\Big|<C_1/K,\qquad \Big|\frac{t^2+s^2}{x}-\frac{[\lambda t]^2+[\lambda s]^2}{[\lambda^2 x]}\Big|<C_2/K^{2/3},$$

for $x\in I_{\lambda 1}(t,s)$, with C_1,C_2 independent of x,t,s,λ,K. From (6.116) and (6.117) we obtain

$$\chi(\lambda,K) := \sup_{x,t,s}\Big|\frac{p_{[\lambda^2x]}([\lambda t],[\lambda s])}{\frac{2}{\pi\lambda^2 x}\mathrm{e}^{-\frac{t^2+s^2}{x}}}-1\Big|\mathbf{1}(x\in I_{\lambda 1}(t,s))<\frac{C}{K^{2/3}},\qquad \forall\, K>K_0. \tag{6.118}$$

Using (6.118) and

$$\Big(1-\frac{z}{\lambda^2}\Big)^{[\lambda^2 x]}\le \mathrm{e}^{z/\lambda^2-z[\lambda^2x]/\lambda^2}\le C\mathrm{e}^{-zx},\qquad 0<z<\lambda^2,$$

we obtain

$$\begin{aligned}\gamma_{\lambda 1}(t,s,z) &\le C\lambda^2\int_0^\infty \mathrm{e}^{-zx}\frac{2}{\pi\lambda^2 x}\mathrm{e}^{-\frac{t^2+s^2}{x}}\Big(1+\chi(\lambda,K)\Big)\mathbf{1}(x\in I_{\lambda 1}(t,s))\,\mathrm{d}x\\ &\le Ch_{\lambda 1}(t,s,z)\ \le\ Ch_4(t,s,z),\qquad K>K_0,\end{aligned}$$

proving (6.114), with $C(K)$ independent of $K>K_0$. Similarly using (6.118) we obtain

$$
\begin{aligned}
&|\gamma_{\lambda 1}(t,s,z) - h_{\lambda 1}(t,s,z)| \le \\
&\quad \le \Big| \int_0^\infty \Big(1 - \frac{z}{\lambda^2}\Big)^{[\lambda^2 x]} \Big\{ \lambda^2 p_{[\lambda^2 x]}([\lambda t],[\lambda s]) - \frac{2}{\pi x} \mathrm{e}^{-\frac{t^2+s^2}{x}} \Big\} \mathbf{1}(x \in I_{\lambda 1}(t,s))\,\mathrm{d}x \Big| \\
&\qquad + 2 \Big| \int_0^\infty \Big\{ \Big(1 - \frac{z}{\lambda^2}\Big)^{[\lambda^2 x]} - \mathrm{e}^{-zx} \Big\} \pi^{-1} x^{-1} \mathrm{e}^{-\frac{t^2+s^2}{x}}\, \mathbf{1}(x \in I_{\lambda 1}(t,s))\,\mathrm{d}x \Big| \\
&\quad \le C \chi(\lambda, K) h_4(t,s,z) + C \int_0^\infty \theta_\lambda(z,x) x^{-1} \mathrm{e}^{-\frac{t^2+s^2}{x}}\,\mathrm{d}x
\end{aligned}
$$

where

$$
\theta_\lambda(z,x) := \Big| \Big(1 - \frac{z}{\lambda^2}\Big)^{[\lambda^2 x]} - \mathrm{e}^{-zx} \Big| \to 0, \qquad \text{as } \lambda \to \infty,
$$

for any $z > 0, x > 0$ fixed, and $|\theta_\lambda(z,x)| \le C\mathrm{e}^{-xz}$ for any $x, z, \lambda > 0$; see above. Therefore by the dominated convergence theorem,

$$
\int_0^\infty \theta_\lambda(z,x) x^{-1} \mathrm{e}^{-\frac{t^2+s^2}{x}}\,\mathrm{d}x \to 0, \qquad \text{as } \lambda \to \infty,
$$

and the last convergence is uniform in $\epsilon < |t| + |s| < 1/\epsilon$, $\epsilon < z < 1/\epsilon$ for any given $\epsilon > 0$. Together with (6.118) this proves (6.113) for the difference $|\gamma_{\lambda 1} - h_{\lambda 1}|$. Relation (6.113) for $|h_{\lambda 0}|$ follows by the mean value theorem, implying

$$
\Big| x^{-1} \mathrm{e}^{-zx - \frac{t^2+s^2}{x}} - y^{-1} \mathrm{e}^{-zy - \frac{t^2+s^2}{y}} \Big| \ \le\ C(\epsilon) |x - y| x^{-1} \mathrm{e}^{-zx - \frac{t^2+s^2}{x}} (1 + x^{-2})
$$

for $0 < x < y$, $0 < z < 1/\epsilon$, $|t| + |s| < 1/\epsilon$. Therefore,

$$
\sup_{\epsilon < |t|+|s| < 1/\epsilon,\, \epsilon < z < 1/\epsilon} |h_{\lambda 0}(t,s,z)| \ \le\ C/\lambda^2 \ =\ o(1),
$$

where

$$
C := \sup_{\epsilon < |t|+|s| < 1/\epsilon,\, z > \epsilon} \int_0^\infty x^{-1} \mathrm{e}^{-zx - \frac{t^2+s^2}{x}} (1 + x^{-2})\,\mathrm{d}x < \infty.
$$

It remains to prove (6.115). Note $\gamma_{\lambda 2}(t,s,z) \le \bar\gamma_2([\lambda t],[\lambda s])$, $0 < z < \lambda^2$, where

$$
\bar\gamma_2(t,s) := \sum p_k(t,s)\, \mathbf{1}(K < k < \sqrt{K(|t|^3 + |s|^3)}), \qquad t, s \in \mathbb{Z}.
$$

Note $K < k < \sqrt{K(|t|^3 + |s|^3)}$ implies

$$
\frac{(|t+s| + |t-s|)^4}{k^2} \ \ge\ \frac{(t+s)^4 + (t-s)^4}{K(|t|^3 + |s|^3)} \ \ge\ \frac{2(t^4 + s^4)}{K(|t|^3 + |s|^3)} \ \ge\ \frac{1}{4K}(|t|^{1/2} + |s|^{1/2})^2.
$$

Hence and using (6.95) we obtain

$$\begin{aligned}\bar\gamma_2(t,s) &\le \sum_{K<k<\sqrt{K(|t|^3+|s|^3)}} p(k,t+s)p(k,t-s) \\ &\le 4 \sum_{K<k<\sqrt{K(|t|^3+|s|^3)}} \exp\Big\{-\frac{|t|^{1/2}+|s|^{1/2}}{4\sqrt{K}}\Big\} \\ &< C(K)\,\mathrm{e}^{-c(K)(|t|^{1/2}+|s|^{1/2})}, \end{aligned} \tag{6.119}$$

where constants $C(K)>0$, $c(K)>0$ depend only on $K<\infty$. This proves (6.115) for $\gamma_{\lambda 2}$. The last bound in (6.119) holds for

$$\bar\gamma_3(t,s) := \sum_{k=0}^{K} p(k,t+s)p(k,t-s) \le (K+1)\mathbf{1}(|t+s|\le K, |t-s|\le K),$$

too, dominating $\gamma_{\lambda 3}(t,s,z) \le \bar\gamma_3([\lambda t],[\lambda s])$, $0<z<\lambda^2$. The remaining bounds in (6.115) follow easily. Lemma 6.5.1 is proved.

7

Ruin probability with claims modeled by $\alpha-$stable aggregated AR(1) process

Abstract. We study the asymptotics of the ruin probability in a discrete time risk insurance model with stationary claims following the aggregated heavy-tailed AR(1) process discussed in Chapter 4. The present work is based on the general characterization of the ruin probability with claims modeled by stationary $\alpha-$stable process in Mikosch and Samorodnitsky (2000, [75]). We prove that for the aggregated AR(1) claims' process, the ruin probability decays with exponent $\alpha(1-H)$, where $H \in [1/\alpha, 1)$ is the asymptotic self-similarity index of the claim process, $1 < \alpha < 2$. This result agrees with the decay rate of the ruin probability with claims modeled by increments of linear fractional motion in [75] and also with other characterizations of long memory of the aggregated AR(1) process with infinite variance in Chapter 4.

7.1 Introduction and the main result

In this Chapter we study the asymptotics of the ruin probability

$$\psi(u) := \mathrm{P}\Big(\sup_{n\geq 1}(\sum_{t=1}^{n} Y(t) - cn) > u\Big), \quad \text{as } u \to \infty, \tag{7.1}$$

where 'claims' $\{Y(t),\, t \in \mathbb{Z}\}$ form a stationary, $\alpha-$stable process of a certain type, $1 < \alpha < 2$, obtained by aggregating independent copies of random-coefficient AR(1) heavy-tailed processes. In (7.1), $c > 0$ is interpreted as a constant deterministic premium rate and u is the initial capital. The above problem was investigated in [75] for stable processes $\{Y(t),\, t \in \mathbb{Z}\}$. Applying large deviations methods for Poisson point processes, authors proved the asymptotics $\psi(u) \sim \psi_0(u)$, where $f(u) \sim g(u)$ means that $f(u)/g(u) \to 1$ as $u \to \infty$, and the function ψ_0 is written in terms of the kernel and the control measure of stochastic integral representation of $\{Y(t),\, t \in \mathbb{Z}\}$ (see (7.15), page 163), below, in the special case when $\{Y(t),\, t \in \mathbb{Z}\}$ is a mixed stable moving average). Using the above result, Mikosch and Samorodnitsky [75] obtained the 'classical' decay rate $\psi(u) \sim C\, u^{-(\alpha-1)}$, see e.g. [34], for a wide class of weakly dependent symmetric $\alpha-$stable (SαS) stationary claims, and a markedly different decay rate $\psi(u) \sim C\, u^{-\alpha(1-H)}$ for increments of fractional SαS motion with self-similarity index $H \in (1/\alpha, 1)$. In view of these findings, Mikosch and Samorodnitsky ([75], p.1817) propose the decay rate of the ruin probability as an alternative characteristic of long-range dependence of a SαS process. See also [5], [6].

The present Chapter complements the results in [75], by obtaining the characteristic decay of the ruin probability when claims are modeled by the mixed SαS process studied in Chapter 4. The latter process arises in the result of aggregation of independent copies of random-coefficient AR(1) processes with heavy-tailed innovations, following the classical scheme of contemporaneous aggregation (see [41]). Aggregation is a common procedure in statistical and econometric modeling and can explain certain empirical 'stylized facts' of financial time series (such as long memory) from simple heterogeneous dynamic models describing the evolution of individual 'agents'. See [29], [39], [102], [103], [104], among others.

In Chapters 3 and 4, we discussed aggregation of infinite variance random-coefficient AR(1) processes and long-memory properties of the limit aggregated process. Let us recall the main results from the Chapter 4. Let $\{X(t), t \in \mathbb{Z}\}$ be a stationary solution of the AR(1) equation

$$X(t) = aX(t-1) + \varepsilon(t), \tag{7.2}$$

where $\{\varepsilon(t), t \in \mathbb{Z}\}$ are i.i.d. r.v.'s in the domain of the (normal) attraction of an $\alpha-$stable law, $0 < \alpha < 2$, and where $a \in (-1, 1)$ is a r.v., independent of $\{\varepsilon(t), t \in \mathbb{Z}\}$ and satisfying some mild additional condition. Let the $X_i(t) = a_i X_i(t-1) + \varepsilon_i(t)$, $i = 1, 2, \ldots$, be independent copies of (7.2). Then the aggregated process $\Big\{N^{-1/\alpha} \sum_{i=1}^N X_i(t),\ t \in \mathbb{Z}\Big\}$ tends, as $N \to \infty$, in the sense of weak convergence of finite-dimensional distributions, to a limit process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ written as a stochastic integral

$$\mathfrak{X}(t) \;=\; \sum_{s\le t}\int_{(-1,1)} a^{t-s} M_s(\mathrm{d}a), \tag{7.3}$$

where $\{M_s, s\in\mathbb{Z}\}$ are i.i.d. copies of an $\alpha-$stable random measure M on $(-1,1)$ with control measure proportional to the distribution $\Phi(\mathrm{d}x) = \mathrm{P}(a\in \mathrm{d}x)$ of r.v. a (see (4.4), page 62). In the case when $1<\alpha<2$ and the mixing distribution Φ is concentrated in the interval $(0,1)$ having a density ϕ such that

$$\phi(a) \;\sim \phi_1\,(1-a)^{\beta} \qquad \text{as } a\uparrow 1, \quad \text{for some } \phi_1>0, \qquad 0<\beta<\alpha-1, \tag{7.4}$$

we proved that the aggregated process in (7.3) has long memory. In particular, it was shown that normalized partial sums of $\{\mathfrak{X}(t), t\in\mathbb{Z}\}$ in (7.3) tend to an $\alpha-$stable stationary increment process $\{\Lambda_{\alpha,\beta}(\tau)\}$, which is self-similar with index $H = 1-(\beta/\alpha)\in(1/\alpha,1)$ and is written as a stochastic integral

$$\begin{aligned}\Lambda_{\alpha,\beta}(\tau) \;&:=\; \int_{(0,\infty)\times\mathbb{R}}\Big(f(x,\tau-s)-f(x,-s)\Big)N(\mathrm{d}x,\,\mathrm{d}s), \\ f(x,t) \;&:=\; \begin{cases}1-\mathrm{e}^{-xt}, & \text{if } x>0 \text{ and } t>0,\\ 0, & \text{otherwise,}\end{cases}\end{aligned} \tag{7.5}$$

with respect to an $\alpha-$stable random measure $N(\mathrm{d}x,\,\mathrm{d}s)$ on $(0,\infty)\times\mathbb{R}$ with control measure $\phi_1 x^{\beta-\alpha}\,\mathrm{d}x\,\mathrm{d}s$. Let us note that (7.5) is different from the $\alpha-$stable fractional motion discussed in [75], which arises in a similar context by aggregating AR(1) processes with *common* infinite-variance innovations; see Chapter 3. Under the same assumptions (7.4), in Chapter 4 we established further long memory properties of $\{\mathfrak{X}(t),\, t\in\mathbb{Z}\}$ in (7.3), namely, a (hyperbolic) decay rate of codifference and the long-range dependence (sample Allen variance) property of Heyde and Yang (see [45]). We also showed that the value $\beta=\alpha-1$ separates long memory and short memory in the above aggregation scheme; indeed, in the case $\beta>\alpha-1$ the aggregated process has the short-range dependence (sample Allen variance) property and its partial sums tend to an $\alpha-$stable Lévy process with independent increments (see Chapter 4).

In the rest of this Chapter, we assume that $\{\mathfrak{X}(t), t\in\mathbb{Z}\}$ is the mixed moving average in (7.3), where $M_s(\mathrm{d}a)$ is a SαS random measure with characteristic function $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta M_s(A)} = \mathrm{e}^{-\omega_\alpha|\theta|^\alpha\Phi(A)}$, $\theta\in\mathbb{R}$, where $1<\alpha<2$, $\omega_\alpha>0$ and $A\subset(0,1)$ is any Borel set. This means that all finite-dimensional distributions of $\{\mathfrak{X}(t), t\in\mathbb{Z}\}$ are SαS. In particular,

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta\mathfrak{X}(0)} = \mathrm{e}^{-\sigma^\alpha|\theta|^\alpha}, \quad \theta\in\mathbb{R}, \quad \text{where} \quad \sigma^\alpha := \omega_\alpha\sum_{k=0}^{\infty}\mathrm{E}|a|^{\alpha k} = \omega_\alpha \mathrm{E}\Big[\frac{1}{1-|a|^\alpha}\Big].$$

Let $C_\alpha > 0$ be the constant determined from the relation

$$\lim_{u\to\infty} u^\alpha \mathrm{P}(\mathfrak{X}(0) > u) \;=\; \frac{1}{2} C_\alpha \sigma^\alpha. \tag{7.6}$$

The constant C_α depends only on α and is explicitly written in [94]

$$C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}.$$

Also define

$$g(z) \;:=\; \sup_{w>0} \frac{1-\mathrm{e}^{-w}}{w+z}, \quad z > 0. \tag{7.7}$$

The function g is continuous in the interval $(0, \infty)$ and satisfies the following conditions

$$\lim_{z\to 0} g(z) = 1, \quad \lim_{z\to\infty} zg(z) = 1. \tag{7.8}$$

The main result of this chapter is the following theorem.

Theorem 7.1.1. *Assume that the mixing distribution* $\Phi(A) = \mathrm{P}(a \in A)$ *is absolutely continuous having a density*

$$\phi(a) \;=\; \varphi(a)(1-a)^\beta, \quad a \in (0,1), \tag{7.9}$$

where $\beta > 0$ *and* φ *is integrable on* $(0,1)$ *and has a limit* $\lim_{a\to 1}\varphi(a) =: \phi_1 > 0$. *Let* $\psi(u)$ *be the ruin probability in* (7.1) *corresponding to* $\{Y_t \equiv \mathfrak{X}(t)\}$.

(i) Let $0 < \beta < \alpha - 1$. *Then*

$$\psi(u) \;\sim\; \frac{C_\alpha K(\alpha,\beta)}{2c^{H\alpha}} u^{-\alpha(1-H)}, \quad u \to \infty, \tag{7.10}$$

where $H = 1 - (\beta/\alpha) \in (1/\alpha, 1)$ *and*

$$K(\alpha,\beta) := \frac{\phi_1}{\alpha}\int_0^\infty z^{\beta-1} g^\alpha(z)\,\mathrm{d}z + \frac{\phi_1}{\beta}\int_0^\infty z^\beta g^\alpha(z)\,\mathrm{d}z. \tag{7.11}$$

(ii) Let $\beta > \alpha - 1$. *Then*

$$\psi(u) \;\sim\; \frac{C_\alpha K(\alpha,\Phi)}{2c} u^{-(\alpha-1)}, \quad u \to \infty, \tag{7.12}$$

where

$$K(\alpha,\Phi) := \frac{1}{\alpha-1}\mathrm{E}\Big[\frac{1}{(1-a)^\alpha}\Big]. \tag{7.13}$$

In what follows, C stands for a constant whose precise value is unimportant and which may change from line to line.

7.2 Proof of Theorem 7.1.1.

The proof of Theorem 7.1.1 is based on Theorem 7.2.1, below, due to [75], Theorem 2.5. For our purpose, we formulate the above mentioned result in a special case of mixed SαS moving average in (7.14). For terminology and properties of stochastic integrals with respect to stable random measures, we refer to [94].

Let $\{Y(t)\} = \{Y(t), t = 1, 2, \ldots\}$ be a stationary SαS process, $1 < \alpha < 2$, having the form

$$Y(t) \;=\; \int_{W\times\mathbb{R}} f(v, x-t) M(\,\mathrm{d}v,\,\mathrm{d}x), \qquad t = 1, 2, \ldots, \tag{7.14}$$

where M is a SαS random measure on a measurable product space $W \times \mathbb{R}$ with control measure $\nu \times$ Leb, ν is a $\sigma-$finite measure on W, Leb is the Lebesgue measure, and $f \in L^\alpha(W \times \mathbb{R})$ is a measurable function with

$$\int_{W\times\mathbb{R}} |f(v,x)|^\alpha \nu(\,\mathrm{d}v)\,\mathrm{d}x < \infty.$$

Introduce

$$m_n \;:=\; C_\alpha^{1/\alpha}\Big(\int_{W\times\mathbb{R}} \Big|\sum_{t=1}^{n} f(v, x-t)\Big|^\alpha \nu(\,\mathrm{d}v)\,\mathrm{d}x\Big)^{1/\alpha}$$

and a function $\psi_0 : (0,\infty) \to (0,\infty)$ by

$$\begin{aligned}\psi_0(u) \;&:=\; \frac{C_\alpha}{2}\int_{W\times\mathbb{R}} \sup_{n\geq 1} \frac{\Big(\sum_{t=1}^{n} f(v,x-t)\Big)_+^\alpha}{(u+nc)^\alpha}\,\nu(\,\mathrm{d}v)\,\mathrm{d}x \\ &+\; \frac{C_\alpha}{2}\int_{W\times\mathbb{R}} \sup_{n\geq 1} \frac{\Big(\sum_{t=1}^{n} f(v,x-t)\Big)_-^\alpha}{(u+nc)^\alpha}\,\nu(\,\mathrm{d}v)\,\mathrm{d}x;\end{aligned} \tag{7.15}$$

where $x_+ := \max(x, 0), x_- := \max(-x, 0)$ and where the constant C_α is the same as in (7.6).

Theorem 7.2.1. *(see [75]). Let $\{Y_t\}$ be given as in* (7.14)*. Assume that $m_n = O(n^\gamma)$ for some $\gamma \in (0,1)$. Then*

$$\psi(u) \;\sim\; \psi_0(u), \qquad \textit{as } u \to \infty.$$

Proof of Theorem 7.1.1. In order to use Theorem 7.2.1, we first rewrite the process in (7.3) in the form of (7.14):

$$\mathfrak{X}(t) \;=\; \int_{(0,1)\times\mathbb{R}} f(a, t-x) M(\,\mathrm{d}a,\,\mathrm{d}x), \tag{7.16}$$

where

$$f(a,x) := a^{[x]}\mathbf{1}(x \ge 0) = \begin{cases} a^{[x]}, & x \ge 0, \\ 0, & x < 0, \end{cases} \qquad (a,x) \in (0,1) \times \mathbb{R},$$

and $M(\mathrm{d}a, \mathrm{d}x)$ is a SαS random measure on $(0,1) \times \mathbb{R}$ with control measure $\Phi \times$ Leb.

Condition $m_n = O(n^\gamma)$ of Theorem 7.2.1 for the process in (7.3) is verified in (4.34), with $\gamma = H = 1 - (\beta/\alpha) \in (1/\alpha, 1)$. Therefore it suffices to show (7.10) with $\psi(u)$ replaced by $\psi_0(u)$ as defined in (7.15). We have

$$\begin{aligned}
\psi_0(u) &= \frac{C_\alpha}{2} \int_{(0,1)\times\mathbb{R}} \sup_{n\ge 1} \frac{\left(\sum_{t=1}^n a^{[t-x]}\mathbf{1}(t \ge x)\right)^\alpha}{(u+nc)^\alpha} \Phi(\mathrm{d}a)\,\mathrm{d}x \\
&= \frac{C_\alpha}{2}\Big(\mathrm{E}\Big[\sum_{x=-\infty}^{0} \sup_{n\ge 1} \frac{\left(\sum_{t=1}^n a^{t-x}\right)^\alpha}{(u+nc)^\alpha}\Big] + \mathrm{E}\Big[\sum_{x=1}^{\infty} \sup_{n\ge x} \frac{\left(\sum_{t=x}^n a^{t-x}\right)^\alpha}{(u+nc)^\alpha}\Big]\Big) \\
&=: \frac{C_\alpha}{2}\Big(I_1 + I_2\Big). \qquad (7.17)
\end{aligned}$$

Consider first the expectation

$$I_2 = \mathrm{E}\Big[\sum_{x=1}^{\infty} \frac{1}{(1-a)^\alpha} \sup_{k\ge 1}\Big(\frac{1-a^k}{u+(k-1+x)c}\Big)^\alpha\Big],$$

which can be rewritten as

$$\begin{aligned}
I_2 &= c^{-\alpha}\int_0^1 y^{-\alpha}\phi(1-y)\,\mathrm{d}y \sum_{x=1}^{\infty}\sup_{k\ge 1}\Big(\frac{1-(1-y)^k}{(u/c)+k-1+x}\Big)^\alpha \\
&= c^{-\alpha}\Big\{\int_0^\epsilon y^{-\alpha}\phi(1-y)\,\mathrm{d}y \sum_{x=1}^{\infty}\sup_{k\ge 1}\Big(\frac{1-(1-y)^k}{(u/c)+k-1+x}\Big)^\alpha \\
&+ \int_\epsilon^1 y^{-\alpha}\phi(1-y)\,\mathrm{d}y \sum_{x=1}^{\infty}\sup_{k\ge 1}\Big(\frac{1-(1-y)^k}{(u/c)+k-1+x}\Big)^\alpha\Big\} \\
&=: c^{-\alpha}\Big\{I_{21} + I_{22}\Big\}. \qquad (7.18)
\end{aligned}$$

Clearly, in view of (7.9), we can replace $\phi(1-y)$ by $\phi_1 y^\beta$ in the integral I_{21}. For notational simplicity, assume that $\phi(1-y) = \phi_1 y^\beta$, $0 < y < \epsilon$. Then $u^\beta I_{21}$ can be rewritten as

$$\begin{aligned}
u^\beta I_{21} &= \phi_1 u^\beta \int_0^\epsilon y^{\beta-\alpha}\,\mathrm{d}y \sum_{x=1}^\infty \sup_{k\ge1}\Big(\frac{1-(1-y)^k}{(u/c)+k-1+x}\Big)^\alpha \\
&= \phi_1 u^\beta \int_0^\epsilon y^{\beta}\,\mathrm{d}y \sum_{x=1}^\infty \sup_{k\ge1}\Big(\frac{1-(1-y)^k}{y((u/c)+x-1)+yk}\Big)^\alpha \\
&= \phi_1 u^\beta \int_0^{\epsilon((u/c)+x-1)} \frac{z^\beta}{((u/c)+x-1)^\beta}\,\mathrm{d}\Big(\frac{z}{(u/c)+x-1}\Big) \\
&\quad \times \sum_{x=1}^\infty \sup_{k\ge1}\Bigg(\frac{1-\Big(1-\frac{z}{(u/c)+x-1}\Big)^k}{z+\frac{zk}{(u/c)+x-1}}\Bigg)^\alpha \\
&= \phi_1 \sum_{x=1}^\infty \frac{u^\beta}{((u/c)+x-1)^{\beta+1}} \int_0^{\epsilon((u/c)+x-1)} z^\beta (g_{u,x}(z))^\alpha\,\mathrm{d}z, \qquad (7.19)
\end{aligned}$$

where

$$g_{u,x}(z) := \sup_{k\ge1} \frac{1-\Big(1-\frac{z}{(u/c)+x-1}\Big)^k}{z+\frac{zk}{(u/c)+x-1}}\mathbf{1}(0<z<\epsilon((u/c)+x-1)). \qquad (7.20)$$

According to Lemma 7.2.2, below, the function $g_{u,x}(z)$ tends to $g(z)$ in (7.7), as $u\to\infty$, and satisfies condition (7.25), therefore, by dominated convergence theorem, the integral in (7.19) tends to

$$\int_0^\infty z^\beta g^\alpha(z)\,\mathrm{d}z < \infty$$

uniformly in $x \ge 1$. We also have that

$$\sum_{x=1}^\infty \frac{u^\beta}{((u/c)+x-1)^{\beta+1}} = \sum_{x=0}^\infty \frac{1}{u}\frac{1}{\Big((1/c)+(x/u)\Big)^{\beta+1}} \to \int_0^\infty \frac{\mathrm{d}x}{((1/c)+x)^{\beta+1}} = \frac{c^\beta}{\beta}.$$

Whence and from (7.19) we obtain that

$$\lim_{u\to\infty} u^\beta I_{21} = \frac{\phi_1 c^\beta}{\beta}\int_0^\infty z^\beta g^\alpha(z)\,\mathrm{d}z. \qquad (7.21)$$

On the other hand,

$$\begin{aligned}
|I_{22}| &\le C\mathrm{E}\Bigg[(1-a)^{-\alpha}\mathbf{1}(0<a<1-\epsilon)\sum_{x=1}^\infty \sup_{k\ge1}\Big(\frac{1-a^k}{(u/c)+k-1+x}\Big)^\alpha\Bigg] \\
&\le C\sum_{x=1}^\infty \Big(\frac{1}{(u/c)+x}\Big)^\alpha = O(u^{-(\alpha-1)})
\end{aligned}$$

implying $\lim_{u\to\infty} u^\beta I_{22} = 0$ thanks to condition $\beta < \alpha - 1$.

Consider the term I_1 in (7.17):

$$\begin{aligned} I_1 &= \mathrm{E}\sum_{x=-\infty}^{0}\sup_{n\geq 1}\frac{\left(\sum_{t=1}^{n}a^{t-x}\right)^{\alpha}}{(u+nc)^{\alpha}} = \mathrm{E}\sum_{x=-\infty}^{0}\sup_{n\geq 1}\frac{a^{(1-x)\alpha}(1-a^n)^{\alpha}}{(1-a)^{\alpha}(u+nc)^{\alpha}} \\ &= c^{-\alpha}\int_0^1 \mathrm{d}y\, y^{-\alpha}\phi(1-y)\sum_{x=-\infty}^{0}(1-y)^{(1-x)\alpha}\sup_{n\geq 1}\left(\frac{1-(1-y)^n}{(u/c)+n}\right)^{\alpha} \\ &= c^{-\alpha}\Big\{\int_0^{\epsilon}\mathrm{d}y\, y^{-\alpha}\phi(1-y)\sum_{x=-\infty}^{0}(1-y)^{(1-x)\alpha}\sup_{n\geq 1}\left(\frac{1-(1-y)^n}{(u/c)+n}\right)^{\alpha} \\ &\quad + \int_{\epsilon}^1 \mathrm{d}y\, y^{-\alpha}\phi(1-y)\sum_{x=-\infty}^{0}(1-y)^{(1-x)\alpha}\sup_{n\geq 1}\left(\frac{1-(1-y)^n}{(u/c)+n}\right)^{\alpha}\Big\} \\ &=: c^{-\alpha}\Big\{I_{11}+I_{12}\Big\}. \end{aligned}$$

For notational simplicity, assume that $\phi(1-y) = \phi_1 y^{\beta}$, $0 < y < \epsilon$. Then $u^{\beta} I_{11}$ can be rewritten as

$$\begin{aligned} u^{\beta}I_{11} &= u^{\beta}\phi_1\int_0^{\epsilon}\mathrm{d}y\, y^{\beta-\alpha}\sum_{x=-\infty}^{0}(1-y)^{(1-x)\alpha}\sup_{n\geq 1}\left(\frac{1-(1-y)^n}{(u/c)+n}\right)^{\alpha} \\ &= u^{\beta}\phi_1\int_0^{\epsilon}\mathrm{d}y\, y^{\beta}\frac{(1-y)^{\alpha}}{1-(1-y)^{\alpha}}\sup_{n\geq 1}\left(\frac{1-(1-y)^n}{(yu/c)+yn}\right)^{\alpha} \\ &= c^{\beta}\phi_1\int_0^{\epsilon u/c}\mathrm{d}z\left(\frac{c}{u}\right)\frac{(1-cz/u)^{\alpha}}{1-(1-cz/u)^{\alpha}}z^{\beta}\sup_{n\geq 1}\left(\frac{1-(1-cz/u)^n}{z+czn/u}\right)^{\alpha} \\ &= c^{\beta}\phi_1\int_0^{\epsilon u/c}\mathrm{d}z\left(\frac{cz}{u}\right)\frac{(1-cz/u)^{\alpha}}{1-(1-cz/u)^{\alpha}}z^{\beta-1}(g_{u,1}(z))^{\alpha}, \end{aligned}$$

Using Lemma 7.2.2, below, and the facts that

$$\lim_{x\to 0} x(1-x)^{\alpha}/(1-(1-x)^{\alpha}) = 1/\alpha$$

and

$$0 \leq x(1-x)^{\alpha}/(1-(1-x)^{\alpha}) \leq 1/\alpha$$

for all $x \in (0,\, 1]$, we have that

$$\lim_{u\to\infty} u^{\beta}I_{11} = \frac{\phi_1 c^{\beta}}{\alpha}\int_0^{\infty} z^{\beta-1}g^{\alpha}(z)\,\mathrm{d}z. \tag{7.22}$$

Next,

$$\begin{aligned} I_{12} &= \mathrm{E}\Big[(1-a)^{-\alpha}\mathbf{1}(0<a<1-\epsilon)\sum_{x=-\infty}^{0}a^{(1-x)\alpha}\sup_{n\geq 1}\left(\frac{1-a^n}{(u/c)+n}\right)^{\alpha}\Big] \\ &\leq c^{\alpha}\mathrm{E}\Big[(1-a)^{-\alpha}\mathbf{1}(0<a<1-\epsilon)\frac{a^{\alpha}}{1-a^{\alpha}}\Big]u^{-\alpha} \\ &= Cu^{-\alpha}. \end{aligned}$$

Since $\beta < \alpha - 1$, we have $\lim_{u\to\infty} u^\beta I_{12} = 0$. This proves part (i).

(ii) We use Theorem 7.2.1 as in part (i). Condition $m_n = O(n^\gamma)$ is proved in (4.35), with $\gamma = 1/\alpha \in (0,1)$. Therefore it suffices to show (7.10) for $\psi_0(u)$. Consider the expectation I_2 in (7.17). Then

$$u^{\alpha-1} I_2 \;=\; u^{\alpha-1} c^{-\alpha} \mathrm{E}\Big[\frac{1}{(1-a)^\alpha} \sum_{x=1}^\infty \frac{1}{((u/c)+x-1)^\alpha}\, q_u^\alpha(a,x)\Big],$$

where

$$q_u(a,x) \;:=\; \sup_{k\ge 1} \frac{1-a^k}{1+\frac{k}{(u/c)+x-1}}.$$

Note $0 \le q_u(a,x) \le 1$ and $q_u(a,x) \to 1$, $u \to \infty$, for any $0 < a < 1$, $x \ge 1$ fixed. Indeed,

$$q_u(a,x) - 1 \;=\; \sup_{k\ge 1} \frac{-a^k - \frac{k}{(u/c)+x-1}}{1+\frac{k}{(u/c)+x-1}} \;=\; -\inf_{k\ge 1} \frac{a^k + \frac{k}{(u/c)+x-1}}{1+\frac{k}{(u/c)+x-1}} \;\to\; 0$$

follows by taking e.g. $k = [\log u]$ in the last infimum. Therefore by the dominated convergence theorem

$$\begin{aligned}\lim_{u\to\infty} u^{\alpha-1} I_2 \;&=\; c^{-\alpha} \lim_{u\to\infty} \mathrm{E}\Big[\frac{1}{(1-a)^\alpha} \sum_{x=1}^\infty \frac{u^{\alpha-1}}{((u/c)+x-1)^\alpha}\Big] \\ &=\; \frac{1}{c(\alpha-1)} \mathrm{E}\Big[\frac{1}{(1-a)^\alpha}\Big] \;=\; c^{-1} K(\alpha,\Phi), \end{aligned} \tag{7.23}$$

where we used the fact that the last expectation is finite.

Next, consider

$$I_1 \;=\; \mathrm{E}\Big[\frac{a^\alpha}{(1-a^\alpha)(1-a)^\alpha} \Big(\sup_{n\ge 1} \frac{1-a^n}{u+nc}\Big)^\alpha\Big].$$

We claim that $I_1 = o(u^{-(\alpha-1)})$ and therefore part (ii) follows from the limit in (7.23). To prove the last claim, split the expectation $I_1 = I_{11} + I_{12}$ according to whether $0 < a < 1-\epsilon$ or $1-\epsilon < a < 1$ holds, similarly to (7.18). It is clear that $I_{11} = O(u^{-\alpha}) = o(u^{-(\alpha-1)})$. Therefore it suffices to estimate I_{12} only. Then using (7.26), below, and the inequality $|1-(1-y)^\alpha| > Cy$, $0 < y < \epsilon$, we obtain

$$\begin{aligned} I_{12} \;&\le\; C \int_0^\epsilon \frac{y^{\beta-\alpha}\,\mathrm{d}y}{1-(1-y)^\alpha} \Big(\sup_{n\ge 1} \frac{1-(1-y)^n}{u+nc}\Big)^\alpha \\ &\le\; C \int_0^\epsilon y^{\beta-1}\,\mathrm{d}y \Big(\sup_{n\ge 1} \frac{1-(1-y)^n}{y(u/c)+ny}\Big)^\alpha \end{aligned}$$

$$\begin{aligned} &\leq \ C\int_0^\epsilon y^{\beta-1}\,\mathrm{d}y\Big(\sup_{n\geq 1}\frac{1-\mathrm{e}^{-ny}}{y(u/c)+ny}\Big)^\alpha \leq C\int_0^\epsilon y^{\beta-1}g^\alpha(yu/c)\,\mathrm{d}y \\ &\leq \ C\int_0^\epsilon \frac{y^{\beta-1}}{(1+yu)^\alpha}\,\mathrm{d}y = Cu^{-\beta}\int_0^{\epsilon u}\frac{z^{\beta-1}}{(1+z)^\alpha}\,\mathrm{d}z, \end{aligned}$$

where the last inequality follows from (7.8). If $\alpha > \beta$, the last integral is bounded and hence $I_{12} = O(u^{-\beta}) = o(u^{-(\alpha-1)})$. On the other hand, if $\beta \geq \alpha$, we easily obtain $I_{21} = O(u^{-\alpha}\log(u)) = o(u^{-(\alpha-1)})$. This concludes the proof of Theorem 7.1.1.

Lemma 7.2.2. *Let $g(z)$, $g_{u,x}(z)$ be defined at (7.7), (7.20), respectively. Then*

$$\lim_{u\to\infty} g_{u,x}(z) \ = \ g(z), \qquad \forall\, z>0,\ \forall\, x\geq 1, \tag{7.24}$$

$$g_{u,x}(z) \ \leq \ Cg(z), \qquad \forall\, z>0,\ \forall\, u\geq 1,\ \forall\, x\geq 1, \tag{7.25}$$

where the constant C is independent of u, x, z. The function $g(z)$ satisfies (7.8).

Proof. Let $\tau_k(y) := (1-(1-y)^k)/(1-\mathrm{e}^{-ky})$, $0<y<1$, $k=1,2,\ldots$. Let us first prove the elementary inequality: for any $0<\epsilon<1$ there exists a constant $C>0$, independent of $0<\epsilon<1$, $k\geq 1$ and such that

$$|\tau_k(y)-1| \ \leq C(\epsilon + k^{-1}), \quad \forall\, 0<y<\epsilon,\ \forall\, k=1,2,\ldots. \tag{7.26}$$

Indeed, let $0<y\leq 1/(2k)$. Since $1-\mathrm{e}^{-x}\geq x/2$, $0<x<1/2$ so

$$|\tau_k(y)-1| \leq 2\frac{|\mathrm{e}^{-ky}-(1-y)^k|}{ky} \leq C\frac{k|\mathrm{e}^{-y}-1+y|}{ky} \leq Cy \leq C/k.$$

Next, let $1/(2k)<y<\epsilon<1$. Then $1-\mathrm{e}^{-ky}\geq 1-\mathrm{e}^{-1/2}>0$ and $\log(1-y)\leq -y(1-\epsilon)$. Therefore

$$\begin{aligned} &|\tau_k(y)-1| \leq C|\mathrm{e}^{-ky}-(1-y)^k| \leq C\sup_{k\geq 1,\,1/2<x\leq\epsilon k}|\mathrm{e}^{k\log(1-\frac{x}{k})}-\mathrm{e}^{-x}| \\ &\leq C\sup_{x>1/2}\Big(\mathrm{e}^{-x(1-\epsilon)}-\mathrm{e}^{-x}\Big) \ \leq C\epsilon \end{aligned}$$

since $\sup_{x\geq 1/2} x\mathrm{e}^{-x(1-\epsilon)}<\infty$. This proves (7.26).

Using (7.26) we can write

$$\begin{aligned} g_{u,x}(z) \ &:= \ \sup_{k\geq 1}\tau_k\Big(\frac{z}{(u/c)+x-1}\Big)\frac{1-\mathrm{e}^{-\frac{zk}{(u/c)+x-1}}}{z+\frac{zk}{(u/c)+x-1}}\mathbf{1}(0<z<\epsilon((u/c)+x-1)) \\ &\leq \ C\sup_{k\geq 1}\frac{1-\mathrm{e}^{-\frac{zk}{(u/c)+x-1}}}{z+\frac{zk}{(u/c)+x-1}} \ \leq \ Cg(z), \end{aligned} \tag{7.27}$$

thus proving the bound in (7.25). The convergence (7.24) follows similarly from

(7.27) and (7.26).

To show (7.8), note that $\omega \mapsto \frac{1-\mathrm{e}^{-\omega}}{z+\omega}$ increases on the interval $(0, \omega_*)$ and decreases on (ω_*, ∞), where $\omega_* = \omega_*(z) > 0$ is the unique solution of $\omega + z + 1 = \mathrm{e}^{\omega}$. Thus, $g(z) = \frac{1}{z+1+\omega_*}$. It is clear that $\omega_* \to 0$, as $z \to 0$, and therefore $\lim_{z\to 0} g(z) = 1$. Moreover, $\omega_* \to \infty$, as $z \to \infty$, and $\omega_* \le \log(1+z)$, implying $\lim_{z\to\infty} zg(z) = \lim_{z\to\infty} \frac{z}{z+1+\omega_*} = 1$. Lemma 7.2.2 is proved.

8

Conclusions

The main conclusions of the thesis research:

• We have extended the aggregation scheme of random-coefficient AR(1) processes from finite variance to infinite variance case. Under assumptions, that innovations belong to the domain of normal attraction of an $\alpha-$stable law and that the density function of a random coefficient is regularly varying at the "unit root" $a = 1$ with exponent $\beta > -1$, [1]

$$\phi(a) \ \sim C(1-a)^{\beta}, \qquad \text{as } a \uparrow 1, \tag{8.1}$$

we found conditions under which the limit aggregated process exists and can be represented as a moving-average (3.22) in common innovations case and a mixed $\alpha-$stable moving-average (4.4) in idiosyncratic innovations case (see Table 8.1., page 175). The long memory properties of the limit aggregated processes depend on parameters β and α, $0 < \alpha \leq 2$. The β is smaller, the dependence in the limit aggregated process is stronger. Smaller β means the mixing distribution [2] is putting more weight near the unit root $a = 1$. Note, that in the case of common innovations, the limit aggerated process is moving average, which is well defined for $1/\alpha - 1 < \beta$. If $\beta > 0$, coefficients of this moving-average are absolutely summable. Therefore, it's partial sums will converge to the process with independent increments and the moving average will admit distributional short memory. In the case of idiosyncratic innovations, the limit aggregated process is

1. Note, that in the Chapter 3, the mixing density (3.3) depends on parameters d_1, d_2. Here we give results for $d_1 := \beta$, assuming, that $a \in [0, 1)$ a.s.

2. The distribution of the random coefficient a is called the mixing distribution.

the mixed $\alpha-$stable moving average, which is well defined for $\beta > 0$. We proved, that for $0 < \alpha \leq 1$, partial sums of the mixed $\alpha-$stable moving average will also converge to the process with independent increments. It follows, that the case $0 < \alpha \leq 1$ can not lead to the long memory. Only for $1 < \alpha \leq 2$ we can (expect to) get long memory. These facts are illustrated in the Figure 8.1 and in the Table 8.1 (page 175) too.

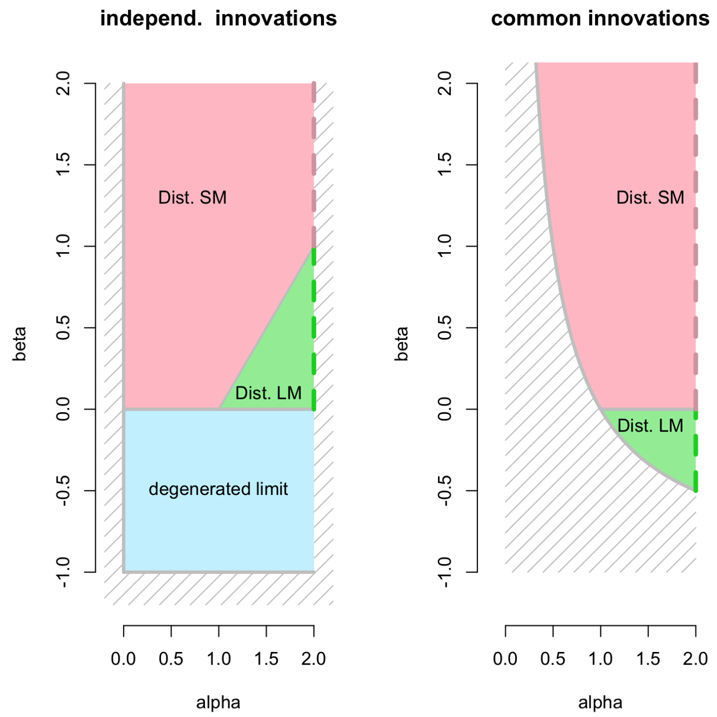


Figure 8.1: Distributional LM areas

• The second aim was to describe the aggregation scheme of *independent* AR(1) processes, which leads to the case of finite variance but not necessary Gaussian [3] or infinite variance but not necessary stable limit aggregated process. For this reason, we discussed the contemporaneous aggregation of independent copies of a triangular array of random-coefficient AR(1) processes with independent innovations belonging to the domain of attraction of an infinitely divisible law W. Under general assumptions on W and the mixing distribution, we showed that the limit aggregated process exists and is represented as a mixed infinitely divisible moving-average (5.4), page 84.

The long memory properties of the limit aggregated process were studied under assumption, that the mixing density is regularly varying at the "unit root" $a = 1$ with exponent $\beta > 0$ (see (8.1)), and that $\mathrm{E}W^2 < \infty$. We showed that the

3. In the scientific literature is described the aggregation scheme of independent AR(1) processes, which leads to the Gaussian case.

partial sums of the mixed infinitely divisible moving-average (5.4) may exhibit four different limit behaviors depending on β and the Lévy triplet (μ, σ, π) of W (see (5.6)). Note, that the behavior of Lévy measure at the origin

$$\lim_{x\to 0} x^{\alpha_0}\pi(\{u > x\}) = c^+, \qquad \lim_{x\to 0} x^{\alpha_0}\pi(\{u \le -x\}) = c^-.$$

is very important for the limits of partial sums. The four limit behaviors of $S_n(\tau) := \sum_{t=1}^{[n\tau]} \mathfrak{X}(t)$ are:

(i) if $0 < \beta < 1,\ \sigma > 0$, the limit is fractional Brownian motion with self-similarity parameter $H = 1 - \beta/2$,

(ii) if $0 < \beta < 1,\ \sigma = 0,\ 1+\beta < \alpha_0 < 2$, the limit is α_0-stable self-similar process with dependent increments and self-similarity parameter $H = 1 - \beta/\alpha_0$,

(iii) if $0 < \beta < 1,\ \sigma = 0,\ 0 < \alpha_0 < 1 + \beta$, the limit is $(1 + \beta)-$stable Lévy process with independent increments,

(iv) if $\beta > 1$, the limit is Brownian motion.

Accordingly, the process $\{\mathfrak{X}(t), t \in \mathbb{Z}\}$ in (5.4) has distributional long memory in cases (i) and (ii) and distributional short memory in case (iii). At the same time, $\{\mathfrak{X}(t),\, t \in \mathbb{Z}\}$ has covariance long memory in all three cases (i)-(iii). Case (iv) corresponds to distributional and covariance short memory. See generalizing Table 8.2, page 176.

• And finally, we extended the aggregation scheme from one-dimensional processes to two-dimensional random fields. We described the aggregation scheme of independent nearest-neighbor random fields with innovations belonging to the domain of attraction of an $\alpha-$stable law and showed that the limit aggregated random field is mixed stable moving average in (6.10). Since the properties of the limit aggregated random field are highly dependent on individual models, we studied partial sums of the limit aggregated field in two special cases. Assuming that individuals are described by 3N and 4N models (see (6.14) and (6.15)), we showed that the partial sums of the limit aggregated random field converge to operator scaling random fields. In order to explain these results and the dependence structure of random fields, we introduced the notion of anisotropic/isotropic long memory for random fields on $\mathbb{Z}^2$, whose partial sums on incommensurate rectangles with sides growing at different rates $O(n)$ and $O(n^{H_1/H_2}), H_1 \ne H_2$, tend to an operator scaling random field on $\mathbb{R}^2$ with two scaling indices H_1, H_2. We proved, that the limit aggregated random field has anisotropic distributional long memory with parameters $H_1 = (1/2+\alpha-\beta)/\alpha$, $H_2 = 2H_1$, if micro behavior is described by 3N model. And the limit aggregated random field will admit isotropic distributional long memory with parameter $H = 2(\alpha - \beta)/\alpha$, if individuals are described by 4N model. See the Table 8.3, page 177.

Note, that the definition of anisotropic/isotropic distributional long memory is new. Using this definition we described the depencence structure of the limit aggregated random field in two special cases. In the future, we expect to prove, that the random field $\{Y(t,s),\ (t,s) \in \mathbb{Z}^2\}$ can have anisotropic distributional long memory with only one combination of parameters H_1, H_2, i.e. if $\{Y(t,s),\ (t,s) \in \mathbb{Z}^2\}$ has anisotropic distributional long memory with parameters H_1, H_2, then, for parameters $\tilde{H}_1 := H_1$ and $\tilde{H}_2 \neq H_2$, the limit of partial sums

$$n^{-\tilde{H}_1} \sum_{t=1}^{[nx]} \sum_{s=1}^{[n^{\tilde{H}_1/\tilde{H}_2} y]} Y(t,s) \quad \to_{\mathrm{fdd}} \quad V(x,y), \qquad (x,y) \in \mathbb{R}_+^2,$$

will have independent increments in some direction and random field will not admit anisotropic distributional long memory with parameters $\tilde{H}_1 := H_1$ and $\tilde{H}_2 \neq H_2$. But this is an open question today.

Aggregation of AR(1) processes, $\varepsilon(t) \in D(\alpha)$, $0 < \alpha \leq 2$	
Common innovations	Idiosyncratic innovations
Individuals: $X_i(t) = a_i X_i(t-1) + \varepsilon(t),\ i = 1, \ldots, N.$	$X_i(t) = a_i X_i(t-1) + \varepsilon_i(t),\ i = 1, \ldots, N.$
Aggregated process: $$\bar{X}_N(t) := \frac{1}{N} \sum_{i=1}^{N} X_i(t), \quad t \in \mathbb{Z}.$$	$$\bar{X}_N(t) := \frac{1}{N^{1/\alpha}} \sum_{i=1}^{N} X_i(t), \quad t \in \mathbb{Z}.$$
The limit aggregated process: if $1/\alpha - 1 < \beta$, $$\mathfrak{X}(t) \ = \ \sum_{j=0}^{\infty} \mathrm{E}[a^j]\varepsilon(t-j), \quad t \in \mathbb{Z},$$ if $-1 < \beta < 1/\alpha - 1$, the moving average is not defined.	The limit aggregated process: if $0 < \beta$, $$\mathfrak{X}(t) \ = \ \sum_{s \leq t} \int_0^1 a^{t-s} M_s(da), \quad t \in \mathbb{Z},$$ where $M_s(\cdot)$, $s \in \mathbb{Z}$, are i.i.d. copies of an $\alpha-$stable random measure. If $-1 < \beta < 0$, $\bar{X}_N(t) \ \to_{\mathrm{fdd}} \ \widetilde{Z}$, where $\widetilde{Z}$ is $\alpha(1+\beta)-$stable r.v., which does not depend on t.
Long memory properties: : if $\beta > 0$, $\mathfrak{X}(t)$ has distributional short memory, if $1/\alpha - 1 < \beta < 0$, $\mathfrak{X}(t)$ has distributional long memory.	Long memory properties: if $\beta > \max(\alpha - 1, 0)$, $\mathfrak{X}(t)$ has distributional short memory, if $0 < \beta < \max(\alpha - 1, 0)$, $\mathfrak{X}(t)$ has distributional long memory.
Finite variance case: $\alpha = 2$, $\beta > -1/2$: $$r(h) \ \sim \ Ch^{-2\beta-1}, \quad \text{as } h \to \infty.$$ Covariance long memory: if $-1/2 < \beta < 0$.	$\alpha = 2$, $\beta > 0$: $$r(h) \ \sim \ Ch^{-\beta}, \quad \text{as } h \to \infty.$$ Covariance long memory: if $0 < \beta < 1$.

Table 8.1: Aggregation of AR(1) processes, $\varepsilon(t) \in D(\alpha)$, $0 < \alpha \leq 2$

<table>
<tr><th colspan="2">Aggregation of AR(1) processes, $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\} \in D(W)$</th></tr>
<tr><th>Common innovations</th><th>Idiosyncratic innovations</th></tr>
<tr><td rowspan="5">OPEN QUESTION</td><td>Individuals:
$$X_i^{(N)}(t) = a_i X_i^{(N)}(t-1) + \varepsilon_i^{(N)}(t), \qquad t \in \mathbb{Z}, \quad i = 1, 2, \ldots, N$$</td></tr>
<tr><td>Aggregated process:
$$\bar{X}_N(t) := \sum_{i=1}^{N} X_i^{(N)}(t), \quad t \in \mathbb{Z}.$$</td></tr>
<tr><td>The limit aggregated process:
for $\beta > 0$,
$$\mathfrak{X}(t) \;=\; \sum_{s \le t} \int_0^1 a^{t-s} M_s(da), \quad t \in \mathbb{Z},$$
where $M_s(\cdot)$, $s \in \mathbb{Z}$, are i.i.d. copies of an infinitely divisible random measure.</td></tr>
<tr><td>Long memory properties (finite variance case, $\mathrm{E}W^2 < \infty$):

if $0 < \beta < 1,\ \sigma = 0,\ 0 < \alpha_0 < 1 + \beta$, or
if $\beta > 1$,
$\mathfrak{X}(t)$ has distributional short memory,

if $0 < \beta < 1,\ \sigma > 0$, or
if $0 < \beta < 1,\ \sigma = 0,\ 1 + \beta < \alpha_0 < 2$,
$\mathfrak{X}(t)$ has distributional long memory,

Covariance function:
$$r(h) \sim C h^{-\beta}, \quad \text{as } h \to \infty.$$
Covariance long memory: if $0 < \beta < 1$.</td></tr>
<tr><td>Long memory properties (infinite variance case):

OPEN QUESTION</td></tr>
</table>

Table 8.2: Aggregation of AR(1) processes, $\{\varepsilon^{(N)}, N \in \mathbb{N}^*\} \in D(W)$

<table>
<tr><th colspan="2">Aggregation of nearest-neighbor random fields, $\varepsilon(t,s) \in D(\alpha)$</th></tr>
<tr><th>Common innovations</th><th>Idiosyncratic innovations</th></tr>
<tr><td rowspan="4">OPEN QUESTION</td><td>Individuals: $(t,s) \in \mathbb{Z}^2$,
$$X_i(t,s) = \sum_{|u|+|v|=1} a_i(u,v) X_i(t+u, s+v) + \varepsilon_i(t,s),\ i = 1, \dots, N,$$</td></tr>
<tr><td>Aggregated field:
$$\bar{X}_N(t,s) := N^{-1/\alpha} \sum_{i=1}^{N} X_i(t,s), \quad (t,s) \in \mathbb{Z}^2.$$</td></tr>
<tr><td>The limit aggregated field:
$$\mathfrak{X}(t,s) = \sum_{(u,v)\in\mathbb{Z}^2} \int_{\boldsymbol{A}} g(t-u, s-v, a) M_{u,v}(\mathrm{d}a), \qquad (t,s) \in \mathbb{Z}^2,$$
where $M_{u,v}(\cdot)$, $(u,v) \in \mathbb{Z}$, are i.i.d. copies of an $\alpha-$stable random measure. $g(t,s,a)$ is a lattice Green function, and $\boldsymbol{A} := \{a(t,s) \in [0,1), \sum_{|t|+|s|=1} a(t,s) < 1\} \subset \mathbb{R}^4$.</td></tr>
<tr><td>Long memory properties:

New notion of long memory for random fields on $\mathbb{Z}^2$ - Anisotropic/isotropic distributional long memory.

3N case:
for $1 < \alpha \le 2$, $0 < \beta < \alpha - 1$,
$\mathfrak{X}(t,s)$ has anisotropic distributional long memory with parameters $H_1 = (1/2 + \alpha - \beta)/\alpha$, $H_2 = 2H_1$,

4N case:
for $1 < \alpha \le 2$, $0 < \beta < \alpha - 1$,
$\mathfrak{X}(t,s)$ has isotropic distributional long memory with parameter $H = 2(\alpha - \beta)/\alpha$.</td></tr>
</table>

Table 8.3: Aggregation of nearest-neighbor random fields, $\varepsilon(t,s) \in D(\alpha)$

Bibliography

[1] M. Abramovitz and I. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York, 1965. 105

[2] V. V. Ahn, J. M. Angulo, and M. D. Ruiz-Medina. Possible long-range dependence in fractional random fields. *J. Statist. Plan. Infer.*, 80:95–110, 1999. 112

[3] V. V. Ahn, N. N. Leonenko, and M. D. Ruiz-Medina. Macroscaling limit theorems for filtered spatiotemporal random fields. *Preprint.* 112

[4] S. Albeverio, S.A. Molchanov, and D. Surgailis. Stratified structure of the universe and burgers' equation - a probabilistic approach. *Probab. Th. Rel. Fields*, 100:457–484, 1994. 112

[5] U.T. Alparslan. Exceedance of power barriers for integrated continuous-time stationary ergodic stable processes. *Adv. Appl. Probab.*, 41:874–892, 2009. 160

[6] U.T. Alparslan and G. Samorodnitsky. Asymptotic analysis of exceedance probability with stationary stable steps generated by dissipative flows. *Scandinavian Actuarial J.*, 78:1–28, 2007. 160

[7] A. Astrauskas. Limit theorems for sums of linearly generated random variables. *Lithuanian Math. J.*, 23:127–134, 1983. 37, 55, 56, 57, 71

[8] A. Astrauskas. On limit theorems for quadratic forms of linear processes. *Lithuanian Math. J.*, 23(4):3–11, 1983. 58, 59

[9] A. Astrauskas, J.B. Levy, and M.S. Taqqu. The asymptotic dependence structure of the linear fractional Lévy motion. *Lithuanian Math. J.*, 31:1–28, 1991. 37

[10] F. Avram and M.S. Taqqu. Weak convergence of sums of moving averages in the $\alpha-$stable domain of attraction. *Ann. Probab.*, 20:483–503, 1992. 57

[11] B. von Bahr and C.-G. Esséen. Inequalities for the rth absolute moment of a sum of random variables, $1 \leq r \leq 2$. *Ann. Math. Statist.*, 36:299–303, 1965. 43

[12] J. Beran. Statistics for Long Memory Processes. In *Monographs on Statistics and Applied Probability.* 112

[13] J. Beran, Y. Feng, S. Ghosh, and R. Kulik. *Long-Memory Processes: Probabilistic Properties and Statistical Methods.* Springer, 2013. 8

[14] J. Beran, M. Schuetzner, and S. Ghosh. From short to long memory: Aggregation and estimation. *Comput. Stat. Data Anal.*, 54:2432–2442, 2010. 8, 28, 29

[15] R.J. Bhansali, L. Giraitis, and P. Kokoszka. Approximations and limit theory for quadratic forms of linear processes. *Stoch. Proc. Appl.*, 117:71–95, 2007. 110

[16] H. Biermé, M.M. Meerschaert, and H.P. Scheffler. Operator scaling stable random fields. *Stoch. Process. Appl.*, 117:312–332, 2007. 111, 112, 127

[17] P. Billingsley. *Convergence of Probability Measures.* Wiley, New York, 1968. 99

[18] K. Bružaitė, D. Surgailis, and M. Vaičiulis. Time-varying fractionally integrated processes with finite or infinite variance and nonstationary long memory. *Acta Appl. Math.*, 96:99–118, 2007. 58

[19] D. Celov. Time series aggregation, disaggregation and long memory. *Doctoral thesis*, 2008. 7, 8, 17

[20] D. Celov, V. Kvedaras, and R. Leipus. Comparison of estimation methods for the density of autoregressive parameter in aggregated AR(1) processes. *Lithuanian Math. J., (spec. issue)*, 47:508–516, 2007. 32

[21] D. Celov, R. Leipus, and A. Philippe. Time series aggregation, disaggregation and long memory. *Lithuanian Math. J.*, 47:379–393, 2007. 8, 67

[22] D. Celov, R. Leipus, and A. Philippe. Asymptotic normality of the mixture density estimator in a disaggregation scheme. *J. Nonparametric Statist.*, 22:425–442, 2010. 10, 28, 29, 31, 86, 106, 110

[23] K.S. Chan and H. Tsai. Temporal aggregation of stationary and nonstationary continuous time processes. *Scandinavian J. Statistics*, 32:583–597, 2005. 16

[24] K.S. Chan and H. Tsai. Temporal aggregation of stationary and nonstationary discrete-time processes. *J. Time. Ser. Anal.*, 26:691–713, 2005. 16

[25] T. T. Chong. The polynomial aggregated AR(1) model. *Econometrics Journal*, 9:98–122, 2006. 28, 29, 31, 32

[26] R. Cioczek-Georges and B. B. Mandelbrot. Stable fractal sums of pulses: the general case. *http://yale.academia.edu/BenoitMandelbrot/Papers/21711.*, 1995. 63

[27] M. Clausel and B. Vedel. Explicit construction of operator scaling Gaussian random fields. *Fractals*, 19:101–111, 2011. 112

[28] D.R. Cox. Long-range dependence: A review, in: H.A. David and H.T. David (Eds.). *Statistics: An Appraisal*, pages 55–74, 1984. Iowa State Univ. Press, Iowa. 27, 34, 36, 39, 41, 53, 63, 112

[29] Z. Ding and C.W.J. Granger. Modeling volatility persistence of speculative returns: a new approach. *J. Econometrics*, 73:185–215, 1996. 16, 160

[30] R.L. Dobrushin. Gaussian and their subordinated self-similar random generalized fields. *Ann. Probab.*, 7:1–28, 1979. 112

[31] R.L. Dobrushin and P. Major. Non-central limit theorems for non-linear functionals of gaussian fields. *Probab. Th. Rel. Fields*, 50:27–52, 1979. 112

[32] P. Doukhan, G. Lang, and D. Surgailis. Asymptotics of weighted empirical processes of linear random fields with long range dependence. *Annales d'Institute de H. Poincaré*, 38:879–896, 2002. 112

[33] P. Doukhan, G. Oppenheim, and M.S. Taqqu. *Theory and Applications of Long-Range Dependence*. Birkhäuser, Boston, 2003. 112

[34] P. Embrechts and N. Veraverbeke. Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance: Mathematics and Economics*, 1:55–72, 1982. 160

[35] W. Feller. *An Introduction to Probability Theory and its Applications*, volume 2. Wiley, New York, 1966. 88, 116, 123

[36] W. Feller. *An Introduction to Probability Theory and its Applications*, volume 1. Wiley, New York, 3 edition, 1971. 42, 116, 140, 146

[37] R. Giacomini and C.W.J. Granger. Aggregation of space time processes. *Journal of Econometrics*, 118:7–26, 2004. 16

[38] L. Giraitis, H.L. Koul, and D. Surgailis. *Large Sample Inference for Long Memory Processes*. Imperial College Press, London, 2012. 112

[39] L. Giraitis, R. Leipus, and D. Surgailis. Aggregation of random coefficient GLARCH(1,1) process. *Econometric Theory*, 26:406–425, 2010. 8, 16, 160

[40] E. Gonçalves and C. Gouriéroux. Aggrégation de processus autoregressifs d'ordre 1. *Annales d'Economie et de Statistique*, 12:127–149, 1988. 8

[41] C.W.J. Granger. Long memory relationship and the aggregation of dynamic models. *J. Econometrics*, 14:227–238, 1980. 7, 8, 17, 18, 24, 52, 84, 160

[42] C.W.J. Granger. *Aggregation of time series variables–A Survey*. Institute for Empirical Macroeconomics, Discussion Paper 1, Federal Reserve Bank of Minneapolis, 1988. 7, 8, 16, 17

[43] X. Guyon. *Random fields on a Network*. Springer, 1995. 25

[44] A. Hankey and H.E. Stanley. Systematic application of generalized homogeneous functions to static scaling, dynamic scaling, and universality. *Phys. Review B*, 6:3515–3542, 1972. 138

[45] C.C. Heyde and Y. Yang. On defining long-range dependence. *J. Appl. Probab.*, 34:939–944, 1997. 34, 36, 39, 41, 53, 61, 63, 69, 78, 161

[46] W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58:13–30, 1963. 116, 148

[47] B.D. Hughes. *Random Walks and Random Environments*, volume 1. Clarendon Press, Oxford, 1995. 145

[48] H. Hult and G. Samorodnitsky. Tail probabilities for infinite series of regularly varying random vectors. *Bernoulli*, 14:838–864, 2008. 44, 46, 125

[49] H. Hurst. Long-term storage capacity of reservoirs. *Transactions of the American Society of Civil Engineers*, 116:770–808, 1951. 33

[50] I. A. Ibragimov and YU. V. Linnik. Independent and Stationary Sequences of Random Variables. *Wolters-Noordhoff, Groningen*, 1971. 71, 101

[51] A.V. Ivanov and N.N. Leonenko. *Statistical Analysis of Random Fields*. Kluwer, Dordrecht, 1989. 25

[52] M. Jirak. Asymptotic behavior of weakly dependent aggregated processes. *Period. Math. Hungar.*, 62(1):39–60, 2012. 8

[53] V. Kazakevičius, R. Leipus, and M.-C. Viano. Stability of random coefficient ARCH models and aggregation schemes. *J. Econometrics*, 120:139–158, 2004. 8, 16

[54] P. Kokoszka and M.S. Taqqu. Fractional FARIMA with stable innovations. *Stochastic Process. Appl.*, 60:19–47, 1995. 52

[55] H.L. Koul and D. Surgailis. Asymptotics of empirical processes of long memory moving averages with infinite variance. *Stochastic Process. Appl.*, 91:309–336, 2001. 37

[56] J. Lamperti. Semi-stable stochastic processes. *Trans. Amer. Math. Soc.*, 104:62–78, 1962. 36, 112

[57] F. Lavancier. Long memory random fields. In *in Dependence in Probability and Statistics, Lecture Notes in Statistics*, volume 187. Bertail, P; Doukhan, P; Soulier, Ph (Eds.), Springer., 2005. 25, 26, 38

[58] F. Lavancier. Invariance principles for non-isotropic long memory random fields. *Stat. Inference Stoch. Process.*, 10:255–282, 2007. 25

[59] F. Lavancier. The V/S test of long-range dependence in random fields. *Electronic Journal of Statistics*, 2:1373–1390, 2008. 25

[60] F. Lavancier. Aggregation of isotropic random fields. *J. Statist. Plan. Infer.*, 141:3862–3866, 2011. 8, 10, 25, 26, 27, 87, 115, 117

[61] F. Lavancier, R. Leipus, and D. Surgailis. Aggregation of anisotropic random-coefficient autoregressive random field. *Preprint*. 10, 25, 113, 115, 117

[62] R. Leipus, G. Oppenheim, A. Philippe, and M.-C. Viano. Orthogonal series density estimation in a disaggregation scheme. *J. Statist. Plan. Inf.*, 136:2547–2571, 2006. 8, 10, 28, 29, 30, 31, 32, 83, 86, 104, 106, 107, 108, 109, 110

[63] R. Leipus and D. Surgailis. Random coefficient autoregression, regime switching and long memory. *Adv. Appl. Probab.*, 35:1–18, 2003. 94

[64] R. Leipus and M.-C. Viano. Aggregation in ARCH models. *Lithuanian Math. J.*, 42:54–70, 2002. 8, 16

[65] N. Leonenko and E. Taufer. Disaggregation of spatial autoregressive processes. *J. Spatial Statistics*, 3:1–20, 2013. 8, 10, 28, 32, 33

[66] N.N. Leonenko. *Random Fields with Singular Spectrum*. Kluwer, Dordrecht, 1999. 112

[67] N.N. Leonenko and A. Olenko. Tauberian and abelian theorems for long-range dependent random fields. *Methodol Comput Appl Probab.*, 14:1–28, 2012. 115

[68] N.N. Leonenko, M.D. Ruiz-Medina, and M.S. Taqqu. Fractional elliptic, hyperbolic and parabolic random fields. *Electronic J. Probab.*, 16:1134–1172, 2011. 112

[69] A. Lewbel. Aggregation and simple dynamics. *American Economic Review*, 84:905–918, 1994. 8, 28, 29

[70] M. Loève. Probability theory. *Van Nostrand, Princeton, NJ.*, 3rd edn., 1963. 78

[71] B. Mandelbrot and J. Van Ness. Fractional brownian motions, fractional noises and applications. *SIAM Review*, 10:422–437, 1968. 33

[72] B. Mandelbrot and M. Taqqu. Robust R/S analysis of long-run serial correlation. *in Proceedings of the 42nd Session of the International Statistical Institute*, pages 69–104, 1979. 33

[73] B. Mandelbrot and J. Wallis. Noah, joseph and operational hydrology. *Water Resources Research*, 4:909–918, 1968. 33

[74] T. Mikosch, S. Resnick, H. Rootzén, and A. Stegeman. Is network traffic approximated by stable lévy motion or fractional brownian motion? *Ann. Appl. Probab.*, 12:23–68, 2002. 94

[75] T. Mikosch and G. Samorodnitsky. Ruin probability with claims modeled by a stationary ergodic stable process. *Ann. Probab.*, 28:1814–1851, 2000. 64, 71, 159, 160, 161, 163

[76] E.W. Montroll and G.H. Weiss. Random walks on lattices. *J. Mathematical Physics*, 6:167–181, 1965. 145

[77] G. Oppenheim and M.-C. Viano. Aggregation of random parameters ornstein-uhlenbeck or AR processes: some convergence results. *J. Time Ser. Anal.*, 25:335–350, 2004. 8, 21, 22, 23, 65

[78] W. Palma. *Long-Memory Time Series.* John Wiley and Sons, Hoboken, New Jersey, 2007. 35, 36

[79] V. Paulauskas. On $\alpha-$covariance, long, short and negative memories for sequences of random variables with infinite variance. *Working paper.* 34

[80] V. Paulauskas. On unit roots for spatially autoregressive models. *J. Multiv. Anal.*, 98:209–226, 2007. 113

[81] K. Perilioglu and D. Puplinskaitė. Asymptotics of the ruin probability with claims modeled by α-stable aggregated AR(1) process. *Turk J Math*, 37:129–138, 2013. 11, 71

[82] A. Philippe, D. Puplinskaitė, and D. Surgailis. Contemporaneous aggregation of a triangular of random coefficient AR(1) processes. *to appear in J. Time Ser. Anal.* 9

[83] V. Pilipauskaitė and D. Surgailis. Large-scale temporal and contemporaneous aggregation of random coefficient AR(1) processes. *Preprint.* 16

[84] D. Puplinskaitė and D. Surgailis. Aggregation of autoregressive random fields and anisotropic long memory. *Preprint.* 10, 87

[85] D. Puplinskaitė and D. Surgailis. Aggregation of random coefficient AR(1) process with infinite variance and common innovations. *Lithuanian Math. J.*, 49:446–463, 2009. 9, 84, 87, 115

[86] D. Puplinskaitė and D. Surgailis. Aggregation of random coefficient AR(1) process with infinite variance and idiosyncratic innovations. *Adv. Appl. Probab.*, 42:509–527, 2010. 9, 40, 62, 84, 87, 115

[87] G. Samorodnitsky M. S. Taqqu R. Cioczek-Georges, B. B. Mandelbrot. Stable fractal sums of pulses: the cylindrical case. *Bernoulli*, 1:201–216, 1995. 63

[88] B.S. Rajput and P. Robinson. Spectral representations of infinitely divisible processes. *Probab. Th. Rel. Fields*, 82:451–487, 1989. 85

[89] M. Reed and B. Simon. *Methods of Modern Mathematical Physics*, volume 2. 100

[90] P. Robinson. Statistical inference for a random coefficient autoregressive model. *Scand. J. Statist.*, 5:163–168, 1978. 7, 8, 17, 28, 29, 84

[91] J. Rosiński. On path properties of certain infinitely divisible processes. *Stochastic Process. Appl.*, 33:73–87, 1989. 59

[92] J. Rosiński. On the structure of stationary stable processes. *Ann. Probab.*, 23:464–473, 1995. 62

[93] G. Samorodnitsky. Long Range Dependence. *Foundations and Trends in Stochastic Systems*, 1 (3):163–257, 2006. 8, 33

[94] G. Samorodnitsky and M.S. Taqqu. Stable Non-Gaussian Random Processes. Chapman and Hall, New York, 1994. 34, 36, 54, 59, 63, 68, 69, 70, 71, 80, 137, 162, 163

[95] K.-I. Sato. *Lévy processes and Infinitely Divisible Distributions.* Cambridge Univ. Press, Cambridge, 1999. 87, 88, 98

[96] D. Surgailis. On the markov property of a class of linear infinitely divisible fields. *Z. Wahrscheinlichkeitsth*, 49:293–311, 1979. 67

[97] D. Surgailis. On infinitely divisible self-similar random fields. *Zeit. Wahrsch. verw. Geb.*, 58:453–477, 1981. 99

[98] D. Surgailis. Zones of attraction of self-similar multiple integrals. *Lithuanian Math. J.*, 22:185–201, 1982. 112

[99] D. Surgailis, J. Rosiński, V. Mandrekar, and S. Cambanis. Stable generalized moving averages. University of North Carolina, 1992. 63, 68

[100] D. Surgailis, J. Rosiński, V. Mandrekar, and S. Cambanis. Stable mixed moving averages. *Probab. Theory Related Fields*, 97:543–558, 1993. 61, 62, 65, 73, 114

[101] P. Whittle. On stationary processes in the plane. *Biometrika*, 41:434–449, 1954. 25

[102] P. Zaffaroni. Contemporaneous aggregation of linear dynamic models in large economies. *J. Econometrics*, 120:75–102, 2004. 8, 9, 18, 19, 20, 21, 23, 24, 39, 41, 57, 63, 65, 160

[103] P. Zaffaroni. Aggregation and memory of models of changing volatility. *J. Econometrics*, 136:237–249, 2007. 8, 16, 160

[104] P. Zaffaroni. Contemporaneous aggregation of GARCH processes. *J. Time Ser. Anal.*, 28:521–544, 2007. 16, 160

Contents

List of Tables

List of Figures