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A Class of Bounded Iterative Sequences of Integers

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Abstract: In this note, we show that, for any real number $\tau \in [\frac{1}{2}, 1)$, any finite set of positive integers K and any integer $s_1 \geq 2$, the sequence of integers $s_1, s_2, s_3, \dots$ satisfying $s_{i+1} - s_i \in K$ if s_i is a prime number, and $2 \leq s_{i+1} \leq \tau s_i$ if s_i is a composite number, is bounded from above. The bound is given in terms of an explicit constant depending on τ, s_1 and the maximal element of K only. In particular, if K is a singleton set and for each composite s_i the integer s_{i+1} in the interval $[2, \tau s_i]$ is chosen by some prescribed rule, e.g., s_{i+1} is the largest prime divisor of s_i, then the sequence $s_1, s_2, s_3, \dots$ is periodic. In general, we show that the sequences satisfying the above conditions are all periodic if and only if either $K = \{1\}$ and $\tau \in [\frac{1}{2}, \frac{3}{4})$ or $K = \{2\}$ and $\tau \in [\frac{1}{2}, \frac{5}{9})$.

Keywords: sequences; prime and composite numbers; periodicity

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1. Introduction

Throughout, we denote by $P(n)$ the largest prime divisor of an integer $n \geq 2$. In [1], for $k \in \mathbb{N}$, the sequence of integers $a_1, a_2, a_3, \dots$, where $a_1 \geq 2$ and, for each $i = 1, 2, 3, \dots$,

$$a_{i+1} = \begin{cases} a_i + k, & \text{if } a_i \text{ is a prime number;} \\ P(a_i), & \text{if } a_i \text{ is a composite number} \end{cases} \tag{1}$$

has been considered. For example, in the case when $a_1 = 2$ and $k = 12$, this sequence $(a_i)_{i=1}^{\infty}$ is

$$2, 14, 7, 19, 31, 43, 55, 11, 23, 35, 7, 19, 31, 43, 55, 11, 23, 35, \dots. \tag{2}$$

Evidently, no two consecutive terms of the sequence $(a_i)_{i=1}^{\infty}$ defined in (1) can be a composite. Deleting all the composite terms and leaving only those elements of $(a_i)_{i=1}^{\infty}$ that are primes, we will obtain a sequence of prime numbers $p_1, p_2, p_3, \dots$, where $p_1 = a_1$ if a_1 is a prime number and $p_1 = a_2$ if a_1 is a composite number, satisfying

$$p_{i+1} = P(p_i + k) \tag{3}$$

for each $i = 1, 2, 3, \dots$. Accordingly, removing the composite terms from (2), we obtain the following sequence of primes $(p_i)_{i=1}^{\infty}$ satisfying (3) with the first term $p_1 = 2$ and $k = 12$:

$$2, 7, 19, 31, 43, 11, 23, 7, 19, 31, 43, 11, 23, \dots.$$

The sequences (1) and (3) are both iterative sequences of integers

$$x, f(x), f(f(x)), f(f(f(x))), \dots,$$

where f is a map from the set $\mathbb{N}$ to itself. The most known sequence of this type is the Collatz sequence defined by $f(x) = 3x + 1$ for x odd and $f(x) = x/2$ for x even; see [2] and some recent papers [3–6] on the original Collatz problem and its variations. The results are very far from the conjecture asserting that the Collatz sequence starting from an arbitrary positive integer is ultimately periodic with the period $1, 4, 2$. Some other versions of iterative integer sequences have been considered in [7] (where $f(x) = \lfloor \alpha x + \beta \rfloor$) and subsequently in [8,9].

In [1], it was shown that the sequence (1) is periodic for any $k \in \mathbb{N}$ and any initial choice of $a_1 \geq 2$. Now, we will give a different proof of this fact by deriving an explicit upper bound on the largest element of this sequence in terms of a_1 and k. Of course, this immediately implies the periodicity of $(a_i)_{i=1}^{\infty}$, because, by (1), for each $i \in \mathbb{N}$, the element a_{i+1} is uniquely determined by its predecessor a_i.

To present our result, we will use the following notation. For a given $k \in \mathbb{N}$, by $N(k)$ we denote the smallest prime number that does not divide k. For odd k, it is clear that

$$N(k) = 2.$$

Here are the first 15 values of $N(k)$ for k even.

k	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
$N(k)$	3	3	5	3	3	5	3	3	5	3	3	5	3	3	7

By the definition of $N(k)$, it follows that

$$N(k) \leq k + 1 \tag{4}$$

for each $k \in \mathbb{N}$ with equality if and only if $k \in \{1, 2\}$. For large k, the upper bound for $N(k)$ is much better than that in (4). Indeed, let $q = N(k)$ be the least prime number not dividing $k > 2$. Then, all the primes smaller than q must divide k. Thus, their product $\prod_{p<q} p$ divides k and hence

$$\sum_{p<q} \log p \leq \log k.$$

Using the asymptotical formula $\sum_{p<x} \log p \sim x$ as $x \to \infty$, we deduce that for any $\varepsilon > 0$ there is a constant $k(\varepsilon) > 0$ such that

$$N(k) \leq (1+\varepsilon) \log k \quad \text{for each } k \geq k(\varepsilon). \tag{5}$$

In fact, by [10] (Theorem 4), the lower bound

$$\sum_{p<q} \log p = \sum_{p \leq q-1} \log p > (q-1)\Big(1 - \frac{1}{2\log(q-1)}\Big)$$

holds for $q \geq 564$, so an explicit $k(\varepsilon)$ in (5) in terms of ε can be determined.

With this notation, we can state our first result:

Theorem 1. *All the elements of the sequence* (1) *are smaller than or equal to*

$$\max\{a_1, N(k)k\} + N(k)k, \tag{6}$$

while all the elements of the sequence (3) *are smaller than or equal to*

$$\max\{p_1, N(k)k\} + (N(k) - 1)k, \tag{7}$$

In particular, the sequences (1) *and* (3) *are both periodic.*

For $k = 1$ and $a_1 = 2$, the sequence (1) is $2, 3, 4, 2, 3, 4, \ldots$, while the right-hand side of (6) is $\max\{2, 2\} + 2 = 4$. For $k = 1$ and $p_1 = 2$, the sequence (3) is $2, 3, 2, 3, 2, 3, \ldots$, whereas the right-hand side of (7) is 3. So, formally, the inequalities (6) and (7) are the best possible. For $k \geq 2$, these bounds can be improved, but we will not go into the details.

More generally, for a fixed real number τ satisfying $\frac{1}{2} \leq \tau < 1$ and a finite set

$$K = \{k_1, \ldots, k_m\} \subset \mathbb{N},$$

with

$$k = \max_{k_j \in K} k_j, \tag{8}$$

we will consider a class of integer sequences $\mathcal{S}(\tau, K)$ consisting of all sequences $\{s_1, s_2, s_3, \dots\}$ satisfying $s_1 \geq 2$ and, for each $i = 1, 2, 3, \dots$,

$$s_{i+1} - s_i \in K \text{ if } s_i \text{ is a prime number;} \tag{9}$$

$$2 \leq s_{i+1} \leq \tau s_i \text{ if } s_i \text{ is a composite number.} \tag{10}$$

Note that the smallest composite number in $\mathbb{N} \setminus \{1\}$ is 4, so some integer s_{i+1} satisfying (10) can always be chosen due to $\tau \geq \frac{1}{2}$ and $s_i \geq 4$.

In particular, if $K = \{k\}$ is a singleton set (this notation is consistent with (8)), then, by (9) and (10), for each $i = 1, 2, 3, \dots$,

$$s_{i+1} = \begin{cases} s_i + k, & \text{if } s_i \text{ is a prime number;} \\ \text{any integer in the interval } [2, \tau s_i], & \text{if } s_i \text{ is a composite number.} \end{cases}$$

It is clear that

$$\mathcal{S}(\tau, K) \subseteq \mathcal{S}(\tau', K) \quad \text{if} \quad \tau < \tau'$$

and

$$\mathcal{S}(\tau, K) \subseteq \mathcal{S}(\tau, K') \quad \text{if} \quad K \subset K'.$$

For each $S \in \mathcal{S}(\tau, K)$, we will show the following:

Theorem 2. *Assume that $\tau \in [\frac{1}{2}, 1)$ and that $K = \{k_1, \dots, k_m\} \subset \mathbb{N}$ has the largest element k. Then, the elements of the sequence $\{s_1, s_2, s_3, \dots\} \in \mathcal{S}(\tau, K)$ (as defined in (9) and (10)) are all smaller than*

$$\max\{s_1, \tau e^{2k}/(1-\tau)\} + e^{2k}. \tag{11}$$

Furthermore, in a particular case, when $K = \{k\}$, all the elements of $S = \{s_1, s_2, s_3, \dots\} \in \mathcal{S}(\tau, K)$ are smaller than or equal to

$$\max\{s_1, \tau N(k)k/(1-\tau)\} + N(k)k, \tag{12}$$

while all the prime elements of S do not exceed

$$\max\{p_1, \tau N(k)k/(1-\tau)\} + (N(k)-1)k, \tag{13}$$

where p_1 is the first prime element of the sequence S.

Note that parts (12) and (13) of Theorem 2 imply Theorem 1. Indeed, the sequence (1) belongs to the class $\mathcal{S}(\tau, K)$, where $K = \{k\}$ is singleton set and $\tau = \frac{1}{2}$. (The largest prime factor of a composite integer $n \geq 4$ does not exceed $n/2$.) Thus, the upper bound (6) follows from (12) with $\tau = \frac{1}{2}$, whereas (7) follows from (13). If $i < j$ is the pair of positive integers with the smallest index i and the smallest difference $j - i$ satisfying $a_i = a_j$, then the sequence (1) is ultimately periodic with period $a_i, a_{i+1}, \dots, a_{j-1}$. (The same is true for the sequence (3) and the first pair of primes in it satisfying $p_i = p_j$.)

Of course, the sequences in $\mathcal{S}(\tau, K)$, although bounded, are not necessarily all periodic. All the cases when they are all periodic are described by the next theorem:

Theorem 3. *Assume that $\tau \in [\frac{1}{2}, 1)$ and $K = \{k_1, \dots, k_m\} \subset \mathbb{N}$. Then, the sequences in $\mathcal{S}(\tau, K)$ are all periodic if and only if one of the following holds:*

(i) $K = \{1\}$ *and* $\frac{1}{2} \leq \tau < \frac{3}{4}$;
(ii) $K = \{2\}$ *and* $\frac{1}{2} \leq \tau < \frac{5}{9}$.

In all other cases, the class $\mathcal{S}(\tau, K)$ contains infinitely many nonperiodic sequences.

In the next section, we will give three auxiliary lemmas. Then, in Sections 3 and 4, we will prove Theorems 2 and 3, respectively. (As we already observed above, Theorem 2 implies Theorem 1.) In the last section, we will show that the class $\mathcal{S}(K, \tau)$ always contains nonperiodic sequences in the case when $K \subset \mathbb{N}$ is infinite.

2. Auxiliary Lemmas

Lemma 1. *For any integers $a, k \geq 2$, the arithmetic progression*

$$a, a+k, \ldots, a+(N(k)-1)k, a+N(k)k \tag{14}$$

contains a composite number. Moreover, if $a \neq N(k)$, then the arithmetic progression

$$a, a+k, \ldots, a+(N(k)-1)k \tag{15}$$

contains a composite number.

For example, for $k = 210$, we have $N(k) = 11$. Selecting $a = 199$, we see that the first 10 numbers in (15)

$$199, 409, 619, 829, 1039, 1249, 1459, 1669, 1879, 2089$$

are all primes, while the eleventh number

$$a + (N(k)-1)k = 199 + 10 \cdot 210 = 2299 = 11^2 \cdot 19$$

is a composite. This shows that for $k = 210$, the list (15) cannot be replaced by the shorter list $a, a+k, \ldots, a+(N(k)-2)k$. See the Wikipediaarticle (https://en.wikipedia.org/wiki/Primes_in_arithmetic_progression) (accessed on 2 January 2024) for some further nontrivial examples of primes that form long (in terms of k) arithmetic progressions with the difference k.

Proof. Consider the list of integers (15) modulo $q = N(k)$. If for some integers i, j satisfying $0 \leq i < j \leq q-1$ the numbers $a + ik$ and $a + jk$ were equal modulo q, then $q \mid (j-i)k$. Because q is a prime and $1 \leq j - i \leq q - 1$, this forces $q \mid k$, which is not the case by the definition of $q = N(k)$. Therefore, the integers (15) are all distinct modulo q, which means that exactly one of them, say $a + \ell k$, $0 \leq \ell \leq q-1$, is divisible by q. This number is composite, unless $a + \ell k = q$. Note that for $\ell \geq 1$, we have

$$N(k) = q = a + \ell k \geq a + k \geq k + 2,$$

which is impossible using (4). Hence, the equality $a + \ell k = q$ occurs only for $\ell = 0$ and $a = q$. This proves the second assertion, because then $a \neq q$. Of course, for $a \neq q$, this also proves the first assertion. On the other hand, if $a = q = N(k)$, then the last number in the list (14), namely, $a + N(k)k = N(k)(1+k)$, is composite. This completes the proof of the first assertion of the lemma. □

The next lemma is (1.12) from [11].

Lemma 2. *For any real numbers $x > 0$ and $y > 1$, the interval $(x, x+y]$ contains at most $2y/\log y$ prime numbers.*

This result of Montgomery and Vaughan is related to the famous Hardy–Littlewood conjecture, which asserts that for the prime-counting function $\pi(x) = \#\{p \leq x\}$ the inequality

$$\pi(x+y) \leq \pi(x) + \pi(y)$$

holds for any integers $x, y \geq 2$, see [12] (p. 54). This inequality has been proved only under some assumptions on x and y; roughly, when x and y are of similar size, see, e.g., [13–16]. More references can be found in [17]. However, in our situation, y can be small compared to x, so the bound with an extra factor 2 as given in Lemma 2 seems to be the best available known result for our purposes. In fact, as a result of Hensley and Richards [18], the conjecture of Hardy and Littlewood is incompatible with the so-called prime k-tuples conjecture, which is widely believed to be true. In view of this, it is not clear at all if the constant 2 in Lemma 2 can be replaced by a constant arbitrarily close to 1.

To state our next lemma, we need the following definition. We say that a finite string of positive integers

$$C = s_1, s_2, \dots, s_t$$

is an *s-cycle* in the class $\mathcal{S}(\tau, K)$ if $s_1 = s$, $s_j \neq s$ for $j = 2, \dots, t$ and the purely periodic sequence

$$C^\infty = s_1, s_2, \dots, s_t, s_1, s_2, \dots, s_t, \dots$$

belongs to the class $\mathcal{S}(\tau, K)$. This means that the elements of the sequence C^∞ are all in $\mathbb{N} \setminus \{1\}$ and satisfy (9) and (10). (Of course, it is sufficient to verify this for $i = 1, \dots, t$, because $s_{t+1} = s_1 = s$ and the sequence C^∞ is periodic.)

For example, consider the case $\tau = \frac{1}{2}$ and $K = \{4\}$. Note that if $s_i = 15$, then, by (10), as s_{i+1} we can select, for instance, 3 or 6. Hence, $C = 3, 7, 11, 15$ and $C' = 3, 7, 11, 15, 6$ are both 3-cycles in the class $\mathcal{S}(\frac{1}{2}, \{4\})$. (Their first element is 3, and 3 is the only element in both strings C, C'.)

Lemma 3. *Assume that for some integer $s \geq 2$, the class $\mathcal{S}(\tau, K)$ has at least two distinct s-cycles. Then, $\mathcal{S}(\tau, K)$ contains infinitely many nonperiodic sequences.*

Proof. Let C and C' be two distinct s-cycles in $\mathcal{S}(\tau, K)$. Take any nonperiodic sequence with two letters of the alphabet $\{C, C'\}$. Then, replace C, C' in it with their corresponding strings of integers, say, $s, s_2, \dots, s_t$ and $s, s'_2, \dots, s'_m$. We claim that the resulting sequence $S \in \mathcal{S}(\tau, K)$ is nonperiodic.

Assume that S is periodic. Then, without the loss of generality, we may assume that some period in it starts with s and ends at a certain integer $s' \neq s$. The next element of S must be s again; so, in the period, we can replace the strings back to the letters C and C'. Because S is periodic, this means that a nonperiodic sequence on $\{C, C'\}$ from a certain place is also represented by a periodic sequence on the same two letters. Consequently, at some stage, say from the gth element, we must have the cycles C and C' both starting from the same element $s_g = s$. As $C \neq C'$, the cycles C and C' cannot be of the same length. Indeed, otherwise, the sequence of C, C', starting from the element s_g, is uniquely determined, and a nonperiodic sequence on these two letters cannot be represented by a periodic one.

Assume that C has more elements than C', i.e., $t > m$. Recall that the cycles C and C' both start from $s_g = s$. But then, as after C' we have C or C', the element s_{g+m} of the sequence S must be s, which is not allowed by the definition of C (s is only the first element of C). The case $t < m$ can be treated with the same argument.

Therefore, the sequence S obtained as a nonperiodic combination of two s-cycles C, C' and then replacing them with their corresponding strings of numbers in $\mathbb{N} \setminus \{1\}$ is indeed nonperiodic.

Finally, observe that, by taking any composite integer s_0 greater than $2s$ and adding it to the beginning of the above constructed nonperiodic sequence

$$S = \{s_1 = s, s_2, s_3, \dots\} \in \mathcal{S}(\tau, K),$$

we will obtain a new nonperiodic sequence $s_0, s_1, s_2, s_3, \dots$ in $\mathcal{S}(\tau, K)$; see the property (10). This completes the proof of the lemma, because there are infinitely many choices of such integers s_0. $\square$

3. Proof of Theorem 2

Let $S = \{s_1, s_2, s_3, \dots\}$ be a sequence from the class $\mathcal{S}(\tau, K)$. For the simplicity of exposition, we present this sequence in a binary alphabet $\{p, c\}$, where the letter p stands for s_i if s_i is prime, and the letter c stands for s_i if s_i is composite. For example, the sequence (2) is

$$p, c, p, p, p, p, c, p, p, c, p, p, p, p, c, p, p, c, \dots.$$

We clearly have $s_i < s_{i+1}$ if the letter p stands for s_i, and $s_i > s_{i+1}$ if the letter c stands for s_i.

Let $\{p_1, p_2, p_3, \ldots\}$ be a subsequence of S obtained from S by deleting its composite elements, so p_i simply enumerate the letters p. If the sequence $\{p_1, p_2, p_3, \ldots\}$ were finite, then we would have $s_{i+1} \in [2, \tau s_i]$ for each sufficiently large i, say, for $i \geq n_0$. But then, for each $i \geq n_0$ from $s_{i+1} \leq \tau s_i < s_i$, we deduce that $s_{n_0} > s_{n_0+1} > s_{n_0+2} > \ldots$ is a decreasing sequence of integers. This is impossible, because $s_j \geq 2$ for all $s_j \in S$. Consequently, the sequence $\{p_1, p_2, p_3, \ldots\}$ is infinite. In the notation with p and c, this means that the sequence S contains infinitely many letters p.

Next, we consider a subsequence $\{q_1, q_2, q_2, \ldots\}$ of $\{p_1, p_2, p_3, \ldots\}$ obtained by removing from $\{p_1, p_2, p_3, \ldots\}$ the primes from consecutive patterns $p, p, \ldots, p$ of all primes except for the first one. In particular, we will have $q_1 = p_1$, while for each q_i, $i \geq 2$, between q_{i-1} and q_i, first there are possibly a few prime elements of S and then there must be one of several composite elements of S.

Now, we will prove (13). (Recall that $K = \{k\}$.) We claim that

$$q_i \leq M := \max\{p_1, \tau N(k)k/(1-\tau)\} \tag{16}$$

for each $i \in \mathbb{N}$.

We will use the induction on i. Of course, (16) trivially holds for $i = 1$ because then $q_1 = p_1$. Assume that (16) is true for some q_{i-1}, where $i \geq 2$. Suppose that between q_{i-1} and q_i there are $\ell \geq 0$ primes (letters p) and then $l \geq 1$ composite elements of S (letters c). By Lemma 1, we have

$$\ell \leq N(k) - 1. \tag{17}$$

Thus, the first composite element is smaller than or equal to $q_{i-1} + N(k)k$. The lth composite element (the one that appears just before q_i, say, s_j) is therefore at most $\tau^{l-1}(q_{i-1} + N(k)k)$. Hence,

$$q_i \leq \tau s_j \leq \tau^l(q_{i-1} + N(k)k) \leq \tau(q_{i-1} + N(k)k).$$

Now, by our inductive assumption $q_{i-1} \leq M$, it remains to verify the inequality

$$\tau(M + N(k)k) \leq M.$$

However, the latter inequality is equivalent to $\tau N(k)k/(1-\tau) \leq M$, which is true by the definition of M in (16). This completes the proof of (16).

Next, note that each p_j is of the form $q_i + uk$ with some integers $i \geq 1$ and $u \geq 0$. Furthermore, we must have $u \in \{0, 1, \ldots, N(k) - 1\}$ by Lemma 1. Hence, by (16), each p_i, $i \in \mathbb{N}$, is smaller than or equal to $M + (N(k) - 1)k$. This completes the proof of (13).

In order to prove (12), we first observe that, by Lemma 1 and the definition of $(q_i)_{i=0}^{\infty}$, each element of the sequence S is smaller than or equal to

$$\max\{s_1, \max\{q_1, q_2, q_3, \ldots\} + N(k)k\} \leq \max\{s_1, M + N(k)k\}.$$

Hence, by the definition of M in (16), all the elements of S do not exceed

$$\max\{s_1, \max\{p_1, \tau N(k)k/(1-\tau)\} + N(k)k\}. \tag{18}$$

Because $p_1 \leq s_1$, (18) does not exceed the right-hand side of (12).

It remains to prove (11) for the set $K = \{k_1, \ldots, k_m\}$ with the largest element k. This time, we claim that

$$q_i \leq M' := \max\{p_1, \tau e^{2k}/(1-\tau)\} \tag{19}$$

for each $i \geq 1$.

It is clear that (19) is true for $i = 1$. Assume that (19) is true for q_{i-1} with $i \geq 2$. As above, suppose that between q_{i-1} and q_i first there are $\ell \geq 0$ prime elements and then $l \geq 1$ composite elements of S. We will show that

$$\ell < \frac{e^{2k}}{k} - 1. \tag{20}$$

By $7k < e^{2k}$, it is clear that (20) holds for $\ell \leq 6$, so assume that $\ell \geq 7$. The inequality (20) also holds for K being a singleton set by (4) and (17) because $k(k+1) < e^{2k}$. Thus, we can assume that $m = |K| \geq 2$. The $\ell + 1$ consecutive elements of S

$$q_{i-1}, q_{i-1} + k_{i_1}, q_{i-1} + k_{i_1} + k_{i_2}, \ldots, q_{i-1} + k_{i_1} + k_{i_2} + \ldots + k_{i_\ell}, \tag{21}$$

where $k_{i_1}, k_{i_2}, \ldots, k_{i_\ell} \in K$, are all prime, and the first composite element of S following them is

$$q_{i-1} + k_{i_1} + k_{i_2} + \ldots + k_{i_\ell} + k_{i_{\ell+1}}, \quad k_{i_{\ell+1}} \in K.$$

If $l > 1$, there are also other composite elements between this element and q_i, but they all appear in descending order. This means that

$$q_i \leq \tau(q_{i-1} + k_{i_1} + k_{i_2} + \ldots + k_{i_\ell} + k_{i_{\ell+1}}). \tag{22}$$

Also, the interval

$$(x, x+y] = (q_{i-1} - 1/2, q_{i-1} + k_{i_1} + k_{i_2} + \ldots + k_{i_\ell}]$$

contains at least $\ell + 1$ prime numbers, for example, $\ell + 1$ distinct primes that are listed in (21). Here, $x = q_{i-1} - 1/2$ and

$$y = k_{i_1} + k_{i_2} + \ldots + k_{i_\ell} + 1/2.$$

Therefore, using $\ell \geq 7$ and (8), we obtain

$$8 \leq \ell + 1 \leq y \leq \ell k + 1/2 < (\ell + 1)k.$$

Hence, by Lemma 2, it follows that

$$\ell + 1 \leq \frac{2y}{\log y} < \frac{2(\ell+1)k}{\log((\ell+1)k)}, \tag{23}$$

because the function $\frac{y}{\log y}$ is increasing for $y \geq e$. Inequality (23) implies $\log((\ell+1)k) < 2k$, which yields (20).

Next, by (20) and (22), we obtain

$$q_i \leq \tau(q_{i-1} + k_{i_1} + k_{i_2} + \ldots + k_{i_\ell} + k_{i_{\ell+1}}) \leq \tau(q_{i-1} + (\ell+1)k) < \tau(q_{i-1} + e^{2k}). \tag{24}$$

Using the inductive assumption $q_{i-1} \leq M'$, from (24) we deduce that $q_i < \tau(M' + e^{2k})$, which is less than or equal to M' by the definition of M' in (19). Hence, $q_i < M'$. This concludes the proof of (19) for each $i \in \mathbb{N}$.

Because the bound on ℓ in (20) is independent of i, the largest prime element of S does not exceed

$$M' + \ell k < M' + \left(\frac{e^{2k}}{k} - 1\right)k = M' + e^{2k} - k. \tag{25}$$

Consider the subsequence $\{s_u, s_{u+1}, s_{u+2}, \ldots\}$ of $(s_i)_{i=1}^{\infty}$, where $s_u = p_1$ and u is the smallest integer with this property. By (25), the largest element of this subsequence is less than

$$(M' + e^{2k} - k) + k = M' + e^{2k} = \max\{p_1, \tau e^{2k}/(1-\tau)\} + e^{2k}.$$

This proves (11) in the case when $u = 1$. Assume that $u > 1$. Then, the largest element of $S = \{s_1, s_2, s_3, \ldots\}$ is either less than $M' + e^{2k}$ (if it is among $\{s_u, s_{u+1}, s_{u+2}, \ldots\}$) or is equal to $\max\{s_1, \ldots, s_{u-1}\} = s_1$. Because $s_1 > p_1$, $M' + e^{2k}$ does not exceed the right-hand side of (11). On the other hand, the element s_1 is also strictly smaller than the right-hand side of (11). Consequently, all the elements of S are smaller than $\max\{s_1, \tau e^{2k}/(1-\tau)\} + e^{2k}$. This finishes the proof of the theorem.

4. Proof of Theorem 3

Consider the case (i). Let $S = s_1, s_2, s_3, \ldots$ be a sequence in the class $\mathcal{S}(\tau, \{1\})$, where $\frac{1}{2} \leq \tau < \frac{3}{4}$. If $s_1 \in \{2, 3, 4\}$, then, by (9) and (10), S is a purely periodic sequence with period

$2,3,4$ or $3,4,2$ or $4,2,3$. We will show that any $S \in \mathcal{S}(\tau, \{1\})$ is ultimately periodic with one of those three periods. The proof is by induction on $s_1 = s$. Assume that $s \geq 5$ and that we already established the periodicity of the sequence S in case it has an element at most $s-1$. For $s \geq 5$, at least one of the numbers $s, s+1$ is composite and the next element of S is smaller than $\frac{3}{4} \cdot (s+1) < s$. The periodicity now follows due to our inductive assumption.

Now, consider the case (ii). Let $S = s_1, s_2, s_3, \ldots$ be a sequence in $\mathcal{S}(\tau, \{2\})$, where $\frac{1}{2} \leq \tau < \frac{5}{9}$. We will show that each S is ultimately periodic with one of the possible periods, $2,4$ or $4,2$ or $3,5,7,9$ or $5,7,9,3$ or $7,9,3,5$ or $9,3,7,5$. If $s_1 \in \{2,4\}$, then, by (9) and (10), S is purely periodic with period $2,4$ or $4,2$. If $s_1 \in \{3,5,7,9\}$, then after several steps we reach $s_u = 9$ (possibly $u = 1$), so the next element s_{u+1} is less than $\frac{5}{9} \cdot 9 = 5$. If s_{u+1} is 2 or 4, then the sequence becomes ultimately periodic with period $2,4$ or $4,2$. Otherwise, $s_{u+1} = 3$. If it is always 3, namely, $s_{u+1+4m} = 3$ for every $m \geq 0$, then the sequence is purely periodic with period $3,5,7,9$ or $5,7,9,3$ or $7,9,3,5$ or $9,3,5,7$. If otherwise $s_{u+1+4m} \in \{2,4\}$ for some $m \geq 0$, then it is ultimately periodic with period $2,4$ or $4,2$. For $s_1 = s \geq 6$, one of the integers $s, s+2, s+4$ is composite, so the next element of S is less than $\frac{5}{9} \cdot (s+4) < s$. Hence, it is at most $s-1$, which concludes the proof by induction on s.

Assume that τ and K are such that neither (i) nor (ii) is satisfied. We first consider the case when the set K contains an element k satisfying $k \geq 3$. In view of $\mathcal{S}(\frac{1}{2}, \{k\}) \subseteq \mathcal{S}(\tau, K)$, it is sufficient to show that $\mathcal{S}(\frac{1}{2}, \{k\})$ contains infinitely many nonperiodic sequences.

Suppose first that $k \geq 6$ is even. Then, $2+k$ is composite. Thus, $2, 2+k$ is a 2-cycle of $\mathcal{S}(\frac{1}{2}, \{k\})$. Moreover, if $1+k/2$ is a composite number, then $2, 2+k, 1+k/2$ is also a 2-cycle of $\mathcal{S}(\frac{1}{2}, \{k\})$. On the other hand, if $1+k/2$ is a prime number, then $k \neq 6$ and $k/2$ is not a prime. In that case, $2, 2+k, k/2$ is a 2-cycle of $\mathcal{S}(\frac{1}{2}, \{k\})$. Therefore, in both cases, for even $k \geq 6$, the class $\mathcal{S}(\frac{1}{2}, \{k\})$ contains at least two distinct 2-cycles. Consequently, by Lemma 3, it contains infinitely many nonperiodic sequences.

Likewise, for $k \geq 9$ odd, $3, 3+k$ and $3, 3+k, (3+k)/2$, where $(3+k)/2$ is composite, are both 3-cycles of $\mathcal{S}(\frac{1}{2}, \{k\})$, so the result follows by Lemma 3. If $(3+k)/2$ is a prime, then $k \neq 9$ and $3, 3+k, (1+k)/2$ is a 3-cycle, because $(1+k)/2 = (3+k)/2 - 1$ is composite and greater than or equal to 6. The result again follows by Lemma 3.

In the remaining cases $k = 3,4,5,7$, we will explicitly present the corresponding 2-cycles in $\mathcal{S}(\frac{1}{2}, \{k\})$. For $k = 7$, in $\mathcal{S}(\frac{1}{2}, \{k\})$ there are two distinct 2-cycles, $2,9$ and $2,9,4$. For $k = 5$, there are two distinct 2-cycles, $2,7,12$ and $2,7,12,6$. For $k = 4$, there are two distinct 2-cycles, $2,6$ and $2,6,3,7,11,15$. Finally, for $k = 3$, in $\mathcal{S}(\frac{1}{2}, \{k\})$ there are two distinct 2-cycles, $2,5,8$ and $2,5,8,4$. In all the above cases, the required result follows from Lemma 3.

Now, it remains to consider the case $K \subseteq \{1,2\}$. Suppose first that $K = \{1,2\}$. Then, the class $\mathcal{S}(\tau, K)$ contains two distinct 2-cycles, $2,4$ and $2,3,4$, so the proof is concluded by Lemma 3. Because the cases (i) and (ii) are already considered, we are left with two possibilities $K = \{1\}$, $\tau \geq \frac{3}{4}$ and $K = \{2\}$, $\tau \geq \frac{5}{9}$. If $K = \{1\}$ and $\tau \geq \frac{3}{4}$, then in $\mathcal{S}(\tau, \{1\})$ there are the following two distinct 2-cycles: $2,3,4$ and $2,3,4,3,4$. Finally, if $K = \{2\}$ and $\tau \geq \frac{5}{9}$, then the class $\mathcal{S}(\tau, \{2\})$ also contains two distinct 3-cycles, for instance, $3,5,7,9$ and $3,5,7,9,5,8$. In both cases, the proof is concluded by Lemma 3 as before.

5. Concluding Remarks

The main result of this paper's Theorem 2 shows that the sequences of the class $\mathcal{S}(K, \tau)$ are all bounded. More precisely, the largest element of $S = \{s_1, s_2, s_3, \ldots\} \in \mathcal{S}(K, \tau)$ is bounded from above in terms of s_1, τ and the maximal element of K no matter how large the finite set K is.

What about the case when the set $K \subset \mathbb{N}$ is infinite, which is possibly a very sparse set? We will show that then no result similar to Theorem 2 is possible, because the class $\mathcal{S}(K, \tau)$ always contains unbounded sequences for any infinite $K \subset \mathbb{N}$ and any $\tau \in (0,1)$.

Indeed, let us start the construction of such $S = \{s_1, s_2, s_3, \ldots\} \in \mathcal{S}(K, \tau)$ from any prime number $s_1 = p$. Because K is infinite, we can choose $k \in K$ so large that

$$\left(\frac{2}{\tau} - 1\right) p < k. \tag{26}$$

Take the least positive integer j for which the number $p + jk$ is composite. By Lemma 1, this j does not exceed $N(k)$. Then, by the rule (9), because $p, p + k, \ldots, p + (j-1)k$ are all primes, the numbers

$$s_2 = p + k, \ldots, s_{j+1} = p + jk$$

can be chosen as the consecutive elements of S. Because s_{j+1} is composite, by the rule (10), as the next element s_{j+2} of S we can choose any integer from the interval

$$\left[\frac{\tau}{2}(p + k), \tau(p + jk)\right]. \tag{27}$$

This is indeed possible, because the right endpoint of the interval (27) is τs_{j+1}, while its left endpoint is

$$\frac{\tau}{2}(p + k) > \frac{\tau}{2} \cdot \frac{2p}{\tau} = p \geq 2$$

due to (26).

Note that the interval (27) is of the form $[u, 2u]$, with $u \geq 2$. Therefore, by Bertrand's postulate, it contains a prime number, say, p'. Let us choose $s_{j+2} = p'$. Because $s_{j+2} = p'$ belongs to the interval (27), using (26), we deduce

$$s_{j+2} = p' \geq \frac{\tau}{2}(p + k) > \frac{\tau}{2} \cdot \frac{2p}{\tau} = p,$$

so $p' > p$.

Now, arguing with $s_{j+2} = p' > p$ as before, namely, choosing $k' \in K$ so large that

$$\left(\frac{2}{\tau} - 1\right) p' < k',$$

we will construct another prime element p'' of S satisfying $p'' > p'$. Continuing this process, we will obtain a sequence $S \in \mathcal{S}(K, \tau)$ containing an infinite subsequence of primes

$$p < p' < p'' < p''' < \ldots.$$

The latter sequence of primes is unbounded, so $S \in \mathcal{S}(K, \tau)$ is unbounded too.

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