

# The existence and uniqueness of the solution of the integral equation driven by fractional Brownian motion

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## Introduction

In this note we consider a non-linear stochastic integral equation (SIE)

$$X_t = \xi + \int_0^t f(X_s) ds + \int_0^t g(X_s) dB_s^H, \quad 0 \leq t \leq T, \quad (1)$$

where  $B^H$  is a fractional Brownian motion (fBm) with the Hurst index  $1/2 < H < 1$ . It is known that almost all sample paths of fBm  $B^H$ ,  $1/2 \leq H < 1$ , have bounded  $p$ -variation for  $p > 1/H$ . Thus the integrals on the right side of (1) will exist pathwise as the Riemann-Stieltjes integrals.

A solution of the stochastic integral equation (1), on a given filtered probability space  $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$  and with respect to the fixed fBm  $(B^H, \mathbf{F})$ ,  $1/2 < H < 1$ , and initial condition  $\xi$ , is an adapted to the filtration  $\mathbf{F}$  continuous process  $X = \{X_t: 0 \leq t \leq T\}$  such that  $X_0 = \xi$  a.s.,  $\mathbf{P}(\int_0^t |f(X_s)| ds + |\int_0^t g(X_s) dB_s^H| < \infty) = 1$  for every  $0 \leq t \leq T$ , and its almost all sample paths satisfy (1).

For  $0 < \alpha \leq 1$ ,  $\mathcal{H}_{1+\alpha}$  will denote the set of all  $C^1$ -functions  $g: \mathbf{R} \rightarrow \mathbf{R}$  such that

$$\sup_x |g'(x)| + \sup_{x \neq y} \frac{|g'(x) - g'(y)|}{|x - y|^\alpha} < \infty.$$

Denote by  $\mathcal{W}_p([a, b])$ ,  $1 < p < 2$ , the class of all functions defined on  $[a, b]$  with bounded  $p$ -variation. Let  $CW_p([a, b])$  denote the subclass of  $\mathcal{W}_p([a, b])$  of all continuous functions.

**Theorem.** *Let  $f$  be a Lipschitz function and  $g \in \mathcal{H}_{1+\alpha}$ ,  $0 < \alpha \leq 1$ . For  $1 \leq p < 1 + \alpha$  there exists a unique solution of the equation (1) with almost all sample paths in the  $CW_p([0, T])$ .*

## Existence and uniqueness of the solution

All facts mentioned bellow about the  $p$ -variation are taken from [1] and [6].

For  $p \geq 1$ , denote by  $v_p(f; [a, b])$  the  $p$ -variation of the function  $f$  on  $[a, b]$  and define  $V_p(f; [a, b]) = v_p^{1/p}(f; [a, b])$ , which is a seminorm on  $\mathcal{W}_p([a, b])$ . Let  $V_{p,\infty}(f; [a, b]) = v_p(f; [a, b]) + \sup_{a \leq x \leq b} |f(x)|$ . Then  $V_{p,\infty}(f; [a, b])$  is a norm and  $\mathcal{W}_p([a, b])$  equipped with this norm is a Banach space.

Note that

$$V_{p,\infty}(f; [a, b]) \leq V_{p,\infty}(f; [a, c]) + V_{p,\infty}(f; [c, b]), \tag{2}$$

where  $a < c < b$ . The rest inequality follows from the inequality

$$V_p(f; [a, b]) \leq V_p(f; [a, c]) + V_p(f; [c, b]). \tag{3}$$

Let  $f \in \mathcal{W}_q([a, b])$  and  $h \in \mathcal{W}_p([a, b])$  with  $p > 0, q > 0, 1/p + 1/q > 1$ . If  $f$  and  $h$  have no common discontinuities then the Riemann – Stieljes integral  $\int_a^b f dh$  exists and the Love–Young inequality

$$V_p\left(\int_a^b f(x) dh(x)\right) \leq C_{p,q} V_{q,\infty}(f; [a, b]) V_p(h; [a, b]) \tag{4}$$

holds, where  $C_{p,q} = \zeta(p^{-1} + q^{-1})$ ,  $\zeta(s)$  denotes the Riemann zeta function, i.e.,  $\zeta(s) = \sum_{n \geq 1} n^{-s}$ .

Let  $f$  be a function on  $[0, T]$  and let  $\lambda = \{\lambda_m: m \geq 1\}$  be a sequence of dyadic partitions  $\lambda_m = \{i2^{-m}: i = 0, \dots, ([T] + 1)2^m\}$  of  $[0, [T] + 1]$ . For  $0 < p < \infty$  and  $0 < t \leq T$ , let

$$v_p(f; \lambda_m)(t) := \max \left\{ \sum_{j=1}^k |f(s_j \wedge t) - f(s_{j-1} \wedge t)|^p: \right. \\ \left. \{0, ([T] + 1)2^m\} \subset \{s_j: j = 0, \dots, k\} \subset \lambda_m \right\},$$

which is the  $p$ -variation over the finite set  $\{i2^{-m} \wedge t: i = 0, \dots, ([T] + 1)2^m\}$ .

Since  $\lambda$  is a sequence of nested partitions, the sequence  $v_p(f; \lambda_m)(t)$ ,  $m \geq 1$ , is non-decreasing for each  $0 < t \leq T$ . For  $0 \leq t \leq T$ , let

$$v_p(f)(t) := \sup_{m \geq 1} v_p(f; \lambda_m)(t) = \lim_{m \rightarrow \infty} v_p(f; \lambda_m)(t).$$

For a stochastic process  $Y = \{Y(t): 0 \leq t \leq T\}$  and each  $0 \leq t \leq T$ ,  $v_p(Y)(t, \omega) := v_p(Y(\cdot, \omega))(t)$  is possibly unbounded but measurable function of  $\omega \in \Omega$ . Let  $Y$  be a cadlag process. If  $v_p(Y)(T) < \infty$  almost surely, then  $\{v_p(Y; [0, t]): 0 \leq t \leq T\}$  is a stochastic process indistinguishable from  $v_p(Y)$ . Moreover,  $\{v_p(Y; [0, t]): 0 \leq t \leq T\}$  is a cadlag stochastic process.

The proof of the Theorem 1 is similar to the proof in the case when an integral equation is driven by a deterministic function of bounded  $p$ -variation (see [2]-[4]). Here we have to prove in addition that a solution is indeed adapted.

The existence of a solution is proved using the Picard iteration method, i.e. we consider the iteration

$$X_t^{n+1} = \xi + \int_0^t f(X_s^n) ds + \int_0^t g(X_s^n) dB_s^H, \quad n \geq 0, \quad (5)$$

where  $X^0 = \xi$ . The integrals on the right side of (5) are clearly well defined for  $n = 0$ . Moreover, the processes  $X^1$  is continuous,  $\mathbf{F}$ -adapted and  $v_p(X^1)(T) < \infty$  a.s. By induction one can prove that for any  $n \geq 1$  the process  $X^n$  has these properties.

First we will prove two lemmas. Define a sequence of stopping times

$$\tau_n = \inf \left\{ t > \tau_{n-1} : V_p(B^H; [\tau_{n-1}, t]) > \frac{1}{4C_{p,p}} \min \{1, L^{-1}, (2|g'|_\infty)^{-1}\} \right. \\ \left. \wedge \left( \tau_{n-1} + \frac{1}{4C_{p,p}} \min \{1, L^{-1}, (2|g'|_\infty)^{-1}\} \right), \quad n \in \mathbf{N}, \quad \tau_0 = 0, \right.$$

where  $L$  is the Lipschitz constant of the function  $f$  in (5) and  $|g'|_\infty = \sup_x |g'(x)|$ .

**Lemma 1.** For any  $m, n \in \mathbf{N}$  the inequality

$$V_p(X^{n+1}; [0, \tau_m]) \leq 2^{m+1} \max \{1, |\xi|\} [1 + |f(0)| + |g(0)|] \quad (6)$$

holds.

*Proof.* For any  $n$  and  $k$ , by the Love-Young inequality (4), we have

$$\begin{aligned} & V_p(X^{n+1}; [\tau_{k-1}, \tau_k]) \\ & \leq V_1 \left( \int_0^{\tau_k} f(X_s^n) ds; [\tau_{k-1}, \tau_k] \right) + V_p \left( \int_0^{\tau_k} g(X_s^n) dB_s^H; [\tau_{k-1}, \tau_k] \right) \\ & \leq \int_{\tau_{k-1}}^{\tau_k} |f(X_s^n)| ds + C_{p,p} V_{p,\infty}(g(X^n); [\tau_{k-1}, \tau_k]) V_p(B^H; [\tau_{k-1}, \tau_k]) \\ & \leq \left[ |f(0)| + L|X^n(\tau_{k-1})| + L \cdot V_p(X^n; [\tau_{k-1}, \tau_k]) \right] (\tau_k - \tau_{k-1}) \\ & \quad + C_{p,p} \left[ 2|g'|_\infty V_p(X^n; [\tau_{k-1}, \tau_k]) + |g(0)| + |g'|_\infty |X^n(\tau_{k-1})| \right] \\ & \quad \times V_p(B^H; [\tau_{k-1}, \tau_k]). \end{aligned} \quad (7)$$

To prove (6) we use the inequality (7) for induction on  $k$ . First we estimate the quantity  $V_p(X^{n+1}; [0, \tau_1])$ . Denote  $R := 2 \max \{1, |\xi|\} (1 + |f(0)| + |g(0)|)$ . By the definition of the stopping time  $\tau_1$  it is obvious that for all  $n \geq 0$

$$\begin{aligned} V_p(X^{n+1}; [0, \tau_1]) & \leq |f(0)| + |g(0)| + |\xi| + \frac{1}{2} V_p(X^n; [0, \tau_1]) \\ & \leq \left( 1 + \frac{1}{2} + \dots + \frac{1}{2^n} \right) (|f(0)| + |g(0)| + |\xi|) \\ & \quad + \frac{1}{2^{n+1}} V_p(X^0; [0, \tau_1]) \end{aligned}$$

$$\leq 2(|f(0)| + |g(0)| + |\xi|) \leq R.$$

Let

$$V_p(X^{n+1}; [\tau_{k-1}, \tau_k]) \leq 2^{k-1}R, \quad n \geq 0.$$

Then by the inequality (7), we get

$$\begin{aligned} & V_p(X^{n+1}; [\tau_k, \tau_{k+1}]) \\ & \leq \left[ |f(0)| + C_{p,p}|g(0)| + (L + C_{p,p}|g'|_\infty) \left( V_p(X^n; [0, \tau_k]) + |\xi| \right) \right. \\ & \quad \left. + (L + 2C_{p,p}|g'|_\infty) \cdot V_p(X^n; [\tau_k, \tau_{k+1}]) \right] \\ & \quad \times \max \left\{ \tau_{k+1} - \tau_k, V_p(B^H; [\tau_k, \tau_{k+1}]) \right\} \\ & \leq |f(0)| + |g(0)| + \frac{1}{2} \left[ \sum_{i=1}^k 2^{i-1}R + |\xi| \right] + \frac{1}{2} V_p(X^n; [\tau_k, \tau_{k+1}]) \\ & \leq 2 \left[ |f(0)| + |g(0)| + \frac{2^k - 1}{2} R + \frac{1}{2} |\xi| \right] \leq 2^k R. \end{aligned} \tag{8}$$

Thus (3) and (8) imply the result.  $\square$

For fixed  $m \geq 1$ , define a sequence of stopping times  $\sigma_m = \gamma_m \wedge \tau_m \wedge \theta_m$ , where

$$\begin{aligned} \theta_m &= m \cdot \mathbf{1}\{|\xi| \leq m\}, \\ \gamma_m &= \inf \left\{ t > \gamma_{m-1} : V_p(B^H; [\gamma_{m-1}, t]) > \frac{1}{8C_{p,p/\alpha}} \right. \\ & \quad \left. \times [2|g'|_\infty + |g'|_\alpha 2^{m+1} \widehat{R}_m]^{-1} \right\} \wedge \left( \gamma_{m-1} + \frac{1}{8L} \right), \\ \widehat{R}_m &= m \cdot [1 + |f(0)| + |g(0)|]. \end{aligned}$$

**Lemma 2.** For fixed  $m$ , we have

$$\begin{aligned} & V_{p,\infty}(X^{n+1} - X^n; [0, \sigma_m]) \\ & \leq \frac{2}{2^n} 2^{m-1} [1 + n + \dots + n^{m-1}] [|f(\xi)| \cdot \sigma_m + |g(\xi)| \cdot V_p(B^H; [0, \sigma_m])]. \end{aligned}$$

*Proof.* Denote  $Z^{n+1} = X^{n+1} - X^n$ ,  $n \geq 0$ . Note that for any  $k, n \in \mathbb{N}$

$$\begin{aligned} & V_{p,\infty}(Z^{n+1}; [\sigma_{k-1}, \sigma_k]) \\ & \leq 2V_p(Z^{n+1} - Z^{n+1}(\sigma_{k-1}); [\sigma_{k-1}, \sigma_k]) + |Z^{n+1}(\sigma_{k-1})|. \end{aligned} \tag{9}$$

By the Love-Young inequality and Lemma 2 in [5], we have

$$\begin{aligned}
& V_p(Z^{n+1} - Z^{n+1}(\sigma_{k-1}); [\sigma_{k-1}, \sigma_k]) \\
& \leq \int_{\sigma_{k-1}}^{\sigma_k} |f(X_s^n) - f(X_s^{n-1})| ds \\
& \quad + C_{p,p/\alpha} V_{p/\alpha,\infty}(g(X^n) - g(X^{n-1}); [\sigma_{k-1}, \sigma_k]) V_p(B^H; [\sigma_{k-1}, \sigma_k]) \\
& \leq L \sup_{\sigma_{k-1} \leq s \leq \sigma_k} |Z_s^n| \cdot (\sigma_k - \sigma_{k-1}) \\
& \quad + C_{p,p/\alpha} \{2|g'|_\infty + |g'|_\alpha \cdot V_p^\alpha(X^{n-1}; [\sigma_{k-1}, \sigma_k])\} \\
& \quad \times V_{p,\infty}(Z^n; [\sigma_{k-1}, \sigma_k]) \cdot V_p(B^H; [\sigma_{k-1}, \sigma_k]). \tag{10}
\end{aligned}$$

It is obvious that

$$V_{p,\infty}(Z^1; [0, \sigma_1]) \leq 2V_p(Z^1; [0, \sigma_1]) \leq 2|f(\xi)| \cdot \sigma_1 + 2|g(\xi)| \cdot V_p(B^H; [0, \sigma_1])$$

and by (10), it follows that

$$\begin{aligned}
V_{p,\infty}(Z^{n+1}; [0, \sigma_1]) & \leq \frac{1}{2^n} V_{p,\infty}(Z^1; [0, \sigma_1]) \\
& \leq \frac{2}{2^n} [|f(\xi)| \cdot \sigma_1 + |g(\xi)| \cdot V_p(B^H; [0, \sigma_1])].
\end{aligned}$$

Denote  $A := |f(\xi)| \cdot \sigma_k + |g(\xi)| \cdot V_p(B^H; [0, \sigma_k])$ . Similarly, we have

$$\begin{aligned}
& V_{p,\infty}(Z^{n+1} - Z^{n+1}(\sigma_{m-1}); [\sigma_{m-1}, \sigma_m]) \\
& \leq \frac{1}{2} V_{p,\infty}(Z^{n+1} - Z^{n+1}(\sigma_{m-1}); [\sigma_{m-1}, \sigma_m]) + \frac{1}{2} |Z^n(\sigma_{m-1})| \\
& \leq \frac{1}{2^n} V_{p,\infty}(Z^1 - Z^1(\sigma_{m-1}); [\sigma_{m-1}, \sigma_m]) + \sum_{i=1}^n \frac{1}{2^i} |Z^{n-i+1}(\sigma_{m-1})| \\
& \leq \frac{2}{2^n} [|f(\xi)|(\sigma_m - \sigma_{m-1}) + |g(\xi)| V_p(B^H; [\sigma_{m-1}, \sigma_m])] \\
& \quad + \sum_{i=1}^n \frac{1}{2^i} \sum_{j=1}^{m-1} V_{p,\infty}(Z^{n-i+1} - Z^{n-i+1}(\sigma_{j-1}); [\sigma_{j-1}, \sigma_j]) \\
& \leq \frac{2A}{2^n} + \sum_{i=1}^n \frac{1}{2^i} \left\{ \frac{2A}{2^{n-i}} + \frac{2A}{2^{n-i}} [1 + n - i] \right. \\
& \quad \left. + \frac{2A}{2^{n-i}} \sum_{j=2}^{m-1} 2^{j-2} [1 + (n-i) + \dots + (n-i)^{j-1}] \right\} \\
& \leq \frac{2A}{2^n} \left\{ 1 + n + n^2 + \sum_{j=2}^{m-1} 2^{j-2} [n + n^2 + \dots + n^j] \right\} \\
& \leq \frac{2A}{2^n} \cdot 2^{m-2} [1 + n + \dots + n^{m-1}].
\end{aligned}$$

Then by (9) and (2), we have

$$\begin{aligned} V_{p,\infty}(Z^{n+1}; [\sigma_{m-1}, \sigma_m]) &\leq \sum_{j=1}^m V_{p,\infty}(Z^{n+1} - Z^{n+1}(\sigma_{j-1}); [\sigma_{j-1}, \sigma_j]) \\ &\leq \frac{2A}{2^n} + \frac{2A}{2^n} n + \sum_{j=3}^m \frac{2A}{2^n} 2^{j-2} [1 + n + \dots + n^{j-1}] \\ &\leq \frac{2A}{2^n} 2^{m-1} [1 + n + \dots + n^{m-1}]. \end{aligned}$$

The proof of the lemma is complete.  $\square$

*Proof of Theorem. Existence of the solution.* By Lemma 2, it follows that there exists a stochastic process  $Y$  with almost all trajectories in  $C\mathcal{W}_p([0, T])$  such that for any fixed  $k$ ,  $V_{p,\infty}(X^{n,\sigma_k} - Y^{\sigma_k}; [0, T]) \rightarrow 0$  as  $n \rightarrow \infty$  since  $V_{p,\infty}(X^n - Y; [0, \sigma_k]) = V_{p,\infty}(X^{n,\sigma_k} - Y^{\sigma_k}; [0, T])$ , where  $X_t^{n,\sigma_k} = X^n(t \wedge \sigma_k)$ ,  $Y_t^{\sigma_k} = Y(t \wedge \sigma_k)$ .

The process  $Y^{\sigma_k}$  is  $\mathbf{F}^{\sigma_k} = \{\mathcal{F}(t \wedge \sigma_k), 0 \leq t \leq T\}$ -adapted. We still have to show that

$$Y_t = \xi + \int_0^t f(Y_s) ds + \int_0^t g(Y_s) dB_s^H \quad t \in [0, \sigma_k]. \tag{11}$$

By the definition of the stopping times  $\sigma_k$  and Lemma 2 in [5], we have

$$\begin{aligned} &V_{p,\infty}\left(Y - \xi - \int_0^\cdot f(Y_s) ds - \int_0^\cdot g(Y_s) dB_s^H; [0, \sigma_k]\right) \\ &\leq V_{p,\infty}(Y - X^n; [0, \sigma_k]) + V_{p,\infty}\left(\int_0^\cdot [f(Y_s) - f(X_s^{n-1})] ds; [0, \sigma_k]\right) \\ &\quad + V_{p,\infty}\left(\int_0^\cdot [g(Y_s) - g(X_s^{n-1})] dB_s^H; [0, \sigma_k]\right) \\ &\leq V_{p,\infty}(Y - X^n; [0, \sigma_k]) + 2Lk \sup_{0 \leq s \leq \sigma_k} |Y_s - X_s^{n-1}| \\ &\quad + 2kC_{p,p/\alpha} [|g'|_\infty + 2^{k+1}|g'|_\alpha \widehat{R}_k] V_{p,\infty}(Y - X^{n-1}; [0, \sigma_k]). \end{aligned}$$

The equality (11) follows from the above inequality.

*Uniqueness of the solution.* Let  $X$  be another adapted, continuous solution of (11). By the Love-Young inequality and definition of the stopping times  $(\sigma_m)$ , we have  $V_{p,\infty}(X - Y; [0, \sigma_1]) \leq \frac{1}{4} V_{p,\infty}(X - Y; [0, \sigma_1])$ . Thus  $X = Y$  on the interval  $[0, \sigma_1]$ . Similarly, we prove that  $V_{p,\infty}(X - Y; [\sigma_{k-1}, \sigma_k]) = 0$ . Thus  $X = Y$  on  $[0, \sigma_m]$ .

Since the sequence of stopping times  $(\sigma_m)$  goes to infinity as  $m \rightarrow \infty$  then we have existence and uniqueness of the solution of equation (1) on the interval  $[0, T]$ .

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## **Integralinės lygties valdomos trupmeninio Brauno judesio sprendinio egzistavimas ir vienatis**

K. Kubilius

Nagrinėjame trupmeninį Brauno judesį, kurio Hursto indeksas  $1/2 < H < 1$ . Rastos sąlygos, kada nagrinėjama lygtis turi vienintelį sprendinį ir sprendinio trajektorijos yra tolydžių, turinčių baigtinę  $p$ -variaciją,  $p > 1/H$ , funkcijų klasėje.