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On a modified Dirichlet type problem for the elliptic equation degenerating on a line

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1. Statement of the problem

We consider the equation

$$Lu := u_{zz} + L_x u = 0 \tag{1}$$

in the cylinder $Q_R = \{(x,z) : x \in B_R,\ 0 < z < H\}$, where $B_R = \{x : |x| < R\} \subset \mathbb{R}^n$. The operator L_x in (1) is defined by the formula

$$L_x u = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + |x|^{-1} \sum_{i=1}^{n} x_i b_i(x) u_{x_i} + c(x)u.$$

We assume that the following conditions are fulfilled:

(i) $a_{ij} \in C^{1+\alpha}(\bar{B}_R)$, both b_i and $c \in C^{0+\alpha}(\bar{B}_R)$ with $0 < \alpha \leqslant 1$ for $i,j = \overline{1,n}$;

(ii) there exists a number $\gamma > 0$, and continuous functions μ_1 and μ_2 such that $0 < \mu_1(|x|) \leqslant \mu_2(|x|)$ for each $|x| \in (0,R]$, $\mu_2(|x|) = O(|x|^{1+\gamma})$ as $|x| \to 0$, and

$$\mu_1(|x|)|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \mu_2(|x|)|\xi|^2 \tag{2}$$

for each $x \in \bar{B}_R$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$;

(iii) $b_i(0) = b_0 < 0$ for $i = \overline{1,n}$;

(iv) there exists a constant $\nu < 0$ such that $c(x) \leqslant \nu$ for each $x \in B_R$.

Observe that according to (ii) equation (1) is elliptic in $Q_R \backslash \{x = 0\}$ and degenerates on the axis $x = 0$ of cylinder Q_R. The Dirichlet problem for a special case of equation (1) with the assumption $b_i(x) \equiv 0$, $i = \overline{1,n}$, is discused in [1]. There is shown that Dirichlet problem is well-posed in usual formulation (without any additional condition at the line of degeneracy). Existence of nontrivial lower terms in equation (1) which is consider here has an essential influense for well-posedness of Dirichlet type problem. It appears that classical Dirichlet problem in the class of functions bounded in Q_R is non-Fredholm one. Therefore, we are in need of an additional condition in the statement of Dirichlet type problem. In this work we shall formulate the problem with a supplementary condition for the asymptotics of solution as $|x| \to 0$.

Let us introduse the following notation of the domains and manifolds: $Q_R^\delta = Q_R \backslash \bar{Q}_\delta$, $B_R^\delta = B_R \backslash \bar{B}_\delta$ $(\delta < R)$, $Q_R^0 = Q_R \backslash \{x = 0\}$, $B_R^0 = B_R \backslash \{x = 0\}$, $S_R = \{x : |x| = R\}$, $\Omega_R = \{(x, z) : x \in S_R, 0 \leqslant z \leqslant H\}$, $B_{0R} = \{(x, 0) : x \in \bar{B}_R\}$, $R_{HR} = \{(x, H) : x \in \bar{B}_R\}$, $\Gamma_R = \Omega_R \cup B_{0R} \cup B_{HR}$, $\Gamma_R^0 = \Gamma_R \backslash \{x = 0\}$.

We shall consider the following problem:

$$Lu = 0 \quad \text{in} \quad Q_R^0, \quad u \in C^{2+\alpha}(Q_R^0) \cap C^0(Q_R^0 \cup \Gamma_R^0), \tag{3}$$

$$u(x, z) = 0 \quad \text{on} \quad B_{0R} \cup B_{HR}, \tag{4}$$

$$u(x, z) = f(x, z) \quad \text{on} \quad \Omega_R, \tag{5}$$

$$\lim_{|x| \to 0} \big(u(x, z) - g(x/|x|, z)\big) = 0, \quad 0 < z < H, \tag{6}$$

where both $f(x, z) \in C_{x,y}^{2+\alpha,2}(\Omega_R)$ and $g(x, z) \in C_{x,y}^{2+\alpha,2}(\Omega_1)$ are given functions such that $f(x, z) = 0$ on $(\Omega_R \cap B_{0R}) \cup (\Omega_R \cap B_{HR})$ and $g(x, z) = 0$ on $(\Omega_1 \cap B_{01}) \cup (\Omega_1 \cap B_{H1})$.

As the main result in this article we shall prove the following theorem.

Theorem 1. *Let assumptions* (i)–(iv) *hold. Then there exists the unique solution of problem* (3)–(6).

2. Auxiliaries

Lemma 1. *Let* $\beta \in (0, \lambda)$ *with* $\lambda = \min\{\alpha, \gamma\}$. *Then*

$$L|x|^\beta = b_0 \beta |x|^{1-\beta}\big(1 + O(|x|^{\lambda-\beta})\big) \quad as\ |x| \to 0. \tag{7}$$

Proof. Notice that condition (2) yields the relations:

$$a_{ij}(x) = O(|x|^{1+\gamma}), \quad \sum_{i,j=1}^{n} a_{ij}(x) x_i x_j = O(|x|^{3+\gamma}) \tag{8}$$

a $|x| \to 0$. In view of (iii) and $b_i \in C^{0+\alpha}(\bar{B}_R)$ we obtain that

$$b_i(x) - b_0 = O(|x|^\alpha), \quad \text{as } |x| \to 0, \ i = \overline{1, n}. \tag{9}$$

By the direct calculation we get the equality

$$\begin{aligned} L|x|^\beta = \beta |x|^{1-\beta} \Big(& b_0 + |x|^{-1} \sum_{i=1}^{n} a_{ii}(x) + (\beta - 2)|x|^{-3} \sum_{i,j=1}^{n} a_{ij}(x) x_i x_j \\ & + |x|^{-2} \sum_{i=1}^{n} x_i^2 (b_i(x) - b_0) + |x| c(x)/\beta \Big). \end{aligned} \tag{10}$$

Now, it is easily seen that (7) follows from (10) in view of (8) and (9). Thus, lemma is proved.

Lemma 2. *Let λ be from Lemma* 1 *and let function $h(x,z) \in C^2(\Omega_1)$. Let us define the function $\widehat{h}(x,z) = h(x/|x|, z)$. Then*

$$L\widehat{h}(x,z) = O(|x|^{\lambda-1}) \quad as \ |x| \to 0. \tag{11}$$

Proof. Introduce the vector $y = (y_1, \dots, y_n)$ with $y_k = x/|x_k|$, $k = \overline{1,n}$. Taiking in account the equality

$$\sum_{i=1}^{n} x_i h_{x_i}(y,z) = 0, \quad (y,z) \in \Omega_1,$$

which holds due to zero-homogenity of function $\widehat{h}(x,z)$ (with respect to x), we obtain that

$$\begin{aligned} L\widehat{h}(x,z) = {} & h_{zz}(y,z) + \sum_{l,k=1}^{n} h_{y_l y_k}(y,z) \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial y_l}{\partial x_i} \frac{\partial y_k}{\partial x_j} \\ & + \sum_{l=1}^{n} h_{y_l}(y,z) \sum_{i=1}^{n} a_{ij}(x) \frac{\partial^2 y_l}{\partial x_i \partial x_j} \\ & + \sum_{l=1}^{n} h_{y_l}(y,z) \sum_{i=1}^{n} y_i (b_i(x) - b_0) \frac{\partial y_l}{\partial x_i} + c(x) h(y,z). \end{aligned}$$

This implies, in accordance with (9), as well as the eimbidding $h(y,z) \in C^2(\Omega_1)$ and evident relations $\partial y_l / \partial x_i = O(|x|^{-1})$, $\partial^2 y_l / \partial x_i \partial x_j = O(|x|^{-2})$, the relation (11). Lemma is proved.

Let us define the class of functions

$$C_1^{1+\alpha}(B_R) = \left\{ v : v \in C^{2+\alpha}(\bar{B}_R^{\delta}) \ \forall \delta > 0, \ |v| < \infty \ \text{in} \ \bar{B}_R \right\}$$

Theorem 2 [2]. *Let conditions* (i)–(iv) *be fulfilled. Then there exists the unique solution of the following problem:*

$$L_x v = 0 \ \ in \ B_R^0, \ \ v \in C_1^{2+\alpha}(B_R), \quad v = \varphi \ \ on \ S_R, \quad \lim_{|x| \to 0} \left(v(x) - \psi(x/|x|) \right) = 0,$$

where $\varphi \in C^{2+\alpha}(S_R)$ and $\psi \in C^{2+\alpha}(S_1)$ both are given functions. Moreover, the estimate $\sup_{B_R^0} |v| \leqslant \max\left\{ \max_{S_R} |\varphi|, \max_{S_1} |\psi| \right\}$ *holds.*

3. Proof of Theorem 1

Let us build the Fourier series expansions for functions f and g:

$$f(x,z)=\sum_{n=1}^{\infty} f_n(x)\sin\gamma_n z, \quad (x,z)\in\Omega_R, \tag{12}$$

$$g(x,z)=\sum_{n=1}^{\infty} g_n(x)\sin\gamma_n z, \quad (x,z)\in\Omega_1, \tag{13}$$

where $\gamma_n=\pi n/H$,

$$f_n(x)=\frac{2}{H}\int_0^H f(x,z)\sin\gamma_n z\,\mathrm{d}z, \quad g_n(x)=\frac{2}{H}\int_0^H g(x,z)\sin\gamma_n z\,\mathrm{d}z. \tag{14}$$

Due to smoothness of functions f and g the estimates

$$|f_n(x)|\leqslant M_1/n^2, \quad |g_n(x)|\leqslant M_2/n^2 \tag{15}$$

hold with $M_1=\max_{\Omega_R}|f_{zz}|$, $M_2=\max_{\Omega_1}|g_{zz}|$. Therefore, series (12) and (13) converge uniformly and absolutely. Besides that, it follows from (14) that $f_n\in C^{2+\alpha}(S_R)$ and $g_n\in C^{2+\alpha}(S_1)$. Hence, according to theorem 2, there exists a unique solution $v_n\in C_1^{2+\alpha}(B_R)$ of the problem: $(L_x-\gamma_n^2)v=0$ in B_R^0, $v=f_n$ on S_R, $\lim_{|x|\to 0}(v(x)-g_n(x/|x|))=0$. Moreover, v_n is such that

$$\sup_{B_R^0}|v_n|\leqslant\max\{\max_{S_R}|f_n|,\ \max_{S_1}|g_n|\}. \tag{16}$$

It is easy to verify that the functions $v_n(x)\sin\gamma_n z$, $n=1,2,\ldots$, represent the set of the partial solutions of equation (1).

We shall show that the solution u of problem (3)–(6) can be expressed in the form of the series

$$u(x,z)=\sum_{n=1}^{\infty} v_n(x)\sin\gamma_n z. \tag{17}$$

In view of (15) and (16) it is evident that series (17) converges uniformly and absolutaly in $\bar{Q}_R^\delta$ with arbitrary δ. Thus, the function u defined by (17) satisfies both boundary value conditions (4) and (5).

Let $Q'\subset Q_R^\delta$ be an arbitrary subdomain with the boundary $\partial Q'$ from the Liapunov's class. According to smoothness of bouth boundary $\partial Q'$ and coefficients of equation (1), and due to condition (iv) and strong elipticity in domain Q' there exists a unique Green's function of Dirichlet problem for equation (1)in domain Q' [3]. Hence, in view of Harnack's theorem [4], series (17) is twice differentiable in Q'. Since subdomain Q' and number δ both are arbitrarily choosen, function (17) satisfies the equation (1) evrywhere in Q_R^0. Moreover, condition (i) and uniform convergence of series (17) in $\bar{Q}_R^\delta$ yield the embidding $u \in C^{2+\alpha}(Q_R^0) \cap C(Q_R^0 \cup \Gamma_R^0)$ (see, e.g., [5], [6]).

To prove (6), introduce the functions

$$\omega_\varepsilon^\pm(x,) = g(x/|x|, z) \pm (k|x|^\beta + \varepsilon),$$

where β is the same constant as in Lemma 1, ε is positive arbitrary number, and k is positive constant which will be choosen below.

Let u_n and $g^{(n)}$ be the partial sums of series (12) and (13), respectively. Observe that due to uniform convergence of series (13) on Ω_1 there exists an integer n_0 such that

$$|g(x/|x|, z) - g^{(n)}(x/|x|, z) < \varepsilon$$

for $n > n_0$ and each $(x, z) \in Q_R^0$. This yield the inequalities

$$\begin{aligned} u_n(x,z) - \omega_\varepsilon^+(x,z) &< u_n(x,z) - g^{(n)}(x/|x|, z) - k|x|^\beta, \\ u_n(x,z) - \omega_\varepsilon^-(x,z) &> u_n(x,z) - g^{(n)}(x/|x|, z) + k|x|^\beta, \end{aligned} \tag{18}$$

which hold evrywhere in Q_R^0 for $n > n_0$. Since $\lim_{|x|\to 0}(u_n(x,z) - g^{(n)}(x/|x|, z)) = 0$, $n = 1, 2, \dots$, it follows from (18) that

$$\overline{\lim_{|x|\to 0}}\,(u_n(x,z) - \omega_\varepsilon^+(x,z)) \leqslant 0, \quad \underline{\lim_{|x|\to 0}}\,(u_n(x,z) - \omega_\varepsilon^-(x,z)) \geqslant 0 \tag{19}$$

for $n > n_0$ and each $z \in (0, H)$.

According to Lemmas 1 and 2 we get the relations

$$L(u_n - \omega_\varepsilon^\pm) = \mp b_0 k\beta |x|^{\beta-1}(1 + O(|x|^{\lambda-\beta})), \quad |x| \to 0,$$

which obviously are uniform with respect to n and ε. Since $b_0 < 0$, $k > 0$, and $0 < \beta < \min\{1, \lambda\}$, these relations imply the existence of a domain Q_ρ^0 such that

$$L(u_n - \omega_\varepsilon^+) > 0, \quad L(u_n - \omega_\varepsilon^-) < 0 \tag{20}$$

evrywhere in Q_ρ^0, if ρ is small enough.

Let $K_1 = \sup\{\sup_{Q_R^0} |u_n|\}$, $K_2 = \max_{Q_1} |g|$. We choose k such that $k > (K_1 + K_2)/\rho^\beta$, then if it is easily seen

$$u_n - \omega_\varepsilon^+ < 0, \quad u_n - \omega_\varepsilon^- > 0 \text{ on } \Gamma_\rho^0. \tag{21}$$

In accordance with Hopf's maximum principle [7] it follows from (19)–(21) that inequalities (21) hold in Q_ρ^0, too, i.e., we have that

$$|u_n(x,z) - g(x/|x|, z)| < k|x|^\beta + \varepsilon \tag{22}$$

for $n > n_0$ and for each $(x, z) \in Q^0_\rho$. Letting $n \to \infty$ in (22) and letting afterwards $|x| \to 0$ in one we get due to arbitrarility of ε the validity of (6).

Thus, the existence of the solution of problem (3)–(6) is proved.

The uniqueness of the solution of this problem follows from the maximum principle which due to (iv) and ellipticity of equation (1) in Q^0_R holds. This completes the proof of Theorem 1.

REMARK 1. If function $g(x, z)$ is independent of x, the solution u of problem (3)–(6) is continuously extendable by the values of this function onto the axis $x = 0$ of cylinder Q_R.

REMARK 2. In fact the assumptions that $f(x, z) \in C^{2+\alpha,2}_{x,y}(\Omega_R)$ and $g(x, z) \in C^{2+\alpha,2}_{x,z}(\Omega_1)$ are too strong for the existence of the solution of problem (3)–(6). Really Theorem 1 holds under the embiddings $f \in C^2(\Omega_R)$, $g \in C^2(\Omega_1)$.

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Apie išsigimstančios tiesėje elipsinės lygties modifikuotąjį Dirichlė tipo uždavinį

S. Rutkauskas

Cilindre nagrinėjama elipsinė lygtis, išsigimstanti cilindro ašyje. Suformuluotas Dirichlė tipo kraštinis uždavinys su papildoma asimptotine sąlyga ieškomajam sprendiniui. Įrodyta šio uždavinio sprendinio egzistencija ir vienatis.