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Ruin probability and Gerber-Shiu function for the discrete time risk model with inhomogeneous claims

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Bankroto tikimybė ir Gerber-Shiu funkcija diskretaus laiko rizikos modeliui su skirtingai pasiskirsčiusiomis žalomis

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Introduction

Problems associated with the calculation of ruin probabilities and Gerber-Shiu function (also called expected discounted penalty function) have received considerable attention in recent years. Huge projects all over Europe have taken place in determining the necessary so called solvency capital which would reduce insurer's ruin probability to 0.5 percent per year. This problem is not only of the theoretical nature. Some practical difficulties arise as well, because a complex set of factors (claim frequency, severity and dependency; premium income; investment income; interactions between factors, lots of other risks) are investigated and a big amount of computations must be performed in order to test methods' applicability, suitability and accuracy in a real life with real data.

As a result of all those extensive calculations, the solvency capital requirement would serve for: reducing the risk that an insurer would be unable to meet claims; reducing the policyholder losses in case the insurer is unable to fulfill its liabilities; providing supervisors early warning so that they can react immediately if capital falls below the required level; and promoting confidence in the insurance market financial stability.

The insurance mathematics theory was begun to develop in the beginning of the 20th century by Lundberg and Cramér who introduced the classical Poisson model describing the basic dynamics of a homogeneous insurance portfolio. This model was later generalized by Sparre Andersen by introducing a more general distribution function for a number process. In this thesis, the generalization of the discrete time risk model introduced by Gerber in 1988 is considered.

The discrete time risk model in Risk theory is designated for the investigations of the insurance company's balance behavior. Its main components are initial balance (or so called initial capital), premiums received and claims paid. One of the indicators about the insurance company's current risk situation is time of ruin, which represents the first time when insurance company becomes insolvent, i.e. its balance becomes negative or null and the company is no longer able to meet its obligations. The probability of ruining until the time moment is called ruin probability. This measure is one of the most widely used in defining the risks. Another risk measure Gerber-Shiu function, or expected discounted penalty function, proposed by Gerber and Shiu in 1998, is a generalization of ruin probabilities. By changing the Gerber-Shiu function parameters, the properties of the claim portfolio are analyzed and some insights about the situation of the portfolio are derived.

Aims and problems

The aim of this thesis is to find the expression of the main risk measures in the discrete time risk model with inhomogeneous claims. The following problems of the discrete time risk model are considered:

- Finding the exact values of the ruin probabilities and the Gerber-Shiu functions.

- Releasing the homogeneous claim assumption and allowing the claim inhomogeneity. However, the claims are still considered to be independent.

- Analyzing the finite and the infinite time horizons. As the claims are distributed nonidentically, the investigation of infinite time risk measures is a challenge.

- Splitting the infinite time case investigations according the initial capital value, which can be equal to zero or be higher than zero.

- Investigating the special case of the discrete time risk model with claims distributed according the geometric law.

- Introducing the special case of the discrete time risk model with initial capital, premiums and claims acquiring the rational values.

Novelty

This thesis is dedicated for finding the exact expressions of ruin probability and Gerber-Shiu function in the discrete time risk model with inhomogeneous claims. The formulas obtained are recursive and enable fast calculation of the finite time ruin probabilities. The investigations of the finite time ruin probability when claims are distributed nonidentically may be found in the literature (for instance, [15], [42]). However, neither the infinite ruin probability nor the Gerber-Shiu function for inhomogeneous claim case are considered in the literature at all, so the results of this work are pioneer in this area. In addition, the finite time Gerber-Shiu function is introduced only one year ago and only in classical models, therefore the extension by inhomogeneous claims is also a new trend in Risk theory. Finally, the model extension with rational claims is also investigated and the finite time ruin probabilities are given.

Main concepts

First of all we will shortly introduce all necessary concepts for the discrete time risk model with inhomogeneous claims. The wider explanations and descriptions are given in Chapter 3.

Definition 1 (Insurer's balance)**.** *The insurer's balance is defined by*

$$U_u^{(j)}(n) = u + n - \sum_{i=1}^{n} Z_{i+j},$$

and the following conditions are satisfied:

- $u = U_u^{(j)}(0) \in \{0\} \cup \mathbb{N} =: \mathbb{N}_0$*,* $j \in \mathbb{N}_0$*,* $n \in \mathbb{N}$*;*
- *claim amounts* $Z_1, Z_2, Z_3, \ldots$ *are independent nonnegative integer valued r.v.s. with corresponding local probabilities and distribution functions (*$j, k \in \mathbb{N}_0$*):*

$$h_k^{(j)} = \mathbb{P}\left(Z_{1+j} = k\right), \quad H^{(j)}(x) = \mathbb{P}(Z_{1+j} \leqslant x) = \sum_{i=0}^{\lfloor x \rfloor} h_i^{(j)}, x \in \mathbb{R}.$$

Definition 2 (Time of ruin)**.** *The time of ruin is defined by*

$$T_u^{(j)} = \begin{cases} \min\left\{n \geqslant 1 : U_u^{(j)}(n) \leqslant 0\right\}, \\ \infty, \ \textit{if} \ \ U_u^{(j)}(n) > 0 \ \ \textit{for all} \ \ n \in \mathbb{N}. \end{cases}$$

Definition 3 (Finite and infinite time ruin probability)**.** *The finite time ruin probability at the moment $t \in \mathbb{N}$ with the initial capital $u \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$ is defined by*

$$\psi^{(j)}(u,t) = \mathbb{P}\left(T_u^{(j)} \leqslant t\right).$$

The infinite time ruin probability, or the ultimate ruin probability, is

$$\psi^{(j)}(u) = \mathbb{P}\left(T_u^{(j)} < \infty\right) = \lim_{t\to\infty} \psi^{(j)}(u,t).$$

Definition 4 (Finite and infinite time Gerber-Shiu function)**.** *The finite and infinite time Gerber-Shiu functions for the model with inhomogeneous claims are*

$$\phi_\delta^{(j)}(u,t) = \mathbb{E}[e^{-\delta T_u^{(j)}} \mathbb{1}_{\{T_u^{(j)} \leqslant t\}}],$$
$$\phi_\delta^{(j)}(u) = \mathbb{E}[e^{-\delta T_u^{(j)}} \mathbb{1}_{\{T_u^{(j)} < \infty\}}],$$

where $\delta \geqslant 0$, $t \in \mathbb{N}$, $u \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$.

Main results

Now we will state all theorems which are proven in Chapters 4, 5 and 6. All theorems consider discrete time risk model with inhomogeneous claims described above except Theorems 3 and 4, where model enhancements are described separately.

The first five theorems consider the ruin probabilities in the finite and infinite time, whereas the last three theorems give the evaluation of Gerber-Shiu function. All theorems except the last one give the exact recursive expression, which allow evaluating the ruin probabilities and Gerber-Shiu function with the desired accuracy. The implementation codes in Maple are given in Appendix.

Theorem 1. *Consider the discrete time risk model with nonnegative non-identically distributed claims. Then the finite time ruin probabilities*

$$\psi^{(j)}(u,t) := \mathbb{P}\left(u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0 \ \ \textit{for some} \quad n \in \{1, 2, \ldots, t\}\right)$$

for all $j, u \in \mathbb{N}_0$ *satisfy the following equations*

$$\begin{aligned}
\psi^{(j)}(u,1) &= 1 - H^{(j)}(u),\\
\psi^{(j)}(u,t) &= \psi^{(j)}(u,1) + \sum_{k=0}^{u} \psi^{(j+1)}(u+1-k,t-1)\ h_k^{(j)},\ t = 2,3,\ldots\ .
\end{aligned}$$

Theorem 2. *Let* $u \in \mathbb{N}_0$*; r.v.s* $Z_1, Z_2, \ldots$ *be independent and integer-valued;* $h_k^{(j)} = \mathbb{P}(Z_{1+j} = k)$ *for* $k \in \mathbb{Z}$*,* $j = 0,1,\ldots$*. Then the probabilities*

$$\tilde{\psi}^{(j)}(u,t) := \mathbb{P}\left(u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0,\ \text{for some}\quad n \in \{1,2,\ldots,t\}\right)$$

satisfy the following equations

$$\begin{aligned}
\tilde{\psi}^{(j)}(u,1) &= \sum_{k>u} h_k^{(j)},\\
\tilde{\psi}^{(j)}(u,t) &= \tilde{\psi}^{(j)}(u,1) + \sum_{k=-\infty}^{u} \tilde{\psi}^{(j+1)}(u+1-k,t-1)\ h_k^{(j)},
\end{aligned}$$

for all $j \in \mathbb{N}_0$ *and* $u \in \mathbb{N}_0$*.*

For the next two theorems, the classical discrete time risk model will be enhanced by taking varying premium income and nonhomogeneous claims which can take rational values. Moreover, premium income and initial insurers' capital may also take rational values. We describe this model below.

Suppose that the insurer capital at each moment $n \in \mathbb{N}$ is

$$U(n) = u + \sum_{i=1}^{n} c_i - \sum_{i=1}^{n} Z_i$$

and the following conditions are satisfied:

- the quantities $u, c_1, c_2, \ldots$ are nonnegative and belong to the set of rational numbers $\mathbb{Q}$;
- the claim amounts $Z_1, Z_2, Z_3, \ldots$ are independent nonnegative rational-valued r.v.s;
- there exists a natural number α for which: $\alpha u \in \mathbb{Z}$, $\alpha c_i \in \mathbb{Z}$ $(i \in \mathbb{N})$ and $\alpha D_k^{(j)} \in \mathbb{Z}$ $(j \in \mathbb{N}_0,\ k \in \mathbb{N})$, where $D_k^{(j)}$ are rational values of r.v. Z_{1+j} acquired with probabilities $h_k^{(j)}$.

Theorem 3. *Consider the discrete time risk model with nonidentically rational-valued claims as described above. The finite time ruin probability*

$$\psi_{\mathbb{Q}}(u,t) := \mathbb{P}\left(u + \sum_{i=1}^{n} c_i - \sum_{i=1}^{n} Z_i \leqslant 0,\ \text{for some}\ \ n \in \{1,\ldots,t\}\right)$$

for all $u, t \in \mathbb{N}_0$ *coincides with probability* $\tilde{\psi}^{(0)}(\alpha u, t)$, *defined in Theorem 2, where integer-valued r.v.s* $\widehat{Z}_{1+j}$, $j \in \mathbb{N}_0$ *are distributed according to the local probabilities*

$$\widehat{h}^{(j)}_{\alpha D_k^{(j)} - \alpha c_j - 1} = \mathbb{P}\left(\widehat{Z}_{1+j} = \alpha D_k^{(j)} - \alpha c_j - 1\right) = h_k^{(j)}, \qquad k \in \mathbb{N}.$$

Theorem 4. *Consider the discrete time risk model with nonidentically rational-valued claims as in Theorem 3. Then the finite-time ruin probability* $\psi_{\mathbb{Q}}(u, t)$ *defined above for all* $u, t = 0, 1, \ldots$ *is equal to the finite-time ruin probability* $\psi^{(0)}\left(\alpha u, \sum_{i=1}^{t} \alpha c_i\right)$ *defined in Theorem 4.1 with integer-valued claim sequence* $\widehat{Z}_i$, $i = 1, 2, \ldots$ *distributed according to the following local probabilities:*

$$\begin{cases} \mathbb{P}(\widehat{Z}_l = \alpha D_k^{(i-1)}) = h_k^{(i-1)}, \ k \in \mathbb{N}, \quad \textit{if} \ \ l = \sum\limits_{m=1}^{i} \alpha c_m, \\ \mathbb{P}(\widehat{Z}_l = k) = \mathbb{1}_{\{k=0\}}, \quad \textit{otherwise.} \end{cases}$$

Now we return to the main discrete time risk model with inhomogeneous claims and state the theorems for finding the infinite time ruin probability and the values of the finite and infinite time Gerber-Shiu function.

Theorem 5. *Consider the discrete time risk model with nonnegative independent nonidentically distributed claims. Then the infinite time ruin probabilities*

$$\psi^{(j)}(u) = \mathbb{P}\left(u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0, \quad \textit{for some } n \in \mathbb{N}\right)$$

for all $j \in \mathbb{N}_0$ *and* $u \in \mathbb{N}_0$ *satisfy the following equation:*

$$\psi^{(j)}(u) = \psi^{(j)}(0) + \sum_{r=1}^{u} (\psi^{(j)}(r) - \psi^{(j+1)}(r) H^{(j)}(u - r)) - \sum_{r=0}^{u-1} (1 - H^{(j)}(r)).$$

Theorem 6. *Consider the discrete time risk model with nonnegative independent nonidentically distributed claims. Then the finite time Gerber-Shiu function for all* $j \in \mathbb{N}_0$, $u \in \mathbb{N}_0$ *and* $\delta \geqslant 0$ *satisfies the following equations:*

$$\begin{aligned} \phi_\delta^{(j)}(u, 1) &= e^{-\delta}\left(1 - H^{(j)}(u)\right), \\ \phi_\delta^{(j)}(u, t) &= \phi_\delta^{(j)}(u, 1) + e^{-\delta} \sum_{k=0}^{u} \phi_\delta^{(j+1)}(u + 1 - k, t - 1)\ h_k^{(j)}, \ t = 2, 3, \ldots. \end{aligned}$$

The following two theorems combined together allow evaluating the infinite time Gerber-Shiu function values in the inhomogeneous claim case. However, the exact evaluation would be complicated without any assumptions about underlying claim distributions.

Theorem 7. *Consider the discrete time risk model with nonnegative independent nonidentically distributed claims. Then the infinite time Gerber-Shiu function* $\phi_\delta^{(j)}(u)$ *for all* $j \in \mathbb{N}_0$, $u \in \mathbb{N}_0$ *and* $\delta \geqslant 0$ *satisfies the following equation:*

$$\begin{aligned}\phi_\delta^{(j+1)}(u)H^{(j)}(0) &= e^\delta \phi_\delta^{(j)}(0) + \sum_{r=1}^{u-1}\left(e^\delta \phi_\delta^{(j)}(r) - \phi_\delta^{(j+1)}(r)H^{(j)}(u-r)\right)\\ &\quad -\sum_{r=0}^{u-1}\left(1-H^{(j)}(r)\right).\end{aligned}$$

Theorem 8. *Consider the discrete time risk model with nonnegative independent nonidentically distributed claims as described above. Then*

$$\begin{aligned}\phi_\delta^{(j)}(0) &= e^{-\delta}\left(1-H^{(j)}(0)\right) + e^{-2\delta}\left(h_0^{(j)}(1-H^{(j+1)}(1))\right)\\ &\quad +\sum_{m=1}^{\infty} e^{-\delta(m+2)} \sum_{\substack{i_1,i_2,\ldots,i_m\in\{0,1,\ldots\}\\ i_1\leqslant 1, i_1+i_2\leqslant 2,\ldots,\\ i_1+\ldots+i_m\leqslant m}} h_0^{(j)} h_{i_1}^{(j+1)} h_{i_2}^{(j+2)} \cdots h_{i_m}^{(j+m)}\\ &\quad \cdot\left(1-H^{(j+m+1)}\left(m+1-\sum_{n=1}^{m} i_n\right)\right),\end{aligned}$$

for all $j \in \mathbb{N}_0$ *and* $\delta \geqslant 0$.

Publications, presentations and conferences

The results are publicated in the following articles:

1. Blaževičius, K., Bieliauskienė, E. and Šiaulys, J. Finite time ruin probability in the inhomogeneous claim case. *Lithuanian Mathematical Journal 50* (2010), 260–270.

2. Bieliauskienė, E. and Šiaulys, J. Infinite time ruin probability in inhomogeneous claims case. *Lietuvos matematikos rinkinys. LMD darbai. 51* (2010), 352–356.

3. Bieliauskienė, E. Bankroto tikimybė skirtingai pasiskirsčiusioms žaloms: geometrinio dėsnio atvejis. *Lietuvos matematikos rinkinys. LMD darbai. 52* (2011), 274–279.

4. Bieliauskienė, E. and Šiaulys, J. Gerber-Shiu function in the discrete inhomogeneous claim case. *International Journal of Computer Mathematics* (accepted).

The results were presented in the following conferences:

1. Bieliauskienė, E. *Infinite time ruin probability in inhomogeneous claim case.* LI Conference of the Lithuanian Mathematical Society, Šiauliai University, 17-18 June 2010, Šiauliai, Lithuania.

2. Bieliauskienė, E. *Computations of Ruin Probabilities in Inhomogeneous Claim Case.* First Conference of Young Scientists, Lithuanian Academy of Science, 8 February 2011, Vilnius, Lithuania.

3. Bieliauskienė, E. *Ruin Probability Analysis in Geometric Inhomogeneous Claim Case.* LII Conference of the Lithuanian Mathematical Society, The General Jonas Žemaitis Military Academy of Lithuania, 16-17 June 2010, Vilnius, Lithuania.

The international summer schools on related topics were attended:

1. Salzburg Institute of Actuarial Studies: Summer School *Risk Management in Insurance*, 26-29 March 2008, Salzburg, Austria.

2. Group Consultatif Actuarial Europeen: Summer School *Solvency II and Enterprise Risk Management*, 21-23 July 2008, Lyon, France.

3. Group Consultatif Actuarial Europeen: Summer School *Testing and Disclosing Own Risk Models for Solvency Assessments (ORSA)*, 25-27 May 2011, Lisbon, Portugal.

Structure of the thesis

The thesis is organized in two parts. In the first part, the literature and the related theory is reviewed, while in the second part, the new ideas, definitions and theorems with proof are presented:

- **Chapters 1, 2** contains the review of the classical models (Cramer-Lundberg, Sparre Andersen, Compound binomial) and their extensions, relating to the area of thesis investigations: inhomogeneous claims, non uniform premium payments, etc.

- In **Chapter 3**, the working environment – discrete time risk model with inhomogeneous claims is described, all necessary definitions of risk measures are introduced.

- In **Chapters 4, 5**, the finite and infinite time ruin probabilities are investigated and their recursive formula with proof is given. The numerical examples are investigated.

- **Chapter 6** is designated for Gerber-Shiu function analysis. First of all the theorems with proof are given. Then the algorithms for practical evaluation are provided. Finally, the numerical examples with grapichal figures are given.

- Finally, **Chapter 7** gives the example of the discrete time risk model when claims are distributed according Geometric law.

1

Outlines of Risk Theory and Classical Risk Models

The development of the modern insurance mathematics theory started in the beginning of the 20th century, when Swedish mathematicians Filip Lundberg [47] and Harald Cramér [14] described the model which currently is known as Cramér-Lundberg or classical Poisson model and which describes the basic dynamics of a homogeneous insurance portfolio.

Later, another model was proposed by Sparre Andersen in 1957 [61] as a generalization of the classical (Poisson) risk theory. Instead of assuming only exponentially distributed independent interclaim times, he introduced a more general distribution function of number process (so called *renewal process*) but retained the assumption of independence.

Finally, the Compound binomial model, first proposed by Gerber [28] in 1988, is a discrete analog of the compound Poisson model in risk theory. In this thesis, the compound binomial model is extended by releasing the assumption about homogeneity of the claims, which are not necessarily identically distributed. This model is described in Chapter 3.

In this chapter, the main models in the classical risk theory are reviewed. First of all the most general case (Sparre Andersen model) is described and the definitions of the main components are given. Then two cases are analyzed: the compound Poisson model and the compound binomial model. Finally, the critical characteristics of the risk model are discussed and their investigations in literature are reviewed.

1.1 Sparre Andersen model

Nowadays the Sparre Andersen model is one of the most popular and used models in nonlife insurance mathematics, which describes the evolution of the insurance company's wealth over time which is measured by the assets it holds. The assets depend on the initial capital, premium income, and incurring claims. We will introduce the model and define the main concepts used.

Definition 1.1 (Renewal counting process)**.** *Let $W_1, W_2, \ldots$ be an independent identically distributed (i.i.d.) sequence of a.s. nonnegative random variables (r.v.s.). Then the random walk*

$$T_0 = 0, \quad T_n = W_1 + \ldots + W_n, \quad n \in \mathbb{N} = \{1, 2, 3, \ldots\},$$

is said to be a renewal sequence and the counting process

$$N(t) = \#\{i \geqslant 1 : T_i \leqslant t\} \quad t \geqslant 0,$$

is the corresponding renewal counting process.

The sequences $T_0, T_1, T_2, \ldots$ and $W_1, W_2, \ldots$ are also referred as the sequences of the arrival and inter-arrival times of the renewal process N, respectively.

Note that, in contrast to the Cramér-Lundberg model (described in Section 1.1.1), the resulting surplus process in Sparre Andersen model is not a Lévy process any more.

Definition 1.2 (Aggregate claim amount process)**.** *The total claim amount process or aggregate claim amount process is a process defined by:*

$$S(t) = \sum_{i=1}^{N(t)} X_i = \sum_{i=1}^{\infty} X_i I_{[0,t]}(T_i), \quad t \geqslant 0,$$

where $X_1, X_2, \ldots$ is a sequel of nonnegative i.i.d. r.v.s.

Definition 1.3 (Surplus process)**.** *The process U*

$$U_u(t) = u + ct - S(t)$$

is called surplus or balance process. Here $u = U_u(0)$ is the initial surplus, c - premium payment rate and $S(t)$ is the total claim amount process.

So the key assumptions of the Surplus process in the **Sparre Andersen model** are the following.

1. Claims happen at the times T_i satisfying $0 \leqslant T_1 \leqslant T_2 \leqslant \ldots$. We call them claim arrivals or claim times.

2. The ith claim arriving at time T_i causes the claim size or claim severity X_i. The sequence $X_1, X_2, \ldots$ constitutes an i.i.d. sequence of nonnegative random variables.

3. The claim sizes (X_i) and the claim arrival moments $T_1, T_2, \ldots$ are mutually independent. In particular, the counting process N and the sequence $X_1, X_2, \ldots$ are independent.

In figure 1.1, we can see the characteristic behaviour of the surplus process $U_u(t)$.

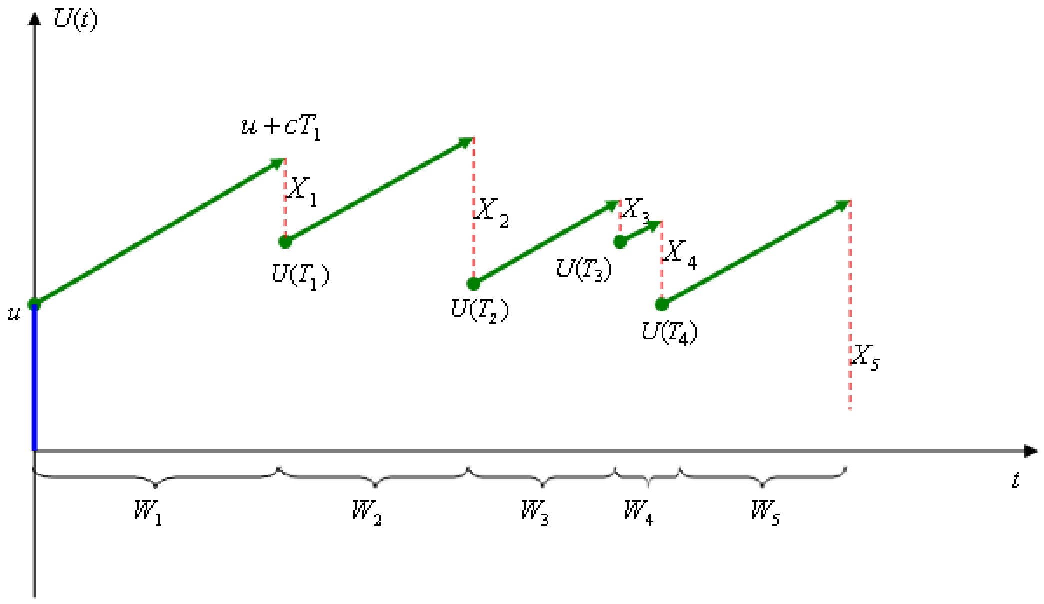


Figure 1.1: The Surplus Process.

The critical characteristics of Sparre Andersen model are ruin probability and Gerber-Shiu function, which are described in Section 1.2. To mention shortly, the explicit result for ultimate ruin probability was derived by Sparre Andersen [61], while the finite time ruin probabilities were considered by Thorin [66], [67], [68], Thorin and Wikstad [69], Brans [9], Dreze [24], Takács [64] and Segerdahl [57].

1.1.1 Cramér-Lundberg model

The Cramér-Lundberg model was one of the first models in the modern insurance mathematics theory, proposed by Lundberg [47] and Cramér [14] in the beginning of the 20th century. It is the special case of the Sparre Andersen model described above with the assumption of claim number process, which is assumed to be a homogeneous Poisson process.

For a definition of the Poisson process, some notation has to be introduced. For any real-valued function f on $[0, \infty)$ we write

$$f(s,t] = f(t) - f(s), \quad 0 \leqslant s < t < \infty.$$

An integer-valued random variable M is said to have a Poisson distribution with parameter $\lambda > 0$ ($M \sim \mathcal{P}(\lambda)$) if it has a distribution

$$P(M = k) = e^{-\lambda} \frac{\lambda_k}{k!}, \quad k = 0, 1, \ldots.$$

We say that the random variable $M = 0$ a.s. has a $\mathcal{P}(0)$ distribution.

It is known and easily proved that a Poisson random variable M has a property that $\lambda = EM = var(M)$, i.e., it is determined only by its mean value (= variance).

Definition 1.4 (Poisson process)**.** *A stochastic process $N = (N(t))_{t \geqslant 0}$ is said to be a Poisson process if the following conditions are held:*

1. *The process starts at zero: $N(0) = 0$ a.s.*

2. *The process has independent increments: for any t_i, $i = 0, \ldots, n$, and $n \geqslant 1$ such that $0 = t_0 < t_1 < \cdots < t_n$, the increments $N(t_{i-1}, t_i]$, $i = 1, \ldots, n$, are mutually independent.*

3. *There exists a nondecreasing right-continuous function $\Lambda : [0, \infty) \to [0, \infty)$ with $\Lambda(0) = 0$ such that the increments $N(s, t]$ for $0 \leqslant s < t <$ have a Poisson distribution $\mathcal{P}(\Lambda(s, t])$. We call Λ the mean value function of N.*

4. *With probability 1, the sample paths $(N(t, \omega))_{t \geqslant 0}$ of the process N are right-continuous for $t \geqslant 0$ and have limits from the left for $t > 0$ (which we will call càdlàg sample paths).*

If the mean function Λ of the Poisson process has the form $\Lambda(t) = \lambda t, \quad t \geqslant 0$, then the process is called *homogeneous Poisson process* and the constant $\lambda > 0$ is called *intensity* of homogeneous Poisson process.

Hence, a homogeneous Poisson process with intensity λ
(1) has a càdlàg (right continuous with left limits) sample paths,
(2) starts at zero,
(3) has independent and stationary increments,
(4) $N(t)$ is $\mathcal{P}(\lambda t)$ distributed for every $t > 0$.

A process on $[0, \infty)$ with properties (1)-(3) is called a Lévy process. The homogeneous Poisson process is one of the prime examples of Lévy processes with applications in various areas such as queuing theory, finance, insurance, stochastic networks, to name a few.

When $N(t)$ is some homogeneous Poisson process with intensity $\lambda > 0$, then the total claim amount process S described by definition (1.2) in the Cramér-Lundberg model is also called a *compound Poisson process.* Cramér-Lundberg model can be also obtained from the Sparre Andersen model by assuming, that the inter-arrival claims $W_1, W_2, \ldots$ described by definition (1.1) are distributed according the Exponential distribution with parameter $\lambda > 0$.

1.1.2 The Compound Binomial Model

The compound binomial model is a fully discrete time model where premiums, claim amounts, and the initial surplus are assumed to be integer valued, but can be used as an approximation to the continuous time compound Poisson model. Even though continuous time risk models are much more popular in the literature than discrete time, the latter risk models have their special features and are closer to reality, because their results may be easier to understand than the analogous results in the continuous time risk model. Moreover, they have computing advantages as well, because formulas are usually of a recursive nature and therefore easier to program, or to compute. Finally, it is well known that explicit expressions for some ruin related quantities do not exist in continuous time risk models with heavily tailed claims. Results for the discrete time risk models can be used as approximations or bounds for the corresponding results in continuous time, see Dickson et al. [22] and Cossette et al. [12] for the approximating procedures. An extensive review of discrete time risk models may be found in Li et al. survey [46].

The infinite time ruin probability in this model was investigated by Gerber [28] and Shiu [59], while finite time ruin probability was considered by Willmot [71].

The model has been also extensively investigated by Dickson [19, 20, 21], Dickson and Waters [23], De Vylder and Goovaerts [16], Gerber [28], Shiu [60], Michel [48]. Several extensions to the compound binomial risk model can be found in Yuen and Guo [74, 75], Cossette et al. [11, 12, 13], Li [43, 44], Landriault [39], Yang et al. [72], and references therein.

At first let us exactly describe the Compound Binomial model. Assumed that in Sparre Andersen model the premium income for each period is constant and equal to one. The number of claims up to time $t \in \mathbb{N}$ is governed by a binomial process $\{N(t); t \in \mathbb{N}\}$ with $N(t) = I_1 + I_2 + \ldots + I_t$, $t \in \mathbb{N}$, $N(0) = 0$, where $I_1, I_2, \ldots$ are independent identically distributed Bernoulli's random variables (r.v.s) with mean $q \in (0, 1)$. So there is one claim or no claims in each time period; the probability of having a claim is q and the probability of no claims is $1 - q$.

In addition, assume that the claim amounts $X_1, X_2, \ldots$ are mutually independent identically distributed positive integer valued r.v.s with common local probabilities $p_k = P(X_1 = k)$, $k \in \mathbb{N}$, and the finite mean $\mu > 0$. The claim amounts are also independent of $\{N(t); t \in \mathbb{N}\}$. The insurance company's balance at time t is described by

$$U_u(t) = u + t - \sum_{i=1}^{N(t)} X_i, \quad t \in \mathbb{N},$$

where $U_u(0) = u \in \mathbb{N} \cup \{0\} =: \mathbb{N}_0$ is the initial surplus. The last equality with the restrictions above is the main equality for the surplus process in the compound binomial model. After some calculations we can prove that the main equation of the compound binomial risk model can be also rewritten in the form

$$U_u(t) = u + t - \sum_{i=1}^{t} Z_i,$$

where $Z_i = I_i X_i$, $i \in \mathbb{N}$ is the claim amount in period i with the local probabilities $h_0 = 1 - q$, $h_k = q p_k$ for $k \in \mathbb{N}$.

In the rest part of the work we consider the model defined by the last equality. Usually the model described in the form of the last equality is called *the discrete time risk model.* The main requirements of this model are the following:

- The initial capital $u \in \mathbb{N}_0$,
- Claim amounts $Z_1, Z_2, \ldots$ are independent identically distributed non-negative integer valued variables,

This model is obtained from the Sparre Andersen model by choosing the inter-arrival times equal to $W_1 = W_2 = \ldots = 1$. If $Z_1, Z_2, \ldots$ are i.i.d. variables, then we have a classical version. If $Z_1, Z_2, \ldots$ are not i.i.d. we obtain some extention of the classical version.

1.2 The critical characteristics

In this section, the main characteristics of the discrete time risk model are presented. The characteristics of the continuous time risk model are defined analogously.

1.2.1 Ruin probability

The actuarial risk management's task is to measure the risk of portfolio and evaluate its performance. One of the risk measures is *ruin probability.*

At each moment n the insurer's capital either may remain positive, or become negative, or vanish to zero. The situation, when the capital falls below or is equal to zero, is called *insolvency* or *ruin*.

Definition 1.5 (The time of ruin)**.** *The first time T_u when insurance company becomes insolvent and is no longer able to meet its obligations is called the time of ruin, i.e.*

$$T_u = \begin{cases} \min\left\{n \geqslant 1 : U_u(n) \leqslant 0\right\}, \\ \infty\,, \;\; \textit{if} \;\; U_u(n) > 0 \;\; \textit{for all} \;\; n \in \mathbb{N}. \end{cases}$$

This definition of ruin is adopted by Gerber [28]. There is another definition, used by Shiu [59], which does not take zero balance as insolvency. Such a case would be investigated analogously to the one described by Gerber.

Definition 1.6 (Finite time ruin probability). *The probability to ruin by the moment $t \in \mathbb{N}$, when initial capital is $u \in \mathbb{N}_0$, is called the finite time ruin probability and is defined by*

$$\psi(u,t) = \mathbb{P}\left(T_u \leqslant t\right).$$

It is evident that

$$\begin{aligned}\psi(u,t) &= \mathbb{P}\left(\bigcup_{n=1}^{t}\left\{u+n-\sum_{i=1}^{n} Z_i \leqslant 0\right\}\right)\\ &= \mathbb{P}\left(u+n-\sum_{i=1}^{n} Z_i \leqslant 0, \quad \text{for some } n \in \{1,2,\ldots,t\}\right)\\ &= \mathbb{P}\left(\max_{1\leqslant n\leqslant t}\sum_{i=1}^{n}(Z_i-1) \geqslant u\right)\end{aligned}$$

Definition 1.7 (Infinite time ruin probability). *The infinite time ruin probability or ultimate ruin probability is defined by*

$$\psi(u) = \mathbb{P}\left(T_u < \infty\right) = \lim_{t\to\infty}\psi(u,t).$$

Now we will review what are the main methods for ruin probability evaluation and some results with exact expressions.

Main methods for finding the ruin probabilities

The numerical evaluation and explicit expression of ruin probability is not an easy task even for homogeneous claim case. There are several approaches, some of them are discussed below.

The first approach uses the idea, that sometimes instead of finding the ruin probabilities $\psi(u)$, $\psi(u,t)$ it is easier to find their **Laplace transforms**:

$$\hat{\psi}[-s] = \int_0^\infty e^{-su}\psi(u)du, \quad \hat{\psi}[-s,-\omega] = \int_0^\infty\int_0^\infty e^{-su-\omega t}\psi(u,t)dudt.$$

When this is done, $\psi(u)$, $\psi(u,t)$ can be calculated numerically by some method of transform inversion, say the fast Fourier transform (FFT). For further details one may refer to Grübel [31], Embrechts, Grübel and Pitts [25], Grübel and Hermesmeier [32], [33], Abate and Whitt [1]. The direct approach is also proposed by Garcia [27].

The **Matrix-analytic** methods are used when the claim size distribution is of phase-type (or matrix-exponential, for example Exponential, Erlang with p phases, Hyperexponential, Coxian distributions). As result, $\psi(u)$ is obtained in terms of a matrix-exponential function e^{Uu} (here U is some suitable matrix) which can be computed by some series expansion or by diagonalization, as solution of the linear differential equations. Below we will present a theorem which gives the $\psi(u)$ expression for phase type distributed claims.

Definition 1.8 (Phase-type distribution)**.** *A phase-type distribution is the distribution of the absorption time in a Markov process with finitely many states, of which one is absorbing and the rest transient. The parameters of a phase-type distribution is the set E of transient states, the restriction $\hat{T}$ of the intensity matrix of the Markov process to E and the row vector $\alpha = (\alpha_i)_{i\in E}$ of initial probabilities. The density and c.d.f. are*

$$b(x) = \alpha e^{\hat{T}x} t, \quad \textit{resp. } B(x) = \alpha e^{\hat{T}x} e,$$

where $t = \hat{T}e$ and $e = (1 \dots 1)'$ is the column vector with 1 at all entries. The couple $(\alpha, \hat{T})$ is called the representation.

Consider Cramér-Lundberg (compound Poisson) model, with λ denoting the Poisson intensity, B the claim size distribution, T_u the time of ruin with initial reserve u, $\{S_t\}$ the claim surplus process, $G_+(\cdot) = P(S_{T_0} \in \cdot, T_0 < \infty)$ the ladder height distribution and $M = sup_{t\geqslant 0} S_t$.

Theorem 1.9 (Neuts [49])**.** *Assume that the claim size distribution B is phase-type with representation $(\alpha, \hat{T})$. Then:*
(a) G_+ is defective phase-type with representation $(\alpha_+, \hat{T})$. where α_+ is given by $\alpha_+ = -\beta\alpha\hat{T}^{-1}$, and M is zero-modified phase-type with representation $(\alpha_+, \hat{T} + t\alpha_+)$.
(b) $\psi(u) = \alpha_+ e^{(\hat{T}+t\alpha_+)u}$.
Note in particular that $\rho = ||G_+|| = \alpha_+ e$.

For further details one may refer to Stanford and Stroinski [62], Gerber [30], Shiu [58], Asmussen [4].

By another approach the $\psi(u)$ and $\psi(u,t)$ are expressed as the solution to a **differential- or integral equation**, and then the solution is carried out by some standard numerical methods. One example where this is feasible is the renewal equation of $\psi(u)$ in the compound Poisson model which is an integral equation of Volterra type.

Theorem 1.10 (Asmussen [3], [5] or Feller [26]). *Consider the compound Poisson model. The ruin probability $\psi(u)$ satisfies the defective renewal equation*

$$\psi(u) = \overline{G}_+(u) + G_+ * \psi(u) = \lambda \int_u^\infty \overline{B}(y)dy + \int_u^\infty \psi(u-y)\lambda\overline{B}(y)dy.$$

Here recall the notation $\overline{G}_+(u) = \int_u^\infty G_+(dx)$, B is claim's c.d.f. and λ is a Poisson process intensity.

Equivalently, the survival probability $Z(u) = 1 - \psi(u)$ satisfies the defective renewal equation

$$Z(u) = 1 - \rho + G_+ * Z(u) = 1 - \rho + \int_0^u Z(u-y)\lambda\overline{B}(y)dy.$$

The **Panjer's recursion**, firstly described by Panjer [50] in 1981 (also investigated by many authors later, for example, Panjer and Willmot [51]), enables to calculate the ruin probabilities recursively without using the classical brute force convolution formula. Actually, the method can be traced back to as early as Euler. The Panjer recursion allows to compute easily the aggregate claims distribution if the counting distribution belongs to so called Panjer's class (comprised from Poisson, Negative Binomial and Binomial distributions). Hence, Panjer's recursion is suitable not only for the compound binomial model, but for some other special cases of Sparre Andersen model as well.

Theorem 1.11 (Panjer's recursion). *Consider a compound distribution $S(n)$, $S(n) = X_1 + \ldots + X_n$ with integer-valued nonnegative claims $X_1, X_2, \ldots$ with local probabilities h_k, $k \in \mathbb{N}_0$, for which, for some real a and b, the probability q_n of having n claims satisfies the following recursion relation*

$$q_n = \left(a + \frac{b}{n}\right) q_{n-1}, \quad n \in \mathbb{N}$$

Then the following relations for the probability of a total claim equal to s hold:

$$\mathbb{P}(S(n) = 0) = \begin{cases} \mathbb{P}(N = 0) & \text{if } h_0 = 0; \\ m_N(\log h_0) & \text{if } h_0 > 0; \end{cases}$$

$$\mathbb{P}(S(n) = s) = \frac{1}{1 - ah_0} \sum_{j=1}^{s} \left(a + \frac{bj}{s}\right) h_j \mathbb{P}(S(n) = s - j), \quad s \in \mathbb{N}.$$

Considering the compound binomial model, the finite time and the infinite time ruin probabilities defined in section 1.1.2 can be easily counted for all $u \in \mathbb{N}_0$ using the following recursive formulas (see, for example, De Vylder and Goovaerts [16], Dickson and Waters [23], Dickson [21], Willmot [71]):

$$\psi(u, 1) = 1 - H(u), \tag{1.1}$$

$$\psi(u, t) = \psi(u, 1) + \sum_{k=0}^{u} \psi(u + 1 - k, t - 1)\, h_k, \ t = 2, 3, \ldots, \tag{1.2}$$

$$\psi(u) = \sum_{k=0}^{u-1} (1 - H(k))\, \psi(u - k) + \sum_{k=u}^{\infty} (1 - H(k))\,, \tag{1.3}$$

where

$$H(x) = \mathbb{P}(Z_1 \leqslant x) = \sum_{i=0}^{\lfloor x \rfloor} h_i, \quad x \in \mathbb{R}.$$

Other exact and approximate recursions are also developed. The review may be found in [17], [18], [63].

In this thesis, the analogous formulas to (1.1), (1.2) and (1.3) for the more general case (discrete time risk model with inhomogeneous claims) are derived.

1.2.2 Gerber-Shiu function

Besides the ruin probability, the *expected discounted penalty function* (or so called Gerber-Shiu function) has been widely acknowledged after its first introduction by Gerber and Shiu in [29]. Similarly to the ruin probability, the Gerber-Shiu function will be defined in the discrete time risk model. The definition in continuous time risk model is analogous.

Definition 1.12 (Infinite time Gerber-Shiu function)**.** *The infinite time Gerber-Shiu function ϕ associated with the risk process is defined the following*

$$\phi_\delta(u) = \mathbb{E}[e^{-\delta T_u} w(U_u(T_u - 1), |U_u(T_u)|)\, \mathbb{1}_{\{T_u < \infty\}}],$$

where $\delta \geqslant 0$*,* $u \in \mathbb{N}$*,* $t \in \mathbb{N}_0$ *and* $w(x, y)$ *is a nonnegative function of* $x \in \mathbb{N}_0$ *and* $y \in \mathbb{N}$ *and* $\mathbb{I}$ *denotes the indicator function, that is,* $\mathbb{I}_A = 1$ *if* A *is true and* $\mathbb{I}_A = 0$ *if* A *is false.*

Definition 1.13 (Finite time Gerber-Shiu function)**.** *Similarly to infinite time Gerber-Shiu function, the finite time Gerber-Shiu function is*

$$\phi_\delta(u, t) = \mathbb{E}[e^{-\delta T_u} w(U_u(T_u - 1), |U_u(T_u)|)\mathbb{I}_{\{T_u \leqslant t\}}],$$

where $\delta \geqslant 0$*,* $u \in \mathbb{N}$*,* $t \in \mathbb{N}_0$ *and* $w(x, y)$ *is a nonnegative function of* $x \in \mathbb{N}_0$ *and* $y \in \mathbb{N}$*.*

This definition is really new, even though it is a natural extension of a widely investigated infinite time Gerber-Shiu function. It was independently introduced by Šiaulys and Kočetova in 2010 [70] and by Kuznetsov and Morales in 2011 [38].

In this article the special case of Gerber-Shiu function with $w(x, y) = 1$ is analyzed. It is obvious, that in such a case $\phi_0(u) = \psi(u)$ and $\phi_0(u, t) = \psi(u, t)$.

A lot of authors investigate Gerber-Shiu function. The most relevant to this work would be Pavlova and Willmot [53], Li [44, 43].

Now we will look deeper, how Gerber and Shiu defined the discounted penalty function in their article "On the Time Value of Ruin" [29]. The following is the extract from this article.

Origins by Gerber and Shiu

Consider a Compound Poisson model. As usually, $u \geqslant 0$ is defined as the insurer's initial surplus. The premiums are received continuously at a constant rate c per unit time. The aggregate claims constitute a compound Poisson process, $\{S(t)\}$, given by the Poisson parameter λ and individual claim amount distribution function $B(x)$ with $B(0) = 0$. That is,

$$S(t) = \sum_{j=1}^{N(t)} X_j,$$

where $\{N(t)\}$ is a Poisson process with intensity λ and i.i.d. variables $X_1, X_2, \ldots$ with common distribution $B(x)$. Then

$$U_u(t) = u + ct - S(t)$$

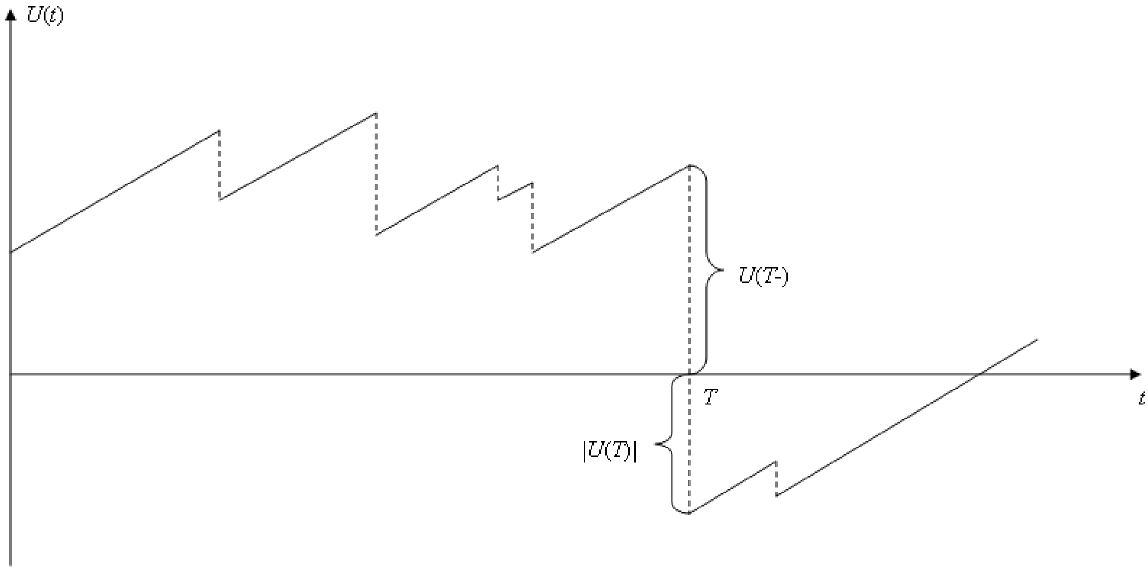


Figure 1.2: The Surplus Immediately before and at Ruin.

is the surplus at time t, $t \geqslant 0$. In addition, it is assumed that $B(x)$ is differentiable, with $B'(x) = p(x)$ being the individual claim amount probability density function.

Let T_u denote the *time of ruin*, $T_u = \inf\{t|U_u(t) < 0\}$ ($T_u = \infty$ if ruin does not occur). The probability of ultimate ruin is considered as a function of the initial surplus $U_u(0) = u \geqslant 0$, $\psi(u) = \mathbb{P}(T_u < \infty|U_u(0) = u)$. Let μ denote the mean of the individual claim amount distribution, i.e.

$$\mu = \int_0^\infty xp(x)dx = \mathbb{E}(X_j).$$

It is assumed $c > \lambda\mu$ to ensure that $\{U_u(t)\}$ has a positive drift; hence $\lim_{t\to\infty} U_u(t) = \infty$ with certainty, and $\psi(u) < 1$, for all $u \geqslant 0$.

The random variable $U_u(T_u-)$ is considered as the surplus immediately before ruin, and $U_u(T_u)$ as the surplus at ruin (see figure 1.2). For given $U_u(0) = u \geqslant 0$, let $f(x, y, t|u)$ denote the joint probability density function of $U_u(T_u-)$, $|U_u(T_u)|$ and T_u. Then

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, t|u)\ dx\ dy\ dt = \mathbb{P}(T_u < \infty|U_u(0) = u) = \psi(u).$$

Because $\psi(u) < 1$, $f(x, y, t|u)$ is a *defective* probability density function. Notice that, for $x > u+ct$, $f(x, y, t|u) = 0$, and that $f(u+ct, y, t|u)\ dx\ dy\ dt = e^{-\lambda t}\lambda p(u + ct + y)\ dy\ dt$.

It is easier to analyze the following function, the study of which is a central theme in Gerber-Shiu paper [29]. For $\delta \geqslant 0$, define

$$f(x, y|u) = \int_0^\infty e^{-\delta t} f(x, y, t|u)\ dt.$$

Here δ can be interpreted as a force of interest or, in the context of Laplace transforms, as a dummy variable. For notational simplicity, the symbol $f(x, y|u)$ does not exhibit the dependence on δ. If $\delta = 0$, this function is the defective joint probability density function of $U_u(T_u-)$ and $|U_u(T_u)|$, given $U_u(0) = u$.

Let $w(x, y)$ be a nonnegative function of $x > 0$ and $y > 0$. We consider, fot $u \geqslant 0$, the function $\phi_\delta(u)$ defined as

$$\begin{aligned} \phi_\delta(u) &= \mathbb{E}[w(U_u(T_u-), |U_u(T_u)|e^{-\delta T_u} \mathbb{I}_{\{T_u<\infty\}}|U_u(0) = u] \\ &= \int_0^\infty \int_0^\infty \int_0^\infty w(x, y)e^{-\delta t} f(x, y, t|u)\ dt\ dx\ dy \\ &= \int_0^\infty \int_0^\infty w(x, y) f(x, y|u)\ dx\ dy. \end{aligned}$$

Note that the symbol $\phi_\delta(u)$ does not exhibit the dependence on the function $w(x, y)$. For $x_0 > 0$ and $y_0 > 0$, if $w(x, y)$ is a "generalized" density function with mass 1 for $(x, y) = (x_0, y_0)$ and 0 for other values of (x, y), then

$$\phi_\delta(u) = f(x_0, y_0|u).$$

Hence the analysis of the function $f(x, y|u)$ is included in the analysis of the function $\phi_\delta(u)$.

If δ is interpreted as a force of interest and w as some kind of penalty when ruin occurs, then $\phi_\delta(u)$ is the expectation of the discounted penalty. If w was interpreted as the benefit amount of an insurance (or reinsurance) payable at the time of ruin, then $\phi_\delta(u)$ is the single premium of the insurance. Hereafter the Gerber-Shiu function $\phi_\delta(u)$ is investigated for $w \equiv 1$ and the theorem below gives the expression for this case.

Theorem 1.14 (Gerber-Shiu function with zero initial surplus)**.** *The Gerber-Shiu function with $w(x, y) \equiv 1$, when initial surplus u is equal to zero, sat-*

isfies

$$E[e^{-\delta T_0}\mathbb{1}_{\{T_0<\infty\}}|U_u(0)=0] = \int_0^\infty\int_0^\infty f(x,y|0)dydx$$
$$= \frac{\lambda}{c}\int_0^\infty e^{-\rho x}[1-B(x)]dx = 1-\frac{\delta}{c\rho},$$

where ρ is a nonnegative root of equation

$$\delta+\lambda-c\xi=\lambda\hat{p}(\xi).$$

Here $\hat{p}(\xi)$ is the Laplace transform of p:

$$\hat{p}(\xi)=\int_0^\infty e^{-\xi x}p(x)dx.$$

For $\delta = 0$, the theorem statement reduces to the famous infinite time ruin probability formula for Lundberg-Cramér model:

$$\psi(0)=\frac{\lambda}{c}\int_0^\infty[1-B(x)]dx=\frac{\lambda\mu}{c}.$$

Li and Garrido approach

Using an approach similar to that of Gerber and Shiu [29], Shuanming Li and José Garrido in 2002 [45] derived a recursive formula in the discrete time risk model for the expected discounted penalty due at ruin, time to ruin (which is analyzed through its p.g.f), the surplus just before ruin and the deficit at ruin. The joint distribution for the compound binomial model is derived by Cheng et al. [10] using martingale techniques and a duality argument, meanwhile Li and Garrido find a recursive formula for the p.g.f. of the time of ruin T_u, the discounted moments of the deficit at ruin and the durplus just before ruin. The results are of independent interest and can give a better understanding of analogous results in the continuous time model, as a limit case of the discrete time model.

Consider the discrete time surplus process

$$U_u(t)=u+t-\sum_{i=1}^{t}Z_i, \quad t\in\mathbb{N},$$

where $u \in \mathbb{N}_0$ is the initial reserve. The Z_i are i.i.d. random variables with local probabilities $p(k) = \mathbb{P}(Z = k)$, for $k \in \mathbb{N}_0$, denoting the total claim amount in period i, occurring at the end of the period. Denote by

$\mu_k = E[Z^k]$ the k-th moment of Z and by $\hat{p}(s) = \sum_{i=0}^{\infty} s^i p(i)$ its probability generating function.

Consider $f_3(x, y, t|u) = P\{U_u(T_u - 1) = x, |U_u(T_u)| = y, T_u = t|U_u(0) = u\}$, $x, y \in \mathbb{N}$, the joint probability function of the surplus just before ruin, deficit at ruin and the time of ruin. Let $v \in (0, 1)$ be the (constant) discount factor over one period with the following relationship to δ in definition (1.12):

$$v = e^{-\delta}.$$

Define $f_2(x, y|u) = \sum_{t=1}^{\infty} v^t f_3(x, y, t|u)$ as a discounted joint p.d.f. of $U(T - 1)$ and $|U(T)|$. Similarly, denote by $f(x|u) = \sum_{y=0}^{\infty} f_2(x, y|u)$. One of the goals is to find the usual conditional probability $f_2(x, y|0)$ and $f_2(x, y|u)$ for any $x, y \in \mathbb{N}$, which give the following relation:

$$f_2(x, y|u) = f(x|u)\frac{p(x + y + 1)}{\sum_{k=x+1}^{\infty} p(k)}, \quad x, y \in \mathbb{N}$$

Let $w(x, y)$, $x, y \in \mathbb{N}_0$ be the nonnegative values for a penalty function. For $0 < v < 1$, define

$$\phi_v(u) = E[v^{T_u} w(U_u(T_u - 1), |U_u(T_u)|)\mathbb{I}_{\{T_u<\infty\}}|U_u(0) = u], \quad u \in \mathbb{N}$$

The quantity $w(U_u(T_u - 1), |U_u(T_u)|)$ can be interpreted as the penalty at the time of ruin for the surplus $U_u(T_u - 1)$ and deficit $|U_u(T_u)|$. Then $\phi_v(u)$ is the expected discounted penalty if v is viewed as a discount rate, or so called Gerber-Shiu function.

Then, given all these conditions, Gerber-Shiu function is equal:

$$\phi_v(0) = v\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} \rho^x w(x, y)p(x + y + 1),$$

and for $u \in \mathbb{N}$,

$$\phi_v(u) = v\sum_{x=0}^{u-1} \phi_v(u-x)\sum_{y=0}^{\infty} \rho^y p(x+y+1)+v\rho^{-u}\sum_{x=u}^{\infty} \rho^x \sum_{y=0}^{\infty} w(x, y)p(x+y+1),$$

where $0 < \rho < 1$ is the root of the equation

$$q(s) := \frac{\hat{p}(s)}{s} = \frac{1}{v}.$$

This recursive expression for $\phi_v(u)$ is obtained using a similar approach to the derivation of the differential equation for $\phi_\delta(u)$ by Gerber and Shiu in [29].

New trends

Since Gerber-Shiu article appearance in 1998, there are three particularly new developments that has influenced and gave new impuls for the development of the risk theory:

- The emphasis on heavy tailed claim distributions.
- The Gerber-Shiu penalty function.
- The possibility to influence the ruin probability by control of the risky investments and possibly reinsurance.

One more interesting development is proposed by Albrecher, Gerber and Yang [2], who investigate the surplus process with downward and upward jumps, modeled by two independent compound Poisson processes.

In the next chapter, some investigations on the extensions of the classical risk theory are discussed.

2

Extensions of Classical Risk Model

Trying to adopt the discrete time risk model and looking at the wider picture, the homogeneous claims' assumption in the classical risk model restricts the applications in practice, because in reality the claims are usually seasonally influenced or dependent on the economic environment. Some authors analyze the processes with random income (see, for example, Yang and Zhang [73], Bao [34], Temnov [65]) or investment income (see, for example, an extensive review of such models by Paulsen [52]).

Now we will look deeper at two works. First of them, written by Ton G. de Kok [15], delivers the recursive formula for inhomogeneous independent claims and arbitrary premium income policies. Another, written by Claude Lefèvre and Philippe Picard [42], became quite popular and gives also very interesting results.

2.1 De Kok results

Ton G. de Kok has investigated a discrete time model with inhomogeneous independent claim size distributions and arbitrary premium income policies. However, his approach was slightly different from the one in this thesis, whereas the recursive scheme derived in [15] paper is based on conditioning on the period claim size in the last period. Meanwhile the recursive formulas in this thesis, as well as de Vylder and Goovaerts [16], are derived by conditioning on the period claim size in the first period. Below you will find the main ideas of his work.

Consider a discrete time ruin problem over N time periods, where the cumulative claim size in period i, $i = 1, \ldots, N$, is a random variable (r.v.), which is denoted by X_i. The cumulative distribution function (c.d.f.) of X_i is given by F_i, i.e.: $F_i(x) = \mathbb{P}\{X_i \leqslant x\}$, $x \geqslant 0$.

$X_1, X_2, \ldots, X_N$ are assumed to be mutually independent. Let $u \geqslant 0$ denote the initial surplus and let P_i, $i = 1, \ldots, N$, denote the premium income in period i. The values P_i are assumed to be given, i.e. the future premium incomes are assumed to be known in advance. The random variables U_i, $i = 1, \ldots, N$, are defined as surplus at the end of period i after claims have been paid:

$$U_i = u + \sum_{j=1}^{i} P_j - \sum_{j=1}^{i} X_j.$$

The ruin probabilities $\psi(u, i)$, $i = 1, \ldots, N$, are defined as follows: $\psi(u, i)$ is the probability that the insurer's surplus, starting from u at time 0, is negative at the end of one or more periods $1, 2, \ldots, i$. For convenience the non-ruin probabilities $\bar{\psi}(u, i)$ are defined by $\bar{\psi}(u, i) = 1 - \psi(u, i)$, $u \geqslant 0$, $i = 1, \ldots, N$. In the sequel De Kok concentrates on expressions for $\bar{\psi}(u, i)$. From the above definitions it may be derived that

$$\bar{\psi}(u, i) = \mathbb{P}\left\{\sum_{k=1}^{j} X_k \leqslant u + \sum_{k=1}^{j} P_k,\ j = 1, \ldots, i\right\}.$$

The quantities $\xi_1, \ldots, \xi_N$, defined as $\xi_i = u + \sum_{j=1}^{i} P_j$, are introduced in order to obtain the following canonical form of the non-ruin probability $\bar{\psi}(u, i)$:

$$\bar{\psi}(u, i) = \mathbb{P}\left\{\sum_{k=1}^{j} X_k \leqslant \xi_j,\ j = 1, \ldots, i\right\}.$$

For the analysis of $\bar{\psi}(u, i)$ it is convenient to introduce the function $G_i(\xi_1, \ldots, \xi_i)$ defined as

$$G_i(\xi_1, \ldots, \xi_i) = \mathbb{P}\left\{\sum_{k=1}^{j} X_k \leqslant \xi_j,\ j = 1, \ldots, i\right\}.$$

Now the artificial random variables $Y_1, \ldots, Y_N$, are introduced that are associated with the above function as follows:

$$\begin{aligned}\mathbb{P}\{Y_1 \leqslant x\} &= \mathbb{P}\{X_1 \leqslant x\},\\ \mathbb{P}\{Y_i \leqslant x\} &= \frac{G_i(\xi_1,\ldots,\xi_{i-1},x)}{G_{i-1}(\xi_1,\ldots,\xi_{i-1})}, \quad x \geqslant 0,\ i=2,\ldots,N. \end{aligned}\tag{2.1}$$

It is easy to see that $Y_1,\ldots,Y_N$, are positive random variables. Note that the dependence of Y_i on the r.v.'s $X_1,\ldots,X_i$ and the constants $\xi_1,\ldots,\xi_i$ is suppressed. With this notation the main theorem recursively characterizes the r.v.'s $Y_i,\ i=1,\ldots,N$.

Now we will present two theorems. First of them follows directly from statement (2.1). The other supplements the first one and is an exact recursive characterization of the canonical non-ruin probability, that enables to derive a fast and accurate approximation scheme to determine the (non-) ruin probability in the discrete time over a finite horizon with independent claim sizes and arbitrary premium incomes in subsequent periods.

Theorem 2.1. *The non-ruin probabilities* $G_i(\xi_1,\ldots,\xi_i)$ *are recursively characterized by*

$$\begin{aligned}G_1(\xi_1) &= \mathbb{P}\{Y_1 \leqslant \xi_1\},\\ G_i(\xi_1,\ldots,\xi_i) &= \mathbb{P}\{Y_i \leqslant \xi_i\}G_{i-1}(\xi_1,\ldots,\xi_{i-1}), \quad i=2,\ldots,N.\end{aligned}\tag{2.2}$$

Theorem 2.2. *The non-ruin probability* $\mathbb{P}\{Y_i \leqslant x\}$ *can be found by*

$$\mathbb{P}\{Y_i \leqslant x\} = \frac{\mathbb{P}\{X_i + Y_{i-1} \leqslant x,\ Y_{i-1} \leqslant \xi_{i-1}\}}{\mathbb{P}\{Y_{i-1} \leqslant \xi_{i-1}\}}, \quad i=2,\ldots,N. \tag{2.3}$$

Proof. The proof is derived by conditioning on the last claim X_n and applying the induction assumption. □

The essence of these theorems is that an expression involving i mutually dependent events was reduced to expressions involving only two mutually dependent events. Since the characterization is exact, it could be exploited to derive exact numerical schemes, e.g. applying fast Furier transforms (cf. Abate and Whitt [1]). In fact the above theorem implies the recursive characterization of the Laplace-Stieltjes transform of $Y_i,\ i=1,\ldots,N$ (cf. Klugman et al. [37]).

Instead of pursuing, possibly numerically intensive, exact methods, De Kok developed a fast recursive scheme that yields approximations for $G_i(\xi_1,\ldots,\xi_i)$

and thereby for the non-ruin probabilities $\bar{\psi}(u,i)$. First of all note that Theorem 2.2 implies that we need an expression for $\mathbb{P}\{Y_i \leqslant x\},\ i = 1,\ldots,N$. Theorem 2.3 implies that

$$\begin{aligned} E[Y_i] &= E[X_i] + E[Y_{i-1}|Y_{i-1} \leqslant \xi_{i-1}],\ i = 1,...,N-1, \\ \sigma^2(Y_i) &= = \sigma^2(X_i) + \sigma^2(Y_{i-1}|Y_{i-1} \leqslant \xi_{i-1}),\ i = 1,...,N-1. \end{aligned}$$

For many well-known probability distributions it is straightforward to compute $E[Y_{i-1}|Y_{i-1} \leqslant \xi_{i-1}]$ and $\sigma^2(Y_{i-1}|Y_{i-1} \leqslant \xi_{i-1})$. Hence we can recursively determine approximations for the first two moments of $Y_i,\ i = 1,...,N$. Besides the two first moments of Y_i, Theorem 2.2 implies that it suffices to compute $\mathbb{P}\{Y_i \leqslant \xi_i\}$. One possible approach is to fit a probability distribution to the first two moments of Y_i, from which $\mathbb{P}\{Y_i \leqslant \xi_i\}$ immediately follows. However, numerical investigations suggest that a more accurate approximation is obtained using the following identity:

$$\mathbb{P}\{X_i + Y_{i-1} \leqslant \xi_i, Y_{i-1} \leqslant \xi_{i-1}\} = \mathbb{P}\{X_i + Y_{i-1} \leqslant \xi_i\}$$

$$-\mathbb{P}\{X_i + (Y_{i-1}|Y_{i-1} \geqslant \xi_{i-1}) \leqslant \xi_i\}\mathbb{P}\{Y_{i-1} \geqslant \xi_{i-1}\}.$$

In this case De Kok suggests to fit convenient c.d.f.'s to the first two moments of X_i, Y_{i-1} and $(Y_{i-1}|Y_{i-1} \geqslant \xi_{i-1})$ as well as to $X_i + Y_{i-1}$ and $X_i + (Y_{i-1}|Y_{i-1} \geqslant \xi_{i-1})$. The better performance of the approximations derived through this equation can be explained from the fact that impact of the conditional random variable $(Y_{i-1}|Y_{i-1} \geqslant \xi_{i-1})$ is explicitly taken into account.

2.2 Picard-Lefèvre formula

Claude Lefèvre and Philippe Picard investigated a nonhomogeneous risk model and derived the finite time ruin probability formula [42] by using a theory about pseudopolynomials of Appell type. They investigated the finite time survival probability with claim arrivals forming a Poisson process, claim sizes being i.i.d. and integer-valued, and premium income modeled by any real function which is nondecreasing and tends to infinity as $t \to \infty$. Below the extract of Picard and Lefèvre article [42] is presented with the model definitions and the formula derivation.

Consider an insurance portfolio which balance is struck at dates $t \in \mathbb{N}_0$. The initial surplus u is of given amount $u \geqslant 0$. The premiums c_t is paid to cover claims for the period $(t, t+1]$. The claims amounts $X_t,\ t \geqslant 1$, correspond to independent random variables, and each X_t has an arithmetic distribution $\{a_i^{(t)}, i \in \mathbb{N}_0\}$ that may depend on t; to avoid trivialities, it is assumed that $a_0^{(t)} \neq 0$.

The surplus U_t at time $t \in \mathbb{N}$ is given by

$$U_t = U_{t-1} + c_{t-1} - X_t \equiv h(t) - S_t,$$

where $h(t) = u + c_0 + \ldots + c_{t-1} = h(t-1+0)$ represents the total premium income over the time interval $(0, t]$, including the initial surplus, and $S_t = X_1 + \ldots + X_t$ is the total claim amount over $(0, t]$. Ruin will occur at the first date T when the surplus becomes strictly negative, i.e.,

$$T = \inf\{t \geqslant 1 : h(t) < S_t\}.$$

We focus our attention on the problem of the evaluation of the probability of (non)ruin over any finite time horizon. If ruin occurs, another statistic of interest is the severity of ruin defined as $S_T - h(T)$ (which is necessarily > 0).

Given any date t_0, let us denote by $S_{t_0,t} = X_{t_0+1} + \ldots + X_t$ the total claim amount over the period $(t_0, t],\ t \geqslant t_0$.

We start by pointing out a simple algebraic structure that underlies its distribution.

Lemma 2.3. *For $t \geqslant t_0 \geqslant 0$,*

$$\mathbb{P}(S_{t_0,t} = n) = a_0^{(t_0,t)} e_n(t_0, t), \quad n \geqslant 0$$

where $a_0^{(t_0,t)} = a_0^{(t_0+1)} \cdots a_0^{(t)}$, and $\mathcal{E}_{t_0} = \{e_n(t_0, .),\ n \geqslant 0\}$ is a family of functions specified below and satisfying border conditions $e_0(t_0, t) = 1$, $e_n(t_0, t_0) = \delta_{n,0}$, and a convolution type property

$$e_n(t_0, t') = \sum_{k=0}^{n} e_k(t_0, t) e_{n-k}(t, t'), \quad \text{when } t' \geqslant t \geqslant t_0.$$

In practice, when $t_0 = 0$ say, the $e_n(0, t)$, for $t \geqslant 1$, will not be computed using their definition, but rather by way of the following recursion. Indeed, we can rewrite

$$e_n(0,t) = \sum_{k=0}^{n} e_k(0,t-1)e_{n-k}(t-1,t) = \sum_{k=0}^{n} \frac{a_{n-k}^{(t)}}{a_0^{(t)}} e_k(0,t-1), \quad n \geqslant 0,$$

$$e_0(0,t) = 1, \quad e_n(0,0) = \delta_{n,0}.$$

Theorem 2.4. *For $t \geqslant 0$,*

$$\mathbb{P}(S_t = n,\ T > t) = a_0^{(0,t)} A_n(t) 1\!\!1_{\{t \geqslant v_n\}}, \quad n \geqslant 0,$$

where v_n denotes the first time where the premium income process reaches or goes beyond the level n, i.e., $v_n = \inf\{t \geqslant 0 : \ h(t+0) \geqslant n\}$, and $\{A_n(.),\ n \geqslant 0\}$ is a family of functions defined by

$$A_n(v_n) = \delta_{n,0}, \quad A_n(t) = \sum_{k=0}^{n} A_k(v_n) e_{n-k}(v_n, t), \quad \textit{if } t \geqslant v_n.$$

In practice, the $A_n(t)$ are more easily computed by:

$$A_n(t) = \sum_{k=0}^{n} \alpha_{n-k} e_k(0,t), \quad n \geqslant 0,$$

where the coefficients α_k are given recursively by

$$\alpha_n = \delta_{n,0} - \sum_{k=1}^{n} \alpha_{n-k} e_k(0, v_n).$$

and $A_n(t) = e_n(0,t)$, for all $n \leqslant u + c_0$.

Corollary 2.5. *For $t \geqslant 0$, (as usual, $[x]$ denotes the integer part of x):*

$$\mathbb{P}(T > t) = a_0^{(0,t)} \sum_{n=0}^{[h(t)]} A_n(t).$$

For any integer $k > h(t)$,

$$\mathbb{P}[T = t,\ \textit{ruin with severity}\ \ k - h(t)] = a_0^{(0,t-1)} \sum_{n=0}^{[h(t-1)]} A_n(t-1) \alpha_{k-n}^{(t)}.$$

Further investigations

Ignatov and Kaishev [35] studied a more relaxed case with dependent claim sizes and exponentially but nonidentically distributed inter-occurrence times and derived explicit two-sided bounds which coincide when the claim arrivals form a Poisson process. See also Ignatov et al. [36], who further improved the formula to be more convenient for numerical evaluations. Also a lot of work is done by Lefèvre and Loisel [40, 41].

3

Discrete Time Risk Model with Inhomogeneous Claims

Trying to adopt the discrete time risk model and looking at the wider picture, the homogeneous claims' assumption in the classical risk model restricts the applications in practice. In this thesis the homogeneity assumption is released and arriving claims are allowed to be not necessarily identically distributed (but still independent).

Now we will describe this discrete time risk model with inhomogeneous claims, which will be used hereafter.

Definition 3.1 (Insurer's balance)**.** *The insurer's balance, as in classical discrete time risk model, is*

$$U_u(t) = u + t - \sum_{i=1}^{t} Z_i,$$

and the following conditions are satisfied:

- $u = U_u(0) \in \mathbb{N}_0$;
- *claim amounts $Z_1, Z_2, Z_3, \ldots$ are independent nonnegative integer valued r.v.s. with corresponding local probabilities and distribution functions ($j, k \in \mathbb{N}_0$):*

$$h_k^{(j)} = \mathbb{P}\left(Z_{1+j} = k\right), \quad H^{(j)}(x) = \mathbb{P}(Z_{1+j} \leqslant x) = \sum_{i=0}^{\lfloor x \rfloor} h_i^{(j)}, x \in \mathbb{R}.$$

It is evident that local probabilities $h_k^{(j)} (j, k \in \mathbb{N}_0)$ or a set of distribution functions $H^{(j)}$ $(j \in \mathbb{N}_0)$ describe fully the distribution of independent nonnegative integer valued r.v.s $Z_1, Z_2, Z_3, \dots$.

Having claim sequence $Z_1, Z_2, Z_3, \dots$ described, we can construct the new shifted sequence of claims $\{Z_{i+j}\}_{i\in\mathbb{N}}$ for every fixed shifting parameter $j \in \mathbb{N}_0$ from the initial claim sequence.

Random variable sequence $Z_{1+j}, Z_{2+j}, \dots$, $(j \in \mathbb{N}_0)$ can be used for creating a shifted discrete time risk model, in which insurer's capital at each moment n is

$$U_u^{(j)}(t) = u + t - \sum_{i=1}^{t} Z_{i+j} = u - S_t^{(j)},$$

where $S_t^{(j)} = \sum_{i=1}^{t} (Z_{i+j} - 1)$. One sample path of this process is given as an example in Figure 3.1.

In case $j = 0$ this shifted model coincides with the initial model and we will call it *base case*.

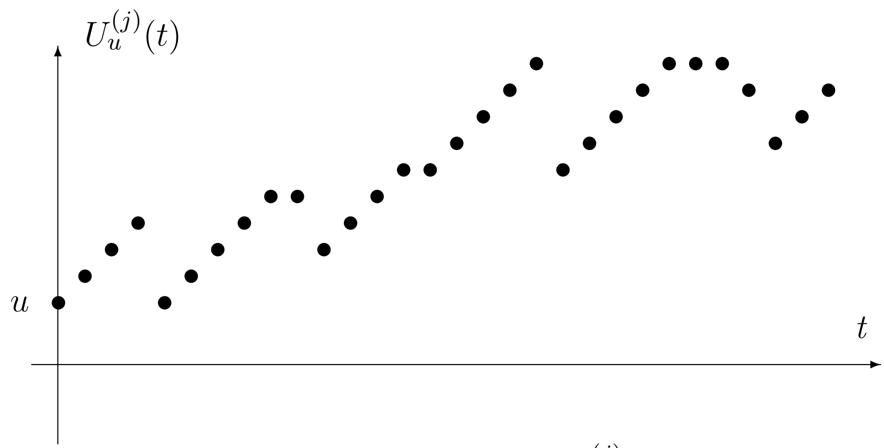


Figure 3.1: A sample path of $U_u^{(j)}(t)$ process.

The premium income in this model, as it can be seen from insurer's balance definition (3.1), is equal to 1. We will also investigate the special case of non uniform premiums payments in Section 4, but this case will be defined separately.

As in classical Risk theory, the ruin time, ruin probabilities and Gerber-Shiu (expected discounted penalty) function is described in this discrete time risk model with inhomogeneous claims by following definitions.

Definition 3.2 (The time of ruin). *The time of ruin for shifted model is* *defined in Gerber [28] style by*

$$T_u^{(j)} = \begin{cases} \min\left\{t \geqslant 1 : U_u^{(j)}(t) \leqslant 0\right\}, \\ \infty\,, \ \textit{if} \ \ U_u^{(j)}(t) > 0 \ \ \textit{for all} \ \ t \in \mathbb{N}. \end{cases}$$

Definition 3.3 (Finite time ruin probability)**.** *The finite-time ruin probability at moment $t \in \mathbb{N}$ with initial capital $u \in \mathbb{N}_0$ usually is defined by*

$$\psi^{(j)}(u,t) = \mathbb{P}\left(T_u^{(j)} \leqslant t\right).$$

It is evident and later will be used that

$$\begin{aligned}
\psi^{(j)}(u,t) &= \mathbb{P}\left(\bigcup_{n=1}^{t}\left\{u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0\right\}\right) \\
&= \mathbb{P}\left(u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0, \text{ for some } \ n \in \{1,2,\ldots,t\}\right) \\
&= \mathbb{P}\left(\max_{1 \leqslant n \leqslant t} \sum_{i=1}^{n} (Z_{i+1} - 1) \geqslant u\right).
\end{aligned}$$

Definition 3.4 (Infinite time ruin probability)**.** *Similarly to finite time, the infinite time ruin probability, or the ultimate ruin probability, is*

$$\psi^{(j)}(u) = \mathbb{P}\left(T_u^{(j)} < \infty\right) = \lim_{t \to \infty} \psi^{(j)}(u,t).$$

Definition 3.5 (Finite and infinite time Gerber-Shiu function)**.** *Let us define the finite and infinite time Gerber-Shiu functions for the model with inhomogeneous claims by*

$$\phi_\delta^{(j)}(u) = \mathbb{E}[e^{-\delta T_u^{(j)}} \mathbb{1}_{\{T_u^{(j)} < \infty\}}],$$

$$\phi_\delta^{(j)}(u,t) = \mathbb{E}[e^{-\delta T_u^{(j)}} \mathbb{1}_{\{T_u^{(j)} \leqslant t\}}],$$

where $\delta \geqslant 0$, $t \in \mathbb{N}$, $u \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$.

All usual concepts are defined including a shifted discrete time risk model parameter, because recursive formulas' derivation (for ruin probabilities, Gerber-Shiu function) depends on analogous values of shifted models as well. For this reason the shifted discrete time risk model calculated values are necessary.

4

Investigation of the Finite Time Ruin Probability

In this chapter, we provide the expression for ruin probability having a simple classical recurrent form. Using this formula the ruin probability until moment t of nonhomogeneous initial model refers to ruin probability until moment $t-1$ for shifted model. Hereby the finite time ruin probability can be easily implemented and calculated with desired accuracy if necessary. Moreover, a model is extended by allowing rationally valued nonidentically distributed claims and nonconstant premium payments. These enhancements enable even wider model applications.

The rest of the chapter contains two parts. In the first part (sections 4.1 and 4.2), a discrete time model with the integer valued claims is investigated and the recursive formula for the finite time ruin probability is obtained. As an example, the tables for the values of the finite time ruin probability for two different claims series are presented. In the second part (sections 4.3 and 4.4), a discrete time risk model with the rationally distributed claims and premiums, which are allowed to be nonconstant over time, is investigated. For such a case, two recursive formulas for the finite time ruin probability are presented in Section 4.3. An example with detailed explanation is shown in Section 4.4.

4.1 Discrete time risk model with inhomogeneous claims

Let us investigate the discrete time risk model with independent inhomogeneous claims as described in Section 3. Below we present two theorems with a proof for the finite time ruin probability in this model evaluation by the recursive formulas.

Theorem 4.1. *Consider the discrete time risk model with nonnegative non-identically distributed claims from Section 3. Then the finite time ruin probabilities*

$$\psi^{(j)}(u,t) := \mathbb{P}\left(u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0 \quad \textit{for some} \quad n \in \{1,2,\ldots,t\}\right)$$

for all $j, u \in \mathbb{N}_0$ satisfy the following equations:

$$\psi^{(j)}(u,1) = 1 - H^{(j)}(u), \tag{4.1}$$

$$\psi^{(j)}(u,t) = \psi^{(j)}(u,1) + \sum_{k=0}^{u} \psi^{(j+1)}(u+1-k, t-1)\ h_k^{(j)},\ t=2,3,\ldots \tag{4.2}$$

Remark 4.2. *The obtained formulas are simple enough and are suitable to calculate the finite time ruin probabilities $\psi^{(0)}(u,t)$, $(u \in \mathbb{N}_0,\ t \in \mathbb{N})$ for the model with the initial claims sequence $Z_1, Z_2, \ldots$. For example, in order to calculate $\psi^{(0)}(2,3)$ we can express it by $\psi^{(1)}(3,2), \psi^{(1)}(2,2)$ and $\psi^{(1)}(1,2)$ according to the second relation of Theorem. Next, in the same way, we can express these quantities by probabilities $\psi^{(2)}(4,1)$, $\psi^{(2)}(3,1), \psi^{(2)}(2,1)$, $\psi^{(2)}(1,1)$ which can be calculated directly from formula (4.1). Computations of the finite time ruin probability via Theorem 4.1 for two risk models are presented in Section 4.2.*

Proof. The proof of Theorem 4.1 is a standard application of conditioning argument, but we present it for readers' convenience. First we observe that

$$\psi^{(j)}(u,1) = \mathbb{P}(u + 1 - Z_{1+j} \leqslant 0) = \mathbb{P}(u + 1 \leqslant Z_{1+j}) = 1 - H^{(j)}(u).$$

When $t \in \{2,3,\ldots\}$, using suitable properties of probability and the fact that all claims are nonnegative we obtain

$$\begin{aligned}
\psi^{(j)}(u,t) &= \mathbb{P}\left(\bigcup_{n=1}^{t}\left\{u+n-\sum_{i=1}^{n} Z_{i+j} \leqslant 0\right\}\right)\\
&= \mathbb{P}\left(Z_{1+j} \geqslant u+1, \bigcup_{n=1}^{t}\left\{u+n-\sum_{i=1}^{n} Z_{i+j} \leqslant 0\right\}\right)\\
&+ \mathbb{P}\left(Z_{1+j} \leqslant u, \bigcup_{n=1}^{t}\left\{u+n-\sum_{i=1}^{n} Z_{i+j} \leqslant 0\right\}\right)\\
&= \mathbb{P}\left(Z_{1+j} \geqslant u+1\right)+\mathbb{P}\left(\bigcup_{k=0}^{u}\left\{Z_{1+j}=k, \bigcup_{n=1}^{t}\left\{u+n-\sum_{i=1}^{n} Z_{i+j} \leqslant 0\right\}\right\}\right)\\
&= \psi^{(j)}(u,1)+\mathcal{J}, \qquad (4.3)
\end{aligned}$$

where the second term of the last expression

$$\begin{aligned}
\mathcal{J} &= \mathbb{P}\left(\bigcup_{k=0}^{u}\bigcup_{n=1}^{t}\left\{Z_{1+j}=k, u+n-\sum_{i=1}^{n} Z_{i+j} \leqslant 0\right\}\right)\\
&= \mathbb{P}\left(\bigcup_{k=0}^{u}\left\{\bigcup_{n=2}^{t}\left\{Z_{1+j}=k, u+n-\sum_{i=1}^{n} Z_{i+j} \leqslant 0\right\}\right\}\right)\\
&= \mathbb{P}\left(\bigcup_{k=0}^{u}\left\{Z_{1+j}=k, \bigcup_{n=2}^{t}\left\{u+n-k-\sum_{i=2}^{n} Z_{i+j} \leqslant 0\right\}\right\}\right).
\end{aligned}$$

Claims $Z_{1+j}, Z_{2+j}, \ldots$ are independent, and vector $(Z_{2+j}, Z_{3+j}, \ldots, Z_{n+j})$ has the same distribution as vector $\left(Z_{1+(j+1)}, Z_{2+(j+1)}, \ldots, Z_{n-1+(j+1)}\right)$, if $n = 2, 3, \ldots$. Therefore

$$\begin{aligned}
\mathcal{J} &= \sum_{k=0}^{u}\mathbb{P}\left(Z_{1+j}=k\right)\mathbb{P}\left(\bigcup_{n=2}^{t}\left\{u+n-k-\sum_{i=2}^{n} Z_{i+j} \leqslant 0\right\}\right)\\
&= \sum_{k=0}^{u} h_k^{(j)}\ \mathbb{P}\left(\bigcup_{n=2}^{t}\left\{u+n-k-\sum_{i=1}^{n-1} Z_{i+(j+1)} \leqslant 0\right\}\right)\\
&= \sum_{k=0}^{u} h_k^{(j)}\ \mathbb{P}\left(\bigcup_{l=1}^{t-1}\left\{u+1-k+l-\sum_{i=1}^{l} Z_{i+(j+1)} \leqslant 0\right\}\right)\\
&= \sum_{k=0}^{u} \psi^{(j+1)}(u+1-k, t-1)\ h_k^{(j)}.
\end{aligned}$$

The last equality and equality (4.3) imply the desired relation (4.2). Theorem 4.1 is proved. □

From the presented proof we observe that the statement of Theorem 4.1 remains valid if r.v.s $Z_1, Z_2, \dots$ are not necessary nonnegative. Below we formulate the statement rigorously. We use the recursion formula presented in the theorem below later, when we consider a discrete time risk model with rational-valued claim amounts. It is clear that r.v.s $Z_1, Z_2, \dots$ which are not necessary nonnegative can not describe claim amounts, and therefore we can not call the probability similar to $\psi(u,t)$ a finite time ruin probability.

Theorem 4.3. *Let $u \in \mathbb{N}_0$; r.v.s $Z_1, Z_2, \dots$ be independent and integer valued; $h_k^{(j)} = \mathbb{P}(Z_{1+j} = k)$ for $k \in \mathbb{Z}$, $j \in \mathbb{N}_0$. Then the probabilities*

$$\tilde{\psi}^{(j)}(u,t) := \mathbb{P}\left(u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0,\ \textit{for some} \quad n \in \{1,2,\dots,t\}\right)$$

satisfy the following equations

$$\begin{aligned} \tilde{\psi}^{(j)}(u,1) &= \sum_{k>u} h_k^{(j)}, \\ \tilde{\psi}^{(j)}(u,t) &= \tilde{\psi}^{(j)}(u,1) + \sum_{k=-\infty}^{u} \tilde{\psi}^{(j+1)}(u+1-k, t-1)\ h_k^{(j)} \end{aligned}$$

for all $j \in \mathbb{N}_0$ and $u \in \mathbb{N}_0$.

Proof. The difference in the proofs of Theorem 4.3 and Theorem 4.1 is very minor. As well as in Theorem 4.1, the probability in the first time period is equal to

$$\tilde{\psi}^{(j)}(u,1) = 1 - H^{(j)}(u) = \sum_{k>u} h_k^{(j)}.$$

When $t \in \{2,3,\dots\}$, using suitable properties of probability we obtain

$$\begin{aligned} \tilde{\psi}^{(j)}(u,t) &= \mathbb{P}\left(\bigcup_{n=1}^{t}\left\{u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0\right\}\right) \\ &= \mathbb{P}\left(Z_{1+j} \geqslant u+1, \bigcup_{n=1}^{t}\left\{u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0\right\}\right) \\ &+ \mathbb{P}\left(Z_{1+j} \leqslant u, \bigcup_{n=1}^{t}\left\{u+n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0\right\}\right) = \mathbb{P}\left(Z_{1+j} \geqslant u+1\right) \\ &+ \mathbb{P}\left(\bigcup_{k=-\infty}^{u}\left\{Z_{1+j} = k, \bigcup_{n=1}^{t}\left\{u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0\right\}\right\}\right) \\ &= \tilde{\psi}^{(j)}(u,1) + \mathcal{I}, \end{aligned} \tag{4.4}$$

where the second term of the last expression

$$\begin{aligned}
\mathcal{I} &= \mathbb{P}\left(\bigcup_{k=-\infty}^{u}\bigcup_{n=1}^{t}\left\{Z_{1+j}=k, u+n-\sum_{i=1}^{n}Z_{i+j}\leqslant 0\right\}\right)\\
&= \mathbb{P}\left(\bigcup_{k=-\infty}^{u}\left\{\bigcup_{n=2}^{t}\left\{Z_{1+j}=k, u+n-\sum_{i=1}^{n}Z_{i+j}\leqslant 0\right\}\right\}\right)\\
&= \mathbb{P}\left(\bigcup_{k=-\infty}^{u}\left\{Z_{1+j}=k, \bigcup_{n=2}^{t}\left\{u+n-k-\sum_{i=2}^{n}Z_{i+j}\leqslant 0\right\}\right\}\right).
\end{aligned}$$

Claims $Z_{1+j}, Z_{2+j}, \ldots$ are independent, and vector $(Z_{2+j}, Z_{3+j}, \ldots, Z_{n+j})$ has the same distribution as vector $\big(Z_{1+(j+1)}, Z_{2+(j+1)}, \ldots, Z_{n-1+(j+1)}\big)$, if $n = 2, 3, \ldots$. Therefore

$$\begin{aligned}
\mathcal{I} &= \sum_{k=-\infty}^{u}\mathbb{P}\left(Z_{1+j}=k\right)\mathbb{P}\left(\bigcup_{n=2}^{t}\left\{u+n-k-\sum_{i=2}^{n}Z_{i+j}\leqslant 0\right\}\right)\\
&= \sum_{k=-\infty}^{u} h_k^{(j)}\,\mathbb{P}\left(\bigcup_{n=2}^{t}\left\{u+n-k-\sum_{i=1}^{n-1}Z_{i+(j+1)}\leqslant 0\right\}\right)\\
&= \sum_{k=-\infty}^{u} h_k^{(j)}\,\mathbb{P}\left(\bigcup_{l=1}^{t-1}\left\{u+1-k+l-\sum_{i=1}^{l}Z_{i+(j+1)}\leqslant 0\right\}\right)\\
&= \sum_{k=-\infty}^{u}\tilde{\psi}^{(j+1)}(u+1-k, t-1)\; h_k^{(j)}.
\end{aligned}$$

The last equality and equality (4.4) imply the statement of Theorem 4.3. □

Remark 4.4. *The relations in Theorem 4.3 can be easily used for the numerical recursive calculation of the quantities $\tilde{\psi}^{(j)}(u,t)$ assuming that there exists some integer K, for which local probabilities $h_k^{(j)} = 0$ with all $k \leqslant K$ and $j \in \mathbb{N}_0$.*

4.2 Two examples

A common situation in nonlife insurance is seasonality in claims, when arriving claims' severity or frequency is dependent on the season. Therefore, the claims' sequence can be constructed to reflect four different claim distributions per annum: winter, spring, summer, and autumn. Each season has its own claim distribution, which does not differ when years pass.

In the first example, the claims are distributed according to the Poisson law $\mathcal{P}(\lambda), \lambda > 0$, that is described by local probabilities $h_k = \lambda^k e^{-\lambda}/k!,\ k \in \mathbb{N}_0$. Therefore, the season fluctuations in the model are warranted by different positive parameters values:

$$Z_1, Z_5, Z_9, \ldots \stackrel{d}{=} \mathcal{P}(0.2); \quad Z_2, Z_6, Z_{10}, \ldots \stackrel{d}{=} \mathcal{P}(0.5);$$
$$Z_3, Z_7, Z_{11}, \ldots \stackrel{d}{=} \mathcal{P}(0.3); \quad Z_4, Z_8, Z_{12}, \ldots \stackrel{d}{=} \mathcal{P}(0.9).$$

As it can be seen from parameters and the properties of the Poisson law, the means of claims $\mathbb{E}Z_i,\ i \in \mathbb{N}$ are lower than 1, which is a premium per unit time interval. Moreover, every fifth claim is repeating in the first example.

The tail behavior of ruin probabilities is investigated as time goes from $t = 1$ to $t = 8$. Moreover, the initial capital varies from $u = 0$ to $u = 6$. Using formulas from Theorem 4.1 we obtain Table 4.1. As we can see from this table, similarly to homogeneous claim case, the ruin probabilities in inhomogeneous claim case are decreasing as the initial capital increase and increasing as time goes pass.

Initial capital	Time period t							
u	1	2	3	4	5	6	7	8
0	0.1813	0.2551	0.2661	0.3059	0.3077	0.3114	0.3123	0.3175
1	0.0175	0.0441	0.0496	0.0752	0.0765	0.0793	0.0800	0.0843
2	0.0011	0.0064	0.0080	0.0179	0.0185	0.0198	0.0201	0.0224
3	0.0001	0.0008	0.0011	0.0041	0.0043	0.0048	0.0049	0.0059
4	0.0000	0.0001	0.0001	0.0001	0.0009	0.0011	0.0011	0.0015
5	0.0000	0.0000	0.0000	0.0002	0.0002	0.0002	0.0002	0.0004
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001

Table 4.1: Finite time ruin probabilities of the model with inhomogeneous claims. Example 1.

As we can see from Theorem 4.1, the discrete time risk model with non-identically distributed claims is also convenient for investigation of the case where one or more claims are distributed with higher means than unity. In the second example, we also take that every fifth claim distribution is repeating and distributed according to the Poisson law with some parameters. However in the second model we take the subsequence of r.v.s distributed according to the Poisson law with parameter for which the average of claim are upper to the premium income:

$$Z_1, Z_5, Z_9, \ldots \stackrel{d}{=} \mathcal{P}(0.2); \quad Z_2, Z_6, Z_{10}, \ldots \stackrel{d}{=} \mathcal{P}(0.5);$$
$$Z_3, Z_7, Z_{11}, \ldots \stackrel{d}{=} \mathcal{P}(0.3); \quad Z_4, Z_8, Z_{12}, \ldots \stackrel{d}{=} \mathcal{P}(2).$$

The tail behavior of ruin probabilities in the second model is investigated as time goes pass from $t = 1$ to $t = 8$, and the initial capital varies from $u = 0$ to $u = 8$. Using formulas from Theorem 4.1 we obtain Table 4.2. We can see from it that, in such dangerous case, the finite time ruin probabilities do not increase dramatically together with t, especially if initial capital is high enough.

Initial capital	Time period t							
u	1	2	3	4	5	6	7	8
0	0.1813	0.2551	0.2661	0.4564	0.4595	0.4650	0.5663	0.5713
1	0.0175	0.0441	0.0496	0.2012	0.2043	0.2098	0.2111	0.2613
2	0.0011	0.0064	0.0080	0.0862	0.0882	0.0919	0.0928	0.1315
3	0.0001	0.0008	0.0011	0.0338	0.0348	0.0368	0.0373	0.0620
4	0.0000	0.0001	0.0001	0.0119	0.0124	0.0133	0.0136	0.0272
5	0.0000	0.0000	0.0000	0.0038	0.0040	0.0044	0.0045	0.0112
6	0.0000	0.0000	0.0000	0.0011	0.0012	0.0013	0.0013	0.0043
7	0.0000	0.0000	0.0000	0.0003	0.0003	0.0004	0.0004	0.0016
8	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0006

Table 4.2: Finite time ruin probabilities of the model with inhomogeneous claims. Example 2.

4.3 Discrete time risk model with rational-valued inhomogeneous claims

The classical discrete time risk model usually investigates the claims which can take only discrete values. Moreover, premium income is constant and equal to one over the time span. In the real world such assumption, however, restricts the model and some additional features are usually desirable. In the literature, various enhancements of the standard model have been investigated. Some authors add investment return (constant or stochastic) to the standard model [55], [54], others investigate nonuniform premium income [55], [56].

In this section, the classical discrete time risk model will be enhanced by taking varying premium income and nonhomogeneous claims which can take rational values. Moreover, premium income and initial insurers' capital may also take rational values. We describe this model below.

Suppose that the insurer capital at each moment $n \in \mathbb{N}_0$ is

$$U(n) = u + \sum_{i=1}^{n} c_i - \sum_{i=1}^{n} Z_i$$

and the following conditions are satisfied:

• the quantities $u, c_1, c_2, \ldots$ are nonnegative and belong to the set of rational numbers $\mathbb{Q}$;

• claim amounts $Z_1, Z_2, Z_3, \ldots$ are independent nonnegative rational-valued r.v.s;

• there exists a natural number α for which: $\alpha u \in \mathbb{Z}$, $\alpha c_i \in \mathbb{Z}$ $(i \in \mathbb{N})$ and $\alpha D_k^{(j)} \in \mathbb{Z}$ $(j, k \in \mathbb{N}_0)$, where $D_k^{(j)}$ are rational values of r.v. Z_{1+j} acquired with probabilities $h_k^{(j)}$.

Theorem 4.5. *Consider the discrete time risk model with nonidentically rational-valued claims as described above. The finite time ruin probability*

$$\psi_{\mathbb{Q}}(u,t) := \mathbb{P}\left(u + \sum_{i=1}^{n} c_i - \sum_{i=1}^{n} Z_i \leqslant 0, \text{ for some } \ n \in \{1, \ldots, t\}\right)$$

for all $u, t \in \mathbb{N}_0$ coincides with probability $\tilde{\psi}^{(0)}(\alpha u, t)$, defined in Theorem 4.3, where integer valued r.v.s $\widehat{Z}_{1+j}$, $j \in \mathbb{N}_0$ are distributed according to the *local probabilities*

$$\widehat{h}^{(j)}_{\alpha D^{(j)}_k-\alpha c_j-1} = \mathbb{P}\left(\widehat{Z}_{1+j} = \alpha D^{(j)}_k - \alpha c_j - 1\right) = h^{(j)}_k, \qquad k \in \mathbb{N}_0.$$

Remark 4.6. *By Theorem 4.5, the finite time ruin probability for a model with rationally distributed claims $\psi^{(0)}_{\mathbb{Q}}(u,t)$ can be calculated using the recurrent formulas presented in Theorem 4.3 applying a suitable model transform. We observe that, according to the restrictions of the model, the transformed r.v.s $\widehat{Z}_{1+j}$ $(j \in \mathbb{N}_0)$ can 'catch' only the finite number of negative values.*

Proof. The proof of the theorem is straightforward, because for the existing special natural α and all fixed $j, n \in \mathbb{N}_0$, we have

$$\alpha U(n) = \alpha u + n - \sum_{i=1}^{n} (\alpha Z_i - \alpha c_i - 1) = \alpha u + n - \sum_{i=1}^{n} \widehat{Z}_i\,.$$

□

Theorem 4.7. *Consider the discrete time risk model with nonidentically rational-valued claims as in Theorem 4.5. Then the finite time ruin probability $\psi_{\mathbb{Q}}(u,t)$ defined above for all $u, t \in \mathbb{N}_0$ is equal to the finite time ruin probability $\psi^{(0)}\left(\alpha u, \sum\limits_{i=1}^{t} \alpha c_i\right)$ defined in Theorem 4.1 with integer valued claim sequence $\widehat{Z}_i$, $i \in \mathbb{N}$ distributed according to the following local probabilities:*

$$\begin{cases} \mathbb{P}(\widehat{Z}_l = \alpha D^{(i-1)}_k) = h^{(i-1)}_k, \; k \in \mathbb{N}_0, \quad \textit{if} \;\; l = \sum\limits_{m=1}^{i} \alpha c_m, \\ \mathbb{P}(\widehat{Z}_l = k) = \mathbb{I}_{\{k=0\}}, \quad \textit{otherwise.} \end{cases}$$

Proof. It is evident that for a special chosen natural α and all fixed $j, n \in \mathbb{N}_0$, we have

$$\alpha U(n) = \alpha u + \sum_{i=1}^{n} \alpha c_i - \sum_{i=1}^{n} \alpha Z_i.$$

Since $\alpha c_i \in \mathbb{N}$ for $i \in \mathbb{N}$, the model under consideration is equivalent to the standard discrete time risk model described in Section 4.1 with integer valued claim sequence

$$\underbrace{E_0, \ldots, E_0}_{\alpha c_1 - 1}, \alpha Z_1, \underbrace{E_0, \ldots, E_0}_{\alpha c_2 - 1}, \alpha Z_2, \ldots, \tag{4.5}$$

where r.v. E_0 has a degenerate law at the origin, i.e.

$$\mathbb{P}(E_0 = k) = \mathbb{1}_{\{k=0\}}.$$

The statement of Theorem follows now immediately from the presented relations. We observe only that the claims Z_t ($t \in \mathbb{N}$) from the initial model transform to r.v.s αZ_t with the serial numbers $\sum_{i=1}^{t} \alpha c_i$ in the transformed model (4.5). Theorem 4.7 is proved. $\square$

4.4 Yet another example

In this section we demonstrate how theorems 4.5 and 4.7 can be applied for a specific situation. Let us consider the discrete time risk model in which every third random claim distribution in the sequence $Z_1, Z_2, Z_3, Z_4, \ldots$ is repeating, and r.v.s Z_1, Z_2 are distributed according to the following laws:

Z_1:

$D_k^{(0)}$	0	0.5	1	1.5
$h_k^{(0)}$	0.6	0.2	0.1	0.1

, Z_2:

$D_k^{(1)}$	0	0.5	1	1.5	2	2.5
$h_k^{(1)}$	0.2	0	0.2	0	0.3	0.3

.

Also we suppose that the premium income c_i is equal to 0.5 for odd i, and is equal to 1.5 for even i.

First, we calculate the ruin probability $\psi_{\mathbb{Q}}(0.5, 2)$ for initial capital $u = 0.5$ and time moment $t = 2$. It is easy to find that the least multiplier which transforms u, c_1, c_2 and values taken by r.v.s $Z_1, Z_2, Z_3, Z_4, \ldots$ into integer numbers is $\alpha = 2$. In order to calculate the desired probability $\psi_{\mathbb{Q}}(0.5, 2)$, we can use Theorem 4.5 as well as Theorem 4.7.

According to Theorem 4.5 for every $t \in \mathbb{N}$ we obtain

$$\psi_{\mathbb{Q}}(0.5, t) = \widetilde{\psi}^{(0)}(1, t) = \mathbb{P}\left(1 + n - \sum_{i=1}^{n} \widehat{Z}_i \leqslant 0 \quad \text{for some} \quad n \in \{1, 2, \ldots, t\}\right),$$

where the r.v. sequence $\widehat{Z}_1, \widehat{Z}_2, \widehat{Z}_3, \widehat{Z}_4, \ldots$ has repeating distributions for every third members and $\widehat{Z}_1, \widehat{Z}_2$ are distributed according to the transformed laws:

$\widehat{Z}_1$:

l	0	1	2	3
$\widehat{h}_l^{(0)}$	0.6	0.2	0.1	0.1

, $\widehat{Z}_2$:

l	-2	-1	0	1	2	3
$\widehat{h}_l^{(1)}$	0.2	0	0.2	0	0.3	0.3

.

Using the recursion formulas of Theorem 4.3 we have

$$\begin{aligned}\widetilde{\psi}^{(0)}(1,2) &= \tilde{\psi}^{(0)}(1,1) + \sum_{k=-\infty}^{1} \tilde{\psi}^{(1)}(2-k,1)h_k^{(0)} \\ &= 1 - H^{(0)}(1) + (1 - H^{(1)}(2))h_0^{(0)} + (1 - H^{(1)}(1))h_1^{(0)}.\end{aligned}$$

After putting numerical values we obtain the required ruin probability

$$\psi_{\mathbb{Q}}(0.5,2) = \tilde{\psi}^{(0)}(1,2) = 0.2 + 0.3 \cdot 0.6 + 0.6 \cdot 0.2 = 0.5.$$

On the other hand, according to Theorem 4.7, the finite time ruin probability $\psi_{\mathbb{Q}}(0.5,2)$ is equal to the finite time ruin probability $\psi^{(0)}(1,4)$ for the discrete time risk model with claims sequence $\widehat{Z}_1, \widehat{Z}_2, \widehat{Z}_3, \widehat{Z}_4, \ldots,$ in which every fifth claim distribution is repeating and the first four claims are distributed according to the following laws:

$\widehat{Z}_1$:

l	0	1	2	3
$h_l^{(0)}$	0.6	0.2	0.1	0.1

, $\widehat{Z}_2$:

l	0
$h_l^{(1)}$	1

, $\widehat{Z}_3$:

l	0
$h_l^{(2)}$	1

,

$\widehat{Z}_4$:

l	0	1	2	3	4	5
$h_l^{(3)}$	0.2	0	0.2	0	0.3	0.3

.

Using the recursion formulas from Theorem 4.1 we obtain

$$\begin{aligned}\psi_{\mathbb{Q}}(0.5,2) &= \psi^{(0)}(1,4) = \psi^{(0)}(1,1) + \psi^{(1)}(2,3)h_0^{(0)} + \psi^{(1)}(1,3)h_1^{(0)} \\ &= \psi^{(0)}(1,1) + h_0^{(0)}\left(\psi^{(2)}(3,2) + \psi^{(1)}(2,1)\right) \\ &\quad + h_1^{(0)}\left(\psi^{(1)}(1,1) + \psi^{(2)}(2,2)\right) \\ &= \psi^{(0)}(1,1) + h_0^{(0)}\left(\psi^{(2)}(3,1) + \psi^{(3)}(4,1)\right) \\ &\quad + h_1^{(0)}\left(\psi^{(2)}(2,1) + \psi^{(3)}(3,1)\right) \\ &= 0.2 + 0.6 \cdot (0 + 0.3) + 0.2 \cdot (0 + 0.6) = 0.5\end{aligned}$$

To sum up, both ways provide the same result, however, due to smaller number of iterations, the faster way of calculating the finite time ruin probability is given by the algorithm provided in Theorem 4.5.

Finally, for the described model we present the Table 4.3 of finite time ruin probabilities $\psi_{\mathbb{Q}}(u,t)$ for initial capitals $0, 0.5, 1, \ldots, 4$ and time periods $1, 2, \ldots, 8$.

So in this chapter we have stated and proved the theorems about finite time ruin probability. In the next chapter we will present the theorems about infinite time ruin probability with proof and example.

Initial capital	Time period t							
u	1	2	3	4	5	6	7	8
0.0	0.4000	0.7600	0.7720	0.7900	0.8007	0.8201	0.8266	0.8387
0.5	0.2000	0.5000	0.5400	0.6108	0.6283	0.6607	0.6721	0.6935
1.0	0.1000	0.2200	0.2880	0.3918	0.4205	0.4722	0.4890	0.5208
1.5	0.0000	0.0900	0.1340	0.2120	0.2451	0.3031	0.3236	0.3624
2.0	0.0000	0.0300	0.0510	0.1092	0.1345	0.1839	0.2040	0.2425
2.5	0.0000	0.0000	0.0120	0.0441	0.0614	0.0997	0.1163	0.1498
3.0	0.0000	0.0000	0.0030	0.0147	0.0250	0.0491	0.0613	0.0873
3.5	0.0000	0.0000	0.0000	0.0045	0.0089	0.0217	0.0295	0.0472
4.0	0.0000	0.0000	0.0000	0.0009	0.0025	0.0086	0.0128	0.0236

Table 4.3: Finite time ruin probabilities of the model with rational-valued claims.

5

Investigation of the Infinite Time Ruin Probability

5.1 Infinite time ruin probability

In Chapter 4, we obtained the following recursive finite time ruin probability formulas:

$$\psi^{(j)}(u,1) = 1 - H^{(j)}(u), \tag{5.1}$$

$$\psi^{(j)}(u,t) = \psi^{(j)}(u,1) + \sum_{k=0}^{u} \psi^{(j+1)}(u+1-k,t-1)\ h_k^{(j)},\ t = 2,3,\ldots \tag{5.2}$$

In this chapter, the discrete time risk model with inhomogeneous claims will be used as it is defined in Chapter 3 and the infinite time ruin probability, or the ultimate ruin probability, will be investigated in more details:

$$\psi(u) = \mathbb{P}\left(T_u < \infty\right) = \lim_{t\to\infty} \psi(u,t).$$

The explicit expression of the infinite time ruin probability in a nonidentically distributed claims model is not investigated in the literature. This section deals with the problem of deriving the infinite time ruin probability of the model with nonidentically distributed claims and gives the partial formula how to calculate it.

Theorem 5.1. *Consider the discrete time risk model with nonnegative independent nonidentically distributed claims as described in Section 3. Then the infinite time ruin probabilities*

$$\psi^{(j)}(u) = \mathbb{P}\left(u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0, \quad \textit{for some } n \in \mathbb{N}\right) \tag{5.3}$$

for all $j \in \mathbb{N}_0$ and $u \in \mathbb{N}_0$, satisfy the following equation

$$\begin{aligned}\psi^{(j)}(u) &= \psi^{(j)}(0) + \sum_{r=1}^{u}(\psi^{(j)}(r) - \psi^{(j+1)}(r)H^{(j)}(u-r)) \\ &- \sum_{r=0}^{u-1}(1 - H^{(j)}(r)). \end{aligned} \tag{5.4}$$

Proof. The following equations are obtained directly from the infinite time ruin probability definition (3.4) and probability properties:

$$\begin{aligned}\psi^{(j)}(u) &= \mathbb{P}\left(u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0 \text{ for some } n \in \mathbb{N}\right) \\ &= \mathbb{P}\left(u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0 \text{ for some } n \in \mathbb{N},\ Z_{1+j} \leqslant u\right) \\ &\quad + \mathbb{P}\left(u + n - \sum_{i=1}^{n} Z_{i+j} \leqslant 0 \text{ for some } n \in \mathbb{N},\ Z_{1+j} > u\right) \\ &= \sum_{k=1}^{u} \mathbb{P}\left(u+n-\sum_{i=1}^{n} Z_{i+j} \leqslant 0 \quad \text{for some } n \in \mathbb{N},\ Z_{1+j}=k\right) \\ &\quad + \mathbb{P}(Z_{1+j} > u) = 1 - H^{(j)}(u) \\ &\quad + \sum_{k=1}^{u} \mathbb{P}\left(u+n-k-\sum_{i=2}^{n} Z_{i+j} \leqslant 0 \quad \text{for some } n \geqslant 2,\ Z_{1+j}=k\right). \end{aligned}$$

Claims $Z_{1+j}, Z_{2+j}, Z_{3+j}, \ldots$ are independent and the distribution of a claim sequence $Z_{1+(j+1)}$,
$Z_{2+(j+1)}, Z_{3+(j+1)}, \ldots$ coincides with a distribution at a sequence Z_{2+j}, Z_{3+j}, $Z_{4+j}, \ldots$. Therefore

$$\begin{aligned} &\psi^{(j)}(u) = 1 - H^{(j)}(u) \\ &+ \sum_{k=1}^{u} \mathbb{P}\left(u + n - k - \sum_{i=2}^{n} Z_{i+j} \leqslant 0 \quad \text{for some } n \geqslant 2\right) \mathbb{P}(Z_{1+j} = k) \\ = \ & 1 - H^{(j)}(u) \\ &+ \sum_{k=1}^{u} \mathbb{P}\left(u + n - k - \sum_{i=1}^{n-1} Z_{i+(j+1)} \leqslant 0 \quad \text{for some } n \geqslant 2\right) h_k^{(j)} \end{aligned}$$

$$\begin{aligned}
&= 1 - H^{(j)}(u) \\
&\quad + \sum_{k=1}^{u} \mathbb{P}\left(u + 1 - k + m - \sum_{i=1}^{m} Z_{i+(j+1)} \leqslant 0 \quad \text{for some } m \in \mathbb{N} \right) h_k^{(j)} \\
&= \sum_{k=1}^{u} \psi^{(j+1)}(u+1-k)\, h_k^{(j)} + 1 - H^{(j)}(u) \\
&= \sum_{r=1}^{u+1} \psi^{(j+1)}(r)\, h_{u+1-r}^{(j)} + 1 - H^{(j)}(u). \qquad (5.5)
\end{aligned}$$

Using equation (5.5) and summing up the infinite time ruin probabilities with the initial capitals from 0 to w, we obtain the following equality

$$\begin{aligned}
\sum_{u=0}^{w} \psi^{(j)}(u) &= \sum_{u=0}^{w} \sum_{r=1}^{u+1} h_{u+1-r}^{(j)}\, \psi^{(j+1)}(r) + \sum_{u=0}^{w} (1 - H^{(j)}(u)) \\
&= \sum_{r=1}^{w+1} \psi^{(j+1)}(r) \sum_{u=r-1}^{w} h_{u+1-r}^{(j)} + \sum_{u=0}^{w} (1 - H^{(j)}(u)) \\
&= \sum_{r=1}^{w+1} \psi^{(j+1)}(r) H^{(j)}(w+1-r) + \sum_{u=0}^{w} (1 - H^{(j)}(u))
\end{aligned}$$

$$= \sum_{r=1}^{w} \psi^{(j+1)}(r) H^{(j)}(w+1-r) + \psi^{(j+1)}(w+1)\, h_0^{(j)} + \sum_{u=0}^{w} (1 - H^{(j)}(u)).$$

Therefore,

$$\psi^{(j+1)}(w+1)\, h_0^{(j)} = \psi^{(j)}(0) + \sum_{r=1}^{w} (\psi^{(j)}(r) - \psi^{(j+1)}(r)\, H^{(j)}(w+1-r)) - \sum_{u=0}^{w} (1 - H^{(j)}(u)).$$

Moreover, it follows from (5.5) that

$$\begin{aligned}
\psi^{(j)}(w) &= \sum_{r=1}^{w+1} \psi^{(j+1)}(r)\, h_{w+1-r}^{(j)} + 1 - H^{(j)}(w) \\
&= h_0^{(j)} \psi^{(j+1)}(w+1) + \sum_{r=1}^{w} h_{w+1-r}^{(j)}\, \psi^{(j+1)}(r) + 1 - H^{(j)}(w).
\end{aligned}$$

Hence

$$h_0^{(j)} \psi^{(j+1)}(w+1) = \psi^{(j)}(w) - \sum_{r=1}^{w} h_{w+1-r}^{(j)}\, \psi^{(j+1)}(r) - (1 - H^{(j)}(w)),$$

and

$$\begin{aligned}
\psi^{(j)}(w) &= \psi^{(j)}(0) + \sum_{r=1}^{w} (\psi^{(j)}(r) - \psi^{(j+1)}(r)\, H^{(j)}(w+1-r)) \\
&\quad + \sum_{r=1}^{w} h_{w+1-r}^{(j)}\, \psi^{(j+1)}(r) - \sum_{r=0}^{w} (1 - H^{(j)}(r)) + (1 - H^{(j)}(w)) \\
&= \psi^{(j)}(0) + \sum_{r=1}^{w} (\psi^{(j)}(r) - \psi^{(j+1)}(r)\, H^{(j)}(w+1-r) \\
&\quad + h_{w+1-r}^{(j)}\, \psi^{(j+1)}(r)) - \sum_{r=0}^{w-1} (1 - H^{(j)}(r)) \\
&= \psi^{(j)}(0) + \sum_{r=1}^{w} (\psi^{(j)}(r) - \psi^{(j+1)}(r)\, H^{(j)}(w-r)) - \sum_{r=0}^{w-1} (1 - H^{(j)}(r)).
\end{aligned}$$

The last expression represents the statement of the Theorem 5.1. □

5.2 Ballot problem

The obtained formula (5.4) for the infinite time ruin probability enables to calculate it with a restriction, that the infinite time ruin probability with initial capital $u = 0$, i.e. $\psi^{(j)}(0)$, is known. This problem is called "ballot problem" and can be solved in several ways. For example, the $\psi^{(j)}(0)$ can be evaluated by the finite time ruin probability $\psi^{(j)}(0, t)$, allowing t to approach to infinity (or stopping that process when the desired accuracy is reached). In addition, some restrictions on claim distributions should be assumed in order the convergence would be consistent, for example the cyclical inhomogeneous claims pattern. In such a case, cyclical inhomogeneous claims pattern ensures that ruin probabilities repeat cyclically as well.

We observe that for the independent inhomogeneous claim sequence $Z_1, Z_2, Z_3, \ldots$ with cycle k the infinite time ruin probability may be evaluated. We say, that claim sequence has cycle k, if:

$$Z_{i+nk} \stackrel{d}{=} Z_{i+(n+1)k} \text{ for each } i = 0, \ldots, k-1,\ n \in \mathbb{N}\ .$$

5.3 Example

In this example, the claim sequence $Z_1, Z_2, Z_3, \dots$ with cycle $k = 3$ is considered. In such a case, for all $u \geqslant 0$ and $k \in \mathbb{N}$

$\psi^{(0)}(u) = \psi^{(3k)}(u)$, $\psi^{(1)}(u) = \psi^{(1+3k)}(u)$ and $\psi^{(2)}(u) = \psi^{(2+3k)}(u)$.

The special case for $\psi^{(0)}(2)$ is presented below in details. From formula (5.4) we obtain:

$$\begin{aligned}\psi^{(2)}(2) &= \psi^{(2)}(0) + \psi^{(2)}(2) - \psi^{(0)}(2)H^{(2)}(0) \\ &\quad +\psi^{(2)}(1) - \psi^{(0)}(1)H^{(2)}(1) - \sum_{r=0}^{1}(1 - H^{(2)}(r)),\end{aligned}$$

if $H^{(j)}(0) > 0$, $j = 0, 1, 2$, then

$$\psi^{(0)}(2) = \frac{1}{H^{(2)}(0)}\left(\psi^{(2)}(0) + \psi^{(2)}(1) - \psi^{(0)}(1)H^{(2)}(1) - \sum_{r=0}^{1}(1 - H^{(2)}(r))\right),$$

Analogously:

$$\psi^{(0)}(1) = \frac{1}{H^{(2)}(0)}\left(\psi^{(2)}(0) - (1 - H^{(2)}(0))\right)$$

and

$$\psi^{(2)}(1) = \frac{1}{H^{(1)}(0)}\left(\psi^{(1)}(0) - (1 - H^{(1)}(0))\right).$$

Let r.v. Z_1 and Z_2 have distributions

Z_1 :

k	0	1
$h_k^{(0)}$	0.5	0.5

, Z_2 :

k	0	1	2	3	4
$h_k^{(1)}$	0.8	0	0	0	0.2

.

and Z_3 is distributed according Poisson law with $\lambda = 0.7$.

Using (5.1) and (5.2) finite time ruin probability formulas we obtain the finite time ruin probability values, which are presented in Table 5.1. It follows from this, that infinite time ruin probabilities when $u = 0$ may be evaluated as: $\psi^{(0)}(0) = 0.725268$, $\psi^{(1)}(0) = 0.56957$ and $\psi^{(2)}(0) = 0.705153$. After obtaining the marginal $\psi^{(j)}(0)$ values, the necessary numerical $\psi^{(0)}(u)$, $\psi^{(1)}(u)$ and $\psi^{(2)}(u)$ values are found and presented in Table 5.2.

t	$\psi^{(0)}(0,t)$	$\psi^{(1)}(0,t)$	$\psi^{(2)}(0,t)$
1	0.5	0.2	0.503415
2	0.6	0.324644	0.503415
3	0.613657	0.324644	0.602732
4	0.613657	0.459715	0.610656
5	0.671062	0.465192	0.610656
...	...	...	...
199	0.725268	0.569578	0.705153
200	0.725268	0.569578	0.705153

Table 5.1: Finite time ruin probability values.

u	$\psi^{(0)}(u)$	$\psi^{(1)}(u)$	$\psi^{(2)}(u)$
0	0.725268	0.569578	0.705153
1	0.406251	0.450536	0.461972
2	0.332169	0.361965	0.313171
3	0.229845	0.302373	0.202456
4	0.130614	0.157318	0.127967
5	0.085316	0.103909	0.081154
6	0.054489	0.066723	0.051594
7	0.034537	0.042255	0.03279
8	0.021937	0.026819	0.020828
9	0.013929	0.017055	0.013235
10	0.008863	0.010802	0.008421

Table 5.2: Infinite time ruin probability values.

6

Investigation of the Gerber-Shiu Function

In this chapter, Gerber-Shiu or so called Expected Discounted Penalty function is analyzed. The results may be summarized the following. Firstly, the recursive formulas for finite and infinite time function values are found. Secondly, the explicit formula for finding Gerber-Shiu value for zero initial capital is given, or so called ballot problem is solved. In the next two parts the direct guidelines how finite and infinite time Gerber-Shiu function values might be calculated practically are given. And finally, in the last part, the numerical examples with figures illustrating Gerber-Shiu function behavior under various conditions are presented.

6.1 Finite and infinite time Gerber-Shiu function

Working in the environment of the discrete time risk model with inhomogeneous claims, described in Chapter 3, below we present three theorems with proof about Gerber-Shiu function evaluation, which may be used for finding numerical values.

Theorem 6.1. *Consider the discrete time risk model with nonnegative independent nonidentically distributed claims as described in Section 3. Then the infinite time Gerber-Shiu function* $\phi_\delta^{(j)}(u)$ *for all* $j \in \mathbb{N}_0,\ u \in \mathbb{N}_0$ *and*

$\delta \geqslant 0$ *satisfies the following equation:*

$$\phi_\delta^{(j+1)}(u)H^{(j)}(0) = e^\delta \phi_\delta^{(j)}(0) + \sum_{r=1}^{u-1}\left(e^\delta \phi_\delta^{(j)}(r) - \phi_\delta^{(j+1)}(r)H^{(j)}(u-r)\right) - \sum_{r=0}^{u-1}\left(1 - H^{(j)}(r)\right). \quad (6.1)$$

Proof. The following equations follow directly from the definition of the Gerber-Shiu function, because the claims $Z_{1+j}, Z_{2+j}, Z_{3+j}, \ldots$ are independent and the distribution of the claim sequence $Z_{1+(j+1)}, Z_{2+(j+1)}, Z_{3+(j+1)}, \ldots$ coincides with a distribution of a sequence $Z_{2+j}, Z_{3+j}, Z_{4+j}, \ldots$.

$$\phi_\delta^{(j)}(u) = \mathbb{E}\left[e^{-\delta T_u^{(j)}} \mathbb{I}_{\{T_u^{(j)}<\infty\}}\right] = \mathbb{E}\left[e^{-\delta T_u^{(j)}} \sum_{m=1}^{\infty} \mathbb{I}_{\{T_u^{(j)}=m\}}\right]$$

$$= \sum_{m=1}^{\infty} \mathbb{E}\left[e^{-\delta m}\mathbb{I}_{\{T_u^{(j)}=m\}}\right] = \sum_{m=1}^{\infty} e^{-\delta m}\mathbb{P}\left(S_1^{(j)}< u, \ldots, S_{m-1}^{(j)} < u, S_m^{(j)} \geqslant u\right)$$

$$= e^{-\delta}\mathbb{P}\left(S_1^{(j)} \geqslant u\right) + \sum_{m=2}^{\infty} e^{-\delta m}$$

$$\cdot \; \mathbb{P}\Big(S_1^{(j)}< u, S_2^{(j)}-S_1^{(j)}< u-S_1^{(j)}, ..., S_{m-1}^{(j)}-S_1^{(j)}< u-S_1^{(j)}, S_m^{(j)}-S_1^{(j)}\geqslant u-S_1^{(j)}\Big)$$

$$= e^{-\delta}\mathbb{P}\left(S_1^{(j)} \geqslant u\right) + \sum_{m=2}^{\infty}\sum_{k=0}^{u} e^{-\delta m} h_k^{(j)}$$

$$\cdot \; \mathbb{P}\left(S_2^{(j)}-S_1^{(j)}< u-k+1, \ldots, S_{m-1}^{(j)}-S_1^{(j)}< u-k+1, S_m^{(j)}-S_1^{(j)}\geqslant u-k+1\right)$$

$$= e^{-\delta}\mathbb{P}\left(S_1^{(j)} \geqslant u\right) + e^{-\delta}\sum_{k=0}^{u} h_k^{(j)} \sum_{l=1}^{\infty} e^{-\delta l}$$

$$\cdot \; \mathbb{P}\left(S_1^{(j+1)}< u-k+1, \ldots, S_{l-1}^{(j+1)} < u-k+1, S_l^{(j+1)} \geqslant u-k+1\right)$$

$$= e^{-\delta}\mathbb{P}\left(S_1^{(j)} \geqslant u\right) + e^{-\delta}\sum_{k=0}^{u} h_k^{(j)} \phi_\delta^{(j+1)}(u-k+1)$$

$$= e^{-\delta}\mathbb{P}\left(S_1^{(j)} \geqslant u\right) + e^{-\delta}\sum_{r=1}^{u+1} h_{u-r+1}^{(j)} \phi_\delta^{(j+1)}(r). \quad (6.2)$$

Hence,

$$\begin{aligned}\sum_{u=0}^{w}\phi_\delta^{(j)}(u) &= \sum_{u=0}^{w} e^{-\delta}\mathbb{P}\left(S_1^{(j)} \geqslant u\right) + e^{-\delta}\sum_{u=0}^{w}\sum_{r=1}^{u+1} h_{u-r+1}^{(j)}\phi_\delta^{(j+1)}(r) \\ &= e^{-\delta}\sum_{u=0}^{w}\left(1 - H^{(j)}(u)\right) + e^{-\delta}\sum_{r=1}^{w+1}\phi_\delta^{(j+1)}(r)\sum_{u=r-1}^{w} h_{u-r+1}^{(j)} \\ &= e^{-\delta}\sum_{u=0}^{w}\left(1-H^{(j)}(u)\right) + e^{-\delta}\sum_{r=1}^{w}\phi_\delta^{(j+1)}(r)H^{(j)}(w-r+1) \\ &\quad + e^{-\delta}\phi_\delta^{(j+1)}(w+1)h_0^{(j)}.\end{aligned}$$

Therefore,

$$\begin{aligned}e^{-\delta}\phi_\delta^{(j+1)}(w+1)h_0^{(j)} &= \phi_\delta^{(j)}(0) + \sum_{r=1}^{w}\left(\phi_\delta^{(j)}(r) - e^{-\delta}\phi_\delta^{(j+1)}(r)H^{(j)}(w-r+1)\right) \\ &\quad - e^{-\delta}\sum_{r=0}^{w}\left(1 - H^{(j)}(r)\right).\end{aligned}$$

In addition, (6.2) implies that

$$e^{-\delta}h_0^{(j)}\phi_\delta^{(j+1)}(w+1) = \phi_\delta^{(j)}(w) - e^{-\delta}\sum_{r=1}^{w} h_{w-r+1}^{(j)}\phi_\delta^{(j+1)}(r) - e^{-\delta}\left(1 - H^{(j)}(w)\right).$$

Hence,

$$\begin{aligned}\phi_\delta^{(j)}(w) &= e^{-\delta}\sum_{r=1}^{w} h_{w-r+1}^{(j)}\phi_\delta^{(j+1)}(r) + e^{-\delta}\left(1 - H^{(j)}(w)\right) + \phi_\delta^{(j)}(0) \\ &\quad + \sum_{r=1}^{w}\left(\phi_\delta^{(j)}(r) - e^{-\delta}\phi_\delta^{(j+1)}(r)H^{(j)}(w-r+1)\right) - e^{-\delta}\sum_{r=0}^{w}\left(1 - H^{(j)}(r)\right) \\ &= \phi_\delta^{(j)}(0) + \sum_{r=1}^{w}\left(\phi_\delta^{(j)}(r) - e^{-\delta}\phi_\delta^{(j+1)}(r)H^{(j)}(w-r)\right) \\ &\quad - e^{-\delta}\sum_{r=0}^{w-1}\left(1 - H^{(j)}(r)\right).\end{aligned}$$

The statement of Theorem 6.1 follows from the expression above. □

Remark 6.2. . *If $\delta = 0$, then from the theorem we may obtain a recursive formula for the infinite time ruin probability $\psi^{(j)}(u) = \mathbb{P}(T_u^{(j)} < \infty)$. Namely, for $u = 0, 1, \ldots$ and $j = 0, 1, \ldots$ we have*

$$\begin{aligned}\psi^{(j+1)}(u)H^{(j)}(0) &= \psi^{(j)}(0)+\sum_{r=1}^{u-1}\Big(\psi^{(j)}(r)-\psi^{(j+1)}(r)H^{(j)}(u-r)\Big)\\ &\quad -\sum_{r=0}^{u-1}\Big(1-H^{(j)}(r)\Big).\end{aligned}$$

This formula was obtained in [7].

Theorem 6.3. *Consider the discrete time risk model with nonnegative independent nonidentically distributed claims as described above. Then the finite time Gerber-Shiu function for all* $j\in\mathbb{N}_0$ *,* $u\in\mathbb{N}_0$ *and* $\delta\geqslant 0$ *satisfies the following equations:*

$$\phi_\delta^{(j)}(u,1) = e^{-\delta}\Big(1-H^{(j)}(u)\Big), \tag{6.3}$$

$$\phi_\delta^{(j)}(u,t) = \phi_\delta^{(j)}(u,1)+e^{-\delta}\sum_{k=0}^{u}\phi_\delta^{(j+1)}(u+1-k,t-1)\ h_k^{(j)},\ t=2,3,\ldots \tag{6.4}$$

Proof. The proof of this theorem refers to the probability properties and direct definition of finite time Gerber-Shiu function. First we observe that

$$\phi_\delta^{(j)}(u,1)=\mathbb{E}\left[e^{-\delta T_u^{(j)}}\mathbb{1}_{\{T_u^{(j)}\leqslant 1\}}\right]=e^{-\delta}(1-H^{(j)}(u)).$$

Then

$$\begin{aligned}\phi_\delta^{(j)}(u,t)&=\mathbb{E}\left[e^{-\delta T_u^{(j)}}\mathbb{1}_{\{T_u^{(j)}\leqslant t\}}\right]=\mathbb{E}\left[e^{-\delta T_u^{(j)}}\sum_{m=1}^{t}\mathbb{1}_{\{T_u^{(j)}=m\}}\right]\\
&= \sum_{m=1}^{t}\mathbb{E}\left[e^{-\delta T_u^{(j)}}\mathbb{1}_{\{S_1^{(j)}<u,\ldots,S_{m-1}^{(j)}<u,S_m^{(j)}\geqslant u\}}\right]=e^{-\delta}\mathbb{P}\left(S_1^{(j)}\geqslant u\right)+\sum_{m=2}^{t}e^{-\delta m}\\
&\cdot\mathbb{P}\left(S_1^{(j)}<u,S_2^{(j)}-S_1^{(j)}<u-S_1^{(j)},\ldots,S_{m-1}^{(j)}-S_1^{(j)}<u-S_1^{(j)},S_m^{(j)}-S_1^{(j)}\geqslant u-S_1^{(j)}\right)\\
&= \phi_\delta^{(j)}(u,1)+\mathcal{K},\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K} &= \sum_{m=2}^{t}\sum_{k=0}^{u} e^{-\delta m} h_k^{(j)} \\
&\cdot \mathbb{P}\left(k<u+1, S_2^{(j)}-S_1^{(j)}<u-k+1, ..., S_{m-1}^{(j)}-S_1^{(j)}<u-k+1, S_m^{(j)}-S_1^{(j)}\geqslant u-k+1\right) \\
&= e^{-\delta}\sum_{k=0}^{u} h_k^{(j)} \sum_{l=1}^{t-1} e^{-\delta l}\mathbb{P}\left(S_1^{(j+1)}< u+1-k, ..., S_{l-1}^{(j+1)}< u+1-k, S_l^{(j+1)}\geqslant u+1-k\right) \\
&= e^{-\delta}\sum_{k=0}^{u} \phi_\delta^{(j+1)}(u+1-k, t-1)\ h_k^{(j)}.
\end{aligned}$$

Hence,

$$\phi_\delta^{(j)}(u,t) = \phi_\delta^{(j)}(u,1) + e^{-\delta}\sum_{k=0}^{u}\phi_\delta^{(j+1)}(u+1-k, t-1)\ h_k^{(j)}.$$

The theorem is proved.

□

Remark 6.4. *When $\delta = 0$ the finite time Gerber-Shiu function coincides with the finite time ruin probability, hence,*

$$\begin{aligned}
\psi^{(j)}(u,1) &= 1 - H^{(j)}(u), \\
\psi^{(j)}(u,t) &= \psi^{(j)}(u,1) + \sum_{k=0}^{u}\psi^{(j+1)}(u+1-k, t-1)\ h_k^{(j)},\ t = 2, 3, \ldots.
\end{aligned}$$

for all $j \in \mathbb{N}_0$ and $u \in \mathbb{N}_0$ This formula can be found in [8].

Note that for the calculation of the infinite time Gerber-Shiu function using recursive formula in Theorem 6.1 we need its value at $u = 0$. We propose two ways of finding the initial Gerber-Shiu function value at $u = 0$.

First way is to approximate it by the finite time Gerber-Shiu function:

$$\phi_\delta^{(j)}(0) = \lim_{t\to\infty}\phi_\delta^{(j)}(0,t), \quad j \in \mathbb{N}_0, \quad \delta \geqslant 0.$$

Second way is to calculate $\phi_\delta^{(j)}(0)$ directly and the assertion below gives the explicit expression.

Theorem 6.5. *Consider the discrete time risk model with nonnegative independent nonidentically distributed claims as described above. Then*

$$\begin{aligned}\phi_\delta^{(j)}(0) &= e^{-\delta}\left(1-H^{(j)}(0)\right)+e^{-2\delta}\left(h_0^{(j)}(1-H^{(j+1)}(1))\right)\\ &+\sum_{m=1}^{\infty}e^{-\delta(m+2)}\sum_{\substack{i_1,i_2,\ldots,i_m\in\{0,1,\ldots\}\\ i_1\leqslant 1,i_1+i_2\leqslant 2,\ldots,\\ i_1+\ldots+i_m\leqslant m}}h_0^{(j)}h_{i_1}^{(j+1)}h_{i_2}^{(j+2)}\cdots h_{i_m}^{(j+m)}\\ &\cdot\left(1-H^{(j+m+1)}\left(m+1-\sum_{n=1}^{m}i_n\right)\right),\end{aligned}\tag{6.5}$$

for all $j\in\mathbb{N}_0$ *and* $\delta\geqslant 0$.

Proof. In order to avoid too many indices we prove this theorem only in the case $j=0$. The general case with $j\in\mathbb{N}_0$ can be considered analogously. From the definitions of the time of ruin and the Gerber-Shiu function (3.5) we obtain:

$$\begin{aligned}\phi_\delta^{(0)}(0) &= \mathbb{E}\left(e^{-\delta T_0^{(0)}}\mathbb{1}_{\{T_0^{(0)}<\infty\}}\right)=\mathbb{E}\left(\sum_{m=1}^{\infty}e^{-\delta m}\mathbb{1}_{\{T_0^{(0)}=m\}}\right)\\ &= \sum_{m=1}^{\infty}e^{-\delta m}\mathbb{P}(T_0^{(0)}=m).\end{aligned}\tag{6.6}$$

We derive that

$$\begin{aligned}\mathbb{P}(T_0^{(0)}=1) &= \mathbb{P}(Z_1\geqslant 1)=\mathbb{P}(Z_1>0)=1-H^{(0)}(0),\\ \mathbb{P}(T_0^{(0)}=2) &= \mathbb{P}(Z_1+Z_2\geqslant 2, Z_1<1)=\mathbb{P}(Z_2\geqslant 2, Z_1=0)\\ &= h_0^{(0)}(1-H^{(1)}(1)),\\ \mathbb{P}(T_0^{(0)}=3) &= \mathbb{P}(Z_1+Z_2+Z_3\geqslant 3, Z_1+Z_2<2, Z_1<1)\\ &= h_0^{(0)}h_0^{(1)}(1-H^{(2)}(2))+h_0^{(0)}h_1^{(1)}(1-H^{(2)}(1)),\\ \mathbb{P}(T_0^{(0)}=4) &= \mathbb{P}(Z_1+Z_2+Z_3+Z_4\geqslant 4, Z_1+Z_2+Z_3<3, Z_1+Z_2<2, Z_1<1)\\ &= h_0^{(0)}h_0^{(1)}h_0^{(2)}(1-H^{(3)}(3))+h_0^{(0)}h_0^{(1)}h_1^{(2)}(1-H^{(3)}(2))\\ &+h_0^{(0)}h_0^{(1)}h_2^{(2)}(1-H^{(3)}(1))+h_0^{(0)}h_1^{(1)}h_0^{(2)}(1-H^{(3)}(2))\\ &+h_0^{(0)}h_1^{(1)}h_1^{(2)}(1-H^{(3)}(1)),\end{aligned}$$

and for any $m\in\mathbb{N}$

$$\mathbb{P}(T_0=m+2)=$$

$$=\sum_{\substack{i_1,i_2,\ldots,i_m\in\{0,1,\ldots\}\\ i_1\leqslant 1,i_1+i_2\leqslant 2,\ldots,\\ i_1+\ldots+i_m\leqslant m}}h_0^{(0)}h_{i_1}^{(1)}h_{i_2}^{(2)}\cdots h_{i_m}^{(m)}\left(1-H^{(m+1)}\left(m+1-\sum_{n=1}^{m}i_n\right)\right).$$

The statement of the Theorem follows from the last expression and equality (6.6). □

Remark 6.6. *The finite time ruin probability* $\psi^{(j)}(0), \quad j \in \mathbb{N}_0$ *can be found using Theorem 6.5 taking* $\delta = 0$ *in formula* (6.5).

Remark 6.7. *Using Theorem 6.1 the infinite time Gerber-Shiu function can be found only if the initial value of Gerber-Shiu function (i.e.* $u = 0$*) is known. This case is described in Theorem 6.5, therefore Theorem 6.1 in combination with Theorem 6.5 can be used for applications.*

6.2 Finding finite time Gerber-Shiu function values

The recursive formulas of the Gerber-Shiu function in the classical discrete time risk model are attractive for their easiness to use and provide accurate results much faster than other standard formulas. However, in comparison to the classical model, the finite and infinite time Gerber-Shiu function formulas for the inhomogeneous claim model have more dimensions and therefore one must be aware while using them. In this and next sections, the extensive guidelines of using these formulas are provided. First of all, we will find the values of the Gerber-Shiu function $\phi_\delta^{(0)}(u,t)$ in inhomogeneous claim case using formulas (6.3) and (6.4).

Suppose we know all local probabilities $h_k^{(j)} = \mathbb{P}\left(Z_{1+j} = k\right), k \in \mathbb{N}_0,\ j \in \mathbb{N}_0$ of the claim sequence $Z_1, Z_2, Z_3, \ldots$ (see Table 6.1).

k	0	1	2	$\ldots$
$\mathbb{P}(Z_1 = k)$	$h_0^{(0)}$	$h_1^{(0)}$	$h_2^{(0)}$	$\ldots$

,

k	0	1	2	$\ldots$
$\mathbb{P}(Z_2 = k)$	$h_0^{(1)}$	$h_1^{(1)}$	$h_2^{(1)}$	$\ldots$

,

k	0	1	2	$\ldots$
$\mathbb{P}(Z_3 = k)$	$h_0^{(2)}$	$h_1^{(2)}$	$h_2^{(2)}$	$\ldots$

, $\ldots$.

Table 6.1: Inhomogeneous claims' local probabilities.

The following algorithm can be used for finding the finite time Gerber-

Shiu function $\phi_\delta^{(0)}(u,t)$ with initial capital $u \in \mathbb{N} \cup \{0\}$, time period $t \in \mathbb{N}$ and $\delta \geqslant 0$.

Step 1. The Gerber-Shiu function in the first time period $\phi_\delta^{(j)}(y,1)$ must be calculated, for $j = 1, 2, \ldots, t-1$ and $y = 1, 2, \ldots, u+t-1$ according to formula (6.3):

$$\phi_\delta^{(j)}(y,1) = e^{-\delta}(1 - H^{(j)}(y)) = e^{-\delta} \sum_{k=y+1}^{\infty} h_k^{(j)}.$$

Step 2. Using the formula (6.4), all values of the finite time Gerber-Shiu function are found until time period $t-1$ and shifting parameter $j = 1$:

$$\begin{aligned}
\phi_\delta^{(t-2)}(y,2) &= \phi_\delta^{(t-2)}(y,1) + e^{-\delta} \sum_{k=0}^{y} \phi_\delta^{(t-1)}(y{+}1{-}k, 1)\ h_k^{(t-2)}, \\
&\quad y{=}1, \ldots, u{+}t{-}2, \\
\phi_\delta^{(t-3)}(y,3) &= \phi_\delta^{(t-3)}(y,1) + e^{-\delta} \sum_{k=0}^{y} \phi_\delta^{(t-2)}(y{+}1{-}k, 2)\ h_k^{(t-3)}, \\
&\quad y{=}1, \ldots, u{+}t{-}3, \\
&\ldots \\
\phi_\delta^{(2)}(y,t-2) &= \phi_\delta^{(2)}(y,1) + e^{-\delta} \sum_{k=0}^{y} \phi_\delta^{(3)}(y{+}1{-}k, t-3)\ h_k^{(2)}, \\
&\quad y = 1, \ldots, u+2, \\
\phi_\delta^{(1)}(y,t-1) &= \phi_\delta^{(1)}(y,1) + e^{-\delta} \sum_{k=0}^{y} \phi_\delta^{(2)}(y{+}1{-}k, t-2)\ h_k^{(1)}, \\
&\quad y = 1, \ldots, u+1.
\end{aligned}$$

Step 3. Finally, using formulas (6.3) and (6.4) we calculate the desired Gerber-Shiu function $\phi_\delta^{(0)}(u,t)$:

$$\phi_\delta^{(0)}(u,t) = e^{-\delta}(1 - H^{(0)}(u)) + e^{-\delta} \sum_{k=0}^{u} \phi_\delta^{(1)}(u+1-k, t-1)\ h_k^{(0)}.$$

We can observe that the calculation of the finite time Gerber-Shiu functions does not require any specific conditions about the distributions of arriving claim or claim structure. Therefore, any pattern of claims can be investigated. In section 6.4, some examples with various claim distributions are presented, which show the dependence of the finite time Gerber-Shiu function on the claim distributions. The implementation of this algorithm in Maple is given in Appendix A.

6.3 Finding the infinite time Gerber-Shiu function values

Similarly as in section above, assume that we have all local probabilities $h_k^{(j)} = \mathbb{P}(Z_{1+j} = k)$, $k \in \mathbb{N}_0$, $j \in \mathbb{N}_0$ of the claim sequence $Z_1, Z_2, \ldots$ with a technical restriction $h_0^{(j)} > 0, j \in \mathbb{N}_0$. The algorithm for computing the infinite time Gerber-Shiu function $\phi_\delta^{(j)}$ is very complicated and we cannot offer it for a general case. Therefore we consider only the cyclically distributed claim structure. Recall that claims $Z_1, Z_2, \ldots$ are cyclically distributed with a cycle length K, i.e. $Z_{i+nK} \stackrel{d}{=} Z_{i+(n+1)K}$ for all $i = 1, 2 \ldots, K$ and $n \in \mathbb{N}_0$.

In the described case the distribution of claim sequence $\{Z_{i+j}\}_{i\in\mathbb{N}}$ coincides with the distribution of the sequence $\{Z_{i+j+nK}\}_{i\in\mathbb{N}}$ for all fixed $j, n \in \mathbb{N}_0$. Then $\phi_\delta^{(j)}(u) = \phi_\delta^{(j+nK)}(u)$ for all $j, n, u \in \mathbb{N}_0$, in particular $\phi_\delta^{(0)}(u) = \phi_\delta^{(K)}(u)$ for $u \in \mathbb{N}_0$. The guidelines below describe how the infinite time Gerber-Shiu function $\phi_\delta^{(0)}(u)$ might be computed using formula (6.1).

Step 1. First of all, the infinite time Gerber-Shiu function values $\phi_\delta^{(j)}(0)$ for shifting parameters $j = 0, 1, \ldots, K-1$ must be found using (6.5) from Theorem 6.5.

Step 2. Having the infinite time Gerber-Shiu function for zero initial capital $\phi_\delta^{(0)}(0)$, $\phi_\delta^{(1)}(0), \ldots,$ $\phi_\delta^{(K-1)}(0)$ calculated, we are able to find the infinite time Gerber-Shiu function for the consecutive values of initial capital using formula (6.1):

$$
\begin{aligned}
\phi_\delta^{(j+1)}(1) &= \frac{1}{H^{(j)}(0)}(e^\delta \phi_\delta^{(j)}(0) - (1 - H^{(j)}(0))),\ j = 0, 1, \ldots, K-1,\\
\phi_\delta^{(0)}(1) &= \phi_\delta^{(K)}(1);\\
\phi_\delta^{(j+1)}(2) &= \frac{1}{H^{(j)}(0)}(e^\delta \phi_\delta^{(j)}(0) + e^\delta \phi_\delta^{(j)}(1) - \phi_\delta^{(j+1)}(1) H^{(j)}(1))\\
&\quad - \sum_{r=0}^{1}(1 - H^{(j)}(r))),\ j = 0, 1, \ldots, K-1,\\
\phi_\delta^{(0)}(2) &= \phi_\delta^{(K)}(2);\\
&\ldots
\end{aligned}
$$

$$\begin{aligned}
&\dots \\
\phi_\delta^{(j+1)}(u) &= \frac{1}{H^{(j)}(0)}(e^\delta \phi_\delta^{(j)}(0) + \sum_{r=1}^{u-1}(e^\delta \phi_\delta^{(j)}(r) - \phi_\delta^{(j+1)}(r) H^{(j)}(u-r)) \\
&\quad - \sum_{r=0}^{u-1}(1 - H^{(j)}(r))), \; j = 0, 1, \dots, K-1, \\
\phi_\delta^{(0)}(u) &= \phi_\delta^{(K)}(u) \,.
\end{aligned}$$

So using these guidelines the finite and infinite Gerber-Shiu function values might be found practically. The implementation of this algorithm in Maple is given in Appendices B and C.

6.4 Numerical examples

In this section, we present some examples of computing numerical values of Gerber-Shiu functions for various discrete time risk models with inhomogeneous claims.

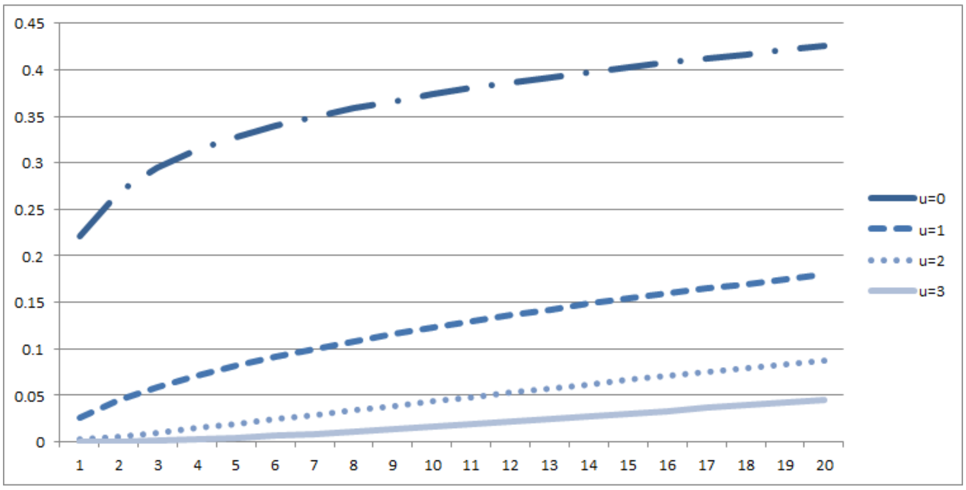


Figure 6.1: Finite time ruin probabilities for Poisson claims with means $i/(3+i)$, $i \in \mathbb{N}$. Here $t = 0, 1, \dots, 20$ and $u = 0, 1, 2, 3$.

Example 1. Let us consider the sequence of inhomogeneous claims $Z_1, Z_2, \dots$, in which claim Z_i is distributed according to the Poisson law with parameter $\lambda_i = i/(3+i)$. Hence, every claim is distributed according to the different law, however, the claim distributions are changing very slightly. In Figure 6.1 we present the special case of Gerber-Shiu function with $\delta = 0$, i.e. the finite time ruin probabilities, calculated for $u = 0, 1, 2, 3$ and with t varying from 1 to 20. Since the expectations are simultaneously increasing but not exceeding 1, the calculated finite time ruin probabilities are changing smoothly as well, without any jumps in the pattern.

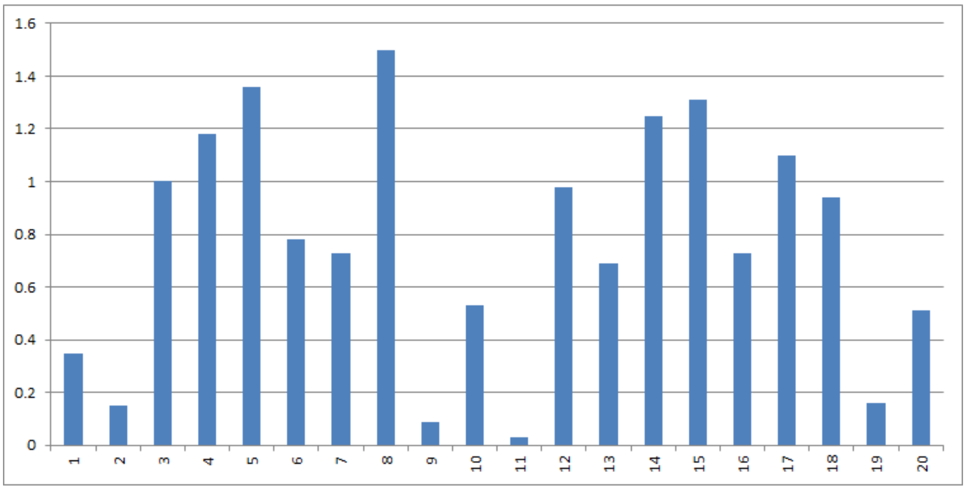


Figure 6.2: Chaotic μ parameters of the second claim sequence in Example 1.

In order to see the contrast, we consider another claim sequence $\hat{Z}_1, \hat{Z}_2, \ldots$, where each $\hat{Z}_i$ is distributed according to the Poisson law with chaotic parameters $\mu_i \in (0, 1.5)$ (see μ values in Figure 6.2). One may notice that the mean of the claim $\hat{Z}_i$ is not necessarily less than 1. Nevertheless, we are investigating finite time Gerber-Shiu function values, so such parameters are appropriate and we put them as the example. The average of the parameters in analyzed period (20 time units) is equal to 0.77. In Figure 6.3, the finite time ruin probabilities are presented, which are higher than the probabilities in Figure 6.1, and one of the reasons of this is that the average of the parameters of sequence $Z_1, Z_2, \ldots, Z_{20}$ is smaller (0.72). Moreover, since the claims are distributed according Poisson law with chaotic parameters $\mu_1, \mu_2, \ldots, \mu_{20}$, the probabilities to ruin do not smoothly increase, but have some jumps, especially in time moments where occurring claim is distributed according a law with high mean.

The main sections of this example implementation in Maple are given in Appendices D and E. The other examples may be implemented by the same algorithms with slight changes.

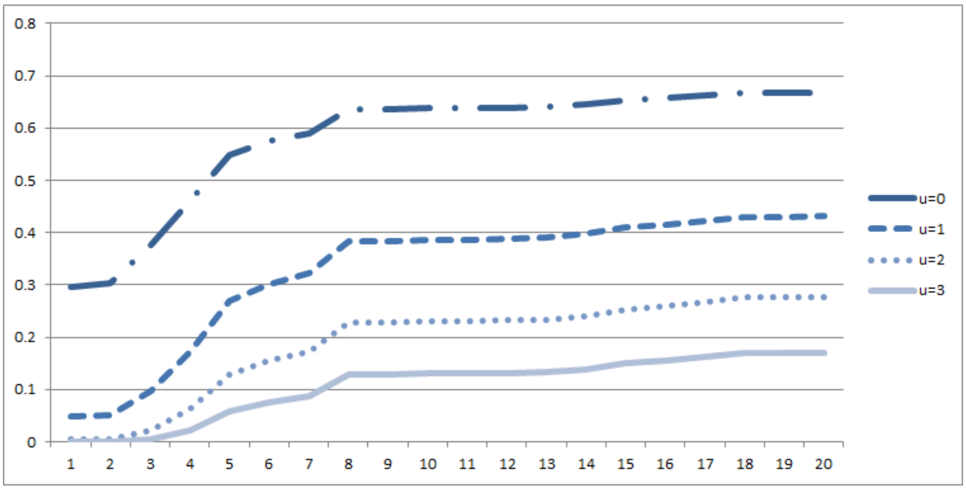


Figure 6.3: Finite time ruin probabilities for Poisson claims with chaotic means. Here $t = 1, 2, \ldots, 20$ and $u = 0, 1, 2, 3$.

Example 2. After finding the values of Gerber-Shiu function where $\delta = 0$ and u differs from 0 to 3, we now look how the finite time Gerber-Shiu function's values depend on δ chosen. The claim sequences are the same as in Example 1. The δ, which is representing the yield force in the model, is chosen to be $0, 0.01, 0.05, 0.1,$ and 0.2. The results are given in Figure 6.4 and Figure 6.5, where the discounted penalty function's sensitivity to the yield is shown.

Example 3. The third example analyzes the behavior of Gerber-Shiu function, when claims $Z_1, Z_2, \ldots$ have a cyclical pattern described in Section 4. The cycle length is selected to be equal to 3, that is each forth claim distribution is the same. Suppose, that the main triplet Z_1, Z_2, Z_3 of claim sequence $Z_1, Z_2, Z_3, Z_4, \ldots$ has the following distributions, presented in Table 6.2, with some positive integer parameter m. It is evident that all r.v.s in the cyclical varying sequence $Z_1, Z_2, \ldots$ have means $0.1m$.

k	0	m
$\mathbb{P}(Z_1 = k)$	0.9	0.1

,

k	0	$2m$
$\mathbb{P}(Z_2 = k)$	0.95	0.05

,

k	0	$4m$
$\mathbb{P}(Z_3 = k)$	0.975	0.025

Table 6.2: Local probability values. Example 3.

In Figure 6.6, the finite time Gerber-Shiu function (with $\delta = 0.05$, $u = 0$)

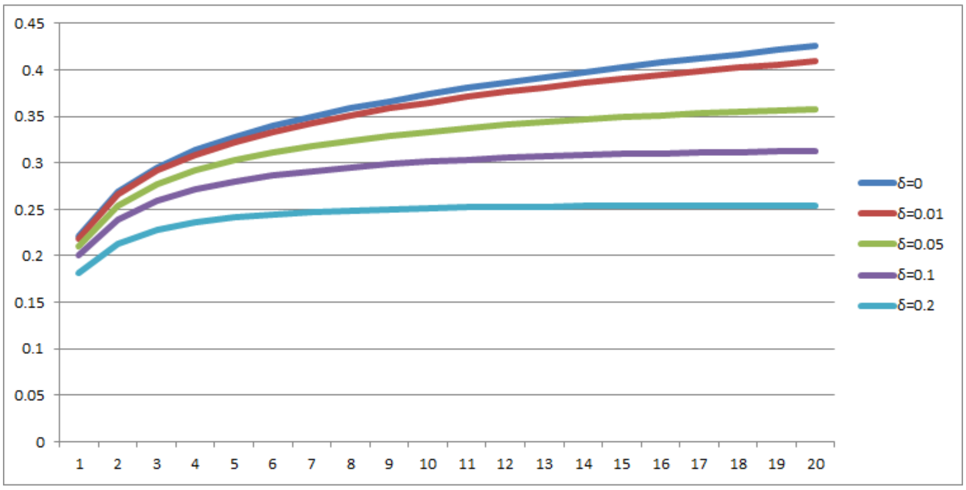


Figure 6.4: Values of the Gerber-Shiu function for Poisson claims with means $i/(3+i)$, $i \in \mathbb{N}$. Here $u = 0$, $t = 1, \ldots, 20$ and $\delta = 0$,0.01,0.05,0.1,0.2.

values are presented for the claim cycle sequences $Z_1, Z_2, \ldots$ with parameters $m = 3, 5, 7$, and 15. It is interesting to note that, even in case of arriving claims higher than premiums (case $m = 15$), the discounted penalty function in ten time periods is not higher than 0.5. This can be explained by quite low positive claim frequency (less than 10%). As expected, Gerber-Shiu function values for claims with lower means are lower as well. Moreover, finite time Gerber-Shiu function increases as the time period increases. Finally, the values of the finite time Gerber-Shiu function converge to the infinite time Gerber-Shiu function, however, the increase is not smooth due to arrived nonidentically distributed claims in each period, as it was already observed in Example 1.

Example 4. Let us consider the cyclically varying claim sequence $Z_1, Z_2, \ldots$ with a cycle length 3 and the same generating r.v. triplet Z_1, Z_2, Z_3 as in Example 3. According to the algorithm from Section 4, we can calculate the infinite time Gerber-Shiu function values $\phi_\delta^{(j)}(0)$ for all positive integer model parameter m, $\delta = 0.05$ and shifting parameter $j = 0, 1, 2$. It is well known that in the case of identically distributed claims the infinite time ruin probability for zero initial capital is equal to the claim mean (see, for example, [19], [21]). However, we can see from Figure 6.7 that in the case of inhomogeneous claims the infinite time Gerber-Shiu function values do not have this property. In addition, it is interesting to note that even if

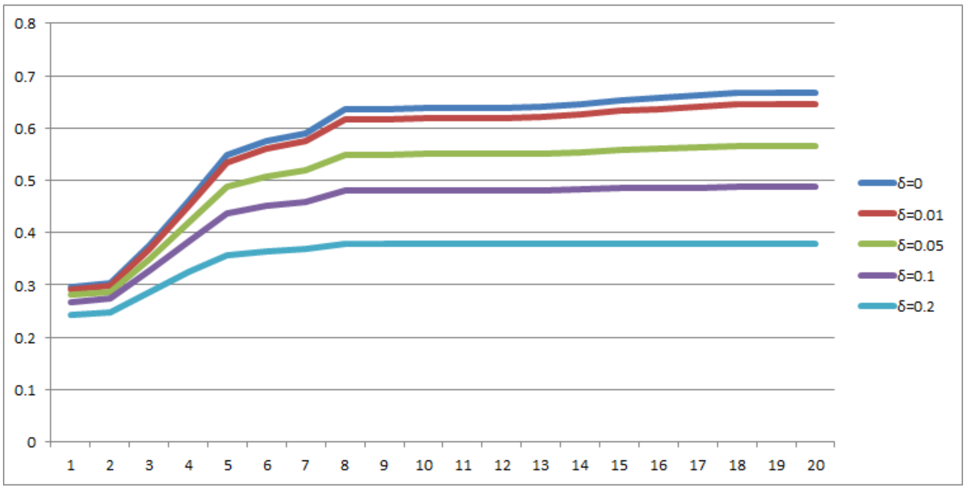


Figure 6.5: Values of the Gerber-Shiu function for Poisson claims with chaotic means. Here $u = 0$, $t = 1, 2, \ldots, 20$ and $\delta = 0,\ 0.01,\ 0.05,\ 0.1,\ 0.2$.

the means of claims are equal ($\mathbb{E}[Z]_1 = \mathbb{E}[Z]_2 = \mathbb{E}[Z]_3 = 0.1m$) the infinite time Gerber-Shiu function values differ when the claim sequence is shifted.

Example 5. In this example, the dependence of the infinite time Gerber-Shiu function on δ is shown. The cyclically varying claim sequence $Z_1, Z_2, \ldots$ from Example 3 with parameter $m = 5$, i.e. $\mathbb{E}[Z]_1 = \mathbb{E}[Z]_2 = \mathbb{E}[Z]_3 = 0.5$, is chosen. In Figure 6.8, the graphs of $\phi_\delta^{(0)}(u)$, for $\delta = 0, 0.01, 0.02, 0.05, 0.1$ are given. As it can be seen from this figure, the difference between infinite time ruin probability ($\delta = 0$) and Gerber-Shiu function value with $\delta = 0.1$ is really large, the latter is almost twice smaller.

Finally, the surfaces of the finite time Gerber-Shiu function values for various δ are given in Figure 6.9. The Figures 6.9(a), 6.9(b), 6.9(c), and 6.9(d) may be compared with each other, showing the relationship among different δ values, $\delta = 0.01, 0.05, 0.10, 0.20$.

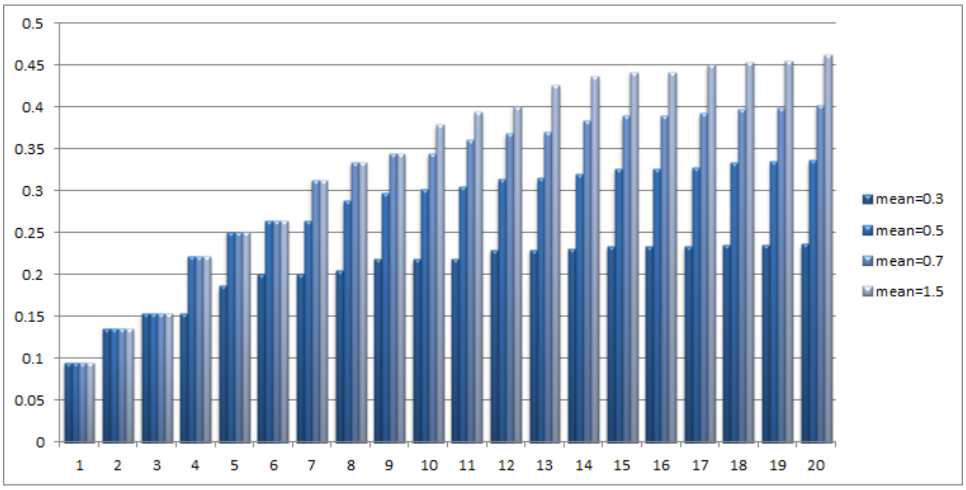


Figure 6.6: Values of the finite time Gerber-Shiu function for cyclically varying claims with coinciding means. Here: $t = 1, 2, \ldots, 20$, $u = 0$, $\delta = 0.05$

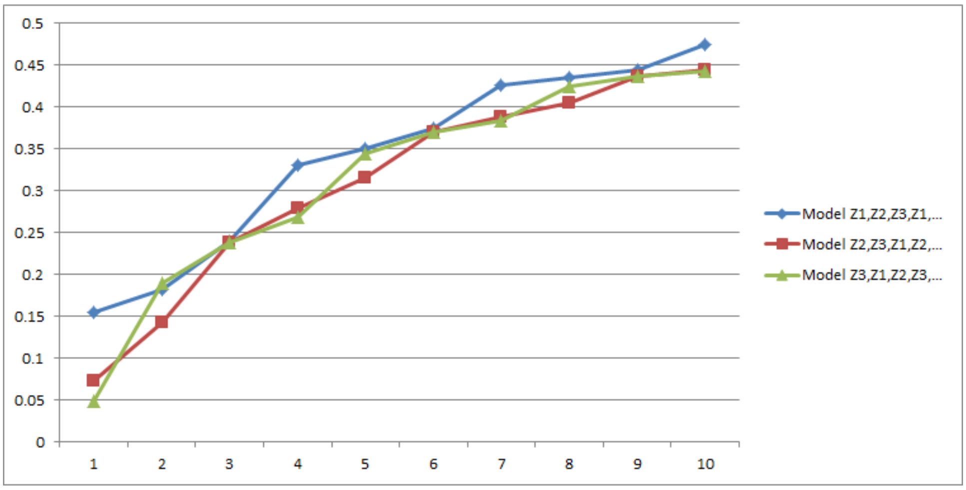


Figure 6.7: Values of $\phi_\delta^{(0)}(0)$, $\phi_\delta^{(1)}(0)$, $\phi_\delta^{(2)}(0)$ with $\delta = 0.05$ for cyclically varying claims. The mean of all claims in the sequences is $0.1\,m$ with $m = 1, 2, \ldots, 10$.

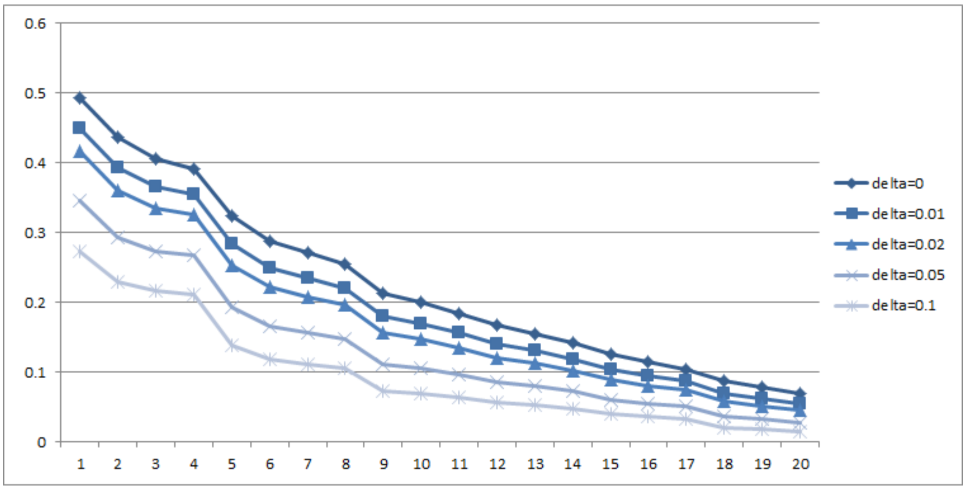
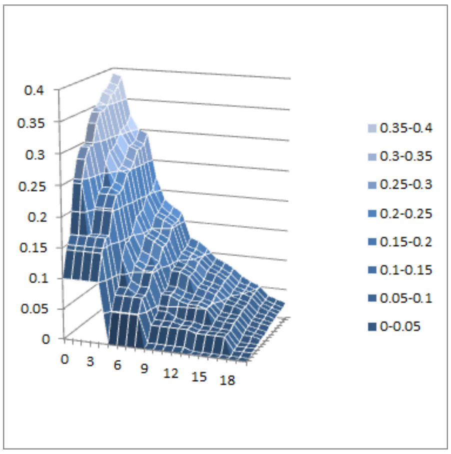
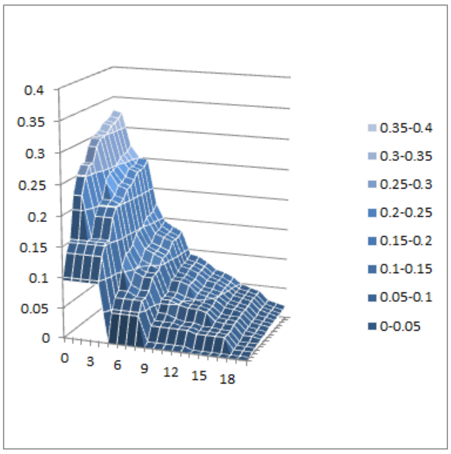


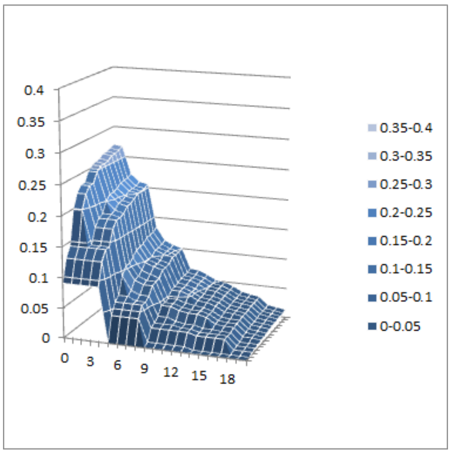
Figure 6.8: Dependence of the infinite time Gerber-Shiu function on $\delta = 0$, 0.01, 0.02, 0.05, 0.1 and $u = 0, 1, \ldots, 19$.



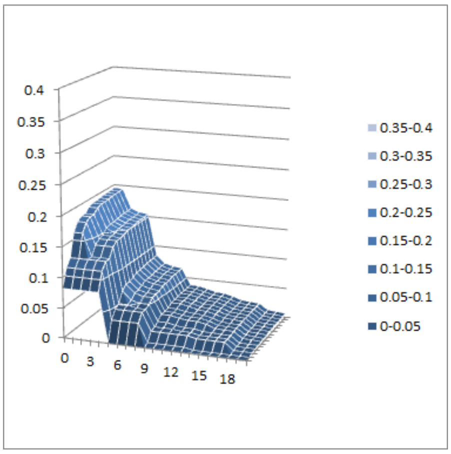
(a) $\delta = 0.01$



(b) $\delta = 0.05$



(c) $\delta = 0.10$



(d) $\delta = 0.20$

Figure 6.9: The surfaces of the finite time Gerber-Shiu function values for different δ values, $u = 0, 1, \ldots, 20$ and $t = 1, \ldots, 15$.

7

Special Case: Geometric Claim Size Distribution

For the finite horizon ruin probability $\psi(u,t)$ in the classical discrete time risk model the only example of something like an explicit expression is the compound Poisson model with constant premium rate $p = 1$ and exponential claim size distribution. However, the formulas (Asmussen, [5] IV.1) are so complicated that they should rather be viewed as basis for numerical methods than as closed-form solutions. In this chapter, we analyze the analogous example for the inhomogeneous claim case, when the claims are distributed according different geometric distributions. This case was investigated and published in [6].

The basis will be the finite time recursive ruin probability formulas derived in Chapter 4:

$$\psi^{(j)}(u,1) = 1 - H^{(j)}(u), \tag{7.1}$$

$$\psi^{(j)}(u,t) = \psi^{(j)}(u,1) + \sum_{k=0}^{u} \psi^{(j+1)}(u+1-k, t-1)\ h_k^{(j)},\ t = 2,3,... \tag{7.2}$$

for each $j, u \in \mathbb{N}_0$.

Let us assume, that the discrete time risk model's claims Z_{1+j}, $j \in \mathbb{N}_0$ are geometrically distributed with parameters $0 < q_j < 1$, i.e. local probabilities for each $k \in \mathbb{N}_0$ are

$$h_k^{(j)} = \mathbb{P}(Z_{1+j} = k) = (1 - q_j) q_j^k,$$

and distribution function for each $u \in \mathbb{N}_0$ is

$$H^{(j)}(u) = \mathbb{P}(Z_{1+j} \leqslant u) = \sum_{k=0}^{u} h_k^{(j)} = 1 - q_j^{u+1}.$$

The mean of the geometrically distributed claim is equal to

$$\mathbb{E}Z_{1+j} = \frac{q_j}{1-q_j}.$$

Now we will investigate two cases of geometric distributions, which parameters differ as time goes by and check how the finite time ruin probabilities are dependent on the parameters chosen.

7.1 Case 1: $q_{1,j} = (2+j)^{-1}, \ j = 0, 1, 2, ...$

First of all let us analyze the finite time ruin probability, when claims $\{Z_{1,1+j}\}_{j \in \mathbb{N}_0}$ are distributed according Geometric law with parameter

$$q_{1,j} = \frac{1}{2+j}.$$

In picture 7.1, one may discover, that in the first moment of time the mean of the claim is 1, i.e. the probability to ruin in the first moment of time is really high. Meanwhile in other moments the claim means are much lower and gradually decrease till zero. It intuitively follows, that the ruin probability is the highest at the first moment. Let's check that theoretically. According to (7.1) and (7.2), the ruin probabilities are:

$$\psi^{(j)}(u, 1) = 1 - H^{(j)}(u) = \frac{1}{(2+j)^{u+1}},$$

$$\psi^{(j)}(u, t) = \frac{1}{(2+j)^{u+1}} + \sum_{k=0}^{u} \psi^{(j+1)}(u+1-k, t-1) \frac{1+j}{(2+j)^{k+1}}.$$

In Risk theory the case with zero initial ($u = 0$) capital or so called ballot problem is often analyzed. Now we will analyze this case with $j = 0$, i.e. claim sequence begins from the first claim. Further this case will be called *base case*. Then the finite time ruin probability is:

$$\psi^{(0)}(0, t) = \frac{1}{2} + \sum_{k=0}^{0} \psi^{(1)}(1-k, t-1) \frac{1}{2^{k+1}} = \frac{1}{2} + \frac{1}{2}\psi^{(1)}(1, t-1).$$

7.1 Case 1: $q_{1,j} = (2+j)^{-1}, \ j = 0, 1, 2, ...$

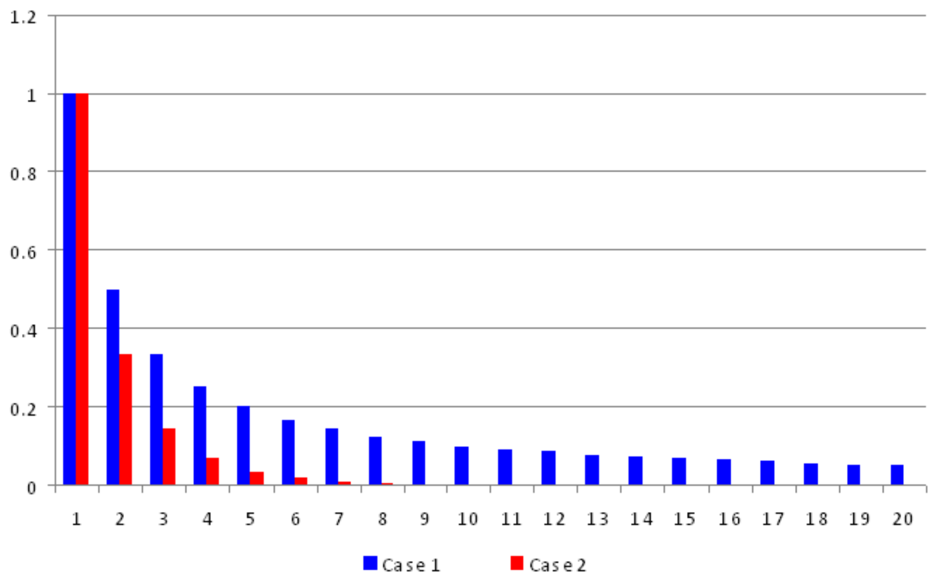


Figure 7.1: The means of the claim sequences from cases 1 and 2.

After calculating the finite time ruin probabilities for the time moments $t \in \mathbb{N}$, we can observe a clear picture how the ruin probabilities with zero initial capital change (ref. picture 7.2).

Applying Risk theory in practice, the ruin probability sensitivity to initial capital increase or decrease is often concerned. After several steps we obtain:

$$\psi^{(j)}(u+1,t) = \frac{1}{(2+j)^{u+2}} + \sum_{k=0}^{u+1} \psi^{(j+1)}(u+2-k,t-1)\frac{1+j}{(2+j)^{k+1}}$$

$$= \frac{1}{(2+j)^{u+2}} + \frac{1+j}{2+j}\psi^{(j+1)}(u+2,t-1) + \sum_{k=1}^{u+1} \psi^{(j+1)}(u+2-k,t-1)\frac{1+j}{(2+j)^{k+1}}$$

$$= \frac{1}{(2+j)^{u+2}} + \frac{1+j}{2+j}\psi^{(j+1)}(u+2,t-1) + \sum_{l=0}^{u} \psi^{(j+1)}(u+1-l,t-1)\frac{1+j}{(2+j)^{l}}$$

$$= \frac{1}{(2+j)^{u+2}} + \frac{1+j}{2+j}\psi^{(j+1)}(u+2,t-1) + (2+j)\left(\psi^{(j)}(u,t) - \frac{1}{(2+j)^{u+1}}\right).$$

Hence, for all $u, j \in \mathbb{N}_0$ and $t \in \mathbb{N}$,

$$\psi^{(j)}(u+1,t) - (2+j)\psi^{(j)}(u,t) = \frac{1+j}{2+j}\psi^{(j+1)}(u+2,t-1) + \left(\frac{1}{(2+j)^{u+2}} - \frac{1}{(2+j)^{u}}\right).$$

In the base case, when $j = 0$, this transforms to:

$$\psi^{(0)}(u+1,t) - 2\psi^{(0)}(u,t) = \frac{1}{2}\psi^{(1)}(u+2,t-1) - \frac{3}{2^{u+2}}.$$

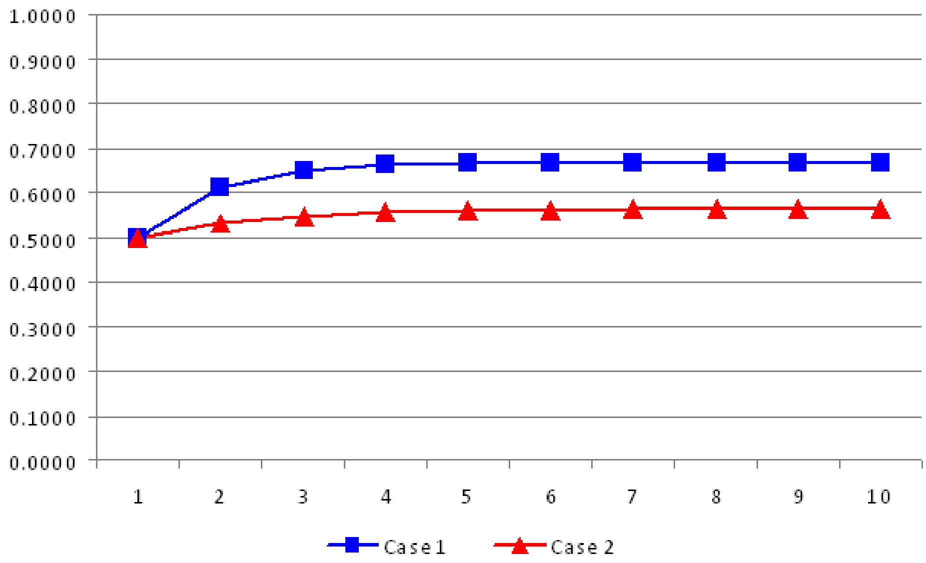


Figure 7.2: Finite time $t = 1, 2, ..., 10$ ruin probabilities with initial capital $u = 0$

7.2 Case 2: $q_{2,j} = 2^{-(j+1)},\ j = 0, 1, 2, ...$

Analogously to Case 1, the geometrically distributed claim sequence $\{Z_{2,1+j}\}_{j \in \mathbb{N}_0}$ is analyzed. This time the parameter is equal to

$$q_{2,j} = 2^{-(j+1)}.$$

The means of this claim sequence are shown in picture 7.1, together with the means of claims in Case 1. It is obvious, that the means from the second sequence are shrinking much faster than first sequence's means. Let us investigate the finite time ruin probabilities. Using 7.1 and 7.2 formulas we obtain

$$\psi^{(j)}(u, 1) = 1 - H^{(j)}(u) = 2^{-(j+1)(u+1)},$$

$$\psi^{(j)}(u, t) = 2^{-(j+1)(u+1)} + \sum_{k=0}^{u} \psi^{(j+1)}(u + 1 - k, t - 1) \cdot (1 - 2^{-(j+1)})2^{-(j+1)k}.$$

In the base case, taking $u = 0$ and $j = 0$, the finite time ruin probability is

$$\psi^{(1)}(0, t) = 2^{-1} + \sum_{k=0}^{0} \psi^{(j+1)}(1 - k, t - 1) \cdot (1 - 2^{-1})2^{-k} = \frac{1}{2} + \frac{1}{2}\psi^{(2)}(1, t - 1).$$

7.2 Case 2: $q_{2,j} = 2^{-(j+1)},\ j = 0, 1, 2, ...$

The calculated numerical values are given in picture 7.2. As we expected, the finite time ruin probabilities are much lower in Case 2 (except the first time moment) than the ruin probabilities in Case 1. This is explained by the differences in the claims' means.

Remarks

In this chapter, the simplest example of the discrete time risk model with nonidentically distributed claims according the geometric law was investigated. However, if in the classical homogeneous claim case the model with geometric claims is extensively analyzed, described and explicit formulas are obtained, in the inhomogeneous claim case is not so simple anymore. The main reason lies in three-dimensional recursive formulas and consequently complex relations.

Nevertheless, the analysis of the model with inhomogeneous geometric claims is possible and we present some examples, which illustrate the ruin probability properties and relations when initial capital increases or decreases by one unit. Finally, the ruin probability with zero initial capital (ballot problem) is investigated.

Conclusions

In this thesis, the classical discrete time risk model is extended by loosening one assumption and letting claims to be not necessarily identically distributed. Such discrete time risk model with inhomogeneous claims is not much discussed in the literature and only a few papers are relevant. Two papers written by De Kok, Picard and Lefèvre, as well as the results in the classical risk theory, are reviewed and the main ideas are highlighted.

The principal objects of investigations in this thesis are the risk measures such as ruin probability and Gerber-Shiu (expected discounted penalty) function. The aim of the work is to be able to find formulas for the exact calculation of these risk measures. In order to fulfill, the following issues were analyzed and theorems proved:

- The recursive formulas for the exact calculation of the finite and infinite time ruin probabilities were derived. In addition to the main case with uniform premiums and discrete claims, the special case of discrete time risk model with rational premiums and claims sizes was investigated as well.

- The recursive formulas for the Gerber-Shiu function calculation in finite and infinite time horizon were obtained. In addition, the special case of infinite time Gerber-Shiu function for initial capital equal to 0 was analyzed in more details and the direct expression was proposed.

- The special case of discrete time risk model application with inhomogeneous claims, distributed according geometric law was discussed; finite time ruin probabilities investigated.

The analysis of Gerber-Shiu function in a discrete time risk model with inhomogeneous claims is new in the literature. On the one hand, the finite time Gerber-Shiu function is introduced only one year ago and only in the classical models. On the other hand, infinite time ruin probability, as well as Gerber-Shiu function, is not investigated in the inhomogeneous claim case at all. The reason for this is that the analysis of the infinite time ruin probability and Gerber-Shiu function is cumbersome without any assumption about underlying claim distributions. In this thesis, as the solution the cyclically distributed claims were proposed. In addition, the case with vanishing claims distributed according geometric law was considered, yet no exact solution could be proposed. So there is a lot of room for further investigations in this area, for example finding the exact and not the recursive solutions for the infinite time ruin probability, in parallel to the existing one in the case of homogeneous claims.

Appendix

In this Appendix, the main programming codes in Maple are presented, which evaluates the Gerber-Shiu function values.

Table A: Implementation of Theorem 6.3. The finite time Gerber-Shiu function.

```
finGS20 := proc(u :: integer, t :: integer, j :: integer, h :: rtable, H
      :: rtable, Finite :: rtable, yield :: rational) :: number;
   g, k, i, x, j1, j2;

 #Calculates the finite time ruin GS function in fast way

     #by taking previous values from the array, not recursively appyling
     the procedure.
 #20 different claim distributions can be applied
  j1 := mod( j, 20);
  if j1 = 0 then j1 := 20 end if ;
  if j1 = 20 then j2 := 1 else j2 := j1 + 1 end if;

    x := (1 − H[ j1, u])·evalf(exp( −yield));
    if t > 1 then
          for k from 0 to u do
              x := x + Finite[u + 1 − k, t − 1, j2]·h[ j1, k]·evalf(exp(
    −yield));
          end do;
    end if;
    return x;
end proc;
```

Table B: Implementation of Theorem 6.1. The infinite time Gerber-Shiu function.

```
infinGS3 := proc(u :: integer, j :: integer, Infinite :: rtable, H :: rtable,
    yield :: rational) :: number;
    k, D1, D2, x, j1, j2;

    #Calculates infinite time ruin probability whith non-homogenous
    claims;

    #Infinite table contains the infinite time ruin probabilities until u-1
    insurer capital.

  j1 := mod(j, 3);
  if j1 = 0 then j1 := 3 end if;
  if j1 = 1 then j2 := 3 else j2 := j1 - 1 end if;
#j1 represents j; j2 represents j-1

 D1 := 0; D2 := 0;
 for k from 0 to u - 1 do
   D1 := D1 + Infinite[j2, k]·evalf(exp(yield)) - 1 + H[j2, k];
   if k > 0 then D2 := D2 + Infinite[j1, k]·H[j2, u - k] end if;
 end do;
x := (D1-D2) / H[j2, 0];
 return x;
end proc;
```

Table C: Implementation of Theorem 6.5. The infinite time Gerber-Shiu function for $u = 0$.

```
infin0 := proc(j :: integer, h :: rtable, H :: rtable, yield :: rational,
    maxsuma :: integer) :: number;

    #"Calculates infinite time ruin probability at zero with non-
    homogenous claims";
 x, n, l, didinamavieta, masyvas, demenusuma, i, sandauginis,
    indeksusuma, suminis;

x := evalf(exp(-yield))·(1 − H[indeks3(j), 0]) + evalf(exp(-2
    · yield))·h[indeks3(j), 0]·(1 − H[indeks3(j + 1), 1]) :

for n from 1 to maxsuma do
  l := 1 :
 didinamavieta := n :
masyvas := rtable(1 ..5000, 1 ..5000, 0) :
while didinamavieta ≠ 0 do
      demenusuma := 0 :
      for i2 from 1 to didinamavieta do
         demenusuma := demenusuma + masyvas[l, i2] :
      end do:

      if demenusuma < didinamavieta
        then
           l := l + 1 :
           for t from 1 to n do
                masyvas[l, t] := masyvas[l − 1, t] :
                if t > didinamavieta then masyvas[l, t] := 0 end if:
            end do:

           masyvas[l, didinamavieta] := masyvas[l, didinamavieta]
    + 1 :
           didinamavieta := n :
         else didinamavieta := didinamavieta − 1 :
       end if:
end do:
suminis := 0 :
for n1 from 1 to l do
    sandauginis := h[indeks3(j), 0] :
    indeksusuma := 0 :
    for n2 from 1 to n do
            sandauginis := sandauginis· h[indeks3(j + n2),
     masyvas[n1, n2]] :
            indeksusuma := indeksusuma + masyvas[n1, n2] :
    end do:
    suminis := suminis + sandauginis·(1 − H[indeks3(j + n + 1), n
      + 1 − indeksusuma]) :
end do:
x := x + suminis·evalf(exp(-yield·(2 + n))) :
end do:

  return x;
end proc;
```

Table D: Introduction of the model parameters. Poisson distribution, a set of δ values.

```
#GS function values for u=0 and X_i~Poisson(i/(3+i))
with(stats) :
poispmf := (x, λ) → statevalf[pf, poisson[λ]](x) :

prob1 := rtable(1 ..20, 0 ..400, [[0, 0, 0], [0, 0, 0], [0, 0, 0]]) :
vidurkis1 := rtable(1 ..20) :
for i from 1 to 20 do
average1[i] := i/(3+i) :
  for g from 0 to 400 do
    prob1[i, g] := poispmf(g, i/(3+i)) :
  end do:
end do:
cdf1 := rtable(1 ..21, 0 ..400, 0) :
  for i from 1 to 20 do
    for g from 0 to 400 do
      cdf1[i, g] := 0 :
      for k from 0 to g do
        cdf1[i, g] := cdf1[i, g] + prob1[i, k] :
      end do;
    end do:
  end do:
  for g from 0 to 400 do cdf1[21, g] := cdf1[1, g]; end do:

yields1 := rtable([0, 1/100, 5/100, 10/100, 20/100]) :
```

Table E: Fast algorithm for the calculation of the finite and infinite time Gerber-Shiu function values using the procedures.

```
values1o1 := rtable(0 ..30, 1 ..30, 1 ..20) :
for t from 1 to 30 do
  for j from 1 to 20 do
    for u from 0 to 30 − t do
    values1o1[u, t, j] := finGS20(u, t, j, prob1, cdf1, values1o1,
     yields1[1]) :
end do end do end do;
```

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