A Weighted Universality Theorem for Periodic Zeta-Functions

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Abstract. The periodic zeta-function \( \zeta(s;a) \), \( s = \sigma + it \) is defined for \( \sigma > 1 \) by the Dirichlet series with periodic coefficients and is meromorphically continued to the whole complex plane. It is known that the function \( \zeta(s;a) \), for some sequences \( a \) of coefficients, is universal in the sense that its shifts \( \zeta(s+i\tau;a) \), \( \tau \in \mathbb{R} \), approximate a wide class of analytic functions. In the paper, a weighted universality theorem for the function \( \zeta(s;a) \) is obtained.

Keywords: Hurwitz zeta-function, Mergelyan theorem, periodic zeta-function, universality.

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1 Introduction

Let \( s = \sigma + it \) be a complex variable, and \( a = \{a_m : m \in \mathbb{N} \} \) be a periodic sequence of complex numbers with minimal period \( k \in \mathbb{N} \). The periodic zeta-function \( \zeta(s;a) \) is defined, for \( \sigma > 1 \), by the Dirichlet series

\[ \zeta(s;a) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}. \]

The Hurwitz zeta-function \( \zeta(s,\alpha) \) with parameter \( \alpha \), \( 0 < \alpha \leq 1 \), is given, for
σ > 1, by the series
\[
\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}
\]
and can be analytically continued to the whole complex plane, except for a simple pole at the point \( s = 1 \) with residue 1. Since, in view of periodicity of \( a \),
\[
\zeta(s; a) = \frac{1}{k^s} \sum_{m=1}^{k} a_m \zeta \left( s, \frac{m}{k} \right),
\]
the periodic zeta-function also has analytic continuation to the whole complex plane, except for a simple pole at the point \( s = 1 \) with residue
\[
\frac{1}{k} \sum_{m=1}^{k} a_m.
\]
If the later quantity is equal to zero, then the function \( \zeta(s; a) \) is entire one.

In 1975, S.M. Voronin discovered [16] the universality of the Riemann zeta-function \( \zeta(s) = \zeta(s, 1) \) on the approximation of a wide class of analytic functions by shifts \( \zeta(s + i\tau), \tau \in \mathbb{R} \). After Voronin’s work, various authors observed that some other zeta-functions also are universal in the Voronin sense. The attention was also devoted to the periodic zeta-function. The first universality results for periodic zeta-function was obtained by B. Bagchi in [1] and [2], and by different methods in [13] and [14]. We will recall Theorem 11.8 from [14]. Denote by \( \mathcal{K} \) the class of compact subsets of the strip \( D = \{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \} \) with connected complements, and by \( H(\mathcal{K}), \mathcal{K} \in \mathcal{K}, \) the class of continuous functions on \( \mathcal{K} \) which are analytic in the interior of \( \mathcal{K} \). Let meas \( A \) stand for the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \).

**Theorem 1.** [14]. Suppose that \( a_m \) is not a multiple of a character \( \text{mod } k \) satisfying \( a_m = 0 \) for \( (m, k) > 1 \). Let \( \mathcal{K} \in \mathcal{K} \) and \( f(s) \in H(\mathcal{K}) \). Then, for every \( \varepsilon > 0 \),
\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in \mathcal{K}} |\zeta(s + i\tau; a) - f(s)| < \varepsilon \right\} > 0.
\]

In [14], also the upper bounds for the density of universality of \( \zeta(s; a) \) were obtained.

We note that the assumptions of Theorem 1 imply that the sequence \( a \) is not multiplicative. We recall that the sequence \( a \) is multiplicative if \( a_{mn} = a_m a_n \) for all coprime \( m, n \in \mathbb{N} \). The universality of \( \zeta(s; a) \) with multiplicative sequence \( a \) was obtained in [11]. Denote by \( H_0(\mathcal{K}), \mathcal{K} \in \mathcal{K}, \) the class of continuous non-vanishing functions on \( \mathcal{K} \), which are analytic in the interior of \( \mathcal{K} \).

**Theorem 2.** [11]. Suppose that the sequence is multiplicative and
\[
\sum_{\alpha=1}^{\infty} \frac{|a_{p^\alpha}|}{p^{\alpha}} \leq c < 1
\]
for all primes \( p \). Let \( \mathcal{K} \in \mathcal{K} \) and \( f(s) \in H_0(\mathcal{K}) \). Then the assertion of Theorem 1 is true.
The universality of periodic zeta-functions is not a simple problem. It turns out, as it was observed in [5], that not all periodic zeta-functions are universal in the Voronin sense. Moreover, in [5], a new restricted universality property for $\zeta(s; a)$ was introduced. For $K \in \mathcal{K}$, the height $h(K)$ of $K$ is defined by

$$ h(K) = \max_{s \in K} \text{Im}(s) - \min_{s \in K} \text{Im}(s). $$

Then in [5] the following theorem has been obtained.

**Theorem 3.** There exists a positive constant $c = c(a)$ such that, for every $K \in \mathcal{K}$ of height $h(K) \leq c$, every $f(s) \in H(K)$ and every $\varepsilon > 0$, the inequality of Theorem 1 is true.

Also, in [5], the necessary and sufficient conditions of the universality for $\zeta(s; a)$ with prime $k$ were obtained. In [15], the universality of the function $\zeta(s; a)$ with prime $k$ satisfying the condition

$$ a_k = \frac{1}{\varphi(k)} \sum_{m=1}^{k-1} a_m, $$

where $\varphi(k)$ is the Euler function, was considered. A joint universality theorem for periodic zeta-functions was proved in [9]. The joint universality of periodic and periodic Hurwitz zeta-functions was studied in [4] and [7].

The aim of this paper is to discuss the weighted universality of the function $\zeta(s; a)$. The universality of this type for the Riemann zeta-function was considered in [6].

Let $w(t)$ be a positive function of bounded variation on $[T_0, \infty)$, $T_0 > 0$, such that the variation $V^b_a w$ on $[a, b]$ satisfies the inequality $V^b_a w \leq cw(a)$ with certain $c > 0$ for any subinterval $[a, b] \subset [T_0, \infty)$. Define $U = U(T, w) = \int_{T_0}^{T} w(t) \, dt$ and suppose that $U(T, w) \to \infty$ as $T \to \infty$. Let $I_A$ stand for the indicator function of the set $A$. Then the following statement holds.

**Theorem 4.** Suppose that the function $w(t)$ satisfies all above conditions, and that the sequence $a$ is as in Theorem 2. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$ \liminf_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau: \sup_{s \in K} |\zeta(s+i\tau; a) - f(s)| < \varepsilon\}}(\tau) \, d\tau > 0. $$

We note that in [6] a certain additional condition generalizing the classical Birkhoff-Khintchine theorem was used. We do not need that condition.

## 2 Limit theorems

Denote by $\mathcal{B}(X)$ the Borel $\sigma$-field of the space $X$, and by $H(D)$ the space of analytic functions on $D$ equipped with the topology of uniform convergence on compacta. This section is devoted to a limit theorem on weakly convergent probability measures in the space $(H(D), \mathcal{B}(H(D)))$.

Let $\gamma \overset{\text{def}}{=} \{ s \in \mathbb{C} : |s| = 1 \}$ be the unit circle on the complex plane. Define $\Omega = \prod_p \gamma_p$, where $\gamma_p = \gamma$ for all primes $p$. With the product topology and pointwise multiplication, the torus $\Omega$ is a compact topological Abelian group. Therefore, the probability Haar measure $m_H$ on $(\Omega, \mathcal{B}(\Omega))$ can be defined. This gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ stand for the projection of $\omega \in \Omega$ to the coordinate space $\gamma_p$. Moreover, let

$$\omega(m) = \prod_{p^a \mid m} \omega^a(p)$$

for $m \in \mathbb{N}$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s, \omega; a)$ by the formula

$$\zeta(s, \omega; a) = \prod_p \left( 1 + \sum_{\alpha=1}^{\infty} \frac{a_{p^\alpha} \omega^\alpha(p)}{p^{\alpha s}} \right).$$

We note that the latter product converges uniformly on compact subsets of $D$ for almost all $\omega \in \Omega$. Moreover, for almost all $\omega \in \Omega$,

$$\zeta(s, \omega; a) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}.$$ 

We start with a weighted limit theorem on the torus. Let, for $A \in \mathcal{B}(\Omega)$,

$$Q_{T,w}(A) = \frac{1}{U} \int_{T_0}^T w(\tau) I_{\{ \tau : (p^{-i\tau} : p \in \mathcal{P}) \in A \}}(\tau) \, d\tau,$$

where $\mathcal{P}$ is the set of all prime numbers.

**Lemma 1.** $Q_{T,w}$ converges weakly to the Haar measure $m_H$ as $T \to \infty$.

**Proof.** Denote by $g_{T,w}(k), k = (k_p : k_p \in \mathbb{Z}, p \in \mathcal{P})$, the Fourier transform of the measure $Q_{T,w}$. Since characters $\chi$ of $\Omega$ are of the form

$$\chi(\omega) = \prod_p \omega^{k_p}(p),$$

where only a finite number of integers $k_p$ are distinct from zero, we have that

$$g_{T,w}(k) = \int_{\Omega} \prod_p \omega^{k_p}(p) \, dQ_{T,w}.$$

Hence, by the definition of $Q_{T,w}$,

$$g_{T,w}(k) = \frac{1}{U} \int_{T_0}^T w(\tau) \prod_p p^{-ik_p\tau} \, d\tau$$

$$= \frac{1}{U} \int_{T_0}^T w(\tau) \exp \left\{ -i\tau \sum_p k_p \log p \right\} \, d\tau, \quad (2.1)$$
where only a finite number of integers $k_p$ are distinct from zero. It is well known that the set \{\log p : p \in \mathcal{P}\} is linearly independent over the field of rational numbers $\mathbb{Q}$. Therefore, in view of (2.1),

\begin{equation}
g_T, w(0) = 1 \tag{2.2}
\end{equation}

and, for \(k \neq 0\), using properties of $w(t)$, we find that

\begin{align*}
g_T, w(k) &= -\frac{1}{U} \sum_p k_p \log p \int_{T_0}^T w(\tau) \exp \left\{ -i \tau \sum_p k_p \log p \right\} \\
&= O\left( |U \sum_p k_p \log p|^{-1} \right).
\end{align*}

This and (2.2) show that

\begin{align*}
\lim_{T \to \infty} g_T, w(k) &= \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0, \end{cases}
\end{align*}

i.e., $g_T, w(k)$, as $T \to \infty$, converges to the Fourier transform of the measure $m_H$. Hence, the lemma follows. \(\Box\)

Let $\theta > \frac{1}{2}$ be a fixed number and, for $m, n \in \mathbb{N}$,

\[ v_n(m) = \exp \left\{ - \left( \frac{m}{n} \right)^\theta \right\}. \]

Define

\[ \zeta_n(s; a) = \sum_{m=1}^\infty \frac{a_m v_n(m)}{m^s} \quad \text{and} \quad \zeta_n(s, \omega; a) = \sum_{m=1}^\infty \frac{a_m \omega(m) v_n(m)}{m^s}, \]

and let the function $u_n : \Omega \to H(D)$ be defined by the formula

\[ u_n(\omega) = \zeta_n(s, \omega; a). \]

Since the series for $\zeta_n(s, \omega; a)$ is absolutely convergent for $\sigma > \frac{1}{2}$ [11], the function $u_n$ is continuous one. We set $\hat{P}_n = m_H u_n^{-1}$, where, for $A \in \mathcal{B}(H(D))$,

\[ \hat{P}_n(A) = m_H u_n^{-1}(A) = m_H (u_n^{-1} A). \]

Define

\[ P_{T, n, w}(A) = \frac{1}{U} \int_{T_0}^T w(\tau) I_{\{ \tau : \zeta_n(s + i\tau; a) \in A \}}(\tau) \, d\tau, \quad A \in \mathcal{B}(H(D)). \]

**Lemma 2.** $P_{T, n, w}$ converges weakly to $\hat{P}_n$ as $T \to \infty$.

**Proof.** Clearly,

\[ u_n \left( p^{-i\tau} : p \in \mathcal{P} \right) = \zeta_n(s + i\tau; a). \]
Therefore,
\[
P_{T,n,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau: (p-i\tau; p\in \mathcal{P}) \in u_{n}^{-1} A\}}(\tau) \, d\tau \\
= Q_{T,w}(u_{n}^{-1} A) = Q_{T,w} u_{n}^{-1}(A).
\]

This, the continuity of \(u_{n}\), Lemma 1 and Theorem 5.1 of [3] prove the lemma. \(\square\)

Now we will approximate \(\zeta(s; a)\) by \(\zeta_{n}(s; a)\). Let, for \(g_{1}, g_{2} \in H(D)\),
\[
\rho(g_{1}, g_{2}) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_{l}} |g_{1}(s) - g_{2}(s)|}{1 + \sup_{s \in K_{l}} |g_{1}(s) - g_{2}(s)|},
\]
where \(\{K_{l} : l \in \mathbb{N}\}\) is a sequence of compact subsets of the strip \(D\) such that \(D = \bigcup_{l=1}^{\infty} K_{l}, K_{l} \subseteq K_{l+1}\) for all \(l \in \mathbb{N}\), and if \(K \subseteq D\) is a compact, then \(K \subseteq K_{l}\) for some \(l\). Then \(\rho\) is a metric on \(H(D)\) which induces its topology of uniform convergence on compacta.

**Lemma 3.** The equality
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) \rho(\zeta(s + i\tau; a), \zeta_{n}(s + i\tau; a)) \, d\tau = 0
\]
holds.

**Proof.** Consider the series
\[
\sum_{m=1}^{\infty} \frac{b_{n}(m)}{m^{s}}, \tag{2.3}
\]
where
\[
b_{n}(m) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \frac{a_{m} l_{n}(s)}{sm^{s}} \, ds, \quad l_{n}(s) = \frac{s}{\theta} \Gamma(s/\theta) n^{s}, \quad n \in \mathbb{N},
\]
\(\Gamma(s)\) is the Euler gamma-function, and \(\theta > \frac{1}{2}\) is as above. Since \(a_{m}\) is uniformly bounded, we find that
\[
b_{n}(m) \ll m^{-\theta}.
\]
Thus, the series (2.3) is absolutely convergent for \(\sigma > \frac{1}{2}\). From this remark, we deduce that
\[
\frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s + z; a) l_{n}(z) \, \frac{dz}{z} = \sum_{m=1}^{\infty} \frac{b_{n}(m)}{m^{s}}, \tag{2.4}
\]
and an application of the Mellin formula shows that
\[
b_{n}(m) = a_{m} \exp \left\{ - \left( \frac{m}{n} \right)^{\theta} \right\}.
\]
Now the series (2.3) coincides with \( \zeta_n(s; a) \). Therefore, by (2.4) and the residue theorem,

\[
\zeta_n(s; a) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z; a) l_n(z) \frac{dz}{z} + \zeta(s; a) + \text{Res}_{z=1-s} \zeta(s + z; a) l_n(z) \frac{1}{z},
\]

(2.5)

where \( \frac{1}{2} < \sigma < 1 \) and \( \sigma > \theta \).

Suppose that \( \sigma \geq \frac{1}{2} \) and \( 2\pi \leq |t| \leq \pi x \). Then, see, for example, [8],

\[
\zeta(s, \alpha) = \sum_{0 \leq m \leq x} \frac{1}{(m + \alpha)^s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma}).
\]

(2.6)

Moreover, by (1.1),

\[
\zeta(s; a) = O \left( \sum_{m=1}^{k} |\zeta(s, \frac{m}{k})| \right).
\]

From this and (2.6), we find similarly as in the proof of Lemma 4 of [10] that, for \( \frac{1}{2} < \sigma < 1 \) and \( \tau \in \mathbb{R} \),

\[
\frac{1}{U} \int_{T_0}^{T+\tau} w(t - \tau) |\zeta(\sigma + it; a)|^2 dt = O(U(1 + |\tau|)^2).
\]

Let \( K \) be a compact subset of the strip \( D \). Then, using (2.5) and the contour integration, we obtain that with \( \hat{\sigma} < 0 \)

\[
\frac{1}{U} \int_{T_0}^{T} w(\tau) \sup_{s \in K} |\zeta(s + i\tau; a) - \zeta_n(s + i\tau; a)| d\tau
\]

\[
= O \left( \int_{-\infty}^{\infty} |l_n(\hat{\sigma} + it)| (1 + |t|)^2 dt \right) + o(1)
\]

as \( T \to \infty \). This and the definition of \( l_n(s) \) prove the lemma. □

Denote by \( P_\zeta \) the distribution of the random element \( \zeta(s, \omega; a) \), i.e,

\[
P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s, \omega; a) \in A), \quad A \in \mathcal{B}(H(D)).
\]

For \( A \in \mathcal{B}(H(D)) \), define

\[
P_{T,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(t) \mathcal{I}_{\{t : \zeta(s + i\tau; a) \in A\}}(\tau) d\tau.
\]

**Theorem 5.** The measure \( P_{T,w} \) converges weakly to \( P_\zeta \) as \( T \to \infty \). Moreover, the support of \( P_\zeta \) is the set \( \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\} \).

**Proof.** On a certain probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), define a random variable \( \eta_T \) by

\[
\mathbb{P}(\eta_T \in A) = \frac{1}{U} \int_{T_0}^{T} w(t) I_A(t) dt, \quad A \in \mathcal{B}(\mathbb{R}).
\]
By Lemma 2, we have that \( P_{T,n,w} \) converges weakly to \( \hat{P}_n \) as \( T \to \infty \). Define

\[
X_{T,n} = X_{T,n}(s) = \zeta_n(s + i\eta T; a).
\]

Then the assertion of Lemma 2 can be written as

\[
X_{T,n} \xrightarrow{D} \hat{X}_n, \quad (2.7)
\]

where \( \xrightarrow{D} \) denotes the convergence in distribution, and \( \hat{X}_n \) is the \( H(D) \)-valued random element having the distribution \( \hat{P}_n \). We will prove that the family of probability measures \( \{ \hat{P}_n : n \in \mathbb{N} \} \) is tight.

Since the series for \( \zeta_n(s; a) \) is absolutely convergent for \( \sigma > \frac{1}{2} \), it is not difficult to see that, for \( \sigma > \frac{1}{2} \),

\[
\lim_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(t)|\zeta_n(\sigma + it; a)|^2 \, dt = \sum_{m=1}^{\infty} \frac{|a_m|^2 v_n^2(m)}{m^{2\sigma}} \leq \sum_{m=1}^{\infty} \frac{|a_m|^2}{m^{2\sigma}} \leq C < \infty. \quad (2.8)
\]

Let \( K_l \) be a compact set from the distribution of the metric \( \rho \). Then using the Cauchy integral formula and (2.8) leads to

\[
\sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) \sup_{s \in K_l} |\zeta_n(s + i\tau; a)| \, d\tau \leq R_l < \infty.
\]

Now let \( \varepsilon > 0 \) be arbitrary and \( M_l = 2^l R_l \varepsilon^{-1} \). Then

\[
\limsup_{T \to \infty} \mathbb{P} \left( \sup_{s \in K_l} |X_{T,n}(s)| > M_l \right)
\]

\[
= \limsup_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I \{ \tau : \sup_{s \in K_l} |\zeta_n(s + i\tau; a)| \geq M_l \} \, d\tau
\]

\[
\leq \sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \frac{1}{M_l U} \int_{T_0}^{T} w(\tau) \sup_{s \in K_l} |\zeta_n(s + i\tau; a)| \, d\tau \leq \frac{\varepsilon}{2^l}.
\]

Therefore, in view of (2.7),

\[
\mathbb{P} \left( \sup_{s \in K_l} |\hat{X}_n(s)| > M_l \right) \leq \frac{\varepsilon}{2^l} \quad (2.9)
\]

for all \( n \in \mathbb{N} \) and \( l \in \mathbb{N} \). Let

\[
H_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K_l} |g(s)| \leq M_l, \ l \in \mathbb{N} \right\}.
\]

Then the set \( H_\varepsilon \) is uniformly bounded on every compact set of \( D \), thus it is compact subset of \( H(D) \). Moreover, by (2.9)

\[
\mathbb{P}(\hat{X}_n(s) \in H_\varepsilon) \geq 1 - \varepsilon
\]
for all $n \in \mathbb{N}$. Hence,
\[
\hat{P}_n(H_\varepsilon) \geq 1 - \varepsilon
\]
for all $n \in \mathbb{N}$, i.e., the family $\{\hat{P}_n\}$ is tight. Therefore, by the Prokhorov theorem [3], it is relatively compact. Hence, every sequence of $\{\hat{P}_n\}$ contains a subsequence $\{\hat{P}_{n_r}\}$ such that $\hat{P}_{n_r}$ converges weakly to a certain probability measure $P$ on $(H(D), \mathcal{B}(H(D)))$, i.e.,
\[
\hat{X}_{n_r} \xrightarrow{\mathcal{D}} P.
\]
Moreover, using Lemma 3, we find that, for every $\varepsilon > 0$,
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau : \rho(\zeta(s + i\tau; a), \zeta_n(s + i\tau, a)) \geq \varepsilon\}}(\tau) \, d\tau \\
\leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{\varepsilon U} \int_{T_0}^T w(\tau) \rho(\zeta(s + i\tau; a), \zeta_n(s + i\tau, a)) \, d\tau = 0.
\]
Now this, (2.7), (2.10) and Theorem 4.2 of [3] show that
\[
X_T(s) = \zeta(s + i\eta T; a) \xrightarrow{\mathcal{D}} P.
\]
Hence, $P_{T, w}$ converges weakly to $P$ as $T \to \infty$. The latter relation also implies, that the measure $P$ in (2.10) is independent of the choice of subsequence $\hat{P}_{n_r}$. Thus
\[
\hat{X}_n \xrightarrow{n \to \infty} P,
\]
or $\hat{P}_n$ converges weakly to $P$. This means that $P_{T, w}$, as $T \to \infty$, converges weakly to the limit measure of $\hat{P}_n$, as $n \to \infty$. It remains to identify the measure $P$.

In [11], the measure
\[
P_T(A) = \frac{1}{T} \text{meas}\{\tau \in [0, T] : \zeta(s + i\tau; a) \in A\}, \quad A \in \mathcal{B}(H(D))
\]
was considered, and it was obtained that $P_T$ converges weakly to $P_\zeta$ as $T \to \infty$. Moreover, in the proving process, it was observed that $P_T$, as $P_{T, w}$, also converges weakly to the limit measure of $\hat{P}_n$ as $n \to \infty$, i.e., to the measure $P$. From these remarks, we have that $P$ coincides with $P_\zeta$. In [11] it is also noted that the support of the measure $P_\zeta$ is the set $\{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$. The theorem is proved. $\Box$

### 3 Universality

The proof of Theorem 4 is quite standard and is based on Theorem 5 and the Mergelyan theorem on the approximation of analytic functions by polynomials [12].
Proof of Theorem 4. By Theorem 5 and the equivalent of weak convergence of probability measures in terms of open sets [3], we have that

$$\liminf_{T \to \infty} \frac{1}{U} \int_{T_0}^T w(t) I_{\{\tau : \zeta(s+i\tau; a) \in G\}}(\tau) \, d\tau \geq P_\zeta(G),$$  \hspace{1cm} (3.1)$$

where

$$G = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - e^{p(s)}| < \frac{\varepsilon}{2} \right\}$$

and $p(s)$ is a polynomial such that

$$\sup_{s \in K} |f(s) - e^{p(s)}| < \frac{\varepsilon}{2}. \hspace{1cm} (3.2)$$

The existence of $p(s)$ is implied by the Mergelyan theorem. By Theorem 5, $e^{p(s)}$ is an element of the support of the measure $P_\zeta$; thus $P_\zeta(G) > 0$ because $G$ is an open neighbourhood of $e^{p(s)}$. Therefore, in view of (3.1) and the definition of $G$,

$$\liminf_{T \to \infty} \frac{1}{U} \int_{T_0}^T w(t) I_{\left\{ \tau : \sup_{s \in K} |\zeta(s+i\tau; a) - e^{p(s)}| < \frac{\varepsilon}{2} \right\}}(\tau) \, d\tau > 0.$$  

From this, the theorem follows since, in virtue of (3.2),

$$\left\{ \tau : \sup_{s \in K} |\zeta(s + i\tau; a) - e^{p(s)}| < \frac{\varepsilon}{2} \right\} \subset \left\{ \tau : \sup_{s \in K} |\zeta(s + i\tau; a) - f(s)| < \varepsilon \right\}.$$  

References


