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VALUE-DISTRIBUTION OF TWISTED L-FUNCTIONS OF NORMALIZED CUSP FORMS

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NORMUOTŲ PARABOLINIŲ FORMŲ L FUNKCIJŲ SĄSŪKŲ REIKŠMIŲ PASISKIRSTYMAS

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Contents

Introduction

In the thesis, the asymptotic behavior of twisted with Dirichlet character L-functions of normalized Hecke eigen cusp form is considered when the modulus of the character increases.

Actuality

L-functions play an important role in analytic number theory. Dirichlet L-functions are an analytic tool for the investigation of the distribution of prime numbers in arithmetic progressions while L-functions of automorphic forms were introduced to study the problems of these forms. The role of L-functions attached to cusp forms was crucial in the proof of the last Fermat theorem [32]. Twists of L-functions attached to automorphic forms with Dirichlet characters are used for the investigation of Fourier coefficients of automorphic forms in arithmetic progressions and other allied problems. The value distribution of arithmetic objects in arithmetic progressions becomes very complicated when the difference of a progression is increasing. The problems of such a kind lead to twists of L-functions with increasing modulus of a character. Therefore, the investigation of twisted L-functions with increasing modulus is an urgent problem of analytic number theory. The twists of L-functions were studied by many famous mathematicians, among them S. Chowla, P. Erdös, P. D. T. A. Elliott, K. Matsumoto, P. Sarnak, H. Iwaniec and others. P. D. T. A. Elliott obtained the first probabilistic results in the field.

Aims and problems

The aim of the thesis is to prove limit theorems in the sense of weak convergence of probability measures for twisted with Dirichlet character L-functions of normalized Hecke eigen cusp forms with respect to increasing modulus of the character, more precisely, to obtain the following theorems:

1. To prove a limit theorem for the modulus of twisted with Dirichlet character L-functions of normalized Hecke eigen cusp forms.

2. To prove a limit theorem for the argument of twisted with Dirichlet character L-functions of normalized Hecke eigen cusp forms.

3. To prove a limit theorem on the complex plane for twisted with Dirichlet character L-functions of normalized Hecke eigen cusp forms.

4. To prove a joint limit theorem for twisted with Dirichlet character L-functions of normalized Hecke eigen cusp forms.

Methods

In the thesis, analytical and probabilistic methods are applied. For the proof of probabilistic limit theorems, the method of characteristic transforms is used. Moreover, some elements of the Dirichlet character theory and of L-functions theory are applied.

Novelty

All results of the thesis are new. Limit theorems for twisted with Dirichlet character L-functions of normalized Hecke eigen cusp forms earlier were not known.

History of the problem and main results

For the definition of the object studied in the thesis, we need some notation and definitions. As usual, denote by $\mathbb{Z}$ the set of all integers, and let

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\}$$

denote the full modular group. Moreover, let U be the upper half-plane together with ∞, i. e.,

$$U = \{z \in \mathbb{C} : z = x+iy, \quad y>0\}.$$

Suppose that $F(z)$ is a holomorphic function on U, and satisfies, for some positive even integer κ and all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

the functional equation

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} F(z).$$

Then, clearly, $F(z)$ is a periodic function, and has the Fourier series expansion at ∞

$$F(z) = \sum_{m=-\infty}^{\infty} c(m) \mathrm{e}^{2\pi i m z}.$$

We say that the function $F(z)$ is holomorphic and vanishing at ∞ if $c(m) = 0$ for $m < 0$ and $m \leq 0$, respectively. We say that $F(z)$ is holomorphic and vanishing at the cusps if the function

$$F\left(\frac{az+b}{cz+d}\right)(cz+d)^{-\kappa},$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

is holomorphic and vanishing at ∞, respectively. In the case when $F(z)$ is holomorphic at the cusps, it is called a modular form of weight κ, and

$$F(z) = \sum_{m=0}^{\infty} c(m)\mathrm{e}^{2\pi i m z}$$

is its Fourier series expansion at ∞. If the modular form $F(z)$ of weight κ is vanishing at cusps, then we call it a cusp form of weight κ. In this case, $F(z)$ has the Fourier series expansion at ∞

$$F(z) = \sum_{m=1}^{\infty} c(m)\mathrm{e}^{2\pi i m z}.$$

The classical example of cusp forms is the Ramanujan cusp form $\Delta(z)$ defined by

$$\begin{aligned} \Delta(z) &= \sum_{m=1}^{\infty} \tau(m)\mathrm{e}^{2\pi i m z} \\ &= \mathrm{e}^{2\pi i z} \prod_{m=1}^{\infty} (1 - \mathrm{e}^{2\pi i m z})^{24}. \end{aligned}$$

The weight of the form $\Delta(z)$ is 12. L. Mordell proved [26] that the Ramanujan function $\tau(m)$ is multiplicative ($\tau(mn) = \tau(m)\tau(n)$ for all $m, n \in \mathbb{N}$, $(m, n) = 1$) and satisfies

$$\tau(p^{k+1}) = \tau(p)\tau(p^k) - p^{11}\tau(p^{k-1})$$

for prime numbers p and integers $k \geq 2$. Also, by the Deligne general result [5],

$$|\tau(p)| \leq 2p^{\frac{11}{2}}.$$

A cusp form $F(z)$ of weight κ is called Hecke eigenform if it is an eigen function of all Hecke operators

$$(T_n f)(z) = n^{\kappa-1} \sum_{d|n} d^{-\kappa} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^z}\right), \quad n \in \mathbb{N}.$$

Then it is known that $c(1) \neq 0$, therefore the form $F(z)$ can be normalized. Thus, a normalized Hecke eigen cusp form has Fourier series expansion at ∞

$$F(z) = \sum_{m=1}^{\infty} c(m) \mathrm{e}^{2\pi i m z}, \quad c(1) = 1.$$

Now let $s = \sigma + it$ be a complex variable, and $F(z)$ be a normalized Hecke eigen cusp form of weight κ. To this form, we can attach the L-function $L(s, F)$ defined by Dirichlet series

$$L(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

By the Weil conjecture,

$$|c(m)| \leq m^{\frac{k-1}{2}} d(m),$$

where $d(m)$ denotes the divisor function

$$d(m) = \sum_{d|m} 1,$$

proved by Deligne [5], we have that the Dirichlet series for $L(s, F)$ converges absolutely for $\sigma > \frac{\kappa+1}{2}$, and defines there an analytic function. Moreover, it is known that the function $L(s, F)$ can be analytically continued to an entire function, and satisfies the functional equation

$$(2\pi)^{-s} \Gamma(s) L(s, F) = (-1)^{\frac{\kappa}{2}} (2\pi)^{s-\kappa} \Gamma(\kappa - s) L(\kappa - s, F),$$

where $\Gamma(s)$ is the Euler gamma-function. The critical strip of $L(s, F)$ is of the form $\{s \in \mathbb{C} : \frac{\kappa-1}{2} < \sigma < \frac{\kappa+1}{2}\}$, and contains non-trivial zeros of $L(s, F)$. These zeros are located symmetrically to the real axis and to the critical line $\sigma = \frac{\kappa}{2}$. The analogue of the Riemann hypothesis for the function $L(s, F)$ says that all non-trivial zeros of $L(s, F)$ lie on the critical line $\sigma = \frac{\kappa}{2}$.

The coefficients $c(m)$ of the Dirichlet series for $L(s, F)$ are multiplicative, and, for primes p and $k \in \mathbb{N} \backslash \{1\}$, satisfy the relation

$$c(p^{k+1}) = c(p) c(p^k) - p^{\kappa-1} c(p^{k-1}).$$

Therefore, the function $L(s, F)$ has, for $\sigma > \frac{\kappa+1}{2}$, the Euler product expansion over primes

$$\begin{aligned} L(s, F) &= \prod_p \left(1 - \frac{c(p)}{p^s} + \frac{1}{p^{2s-\kappa+1}}\right) \\ &= \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1}, \end{aligned} \tag{0.1}$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers satisfying $\alpha(p)+\beta(p)=c(p)$, and

$$|\alpha(p)| \le p^{\frac{\kappa-1}{2}}, \quad |\beta(p)| \le p^{\frac{\kappa-1}{2}}.$$

The function $L(s,F)$, as other L-functions, has a probabilistic limit distribution in the following sense. Let $\mathcal{B}(S)$ denote the class of Borel sets of a space S, and

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for each prime p. By the Tikhonov theorem, with the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and this leads to the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of element $\omega \in \Omega$ to the coordinate space γ_p.

Let $D = \{s \in \mathbb{C} : \sigma > \frac{\kappa}{2}\}$. Denote by $H(D)$ the space of analytic functions on D equipped with the topology of uniform convergence on compacta, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define the $H(D)$-valued random element $L(s,\omega,F)$ by the formula

$$L(s,\omega,F) = \prod_p \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}.$$

Let $meas\{A\}$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Denote by P_L the distribution of the random element $L(s,\omega,F)$, i.e.,

$$P_L(A) = m_H(\omega \in \Omega : L(s,\omega,F) \in A), \quad A \in \mathcal{B}(H(D)).$$

Then the following limit theorem holds [12].

Theorem 0.1 *The probability measure*

$$\frac{1}{T} meas\big\{\tau \in [0,T] : L(s+i\tau, F) \in A\big\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the measure P_L as $T \to \infty$.

Now let χ be a Dirichlet character modulo q. Then the twisted L-function $L(s,F,\chi)$ attached to the form $F(z)$ is defined, for $\sigma > \frac{\kappa+1}{2}$, by the Dirichlet series

$$L(s,F,\chi) = \sum_{m=1}^{\infty} \frac{c(m)\chi(m)}{m^s},$$

and can be analytically continued to an entire function. Also, in the half-plane $\sigma > \frac{\kappa+1}{2}$, the function $L(s,F,\chi)$ can be presented by the Euler product over primes

$$L(s,F,\chi) = \prod_p \Big(1 - \frac{\alpha(p)\chi(p)}{p^s}\Big)^{-1} \Big(1 - \frac{\beta(p)\chi(p)}{p^s}\Big)^{-1}, \tag{0.2}$$

where the complex numbers $\alpha(p)$ and $\beta(p)$ are the same as in (0.1).

A similar result is also true for the twist $L(s,F,\chi)$. Define the $H(D)$-valued random element $L(s,\omega,F,\chi)$ by the formula

$$L(s,\omega,F,\chi)=\prod_p\Big(1-\frac{\alpha(p)\chi(p)\omega(p)}{p^s}\Big)^{-1}\Big(1-\frac{\beta(p)\chi(p)\omega(p)}{p^s}\Big)^{-1},$$

and denote by $P_{L,\chi}$ its distribution

$$P_{L,\chi}(A)=m_H(\omega\in\Omega: L(s,\omega,F,\chi)\in A),\quad A\in\mathcal{B}(H(D)).$$

Then in [25] the following statement has been obtained.

Theorem 0.2 *The probability measure*

$$\frac{1}{T}meas\Big\{\tau\in[0,T]: L(s+i\tau,F,\chi)\in A\Big\},\quad A\in\mathcal{B}(H(D)),$$

converges weakly to the measure $P_{L,\chi}$ as $T\to\infty$.

Theorems 0.1 and 0.2 were applied to obtain the universality of the functions $L(s,F)$ and $L(s,F,\chi)$.

Let

$$D_0=\Big\{s\in\mathbb{C}:\frac{\kappa}{2}<\sigma<\frac{\kappa+1}{2}\Big\}.$$

Theorem 0.3 ([24]) *Let K be a compact subset of the strip D_0 with connected complement, and let $f(s)$ be a continuous and non-vanishing function on K, and analytic in the interior of K. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}meas\Big\{\tau\in[0,T]:\sup_{s\in K}|L(s+i\tau,F)-f(s)|<\varepsilon\Big\}>0.$$

In [25], a version of Theorem 0.3 has been proved for the function $L(s,F,\chi)$.

Note that, in Theorem 0.2, the modulus q of the character χ is fixed. It turns out that it is possible to characterize the asymptotic behavior of the function $L(s,F,\chi)$ by probabilistic limit theorems when the modulus q is not fixed and increases, i.e., to study the asymptotic behavior of $L(s,F,\chi)$ with respect to the parameter q.

Before the statement of the results of the thesis, we will present a short survey of probabilistic results with respect of some parameter in analytic number theory.

The first results in this direction were obtained for Dirichlet L-functions $L(s,\chi)$ defined, for $\sigma>1$, by

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s},$$

and by analytic continuation elsewhere. We recall that the character χ modulo q is principal if $\chi(m) = 1$ for all $(m,q) = 1$, and is denoted by χ_0. If $\chi \neq \chi_0$, than the function $L(s,\chi)$ is entire, while $L(s,\chi_0)$ has a simple pole at $s = 1$ with residue

$$\prod_{p|q}\Big(1-\frac{1}{p}\Big).$$

The first result for Dirichlet L-function $L(s,\chi)$ with an increasing modulus was obtained by Chowla and Erdös [4], who proved a limit theorem for $L(1,\chi)$ with real character χ. The later progress in the field belongs to Elliott. Now we suppose that q is a prime number.

For $Q \geq 2$, denote

$$M_Q = \sum_{q\leq Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\neq\chi_0}} 1.$$

It is known [20], Lemma 2.9.7, that

$$M_Q = \frac{Q^2}{2\log Q} + \mathrm{O}\Big(\frac{Q^2}{\log^2 Q}\Big).$$

For brevity, we introduce the notation

$$\mu_Q(\ldots) = M_Q^{-1}\sum_{q\leq Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\neq\chi_0\\ \ldots}} 1,$$

where in place of dots a condition satisfied by a pair $(q, \chi(\mathrm{mod}\ q))$ is to be written. Let

$$\varepsilon(\chi) = \prod_{m=1}^{\infty}\Big(1-\frac{\chi(m)}{m^s}\Big)\mathrm{e}^{-\chi(m)m^{-s}}.$$

Then in [6], the following theorem was proved.

Theorem 0.4 *Suppose that $\sigma > \frac{1}{2}$. Then the distribution function*

$$\mu_Q\big(|\varepsilon(\chi)|^{-1}|L(s,\chi)| < x\big)$$

converges weakly to some distribution function as $Q \to \infty$.

When $L(s,\chi) \neq 0$, $\frac{1}{2} < \sigma \leq 1$, let $\arg L(s,\chi)$ denote a value of the argument of $L(s,\chi)$ defined by continuous displacement from the point $s = 2$ along an arc on which $L(s,\chi)$ does not vanish. Thus, $\arg L(s,\chi)$ is only defined to within the addition of an integer multiple of $2\pi i$. In [7], a limit theorem for $\arg L(s,\chi)$ has been obtained. In order to state this theorem, we recall the definition and convergence of the distribution functions mod 1.

A function $G(x)$ is said to be a distribution function mod 1 if and only if it satisfies the following three conditions:

1). it increases in the wide sense;

2). it is right continuous, that is, $G(x+0) = G(x)$ for all $x \in \mathbb{R}$;

3). $G(x) = 1$ if $x \geq 1$, and $G(x) = 0$ if $x < 0$.

A distribution function mod 1 $G_n(x)$, $n \in \mathbb{N}$, converges weakly mod 1 as $n \to \infty$ if there exists a distribution function $G(x)$ (mod 1) such that at all points x_1, x_2, $0 \leq x_1 \leq x_2 < 1$, which are continuity points of $G(z)$ we have

$$\lim_{n\to\infty} \big(G_n(x_2) - G_n(x_1)\big) = G(x_2) - G(x_1).$$

Thus, in the range $0 \leq x < 1$, the limit function $G(x)$ is determined only up to an additive constant.

Now we state the main result of [7].

Theorem 0.5 *At each point s in the half-plane $\{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$,*

$$\mu_Q\Big(\frac{1}{2\pi}\arg L(s,\chi) \leq x(\mathrm{mod}\ 1)\Big)$$

converges weakly to a continuous distribution function mod 1 *as $Q \to \infty$. The Fouries transform of the limit distribution function is of the form*

$$\prod_p \left(1 + \sum_{m=1}^{\infty} \left(\begin{array}{c} -\frac{k}{2} \\ m \end{array} \right) \left(\begin{array}{c} \frac{k}{2} \\ m \end{array} \right) \frac{1}{p^{2m\sigma}} \right), \quad k \in \mathbb{Z}.$$

E. Stankus generalized [28] Theorems 0.4 and 0.5 for probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Let P be a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. We remind that the function

$$w(\tau, k) \stackrel{def}{=} \int\limits_{\mathbb{C}\setminus\{0\}} |z|^{i\tau} \mathrm{e}^{ik\arg z} \mathrm{d}P, \quad \tau \in \mathbb{R}, \quad k \in \mathbb{Z},$$

is called the characteristic transform of the measure P. It is known [20] that the measure P is uniqnelly determined by $w(\tau, k)$. Let $c_{\tau,k}(m)$ be a multiplicative function defined, for a prime p and $m \in \mathbb{N}$, by

$$c_{\tau,k}(p^m) = \frac{\xi(\xi+1)\dots\xi(\xi+k-1)}{m!}$$

with $\xi = \frac{i\tau+k}{2}$, and

$$w_P(\tau, k) = \sum_{m=1}^{\infty} \frac{c_{\tau,k}(m) c_{\tau,-k}(m)}{m^{2\sigma}}, \quad \sigma > \frac{1}{2}.$$

Theorem 0.6 ([28]) *Suppose that* $\sigma > \frac{1}{2}$. *Then*

$$\mu_Q\bigl(L(s,\chi) \in A\bigr), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the probability measure P *on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *defined by the characteristic transform* $w_P(\tau, k)$.

Similar results to Theorems 0.4, 0.5 and 0.6 for real characters were obtained in [8] and [29], respectively.

The above limit theorems are examples of limit theorem with respect to some parameter. Now we will present some allied results. Let $\omega(p)$ is defined as in Theorem 0.1. For $\sigma > \frac{1}{2}$, define

$$L(s,\omega) = \prod_p \Bigl(1 - \frac{\omega(p)}{p^s}\Bigr)^{-1}.$$

Theorem 0.7 ([1]) *We have that*

$$\frac{1}{p-1}\#\Bigl\{\chi : \chi \textit{ is a Dirichlet character} \bmod p, \textit{and} \quad L(s,\chi) \in A\Bigr\},$$

$A \in \mathcal{B}(H(D))$, *converges weakly to the distribution of the random element* $L(s,\omega)$ *as* $p \to \infty$ *through the sequence of primes.*

Theorem 0.7 was used [1] to prove the universality of $L(s,\chi)$ in χ-aspect. This was also done independently by S. M. Gonek [10] and K. M. Eminyan [11].

Theorem 0.8 ([1]) *Let* $K \subset D$ *be a compact subset with connected complement, and let* $f(s)$ *be a non-vanishing continuous function on* K *which is analytic in the interior of* K. *Then, for every* $\varepsilon > 0$,

$$\liminf_{p\to\infty} \frac{1}{p-1} \quad \# \quad \Bigl\{\chi : \chi \textit{ is a Dirichlet character} \bmod p, \\ \textit{and} \sup_{s\in K} |L(s,\chi) - f(s)| < \varepsilon\Bigr\} > 0.$$

Also, there exist results on estimates of distribution functions in parameter aspect related to various L-functions. We remind that d is a fundamental discriminant if the following statements holds:

$$d \equiv 1 (\text{mod } 4) \text{ and is square-free,}$$
$$d = 4m, \text{where } m \equiv 2 \text{ or } 3 \ (\text{mod } 4) \text{ and } m \text{ is square-free.}$$

Let

$$D_x = \#\bigl\{d \le x : d \text{ is a fundamental discriminant}\bigr\},$$

and

$$\Phi_x(\tau) = \frac{1}{D_x} \quad \# \quad \bigl\{d \le x : d - \text{ fundamentalusis diskriminantas ir}$$

$$L(1,\chi_d) > \mathrm{e}^{\gamma_0}\tau\};$$

where γ_0 is the Euler constant

$$\gamma_0 = \lim_{n\to\infty}\Big(\sum_{m\le n}\frac{1}{m} - \log n\Big),$$

and χ_d is the character mod d. Then the following formula is known.

Theorem 0.9 ([12]) *Uniformly in $\tau \le \log\log x$,*

$$\Phi_x(\tau) = \exp\Big\{-\frac{\mathrm{e}^{\tau-C}}{\tau}\Big(1+\mathrm{O}\Big(\frac{1}{\tau}\Big)\Big)\Big\},$$

where C has an explicit integral representation, $C = 0.8187\ldots$.

Recently, Y. Lamzouri introduced [17] a new probabilistic model and applied it for the investigation of value distribution of L-functions with respect to a parameter.

Let $d \in \mathbb{N}$ and $\mathcal{P}$ be the set of all prime numbers. For $p \in \mathcal{P}$ and $1 \le j \le d$, let $\theta_j(p)$ be random variables on a certain probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with values on $[-\pi;\pi]$ and satisfying the following conditions:

1).$\mathbb{E}(\mathrm{e}^{i\theta_j(p)}) = 0$ for all $p \in \mathcal{P}$ and $1 \le j \le d$, where $\mathbb{E}(X)$ denotes the expectation of the random element X;

2). the random variables $\theta_j(p)$ and $\theta_k(q)$ are independent for $p \ne q$, p, $q \in \mathcal{P}$;

3). the random variables

$$X(p) \overset{def}{=} \sum_{j=1}^{d}\frac{\mathrm{e}^{i\theta_j(p)}}{d}$$

are identically distributed for every $p \in \mathcal{P}$;

4). there exists an absolute constant $\alpha > 0$ such that, for all $p \in \mathcal{P}$ and all $\varepsilon > 0$,

$$\mathbb{P}\Big(|\theta_1(p)| \le \varepsilon, \ldots, |\theta_d(p)| \le \varepsilon\Big) \gg \varepsilon^{\alpha}.$$

In [17], the following Euler products

$$L(1,X) = \prod_{p}\prod_{j=1}^{d}\left(1 - \frac{\mathrm{e}^{i\theta_j(p)}}{p}\right)^{-1}$$

are considered. Define

$$\Phi(\tau) = \mathbb{P}\big(L(1,X) > (\mathrm{e}^{\gamma_0}\tau)^d\big).$$

Then in [17], it was proved that, for $\tau \gg 1$,

$$\Phi(\tau) = \exp\left\{-\frac{\mathrm{e}^{\tau-A_X}}{\tau}\Big(1+\mathrm{O}\Big(\frac{1}{\sqrt{\tau}}\Big)\Big)\right\},$$

where

$$A_X = 1 + \int\limits_0^\infty \frac{f(t)\mathrm{d}t}{t^2},$$

$$f(t) = \begin{cases} \log \mathbb{E}(\mathrm{e}^{(ReX)t}) & \text{if } 1 \le t < 1, \\ \log \mathbb{E}(\mathrm{e}^{(ReX)t}) - t & \text{if } t \ge 1, \end{cases}$$

and X is a random variable having the same distribution as the $X(p)$.

Note that the latter result covers Theorem 0.9. Moreover, it can be applied to a wider class of L-functions. We will state one theorem related to symmetric power L-functions.

Suppose that $q \in \mathbb{N}$. Then the subgroup of $SL(2, \mathbb{Z})$

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : c \equiv 0 (\mathrm{mod}\ q) \right\}$$

is called a Hecke subgroup. If the equation

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^\kappa F(z)$$

is satisfied for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$, the cusp form $F(z)$ is called a cusp form of weight κ and level q. Denote the space of such forms by $S_\kappa(q)$.

Now we introduce symmetric power L-functions of $F \in S_\kappa(q)$. Denote by $c(m)$ the Fourier coefficients of F, and $c_m = c(m)m^{-\frac{\kappa-1}{2}}$. For any prime number p, define α_p such that

$$c_p\nu = \alpha_p^\nu + \alpha_p^{\nu-2} + \ldots + \alpha_p^{-\nu}, \quad \nu \ge 1,$$

and $|\alpha(p)| = 1$. Then, for $m \in \mathbb{N}$, the symmetric mth power L-function $L(s, sym^m F)$ attached to F is defined, for $\sigma > 1$, by

$$L(s, sym^m F) = \prod_p \prod_{j=0}^m \Big(1 - \frac{\alpha_p^{m-2j}}{p^s}\Big)^{-1},$$

and by analytic continuation elsewhere.

Suppose that q is prime, and define

$$\Phi_q(sym^m, \tau) = \left(\sum_{F \in S_2(q)} \omega_F\right)^{-1} \sum_{\substack{F \in S_2(q) \\ L(1, sym^m F) \ge (\mathrm{e}^{\gamma_0}\tau)^{\mathrm{m}+1}}} \omega_F,$$

where $\omega_F = \frac{1}{4\pi \|F\|}$, and $\|F\|$ the norm of F. Then the following statement is true.

Let

$$\log_k q = \underbrace{\log \ldots \log}_{k} q.$$

Theorem 0.10 ([17]) *For $m \in \mathbb{N}$, uniformly in the region $\tau \le \log_2 q - \log_3 q - 2\log_4 q$*

$$\Phi_q(sym^m, \tau) = \exp\left\{ -\frac{\mathrm{e}^{\tau - A_m}}{\tau}\Big(1 + \mathrm{O}\Big(\frac{1}{\sqrt{\tau}}\Big)\Big)\right\},$$

where

$$A_m = 1 + \int_0^1 h_m(t)\frac{\mathrm{d}t}{t^2} + \int_1^\infty (h_m(t) - t)\frac{\mathrm{d}t}{t^2}$$

and

$$h_m(t) = \log\left(\frac{2}{\pi}\int_0^\pi \exp\Big\{\frac{t}{m+1}\sum_{j=0}^m \cos\big(\theta(m-2j)\big)\Big\}\sin^2\theta \mathrm{d}\theta\right).$$

There are also known other results on value distribution of L-functions with respect to a parameter.

Now we return to the function $L(s, F, \chi)$ and present the results of the thesis.
In Chapter 1, a limit theorem for

$$P_{Q,\mathbb{R}}(A) \stackrel{def}{=} \mu_Q\Big(|L(s, F, \chi)| \in A\Big), \quad A \in \mathcal{B}(\mathbb{R}),$$

is proved. For $\tau \in \mathbb{R}$, let

$$\eta = \eta(\tau) = \frac{i\tau}{2},$$

and, for prime p and $k \in \mathbb{N}$,

$$d_\tau(p^k) = \frac{\eta(\eta+1)\dots(\eta+k-1)}{k!}, \quad d_\tau(1) = 1.$$

Define

$$a_\tau(p^k) = \sum_{l=0}^k d_\tau(p^l)\alpha^l(p)d_\tau(p^{k-l})\beta^{k-l}(p)$$

and

$$b_\tau(p^k) = \sum_{l=0}^k d_\tau(p^l)\overline{\alpha}^l(p)d_\tau(p^{k-l})\overline{\beta}^{k-l}(p),$$

where $\alpha(p)$ and $\beta(p)$ are the coefficients of the Euler product for $L(s, F)$, and $\overline{z}$ denotes the conjugate of z. Moreover, for $m \in \mathbb{N}$, let

$$a_\tau(m) = \prod_{p^l \| m} a_\tau(p^l)$$

and

$$b_\tau(m) = \prod_{p^l \| m} b_\tau(p^l),$$

where $p^l \| m$ means that $p^l | m$ but $p^{l+1} \nmid m$.

Let $P_{\mathbb{R}}$ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by the characteristic transforms

$$w_0(\tau) = w_1(\tau) = \sum_{m=1}^{\infty} \frac{a_\tau(m) b_\tau(m)}{m^{2\sigma}}, \quad \tau \in \mathbb{R}, \quad \sigma > \frac{\kappa+1}{2}.$$

Then the main result of Chapter 1 is the following theorem [22].

Theorem 1.1 *Suppose that $\sigma > \frac{\kappa+1}{2}$. Then the measure $P_{Q,\mathbb{R}}$ converges weakly to $P_{\mathbb{R}}$ as $Q \to \infty$.*

Chapter 2 of the thesis is devoted to the value-distribution of the argument of the twist $L(s, F, \chi)$. The function $L(s, F, \chi)$ has no zeros in the half-plane $\sigma > \frac{\kappa+1}{2}$. We define $\arg L(s, F, \chi)$ from the principal value at $s = \frac{\kappa+3}{2}$ by continuous variation along the path connecting the points $\frac{\kappa+2}{3}$, $\frac{\kappa+2}{3} + it$ and $\sigma + it$. In Chapter 2, we consider

$$P_{Q,\gamma}(A) \overset{def}{=} \mu_Q\Big(\exp\{i \arg L(s, F, \chi)\} \in A \Big), \quad A \in \mathcal{B}(\gamma),$$

where γ is the unit circle on the complex plane. For $k \in \mathbb{Z}$, let

$$\theta = \theta(k) = \frac{k}{2},$$

and, for prime p and $l \in \mathbb{N}$,

$$d_k(p^l) = \frac{\theta(\theta+1)\dots(\theta+l-1)}{l!}, \quad d_k(1) = 1.$$

Similarly as in Theorem 1.1, define, for $m \in \mathbb{N}$,

$$a_k(m) = \prod_{p^l \| m} a_k(p^l)$$

and

$$b_k(m) = \prod_{p^l \| m} b_k(p^l),$$

where

$$a_k(p^l) = \sum_{j=0}^{l} d_k(p^j) \alpha^j(p) d_k(p^{l-j}) \beta^{l-j}(p)$$

and

$$b_k(p^l) = \sum_{j=0}^{l} d_{-k}(p^l) \overline{\alpha}^j(p) d_{-k}(p^{l-j}) \overline{\beta}^{l-j}(p).$$

Moreover, let P_γ be a probability measure on $(\gamma, \mathcal{B}(\gamma))$ defined by the Fourier transform

$$f(k) \overset{def}{=} \int_\gamma x^k \mathrm{d}P_\gamma = \sum_{m=1}^{\infty} \frac{a_k(m) b_k(m)}{m^{2\sigma}}, \quad \sigma > \frac{\kappa+1}{2}.$$

Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\gamma, \mathcal{B}(\gamma))$. We recall that the weak convergence of P_n to P as $n \to \infty$ is equivalent to the convergence

$$P_n(A) \underset{n\to\infty}{\longrightarrow} P(A)$$

for all arcs $A \subset \gamma$ with end points having P-measure zero.

Then we have the following theorem [23].

Theorem 2.1 *Suppose that $\sigma > \frac{\kappa+1}{2}$. Then $P_{Q,\gamma}$ converges weakly to P_γ as $Q \to \infty$.*

Theorem 2.1 also can be stated in the following form [23].

Theorem 2.2 *Suppose that $\sigma > \frac{\kappa+1}{2}$. Then*

$$\mu_Q\Big(\frac{1}{2\pi} \arg L(s, F, \chi) \le x (\text{mod } 1)\Big)$$

converges weakly mod 1 *to the distribution function* mod 1 *defined by the Fourier transform $f(k)$, $k \in \mathbb{Z}$, as $Q \to \infty$.*

In Chapter 3 of the thesis, we connect Theorems 1.1 and 2.1, and prove a limit theorem with increasing modulus on the complex plane for the function $L(s, F, \chi)$.

Note that the function

$$w(\tau, k) \overset{def}{=} \int\limits_{\mathbb{C}\setminus\{0\}} |z|^{i\tau} \mathrm{e}^{ik \arg z} \mathrm{d}P, \quad \tau \in \mathbb{R}, \quad k \in \mathbb{Z},$$

is called a characteristic transform of the probability measure P on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. The measure P is uniquely determine by its characteristic transform $w(\tau, k)$.

Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. We say that P_n converges weakly in the sense of $\mathbb{C}$ to P as $n \to \infty$ if P_n converges weakly to P as $n \to \infty$, and additionally,

$$\lim_{n\to\infty} P_n(\{0\}) = P(\{0\}).$$

For $\tau \in \mathbb{R}$ and $k \in \mathbb{Z}$, let

$$\xi = \xi(\tau, \pm k) = \frac{i\tau \pm k}{2},$$

and, for primes p and $l \in \mathbb{N}$,

$$d_{\tau,\pm k}(p^l) = \frac{\xi(\xi+1)\dots(\xi+l-1)}{l!}, \quad d_{\tau,\pm k}(1) = 1.$$

Define

$$a_{\tau,k}(p^l) = \sum_{j=0}^{l} d_{\tau,k}(p^l)\alpha^j(p) d_{\tau,k}(p^{l-j})\beta^{l-j}(p),$$

and

$$b_{\tau,k}(p^l) = \sum_{j=0}^{l} d_{\tau,-k}(p^l)\overline{\alpha}^j(p) d_{\tau,-k}(p^{l-j})\overline{\beta}^{l-j}(p),$$

and, for $m \in \mathbb{N}$, let

$$a_{\tau,k}(m) = \prod_{p^l \| m} a_{\tau,k}(p^l)$$

and

$$b_{\tau,k}(m) = \prod_{p^l \| m} b_{\tau,k}(p^l).$$

Let $P_{\mathbb{C}}$ be probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ defined by the characteristic transform

$$w(\tau, k) = \sum_{m=1}^{\infty} \frac{a_{\tau,k}(m) b_{\tau,k}(m)}{m^{2\sigma}}, \quad \sigma > \frac{\kappa+1}{2}.$$

Define

$$P_{Q,\mathbb{C}}(A) = \mu_Q\Big(L(s,F,\chi) \in A\Big), \quad A \in \mathcal{B}(\mathbb{C}).$$

Then the main result of Chapter 3 is contained in the following theorem [15].
Theorem 3.1 *Suppose that $\sigma > \frac{\kappa+1}{2}$. Then $P_{Q,\mathbb{C}}$ converges weakly in sense of $\mathbb{C}$ to the measure $P_{\mathbb{C}}$ as $Q \longrightarrow \infty$.*

The last, Chapter 4, of the thesis deals with r-dimensional version of Theorem 1.1.

Let $r \in \mathbb{N} \backslash \{1\}$, and P be a probability measure on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$. Denote by P_j, $j = 1, \dots, r$, P_{j_1,j_2}, $j_2 > j_1 = 1, \dots, r-1, \dots$, $P_{1,\dots,j-1,j+1,\dots,r}$, $j = 1, \dots, r$, one-dimensional, two-dimensional, ..., $(r-1)$-dimensional marginal measures of the measure P. The functions

$$\begin{aligned}
&w_{k_j}(\tau_j) \\
&= \int\limits_{\mathbb{R}\backslash\{0\}} |x_j|^{i\tau_j} \operatorname{sgn}^{k_j} x_j \mathrm{d}P_j, \quad \tau_j \in \mathbb{R}, \quad k_j = 0,1, \quad j = 1, \dots, r, \\
&w_{k_{j_1},k_{j_2}}(\tau_{j_1}, \tau_{j_2}) \\
&= \int\limits_{\mathbb{R}\backslash\{0\}} \int\limits_{\mathbb{R}\backslash\{0\}} |x_{j_1}|^{i\tau_{j_1}} \operatorname{sgn}^{k_{j_1}} x_{j_1} |x_{j_2}|^{i\tau_{j_2}} \operatorname{sgn}^{k_{j_2}} x_{j_2} \mathrm{d}P_{j_1,j_2}, \\
&\quad \tau_{j_1}, \tau_{j_2} \in \mathbb{R}, k_{j_1}, k_{j_2} = 0,1, \quad j_2 > j_1 = 1, \dots, r-1, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r) \\
&= \int\limits_{\mathbb{R}\backslash\{0\}} \dots \int\limits_{\mathbb{R}\backslash\{0\}} |x_1|^{i\tau_1} \operatorname{sgn}^{k_1} x_1 \dots |x_{j-1}|^{i\tau_{j-1}} \operatorname{sgn}^{k_{j-1}} x_{j-1} \\
&\quad \times |x_{j+1}|^{i\tau_{j+1}} \operatorname{sgn}^{k_{j+1}} x_{j+1} \dots |x_r|^{i\tau_r} \operatorname{sgn}^{k_r} x_r \mathrm{d}P_{1,\dots,j-1,j+1,\dots,r}, \\
&\quad \tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r \in \mathbb{R}, \quad k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r = 0,1, \\
&\qquad j = 1, \dots, r, \\
&w_{k_1,\dots,k_r}(\tau_1, \dots, \tau_r) \\
&= \int\limits_{\mathbb{R}\backslash\{0\}} \dots \int\limits_{\mathbb{R}\backslash\{0\}} |x_1|^{i\tau_1} \operatorname{sgn}^{k_1} x_1 \dots |x_r|^{i\tau_r} \operatorname{sgn}^{k_r} x_r \mathrm{d}P,
\end{aligned}$$

$$\tau_1, \dots, \tau_r \in \mathbb{R}, \quad k_1, \dots, k_r = 0, 1,$$

are called the characteristic transforms of the probability measure P on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$. The measure P is uniquely determined by its characteristic transforms [18].

For $j = 1, \dots, r$, let $F_j(z)$ be a holomorphic normalized Hecke eigen cusp form of weight κ_j for the full modular group with the Fourier series expansion

$$F_j(z) = \sum_{m=1}^{\infty} c_j(m) \mathrm{e}^{2\pi i m z}, \quad c_j(1) = 1,$$

and let $L(s, F_j)$ be a corresponding L-function,

$$L(s, F_j) = \sum_{m=1}^{\infty} \frac{c_j(m)}{m^s}, \quad \sigma > \frac{\kappa_j + 1}{2},$$

with the Euler product over primes

$$L(s, F_j) = \prod_p \Big(1 - \frac{\alpha_j(p)}{p^s}\Big)^{-1} \Big(1 - \frac{\beta_j(p)}{p^s}\Big)^{-1},$$

where $\alpha_j(p)$ and $\beta_j(p)$ are complex conjugate numbers satisfying $\alpha_j(p) + \beta_j(p) = c_j(p)$.

In Chapter 4, the value distribution of a collection of twisted L-functions $L(s_1, F_1, \chi)$, ..., $L(s_r, F_r, \chi)$, $s_j = \sigma_j + it_j$, where χ is a Dirichlet character modulo q, q is a prime number, and

$$\begin{aligned} L(s_j, F_j, \chi) &= \sum_{m=1}^{\infty} \frac{c_j(m)\chi(m)}{m^{s_j}} \\ &= \prod_p \Big(1 - \frac{\alpha_j(p)\chi(p)}{p^{s_j}}\Big)^{-1} \Big(1 - \frac{\beta_j(p)\chi(p)}{p^{s_j}}\Big)^{-1}, \quad \sigma_j > \frac{\kappa_j + 1}{2}, \end{aligned}$$

is discussed.

For $j = 1, \dots, r$, define

$$a_{j;\tau}(m) = \prod_{p^k \| m} a_{j;\tau}(p^k)$$

and

$$b_{j;\tau}(m) = \prod_{p^k \| m} b_{j;\tau}(p^k),$$

where

$$a_{j;\tau}(p^k) = \sum_{l=0}^{k} d_\tau(p^l) \alpha_j^l(p) d_\tau(p^{k-l}) \beta_j^{k-l}(p)$$

and

$$b_{j;\tau}(p^k) = \sum_{l=0}^{k} d_\tau(p^l) \overline{\alpha}_j^l(p) d_\tau(p^{k-l}) \overline{\beta}_j^{k-l}(p).$$

Let $P_{\mathbb{R}^r}$ be the probability measure on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$ defined by the characteristic transforms

$$
\begin{cases}
w_{k_j}(\tau_j) = \sum\limits_{m=1}^{\infty} \frac{a_{j;\tau_j}(m) b_{j;\tau_j}(m)}{m^{2\sigma_j}}, \quad j = 1, \dots, r, \\
w_{k_{j_1}, k_{j_2}}(\tau_{j_1}, \tau_{j_2}) \\
\quad = \sum\limits_{m=1}^{\infty} \sum\limits_{m_1 m_2 = m} \frac{a_{j_1;\tau_{j_1}}(m_1) a_{j_2;\tau_{j_2}}(m_2)}{m_1^{s_{j_1}} m_2^{s_{j_2}}} \sum\limits_{n_1 n_2 = m} \frac{b_{j_1;\tau_{j_1}}(n_1) b_{j_2;\tau_{j_2}}(n_2)}{n_1^{\overline{s}_{j_1}} n_2^{\overline{s}_{j_2}}}, \\
\quad j_2 > j_1 = 1, \dots, r-1, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
w_{k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r) \\
\quad = \sum\limits_{m=1}^{\infty} \sum\limits_{m_1 \dots m_{j-1} m_{j+1} \dots m_r = m} \frac{a_{1;\tau_1}(m_1) \dots a_{j-1;\tau_{j-1}}(m_{j-1}) a_{j+1;\tau_{j+1}}(m_{j+1}) \dots a_{r;\tau_r}(m_r)}{m_1^{s_1} \dots m_{j-1}^{s_{j-1}} m_{j+1}^{s_{j+1}} \dots m_r^{s_r}} \\
\quad \times \sum\limits_{n_1 \dots n_{j-1} n_{j+1} \dots n_r = m} \frac{b_{1;\tau_1}(n_1) \dots b_{j-1;\tau_{j-1}}(n_{j-1}) b_{j+1;\tau_{j+1}}(n_{j+1}) \dots b_{r;\tau_r}(n_r)}{n_1^{\overline{s}_1} \dots n_{j-1}^{\overline{s}_{j-1}} n_{j+1}^{\overline{s}_{j+1}} \dots n_r^{\overline{s}_r}}, \\
\quad j = 1, \dots, r, \\
w_{k_1, \dots, k_r}(\tau_1, \dots, \tau_r) \\
\quad = \sum\limits_{m=1}^{\infty} \sum\limits_{m_1 \dots m_r = m} \frac{a_{1;\tau_1}(m_1) \dots a_{r;\tau_r}(m_r)}{m_1^{s_1} \dots m_r^{s_r}} \sum\limits_{n_1 \dots n_r = m} \frac{b_{1;\tau_1}(n_1) \dots b_{r;\tau_r}(n_r)}{n_1^{\overline{s}_1} \dots n_r^{\overline{s}_r}}, \\
\quad \sigma_j > \frac{\kappa_j + 1}{2}, \quad j = 1, \dots, r.
\end{cases}
$$

The main result of Chapter 4 is the following limit theorem [16].
Let

$$P_{Q,\mathbb{R}^r}(A) = \mu_Q\Big(\big(|L(s_1, F_1, \chi)|, \dots, |L(s_r, F_r, \chi)| \big) \in A \Big), \quad A \in \mathcal{B}(\mathbb{R}^r).$$

Theorem 4.1. *Suppose that $\sigma_j > \frac{\kappa_j+1}{2}$, $j = 1, \dots, r$. Then $P_{Q,\mathbb{R}^r}$ converges weakly to the measure $P_{\mathbb{R}^r}$ as $Q \to \infty$.*

In the thesis, we consider the weak convergence of probability measures defined by the twists of $L(s, F, \chi)$ in the half-plane of absolute convergence of the series defining $L(s, F, \chi)$. We believe that the results can be extended to the region $\sigma > \frac{\kappa}{2}$, however, in our opinion, this problem is very difficult. The principal difference from the case of Dirichlet L-functions is that the series defining $L(s, F, \chi)$, $\chi \neq \chi_0$, does not converge for $\sigma > \frac{\kappa}{2}$.

Outline of the thesis

The thesis consists of the introduction, four chapters, conclusions, bibliography and notation. In the introduction, a short review on the actuality of the research field is given, the aims and problems of the thesis are stated, the used methods and novelty of results obtained are shortly discussed. Also, Introduction contains a short history of results with respect to a parameter in the theory of L-functions, and the main results of the thesis. In Chapter 1, a limit theorem with increasing modulus for $|L(s, F, \chi)|$ is proved. Chapter 2 is devoted to a limit theorems of the above type for $\arg L(s, F, \chi)$. Chapter 3 contains a limit theorem with increasing modulus for $L(s, F, \chi)$. In Chapter 4, a joint limit theorem for a collection $|L(s, F_1, \chi)|$, ..., $|L(s, F_r, \chi)|$ is obtained.

Approbation

The results of the thesis were presented at the Conferences of Lithuanian Mathematical Society (2008–2011), at 10^{th} International Vilnius Conference on Probability Theory and Mathematical Statistics (Vilnius, Lithuania, 28^{th} June–2^{nd} July, 2010), at the 16^{th} International Conference Mathematical Modeling and Analysis (Sigulda, Latvia, May 25–28, 2011), at the International Conference 27^{th} Journées Arithmétiques (Vilnius, Lithuania, 27^{th} June–1^{st} July, 2011), as well as at the seminar of number theory of Vilnius University and the seminar of the Faculty of Mathematics and Informatics of Šiauliai University.

Principal publications

The main results of the thesis are published in the following papers:

1. A. Kolupayeva, Value-distribution of twisted L-functions of normalized cusp forms, *Lietuvos Matematikos Rinkinys. LMD darbai.*, **51** (spec. issue) (2010), 35–40.

2. A. Kolupayeva , Value-distribution of twisted automorphic L-functions. III, *Šiauliai Math. Semin.*, **6** (14) (2011), 21–33.

3. A. Kolupayeva, A. Laurinčikas, Value-distribution of twisted automorphic L-functions, *Lith. Math. J.*, **48** (2) (2008), 203–211.

4. A. Kolupayeva, A. Laurinčikas, Value-distribution of twisted automorphic L-functions II, *Lith. Math. J.*, **50** (3) (2010), 284–292.

5. A. Kolupayeva, A. Laurinčikas, Value-distribution of twisted L-functions of normalized cusp forms, *submitted.*

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Chapter 1

A limit theorem for the modulus of twisted L-functions of normalized cusp forms

Let $F(z)$ be a holomorphic normalized Hecke eigen cusp form of weight κ for the full modular group with the Fourier series expansion

$$F(z)=\sum_{m=1}^{\infty} c(m)\mathrm{e}^{2\pi imz}, \quad c(1)=1.$$

The L-function $L(s,F)$, $s=\sigma+it$, attached to $F(z)$ is defined, for $\sigma>\frac{\kappa+1}{2}$, by the Dirichlet series

$$L(s,F)=\sum_{m=1}^{\infty}\frac{c(m)}{m^s},$$

and has analytic continuation to an entire function. Moreover, for $\sigma>\frac{\kappa+1}{2}$, the function $L(s,F)$, has the Euler product expansion over primes

$$L(s,F)=\prod_p\Big(1-\frac{\alpha(p)}{p^s}\Big)^{-1}\Big(1-\frac{\beta(p)}{p^s}\Big)^{-1}, \tag{1.1}$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers satisfying $\alpha(p)+\beta(p)=c(p)$.

Let $\chi(m)$ be a Dirichlet character modulo q, where q is a prime number. Then the twisted L-function $L(s,F,\chi)$ associated to the form $F(z)$ is defined, for $\sigma>\frac{\kappa+1}{2}$, by the Dirichlet series

$$L(s,F,\chi)=\sum_{m=1}^{\infty}\frac{c(m)\chi(m)}{m^s},$$

and has analytic continuation to an entire function. Moreover, similarly to the case of the function $L(s,F)$, $L(s,F,\chi)$ has the Euler product expansion over primes

$$L(s,F,\chi)=\prod_p\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)^{-1}\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)^{-1},\quad \sigma>\frac{\kappa+1}{2}. \tag{1.2}$$

This chapter is devoted to a limit theorem for the modulus $|L(s,F,\chi)|$ of the function $L(s,F,\chi)$ when q increases to infinity.

1.1. Statement of the main theorem

For $Q\geq 2$, denote

$$M_Q=\sum_{q\leq Q}\sum_{\substack{\chi=\chi(\text{mod }q)\\ \chi\neq\chi_0}}1,$$

where χ_0 , as usual, denotes the principal character modulo q. Moreover, for brevity, define

$$\mu_Q(\dots)=M_Q^{-1}\sum_{q\leq Q}\sum_{\substack{\chi=\chi(\text{mod }q)\\ \chi\neq\chi_0\\ \dots}}1,$$

where in place of dots a condition satisfied by a pair $(q,\chi(\text{mod }q))$ is to be written.

For $\tau\in\mathbb{R}$, let

$$\eta=\eta(\tau)=\frac{i\tau}{2},$$

and, for a prime p and $k\in\mathbb{N}$, let

$$d_\tau(p^k)=\frac{\eta(\eta+1)\dots(\eta+k-1)}{k!},\quad d_\tau(1)=1.$$

Now define

$$a_\tau(p^k)=\sum_{l=0}^{k}d_\tau(p^l)\alpha^l(p)d_\tau(p^{k-l})\beta^{k-l}(p)$$

and

$$b_\tau(p^k)=\sum_{l=0}^{k}d_\tau(p^l)\overline{\alpha}^l(p)d_\tau(p^{k-l})\overline{\beta}^{k-l}(p),$$

where the complex numbers $\alpha(p)$ and $\beta(p)$ are defined by (1.1), and $\overline{z}$ denotes the conjugate of a complex number z. For $m\in\mathbb{N}$, we set

$$a_\tau(m)=\prod_{p^l\|m}a_\tau(p^l)$$

and

$$b_\tau(m) = \prod_{p^l \| m} b_\tau(p^l),$$

where $p^l \| m$ means that $p^l | m$ but $p^{l+1} \nmid m$. Then, $a_\tau(m)$ and $b_\tau(m)$ are arithmetic multiplicative functions.

Let $P_{\mathbb{R}}$ be the probability measure on the space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by the characteristic transforms

$$w_0(\tau) = w_1(\tau) = \sum_{m=1}^{\infty} \frac{a_\tau(m) b_\tau(m)}{m^{2\sigma}}, \quad \tau \in \mathbb{R}, \quad \sigma > \frac{\kappa + 1}{2},$$

and

$$P_{Q,\mathbb{R}}(A) = \mu_Q\big(|L(s, F, \chi)| \in A\big), \quad A \in \mathcal{B}(\mathbb{R}).$$

Theorem 1.1 *Suppose that $\sigma > \frac{\kappa+1}{2}$. Then the measure $P_{Q,\mathbb{R}}$ converges weakly to $P_{\mathbb{R}}$ as $Q \to \infty$.*

For the proof of Theorem 1.1, we apply the method of characteristic transforms of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

1.2. Characteristic transforms of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

For convienence of the reader, in this section we remind the theory of characteristic transforms of probability measures on the space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Let $F(x)$ be a distribution function. In [33], V. M. Zolotarev introduced the characteristic transforms of the function $F(x)$ as a multiplicative analogue of the characteristic function of $F(x)$. Let

$$\mathrm{sgn} x = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Then the pair of function

$$w_k(\tau) = \int_{\substack{-\infty \\ x \neq 0}}^{\infty} |x|^{i\tau} \mathrm{sgn}^k x \mathrm{d}F(x), \quad k = 0, 1,$$

are called the characteristic transforms of $F(x)$. V. M. Zolotarev proved [33] the uniqueness and continuity theorems for characteristic transforms. In [21], the Zolotarev theory was rewritten for probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and we apply it for the proof of Theorem 1.1.

Let P be a probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The characteristic transforms $w_k(\tau)$, $k = 0, 1$, of the measure P are defined by

$$w_k(\tau) = \int\limits_{\mathbb{R}\setminus\{0\}} |x|^{i\tau} \mathrm{sgn}^k x \mathrm{d}P, \quad \tau \in \mathbb{R}, \quad k = 0, 1.$$

Lemma 1.2 *The probability measure P is uniquely determined by its characteristic transforms $w_k(\tau)$, $k = 0, 1$.*

The lemma is proved in [21], Theorem 4.

Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We say that P_n, as $n \to \infty$, converges m-weakly to P if P_n converges weakly to P as $n \to \infty$, and, additionally,

$$\lim_{n\to\infty} P_n(\{0\}) = P(\{0\}).$$

The next two lemmas are continuity theorems for probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ in terms of characteristic transforms.

Lemma 1.3 *Suppose that P_n converges m-weakly to the measure P as $n \to \infty$. Then*

$$\lim_{n\to\infty} w_{kn}(\tau) = w_k(\tau), \quad \tau \in \mathbb{R}, \quad k = 0, 1,$$

where $w_{kn}(\tau)$ and $w_k(\tau)$, $k = 0, 1$, are the characteristic transforms of the measures P_n and P, respectively.

The lemma is given in [21], Theorem 5.

Lemma 1.4 *Denote by $w_{kn}(\tau)$, $k = 0, 1$, the characteristic transforms of the probability measure P_n, $n \in \mathbb{N}$, on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and suppose that*

$$\lim_{n\to\infty} w_{kn}(\tau) = w_k(\tau), \quad \tau \in \mathbb{R}, \quad k = 0, 1,$$

where the functions $w_1(\tau)$ and $w_0(\tau)$ are continuous at $\tau = 0$. Then on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, there exists a probability measure P such that the measure P_n converges m-weakly to P as $n \to \infty$. In this case, $w_k(\tau)$, $k = 0, 1$, are the characteristic transforms of the measure P.

The lemma is Theorem 6 from [21].

1.3. Characteristic transforms of $P_{Q,\mathbb{R}}$

Let $0 < \delta < \frac{1}{2}$ be a fixed number, and let

$$R = \{s \in \mathbb{C} : \sigma \geq \frac{\kappa + 1}{2} + \delta\}.$$

From the definitions of $P_{Q,\mathbb{R}}$ and characteristic transforms of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we find that the characteristic transforms $w_{kQ}(\tau)$ of the measure $P_{Q,\mathbb{R}}$ are of the form

$$\begin{aligned} w_{kQ}(\tau) &= \int\limits_{\mathbb{R}\setminus\{0\}} |x|^{i\tau}\operatorname{sgn}^k x \mathrm{d}P \\ &= \frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}} \big|L(s,F,\chi)\big|^{i\tau} \\ &\overset{def}{=} w_Q(\tau), \quad \tau\in\mathbb{R}, \quad k=0,1. \end{aligned} \tag{1.3}$$

Thus, since $|L(s,F,\chi)| \ge 0$, we have only one function $w_Q(\tau)$. Moreover,

$$\begin{aligned} \big|L(s,F,\chi)\big|^{i\tau} &= \big(L(s,F,\chi)\overline{L(s,F,\chi)}\big)^{\frac{i\tau}{2}} \\ &= \big(L(s,F,\chi)\big)^{\frac{i\tau}{2}}\big(\overline{L(s,F,\chi)}\big)^{\frac{i\tau}{2}}. \end{aligned}$$

Therefore, using (1.2), we find that, for $s \in R$,

$$\begin{aligned} &\big|L(s,F,\chi)\big|^{i\tau} \\ &= \bigg(\prod_p \Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)^{-1}\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)^{-1}\bigg)^{-\frac{i\tau}{2}} \\ &\quad\times\bigg(\prod_p \Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)^{-1}\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)^{-1}\bigg)^{-\frac{i\tau}{2}} \\ &= \exp\bigg\{-\frac{i\tau}{2}\sum_p\bigg(\log\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)+\log\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)\bigg) \\ &\quad -\frac{i\tau}{2}\sum_p\bigg(\log\Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)+\log\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)\bigg)\bigg\} \\ &= \prod_p \exp\bigg\{-\frac{i\tau}{2}\bigg(\log\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)+\log\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)\bigg)\bigg\} \\ &\quad\times\prod_p \exp\bigg\{-\frac{i\tau}{2}\bigg(\log\Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)+\log\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)\bigg)\bigg\} \\ &= \prod_p \Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)^{-\frac{i\tau}{2}}\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)^{-\frac{i\tau}{2}} \\ &\quad\times\prod_p \Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)^{-\frac{i\tau}{2}}\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)^{-\frac{i\tau}{2}}. \end{aligned} \tag{1.4}$$

Here the multi-valued functions $\log(1-z)$ and $\big(1-z\big)^{-\frac{i\tau}{2}}$ in the region $|z|<1$ are defined by continuous variation along any path in this region from the values $\log(1-z)\big|_{z=0}=0$

and $\left(1-z\right)^{-\frac{i\tau}{2}}\big|_{z=0}=1$, respectively.

Using the definition of $d_\tau(p^k)$, we have that, for $|z|<1$,

$$(1-z)^{-\eta}=\sum_{k=0}^{\infty} d_\tau(p^k)z^k.$$

Therefore, for $s\in R$,

$$\begin{aligned}\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)^{-\frac{i\tau}{2}} &=\sum_{k=0}^{\infty}\frac{d_\tau(p^k)\alpha^k(p)\chi(p^k)}{p^{ks}},\\ \Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)^{-\frac{i\tau}{2}} &=\sum_{k=0}^{\infty}\frac{d_\tau(p^k)\beta^k(p)\chi(p^k)}{p^{ks}},\\ \Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)^{-\frac{i\tau}{2}} &=\sum_{k=0}^{\infty}\frac{d_\tau(p^k)\overline{\alpha}^k(p)\overline{\chi}(p^k)}{p^{k\overline{s}}}\end{aligned}$$

and

$$\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)^{-\frac{i\tau}{2}}=\sum_{k=0}^{\infty}\frac{d_\tau(p^k)\overline{\beta}^k(p)\overline{\chi}(p^k)}{p^{k\overline{s}}}.$$

Substituting these formulas in (1.4), we obtain that, for $s\in R$,

$$\begin{aligned}&\big|L(s,F,\chi)\big|^{i\tau}\\ &=\prod_p\sum_{k=0}^{\infty}\frac{d_\tau(p^k)\alpha^k(p)\chi(p^k)}{p^{ks}}\sum_{l=0}^{\infty}\frac{d_\tau(p^l)\beta^l(p)\chi(p^l)}{p^{ls}}\\ &\qquad\times\prod_p\sum_{k=0}^{\infty}\frac{d_\tau(p^k)\overline{\alpha}^k(p)\overline{\chi}(p^k)}{p^{k\overline{s}}}\sum_{l=0}^{\infty}\frac{d_\tau(p^l)\overline{\beta}^l(p)\overline{\chi}(p^l)}{p^{l\overline{s}}}\\ &=\prod_p\sum_{k=0}^{\infty}\frac{\widehat{a}_\tau(p^k)}{p^{ks}}\prod_p\sum_{l=0}^{\infty}\frac{\widehat{b}_\tau(p^l)}{p^{l\overline{s}}}\\ &=\sum_{m=1}^{\infty}\frac{\widehat{a}_\tau(m)}{m^s}\sum_{n=1}^{\infty}\frac{\widehat{b}_\tau(n)}{n^{\overline{s}}},\end{aligned}\tag{1.5}$$

where $\widehat{a}_\tau(m)$ and $\widehat{b}_\tau(m)$ are multiplicative functions given, for primes p and $k\in\mathbb{N}$, by

$$\widehat{a}(p^k)=\sum_{l=0}^{k}d_\tau(p^l)\alpha^l(p)\chi(p^l)d_\tau(p^{k-l})\beta^{k-l}(p)\chi(p^{k-l})\tag{1.6}$$

and

$$\widehat{b}(p^k)=\sum_{l=0}^{k}d_\tau(\beta^l)\overline{\alpha}^l(p)\overline{\chi}(p^l)d_\tau(p^{k-l})\overline{\beta}^{k-l}(p)\overline{\chi}(p^{k-l}).\tag{1.7}$$

By the multiplicativity of $\widehat{a}_\tau(m)$ and $\widehat{b}_\tau(m)$, and by (1.6) and (1.7), we find that

$$\begin{aligned}
\widehat{a}_\tau(m) &= \prod_{p^k\|m} \widehat{a}_\tau(p^k) \\
&= \prod_{p^k\|m} \sum_{l=0}^{k} d_\tau(p^l)\alpha^l(p)\chi(p^l)d_\tau(p^{k-l})\beta^{k-l}(p)\chi(p^{k-l}) \\
&= \prod_{p^k\|m} \chi(p^k) \sum_{l=0}^{k} d_\tau(p^l)\alpha^l(p)d_\tau(p^{k-l})\beta^{k-l}(p) \\
&= a_\tau(m)\chi(m)
\end{aligned}$$

and

$$\begin{aligned}
\widehat{b}_\tau(m) &= \prod_{p^k\|m} \widehat{b}_\tau(p^k) \\
&= \prod_{p^k\|m} \sum_{l=0}^{k} d_\tau(p^l)\overline{\alpha}^l(p)\overline{\chi}(p^l)d_\tau(p^{k-l})\overline{\beta}^{k-l}(p)\overline{\chi}(p^{k-l}) \\
&= \prod_{p^k\|m} \overline{\chi}(p^k) \sum_{l=0}^{k} d_\tau(p^l)\overline{\alpha}^l(p)d_\tau(p^{k-l})\overline{\beta}^{k-l}(p) \\
&= b_\tau(m)\overline{\chi}(m),
\end{aligned}$$

where the multiplicative functions $a_\tau(m)$ and $b_\tau(m)$ are defined in Section 1.1. Thus, in view of (1.5) and (1.3), we have that

$$w_Q(\tau) = \frac{1}{M_Q} \sum_{q\le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}} \sum_{m=1}^{\infty} \frac{a_\tau(m)\chi(m)}{m^s} \sum_{n=1}^{\infty} \frac{b_\tau(n)\overline{\chi}(n)}{n^{\overline{s}}}. \tag{1.8}$$

1.4. Asymptotics for the function $w_Q(\tau)$

In this section, we obtain on asymptotic formula for the characteristic transform $w_Q(\tau)$ of the measure $P_{Q,\mathbb{R}}$. Let $c > 0$ be an arbitrary constant.

Theorem 1.5 *Suppose that $Q \to \infty$. Then, uniformly in $|\tau| \le c$ and $s \in R$,*

$$w_Q(\tau) = \sum_{m=1}^{\infty} \frac{a_\tau(m)b_\tau(m)}{m^{2\sigma}} + \mathrm{o}(1).$$

Before the proof of Theorem 1.5, we present some lemmas.

Lemma 1.6 *Suppose that* $|\tau| \leq c$. *Then*

$$|d_\tau(p^k)| \leq (k+1)^{c_1},$$

where the constant c_1 *depends on* c, *only.*

Proof. The definition of $d_\tau(p^k)$ gives

$$\begin{aligned}
\big|d_\tau(p^k)\big| &\leq \frac{|\eta|(|\eta|+1)\dots(|\eta|+k-1)}{k!} \\
&\leq \prod_{v=1}^{k}\Big(1+\frac{|\eta|}{v}\Big) \\
&\leq \exp\Big\{|\eta|\sum_{v=1}^{k}\frac{1}{v}\Big\} \\
&\leq \exp\Big\{|\eta|(1+\log\, k)\Big\} \\
&\leq \exp\Big\{c(1+\log\, k)\Big\} \\
&\leq (k+1)^{c_1}.
\end{aligned}$$

Lemma 1.7 *Suppose that* $(m, q) = 1$. *Then*

$$\sum_{\chi=\chi(\mathrm{mod}q)} \chi(m)\overline{\chi}(n) = \begin{cases} q-1 & \textit{if } m \equiv n(\mathrm{mod}\ q), \\ 0 & \textit{if } m \not\equiv n(\mathrm{mod}\ q). \end{cases}$$

Proof of the lemma can be found, for example, in [3], [14], [27].
Denote by

$$d(m) = \sum_{d|m} 1$$

the classical divisor function.

Lemma 1.8 *For every* $\varepsilon > 0$, *the estimate*

$$d(m) = \mathrm{O}_\varepsilon(m^\varepsilon)$$

is true.

The estimate of the lemma is given, for example, in [27].

Lemma 1.9 *Let* $Q \geq 2$. *Then*

$$M_Q = \frac{Q^2}{2\mathrm{log}Q} + \mathrm{O}\Big(\frac{Q^2}{\mathrm{log}^2 Q}\Big).$$

The proof of the lemma is given in [20], Lemma 2.9.7.

Lemma 1.10 *Let $x \geq 2$. Then*

$$\sum_{p \leq x} 1 = \mathrm{O}\Big(\frac{x}{\log\, x}\Big).$$

The estimate of lemma follows from the asymptotic law of prime numbers, see, for example, [14], [31].

Proof of Theorem 1.5. Lemma 1.6 and the Deligne estimates [5]

$$|\alpha(p)| \leq p^{\frac{\kappa-1}{2}}$$

and

$$|\beta(p)| \leq p^{\frac{\kappa-1}{2}},$$

for $|\tau| \leq c$, imply the bounds

$$\begin{aligned} \big|a_\tau(p^k)\big| &\leq \sum_{l=0}^{k} (l+1)^{c_1} p^{\frac{l(\kappa-1)}{2}} (k-l+1)^{c_1} p^{\frac{(k-l)(\kappa-1)}{2}} \\ &\leq (k+1)^{1+2c_1} p^{\frac{k(\kappa-1)}{2}} \\ &= (k+1)^{c_2} p^{\frac{k(\kappa-1)}{2}}, \end{aligned}$$

where $c_2 = 1 + 2c_1$ and depends on c, only. Therefore, the multiplicativity of the arithmetic functions $a_\tau(m)$ and $b_\tau(m)$, and the formula

$$d(m) = \prod_{p^k \| m} (k+1),$$

show that, for $|\tau| \leq c$,

$$\begin{aligned} \big|a_\tau(m)\big| &= \prod_{p^k \| m} \big|a_\tau(p^k)\big| \\ &\leq \prod_{p^k \| m} (k+1)^{c_2} p^{\frac{k(\kappa-1)}{2}} \\ &= m^{\frac{\kappa-1}{2}} d^{c_2}(m). \end{aligned} \tag{1.9}$$

and

$$\big|b_\tau(m)\big| \leq m^{\frac{\kappa-1}{2}} d^{c_2}(m). \tag{1.10}$$

Let $r = \log\, Q$. Then Lemma 1.8, and (1.9) and (1.10) imply, for $|\tau| \leq c$, $s \in R$ and every $\varepsilon > 0$, the estimates

$$\begin{aligned} \sum_{m > r} \frac{a_\tau(m)\chi(m)}{m^s} &= \mathrm{O}\Big(\sum_{m>r} \frac{m^{\frac{\kappa-1}{2}} d^{c_2}(m)}{m^{\frac{\kappa+1}{2}+\delta}}\Big) \\ &= \mathrm{O}_\varepsilon\Big(\sum_{m>r} \frac{1}{m^{1+\delta-\varepsilon}}\Big) \end{aligned}$$

$$= \mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big)$$

and

$$\sum_{m>r} \frac{b_\tau(m)\chi(m)}{m^{\overline{s}}} = \mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big).$$

From this and from (1.8) we find that, for $|\tau| \le c$, $s \in R$ and every $\varepsilon > 0$,

$$\begin{aligned} w_Q(\tau) &= \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \Bigg(\bigg(\sum_{m \le r} \frac{a_\tau(m)\chi(m)}{m^s} + \mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big) \bigg) \\ &\quad \times \bigg(\sum_{n \le r} \frac{b_\tau(n)\overline{\chi}(n)}{n^{\overline{s}}} + \mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big) \bigg) \Bigg) \\ &= \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \bigg(\sum_{m \le r} \frac{a_\tau(m)\chi(m)}{m^s} \sum_{n \le r} \frac{b_\tau(n)\overline{\chi}(n)}{n^{\overline{s}}} \bigg) \\ &\quad + \mathrm{O}_\varepsilon\Bigg(r^{-\delta+\varepsilon} \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \Big| \sum_{m \le r} \frac{a_\tau(m)\chi(m)}{m^s} \Big| \\ &\quad + \Big| \sum_{n \le r} \frac{b_\tau(n)\overline{\chi}(n)}{n^{\overline{s}}} \Big| \Bigg) + \mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big). \end{aligned} \tag{1.11}$$

Moreover, by (1.9) and (1.10), and Lemma 1.8, for $|\tau| \le c$ and $s \in R$,

$$\begin{aligned} \sum_{m \le r} \frac{a_\tau(m)\chi(m)}{m^s} &= \mathrm{O}\Big(\sum_{m \le r} \frac{d^{c_2}(m)}{m^{1+\delta}} \Big) \\ &= \mathrm{O}(1) \end{aligned}$$

and

$$\sum_{n \le r} \frac{b_\tau(n)\overline{\chi}(n)}{n^{\overline{s}}} = \mathrm{O}(1).$$

Therefore,

$$\begin{aligned} &r^{-\delta+\varepsilon} \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \Big(\Big| \sum_{m \le r} \frac{a_\tau(m)\chi(m)}{m^s} \Big| \\ &\quad + \Big| \sum_{n \le r} \frac{b_\tau(n)\overline{\chi}(n)}{n^{\overline{s}}} \Big| \Big) = \mathrm{O}\big(r^{-\delta+\varepsilon}\big). \end{aligned} \tag{1.12}$$

Clearly, we have that

$$\frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \Big(\sum_{m \le r} \frac{a_\tau(m)\chi(m)}{m^s} \sum_{n \le r} \frac{b_\tau(n)\overline{\chi}(n)}{n^{\overline{s}}} \Big)$$

$$= \sum_{m\le r}\frac{a_\tau(m)}{m^s}\sum_{n\le r}\frac{b_\tau(n)}{n^{\overline{s}}}\frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\chi(m)\overline{\chi}(n). \tag{1.13}$$

If $m = n < r$, then using Lemma 1.10 yields

$$\begin{aligned}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\chi(m)\overline{\chi}(n) &= \sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\left|\chi(m)\right|^2\\ &= M_Q - \sum_{\substack{q\mid m\\ q\le r}}(q-2)\\ &= M_Q + \mathrm{O}\Big(\sum_{q\le r} q\Big)\end{aligned}$$

$$M_Q + \mathrm{O}\Big(r\sum_{q\le r}1\Big) = M_Q + \mathrm{O}(r^2).$$

Thus, this case, in view of Lemma 1.9, for $|\tau| \le c$ and $s \in R$, contributes to (1.11), as $Q \to \infty$,

$$\sum_{m\le r}\frac{a_\tau(m)b_\tau(m)}{m^{2\sigma}} + \mathrm{o}(1) = \sum_{m=1}^{\infty}\frac{a_\tau(m)b_\tau(m)}{m^{2\sigma}} + \mathrm{o}(1). \tag{1.14}$$

In the case $m \ne n$, m, $n \le r$, we apply Lemma 1.7. We have by Lemma 1.10 and the definition of r that

$$\begin{aligned}&\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\chi(m)\overline{\chi}(n)\\ &= \sum_{q\le Q}\sum_{\chi=\chi(\mathrm{mod}\ q)}\chi(m)\overline{\chi}(n) - \sum_{q\le Q}\sum_{\chi=\chi_0(\mathrm{mod}\ q)}\chi(m)\overline{\chi}(n)\\ &= \sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ q\mid(m-n)}}\chi(m)\overline{\chi}(n) + \sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ q\nmid(m-n)}}\chi(m)\overline{\chi}(n)\\ &\quad - \sum_{q\le Q}\sum_{\chi=\chi_0(\mathrm{mod}\ q)}\chi(m)\overline{\chi}(n)\\ &= \mathrm{O}\sum_{q\le r} q + \mathrm{O}\Big(\sum_{q\le Q}1\Big)\\ &= \mathrm{O}(r^2) + \mathrm{O}\Big(\frac{Q}{\log Q}\Big)\\ &= \mathrm{O}\Big(\frac{Q}{\log Q}\Big).\end{aligned}$$

Therefore, by (1.9), (1.10), and Lemma 1.8, for $|\tau| \le c$ and $s \in R$,

$$
\begin{aligned}
&\frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\sum_{\substack{m\le r\\ m\ne n}}\sum_{n\le r}\frac{a_\tau(m)\chi(m)}{m^s}\frac{b_\tau(n)\chi(n)}{n^{\overline{s}}}\\
&\ll\left|\sum_{\substack{m\le r\\ m\ne n}}\sum_{n\le r}\frac{a_\tau(m)}{m^s}\frac{b_\tau(n)}{n^{\overline{s}}}\right|\frac{1}{M_Q}\left|\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\chi(m)\overline{\chi}(n)\right|\\
&\ll\frac{1}{Q}\sum_{m\le r}\frac{|a_\tau(m)|}{m^\sigma}\sum_{n\le r}\frac{|b_\tau(n)|}{n^\sigma}\\
&\ll\frac{1}{Q}\left(\sum_{m\le r}\frac{d^{c_2}(m)}{m^{1+\delta}}\right)^2\ll\frac{1}{Q}.
\end{aligned}
$$

Now this, (1.11)–(1.14) show that, uniformly in $|\tau|\le c$ and $s\in R$,

$$
w_Q(\tau)=\sum_{m=1}^{\infty}\frac{a_\tau(m)b_\tau(m)}{m^{2\sigma}}+\mathrm{o}(1)
$$

as $Q\to\infty$.

1.5. Proof of Theorem 1.1

We apply Lemma 1.4. From Theorem 1.5 and (1.3), it follows that the characteristic transform of the measure $P_{Q,\mathbb{R}}$ converges, uniformly in $|\tau|\le c$ and $s\in R$, to the function

$$
\sum_{m=1}^{\infty}\frac{a_\tau(m)b_\tau(m)}{m^{2\sigma}}
$$

as $Q\to\infty$. Since $\delta>0$ is arbitrary, we have that, for $\sigma>\frac{\kappa+1}{2}$, $w_{kQ}(\tau)$ converges, uniformly in $|\tau|\le c$, to

$$
\sum_{m=1}^{\infty}\frac{a_\tau(m)b_\tau(m)}{m^{2\sigma}} \tag{1.15}
$$

as $Q\to\infty$, $k=0,1$. The functions $a_\tau(m)$ and $b_\tau(m)$ are continuons at $\tau=0$. Thus, all hypotheses of Lemma 1.4 are satified, and we obtain that the measure $P_{Q,\mathbb{R}}$ converges weakly to the measure $P_{\mathbb{R}}$ defined by the characteristic transforms

$$
w_0(\tau)=w_1(\tau)=\sum_{m=1}^{\infty}\frac{a_\tau(m)b_\tau(m)}{m^{2\sigma}},\quad \tau\in\mathbb{R},\quad \sigma>\frac{\kappa+1}{2}.
$$

The theorem is proved.

Chapter 2

A limit theorem for the argument of twisted L-functions of normalized cusp forms

In this chapter, we consider the value-distribution of the argument of the twist $L(s, F, \chi)$ in the half-plane $\sigma > \frac{\kappa+1}{2}$. Define $\arg L(s, F, \chi)$ from the principal value $\arg L(\frac{\kappa+3}{2}, F, \chi)$ by continuous variation along the straight line segments $\left[\frac{\kappa+3}{2}, \frac{\kappa+3}{2}+it\right]$ and $\left[\frac{\kappa+3}{2}, \sigma+it\right]$. In view of the Euler product

$$L(s, F, \chi) = \prod_p \left(1 - \frac{\alpha(p)\chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\chi(p)}{p^s}\right)^{-1}, \quad \sigma > \frac{\kappa+1}{2}, \tag{2.1}$$

and the Deligne estimates [5]

$$|\alpha(p)| \leq p^{\frac{\kappa-1}{2}}, \quad |\beta(p)| \leq p^{\frac{\kappa-1}{2}}, \tag{2.2}$$

we have that $L(s, F, \chi) \neq 0$ for $\sigma > \frac{\kappa+1}{2}$. Thus, $\arg L(s, F, \chi)$ is well-defined. From this definition, it follows that $\arg L(s, F, \chi)$ is defined up to multiple of $2\pi i$.

2.1. Statement of the results

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ denote the unit circle on the complex plane. For $k \in \mathbb{Z}$, let

$$\theta = \theta(k) = \frac{k}{2},$$

and, for prime p and $l \in \mathbb{N}$,

$$d_k(p^l) = \frac{\theta(\theta+1)\dots(\theta+l-1)}{l!}, \quad d_k(1) = 1.$$

Define, for $m \in \mathbb{N}$,

$$a_k(m) = \prod_{p^l \| m} a_k(p^l)$$

and

$$b_k(m) = \prod_{p^l \| m} b_k(p^l),$$

where

$$a_k(p^l) = \sum_{j=0}^{l} d_k(p^j)\alpha^j(p)d_k(p^{l-j})\beta^{l-j}(p)$$

and

$$b_k(p^l) = \sum_{j=0}^{l} d_{-k}(p^j)\overline{\alpha}^j(p)d_{-k}(p^{l-j})\overline{\beta}^{l-j}(p).$$

Moreover, let P_γ be the probability measure on $(\gamma, \mathcal{B}(\gamma))$ defined by the Fourier transform

$$f(k) \stackrel{def}{=} \int_\gamma x^k \mathrm{d}P_\gamma = \sum_{m=1}^{\infty} \frac{a_k(m)b_k(m)}{m^{2\sigma}}, \quad \sigma > \frac{\kappa+1}{2}, \tag{2.3}$$

and

$$P_{Q,\gamma}(A) = \mu_Q\Big(\exp\big\{i \arg L(s,F,\chi)\big\} \in A\Big), \quad A \in \mathcal{B}(\gamma).$$

Let P_n and P, $n \in \mathbb{N}$, be probability measures on $(\gamma, \mathcal{B}(\gamma))$. We recall that the weak convergence of P_n to P as $n \to \infty$ is equivalent to the convergence

$$P_n(A) \underset{n\to\infty}{\longrightarrow} P(A)$$

for all arcs $A \subset \gamma$ with end points having P-measure zero [2].

Theorem 2.1 *Suppose that $\sigma > \frac{\kappa+1}{2}$. Then $P_{Q,\gamma}$ converges weakly to P_γ as $Q \to \infty$.*

A distribution function $G(x)$ is said to be a distribution function mod 1 if $G(x) = 1$ if $x \geq 1$, and $G(x) = 0$ if $x < 0$. A distribution function mod 1 $G_n(x)$, $n \in \mathbb{N}$, converges weakly mod 1 if there exists a distribution function mod 1 $G(x)$ such that at all continuity points x, y, $0 \leq x \leq y < 1$, of $G(x)$

$$\lim_{n\to\infty}(G_n(y) - G_n(x)) = G(y) - G(x).$$

Thus, by this definition, the limit distribution function mod 1 $G(x)$ is determined only up to an additive constant.

Let

$$G_Q(x) = \mu_Q\Big(\frac{1}{2\pi} \arg L(s,F,\chi) \leq x (\text{mod } 1)\Big).$$

Theorem 2.2 *Suppose that* $\sigma > \frac{\kappa+1}{2}$. *Then* $G_Q(x)$ *converges weakly* mod 1 *to the distribution function* mod 1 *defined by the Fourier transform* $f(k)$, $k \in \mathbb{Z}$, *given by* (2.3) *as* $Q \to \infty$.

For the proof of Theorem 2.1 and 2.2, we apply the method of Fourier transforms.

2.2. Fourier transforms of probability measures on $(\gamma, \mathcal{B}(\gamma))$

Let P be a probability measure on $(\gamma, \mathcal{B}(\gamma))$. The Fourier transforms $f(k)$, $k \in \mathbb{Z}$, of P is defined by

$$f(k) = \int_{\gamma} x^k \mathrm{d}P, \quad k \in \mathbb{Z}.$$

Lemma 2.3 *The probability measure* P *is uniquely determined by its Fourier transforms* $f(k)$.

Now let P_n, $n \in \mathbb{N}$, be a probability measure on $(\gamma, \mathcal{B}(\gamma))$.

Lemma 2.4 *Denote by* $f_n(k)$, $k \in \mathbb{Z}$, *the Fourier transform of the measure* P_n, *and suppose that*

$$\lim_{n\to\infty} f_n(k) = f(k), \quad k \in \mathbb{Z}.$$

Then on $(\gamma, \mathcal{B}(\gamma))$, *there exists a a probability measure* P *such that the measure* P_n *converges to* P *as* $n \to \infty$. *In this case,* $f(k)$ *is the Fourier transform of the measure* P.

The theory of weak convergence of probability measures on $(\gamma, \mathcal{B}(\gamma))$ is given in [2].

2.3. Fourier transforms of distributions functions mod 1

Let $G(x)$ be a distribution function mod 1. The Fourier transform $f(k)$, $k \in \mathbb{Z}$, of $G(x)$ is defined by

$$f(k) = \int_0^1 \mathrm{e}^{2\pi ikx} \mathrm{d}G(x), \quad k \in \mathbb{Z}.$$

Lemma 2.5 *The distribution function* mod 1 $G(x)$ *is uniquely determined by its Fourier transform* $f(k)$.

Let $G_n(x)$, $n \in \mathbb{N}$, be a distribution function mod 1.

Lemma 2.6 *Denote by* $f_n(k)$, $k \in \mathbb{Z}$, *the Fourier transform of the distribution function* mod 1 $G_n(x)$, *and suppose that*

$$\lim_{n\to\infty} f_n(k) = f(k), \quad k \in \mathbb{Z}.$$

Then there exists a distribution function mod 1 $G(x)$ *such that* $G_n(x)$ *converges weakly* mod 1 *to* $G(x)$ *as* $n \to \infty$. *In this case,* $f(k)$ *is the Fourier transform of* $G(x)$.

The theory of weak convergence mod 1 for distribution functions mod 1 is given in [9].

2.4. Fourier transform of $P_{Q,\gamma}$

Denote by $f_Q(k)$, $k \in \mathbb{Z}$, the Fourier transform of $P_{Q,\gamma}$, i.e.,

$$f_Q(k) = \int_\gamma x^k \mathrm{d}P_{Q,\gamma}.$$

Thus, by the definition of $P_{Q,\gamma}$, we have that

$$f_Q(k) = \frac{1}{M_Q} \sum_{q\le Q} \sum_{\substack{\chi=\chi(\text{mod } q)\\ \chi\ne\chi_0}} \mathrm{e}^{ik\arg L(s,F,\chi)}. \tag{2.4}$$

It is easily seen that

$$\begin{aligned} \mathrm{e}^{i\arg L(s,F,\chi)} &= \left(|L(s,F,\chi)|\mathrm{e}^{i\arg L(s,F,\chi)}|L(s,F,\chi)|^{-1}\mathrm{e}^{i\arg L(s,F,\chi)}\right)^{\frac{1}{2}} \\ &= \left(\frac{L(s,F,\chi)}{\overline{L(s,F,\chi)}}\right)^{\frac{1}{2}}. \end{aligned} \tag{2.5}$$

Let R be the same region as in Chapter 1. Then, for $s \in R$, the Euler product (2.1) and (2.5) imply

$$\begin{aligned} &\mathrm{e}^{ik\arg L(s,F,\chi)} \\ &= \left(\prod_p (1-\frac{\alpha(p)\chi(p)}{p^s})^{-1}(1-\frac{\beta(p)\chi(p)}{p^s})^{-1}\right)^{\frac{k}{2}} \\ &\times\left(\prod_p (1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}})^{-1}(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}})^{-1}\right)^{\frac{k}{2}} \end{aligned}$$

$$
\begin{aligned}
&= \exp\Bigg\{ -\frac{k}{2}\sum_p \Big(\log\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)+\log\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)\Big) \\
&\quad +\frac{k}{2}\sum_p \Big(\log\Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)+\log\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)\Big)\Bigg\} \\
&= \prod_p \exp\Bigg\{ -\frac{k}{2}\Big(\log\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)+\log\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)\Big)\Bigg\} \\
&\quad \times\prod_p \exp\Bigg\{ \frac{k}{2}\Big(\log\Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)+\log\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)\Big)\Bigg\} \\
&= \prod_p \Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)^{-\frac{k}{2}}\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)^{-\frac{k}{2}} \\
&\quad \times\prod_p \Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)^{\frac{k}{2}}\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)^{\frac{k}{2}}. \qquad (2.6)
\end{aligned}
$$

Here, as in Chapter 1, the multi-valued functions $\log(1-z)$ and $\big(1-z\big)^{\pm\frac{k}{2}}$ in the region $|z|<1$ are defined by continuation variation along any path in this region from the values $\log(1-z)\big|_{z=0}=0$ and $\big(1-z\big)^{\pm\frac{k}{2}}\big|_{z=0}=1$, respectively.

Using the definition of $d_k(p^l)$, we have that, for $|z|<1$,

$$(1-z)^{-\theta}=\sum_{l=0}^{\infty} d_k(p^l)z^l$$

and

$$(1-z)^{\theta}=\sum_{l=0}^{\infty} d_{-k}(p^l)z^l.$$

Hence, for $s\in R$,

$$
\begin{aligned}
\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)^{-\frac{k}{2}} &= \sum_{l=0}^{\infty}\frac{d_k(p^l)\alpha^l(p)\chi(p^l)}{p^{ls}}, \\
\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)^{-\frac{k}{2}} &= \sum_{l=0}^{\infty}\frac{d_k(p^l)\beta^l(p)\chi(p^l)}{p^{ls}}, \\
\Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)^{\frac{k}{2}} &= \sum_{l=0}^{\infty}\frac{d_{-k}(p^l)\overline{\alpha}^l(p)\overline{\chi}(p^l)}{p^{l\overline{s}}}
\end{aligned}
$$

and

$$\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)^{\frac{k}{2}} = \sum_{l=0}^{\infty}\frac{d_{-k}(p^l)\overline{\beta}^l(p)\overline{\chi}(p^l)}{p^{l\overline{s}}}.$$

Substituting the latter expressions in (2.6), we find that, for $s \in R$,

$$
\begin{aligned}
&e^{ik\arg L(s,F,\chi)}\\
&= \prod_p \sum_{l=0}^{\infty} \frac{d_k(p^l)\alpha^l(p)\chi(p^l)}{p^{ls}} \sum_{l=0}^{\infty} \frac{d_k(p^l)\beta^l(p)\chi(p^l)}{p^{ls}}\\
&\quad \times \prod_p \sum_{l=0}^{\infty} \frac{d_{-k}(p^l)\overline{\alpha}^l(p)\overline{\chi}(p^l)}{p^{l\overline{s}}} \sum_{l=0}^{\infty} \frac{d_{-k}(p^l)\overline{\beta}^l(p)\overline{\chi}(p^l)}{p^{l\overline{s}}}\\
&= \prod_p \sum_{l=0}^{\infty} \frac{\widehat{a}_k(p^l)}{p^{ls}} \prod_p \sum_{l=0}^{\infty} \frac{\widehat{b}_k(p^l)}{p^{l\overline{s}}}\\
&= \sum_{m=1}^{\infty} \frac{\widehat{a}_k(m)}{m^s} \sum_{n=1}^{\infty} \frac{\widehat{b}_k(n)}{n^{\overline{s}}},
\end{aligned} \tag{2.7}
$$

where $\widehat{a}_k(m)$ and $\widehat{b}_k(m)$ are multiplicative functions given, for primes p and $l \in \mathbb{N}$, by

$$
\widehat{a}_k(p^l) = \sum_{j=0}^{l} d_k(p^j)\alpha^j(p)\chi(p^j)d_k(p^{l-j})\beta^{l-j}(p)\chi(p^{l-j}) \tag{2.8}
$$

and

$$
\widehat{b}_k(p^l) = \sum_{j=0}^{l} d_{-k}(p^j)\overline{\alpha}^j(p)\overline{\chi}(p^j)d_{-k}(p^{l-j})\overline{\beta}^{l-j}(p)\overline{\chi}(p^{l-j}). \tag{2.9}
$$

The multiplicativity of the functions $\widehat{a}_k(m)$ and $\widehat{b}_k(m)$, and the complete multiplicativity of the character χ together with (2.8) and (2.9) imply

$$
\begin{aligned}
\widehat{a}_k(m) &= \prod_{p^l \| m} \widehat{a}_k(p^l)\\
&= \prod_{p^l \| m} \sum_{j=0}^{l} d_k(p^j)\alpha^j(p)\chi(p^j)d_k(p^{l-j})\beta^{l-j}(p)\chi(p^{l-j})\\
&= \prod_{p^l \| m} \chi(p^j) \sum_{j=0}^{l} d_k(p^j)\alpha^j(p)d_k(p^{l-j})\beta^{l-j}(p)\\
&= a_k(m)\chi(m)
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{b}_k(m) &= \prod_{p^l \| m} \widehat{b}_k(p^l)\\
&= \prod_{p^l \| m} \sum_{j=0}^{l} d_{-k}(p^j)\overline{\alpha}^j(p)\overline{\chi}(p^j)d_{-k}(p^{l-j})\overline{\beta}^{l-j}(p)\overline{\chi}(p^{l-j})\\
&= \prod_{p^l \| m} \overline{\chi}(p^j) \sum_{j=0}^{l} d_{-k}(p^j)\overline{\alpha}^j(p)d_{-k}(p^{l-j})\overline{\beta}^{l-j}(p)
\end{aligned}
$$

$$= \quad b_k(m)\overline{\chi}(m),$$

where the multiplicative functions $a_k(m)$ and $b_k(m)$ are defined in Section 2.1. From this, in view of (2.4) and (2.7), we obtain that

$$\begin{aligned} f_Q(k) &= \frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\sum_{m=1}^{\infty}\frac{a_k(m)\chi(m)}{m^s} \\ &\quad \times\sum_{n=1}^{\infty}\frac{b_k(n)\overline{\chi}(n)}{n^{\overline{s}}}. \end{aligned} \tag{2.10}$$

2.5. Asymptotics for $f_Q(k)$

For the proof of Theorems 2.1 and 2.2, we need an asymptotic formula for the Fourier transform $f_Q(k)$ as $Q \to \infty$.

Theorem 2.7 *Suppose that $Q \to \infty$. Then, for any $k \in \mathbb{Z}$, uniformly in $s \in R$,*

$$f_Q(k) = \sum_{m=1}^{\infty}\frac{a_k(m)b_k(m)}{m^{2\sigma}} + \mathrm{o}(1).$$

Proof. Repeating the proof of Lemma 1.6, we find that

$$\left|d_k(p^l)\right| \le \left(l+1\right)^c,$$

where the constant c depends on k, only. Therefore, the Deligne estimates (2.2), and the definition of $a_k(p^l)$ and $b_k(p^l)$ show that

$$\begin{aligned} \left|a_k(p^l)\right| &\le \sum_{j=0}^{l}(l+1)^c p^{\frac{l(\kappa-1)}{2}}(l-j+1)^c p^{\frac{(l-j)(\kappa-1)}{2}} \\ &\le (l+1)^{1+2c}p^{\frac{l(\kappa-1)}{2}} \end{aligned}$$

and

$$\left|b_k(p^l)\right| \le (l+1)^{1+2c}p^{\frac{l(\kappa-1)}{2}}.$$

Therefore, by multiplicativity of the functions $a_k(m)$ and $b_k(m)$, taking into account the formula

$$d(m) = \sum_{p^l\|m}(l+1),$$

we find that

$$\left|a_k(m)\right| = \prod_{p^l\|m}\left|a_k(p^l)\right|$$

$$\begin{aligned} &\leq \prod_{p^l \| m} (l+1)^{c_1} p^{\frac{l(\kappa-1)}{2}} \\ &= m^{\frac{\kappa-1}{2}} d^{c_1}(m) \end{aligned} \tag{2.11}$$

and

$$\big|b_k(m)\big| \leq m^{\frac{\kappa-1}{2}} d^{c_1}(m), \tag{2.12}$$

where $c_1 = 1 + 2c$ depends on k, only.

Let, as in Chapter 1, $r = \log\ Q$. Then, using Lemma 1.8, and estimate (2.11) and (2.12), we obtain that, for every $k \in \mathbb{Z}$ and $\varepsilon > 0$, uniformly in $s \in R$,

$$\begin{aligned} \sum_{m>r} \frac{a_k(m)\chi(m)}{m^s} &= \mathrm{O}\Big(\sum_{m>r} \frac{m^{\frac{\kappa-1}{2}} d^{c_1}(m)}{m^{\frac{\kappa+1}{2}+\delta}}\Big) \\ &= \mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big) \end{aligned}$$

and

$$\sum_{m>r} \frac{b_k(m)\chi(m)}{m^{\overline{s}}} = \mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big).$$

This and (2.10) show that, for every $k \in \mathbb{Z}$ and $\varepsilon > 0$, uniformly in $s \in R$,

$$\begin{aligned} f_Q(k) &= \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \neq \chi_0}} \left(\left(\sum_{m \leq r} \frac{a_k(m)\chi(m)}{m^s} + \mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big) \right) \right. \\ &\quad \left. \times \left(\sum_{n \leq r} \frac{b_k(n)\overline{\chi}(n)}{n^{\overline{s}}} + \mathrm{O}_\varepsilon\big(r^{-\delta+\varepsilon}\big) \right) \right) \\ &= \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \neq \chi_0}} \left(\sum_{m \leq r} \frac{a_k(m)\chi(m)}{m^s} \sum_{n \leq r} \frac{b_k(n)\overline{\chi}(n)}{n^{\overline{s}}} \right) \\ &\quad + \mathrm{O}_\varepsilon\Big(r^{-\delta+\varepsilon} \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \neq \chi_0}} \left(\left| \sum_{m \leq r} \frac{a_k(m)\chi(m)}{m^s} \right| \right. \\ &\quad \left. + \left| \sum_{n \leq r} \frac{b_k(n)\overline{\chi}(n)}{n^{\overline{s}}} \right| \right) \Big) + \mathrm{O}_\varepsilon\big(r^{-2(\delta-\varepsilon)}\big). \end{aligned} \tag{2.13}$$

Moreover, by (2.11), (2.12) and Lemma 1.8, for every $k \in \mathbb{Z}$ and $s \in R$,

$$\sum_{m \leq r} \frac{a_k(m)\chi(m)}{m^s} = \mathrm{O}\Big(\sum_{m \leq r} \frac{d^c(m)}{m^{1+\delta}}\Big) = \mathrm{O}(1)$$

and

$$\sum_{n \leq r} \frac{b_k(n)\overline{\chi}(n)}{n^{\overline{s}}} = \mathrm{O}(1).$$

Hence, for every $k \in \mathbb{Z}$, $\varepsilon > 0$ and $s \in R$,

$$r^{-\delta+\varepsilon} \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\text{mod } q) \\ \chi \ne \chi_0}} \Bigg(\Bigg| \sum_{m \le r} \frac{a_k(m)\chi(m)}{m^s} \Bigg| + \Bigg| \sum_{n \le r} \frac{b_k(n)\overline{\chi}(n)}{n^{\overline{s}}} \Bigg| = \mathrm{O}\big(r^{-\delta+\varepsilon}\big). \tag{2.14}$$

From (2.13) and (2.14), we find that, for every $k \in \mathbb{Z}$, $\varepsilon > 0$ and $s \in R$,

$$\begin{aligned} f_Q(k) &= \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\text{mod } q) \\ \chi \ne \chi_0}} \Bigg(\sum_{m \le r} \frac{a_k(m)\chi(m)}{m^s} \sum_{n \le r} \frac{b_k(n)\overline{\chi}(n)}{n^{\overline{s}}} \Bigg) + \mathrm{O}\big(r^{-\delta+\varepsilon}\big) \\ &= \sum_{m \le r} \frac{a_k(m)}{m^s} \sum_{n \le r} \frac{b_k(n)}{n^{\overline{s}}} \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\text{mod } q) \\ \chi \ne \chi_0}} \chi(m)\overline{\chi}(n) + \mathrm{O}\big(r^{-\delta+\varepsilon}\big). \end{aligned} \tag{2.15}$$

In Section 1.4, it was obtained that, for $m = n$,

$$\sum_{q \le Q} \sum_{\substack{\chi=\chi(\text{mod } q) \\ \chi \ne \chi_0}} \chi(m)\overline{\chi}(n) = M_Q + \mathrm{O}(r^2),$$

while, for $m \ne n$,

$$\sum_{q \le Q} \sum_{\substack{\chi=\chi(\text{mod } q) \\ \chi \ne \chi_0}} \chi(m)\overline{\chi}(n) = \mathrm{O}\Big(\frac{Q}{\log Q}\Big).$$

Therefore, from (2.15) and Lemma 1.9 we deduce that, for every k and $s \in R$,

$$f_Q(k) = \sum_{m=1}^{\infty} \frac{a_k(m)b_k(m)}{m^{2\sigma}} + \mathrm{o}(1)$$

as $Q \to \infty$.

2.6. Proof of Theorems 2.1 and 2.2

Theorem 2.1 is a straightforward consequence of Lemma 2.4 and Theorem 2.7.

Proof of Theorem 2.2. We have that the Fourier transform $f_Q(k)$, $k \in \mathbb{Z}$, of the function $G_Q(x)$ is

$$f_Q(k) = \int_0^1 \mathrm{e}^{2\pi ikx} \mathrm{d}G_Q(x) = \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\text{mod } q) \\ \chi \ne \chi_0}} \mathrm{e}^{ik\arg L(s,F,\chi)}.$$

Therefore, the theorem follows from Lemma 2.6 and Theorem 2.7.

Chapter 3

A limit theorem on the complex plane for twisted L-functions of normalized cusp forms

In this chapter, we generalize the results of Chapters 1 and 2, and we prove a limit theorem for $L(s, F, \chi)$ on the complex plane $\mathbb{C}$.

3.1. Statement of the results

For $\tau \in \mathbb{R}$ and $k \in \mathbb{Z}$, let

$$\xi = \xi(\tau, \pm k) = \frac{i\tau \pm k}{2},$$

and, for primes p and $l \in \mathbb{N}$,

$$d_{\tau,\pm k}(p^l) = \frac{\xi(\xi+1)\ldots(\xi+l-1)}{l!}, \quad d_{\tau,\pm k}(1) = 1.$$

Similarly, as in previous chapters, we define

$$a_{\tau,k}(p^l) = \sum_{j=0}^{l} d_{\tau,k}(p^l)\alpha^j(p) d_{\tau,k}(p^{l-j})\beta^{l-j}(p)$$

and

$$b_{\tau,k}(p^l) = \sum_{j=0}^{l} d_{\tau,-k}(p^l)\overline{\alpha}^j(p) d_{\tau,-k}(p^{l-j})\overline{\beta}^{l-j}(p),$$

and for $m \in \mathbb{N}$, let

$$a_{\tau,k}(m) = \prod_{p^l \| m} a_{\tau,k}(p^l)$$

and

$$b_{\tau,k}(m) = \prod_{p^l \| m} b_{\tau,k}(p^l).$$

Thus, $a_{\tau,k}(m)$ and $b_{\tau,k}(m)$ are multiplicative arithmetical functions with respect to m.

Let P be a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Then the function

$$w(\tau,k) = \int\limits_{\mathbb{C}\setminus\{0\}} |z|^{i\tau} \mathrm{e}^{ik \arg z} \mathrm{d}P, \quad \tau \in \mathbb{R}, \quad k \in \mathbb{Z}, \tag{3.1}$$

is a characteristic transform of the measure P.

Let $P_{\mathbb{C}}$ be a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ defined by the characteristic transform

$$w(\tau,k) = \sum_{m=1}^{\infty} \frac{a_{\tau,k}(m) b_{\tau,k}(m)}{m^{2\sigma}}, \quad \tau \in \mathbb{R}, \quad k \in \mathbb{Z}, \quad \sigma > \frac{\kappa+1}{2}.$$

Define

$$P_{Q,\mathbb{C}}(A) = \mu_Q\Big(L(s,F,\chi) \in A\Big), \quad A \in \mathcal{B}(\mathbb{C}).$$

Theorem 3.1 *Suppose that $\sigma > \frac{\kappa+1}{2}$. Then $P_{Q,\mathbb{C}}$ converges weakly in the sense of $\mathbb{C}$ to $P_{\mathbb{C}}$ as $Q \to \infty$.*

For the proof of Theorem 3.1, we apply the method of characteristic transforms of probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$.

3.2. Characteristic transforms of probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$

In this section, we state the results from [19], [20] on characteristic transforms of probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$.

Let P be a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, and let $w(\tau,k)$ be its characteristic transform defined by (3.1).

Lemma 3.2 *The probability measure P is uniquely determined by its characteristic transform $w(\tau,k)$.*

Now let P_n, $n \in \mathbb{N}$, and P, be probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. We say that P_n, as $n \to \infty$, converges weakly in the sense of $\mathbb{C}$ to P if P_n converges weakly to P as $n \to \infty$, and, additionally,

$$\lim_{n\to\infty} P_n(\{0\}) = P(\{0\}).$$

The next two lemmas are devoted to weak convergence of probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Denote by $w_n(\tau, k)$ the characteristic transform of the measure P_n.

Lemma 3.3 *Suppose that P_n converges weakly to the measure P as $n \to \infty$. Then*

$$\lim_{n\to\infty} w_n(\tau, k) = w(\tau, k), \quad \tau \in \mathbb{R}, \quad k \in \mathbb{Z}.$$

Lemma 3.4 *Suppose that*

$$\lim_{n\to\infty} w_n(\tau, k) = w(\tau, k), \quad \tau \in \mathbb{R}, \quad k \in \mathbb{Z},$$

where the function $w(\tau, 0)$ is continuous at $\tau = 0$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure P such that P_n converges weakly in the sense of $\mathbb{C}$ to P as $n \to \infty$. In this case, $w(\tau, k)$ is the characteristic transform of the measure P.

3.3. Characteristic transform of $P_{Q,\mathbb{C}}$

Denote by $w_Q(\tau, k)$ the characteristic transform of $P_{Q,\mathbb{C}}$. By the definition of $P_{Q,\mathbb{C}}$, we have that

$$\begin{aligned} w_Q(\tau, k) &= \int\limits_{\mathbb{C}\setminus\{0\}} |z|^{i\tau} \mathrm{e}^{ik \arg z} \mathrm{d}P_{Q,\mathbb{C}} \\ &= \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \ne \chi_0}} |L(s, F, \chi)|^{i\tau} \mathrm{e}^{ik \arg L(s,F,\chi)}, \end{aligned} \tag{3.2}$$

$\tau \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Let R, as in previous chapters, denote the half-plane $\{s \in \mathbb{C} : \sigma \ge \frac{\kappa+1}{2} + \delta\}$, $\delta > 0$. We recall that $L(s, F, \chi) \ne 0$ for $s \in R$.

Since

$$|L(s, F, \chi)| = \left(L(s, F, \chi)\overline{L(s, F, \chi)}\right)^{\frac{1}{2}},$$

and

$$\mathrm{e}^{i \arg L(s,F,\chi)} = \left(\frac{L(s, F, \chi)}{\overline{L(s, F, \chi)}}\right)^{\frac{1}{2}},$$

from the Euler product

$$L(s,F,\chi)=\prod_p\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)^{-1}\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)^{-1},$$

we find that, for $s\in R$,

$$\begin{aligned}
&\big|L(s,F,\chi)\big|^{i\tau}\mathrm{e}^{ik\arg L(s,F,\chi)}\\
&=\exp\Bigg\{-\frac{i\tau}{2}\sum_p\Big(\log\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)+\log\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)\Big)\\
&-\frac{i\tau}{2}\sum_p\Big(\log\Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)+\log\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)\Big)\\
&-\frac{k}{2}\sum_p\Big(\log\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)+\log\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)\Big)\\
&+\frac{k}{2}\sum_p\Big(\log\Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)+\log\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)\Big)\Bigg\}\\
&=\prod_p\exp\Bigg\{-\frac{i\tau}{2}\Big(\log\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)+\log\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)\Big)\\
&-\frac{k}{2}\Big(\log\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)+\log\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)\Big)\Bigg\}\\
&\times\prod_p\exp\Bigg\{-\frac{i\tau}{2}\Big(\log\Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)+\log\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)\Big)\\
&-\frac{k}{2}\Big(\log\Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)+\log\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)\Big)\Bigg\}\\
&=\prod_p\Big(1-\frac{\alpha(p)\chi(p)}{p^s}\Big)^{-\frac{i\tau+k}{2}}\Big(1-\frac{\beta(p)\chi(p)}{p^s}\Big)^{-\frac{i\tau+k}{2}}\\
&\times\prod_p\Big(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)^{-\frac{i\tau-k}{2}}\Big(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\Big)^{-\frac{i\tau-k}{2}}.
\end{aligned}\tag{3.3}$$

Here the multi-valued functions $\log(1-z)$ and $(1-z)^{-w}$, $w\in\mathbb{C}\backslash\{0\}$, in the region $|z|<1$ are defined by continuous variation along any path in this region from the values $\log(1-z)\big|_{z=0}=0$ and $(1-z)^{-w}\big|_{z=0}=1$, respectively.

In the above notation, we have that , for $|z|<1$,

$$(1-z)^{-\xi}=\sum_{l=0}^{\infty}d_{\tau,\pm k}(p^l)z^l.$$

Therefore, (3.3) shows that, for $s\in R$,

$$\big|L(s,F,\chi)\big|^{i\tau}\mathrm{e}^{ik\arg L(s,F,\chi)}$$

$$
\begin{aligned}
&= \prod_p \sum_{j=0}^{\infty} \frac{d_{\tau,k}(p^j)\alpha^j(p)\chi(p^j)}{p^{js}} \sum_{l=0}^{\infty} \frac{d_{\tau,k}(p^l)\beta^l(p)\chi(p^l)}{p^{ls}} \\
&\quad \times \prod_p \sum_{j=0}^{\infty} \frac{d_{\tau,-k}(p^j)\overline{\alpha}^j(p)\overline{\chi}(p^j)}{p^{j\overline{s}}} \sum_{l=0}^{\infty} \frac{d_{\tau,-k}(p^l)\overline{\beta}^l(p)\overline{\chi}(p^l)}{p^{l\overline{s}}} \\
&= \prod_p \sum_{j=0}^{\infty} \frac{\widehat{a}_{\tau,k}(p^j)}{p^{js}} \prod_p \sum_{l=0}^{\infty} \frac{\widehat{b}_{\tau,k}(p^l)}{p^{l\overline{s}}} \\
&= \sum_{m=1}^{\infty} \frac{\widehat{a}_{\tau,k}(m)}{m^s} \sum_{n=1}^{\infty} \frac{\widehat{b}_{\tau,k}(n)}{n^{\overline{s}}}, \qquad (3.4)
\end{aligned}
$$

where $\widehat{a}_{\tau,k}(m)$ and $\widehat{b}_{\tau,k}(m)$ are multiplicative functions defined, for primes p and $l \in \mathbb{N}$, by

$$\widehat{a}_{\tau,k}(p^l) = \sum_{j=0}^{l} d_{\tau,k}(p^j)\alpha^j(p)\chi(p^j)d_{\tau,k}(p^{l-j})\beta^{l-j}(p)\chi(p^{l-j}) \qquad (3.5)$$

and

$$\widehat{b}_{\tau,k}(p^l) = \sum_{j=0}^{l} d_{\tau,-k}(p^j)\overline{\alpha}^j(p)\overline{\chi}(p^j)d_{\tau,-k}(p^{l-j})\overline{\beta}^{l-j}(p)\overline{\chi}(p^{l-j}). \qquad (3.6)$$

Let c be an arbitrary positive constant. For $|\tau| \le c$ and $l \in \mathbb{N}$,

$$
\begin{aligned}
\left|d_{\tau,\pm k}(p^l)\right| &\le \frac{|\xi|(|\xi|+1)\dots(|\xi|+l-1)}{l!} \\
&\le \prod_{v=1}^{l}\left(1+\frac{|\xi|}{v}\right) \\
&\le \exp\left\{|\xi|\sum_{v=1}^{l}\frac{1}{v}\right\} \\
&\le \left(l+1\right)^{c_1}
\end{aligned}
$$

with a suitable positive constant c_1 depending on c and k, only. This, estimates (2.2), and equalities (3.5) and (3.6)) imply, for $|\tau| \le c$ and $l \in \mathbb{N}$, the bounds

$$\left|\widehat{a}_{\tau,k}(p^l)\right| \le (l+1)^{c_2} p^{\frac{l(\kappa-1)}{2}}$$

and

$$\left|\widehat{b}_{\tau,k}(p^l)\right| \le (l+1)^{c_2} p^{\frac{l(\kappa-1)}{2}}$$

with a positive constant depending on c and k. Therefore, by the multiplicativity of $\widehat{a}_{\tau,k}(m)$ and $\widehat{b}_{\tau,k}(m)$, we find that

$$\left|\widehat{a}_{\tau,k}(m)\right| = \prod_{p^l \| m} \left|\widehat{a}_{\tau,k}(p^l)\right|$$

$$\begin{aligned} &\leq m^{\frac{\kappa-1}{2}} \prod_{p^l \| m} (l+1)^{c_2} \\ &= m^{\frac{\kappa-1}{2}} d^{c_2}(m) \end{aligned} \tag{3.7}$$

and

$$\left|\widehat{b}_{\tau,k}(m)\right| \leq m^{\frac{\kappa-1}{2}} d^{c_2}(m). \tag{3.8}$$

3.4. Asymptotics for $w_Q(\tau, k)$

In this section, we give an asymptotic formula for the characteristic transform $w_Q(\tau, k)$ of the probability measure $P_{Q,\mathbb{C}}$. By (3.2) and (3.4), we have that, for $s \in R$,

$$w_Q(\tau, k) = \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi=\chi(\text{mod } q) \\ \chi \neq \chi_0}} \sum_{m=1}^{\infty} \frac{\widehat{a}_{\tau,k}(m)}{m^s} \sum_{n=1}^{\infty} \frac{\widehat{b}_{\tau,k}(n)}{n^{\overline{s}}}, \tag{3.9}$$

where, for the multiplicative functions $\widehat{a}_{\tau,k}(m)$ and $\widehat{b}_{\tau,k}(m)$,the estimates (3.7) and (3.8) are satisfied. We use the same notation $r = \log Q$ as in previous chapters. Then (3.7), (3.8) and Lemma 1.8 show that, uniformly in $s \in R$, $|\tau| \leq c$, and any fixed $k \in \mathbb{Z}$ and $\varepsilon > 0$,

$$\begin{aligned} \sum_{m>r} \frac{\widehat{a}_{\tau,k}(m)}{m^s} &= O\Big(\sum_{m>r} \frac{m^{\frac{\kappa-1}{2}} d^{c_2}(m)}{m^{\frac{\kappa+1}{2}+\delta}}\Big) \\ &= O_{\varepsilon}\Big(\sum_{m>r} \frac{1}{m^{1+\delta-\varepsilon}}\Big) \\ &= O_{\varepsilon}\big(r^{-\delta+\varepsilon}\big), \end{aligned}$$

and

$$\sum_{n>r} \frac{\widehat{b}_{\tau,k}(n)}{n^{\overline{s}}} = O_{\varepsilon}\big(r^{-\delta+\varepsilon}\big).$$

Substituting this in (3.9), we obtain that, for $s \in R$, $|\tau| \leq c$ and any fixed k,

$$\begin{aligned} w_Q(\tau, k) &= \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi=\chi(\text{mod } q) \\ \chi \neq \chi_0}} \Big(\sum_{m \leq r} \frac{\widehat{a}_{\tau,k}(m)}{m^s} + \mathrm{O}_{\varepsilon}(r^{-\delta+\varepsilon})\Big) \\ &\quad \times \Big(\sum_{n \leq r} \frac{\widehat{b}_{\tau,k}(n)}{n^{\overline{s}}} + \mathrm{O}_{\varepsilon}(r^{-\delta+\varepsilon})\Big) \\ &= \frac{1}{M_Q} \sum_{q \leq Q} \sum_{\substack{\chi=\chi(\text{mod } q) \\ \chi \neq \chi_0}} \Big(\sum_{m \leq r} \frac{\widehat{a}_{\tau,k}(m)}{m^s} \sum_{n \leq r} \frac{\widehat{b}_{\tau,k}(n)}{n^{\overline{s}}}\Big) \end{aligned}$$

$$
\begin{aligned}
&+\mathrm{O}_\varepsilon\Bigg(\frac{r^{-\delta+\varepsilon}}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\Big(\Big|\sum_{m\le r}\frac{\widehat{a}_{\tau,k}(m)}{m^s}\Big| \\
&+\Big|\sum_{n\le r}\frac{\widehat{b}_{\tau,k}(n)}{n^{\overline{s}}}\Big|\Big) + \mathrm{O}_\varepsilon(r^{-2(\delta-\varepsilon)})\\
= &\ \frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\Bigg(\sum_{m\le r}\frac{\widehat{a}_{\tau,k}(m)}{m^s}\sum_{n\le r}\frac{\widehat{b}_{\tau,k}(n)}{n^{\overline{s}}}\Bigg)\\
&+\mathrm{O}_\varepsilon(r^{-\delta+\varepsilon}).
\end{aligned} \tag{3.10}
$$

Here we have used the estimates

$$
\begin{aligned}
\sum_{m\le r}\frac{\widehat{a}_{\tau,k}(m)}{m^s} &= \mathrm{O}_\varepsilon\Big(\sum_{m\le r}\frac{m^{\frac{\kappa-1}{2}}d^{c_2}(m)}{m^{\frac{\kappa+1}{2}+\delta}}\Big)\\
&= \mathrm{O}_\varepsilon\Big(\sum_{m\le r}\frac{1}{m^{1+\delta-\varepsilon}}\Big)\\
&= \mathrm{O}_\varepsilon\Big(\sum_{m=1}^{\infty}\frac{1}{m^{1+\delta-\varepsilon}}\Big)\\
&= \mathrm{O}_\varepsilon(1)
\end{aligned}
$$

and

$$
\sum_{n\le r}\frac{\widehat{b}_{\tau,k}(n)}{n^{\overline{s}}} = \mathrm{O}_\varepsilon(1)
$$

which are uniform in $s\in R$ and $|\tau|\le c$.

By (3.5), (3.6), using the multiplicativity of $\widehat{a}_{\tau,k}(m)$ and $\widehat{b}_{\tau,k}(m)$ as well the notation for $a_{\tau,k}(m)$ and $b_{\tau,k}(m)$, we find that

$$
\begin{aligned}
\widehat{a}_{\tau,k}(m) &= \prod_{p^l\|m}\sum_{j=0}^{l} d_{\tau,k}(p^j)\alpha^j(p)\chi(p^j)d_{\tau,k}(p^{l-j})\beta^{l-j}(p)\chi(p^{l-j})\\
&= \prod_{p^l\|m}\chi(p^l)\sum_{j=0}^{l} d_{\tau,k}(p^j)\alpha^j(p)d_{\tau,k}(p^{l-j})\beta^{l-j}(p)\\
&= a_{\tau,k}(m)\chi(m)
\end{aligned}
$$

and

$$
\widehat{b}_{\tau,k}(m) = b_{\tau,k}(m)\overline{\chi}(m).
$$

Therefore, by (3.10), for $s\in R$, $|\tau|\le c$ and any fixed $k\in\mathbb{Z}$, and $\varepsilon>0$,

$$
w_Q(\tau,k) = \frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\chi(m)\overline{\chi}(n)\Bigg(\sum_{m\le r}\frac{a_{\tau,k}(m)}{m^s}\sum_{n\le r}\frac{b_{\tau,k}(n)}{n^{\overline{s}}}\Bigg)
$$

$$+\mathrm{O}_\varepsilon\bigl(r^{-\delta+\varepsilon}\bigr). \tag{3.11}$$

Since, for $m = n$,

$$\sum_{q\le Q}\sum_{\substack{\chi=\chi(\bmod\ q)\\ \chi\ne\chi_0}}\chi(m)\overline{\chi}(n) = M_Q + \mathrm{O}(r^2),$$

and, for $m \ne n$,

$$\sum_{q\le Q}\sum_{\substack{\chi=\chi(\bmod\ q)\\ \chi\ne\chi_0}}\chi(m)\overline{\chi}(n) = \mathrm{O}\Bigl(\frac{Q}{\log\ Q}\Bigr),$$

see Section 1.4, we have in view of (3.11) that, for $s \in R$, $|\tau| \le c$ and any fixed $k \in \mathbb{Z}$,

$$\begin{aligned} w_Q(\tau,k) &= \sum_{m\le r}\frac{a_{\tau,k}(m)b_{\tau,k}(m)}{m^{2\sigma}} + \mathrm{O}_\varepsilon(r^{-\delta+\varepsilon}) + \mathrm{O}\Bigl(\frac{r^2}{Q^2}\Bigr) + \mathrm{O}\Bigl(\frac{1}{Q}\Bigr) \\ &= \sum_{m=1}^{\infty}\frac{a_{\tau,k}(m)b_{\tau,k}(m)}{m^{2\sigma}} + \mathrm{o}(1) \end{aligned}$$

as $Q \to \infty$.

3.5. Proof of Theorem 3.1

In Section 3.3, it was obtained that, for $s \in R$, $|\tau| \le c$ and any fixed k,

$$w_Q(\tau,k) = \sum_{m=1}^{\infty}\frac{a_{\tau,k}(m)b_{\tau,k}(m)}{m^{2\sigma}} + \mathrm{o}(1) \tag{3.12}$$

as $Q \to \infty$. The functions $a_{\tau,k}(m)$ and $b_{\tau,k}(m)$ are continuous in τ. Therefore, the uniform convergence for $|\tau| \le c$ of the series

$$w(\tau,k) = \sum_{m=1}^{\infty}\frac{a_{\tau,k}(m)b_{\tau,k}(m)}{m^{2\sigma}}$$

shows that the function $w(\tau,0)$ is continuous at $\tau = 0$. Therefore, (3.12) together with Lemma 3.4 proves the theorem.

Chapter 4

A joint limit theorem for twisted L-functions of normalized cusp forms

The aim of this chapter is a generalization of Theorem 1.1 to the space $\mathbb{R}^r$, $r \in \mathbb{N} \backslash \{1\}$.

For $j = 1, \ldots, r$, let $F_j(z)$ be a holomorphic normalized Hecke eigen cusp form of weight κ_j for the full modular group with the Fourier series expansion

$$F_j(z) = \sum_{m=1}^{\infty} c_j(m) \mathrm{e}^{2\pi i m z}, \quad c_j(1) = 1,$$

and let $L(s, F_j)$ be a corresponding L-function,

$$L(s, F_j) = \sum_{m=1}^{\infty} \frac{c_j(m)}{m^s}, \quad \sigma > \frac{\kappa_j + 1}{2},$$

with the Euler product over primes

$$L(s, F_j) = \prod_p \Big(1 - \frac{\alpha_j(p)}{p^s}\Big)^{-1} \Big(1 - \frac{\beta_j(p)}{p^s}\Big)^{-1}, \quad \sigma > \frac{\kappa_j + 1}{2},$$

where $\alpha_j(p)$ and $\beta_j(p)$ are complex conjugate numbers satisfying $\alpha_j(p) + \beta_j(p) = c_j(p)$.

Let, as in previous chapters, χ be a Dirichlet character modulo q, and q be a prime number. In this chapter, we prove a limit theorem with increasing modulus for the vector $(|L(s_1, F_1, \chi)|, \ldots, |L(s_r, F_r, \chi)|)$, $s_j = \sigma_j + it_j$, where for $\sigma_j > \frac{\kappa_j+1}{2}$,

$$L(s_j, F_j, \chi) \quad = \quad \sum_{m=1}^{\infty} \frac{c_j(m)\chi(m)}{m^{s_j}}$$

$$= \prod_p \Big(1 - \frac{\alpha_j(p)\chi(p)}{p^{s_j}}\Big)^{-1} \Big(1 - \frac{\beta_j(p)\chi(p)}{p^{s_j}}\Big)^{-1}, \quad j = 1, \dots, r.$$

4.1 Statement of the theorem

For $j = 1, \dots, r$, define

$$a_{j;\tau}(m) = \prod_{p^k \| m} a_{j;\tau}(p^k),$$

and

$$b_{j;\tau}(m) = \prod_{p^k \| m} b_{j;\tau}(p^k),$$

where

$$a_{j;\tau}(p^k) = \sum_{l=0}^{k} d_\tau(p^l)\alpha_j^l(p) d_\tau(p^{k-l})\beta_j^{k-l}(p)$$

and

$$b_{j;\tau}(p^k) = \sum_{l=0}^{k} d_\tau(p^l)\overline{\alpha}_j^l(p) d_\tau(p^{k-l})\overline{\beta}_j^{k-l}(p),$$

and $d_\tau(p^k)$ is defined in Chapter 1.

Let $P_{\mathbb{R}^r}$ be the probability measure on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$ defined by the characteristic transforms

$$w_{k_j}(\tau_j) = \sum_{m=1}^{\infty} \frac{a_{j;\tau_j}(m) b_{j;\tau_j}(m)}{m^{2\sigma_j}}, \quad j = 1, \dots, r,$$

$$w_{k_{j_1},k_{j_2}}(\tau_{j_1}, \tau_{j_2}) = \sum_{m=1}^{\infty} \sum_{m_1 m_2 = m} \frac{a_{j_1;\tau_{j_1}}(m_1) a_{j_2;\tau_{j_2}}(m_2)}{m_1^{s_{j_1}} m_2^{s_{j_2}}} \sum_{n_1 n_2 = m} \frac{b_{j_1;\tau_{j_1}}(n_1) b_{j_2;\tau_{j_2}}(n_2)}{n_1^{\bar{s}_{j_1}} n_2^{\bar{s}_{j_2}}},$$

$j_2 > j_1 = 1, \dots, r-1,$

....

$$w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r) = \sum_{m=1}^{\infty} \sum_{\substack{m_1 \dots m_{j-1} \\ \times m_{j+1} \dots m_r = m}} \frac{a_{1;\tau_1}(m_1) \dots a_{j-1;\tau_{j-1}}(m_{j-1}) a_{j+1;\tau_{j+1}}(m_{j+1}) \dots a_{r;\tau_r}(m_r)}{m_1^{s_1} \dots m_{j-1}^{s_{j-1}} m_{j+1}^{s_{j+1}} \dots m_r^{s_r}}$$
$$\times \sum_{\substack{n_1 \dots n_{j-1} \\ \times n_{j+1} \dots n_r = m}} \frac{b_{1;\tau_1}(n_1) \dots b_{j-1;\tau_{j-1}}(n_{j-1}) b_{j+1;\tau_{j+1}}(n_{j+1}) \dots b_{r;\tau_r}(n_r)}{n_1^{\bar{s}_1} \dots n_{j-1}^{\bar{s}_{j-1}} n_{j+1}^{\bar{s}_{j+1}} \dots n_r^{\bar{s}_r}},$$

$j = 1, \dots, r,$

$$w_{k_1,\dots,k_r}(\tau_1, \dots, \tau_r) = \sum_{m=1}^{\infty} \sum_{m_1 \dots m_r = m} \frac{a_{1;\tau_1}(m_1) \dots a_{r;\tau_r}(m_r)}{m_1^{s_1} \dots m_r^{s_r}} \sum_{n_1 \dots n_r = m} \frac{b_{1;\tau_1}(n_1) \dots b_{r;\tau_r}(n_r)}{n_1^{\bar{s}_1} \dots n_r^{\bar{s}_r}},$$

where $\sigma_j > \frac{\kappa_j+1}{2}$, $j = 1, \dots, r$.

Define

$$P_{Q,\mathbb{R}^r}(A) = \mu_Q\Big(\big(|L(s_1, F_1, \chi)|, \dots, |L(s_r, F_r, \chi)|\big) \in A\Big), \quad A \in \mathcal{B}(\mathbb{R}^r).$$

Theorem 4.1 *Suppose that* $\sigma_j > \frac{\kappa_j+1}{2}$, $j = 1, \dots, r$. *Then* $P_{Q,\mathbb{R}^r}$ *converges weakly to the measure* $P_{\mathbb{R}^r}$ *as* $Q \to \infty$.

For the proof of Theorem 4.1, we apply the method of characteristic transforms of probability measures on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$.

4.2. Characteristic transform of probability measures on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$

Let P be a probability measure on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$. Denote by P_j, $j = 1, \dots, r$, P_{j_1,j_2}, $j_2 > j_1 = 1, \dots, r-1, \dots,$ $P_{1,\dots,j-1,j+1,\dots,r}$, $j = 1, \dots, r$, one-dimensional, two-dimensional, ..., $(r-1)$-dimensional marginal measures of the measure P. That is,

$$
\begin{aligned}
&P_j(A_j)\\
&\quad = P(\underbrace{\mathbb{R}\times\ldots\times\mathbb{R}}_{j-1}\times A_j\times\mathbb{R}\times\ldots\times\mathbb{R}),\quad A_j\in\mathcal{B}(\mathbb{R}),\quad j=1,\ldots,r,\\
&P_{j_1,j_2}(A_{j_1}\times A_{j_2})\\
&\quad = P(\underbrace{\underbrace{\mathbb{R}\times\ldots\times\mathbb{R}}_{j_1-1}\times A_{j_1}\times\mathbb{R}\times\ldots\times\mathbb{R}}_{j_2-1}\times A_{j_2}\times\mathbb{R}\times\ldots\times\mathbb{R}),\\
&\qquad A_{j_1},A_{j_2}\in\mathcal{B}(\mathbb{R}),\quad j_2>j_1=1,\ldots,r-1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&P_{1,\ldots,j-1,j+1,\ldots,r}(A_1\times\ldots\times A_{j-1}\times A_{j+1}\times\ldots\times A_r)\\
&\quad = P(A_1\times\ldots\times A_{j-1}\times\mathbb{R}\times A_{j+1}\times\ldots\times A_r),\\
&\qquad A_1,\ldots,A_{j-1},A_{j+1},\ldots A_r\in\mathcal{B}(\mathbb{R}),\quad j=1,\ldots,r.
\end{aligned}
$$

Then the functions

$$
\begin{aligned}
&w_{k_j}(\tau_j)\\
&\quad = \int\limits_{\mathbb{R}\setminus\{0\}}|x_j|^{i\tau_j}\operatorname{sgn}^{k_j}x_j\mathrm{d}P_j,\quad \tau_j\in\mathbb{R},\quad k_j=0,1,\quad j=1,\ldots,r,\\
&w_{k_{j_1},k_{j_2}}(\tau_{j_1},\tau_{j_2})\\
&\quad = \int\limits_{\mathbb{R}\setminus\{0\}}\int\limits_{\mathbb{R}\setminus\{0\}}|x_{j_1}|^{i\tau_{j_1}}\operatorname{sgn}^{k_{j_1}}x_{j_1}|x_{j_2}|^{i\tau_{j_2}}\operatorname{sgn}^{k_{j_2}}x_{j_2}\mathrm{d}P_{j_1,j_2},\\
&\qquad \tau_{j_1},\tau_{j_2}\in\mathbb{R},k_{j_1},k_{j_2}=0,1,\quad j_2>j_1=1,\ldots,r-1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&w_{k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_r}(\tau_1,\ldots,\tau_{j-1},\tau_{j+1},\ldots,\tau_r)\\
&\quad = \int\limits_{\mathbb{R}\setminus\{0\}}\ldots\int\limits_{\mathbb{R}\setminus\{0\}}|x_1|^{i\tau_1}\operatorname{sgn}^{k_1}x_1\ldots|x_{j-1}|^{i\tau_{j-1}}\operatorname{sgn}^{k_{j-1}}x_{j-1}\\
&\qquad \times|x_{j+1}|^{i\tau_{j+1}}\operatorname{sgn}^{k_{j+1}}x_{j+1}\ldots|x_r|^{i\tau_r}\operatorname{sgn}^{k_r}x_r\mathrm{d}P_{1,\ldots,j-1,j+1,\ldots,r},\\
&\qquad \tau_1,\ldots,\tau_{j-1},\tau_{j+1},\ldots,\tau_r\in\mathbb{R},\quad k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_r=0,1,\\
&\qquad\quad j=1,\ldots,r,\\
&w_{k_1,\ldots,k_r}(\tau_1,\ldots,\tau_r)\\
&\quad = \int\limits_{\mathbb{R}\setminus\{0\}}\ldots\int\limits_{\mathbb{R}\setminus\{0\}}|x_1|^{i\tau_1}\operatorname{sgn}^{k_1}x_1\ldots|x_r|^{i\tau_r}\operatorname{sgn}^{k_r}x_r\mathrm{d}P,\\
&\qquad \tau_1,\ldots,\tau_r\in\mathbb{R},\quad k_1,\ldots,k_r=0,1,
\end{aligned}
$$

are called the characteristic transforms of the probability measure P on $(\mathbb{R}^r,\mathcal{B}(\mathbb{R}^r))$. They were introduced in [18], where, in place of probability measures, the distribution functions were used. Obviously, the results of [18] remain valid for probability measures. Thus, we have the following statements.

Lemma 4.2 *The probability measure P on $(\mathbb{R}^r,\mathcal{B}(\mathbb{R}^r))$ is uniquely determined by its characteristic transforms* $\big\{w_{k_j}(\tau_j),\, w_{k_{j_1},k_{j_2}}(\tau_{j_1},\tau_{j_2}),\,\ldots,\, w_{k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_r}(\tau_1,\,\ldots,\,\tau_{j-1},\,\tau_{j+1},\,\ldots,\,\tau_r),\, w_{k_1,\ldots,k_r}(\tau_1,\ldots,\tau_r)\big\}$.

Lemma 4.3 *Let P_n, $n \in \mathbb{N}$, be a probability measure on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$, and let $\big\{w_{k_j;n}(\tau_j)$, $w_{k_{j_1},k_{j_2};n}(\tau_{j_1}, \tau_{j_2})$, …, $w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r;n}(\tau_1, \dots, \tau_{j-1},\ \tau_{j+1},\ \dots,\ \tau_r)$, $w_{k_1,\dots,k_r;n}(\tau_1,$ …, $\tau_r)\big\}$ be the corresponding characteristic transforms. Suppose that*

$$
\begin{aligned}
&\lim_{n\to\infty} w_{k_j;n}(\tau_j)\\
&\quad = w_{k_j}(\tau_j) \quad j = 1, \dots, r,\\
&\lim_{n\to\infty} w_{k_{j_1},k_{j_2};n}(\tau_{j_1}, \tau_{j_2})\\
&\quad = w_{k_{j_1},k_{j_2}}(\tau_{j_1}, \tau_{j_2}), \quad j_2 > j_1 = 1, \dots, r-1,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\lim_{n\to\infty} w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r;n}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r)\\
&\quad = w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r), \quad j = 1, \dots, r,\\
&\lim_{n\to\infty} w_{k_1,\dots,k_r;n}(\tau_1, \dots, \tau_r)\\
&\quad = w_{k_1,\dots,k_r}(\tau_1, \dots, \tau_r)
\end{aligned}
$$

for all $\tau_j \in \mathbb{R}$, τ_{j_1}, $\tau_{j_2} \in \mathbb{R}$, …, τ_1, …, τ_{j-1}, τ_{j+1}, …, $\tau_r \in \mathbb{R}$, τ_1, …, $\tau_r \in \mathbb{R}$, and that the functions $w_{k_j}(\tau_j)$, $w_{k_{j_1},k_{j_2}}(\tau_{j_1}, \tau_{j_2})$, …, $w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r)$, $w_{k_1,\dots,k_r}(\tau_1, \dots \tau_r)$ are continuous at $\tau_j = 0$; $\tau_{j_1} = 0$, $\tau_{j_2} = 0$; …, $\tau_1 = 0$,…, $\tau_{j-1} = 0$,$\tau_{j+1} = 0$,…, $\tau_r = 0$; $\tau_1 = 0, \dots, \tau_r = 0$, respectively. Then, on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$, there exists a probability measure P such that P_n converges weakly to P as $n \to \infty$. In this case, $\big\{w_{k_j}(\tau_j)$, $w_{k_{j_1},k_{j_2}}(\tau_{j_1}, \tau_{j_2})$, …, $w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r)$, $w_{k_1,\dots,k_r}(\tau_1, \dots, \tau_r)\big\}$ are the characteristic transforms of the measure P.

Proofs of Lemmas 4.2 and 4.3 are given in [18].

We note that the convergence of characteristic transforms gives more than the weak convergence of P_n to P, $P_n(A)$ also converges to $P(A)$ for some special sets $A \in \mathcal{B}(\mathbb{R}^r)$ involving $\{0\}$ as coordinate sets, for example, for $A = \underbrace{\{0\} \times \ldots \times \{0\}}_{r}$, however, as it is stated in Lemma 4.3, we limit ourselves only by the weak convergence.

4.3. Characteristic transform of $P_{Q,\mathbb{R}^r}$

In virtue of Lemma 4.3, we have to consider the characteristic transforms $\big\{w_{k_j;Q}(\tau_j)$, $w_{k_{j_1},k_{j_2};Q}(\tau_{j_1}, \tau_{j_2})$, …, $w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r;Q}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1},\ \dots,\ \tau_r)$, $w_{k_1,\dots,k_r;Q}(\tau_1,$ …, $\tau_r)\big\}$ of the measure $P_{Q,\mathbb{R}^r}$ as $Q \to \infty$.

As in previous chapters,the Euler product and Deligne estimates imply that $L(s_j, F_j, \chi) \neq 0$ for $\sigma_j > \frac{\kappa_j+1}{2}$, $j = 1, \dots, r$. Moreover, since the measure $P_{Q,\mathbb{R}^r}$ is defined by the modulus of $L(s_j, F_j, \chi)$, its characteristic transforms do not depend on k_j; k_{j_1}, k_{j_2}; …;

$k_1, \dots, k_r$. Therefore, the definition of $P_{Q,\mathbb{R}^r}$ and of characteristic transforms yield

$$\begin{cases} w_{k_j;Q}(\tau_j) \\ \quad = \frac{1}{M_Q} \sum\limits_{q \le Q} \sum\limits_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \left|L(s_j, F_j, \chi)\right|^{i\tau_j}, \quad j = 1, \dots, r; \\ w_{k_{j_1}, k_{j_2};Q}(\tau_{j_1}, \tau_{j_2}) \\ \quad = \frac{1}{M_Q} \sum\limits_{q \le Q} \sum\limits_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \left|L(s_{j_1}, F_{j_1}, \chi)\right|^{i\tau_{j_1}} \left|L(s_{j_2}, F_{j_2}, \chi)\right|^{i\tau_{j_2}}, \\ \qquad j_2 > j_1 = 1, \dots, r-1; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r;Q}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r) \\ \quad = \frac{1}{M_Q} \sum\limits_{q \le Q} \sum\limits_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \left|L(s_1, F_1, \chi)\right|^{i\tau_1} \times \dots \left|L(s_{j-1}, F_{j-1}, \chi)\right|^{i\tau_{j-1}} \\ \qquad \times \left|L(s_{j+1}, F_{j+1}, \chi)\right|^{i\tau_{j+1}} \dots \left|L(s_r, F_r, \chi)\right|^{i\tau_r}, \quad j = 1, \dots, r; \\ w_{k_1,\dots,k_r;Q}(\tau_1, \dots, \tau_r) \\ \quad = \frac{1}{M_Q} \sum\limits_{q \le Q} \sum\limits_{\substack{\chi=\chi(\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \left|L(s_1, F_1, \chi)\right|^{i\tau_1} \dots \left|L(s_r, F_r, \chi)\right|^{i\tau_r}. \end{cases} \tag{4.1}$$

Let $0 < \delta < \frac{1}{2}$ be fixed, and let $R_j = \{s \in \mathbb{C} : \sigma_j \ge \frac{\kappa_j + 1}{2} + \delta\}$. Then in Chapter 1, Theorem 1.5, it was obtained that, uniformly in $|\tau| \le c$ and $s_j \in R_j$,

$$w_{k_j;Q}(\tau_j) = \sum_{m=1}^{\infty} \frac{a_{j;\tau_j}(m) b_{j;\tau_j}(m)}{m^{2\sigma_j}} + \mathrm{o}(1), \tag{4.2}$$

as $Q \to \infty$.

So, it remains to consider the functions $w_{k_{j_1},k_{j2};Q}(\tau_{j_1}, \tau_{j_2})$, $w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r;Q}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r)$, …, $w_{k_1,\dots,k_r;Q}(\tau_1, \dots, \tau_r)$. Clearly, it suffices to limit ourselves by the investigations, say, of the function $w_{k_1,\dots,k_r}(\tau_1, \dots, \tau_r)$ because other characteristic transforms are considered similarly.

It follows by (1.5) that, for $s_j \in R_j$,

$$\left|L(s_j, F_j, \chi)\right|^{i\tau_j} = \sum_{m=1}^{\infty} \frac{\widehat{a}_{j;\tau_j}(m)}{m^{s_j}} \sum_{n=1}^{\infty} \frac{\widehat{b}_{j;\tau_j}(n)}{n^{\overline{s}_j}}, \tag{4.3}$$

where $\widehat{a}_{j;\tau}(m)$ and $\widehat{b}_{j;\tau}(m)$ are multiplicative functions given, for primes p and $k \in \mathbb{N}$, by

$$\widehat{a}_{j;\tau}(p^k) = \sum_{l=0}^{k} d_{\tau_j}(p^l) \alpha_j^l(p) \chi(p^l) d_{\tau_j}(p^{k-l}) \beta_j^{k-l}(p) \chi(p^{k-l}) \tag{4.4}$$

and

$$\widehat{b}_{j;\tau}(p^k) = \sum_{l=0}^{k} d_{\tau_j}(p^l) \overline{\alpha}_j^l(p) \overline{\chi}(p^l) d_{\tau_j}(p^{k-l}) \overline{\beta}_j^{k-l}(p) \overline{\chi}(p^{k-l}), \tag{4.5}$$

$j = 1, \ldots, r$. Moreover, for $|\tau_j| \le c$, where $c > 0$ is an arbitrary constant,

$$\left|\widehat{a}_{j,\tau}(m)\right| \le m^{\frac{\kappa_j - 1}{2}} d^{c_1}(m), \tag{4.6}$$

and

$$\left|\widehat{b}_{j;\tau}(m)\right| \le m^{\frac{\kappa_j - 1}{2}} d^{c_1}(m), \tag{4.7}$$

$m \in \mathbb{N}$ and $j = 1, \ldots, r$, with a suitable constant $c_1 > 0$ depending on c, only.

Now, in view of (4.1) and (4.3), we have that, for $s_j \in R_j$,

$$\begin{aligned} w_{k_1,\ldots,k_r;Q}(\tau_1, \ldots, \tau_r) &= \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \sum_{m_1=1}^{\infty} \frac{\widehat{a}_{1;\tau_1}(m_1)}{m_1^{s_1}} \sum_{n_1=1}^{\infty} \frac{\widehat{b}_{1;\tau_1}(n_1)}{n_1^{\bar{s}_1}} \\ &\quad \times \ldots \times \sum_{m_r=1}^{\infty} \frac{\widehat{a}_{r;\tau_r}(m_r)}{m_r^{s_r}} \sum_{n_r=1}^{\infty} \frac{\widehat{b}_{r;\tau_r}(n_r)}{n_r^{\bar{s}_r}}, \end{aligned} \tag{4.8}$$

where $\widehat{a}_{j;\tau_j}(m)$ and $\widehat{b}_{j;\tau_j}(m)$ are multiplicative functions defined by (4.3) and (4.4), and satisfying estimates (4.6) and (4.7), $j = 1, \ldots, r$.

4.4. Asymptotics of $w_{k_1,\ldots,k_r}(\tau_1, \ldots, \tau_r)$

Let $N = \log\ Q$. Then the estimate $d(m) = O_\varepsilon(m^\varepsilon)$ with arbitrary $\varepsilon > 0$ and estimates (4.6) and (4.7) imply, for $|\tau_j| \le c$ and $s_j \in R_j$, the estimates

$$\begin{aligned} \sum_{m_j > N} \frac{\widehat{a}_{j;\tau_j}(m)}{m_j^{s_j}} &= O\left(\sum_{m_j > N} \frac{m_j^{\frac{\kappa_j - 1}{2}} d^{c_1}(m)}{m^{\frac{\kappa_j - 1}{2} + \delta}} \right) \\ &= O_\varepsilon\left(\sum_{m_j > N} \frac{1}{m^{1+\delta-\varepsilon}} \right) \\ &= O_\varepsilon\left(N^{-\delta+\varepsilon}\right), \end{aligned}$$

and

$$\sum_{n_j > N} \frac{\widehat{b}_{j;\tau_j}(n)}{n_j^{\bar{s}_j}} = O_\varepsilon\left(N^{-\delta+\varepsilon}\right),$$

$j = 1, \ldots, r$. From this and (4.8), we find that, for $|\tau_j| \le c$ and $s_j \in R_j$, $j = 1, \ldots, r$,

$$\begin{aligned} & w_{k_1,\ldots,k_r;Q}(\tau_1, \ldots, \tau_r) \\ & = \frac{1}{M_Q} \sum_{q \le Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \left(\sum_{m_1 \le N} \frac{\widehat{a}_{1;\tau_1}(m_1)}{m_1^{s_1}} + O_\varepsilon\left(N^{-\delta+\varepsilon}\right) \right) \end{aligned}$$

$$\times\left(\sum_{n_1\le N}\frac{\widehat{b}_{1;\tau_1}(n_1)}{n_1^{\overline{s}_1}}+O_\varepsilon\big(N^{-\delta+\varepsilon}\big)\right)\dots$$
$$\times\left(\sum_{m_r\le N}\frac{\widehat{a}_{r;\tau_r}(m_r)}{m_r^{s_r}}+O_\varepsilon\big(N^{-\delta+\varepsilon}\big)\right)\left(\sum_{n_r\le N}\frac{\widehat{b}_{r;\tau_r}(n_r)}{n_r^{\overline{s}_r}}+O_\varepsilon\big(N^{-\delta+\varepsilon}\big)\right)$$
$$=\frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\left(\sum_{m_1\le N}\frac{\widehat{a}_{1;\tau_1}(m_1)}{m_1^{s_1}}\sum_{n_1\le N}\frac{\widehat{b}_{1;\tau_1}(n_1)}{n_1^{\overline{s}_1}}\times\dots\right.$$
$$\left.\times\sum_{m_r\le N}\frac{\widehat{a}_{r;\tau_r}(m_r)}{m_r^{s_r}}\sum_{n_r\le N}\frac{\widehat{b}_{r;\tau_r}(n_r)}{n_r^{\overline{s}_r}}\right)+O_\varepsilon\big(N^{-\delta+\varepsilon}\big),\tag{4.9}$$

since by (4.6) and (4.7), for $|\tau_j|\le c$ and $s_j\in R_j$,

$$\begin{aligned}\sum_{m_j\le N}\frac{\widehat{a}_{j;\tau_j}(m_j)}{m_j^{s_j}} &= \mathrm{O}\left(\sum_{m_j\le N}\frac{d^{c_1}(m_j)}{m_j^{1+\delta}}\right)\\ &= \mathrm{O}\left(\sum_{m_j=1}^{\infty}\frac{d^{c_1}(m_j)}{m_j^{1+\delta}}\right)\\ &= \mathrm{O}(1)\end{aligned}$$

and

$$\sum_{n_j\le N}\frac{\widehat{b}_{j;\tau_j}(n_j)}{n_j^{\overline{s}_j}}=\mathrm{O}(1),$$

$j=1,\dots,r$. The multiplicativity of $\widehat{a}_{j;\tau_j}(m)$ and $\widehat{b}_{j;\tau_j}(m)$ together with (4.4) and (4.5) shows that

$$\begin{aligned}\widehat{a}_{j;\tau_j}(m) &= \prod_{p^k\|m}\widehat{a}_{j;\tau_j}(m)\\ &= \prod_{p^k\|m}\sum_{l=0}^{k}d_{\tau_j}(p^l)\alpha_j^l(p)\chi(p^l)d_{\tau_j}(p^{k-l})\beta_j^{k-l}(p)\chi(p^{k-l})\\ &= \prod_{p^k\|m}\chi(p^k)\sum_{l=0}^{k}d_{\tau_j}(p^l)\alpha_j^l(p)d_{\tau_j}(p^{k-l})\beta_j^{k-l}(p)\\ &= \prod_{p^k\|m}\chi(p^k)a_{j;\tau_j}(p^k)\\ &= a_{j;\tau_j}(m)\chi(m)\end{aligned}\tag{4.10}$$

and

$$\widehat{b}_{j;\tau_j}(m)=b_{j;\tau_j}(m)\overline{\chi}(m),\tag{4.11}$$

$j=1,\dots,r$. Therefore, the main term on the right-hand side of (4.9) can be written in the form

$$\begin{aligned}
&\sum_{m_1\le N}\frac{a_{1;\tau_1}(m_1)}{m_1^{s_1}}\sum_{n_1\le N}\frac{b_{1;\tau_1}(n_1)}{n_1^{\overline{s}_1}}\dots\sum_{m_r\le N}\frac{a_{r;\tau_r}(m_r)}{m_r^{s_r}}\sum_{n_r\le N}\frac{b_{r;\tau_r}(n_r)}{n_r^{\overline{s}_r}}\\
&\quad\times\frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\chi(m_1\dots m_r)\overline{\chi}(n_1\dots n_r)\\
&=\sum_{m\le N^r}\sum_{m_1\dots m_r=m}\frac{a_{1;\tau_1}(m_1)\dots a_{r;\tau_r}(m_r)}{m_1^{s_1}\dots m_r^{s_r}}\\
&\quad\times\sum_{n\le N^r}\sum_{n_1\dots n_r=n}\frac{b_{1;\tau_1}(n_1)\dots b_{r;\tau_r}(n_r)}{n_1^{\overline{s}_1}\dots n_r^{\overline{s}_r}}\\
&\quad\times\frac{1}{M_Q}\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\chi(m)\overline{\chi}(n).
\end{aligned}\tag{4.12}$$

If $m=n$, then

$$\begin{aligned}
\sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}\chi(m)\overline{\chi}(n) &= \sum_{q\le Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\ q)\\ \chi\ne\chi_0}}|\chi(m)|^2\\
&= M_Q-\sum_{\substack{q|m\\ q\le N^r}}(q-2)\\
&= M_Q+\mathrm{O}\bigg(\sum_{q\le N^r}q\bigg)\\
&= M_Q+\mathrm{O}\big(N^{2r}\big).
\end{aligned}\tag{4.13}$$

Moreover, in view of (4.6), (4.7) and (4.10), (4.11), using the estimate

$$\sum_{d_1\dots d_r=m}1=\mathrm{O}_\varepsilon(m^\varepsilon)$$

with arbitrary $\varepsilon>0$, we obtain that , for $|\tau_j|\le c$ and $s_j\in R_j$, $j=1,\dots,r$,

$$\begin{aligned}
&\sum_{m_1\dots m_r=m}\frac{a_{1;\tau_1}(m_1)\dots a_{r;\tau_r}(m_r)}{m_1^{s_1}\dots m_r^{s_r}}\sum_{n_1\dots n_r=m}\frac{b_{1;\tau_1}(n_1)\dots b_{r;\tau_r}(n_r)}{n_1^{\overline{s}_1}\dots n_r^{\overline{s}_r}}\\
&=\mathrm{O}\bigg(\sum_{m_1\dots m_r=m}\frac{d^{c_1}(m_1)\dots d^{c_1}(m_r)}{m^{1+\delta}}\sum_{n_1\dots n_r=m}\frac{d^{c_1}(n_1)\dots d^{c_1}(n_r)}{m^{1+\delta}}\bigg)\\
&=\mathrm{O}\bigg(\frac{m^{\frac{\varepsilon}{2}}}{m^{2+2\delta}}\Big(\sum_{m_1\dots m_r=m}1\Big)^2\bigg)\\
&=\mathrm{O}\bigg(\frac{m^{\varepsilon}}{m^{2+2\delta}}\bigg).
\end{aligned}$$

Therefore, by Lemma 1.9 and (4.13), the case $m = n$ contributes to (4.9)

$$\sum_{m=1}^{\infty} \sum_{m_1 \ldots m_r = m} \frac{a_{1;\tau_1}(m_1) \ldots a_{r;\tau_r}(m_r)}{m_1^{s_1} \ldots m_r^{s_r}} \sum_{n_1 \ldots n_r = m} \frac{b_{1;\tau_1}(n_1) \ldots b_{r;\tau_r}(n_r)}{n_1^{\bar{s}_1} \ldots n_r^{\bar{s}_r}} \\ + \mathrm{o}(1), \tag{4.14}$$

uniformly in $|\tau_j| \le c$, and $s_j \in R_j$, $j = 1, \ldots, r$,

Now consider the case $m \ne n$. Using Lemmas 1.7, 1.9 and 1.10, we find that

$$\begin{aligned}
&\sum_{q \le Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \chi(m)\overline{\chi}(n) \\
&= \sum_{q \le Q} \sum_{\chi = \chi(\mathrm{mod} q)} \chi(m)\overline{\chi}(n) - \sum_{q \le Q} \chi_0(m)\overline{\chi_0}(n) \\
&= \sum_{q \le Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\ q) \\ q | (m-n)}} \chi(m)\overline{\chi}(n) + \sum_{q \le Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\ q) \\ q \nmid (m-n)}} \chi(m)\overline{\chi}(n) \\
&\quad - \sum_{q \le Q} \chi_0(m)\overline{\chi_0}(n) \\
&= \mathrm{O}\Big(\sum_{q \le N^r} q \Big) + \mathrm{O}\Big(\sum_{q \le Q} 1 \Big) \\
&= \mathrm{O}\Big(N^{2r} \Big) + \mathrm{O}\Big(\frac{Q}{\log Q} \Big) \\
&= \mathrm{O}\Big(\frac{Q}{\log\ Q} \Big).
\end{aligned}$$

Therefore, this, (4.9), (4.12) and (4.14) show that, uniformly in $|\tau_j| \le c$, and $s_j \in R_j$, $j = 1, \ldots, r$,

$$\begin{aligned}
w_{k_1,\ldots,k_r;Q}(\tau_1, \ldots, \tau_r) \quad = \quad & \sum_{m=1}^{\infty} \sum_{m_1 \ldots m_r = m} \frac{a_{1;\tau_1}(m_1) \ldots a_{r;\tau_r}(m_r)}{m_1^{s_1} \ldots m_r^{s_r}} \\
& \times \sum_{n_1 \ldots n_r = m} \frac{b_{1;\tau_1}(n_1) \ldots b_{r;\tau_r}(n_r)}{n_1^{\bar{s}_1} \ldots n_r^{\bar{s}_r}} + \mathrm{o}(1)
\end{aligned} \tag{4.15}$$

as $Q \to \infty$.

4.5. Proof of Theorem 4.1

We apply Lemma 4.3. Reasoning similarly to the proof of the asymptotic formula (4.15), we find that, uniformly in $|\tau_j| \le c$ and $s_j \in R_j$, $j = 1, \dots, r$, as $Q \to \infty$,

$$
\begin{cases}
w_{k_j;Q}(\tau_j) \\
\quad = \sum\limits_{m=1}^{\infty} \frac{a_{j;\tau_j}(m) b_{j;\tau_j}(m)}{m^{2\sigma_j}} + \mathrm{o}(1), \quad j = 1, \dots, r, \\
w_{k_{j_1},k_{j_2};Q}(\tau_{j_1}, \tau_{j_2}) \\
\quad = \sum\limits_{m=1}^{\infty} \sum\limits_{m_1 m_2 = m} \frac{a_{j_1;\tau_{j_1}}(m_1) a_{j_2;\tau_{j_2}}(m_2)}{m_1^{s_{j_1}} m_2^{s_{j_2}}} \sum\limits_{n_1 n_2 = m} \frac{b_{j_1;\tau_{j_1}}(n_1) b_{j_2;\tau_{j_2}}(n_2)}{n_1^{\bar{s}_{j_1}} n_2^{\bar{s}_{j_2}}} + \mathrm{o}(1), \\
j_2 > j_1 = 1, \dots, r-1, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
w_{k_1,\dots,k_{j-1},k_{j+1},\dots,k_r;Q}(\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r) \\
\quad = \sum\limits_{m=1}^{\infty} \sum\limits_{m_1 \dots m_{j-1} m_{j+1} \dots m_r = m} \frac{a_{1;\tau_1}(m_1) \dots a_{j-1;\tau_{j-1}}(m_{j-1}) a_{j+1;\tau_{j+1}}(m_{j+1}) \dots a_{r;\tau_r}(m_r)}{m_1^{s_1} \dots m_{j-1}^{s_{j-1}} m_{j+1}^{s_{j+1}} \dots m_r^{s_r}} \\
\qquad \times \sum\limits_{n_1 \dots n_{j-1} n_{j+1} \dots n_r = m} \frac{b_{1;\tau_1}(n_1) \dots b_{j-1;\tau_{j-1}}(n_{j-1}) b_{j+1;\tau_{j+1}}(n_{j+1}) \dots b_{r;\tau_r}(n_r)}{n_1^{\bar{s}_1} \dots n_{j-1}^{\bar{s}_{j-1}} n_{j+1}^{\bar{s}_{j+1}} \dots n_r^{\bar{s}_r}} + \mathrm{o}(1), \\
j = 1, \dots, r, \\
w_{k_1,\dots,k_r}(\tau_1, \dots, \tau_r) \\
\quad = \sum\limits_{m=1}^{\infty} \sum\limits_{m_1 \dots m_r = m} \frac{a_{1;\tau_1}(m_1) \dots a_{r;\tau_r}(m_r)}{m_1^{s_1} \dots m_r^{s_r}} \sum\limits_{n_1 \dots n_r = m} \frac{b_{1;\tau_1}(n_1) \dots b_{r;\tau_r}(n_r)}{n_1^{\bar{s}_1} \dots n_r^{\bar{s}_r}} + \mathrm{o}(1).
\end{cases}
\tag{4.16}
$$

Thus, we have that the characteristic transforms of the measure $P_{Q,\mathbb{R}^r}$ converge to continuous functions as $Q \to \infty$. Therefore, by Lemma 4.3, on $(\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$, there exists a probability measure $P_{\mathbb{R}^r}$ such that $P_{Q,\mathbb{R}^r}$ converges weakly to $P_{\mathbb{R}^r}$ as $Q \to \infty$. Moreover, the measure $P_{\mathbb{R}^r}$ is defined the characteristic transforms given by the right-hand sides without o(1) of (4.16).

Conclusions

Let F be a normalized Hecke eigen cusp form for the full modular group, $L(s, F)$ be the L-function attached to the form F, and let $L(s, F, \chi)$ denote a twist of $L(s, F)$ with Dirichlet character χ modulo q, where q is a prime number.

For the function $L(s, F, \chi)$, the following asymptotic properties are true when $q \to \infty$:

1). a limit theorem for $|L(s, F, \chi)|$;

2). a limit theorem for $\arg L(s, F, \chi)$;

3). a limit theorem for $L(s, F, \chi)$ on the complex plane;

4). a joint limit theorem for a collection $|L(s_1, F_1, \chi)|$, ..., $|L(s_r, F_r, \chi)|$.

All limit theorems are understood in the sense of weak convergence of probability measures.

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Notation

j, k, l, m, n	natural numbers
p	prime number
(m, n)	greatest common divisor of natural m and n
$\mathbb{N}$	set of all natural numbers
$\mathbb{Z}$	set of all integer numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
$s = \sigma + it, \quad z = u + iv$	complex variables
$i = \sqrt{-1}$	imaginary unity
$meas\{A\}$	Lebesgue measure of the set A
$\sharp\{A\}$	number of elements of the set A
χ	Dirichlet character
$L(s, \chi)$	Dirichlet L-function
$SL(2, \mathbb{Z})$	full modular group
$F(z)$	Cusp form
$\Gamma(s)$	gamma-function
γ_0	Euler constant defined by $\gamma_0 = -\int\limits_0^\infty \mathrm{e}^{-x} \log x \mathrm{d}x = 0.5772\ldots$
$f(x) = \mathrm{O}(g(x)), \quad x \in I$	means that $\lvert f(x)\rvert \leq Cg(x)$, $x \in I$