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On Universality of Some Beurling Zeta-Functions

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Abstract: Let $\mathcal{P}$ be the set of generalized prime numbers, and $\zeta_{\mathcal{P}}(s)$, $s = \sigma + it$, denote the Beurling zeta-function associated with $\mathcal{P}$. In the paper, we consider the approximation of analytic functions by using shifts $\zeta_{\mathcal{P}}(s + i\tau)$, $\tau \in \mathbb{R}$. We assume the classical axioms for the number of generalized integers and the mean of the generalized von Mangoldt function, the linear independence of the set $\{\log p : p \in \mathcal{P}\}$, and the existence of a bounded mean square for $\zeta_{\mathcal{P}}(s)$. Under the above hypotheses, we obtain the universality of the function $\zeta_{\mathcal{P}}(s)$. This means that the set of shifts $\zeta_{\mathcal{P}}(s + i\tau)$ approximating a given analytic function defined on a certain strip $\widehat{\sigma} < \sigma < 1$ has a positive lower density. This result opens a new chapter in the theory of Beurling zeta functions. Moreover, it supports the Linnik–Ibragimov conjecture on the universality of Dirichlet series. For the proof, a probabilistic approach is applied.

Keywords: Beurling zeta-function; generalized integers; generalized primes; Haar measure; random element; universality; weak convergence

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1. Introduction

A positive integer $q > 1$ is called prime if it has only two divisors, q and 1. Thus, $2, 3, 5, 7, 11, \ldots$ are prime numbers. Integer numbers $k > 1$ that have divisors different from k and 1 are called composite. It is well known that the set of all primes is infinite, and this was first proved by Euclid. By the fundamental theorem of arithmetic, every integer $k > 1$ has a unique representation as a product of prime numbers. Thus,

$$k = q_1^{\alpha_1} \cdots q_r^{\alpha_r}, \quad \alpha_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

and q_j is the jth prime number, $j = 1, \ldots, r$, with some $r \in \mathbb{N}$.

Investigations of the number of prime numbers

$$\pi(x) \stackrel{\text{def}}{=} \sum_{q \leqslant x} 1, \quad x \to \infty,$$

were more complicated. We recall that $a = O(b)$, $a \in \mathbb{C}$, $b > 0$, means that there exists a constant $c > 0$ such that $|a| \leqslant cb$. Comparatively recently, in 1896 Hadamard [1] and de la Vallée-Poussin [2] proved independently the asymptotic formula

$$\pi(x) = \int_2^x \frac{\mathrm{d}u}{\log u} + O\left(x e^{-c\sqrt{\log x}}\right), \quad c > 0.$$

For this, they applied the Riemann idea [3] of using the function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_q \left(1 - \frac{1}{q^s}\right)^{-1}, \quad s = \sigma + it,\ \sigma > 1,$$

now called the Riemann zeta-function. The distribution low of prime numbers was found.

Prime numbers have generalizations. The system $\mathcal{P}$ of real numbers $1 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n \leqslant \cdots$ such that $\lim_{n\to\infty} p_n = \infty$ are called generalized prime numbers. Generalized prime numbers were introduced by Beurling in [4], and are studied by many authors. The system $\mathcal{P}$ generates the associated system $\mathcal{N}_{\mathcal{P}}$ of generalized integers consisting of finite products of the form

$$p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \quad \alpha_j \in \mathbb{N}_0,\ j = 1, \dots r,$$

with some $r \in \mathbb{N}$.

The main problem in the theory of generalized primes is the asymptotic behavior of the function

$$\pi_{\mathcal{P}}(x) \stackrel{\text{def}}{=} \sum_{p \leqslant x,\, p \in \mathcal{P}} 1, \quad x \to \infty.$$

The function $\pi_{\mathcal{P}}(x)$ is closely connected to the number of generalized integers

$$\mathcal{N}_{\mathcal{P}}(x) \stackrel{\text{def}}{=} \sum_{m \leqslant x,\, m \in \mathcal{N}_{\mathcal{P}}} 1, \quad x \to \infty.$$

In these definitions, the sums are taking counting multiplicities of p and m. Distribution results for generalized numbers were obtained by Beurling [4], Borel [5], Diamond [6–8], Malvin [9], Nyman [10], Ryavec [11], Hilberdink and Lapidus [12], Stankus [13], Zhang [14], and others. The important place in generalized number theory is devoted to making relations between $\mathcal{N}_{\mathcal{P}}(x)$ and $\pi_{\mathcal{P}}(x)$. We mention some of them. From a general Landau's theorem for prime ideals [15], we have the estimate

$$\mathcal{N}_{\mathcal{P}}(x) = ax + O\left(x^{\beta}\right), \quad a > 0,\ 0 \leqslant \beta < 1, \tag{1}$$

that implies

$$\pi_{\mathcal{P}}(x) = \int_2^x \frac{\mathrm{d}u}{\log u} + O\left(x\mathrm{e}^{-c\sqrt{\log x}}\right), \quad c > 0.$$

Nyman proved [10] that the estimates

$$\mathcal{N}_{\mathcal{P}}(x) = ax + O\left(\frac{x}{(\log x)^{\alpha}}\right), \quad \alpha > 0, \tag{2}$$

and

$$\pi_{\mathcal{P}}(x) = \int_2^x \frac{\mathrm{d}u}{\log u} + O\left(\frac{x}{(\log x)^{\alpha_1}}\right), \quad \alpha_1 > 0,$$

with arbitrary $\alpha > 0$ and $\alpha_1 > 0$ are equivalent. Beurling observed [4] that the relation

$$\pi_{\mathcal{P}}(x) \sim \frac{x}{\log x}, \quad x \to \infty,$$

is implied by (2) with $\alpha > 3/2$.

It is important to stress that Beurling began to use zeta-functions for investigations of the function $\pi_{\mathcal{P}}(x)$. These zeta-functions $\zeta_{\mathcal{P}}(s)$, now called Beurling zeta-functions, are defined in some half-plane $\sigma > \sigma_0$, by the Euler product

$$\zeta_{\mathcal{P}}(s) = \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or by the Dirichlet series

$$\zeta_{\mathcal{P}}(s) = \sum_{m\in\mathcal{N}_{\mathcal{P}}} \frac{1}{m^s},$$

where σ_0 depends on the system $\mathcal{P}$.

Suppose that (1) is true. Then, the partial summation shows that the series for $\zeta_{\mathcal{P}}(s)$ is absolutely convergent for $\sigma > 1$,

$$\zeta_{\mathcal{P}}(s) = s\int_1^{\infty} \frac{\mathcal{N}_{\mathcal{P}}(x)}{x^{s+1}}\, dx, \tag{3}$$

the function $\zeta_{\mathcal{P}}(s)$ is analytic for $\sigma > 1$, and the equality

$$\sum_{m\in\mathcal{N}_{\mathcal{P}}} \frac{1}{m^s} = \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

is valid.

Analytic continuation for the function $\zeta_{\mathcal{P}}(s)$ is not an easy problem. If (1) is true, then (3) implies

$$\zeta_{\mathcal{P}}(s) = \frac{a\, s}{s-1} + s\int_1^{\infty} \frac{R(x)}{x^{s+1}}\, dx, \quad R(x) = O\left(x^{\beta}\right),\ 0 \leqslant \beta < 1.$$

This gives analytic continuation for $\zeta_{\mathcal{P}}(s)$ to the half-plane $\sigma > \beta$, except for the point $s = 1$ which is a simple pole with residue a.

Beurling zeta-functions are attractive analytic objects; investigations of their properties lead to interesting results, and require new methods. Various authors put much effort into showing that the Beurling zeta-functions have similar properties to classical ones. We mention a recent paper [16] containing deep zero-distribution results for $\zeta_{\mathcal{P}}(s)$.

In this paper, we investigate the analytic properties of the function $\zeta_{\mathcal{P}}(s)$. The approximation of analytic functions is one of the most important chapters of function theory. It is well known that the Riemann zeta-function $\zeta(s)$ is universal in the sense of approximation of analytic functions. More precisely, this means that every non-vanishing analytic function defined on the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ can be approximated with desired accuracy by using shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. Universality of $\zeta(s)$ and other zeta-functions has deep theoretical (zero-distribution, functional independence, set denseness, moment problem, ...) and practical (approximation problem, quantum mechanics) applications. On the other hand, the universality theory of zeta-functions has some interior problems (effectivization, description of a class of universal functions, Linnik–Ibragimov conjecture, see Section 1.6 of [17], ...); therefore, investigations of universality are continued, see [17–23].

Our purpose is to prove the universality of the function $\zeta_{\mathcal{P}}(s)$ with a certain system $\mathcal{P}$. We began studying the approximation of analytic functions by shifts $\zeta_{\mathcal{P}}(s + i\tau)$ in [24]. Suppose that the estimate (1) is valid. Let

$$M_{\mathcal{P}}(\sigma, T) = \int_0^T |\zeta_{\mathcal{P}}(\sigma + it)|^2\, dt,$$

$$\widehat{\sigma} = \inf\left\{\sigma : M_{\mathcal{P}}(\sigma, T) \ll_{\sigma} T, \quad \sigma > \max\left(\frac{1}{2}, \beta\right)\right\}.$$

Suppose that $\widehat{\sigma} < 1$ and define

$$D = D_{\mathcal{P}} = \{s \in \mathbb{C} : \widehat{\sigma} < \sigma < 1\}.$$

Here, and in the sequel, the notation $a \ll_c b$, $a \in \mathbb{C}$, $b > 0$, shows that there exists a constant $c = c(\varepsilon) > 0$ such that $|a| \leqslant cb$. Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta, and by $\text{meas}A$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The main result of [24] is the following theorem.

Theorem 1. *Suppose that the system $\mathcal{P}$ satisfies the axiom (1). Then there exists a closed non-empty subset $F_{\mathcal{P}} \subset H(D)$ such that, for every compact set $K \subset D$, $f(s) \in F_{\mathcal{P}}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{s \in K}|\zeta_{\mathcal{P}}(s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{s \in K}|\zeta_{\mathcal{P}}(s + i\tau) - f(s)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Theorem 1 demonstrates good approximation properties of the function $\zeta_{\mathcal{P}}(s)$; however, the set $F_{\mathcal{P}}$ of approximated functions is not explicitly given. The aim of this paper, using certain additional information on system $\mathcal{P}$, is to identify the set $F_{\mathcal{P}}$.

A new approach for analytic continuation of the function $\zeta_{\mathcal{P}}(s)$ involving the generalized von Mangoldt function

$$\Lambda_{\mathcal{P}}(m) = \begin{cases} \log p & \text{if } m = p^k, p \in \mathcal{P}, k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\psi_{\mathcal{P}}(x) = \sum_{\substack{m \leqslant x \\ m \in \mathcal{N}_{\mathcal{P}}}} \Lambda_{\mathcal{P}}(m)$$

was proposed in [12]. Let, for $\alpha \in [0, 1)$ and every $\varepsilon > 0$,

$$\psi_{\mathcal{P}}(x) = x + O\big(x^{\alpha+\varepsilon}\big). \tag{4}$$

Then, in [12], it was obtained that the function $\zeta_{\mathcal{P}}(s)$ is analytic in the half-plane $\sigma > \alpha$, except for a simple pole at the point $s = 1$. It turns out that estimates of type (4) are useful for the characterization of the system $\mathcal{P}$. It is known [12] that (1) does not imply the estimate

$$\psi_{\mathcal{P}}(x) = x + O\left(x^{\beta_1}\right) \tag{5}$$

with $\beta_1 < 1$. Therefore, together with (1), we suppose that estimate (5) is valid.

Let $\mathcal{K}$ be the class of compact subsets of strip D with the connected complement, and $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Moreover, let

$$L(\mathcal{P}) = \{\log p : p \in \mathcal{P}\}.$$

Note, that the following theorem supports the Linnik–Ibragimov conjecture.

Theorem 2. *Suppose that the system $\mathcal{P}$ satisfies the axioms (1) and (5), and $L(\mathcal{P})$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|\zeta_{\mathcal{P}}(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|\zeta_{\mathcal{P}}(s+i\tau) - f(s)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many at $\varepsilon > 0$.

Notice that the requirement on the set $L(\mathcal{P})$ is sufficiently strong, it shows that the numbers of the system $\mathcal{P}$ must be different. The simplest example is the system

$$\mathcal{P} = \{q + \alpha : q \text{ is prime}\},$$

where α is a transcendental number.

An example of $\mathcal{P}$ with a bounded mean square is given in [25].

For the proof of Theorem 2, we will build the probabilistic theory of the function $\zeta_{\mathcal{P}}(s)$ in the space of analytic functions $H(D)$.

The paper is organized as follows. In Section 2, we introduce a certain probability space, and define the $H(D)$ valued random element. Section 3 is devoted to the ergodicity of one group of transformations. In Section 4, we approximate the mean of the function $\zeta_{\mathcal{P}}(s)$ by an absolutely convergent Dirichlet series. Section 5 is the most important. In this section, we prove a probabilistic limit theorem for the function $\zeta_{\mathcal{P}}(s)$ on a weakly convergent probability measure in the space $H(D)$, and identify the limit measure. Section 6 gives the explicit form for the support of the limit measure of Section 5. In Section 7, the universality of the function $\zeta_{\mathcal{P}}(s)$ is proved.

2. Random Element

Define the Cartesian product

$$\Omega_{\mathcal{P}} = \prod_{p\in\mathcal{P}} \{s \in \mathbb{C} : |s| = 1\}.$$

The set $\Omega_{\mathcal{P}}$ consists of all functions $\omega : \mathcal{P} \to \{s \in \mathbb{C} : |s| = 1\}$. In $\Omega_{\mathcal{P}}$, the operation of pointwise multiplication and product topology can be defined, and this makes $\Omega_{\mathcal{P}}$ a topological group. Since the unit circle is a compact set, the group $\Omega_{\mathcal{P}}$ is compact. Denote by $\mathcal{B}(\mathbb{X})$, the Borel σ-field of the space $\mathbb{X}$. Then, the compactness of $\Omega_{\mathcal{P}}$ implies the existence of the probability Haar measure $m_{\mathcal{P}}$ on $(\Omega_{\mathcal{P}}, \mathcal{B}(\Omega_{\mathcal{P}}))$, and we have the probability space $(\Omega_{\mathcal{P}}, \mathcal{B}(\Omega_{\mathcal{P}}), m_{\mathcal{P}})$.

Denote the elements of $\Omega_{\mathcal{P}}$ by $\omega = (\omega(p) : p \in \mathcal{P})$. Since the Haar measure $m_{\mathcal{P}}$ is the product of Haar measures on unit circles, $\{\omega(p) : p \in \mathcal{P}\}$ is a sequence of independent complex-valued random variables uniformly distributed on the unit circle.

Extend the functions $\omega(p)$, $p \in \mathcal{P}$, to the generalized integers $\mathcal{N}_{\mathcal{P}}$. Let

$$m = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \in \mathcal{N}_{\mathcal{P}}.$$

Then we put

$$\omega(m) = \omega^{\alpha_1}(p_1) \cdots \omega^{\alpha_r}(p_r). \tag{6}$$

Now, for $s \in D$ and $\omega \in \Omega_{\mathcal{P}}$, define

$$\zeta_{\mathcal{P}}(s,\omega) = \sum_{m\in\mathcal{N}_{\mathcal{P}}} \frac{\omega(m)}{m^s}.$$

Lemma 1. *Under the hypotheses of Theorem 2, $\zeta_{\mathcal{P}}(s,\omega)$ is an $H(D)$-valued random element defined on the probability space $(\Omega_{\mathcal{P}}, \mathcal{B}(\Omega_{\mathcal{P}}), m_{\mathcal{P}})$.*

Proof. Fix $\sigma_0 > \widehat{\sigma}$, and consider

$$a_m(\omega) = \frac{\omega(m)}{m^{\sigma_0}}, \quad m \in \mathcal{N}_{\mathcal{P}}.$$

Then $\{a_m : m \in \mathcal{N}_{\mathcal{P}}\}$ is a sequence of complex-valued random variables on $(\Omega_{\mathcal{P}}, \mathcal{B}(\Omega_{\mathcal{P}}), m_{\mathcal{P}})$. Denote by $\bar{z}$ the complex conjugate of $z \in \mathbb{C}$. Suppose that $m_1 \neq m_2$, $m_1, m_2 \in \mathcal{N}_{\mathcal{P}}$. Since the set $L(\mathcal{P})$ is linearly independent over $\mathbb{Q}$, in the product $\omega(m_1)\overline{\omega(m_2)}$, there exists at least one factor $\omega^{\alpha}(p)$, $p \in \mathcal{P}$, with integer $\alpha \neq 0$. Therefore, denoting by $\mathbb{E}\xi$ the expectation of the random variable ξ, we have

$$\mathbb{E}|a_m(\omega)|^2 = \frac{1}{m^{2\sigma_0}}, \quad m \in \mathcal{N}_{\mathcal{P}}, \tag{7}$$

$$\mathbb{E}a_{m_1}(\omega)\overline{a_{m_2}(\omega)} = \frac{1}{m_1^{\sigma_0} m_2^{\sigma_0}} \int\limits_{\Omega_{\mathcal{P}}} \omega(m_1)\overline{\omega(m_2)}\, \mathrm{d}m_{\mathcal{P}} = 0, \quad m_1 \neq m_2,$$

because the integral includes the factor

$$\int\limits_{\gamma} \omega^{\alpha}(p)\, \mathrm{d}m_{\gamma} = \int\limits_0^1 \mathrm{e}^{2\pi i \alpha u}\, \mathrm{d}u = 0,$$

where γ is the unit circle on $\mathbb{C}$, and m_{γ} the Haar measure on γ. This and (7) show that $\{a_m\}$ is a sequence of pairwise orthogonal complex-valued random variables and the series

$$\sum_{m \in \mathcal{N}_{\mathcal{P}}} \mathbb{E}|a_m|^2 \log^2 m$$

is convergent. Hence, by the classical Rademacher theorem, see [26], the series

$$\sum_{m \in \mathcal{N}_{\mathcal{P}}} \frac{\omega(m)}{m^{\sigma_0}}$$

converges for almost all ω with respect to the measure $m_{\mathcal{P}}$. Therefore, by a property of the Dirichlet series, see [22], the series

$$\sum_{m \in \mathcal{N}_{\mathcal{P}}} \frac{\omega(m)}{m^{s}} \tag{8}$$

converges uniformly on compact sets of the half-plane $\sigma > \sigma_0$ for almost all $\omega \in \Omega_{\mathcal{P}}$.

Now, let

$$\sigma_k = \widehat{\sigma} + \frac{1}{k}, \quad k \in \mathbb{N},$$

and $D_k = \{s \in \mathbb{C} : \sigma > \sigma_k\}$. Denote by the set $\Omega_k \subset \Omega_{\mathcal{P}}$ such that the series (8) converges uniformly on compact sets of D_k for almost all $\omega \in \Omega_k$. Then, by the above remark,

$$m_{\mathcal{P}}(\Omega_k) = 1. \tag{9}$$

On the other hand, taking

$$\widehat{\Omega} = \bigcap_k \Omega_k,$$

we obtain from (9) that $m_{\mathcal{P}}(\widehat{\Omega}) = 1$, and the series (8) converges uniformly on compact sets of the half-plane $\sigma > \widehat{\sigma}$ of the strip D. Hence, $\zeta_{\mathcal{P}}(s,\omega)$ is the $H(D)$-valued random element on $(\Omega_{\mathcal{P}}, \mathcal{B}(\Omega_{\mathcal{P}}), m_{\mathcal{P}})$. $\square$

Lemma 2. *For almost all ω, the product*

$$\prod_{p\in\mathcal{P}}\left(1-\frac{\omega(p)}{p^s}\right)^{-1}$$

converges uniformly on compact subsets of the half-plane $\sigma > \widehat{\sigma}$, and the equality

$$\zeta_{\mathcal{P}}(s,\omega)=\prod_{p\in\mathcal{P}}\left(1-\frac{\omega(p)}{p^s}\right)^{-1}$$

holds.

Proof. The series $\zeta_{\mathcal{P}}(s,\omega)$ is absolutely convergent for $\sigma > 1$. Therefore, the equality of the lemma, in view of (6), is valid for $\sigma > 1$. By proof of Lemma 1, the function $\zeta_{\mathcal{P}}(s,\omega)$, for almost all $\omega \in \Omega_{\mathcal{P}}$, is analytic in the half-plane $\sigma > \widehat{\sigma}$. Therefore, by analytic continuation, it suffices to show that the product of the lemma, for almost all $\omega \in \Omega_{\mathcal{P}}$, converges uniformly on compact subsets of the strip D.

Write

$$\prod_{p\in\mathcal{P}}\left(1-\frac{\omega(p)}{p^s}\right)^{-1}=\prod_{p\in\mathcal{P}}(1+a_p(s,\omega)) \tag{10}$$

with

$$a_p(s,\omega)=\sum_{\alpha=1}^{\infty}\frac{\omega^{\alpha}(p)}{p^{\alpha s}}.$$

We observe that the convergence of product (10) follows from that of the series

$$\sum_{p\in\mathcal{P}} a_p(s,\omega) \qquad \text{and} \qquad \sum_{p\in\mathcal{P}} |a_p(s,\omega)|^2.$$

Set

$$b_p(s,\omega)=\frac{\omega(p)}{p^s}.$$

Then

$$a_p(s,\omega)-b_p(s,\omega)=\sum_{\alpha=2}^{\infty}\frac{\omega^{\alpha}(p)}{p^{\alpha s}} \ll \frac{1}{p^{2\sigma}}, \quad \sigma > \widehat{\sigma}.$$

Hence, the series

$$\sum_{p\in\mathcal{P}} |a_p(s,\omega)-b_p(s,\omega)| \tag{11}$$

is convergent for all $\omega \in \Omega_{\mathcal{P}}$ with every $\sigma = \sigma_0$, $\sigma_0 > \widehat{\sigma}$, thus, uniformly convergent on compact subsets of the half-plane $\sigma > \widehat{\sigma}$. To prove the convergence for the series

$$\sum_{p\in\mathcal{P}} b_p(s,\omega),$$

we apply the same arguments as in the proof of Lemma 1. For fixed $\sigma > \widehat{\sigma}$, we have

$$\mathbb{E}|b_p(\sigma,\omega)|^2=\frac{1}{p^{2\sigma}}$$

and for $p,q \in \mathcal{P}$, $p \neq q$,

$$\mathbb{E}b_p(\sigma,\omega)\overline{b_q(\sigma,\omega)}=\frac{1}{p^{\sigma}q^{\sigma}}\int\limits_{\Omega_{\mathcal{P}}}\omega(p)\overline{\omega(q)}\,\mathrm{d}m_{\mathcal{P}}=0.$$

Thus, the series

$$\sum_{p\in\mathcal{P}} \mathbb{E}|b_p(\sigma,\omega)|^2 \log^2 p$$

is convergent, and the Rademacher theorem implies that the series

$$\sum_{p\in\mathcal{P}} b_p(\sigma,\omega)$$

converges for almost all $\omega \in \Omega_{\mathcal{P}}$. Hence, this series, for almost all $\omega \in \Omega_{\mathcal{P}}$, converges uniformly on compact subsets of the half-plane $\sigma > \widehat{\sigma}$. This, together with a convergence property of the series (11), shows that the series

$$\sum_{p\in\mathcal{P}} a_p(s,\omega),$$

for almost all $\omega \in \Omega_{\mathcal{P}}$, converges uniformly on compact subsets of the half-plane $\sigma > \widehat{\sigma}$, and it remains to prove the same for the series

$$\sum_{p\in\mathcal{P}} |a_p(s,\omega)|^2. \tag{12}$$

Clearly, for all $\omega \in \Omega_{\mathcal{P}}$,

$$|a_p(s,\omega)|^2 \ll \frac{1}{p^{2\sigma}}, \quad \sigma > \widehat{\sigma}.$$

Hence, the series (12), for all $\omega \in \Omega_{\mathcal{P}}$, converges uniformly on compact subsets of the half-plane $\sigma > \widehat{\sigma}$. □

3. Ergodicity

For $\tau \in \mathbb{R}$, let

$$\kappa_\tau = \left(p^{-i\tau} : p \in \mathcal{P}\right),$$

and

$$g_\tau(\omega) = \kappa_\tau \omega, \quad \omega \in \Omega_{\mathcal{P}}.$$

Since the Haar measure $m_{\mathcal{P}}$ is invariant with respect to shifts by elements of $\Omega_{\mathcal{P}}$, i.e., for all $A \in \mathcal{B}(\Omega_{\mathcal{P}})$ and $\omega \in \Omega_{\mathcal{P}}$,

$$m_{\mathcal{P}}(A) = m_{\mathcal{P}}(\omega A) = m_{\mathcal{P}}(A\omega),$$

$g_\tau(m)$ is a measurable measure preserving transformation on $\Omega_{\mathcal{P}}$. Thus, we have the one-parameter group $G_\tau = \{g_\tau : \tau \in \mathbb{R}\}$ of transformations of $\Omega_{\mathcal{P}}$. A set $A \in \mathcal{B}(\Omega_{\mathcal{P}})$ is called invariant with respect to G_τ if, for every $\tau \in \mathbb{R}$, the sets A and $A_\tau = g_\tau(A)$ differ one from another at most by a set of $m_{\mathcal{P}}$-measure zero. It is well known that all invariant sets form a σ-field which is a subfield of $\mathcal{B}(\Omega_{\mathcal{P}})$. The group G_τ is called ergodic if its σ-field of invariant sets consists only of sets $m_{\mathcal{P}}$-measure 0 or 1.

Lemma 3. *Under the hypotheses of Theorem 2, the group G_τ is ergodic.*

Proof. Let $A \in \mathcal{B}(\Omega_{\mathcal{P}})$ be a fixed invariant set of G_τ. Denote by $\mathrm{I}_A(\omega)$ the indicator function of the set A. Then, for almost all $\omega \in \Omega_{\mathcal{P}}$,

$$\mathrm{I}_A(g_\tau(\omega)) = \mathrm{I}_A(\omega). \tag{13}$$

Characters χ of the group $\Omega_{\mathcal{P}}$ are of the form

$$\chi(\omega) = \prod_{p\in\mathcal{P}}^{*} \omega^{k_p}(p), \tag{14}$$

where $*$ indicates that only a finite number of integers k_p are distinct from zero. Suppose that χ is a nontrivial character, i.e., $\chi(\omega) \not\equiv 1$ for all $\omega \in \Omega_{\mathcal{P}}$. Then, we have

$$\chi(g_\tau) = \prod_{p\in\mathcal{P}}^{*} p^{-ik_p\tau} = \exp\left\{-i\tau \sum_{p\in\mathcal{P}}^{*} k_p \log p\right\}.$$

Since the set $L(\mathcal{P})$ is linearly independent over $\mathbb{Q}$, and χ is a nontrivial character,

$$\sum_{p\in\mathcal{P}}^{*} k_p \log p \neq 0.$$

Thus, there exists a real number $a \neq 0$ such that

$$\chi(g_\tau) = \mathrm{e}^{-i\tau a}.$$

Hence, there is $\tau_0 \in \mathbb{R}$ satisfying $\chi(g_{\tau_0}) \neq 1$.

Now, we deal with Fourier analysis on $\Omega_{\mathcal{P}}$. Denote by $\widehat{g}$ the Fourier transform of a function g, i.e.,

$$\widehat{g}(\chi) = \int_{\Omega_{\mathcal{P}}} g(\omega)\chi(\omega)\,\mathrm{d}m_{\mathcal{P}}.$$

In virtue of (13), we find

$$\widehat{\mathrm{I}}_A(\chi) = \int_{\Omega_{\mathcal{P}}} \mathrm{I}_A(\omega)\chi(\omega)\,\mathrm{d}m_{\mathcal{P}} = \chi(g_{\tau_0}) \int_{\Omega_{\mathcal{P}}} \chi(\omega)\mathrm{I}_A(\omega)\,\mathrm{d}m_{\mathcal{P}} = \chi(g_{\tau_0})\widehat{\mathrm{I}}_A(\chi).$$

Hence, in view of inequality $\chi(g_{\tau_0}) \neq 1$, we obtain

$$\widehat{\mathrm{I}}_A(\chi) = 0. \tag{15}$$

Consider the case of the trivial character χ_0 of the group $\Omega_{\mathcal{P}}$. We set $\widehat{\mathrm{I}}_A(\chi_0) = c$. Then, the orthogonality of characters implies that

$$\widehat{c}(\chi) = \int_{\Omega_{\mathcal{P}}} c(\chi)\chi(\omega)\,\mathrm{d}m_{\mathcal{P}} = c\int_{\Omega_{\mathcal{P}}} \chi(\omega)\,\mathrm{d}m_{\mathcal{P}} = \begin{cases} c & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0. \end{cases}$$

Therefore, using (15) yields the equality

$$\widehat{\mathrm{I}}_A(\chi) = \widehat{c}(\chi). \tag{16}$$

It is well known that a function is completely determined by its Fourier transform. Thus, by (16), we have that for almost all $\omega \in \Omega_{\mathcal{P}}$, $\mathrm{I}_A(\omega) = c$. However, as $\mathrm{I}_A(\omega)$ is the indicator function, it follows that $c = 0$ or 1. In other words, for almost all $\omega \in \Omega_{\mathcal{P}}$, $\mathrm{I}_A(\omega) = 0$ or $\mathrm{I}_A(\omega) = 1$. Thus, $m_{\mathcal{P}}(A) = 0$ or $m_{\mathcal{P}}(A) = 1$. The lemma is proved. □

We apply Lemma 3 for the estimation of the mean square for $\zeta_{\mathcal{P}}(s,\omega)$.

Lemma 4. *Under hypotheses of Theorem 2, for fixed $\widehat{\sigma} < \sigma < 1$ and almost all $\omega \in \Omega_{\mathcal{P}}$,*

$$\int_{-T}^{T} |\zeta_{\mathcal{P}}(\sigma+it,\omega)|^2\,\mathrm{d}t \ll_{\mathcal{P},\sigma} T, \quad T \to \infty.$$

Proof. Let $a_m(\sigma,\omega)$, $m \in \mathcal{N}_\mathcal{P}$, be the same as the proof of Lemma 1. The random variables $a_m(\sigma,\omega)$ are pairwise orthogonal, and

$$\mathbb{E}|a_m(\sigma,\omega)|^2 = \frac{1}{m^{2\sigma}}.$$

Therefore,

$$\mathbb{E}|\zeta_\mathcal{P}(\sigma,\omega)|^2 = \mathbb{E}\left|\sum_{m\in\mathcal{N}_\mathcal{P}} a_m(\sigma,\omega)\right|^2 = \sum_{m\in\mathcal{N}_\mathcal{P}} \mathbb{E}|a_m(\sigma,\omega)|^2 = \sum_{m\in\mathcal{N}_\mathcal{P}} \frac{1}{m^{2\sigma}} < \infty. \tag{17}$$

Let $g_\tau(\omega)$ be the transformation from the proof of Lemma 3. Then, by the definition of g_τ,

$$|\zeta_\mathcal{P}(\sigma, g_t(\omega))|^2 = |\zeta_\mathcal{P}(\sigma, g_t\omega)|^2 = |\zeta_\mathcal{P}(\sigma+it,\omega)|^2.$$

We recall that a strongly stationary random process $X(t,\omega)$, $t \in \mathcal{T}$, on $(\Omega,\mathcal{A},P)$ is called ergodic if its σ-field of invariant sets consists of sets of P-measure 0 or 1. Since the group G_τ is ergodic, the stationary process $|\zeta_\mathcal{P}(\sigma+it,\omega)|^2$ is ergodic, for details, see [22]. Therefore, we can apply the classical Birkhoff–Khintchine ergodic theorem, see [27]. This gives, by (17),

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\zeta_\mathcal{P}(\sigma+it,\omega)|^2\,\mathrm{d}t = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\zeta_\mathcal{P}(\sigma,g_t(\omega))\,\mathrm{d}t = \mathbb{E}|\zeta_\mathcal{P}(\sigma,\omega)|^2 < \infty.$$

□

4. Approximation in the Mean

In this section, we approximate the functions $\zeta_\mathcal{P}(s)$ and $\zeta_\mathcal{P}(s,\omega)$ by absolutely convergent Dirichlet series. Let $\eta > 1-\widehat{\sigma}$ be a fixed number, and, for $m \in \mathcal{N}_\mathcal{P}$ and $n \in \mathbb{N}$,

$$a_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^\eta\right\}.$$

Then the series

$$\zeta_{\mathcal{P},n}(s) = \sum_{m\in\mathcal{N}_\mathcal{P}}\frac{a_n(m)}{m^s} \quad \text{and} \quad \zeta_{\mathcal{P},n}(s,\omega) = \sum_{m\in\mathcal{N}_\mathcal{P}}\frac{a_n(m)\omega(m)}{m^s}, \quad \omega \in \Omega_\mathcal{P},$$

are absolutely convergent for $\sigma > \widehat{\sigma}$ and for every fixed $n \in \mathbb{N}$. We will approximate $\zeta_\mathcal{P}(s)$ and $\zeta_\mathcal{P}(s,\omega)$ by $\zeta_{\mathcal{P},n}(s)$ and $\zeta_{\mathcal{P},n}(s,\omega)$, respectively, in the mean. Recall a metric in the space $H(D)$ inducing its topology. Let $\{K_l : l \in \mathbb{N}\} \subset D$ be a sequence of embedded compact sets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

and every compact set $K \subset D$ lies in some K_l. Then

$$\rho(g_1,g_2) = \sum_{l=1}^{\infty} 2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|}, \quad g_1,g_2 \in H(D),$$

is the desired metric in $H(D)$.

In [24], the following statement has been obtained.

Lemma 5. *Suppose that* (1) *is valid. Then*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho(\zeta_{\mathcal{P}}(s+i\tau),\zeta_{\mathcal{P},n}(s+i\tau))\,\mathrm{d}\tau = 0.$$

Denote by $\Omega_{\mathcal{P},1}$ a subset of $\Omega_{\mathcal{P}}$ such that a product

$$\prod_{p\in\mathcal{P}}\left(1-\frac{\omega(p)}{p^s}\right)^{-1}$$

converges uniformly on compact subsets of D for $\omega \in \Omega_{\mathcal{P},1}$, and by $\Omega_{\mathcal{P},2}$ a subset of $\Omega_{\mathcal{P}}$ such that, for $\omega \in \Omega_{\mathcal{P},2}$, the estimate

$$\int_{-T}^{T}|\zeta_{\mathcal{P}}(\sigma+it,\omega)|^2\,\mathrm{d}t \ll_\sigma T$$

holds for $\widehat{\sigma} < \sigma < 1$. Then, by Lemmas 3 and 4, $m_{\mathcal{P}}(\Omega_{\mathcal{P},j}) = 1$, $j = 1,2$. Let

$$\widehat{\Omega}_{\mathcal{P}} = \Omega_{\mathcal{P},1}\cap\Omega_{\mathcal{P},2}.$$

Then again $m_{\mathcal{P}}(\widehat{\Omega}_{\mathcal{P}}) = 1$.

Lemma 6. *Under the hypotheses of Theorem 2, for $\omega \in \widehat{\Omega}_{\mathcal{P}}$ the equality*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho(\zeta_{\mathcal{P}}(s+i\tau,\omega),\zeta_{\mathcal{P},n}(s+i\tau,\omega))\,\mathrm{d}\tau = 0$$

holds.

Proof. Denote

$$l_n(s) = \eta^{-1}\Gamma(\eta^{-1}s)n^s, \quad n\in\mathbb{N},$$

where $\Gamma(s)$ is the Euler gamma function. Then the classical Mellin formula

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(z)b^{-z}\,\mathrm{d}z = \mathrm{e}^{-b}, \quad a,b>0,$$

implies, for $m\in\mathcal{N}_{\mathcal{P}}$,

$$\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} m^{-z}l_n(z)\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\Gamma(z)\left(\frac{m}{n}\right)^{-z}\mathrm{d}z = a_n(m).$$

Therefore, for $\sigma > \widehat{\sigma}$ and $\omega \in \widehat{\Omega}_{\mathcal{P}}$,

$$\begin{aligned}\zeta_{\mathcal{P},n}(s,\omega) &= \sum_{m\in\mathcal{N}_{\mathcal{P}}}\frac{a_n(m)\omega(m)}{m^s} = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\left(\sum_{m\in\mathcal{N}_{\mathcal{P}}}\frac{\omega(m)}{m^{s+z}}\right)l_n(z)\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\zeta_{\mathcal{P}}(s+z,\omega)l_n(z)\,\mathrm{d}z. \qquad (18)\end{aligned}$$

The definition of the metric ρ implies that it is sufficient to show that, for every compact set $K \subset D$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} |\zeta_{\mathcal{P}}(s+i\tau,\omega) - \zeta_{\mathcal{P},n}(s+i\tau,\omega)| \, \mathrm{d}\tau = 0. \tag{19}$$

Thus, let $K \subset D$ be a compact set. Then there exists $\varepsilon > 0$ satisfying for $\sigma + it \in K$ the inequalities $\widehat{\sigma} + \varepsilon \leqslant \sigma \leqslant 1 - \varepsilon/2$. Take $\eta = 1$ and $\eta_1 = \widehat{\sigma} - \varepsilon/2 - \sigma$ with the above σ. Then $\eta_1 < 0$ and $\eta_1 \geqslant \widehat{\sigma} + \varepsilon/2 - 1 + \varepsilon/2 = \widehat{\sigma} - 1 + \varepsilon > -1$. Consequently, the integrand in (18) has only a simple pole $z = 0$ in the strip $\eta_1 < \mathrm{Re} z < \eta$. Hence, the residue theorem and (18) show that, for $s \in K$,

$$\zeta_{\mathcal{P},n}(s,\omega) - \zeta_{\mathcal{P}}(s,\omega) = \frac{1}{2\pi i} \int_{\eta_1 - i\infty}^{\eta_1 + i\infty} \zeta_{\mathcal{P}}(s+z,\omega) l_n(z) \, \mathrm{d}z.$$

Thus, for $s \in K$,

$$\begin{aligned}
&\zeta_{\mathcal{P},n}(s+i\tau,\omega) - \zeta_{\mathcal{P}}(s+i\tau,\omega) \\
&\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau + it + iu, \omega\Big) l_n\Big(\widehat{\sigma} + \frac{\varepsilon}{2} - \sigma + iu\Big) \, \mathrm{d}u \\
&\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau + iu, \omega\Big) l_n\Big(\widehat{\sigma} + \frac{\varepsilon}{2} - s + iu\Big) \, \mathrm{d}u \\
&\quad \ll \int_{-\infty}^{\infty} \Big|\zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau + iu, \omega\Big)\Big| \sup_{s\in K} \Big| l_n\Big(\widehat{\sigma} + \frac{\varepsilon}{2} - s + iu\Big)\Big| \, \mathrm{d}u.
\end{aligned} \tag{20}$$

It is well known that, for the gamma-function $\Gamma(\sigma + it)$, the estimate

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0, \tag{21}$$

is valid uniformly for $\sigma \in [\sigma_1, \sigma_2]$ with every $\sigma_1 < \sigma_2$. Therefore, (20) implies

$$\begin{aligned}
&\frac{1}{T} \int_0^T \sup_{s\in K} |\zeta_{\mathcal{P}}(s+i\tau,\omega) - \zeta_{\mathcal{P},n}(s+i\tau,\omega)| \, \mathrm{d}\tau \\
&\quad \ll \int_{-\infty}^{\infty} \left(\frac{1}{T} \int_0^T \Big|\zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau + iu, \omega\Big)\Big| \, \mathrm{d}\tau \right) \sup_{s\in K} \Big| l_n\Big(\widehat{\sigma} + \frac{\varepsilon}{2} - s + iu\Big)\Big| \, \mathrm{d}u \stackrel{\mathrm{def}}{=} I.
\end{aligned} \tag{22}$$

By Lemma 4, for $\omega \in \widehat{\Omega}_{\mathcal{P}}$,

$$\int_{-T}^{T} \Big|\zeta_{\mathcal{P}}\Big(\widehat{\sigma} + \frac{\varepsilon}{2} + i\tau, \omega\Big)\Big|^2 \, \mathrm{d}\tau \ll_{\varepsilon} T.$$

Hence,

$$\frac{1}{T}\int_0^T \left|\zeta_{\mathcal{P}}\left(\widehat{\sigma}+\frac{\varepsilon}{2}+i\tau+iu,\omega\right)\right| \mathrm{d}\tau \leqslant \left(\frac{1}{T}\int_0^T \left|\zeta_{\mathcal{P}}\left(\widehat{\sigma}+\frac{\varepsilon}{2}+i\tau+iu,\omega\right)\right|^2 \mathrm{d}\tau\right)^{1/2}$$
$$\leqslant \left(\frac{1}{T}\int_{-|u|}^{T+|u|} \left|\zeta_{\mathcal{P}}\left(\widehat{\sigma}+\frac{\varepsilon}{2}+i\tau,\omega\right)\right|^2 \mathrm{d}\tau\right)^{1/2}$$
$$\ll_\varepsilon \left(\frac{T+|u|}{T}\right)^{1/2} \ll_\varepsilon (1+|u|)^{1/2}. \tag{23}$$

In view of (21), for $s \in K$,

$$l_n\left(\widehat{\sigma}+\frac{\varepsilon}{2}-s+iu\right) \ll n^{\widehat{\sigma}+\varepsilon/2-\sigma}\exp\{-c|u-t|\} \ll_K n^{-\varepsilon/2}\exp\{-c_1|u|\}, \quad c_1>0.$$

This and (23) give

$$I \ll_{\varepsilon,K} n^{-\varepsilon/2}\int_{-\infty}^{\infty}(1+|u|)^{1/2}\exp\{-c_1|u|\}\,\mathrm{d}u \ll_{\varepsilon,K} n^{-\varepsilon/2},$$

and (19) is proved. □

5. Limit Theorems

In previous sections, we gave preparatory results for the proof of a limit theorem for $\zeta_{\mathcal{P}}(s)$ in the space of analytic functions $H(D)$. In this section, we consider the weak convergence for

$$P_{T,\mathcal{P}}(A) = \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \zeta_{\mathcal{P}}(s+i\tau) \in A\}$$

and

$$\widehat{P}_{T,\mathcal{P}}(A) = \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \zeta_{\mathcal{P}}(s+i\tau,\omega) \in A\}$$

as $T \to \infty$, where $A \in \mathcal{B}(H(D))$, $\omega \in \widehat{\Omega}_{\mathcal{P}}$.

We start with a limit lemma on $\Omega_{\mathcal{P}}$. For $A \in \mathcal{B}(\Omega_{\mathcal{P}})$, define

$$P_{T,\mathcal{P}}^{\Omega_{\mathcal{P}}}(A) = \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \left(p^{-i\tau} : p \in \mathcal{P}\right) \in A\right\}.$$

Lemma 7. *Suppose that the set $L(\mathcal{P})$ is linearly independent over $\mathbb{Q}$. Then $P_{T,\mathcal{P}}^{\Omega_{\mathcal{P}}}$ converges weakly to the Haar measure $m_{\mathcal{P}}$ as $T \to \infty$.*

Proof. In the proof of Lemma 3, we have seen that characters of the group $\Omega_{\mathcal{P}}$ are given by (14). Therefore, the Fourier transform $F_{T,\mathcal{P}}(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z}, p \in \mathcal{P})$ of $P_{T,\mathcal{P}}^{\Omega_{\mathcal{P}}}$ is defined by

$$F_{T,\mathcal{P}}(\underline{k}) = \int_{\Omega_{\mathcal{P}}} \sideset{}{^*}\prod_{p\in\mathcal{P}} \omega^{k_p}(p)\,\mathrm{d}P_{T,\mathcal{P}}^{\Omega_{\mathcal{P}}} = \frac{1}{T}\int_0^T \left(\sideset{}{^*}\prod_{p\in\mathcal{P}} p^{-i\tau k_p}\right)\mathrm{d}\tau$$
$$= \frac{1}{T}\int_0^T \exp\left\{-i\tau \sideset{}{^*}\sum_{p\in\mathcal{P}} k_p\log p\right\}\mathrm{d}\tau. \tag{24}$$

We have to show that

$$\lim_{T\to\infty} F_{T,\mathcal{P}}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}. \end{cases} \tag{25}$$

For this, we apply the linear independence of the set $L(\mathcal{P})$. We have

$$A_{\mathcal{P}}(\underline{k}) \stackrel{\text{def}}{=} \sum_{p\in\mathcal{P}}^{*} k_p \log p = 0$$

if and only if $\underline{k}_p = \underline{0}$. Thus, (24),

$$F_{T,\mathcal{P}}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ \frac{1-\exp\{-iT\,A_{\mathcal{P}}(\underline{k})\}}{iT\exp\{-i\,A_{\mathcal{P}}(\underline{k})\}} & \text{otherwise,} \end{cases}$$

and (25) take place. □

The next lemma is devoted to the functions $\zeta_{\mathcal{P},n}(s)$ and $\zeta_{\mathcal{P},n}(s,\omega)$. For $A \in \mathcal{B}(H(D))$, set

$$P_{T,\mathcal{P},n}(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \zeta_{\mathcal{P},n}(s+i\tau) \in A\}$$

and

$$\widehat{P}_{T,\mathcal{P},n}(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \zeta_{\mathcal{P},n}(s+i\tau,\omega) \in A\}.$$

Lemma 8. *Suppose that the set $L(\mathcal{P})$ is linearly independent over $\mathbb{Q}$. Then, on $(H(D),\mathcal{B}(H(D)))$ there exists a probability measure $P_{\mathcal{P},n}$ such that both the measures $P_{T,\mathcal{P},n}$ and $\widehat{P}_{T,\mathcal{P},n}$ converge weakly to $P_{\mathcal{P},n}$ as $T\to\infty$.*

Proof. We use a property of the preservation of weak convergence under continuous mappings. Consider the mapping $v_{\mathcal{P},n} : \Omega_{\mathcal{P}} \to H(D)$ given by

$$v_{\mathcal{P},n}(\omega) = \zeta_{\mathcal{P},n}(s,\omega).$$

Since the series for $\zeta_{\mathcal{P},n}(s,\omega)$ is absolutely convergent for $\sigma > \widehat{\sigma}$, the mapping $v_{\mathcal{P},n}$ is continuous. Moreover, for $A \in \mathcal{B}(H(D))$,

$$P_{T,\mathcal{P},n}(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \Big(p^{-i\tau} : p \in \mathcal{P}\Big) \in v_{\mathcal{P},n}^{-1}A\Big\} = P_{T,\mathcal{P}}^{\Omega_{\mathcal{P}}}\Big(v_{\mathcal{P},n}^{-1}A\Big).$$

Thus, denoting by $P_{T,\mathcal{P}}^{\Omega_{\mathcal{P}}} v_{\mathcal{P},n}^{-1}$ the measure given by the latter equality, we obtain that $P_{T,\mathcal{P},n} = P_{T,\mathcal{P}}^{\Omega_{\mathcal{P}}} v_{\mathcal{P},n}^{-1}$. This equality continuity of $v_{\mathcal{P},n}$, and the principle of preservation of weak convergence, see Theorem 5.1 of [28], show that $P_{T,\mathcal{P},n}$ converges weakly to the measure $Q_{\mathcal{P},n} \stackrel{\text{def}}{=} m_{\mathcal{P}} v_{\mathcal{P},n}^{-1}$ as $T\to\infty$.

Define one more mapping $\widehat{v}_{\mathcal{P},n} : \Omega_{\mathcal{P}} \to H(D)$ by

$$\widehat{v}_{\mathcal{P},n}(\widehat{\omega}) = \zeta_{\mathcal{P},n}(s,\omega\widehat{\omega}), \quad \widehat{\omega} \in \Omega_{\mathcal{P}}.$$

Then, repeating the above arguments, we find that $\widehat{P}_{T,\mathcal{P},n}$ converges weakly to $\widehat{Q}_{\mathcal{P},n} \stackrel{\text{def}}{=} m_{\mathcal{P}}\widehat{v}_{\mathcal{P},n}^{-1}$. Let $v_{\mathcal{P}}(\widehat{\omega}) = \omega\widehat{\omega}$. Then, by invariance of the measure $m_{\mathcal{P}}$, we have

$$\widehat{Q}_{\mathcal{P},n} = m_{\mathcal{P}}(v_{\mathcal{P},n}v_{\mathcal{P}})^{-1} = \Big(m_{\mathcal{P}}v_{\mathcal{P}}^{-1}\Big)v_{\mathcal{P},n}^{-1} = m_{\mathcal{P}}v_{\mathcal{P},n}^{-1} = Q_{\mathcal{P},n}.$$

Thus, $P_{T,\mathcal{P},n}$ and $\widehat{P}_{T,\mathcal{P},n}$ converge weakly to the same measure $Q_{\mathcal{P},n}$ as $T\to\infty$. □

Next, we study the family of probability measures $\{Q_{\mathcal{P},n} : n \in \mathbb{N}\}$. We recall some notions. A family of probability measures $\{P\}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is called tight if, for every $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{X}$ such that

$$P(K) > 1 - \varepsilon$$

for all P, and $\{P\}$ is relatively compact if every sequence $\{P_k\} \subset \{P\}$ has a subsequence $\{P_{n_k}\}$ weakly convergent to a certain probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ as $k \to \infty$. By the classical Prokhorov theorem, see Theorem 6.1 of [28], every tight family of probability measures is relatively compact.

Lemma 9. *Under the hypotheses of Theorem 2, the family $\{Q_{\mathcal{P},n} : n \in \mathbb{N}\}$ is relatively compact.*

Proof. In view of the above remark, it suffices to prove the tightness of $\{Q_{\mathcal{P},n}\}$. Let $K \subset D$ be a compact. Then, using the Cauchy integral formula and absolute convergence of the series for $\zeta_{\mathcal{P},n}(s)$, we obtain $\sigma_\kappa > \widehat{\sigma}$

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} |\zeta_{\mathcal{P},n}(s+i\tau)|^2 \, d\tau \ll \sup_{n\in\mathbb{N}} \sum_{m\in\mathcal{N}_\mathcal{P}} \frac{a_n^2(m)}{m^{2\sigma_\kappa}} \ll \sum_{m\in\mathcal{N}_\mathcal{P}} \frac{1}{m^{2\sigma_\kappa}} \stackrel{\text{def}}{=} V_\kappa < \infty. \tag{26}$$

Suppose that ξ_T is a random variable on a certain probability space $(\Xi, \mathcal{A}, \mu)$ uniformly distributed in the interval $[0, T]$. Define the $H(D)$-valued random element

$$Y_{T,\mathcal{P},n} = Y_{T,\mathcal{P},n}(s) = \zeta_{\mathcal{P},n}(s + i\xi_T).$$

Then, denoting by $\xrightarrow{\mathcal{D}}$ the convergence in distribution by Lemma 8, we obtain

$$Y_{T,\mathcal{P},n} \xrightarrow[T\to\infty]{\mathcal{D}} Y_{\mathcal{P},n}, \tag{27}$$

where $Y_{\mathcal{P},n}(s)$ is the $H(D)$-valued random element with the distribution $Q_{\mathcal{P},n}$. Since the convergence in $H(D)$ is uniform on compact sets, (27) implies

$$\sup_{s\in K} |Y_{T,\mathcal{P},n}(s)| \xrightarrow[T\to\infty]{\mathcal{D}} \sup_{s\in K} |Y_{\mathcal{P},n}(s)|. \tag{28}$$

Now, let $K = K_l$, where $\{K_l\}$ is a sequence of compact sets of D from the definition of the metric ρ. Fix $\varepsilon > 0$, and set $R_l = 2^l \varepsilon^{-1} \sqrt{V_l}$ where $V_l = V_{\kappa_l}$. Therefore, relation (26), and the Chebyshev type inequality yield

$$\begin{aligned}
\limsup_{T\to\infty} \mu\left\{ \sup_{s\in K_l} |Y_{T,\mathcal{P},n}(s)| > R_l \right\} &\leqslant \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{TR_l} \int_0^T \sup_{s\in K_l} |\zeta_{\mathcal{P},n}(s+i\tau)| \, d\tau \\
&\leqslant \sup_{s\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{R_l} \left(\frac{1}{T} \int_0^T \sup_{s\in K_l} |\zeta_{\mathcal{P},n}(s+i\tau)|^2 \, d\tau \right)^{1/2} = \frac{\varepsilon}{2^l}.
\end{aligned}$$

Hence, in view of (28),

$$\mu\left\{ \sup_{s\in K_l} |Y_{\mathcal{P},n}(s)| > R_l \right\} \leqslant \frac{\varepsilon}{2^l}. \tag{29}$$

Define the set

$$H(\varepsilon) = \left\{ g \in H(D) : \sup_{s\in K_l} |g(s)| \leqslant R_l, l \in \mathbb{N} \right\}.$$

Then $H(\varepsilon)$ is a compact set in $H(D)$. Moreover, inequality (29) implies that

$$\mu\{Y_{\mathcal{P},n} \in H(\varepsilon)\} = 1 - \mu\{Y_{\mathcal{P},n} \notin H(\varepsilon)\} \geqslant 1 - \varepsilon \sum_{l=1}^{\infty} \frac{1}{2^l} = 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Since $Q_{\mathcal{P},n}$ is the distribution of $Y_{\mathcal{P},n}$, this shows that

$$Q_{\mathcal{P},n}(H(\varepsilon)) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$. The lemma is proved. □

Now, we are ready to consider the weak convergence for $P_{T,\mathcal{P}}$ and $\widehat{P}_{T,\mathcal{P}}$. For convenience, we recall one general statement.

Proposition 1. *Suppose that a metric space $(\mathbb{X}, d)$ is separable, and the $\mathbb{X}$-valued random elements x_{mn} and y_n, $m, n \in \mathbb{N}$ are defined on the same probability space $(\Xi, \mathcal{A}, \mu)$. Suppose that*

$$x_{mn} \xrightarrow[n\to\infty]{\mathcal{D}} x_m, \quad x_m \xrightarrow[m\to\infty]{\mathcal{D}} x,$$

and, for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mu\{d(x_{mn}, y_n) \geqslant \varepsilon\} = 0$$

Then

$$y_n \xrightarrow[n\to\infty]{\mathcal{D}} x.$$

Proof. The proposition is Theorem 4.2 of [28], where its proof is given. □

Lemma 10. *Under the hypotheses of Theorem 2, on $(H(D), \mathcal{B}(H(D)))$ there exists a probability measure $P_{\mathcal{P}}$ such that both the measures $P_{T,\mathcal{P}}$ and $\widehat{P}_{T,\mathcal{P}}$ converge weakly to $P_{\mathcal{P}}$ as $T \to \infty$.*

Proof. Let ξ_T be the same random variable as in the proof of Lemma 9. By Lemma 9, there exists a sequence $\{Q_{\mathcal{P},n_m}\} \subset \{Q_{\mathcal{P},n}\}$ and the probability measure $Q_{\mathcal{P}}$ on $(H(D), \mathcal{B}(H(D)))$ such that $Q_{\mathcal{P},n_m}$ converges weakly to $Q_{\mathcal{P}}$ as $m \to \infty$. In other words, in the notation of the proof of Lemma 9,

$$Y_{\mathcal{P},n_m} \xrightarrow[m\to\infty]{\mathcal{D}} Q_{\mathcal{P}}. \tag{30}$$

On $(\Xi, \mathcal{A}, \mu)$, define one more $H(D)$-valued random element

$$Y_{T,\mathcal{P}} = Y_{T,\mathcal{P}}(s) = \zeta_{\mathcal{P}}(s + i\xi_T).$$

Then the application of Lemma 5 gives, for $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{m\to\infty} \limsup_{T\to\infty} \mu\{\rho(Y_{T,\mathcal{P}}, Y_{T,\mathcal{P},n_m}) \geqslant \varepsilon\} \\
&\quad = \lim_{m\to\infty} \limsup_{T\to\infty} \frac{1}{T} \mathrm{meas}\{\tau \in [0,T] : \rho(\zeta_{\mathcal{P}}(s+i\tau), \zeta_{\mathcal{P},n_m}(s+i\tau)) \geqslant \varepsilon\} \\
&\quad \leqslant \lim_{m\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_0^T \rho(\zeta_{\mathcal{P}}(s+i\tau), \zeta_{\mathcal{P},n_m}(s+i\tau))\, \mathrm{d}\tau = 0.
\end{aligned}$$

This, and relations (27) and (30) show that all conditions of Proposition 1 are fulfilled. Thus, we have

$$Y_{T,\mathcal{P}} \xrightarrow[T\to\infty]{\mathcal{D}} Q_{\mathcal{P}}, \tag{31}$$

$P_{T,\mathcal{P}}$ converges weakly to $Q_{\mathcal{P}}$ as $T \to \infty$. Since the family $\{Q_{\mathcal{P},n}\}$ is relatively compact, relation (31), in addition, implies that

$$Y_{\mathcal{P},n} \xrightarrow[n\to\infty]{\mathcal{D}} Q_{\mathcal{P}}. \tag{32}$$

It remains to prove weak convergence for $\widehat{P}_{T,\mathcal{P}}$. On $(\Xi, \mathcal{A}, \mu)$, define the $H(D)$-valued random elements

$$\widehat{Y}_{T,\mathcal{P},n} = \widehat{Y}_{T,\mathcal{P},n}(s) = \zeta_{\mathcal{P},n}(s + i\xi_T, \omega)$$

and

$$\widehat{Y}_{T,\mathcal{P}} = \widehat{Y}_{T,\mathcal{P}}(s) = \zeta_{\mathcal{P}}(s + i\xi_T, \omega).$$

Lemma 8 implies the relation

$$\widehat{Y}_{T,\mathcal{P},n} \xrightarrow[T\to\infty]{\mathcal{D}} Q_{\mathcal{P}}, \tag{33}$$

while, in view of Lemma 6, for $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mu\left\{\rho\left(\widehat{Y}_{T,\mathcal{P}}, \widehat{Y}_{T,\mathcal{P},n}\right) \geqslant \varepsilon\right\} \\
&\qquad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_0^T \rho(\zeta_{\mathcal{P}}(s + i\tau, \omega), \zeta_{\mathcal{P},n}(s + i\tau, \omega))\, \mathrm{d}\tau = 0.
\end{aligned}$$

This, (32), (33) and Lemma 10 yield the relation

$$\widehat{Y}_{T,\mathcal{P}} \xrightarrow[T\to\infty]{\mathcal{D}} Q_{\mathcal{P}}.$$

Thus, $\widehat{P}_{T,\mathcal{P}}$, as $T \to \infty$, also converges weakly to $Q_{\mathcal{P}}$. □

It remains to identify the measure $Q_{\mathcal{P}}$. Denote by $P_{\zeta_{\mathcal{P}}}$ the distribution of the random element $\zeta_{\mathcal{P}}(s, \omega)$, i.e.,

$$P_{\zeta_{\mathcal{P}}}(A) = m_{\mathcal{P}}\{\omega \in \Omega_{\mathcal{P}} : \zeta_{\mathcal{P}}(s, \omega) \in A\}.$$

Theorem 3. *Under hypotheses of Theorem 2, $P_{T,\mathcal{P}}$ converges weakly to the measure $P_{\zeta_{\mathcal{P}}}$ as $T \to \infty$.*

Proof. We will show that the limit measure $Q_{\mathcal{P}}$ in Lemma 10 coincides with $P_{\zeta_{\mathcal{P}}}$.

We apply the equivalent of weak convergence of probability measures in terms of continuity sets, see Theorem 2.1 of [28]. Let A be a continuity set of the measure $Q_{\mathcal{P}}$, i.e., $Q_{\mathcal{P}}(\partial A) = 0$, where ∂A denotes the boundary of A. Then, Lemma 10 implies that

$$\lim_{T\to\infty} \widehat{P}_{T,\mathcal{P}}(A) = Q_{\mathcal{P}}(A). \tag{34}$$

On $(\Omega_{\mathcal{P}}, \mathcal{B}(\Omega_{\mathcal{P}}))$, define the random variable

$$\xi_{\mathcal{P}}(\omega) = \begin{cases} 0 & \text{if } \zeta_{\mathcal{P}}(s, \omega) \notin A, \\ 1 & \text{otherwise.} \end{cases}$$

Return to the group G_τ of Lemma 3. Since, by Lemma 3, the group G_τ is ergodic, the process $\xi(g_\tau(\omega))$ is ergodic, and application of the Birkhoff–Khintchine theorem [27] gives

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \xi_{\mathcal{P}}(g_\tau(\omega))\, \mathrm{d}\tau = \mathbb{E}\xi_{\mathcal{P}}(\omega) \tag{35}$$

for almost all $\omega \in \Omega_{\mathcal{P}}$. However, the definition of the random variable $\xi_T(\omega)$ implies that, for almost all $\omega \in \Omega_{\mathcal{P}}$,

$$\frac{1}{T}\int_0^T \xi_{\mathcal{P}}(g_\tau(\omega))\,\mathrm{d}\tau = \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \zeta_{\mathcal{P}}(s, g_\tau(\omega)) \in A\}$$
$$= \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \zeta_{\mathcal{P}}(s+i\tau, \omega) \in A\}.$$

Thus, by (34),

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \xi_{\mathcal{P}}(g_\tau(\omega))\,\mathrm{d}\tau = Q_{\mathcal{P}}(A). \tag{36}$$

Moreover,

$$\mathbb{E}\xi(\omega) = \int_{\Omega_{\mathcal{P}}} \xi_{\mathcal{P}}(\omega)\mathrm{d}m_{\mathcal{P}} = P_{\zeta_{\mathcal{P}}}(A).$$

This, (35) and (36) prove that $Q_{\mathcal{P}}(A) = P_{\zeta_{\mathcal{P}}}(A)$ for all continuity sets A of the measure $Q_{\mathcal{P}}$. It is well known that all continuity sets constitute a determining class. Hence, we have $Q_{\mathcal{P}} = P_{\zeta_{\mathcal{P}}}$, and the theorem is proved. □

6. Support

For the proof of Theorem 2, the explicitly given support of the measure $P_{\zeta_{\mathcal{P}}}$ is needed. We recall that the support of $P_{\zeta_{\mathcal{P}}}$ is a minimal closed set $S_{\mathcal{P}} \subset H(D)$ such that $P_{\zeta_{\mathcal{P}}}(S_{\mathcal{P}}) = 1$. Every open neighbourhood of elements $S_{\mathcal{P}}$ has a positive $P_{\zeta_{\mathcal{P}}}$-measure.

Define the set

$$S_{\mathcal{P}} = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Proposition 2. *Under the hypotheses of Theorem 2, the support of the measure $P_{\zeta_{\mathcal{P}}}$ is the set $S_{\mathcal{P}}$.*

A proof of Proposition 2 is similar to that in the case of the Riemann zeta-function. Therefore, we will state without proof only the lemmas because their proofs word for word coincide with analogical assertions from [22].

We start with some estimations over generalized primes $p \in \mathcal{P}$.

Lemma 11. *Suppose that the estimate (5) is valid. Then, for $x \to \infty$,*

$$\sum_{\substack{p\leqslant x\\ p\in\mathcal{P}}} \frac{1}{p} = \log\log x + a + O\Big(x^{\beta_2-1}\Big),$$

where a is a constant, and $0 \leqslant \beta_2 < 1$.

Proof. We have

$$\psi_1(x) \stackrel{\mathrm{def}}{=} \sum_{\substack{p\leqslant x\\ p\in\mathcal{P}}} \log p = \psi(x) - \sum_{\substack{p^\alpha\leqslant x\\ p\in\mathcal{P}}} \sum_{2\leqslant\alpha\leqslant(\log x)/(\log 2)} \log p$$
$$= \psi(x) + O\Big(\psi\Big(x^{1/2}\Big)\log x\Big) = x + r(x),$$

where

$$r(x) = O\Big(x^{\beta_2}\log x\Big)$$

with

$$\beta_2 = \max\Big(\beta_1, \frac{1}{2}\Big).$$

From this, by partial summation, we obtain

$$\sum_{\substack{p\leqslant x\\ p\in\mathcal{P}}}\frac{1}{p}=\frac{1}{x\log x}\sum_{\substack{p\leqslant x\\ p\in\mathcal{P}}}\log p+\int\limits_{p_1}^{x}\left(\frac{1}{u^2\log u}+\frac{1}{u^2\log^2 u}\right)\psi_1(u)\,\mathrm{d}u$$
$$=\frac{1}{\log x}+\log\log x-\frac{1}{\log x}+c_1+\int\limits_{p_1}^{x}\left(\frac{1}{u^2\log u}+\frac{1}{u^2\log^2 u}\right)r(u)\,\mathrm{d}u$$
$$=\log\log x+c_1+\int\limits_{p_1}^{\infty}\left(\frac{1}{u^2\log u}+\frac{1}{u^2\log^2 u}\right)r(u)\,\mathrm{d}u$$
$$-\int\limits_{x}^{\infty}\left(\frac{1}{u^2\log u}+\frac{1}{u^2\log^2 u}\right)r(u)\,\mathrm{d}u$$
$$=\log\log x+c_2+O\left(\int\limits_{x}^{\infty}u^{\beta_1-2}\,\mathrm{d}u\right)=\log\log x+c_2+O(\left(x^{\beta_2-1}\right).$$

□

In what follows, we will use some properties of functions of exponential type. We recall a function $g(s)$ analytic in the region $|\arg s|\leqslant\theta_0$, $0<\theta_0\leqslant\pi$ is of exponential type if uniformly in θ, $\theta\leqslant\theta_0$,

$$\limsup_{r\to\infty}\frac{\log|g(re^{i\theta})|}{r}<\infty.$$

Lemma 12. *Suppose that $g(s)$ is an entire function of exponential type, (5) holds, and*

$$\limsup_{r\to\infty}\frac{\log|g(r)|}{r}>-1.$$

Then

$$\sum_{p\in\mathcal{P}}|g(\log p)|=\infty.$$

Proof. We use the formula of Lemma 11, and repeat word for word the proof of Theorem 6.4.14 of [22]. □

Let $s\in D$, and $|a_p|=1$. For brevity, we set

$$g_{\mathcal{P}}(s,a_p)=\log\left(1-\frac{a_p}{p^s}\right),\quad p\in\mathcal{P},$$

where

$$\log\left(1-\frac{a_p}{p^s}\right)=-\frac{a_p}{p^s}-\frac{a_p^2}{2p^{2s}}-\cdots.$$

Lemma 13. *Suppose that (5) holds. Then the set of all convergent series*

$$\sum_{p\in\mathcal{P}}g_{\mathcal{P}}(s,a_p)$$

is dense in the space $H(D)$.

Proof. The object connected to the system $\mathcal{P}$ is only Lemma 12. Other arguments of the proof are the same as those applied in the proof of Lemma 6.5.4 from [22]. □

Recall that the support of the distribution of a random element X is called a support of X, and is denoted by S_X.

For convenience, we state a lemma on the support of a series of random elements.

Lemma 14. *Let $\{\xi_m\}$ be a sequence of independent $H(D)$-valued random elements on a certain probability space $(\Xi, \mathcal{A}, \mu)$; the series*

$$\sum_{m=1}^{\infty} \xi_m$$

is convergent almost surely. Then, the support of the sum of this series is the closure of the set of all $g \in H(D)$ which may be written as a convergent series

$$g = \sum_{m=1}^{\infty} g_m, \quad g_m \in S_{\xi_m}.$$

Proof. The lemma is Theorem 1.7.10 of [22], where its proof is given. □

Proof of Proposition 2. By the definition, $\{\omega(p) : p \in \mathcal{P}\}$ is a sequence of independent complex-valued random variables. Therefore, $\{g_{\mathcal{P}}(s, \omega(p))\}$ is a sequence of independent $H(D)$-valued random elements. Since the support of each $\omega(p)$ is the unit circle, the support of $g_{\mathcal{P}}(s, \omega(p))\}$ is the set

$$\left\{ g \in H(D) : g(s) = -\log\left(1 - \frac{a}{p^s}\right), |a| = 1 \right\}.$$

Therefore, in view of Lemma 14, the support of the $H(D)$-valued random element

$$\log \zeta_{\mathcal{P}}(s, \omega) = -\sum_{p \in \mathcal{P}} \log\left(1 - \frac{\omega(p)}{p^s}\right)$$

is the closure of the set of all convergent series

$$\sum_{p \in \mathcal{P}} g_{\mathcal{P}}(s, a_p)$$

with $|a_p| = 1$. By Lemma 13, the set of the latter series is dense in $H(D)$. Define $u : H(D) \to H(D)$ by $u(g) = e^g$, $g \in H(D)$. The mapping u is continuous, $u(\log \zeta_{\mathcal{P}}(s, \omega)) = \zeta_{\mathcal{P}}(s, \omega)$ and $u(H(D)) = S_{\mathcal{P}} \setminus \{0\}$. This shows that $S_{\mathcal{P}} \setminus \{0\}$ lies in the support of $\zeta_{\mathcal{P}}(s, \omega)$. Since the support is a closed set, we obtain that the support of $\zeta_{\mathcal{P}}(s, \omega)$ contains the closure of $S_{\mathcal{P}} \setminus \{0\}$, i.e.,

$$S_{\zeta_{\mathcal{P}}} \supset S_{\mathcal{P}}. \tag{37}$$

On the other hand, the random element $\zeta_{\mathcal{P}}(s, \omega)$ is convergent for almost all $\omega \in \Omega_{\mathcal{P}}$, a product of non-zeros multipliers. Therefore, by the classical Hurwitz theorem, see [29],

$$S_{\zeta_{\mathcal{P}}} \subset S_{\mathcal{P}}.$$

This inclusion together with (37) proves the proposition. □

7. Proof of Universality

In this section, we prove Theorem 2. Its proof is based on Theorem 3, Proposition 2 and the Mergelyan theorem [30] on the approximation of analytic functions by polynomials on compact sets with connected complements.

Proof of Theorem 2. Let $p(s)$ be a polynomial, K and ε defined in Theorem 2, and

$$\mathcal{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} \left| g(s) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\}.$$

Then, the set $\mathcal{G}_\varepsilon$ is an open neighborhood of an element $e^{p(s)} \in S_\mathcal{P}$. Since, in view of Proposition 2, $S_\mathcal{P}$ is the support of the measure $P_{\zeta_\mathcal{P}}$, by a property of supports, we have

$$P_{\zeta_\mathcal{P}}(\mathcal{G}_\varepsilon) > 0. \tag{38}$$

Since $f(s) \in H_0(K)$, we may apply the mentioned Mergelyan theorem and choose the polynomial $p(s)$ satisfying

$$\sup_{s\in K}\left|f(s) - e^{p(s)}\right| < \frac{\varepsilon}{2}.$$

This shows that the set $\mathcal{G}_\varepsilon$ lies in

$$\widehat{\mathcal{G}}_\varepsilon = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Thus, by (38), we have

$$P_{\zeta_\mathcal{P}}(\widehat{\mathcal{G}}_\varepsilon) > 0. \tag{39}$$

Theorem 3 and the equivalent of weak convergence in terms of open sets yield

$$\liminf_{T\to\infty} P_{T,\mathcal{P}}(\widehat{\mathcal{G}}_\varepsilon) \geqslant P_{\zeta_\mathcal{P}}(\widehat{\mathcal{G}}_\varepsilon).$$

This, (39), and the definitions of $P_{T,\mathcal{P}}$ and $\widehat{\mathcal{G}}_\varepsilon$ prove the first statement of the theorem.

To prove the second statement of the theorem, we observe that the boundary $\partial\widehat{\mathcal{G}}_\varepsilon$ of the set $\widehat{\mathcal{G}}_\varepsilon$ lies in the set

$$\left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon \right\}.$$

Hence, the boundaries $\partial\widehat{\mathcal{G}}_{\varepsilon_1}$ and $\partial\widehat{\mathcal{G}}_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. Therefore, $P_{\zeta_\mathcal{P}}(\partial\widehat{\mathcal{G}}_\varepsilon) > 0$ for countably many $\varepsilon > 0$. In other words, the set $\widehat{\mathcal{G}}_\varepsilon$ is a continuity set of the measure $P_{\zeta_\mathcal{P}}$ for all but at most countably many $\varepsilon > 0$. This, (39), Theorem 3 and the equivalent of weak convergence in terms of continuity sets prove the second statement of the theorem. □

8. Conclusions

In the paper, we considered the set $\mathcal{P}$ of generalized prime numbers satisfying

$$\sum_{\substack{m\leqslant x\\ m\in\mathcal{N}_\mathcal{P}}} 1 = ax + O\left(x^\beta\right), \quad a > 0,\ 0 \leqslant \beta < 1,$$

and

$$\sum_{\substack{m\leqslant x\\ m\in\mathcal{N}_\mathcal{P}}} \Lambda_\mathcal{P}(m) = x + O\left(x^{\beta_1}\right), \quad 0 \leqslant \beta_1 < 1,$$

where $\mathcal{N}_\mathcal{P}$ is the set of generalized integers and $\Lambda_\mathcal{P}(m)$ is the generalized von Mangoldt function corresponding to the set $\mathcal{P}$. Assuming that the set $\{\log p : p \in \mathcal{P}\}$ is linearly independent over $\mathbb{Q}$, and the Beurling zeta-function

$$\zeta_\mathcal{P}(s) = \sum_{m\in\mathcal{N}_\mathcal{P}} \frac{1}{m^s}, \quad s = \sigma + it,\ \sigma > 1,$$

has the bounded mean square for $\sigma > \widehat{\sigma}$ with some $\beta < \widehat{\sigma} < 1$, we obtained universality of $\zeta_\mathcal{P}(s)$, i.e., that every non-vanishing analytic function can be approximated by shifts $\zeta_\mathcal{P}(s + i\tau)$, $\tau \in \mathbb{R}$.

In the future, we are planning to obtain a more complicated discrete version of Theorem 2, i.e., to prove the approximation of analytic functions by discrete shifts $\zeta_{\mathcal{P}}(s + ikh)$, $h > 0$, $k \in \mathbb{N}_0$.

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